Sequences

Convergence
For $(x_n) \in \mathbb{R}$. $x_n \to a \in \mathbb{R}$ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |x_n - a| < \varepsilon.$$

- Every convergent sequence is bounded.

Cauchy Convergence
$(x_n) \in \mathbb{R}$ is Cauchy if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \quad \text{for all } n, m \geq N.$$

Cauchy Theorem
(x_n) is Cauchy $\iff$ (x_n) is convergent.

Triangle Inequality

$$|x+y| \leq |x| + |y|$$

$$|x-y| \geq ||x| - |y||$$

$$|x-y| \leq |x-z| + |y-z|$$

Subsequences
A subsequence of (x_n) is a sequence $(x_{n_k})_{k\in\mathbb{N}}$ which is a selection of some (possibly all) of the x_n 's taken in order.

Bolzano-Weierstrass
Every bounded $(x_n) \in \mathbb{R}$ has a convergent subsequence.

limsup liminf
If $(x_n) \in \mathbb{R}$ is bounded:

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right),$$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right).$$

also

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = L \implies (x_n) \to L$$

Monotone Convergence Theorem
For $(x_n) \in \mathbb{R}$
monotone increasing and bounded above:
$\implies \lim_{n\to\infty} x_n = \sup_{n\geq 1} x_n$
monotone decreasing and bounded below:
$\implies \lim_{n\to\infty} x_n = \inf_{n\geq 1} x_n$

Series

Let $S = \sum_{k=1}^{\infty} a_k$, the partial sum is

$$s_n = \sum_{k=1}^{n} a_k.$$

S converges if and only if (s_n) converges where

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$$

S is absolutely convergent if $\sum_{k=1}^{\infty} |a_k|$ is convergent, otherwise S is just conditionally convergent.

Cauchy criterion for series
Let $S = \sum_{k=1}^{\infty} a_k$.
S is convergent if and only if $\forall \varepsilon > 0$, $\exists N$

$$m \geq n \geq N \implies \left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon$$

Absolute Convergence
Let $S = \sum_{k=1}^{\infty} a_k$ be absolutely convergent:
(a) The series S is convergent.
(b) (Rearrangement) Let $z : \mathbb{N} \to \mathbb{N}$ be a bijection. Then the series $\sum_{k=1}^{\infty} a_{z(k)}$ is convergent and

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{z(k)}$$

Conditional Convergence
Let $S = \sum_{k=1}^{\infty} a_k$ be conditionally convergent. There exist rearrangements $z : \mathbb{N} \to \mathbb{N}$ (where z is a bijection) such that for $S_z = \sum_{k=1}^{\infty} a_{z(k)}$
(a) $S_z = r \in \mathbb{R}$ for any r.
(b) S_z diverges to $+\infty$.
(c) S_z diverges to $-\infty$.
(d) The partial sums of S_z oscillate between any two real numbers.

Ratio Test
Let $S = (r_n)$. S converges if and only if

$$\lim_{n\to\infty} |a_{n+1}/a_n| < 1$$

Continuous Functions

Continuity
For $D = dom(f) \subset \mathbb{R}$ let $f : D \to \mathbb{R}$.
f is continuous at $a \in D$ if and only if

$\exists\, (x_n) \in D$ such that

$$\lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = f(a)$$

$$AND/OR$$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|x-a| < \delta \implies |f(x) - f(a)| < \epsilon$$

Uniform Continuity
Let $I \subset \mathbb{R}$ be an *interval* and $f : I \to \mathbb{R}$. f is uniformly continuous on I if $\forall \epsilon > 0$, $\exists \delta > 0$ such that for $x, y \in I$

$$|x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Prooving Uniform Continuity

- Any f defined on a closed interval is uniformly continuous if it is continuous.
- If I is an open interval, f is differentiable on I and f' is bounded, then f is uniformly continuous.
- If I is any interval, f is uniformly continuous on I if and only if whenever $s_n, t_n \in I$ are such that $|s_n - t_n| \to 0$, then $|f(s_n) - f(t_n)| \to 0$.

Combining Continuous Functions
Let $f, g : D \to \mathbb{R}$ be continuous on D, and let $\alpha \in \mathbb{R}$ then αf, $f+g$, fg, $f \circ g$ are continuous on D.

Mean Value Theorem
If f is continuous on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b)$ such that

$$(b-a)f'(c) = f(b) - f(a)$$

Intermediate Value Theorem
Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$.

$$f(a)f(b) < 0 \implies \exists c \in (a,b) : f(c) = 0$$

Extreme value theorem
Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. f is *bounded* on $[a,b]$ and $\exists c, d \in [a,b]$ such that

$$f(c) = \inf_x f(x)$$

$$f(d) = \sup_x f(x)$$

Sequences of Functions

Pointwise Convergence
Let E be a nonempty subset of $\mathbb{R}$.
$f_n : E \to \mathbb{R}$ converges pointwise on E if and only if $\forall \epsilon > 0$, some $x \in E$ and $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon.$$

The limit function does not have to be the same for all x.
If f_n is continuous/differentiable/integrable the pointwise limit is not necessarily continuous/differentiable/integrable.

Uniform Convergence
Let E be a nonempty subset of $\mathbb{R}$.
$f_n : E \to \mathbb{R}$ converges uniformly on E to a function f if and only if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$. $f(x)$ is the same for all x.
Equivalently
- $\sup_{x\in E} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.
- There exists a sequence $a_n \to 0$ such that $|f_n(x) - f(x)| \leq a_n$ for all $x \in E$.

If f is uniformly convergent only for $x \in [a,b]$, we can write $\mathbb{R} = \bigcup_{n=1}^{\infty}[-n,n]$ and we have that f is pointwise convergent on $\mathbb{R}$.

Unif. Conv. $\implies$ Continuity
If $f_n \to f$ uniformly on E and each f_n is continuous at some $x_0 \in E$, then f is continuous at x_0.

Unif. Conv. $\implies$ Integrable Limits
Suppose that $f_n \to f$ uniformly on a closed interval $[a,b]$. If each f_n is integrable on $[a,b]$, then so is f and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right) dx.$$

And
$$\int_a^x f_n(t)dt \to \int_a^x f(t)dt$$
uniformly for $x \in [a,b]$.

Unif. Conv. and Derivative of Limit

Let each f_n converge at some $x_0 \in (a,b)$ and be differentiable on (a,b).
If f_n' converges uniformly on (a,b) then f_n converges uniformly on (a,b) and
$$\lim_{n\to\infty} f_n'(x) = \left(\lim_{n\to\infty} f_n(x)\right)'$$
for each $x \in (a,b)$.

Series of Functions

Convergence of Series of Functions

Let f_k be a sequence of real functions defined on some set E
$$s_n(x) = \sum_{k=1}^{n} f_k(x), \quad x \in E, n \in \mathbb{N}.$$
1. The series $\sum_{k=1}^{\infty} f_k$ is said to converge pointwise if and only if the sequence $s_n(x)$ converges pointwise on E
2. The series $\sum_{k=1}^{\infty} f_k$ is said to converge uniformly if and only if the sequence $s_n(x)$ converges uniformly on E
3. The series $\sum_{k=1}^{\infty} f_k$ is said to converge absolutely (pointwise) on E if and only if $\sum_{k=1}^{\infty} |f_k(x)|$ converges for each $x \in E$.

Continuity of Limits of Series

Suppose that $x_0 \in E$ and that each f_k is continuous at $x_0 \in E$.
If $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on E, then f is continuous at $x_0 \in E$.

Term-by-term Integration

If f_k is integrable on a closed interval $[a,b]$ and $f = \sum_{k=1}^{\infty} f_k$ converges uniformly (i.e. is continuous) on $[a,b]$,
then f is integrable on $[a,b]$ and
$$\int_a^b \sum_{k=1}^{\infty} f_k(x)dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)dx.$$

Term-by-term Differentiation

Suppose that E is and open interval. If:
- each f_k is differentiable on E.
- $\sum_{k=1}^{\infty} f_k(x_0)$ converges at some $x_0 \in E$,
- $g = \sum_{k=1}^{\infty} f_k'$ converges uniformly on E,

then $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on E, is differentiable on E, and
$$f'(x) = \left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f_k'(x) = g(x)$$
for $x \in E$.

Weierstrass M-Test

Let E be a nonempty subset of $\mathbb{R}$ and $f_k : E \to \mathbb{R}$.
$$\sum_{k=1}^{\infty} M_k < \infty \text{ and } |f_k(x)| \le M_k$$
Then $f = \sum_{k=1}^{\infty} f_k$ converges absolutely and uniformly on E.

Power Series

Let $(a_n) \in \mathbb{R}$ and $c \in \mathbb{R}$.
$$\sum_{n=0}^{\infty} a_n(x-c)^n$$
is a power series.
Often $c = 0$. The series will always converge to x when $x = c$.

Radius of Convergence

The radius of convergence, R, determines the interval around c where the power series converges.
Power series converge absolutely within R, but convergence at the boundary points $(c \pm R)$ must be checked separately.
To calculate R:
- Ratio Test:
$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
The radius of convergence is $R = \frac{1}{L}$. If $L = 0$, the radius is infinite (the series converges everywhere). If $L = \infty$, the radius is zero (it converges only at c).
- Root Test:
$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$

A special case of a power series is the geometric series, where $c = 0$ and $a_n = r^n$. It converges when $|r| < 1$ and
$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Continuity of Power Series

Let $R > 0$ and $0 < r < R$.
A power series converges uniformly and absolutely on $|x-c| \le r$ to a continuous function f. Hence
$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$
Is continuous on $(c-R, c+R)$.

Differentiability of Power Series
$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$
is infinitely differentiable on $|x-c| < R$, and for such x,
$$f'(x) = \sum_{n=0}^{\infty} na_n(x-c)^{n-1}$$
The series converges absolutely and uniformly on $[c-r, c+r]$ for any $r < R$. Moreover
$$a_n = \frac{f^{(n)}(c)}{n!}.$$
$f'(x)$ has the same R as $f(x)$.

Step Functions

$\phi : \mathbb{R} \to \mathbb{R}$ is a step function if there exist real numbers $x_0 < x_1 < \cdots < x_n$ such that
$$\phi(x) = \sum_{j=1}^{n} c_j \chi_{(x_{j-1}, x_j)}(x)$$
$$\int \phi := \sum_{j=1}^{n} c_j (x_j - x_{j-1}).$$

Lebesgue Integrals

A function $f : I \to \mathbb{R}$ is Lebesgue integrable on an interval I if there exist numbers c_j and bounded intervals such that
$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$$
(i.e. the sum is absolutely convergent)
$$f(x) = \sum_{j=1}^{\infty} c_j \chi_{J_j}(x)$$
for all $x \in I$ at which
$$\sum_{j=1}^{\infty} |c_j| \chi_{J_j}(x) < \infty$$
We denote by $\int_I f$ the number
$$\int_I f = \sum_{j=1}^{\infty} c_j \lambda(J_j)$$

Prooving Integrability

- f is continuous on $[a,b] \implies f$ is integrable on $[a,b]$.
- g is integrable on $[0,\infty)$ if and only if g is integrable on $[0,v)$ for all $v < \infty$ and
$$\sup_{v>0} \int_0^v |g(x)|dx < \infty$$
- f be a measurable function on I and assume that $|f(x)| \le g(x)$ for almost every $x \in I$, where g is an integrable function on I. Then f is integrable on I.

Properties of Lebesgue Integral

Suppose f and g are Lebesgue integrable on I and $\alpha, \beta \in \mathbb{R}$
(a) $\alpha f + \beta g$ is integrable on I and
$$\int_I (\alpha f + \beta g) = \alpha \int_I f + \beta \int_I g.$$
(b) If $f \ge 0$ on I then $\int_I f \ge 0$; if $f \ge g$ on I then $\int_I f \ge \int_I g$.
(c) $|f|$ is integrable on I and $\left|\int_I f\right| \le \int_I |f|$.
(d) $\max\{f,g\}$ and $\min\{f,g\}$ are integrable on I.
(e) If one of the functions is bounded then the product fg is integrable on I.
(f) If $f \ge 0$ with $\int_I f = 0$ then any function h such that $0 \le h \le f$ on I is integrable on I.

Integration on a Sub-Interval

Let I and J be two intervals such that $J \subset I$.
(a) If f is integrable on I then f is also

integrable on the subinterval J.
(b) If f is integrable on J and simultaneously $f(x) = 0$ for all $x \in I \backslash J$ then f is integrable on I and

$$\int_J f = \int_I f$$

(c) If f is integrable on I and $f(x) \geq 0$ for all $x \in I$ then

$$\int_J f \leq \int_I f.$$

(d) Suppose that I can be written as the union of disjoint intervals $I_n, n = 1, 2, 3, \ldots$ and let f be integrable on each of the intervals I_n. Then f is integrable on I if and only if

$$\sum_{n=1}^{\infty} \int_{I_n} |f| < \infty.$$

If this holds then

$$\int_I f = \sum_{n=1}^{\infty} \int_{I_n} f.$$

Adding Integrals on a Sub-Interval
If any two of these integrals

$$\int_a^b f, \quad \int_b^c f, \quad \int_a^c f,$$

exist then so does the third and

$$\int_a^b f + \int_b^c f = \int_a^c f$$

must hold.

Integrating Infinite Series

Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions each of which is integrable on I.
(a) Assume that

$$\sum_{n=1}^{\infty} \int_I |f_n| < \infty.$$

Let f be a function on the interval I such that

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{for all } x \in I$$

such that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$
Then f is integrable on I and its integral on I equals to

$$\int_I f = \sum_{n=1}^{\infty} \int_I f_n$$

(b) Assume that each $f_n \geq 0$ on I and let $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for all $x \in I$ (we allow for the possibility that at some points this sum is infinite). Then f is integrable on I if and only if

$$\sum_{n=1}^{\infty} \int_I f_n < \infty$$

Monotone Convergence Theorem (Integrals)
Suppose that (f_n) is a monotone nondecreasing sequence of integrable functions on an interval I. That is $f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots$ for all $x \in I$. For all $x \in I$ let

$$f(x) = \lim_{n\to\infty} f_n(x),$$

where we allow for the possibility that at some points this limit is infinite. Then f is integrable on I of and only if

$$\sup_{n\in\mathbb{N}} \int_I f_n = \lim_{n\to\infty} \int_I f_n < \infty$$

Also:

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

Fatoux Lemma
Let (f_n) be a sequence of nonnegative integrable functions on an interval I. Let

$$f(x) = \liminf_{n\to\infty} f_n(x), \quad \text{for all } x \in I.$$

If $\liminf_{n\to\infty} \int_I f_n < \infty$ then f is integrable on I and

$$\int_I f \leq \liminf_{n\to\infty} \int_r.$$

Dominated convergence theorem
Let (f_n) be a sequence of on integrable functions on an interval I and assume that

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \text{for all } x \in I.$$

Assume also that the sequence (f_n) is dominated by some integrable function g, that is

$$|f_n(x)| \leq g(x), \quad \text{for all } x \in I$$

$$\text{and } n = 1, 2, \ldots, \quad \int_I g < \infty.$$

Then the function f is integrable on I and

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

Uniform Convergence and Integrals
Suppose $f_n : (a, b) \to \mathbb{R}$ are integrable and $f_n \to f$ uniformly. Then f is integrable on (a, b) and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

Riemann Integrals

Let $f : [a, b] \to \mathbb{R}$. f is Riemann-integrable if either

- $\forall \epsilon > 0$ there exist step functions ϕ and ψ such that

$$\phi \leq f \leq \psi$$

and

$$\int \psi - \int \phi < \epsilon.$$

- $\forall \epsilon > 0$ there exist $a = x_0 < \cdots < x_n = b$ and $I_j = (x_{j-1}, x_j)$ such that,
if M_j and m_j denote the supremum and infimimum values of f on I_j, then

$$\sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \epsilon.$$

equivelently

$$\sum_{j=1}^{n} \sup_{x,y\in I_j} |f(x) - f(y)| \lambda(I_j) < \epsilon$$

Let $g : [a, b] \to \mathbb{R}$ and $f(x) = g(x)$ for $x \in [a, b]$ and $f(x) = 0$ otherwise.

- If g is continuous on $[a, b]$ or (a, b), then f is Riemann-integrable.
- If g is a monotone function then f is Riemann-integrable.

Riemann and Lebesgue Integrable
Let $I = (a, b)$, if there exists points $a = x_0 < x_1 < \ldots x_n = b$ such that $f : I \to \mathbb{R}$ is bounded and continuous on each subinterval (x_j, x_{j+1}). Then f is both Riemann and Lebesgue integrable.

FTCs

- Let $g : I \to \mathbb{R}$ be integrable on an interval I. For all $x \in I$ and some fixed $x_0 \in I$ define

$$G(x) = \int_{x_0}^{x} g.$$

Suppose g is continuous at x for some $x \in I$. Then G is differentiable at x and $G'(x) = g(x)$.

- Suppose $f : I \to \mathbb{R}$ has continuous derivative f' on the interval I. Then $\forall a, b \in I$

$$\int_a^b f' = f(b) - f(a)$$

Fourier Series

Integration on $\mathbb{C}$
Let $f = g + ih$ where $g, h : [a, b] \to \mathbb{R}$. f is Lebesgue integrable if g and h are Lebesgue integrable and

$$\int_a^b f = \int_a^b g + i \int_a^b h$$

The Space L^2
The space $L^2 = L^2([a, b])$ is the set of measurable functions $f : [a, b] \to \mathbb{C}$ so that the function $x \mapsto |f(x)|^2$ is Lebesgue integrable, i.e.

$$\|f\|_2^2 := \int_a^b |f(x)|^2 dx < \infty.$$

The quantity $\|f\|_2$ is called the L^2-norm of f. If $\|f\|_2 = 1$, then we say that f is L^2-normalized.

Inner Product
For two functions $f, g \in L^2([a,b])$ we define their inner product by

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

Properties:
For $f, g, h \in L^2$ and $\lambda \in \mathbb{C}$:
- Sesquilinearity:

$$\langle f + \lambda g, h\rangle = \langle f, h\rangle + \lambda\langle g, h\rangle,$$
$$\langle h, f + \lambda g\rangle = \langle h, f\rangle + \bar{\lambda}\langle h, g\rangle.$$

- Antisymmetry: $\langle f, g\rangle = \overline{\langle g, f\rangle}$
- Positivity: $\|f\|_2^2 = \langle f, f\rangle \geq 0$ (and > 0 unless f is zero almost everywhere)

Cauchy-Schwarz Inequality
Let $f, g \in L^2([a,b])$. Then the function $x \mapsto f(x)\overline{g(x)}$ is Lebesgue integrable and we have

$$|\langle f, g\rangle| \leq \|f\|_2 \|g\|_2.$$

Minkowski's Inequality
For $f, g \in L^2([a,b])$,

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2.$$

Convergence in L^2
Let $(f_n) \in L^2([a,b])$. $f_n \to f$ in L^2 if

$$\|f_n - f\|_2 = \left(\int_a^b |f_n(x) - f(x)|^2 dx\right)^{1/2}$$

converges to zero.

Orthonormal Systems
A sequence $(\phi_n)_n$ of L^2 functions on $[a,b]$ is called an orthonormal system on $[a,b]$ if

$$\langle \phi_n, \phi_m\rangle = \int_a^b \phi_n(x)\overline{\phi_m(x)}dx$$
$$= \delta_{nm} = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m. \end{cases}$$

Orthogonal Projection
Let $(\phi_n)_n$ be an orthonormal system on $[a,b]$ and $f \in L^2$. Consider

$$s_N(x) = \sum_{n=1}^N \langle f, \phi_n\rangle \phi_n(x).$$

Denote the linear span of the functions $(\phi_n)_{n=1,\dots,N}$ by X_N. Then

$$\|f - s_N\|_2 \leq \|f - g\|_2$$

holds for all $g \in X_N$ with equality if and only if $g = s_N$. Hence s_N is the "best L^2 approximation" of f in X_N.

Bessel's Inequality
If $(\phi_n)_n$ is an orthonormal system on $[a,b]$ and $f \in L^2$, then

$$\sum_n |\langle f, \phi_n\rangle|^2 \leq \|f\|_2^2$$

Riemann-Lebesgue lemma, L^2 version
Let $(\phi_n)_{n=1,2,\dots}$ be an orthonormal system and $f \in L^2$, then

$$\lim_{n\to\infty} \langle f, \phi_n\rangle = 0.$$

Complete Orthonormal Systems
An orthonormal system $(\phi_n)_n$ on $[a,b]$ is called complete if

- $$\sum_n |\langle f, \phi_n\rangle|^2 = \|f\|_2^2$$

  for all $f \in L^2$.

- *Convergence Characterisation*
  Let $(s_N)_N$ an orthonormal projection. Then $(\phi_n)_n$ is complete if and only if $(s_N)_N$ converges to f in the L^2-norm for every $f \in L^2$.

Trigonometric Polynomials
A trigonometric polynomial is defined

$$f(x) = \sum_{n=-N}^N c_n e^{2\pi i n x} \quad (x \in \mathbb{R}),$$

where $c_n \in \mathbb{C}$. If c_N or c_{-N} is non-zero, then N is called the degree of f. Trigonometric polynomials are continuous functions.
Properties

$$\text{1-periodicity: } f(x) = f(x+1)$$

$$f(x) = a_0 + \sum_{n=1}^N (a_n \cos(2\pi n x) + b_n \sin(2\pi n x))$$

Trig.Polynomials and Orthonormla Systems
$(e^{2\pi i n x})_{n\in\mathbb{Z}}$ forms an orthonormal system on $[0,1]$. In particular,
(i) for all $n \in \mathbb{Z}$,

$$\int_0^1 e^{2\pi i n x}dx = \begin{cases} 0, & \text{if } n \neq 0, \\ 1, & \text{if } n = 0. \end{cases}$$

(ii) if $f(x) = \sum_{n=-N}^N c_n e^{2\pi i n x}$ is a trigonometric polynomial, then

$$c_n = \langle f, \phi_n\rangle = \int_0^1 f(t)e^{-2\pi i n t}dt.$$

Fourier Series
For a 1-periodic integrable function f and $n \in \mathbb{Z}$ we define the nth Fourier coefficient by

$$\widehat{f}(n) = \int_0^1 f(t)e^{-2\pi i n t}dt = \langle f, \phi_n\rangle.$$

(The integral on the right exists since f is integrable and $|\phi_n| \leq 1$.) The doubly infinite series

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi i n x}$$

is called the Fourier series of f.

Doubly Infinite Series
These are series of the form

$$\sum_{n=-\infty}^{\infty} a_n$$

They are convergent if both the series $\sum_{n=1}^\infty a_n$ and $\sum_{n=0}^\infty a_{-n}$ are convergent. In the case of convergence, the limit of the doubly infinite series is defined as the sum of the limits of these two series.

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \hat{f}(0) + 2 \cdot \sum_{n=1}^{\infty} |\hat{f}(n)|^2$$

The series $\sum_{n=-\infty}^\infty a_n$ converges in the principal value sense, if the sequence of partial sums $\left(\sum_{n=-N}^N a_n\right)_{N=1,2,\dots}$ converges.

Partial Sum of 1-periodic Functions
For a 1-periodic integrable function f we define the partial sums

$$S_N f(x) = \sum_{n=-N}^N \widehat{f}(n)e^{2\pi i n x}.$$

Note that since $(\phi_n)_n$ is an orthonormal system, $S_N f$ is exactly the orthogonal projection of f ont the space of trigonometric polynomials of degree $\leq N$.

Convolutions
For two 1-periodic functions $f, g \in L^2$ we define their convolution by

$$f * g(x) = \int_0^1 f(t)g(x-t)dt.$$

For 1-periodic functions $f, g \in L^2$,

$$f * g = g * f.$$

Dirichlet Function

$$\mathbf{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Dirichlet Kernel
It turns out that the partial sum $S_N f$ can be written in terms of a convolution:

$$S_N f(x) = \sum_{n=-N}^N \int_0^1 f(t)e^{-2\pi i n t}dt e^{2\pi i n x}$$
$$= \int_0^1 f(t) \sum_{n=-N}^N e^{2\pi i n(x-t)}dt$$
$$= f * D_N(x).$$

where

$$D_N(x) = \sum_{n=-N}^N e^{2\pi i n x}.$$

The sequence of functions $(D_N)_N$ is called Dirichlet kernel and can also be written

$$D_N(x) = \frac{\sin\left(2\pi\left(N + \frac{1}{2}\right)x\right)}{\sin(\pi x)}$$

Fejer Kernel
We define the Fejér kernel by

$$K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x).$$

The intuition here is that the additional average should "smooth things out" so that hopefully $f * K_N$ will have better convergence properties. This turns out to work. We have

$$K_N(x) = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)}\right)^2$$

Fejér Theorem
For every 1-periodic continuous function f,

$$K_N * f \to f$$

uniformly on $\mathbb{R}$ as $N \to \infty$.
Every 1-periodic continuous function can be uniformly approximated by trigonometric polynomials. That is, for every 1-periodic continuous f there exists a sequence $(f_n)_n$ of trigonometric polynomials so that $f_n \to f$ uniformly.
Approximation of Unity
An approximation of unity is a sequence $(k_n)_n$ that approximates unity:

$$\lim_{n\to\infty} k_n * f = f$$

for every continuous, 1-periodic f.

- That is $f * k_n$ converges uniformly to f on $\mathbb{R}$

$$\sup_{x\in\mathbb{R}} |f * k_n(x) - f(x)| \to 0$$

- There is no unity for the convolution of functions. More precisely, there exists no continuous function k such that $k * f = f$ for all continuous, 1-periodic f.

- Let $(k_n)_n$ be a sequence of 1-periodic integrable functions such that
  1. $k_n(x) \geq 0$ for all $x \in \mathbb{R}$.
  2. $\int_{-1/2}^{1/2} k_n(t)dt = 1$.
  3. For all $1/2 \geq \delta > 0$ we have

  as $n \to \infty$. $$\int_{-\delta}^{\delta} k_n(t)dt \to 1$$

  Then $(k_n)_n$ is an approximation of unity.

The Fejér kernel$(K_N)_N$ is an approximation of unity.

Limit of 1-periodic Fucntions
Let f be a 1-periodic and continuous function. Then

$$\lim_{N\to\infty} \|S_N f - f\|_2 = 0.$$

Assume that f is differentiable at x. Then $S_N f(x) \to f(x)$ as $N \to \infty$.

Completenes of the Trigonometric System
For every 1-periodic L^2 function f we have

$$\lim_{N\to\infty} \|S_N f - f\|_2 = 0.$$

In other words, the Fourier series of f converges to f in the L^2 sense.
Parseval's Theorem
If f, g are 1 -periodic L^2 functions, then

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}.$$

In particular,

$$\int_a^b |f(x)|^2 \mathrm{dx} = \|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$$

Abel Summation
The series $\sum_{n=0}^{\infty} a_n$ with $a_n \in \mathbb{C}$ is Abel summable to S if the series $A(r) = \sum_{n=0}^{\infty} a_n r^n$ converges for every $r \in (0,1)$ and the limit $\lim_{r\to 1-} A(r)$ exists and equals S.
Cesaro Summation
Given the sequence a_k, form the partial sums $s_n = \sum_{k=1}^{n} a_k$ and let

$$\sigma_N = \frac{s_1 + \cdots + s_N}{N}.$$

σ_N is called the Nth Cesàro mean of the sequence s_k or the Nth Cesàro sum of the series $\sum_{k=1}^{\infty} a_k$.
If σ_N converges to a limit S we say that the series $\sum_{k=1}^{\infty} a_k$ is Cesàro summable to S and

$$|\sigma_N - S| < \epsilon|$$

Misc.

Sup Inf
*Supremum (*sup*)*
1. $\forall x \in S, x \leq \sup(S)$.
2. $\forall \epsilon > 0, \exists x \in S, \sup(S) - x < \epsilon$.

*Infimum (*inf*)*
1. $\forall x \in S, x \geq \inf(S)$.
2. $\forall \epsilon > 0, \exists x \in S, x - \inf(S) < \epsilon$.

Nested Interval Property
A sequence $(I_n)_{n\in\mathbb{N}}$ of sets nested if

$$I_1 \supset I_2 \supset I_3 \supset \dots$$

If each (I_n) nonempty, closed and bounded then

$$E = \bigcap_{n\in\mathbb{N}} I_n = \{x \in \mathbb{R} : x \in I_n \text{ for all } n \in \mathbb{N}\}$$

is nonempty. If $\lambda(I_n) \to 0$ then E contains exactly one number.
Subcovers
Let $E = [a, b]$ for some real numbers $a \leq b$. Suppose that $(I_\alpha)_{\alpha\in\mathcal{A}}$ is a collection of open intervals that cover E, that is

$$E \subset \bigcup_{\alpha\in\mathcal{A}} I_\alpha.$$

Then there exist a finite set of indices $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset A$ such that

$$E \subset I_{\alpha_1} \cup I_{\alpha_2} \cup \cdots \cup I_{\alpha_n}.$$

$(I_{\alpha_i})_{i=1,2,\dots,n}$ is a finite subcover of E.
Countability
A set E is countable if there exists a bijection $f : E \to \mathbb{N}$.
A countable union of countable sets is countable.
The Exponential Function

$$E(x) := \sum_{n=0} \frac{x^n}{n!}$$

Notes

- $xy \leq \frac{1}{2}\left(x^2 + y^2\right)$ for $x, y \geq 0$
- $|z|^2 = z\bar{z}$ for $z \in \mathbb{C}$
- $\int uv\ dx = u\int v\ dx - \int(u'\int v\ dx)\ dx$
- $\int \chi_I := \lambda(I)$
- $|f_n(x) - f(x)| < \epsilon$ and $\sup_x f(x) = M \implies \sup_x f_n(x) \leq M + \epsilon$
- $e^{\pi i} = -1$
- $|sin(x)| \leq |x|$
- $\sum_{j=1}^{\infty} \frac{1}{n(n+1)} = 1 < \infty$
- $\frac{1}{2+x} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n$
- $\frac{d}{dx}\sin(x) = \cos(x)$

Examples

Show that g is integrable F^2 version
Q. Suppose that $f \in L^2([0,1])$. Show that the function $g(x) = f(x)x^{-1/4}\chi_{(0,1]}(x)$ is integrable on $[0,1]$.
A. Let

$$k_n(x) = x^{-1/4}\chi_{(1/n,1]}(x), \quad n = 1, 2, \dots$$

As each k_n is bounded and piece-wise continuous it is measurable and $\int_0^1 k_n^2 < \infty$. It follows that since $k_n \to k$ then k is measurable and k^2 is integrable on $[0,1]$ if and only if

$$\sup_n \int_0^1 k_n^2 < \infty$$

which can be verified by a direct calculation via fundamental theorem of calculus.

Find the limit
Q. Find the limit of some weird function $f_n(x)$

- Guess the limit $f(x)$ using school math assumptions
- Use $f_n(x) - f(x)| < \epsilon$ to work out what value we need for N. If we just need "some large N say "We can choose N such that..."
- Show that for $n \geq N$ the limit is $f(x)$.

Is the Convergence Uniform?
Consider the possible values of $f_n(x)$, if it can take a value that the $f(x)$ cannot, then use the value of x that gives this value as a counter example (working backwards).

Is Cesaro Summable?
Q. If $\sum_{k=0}^{\infty} a_k$converges to L, show that it is Cesáro summable to L
Let $\varepsilon > 0$. Pick $N_1 \in \mathbb{N}$ such that $k \geq N_1$ implies that $|s_k - L| < \frac{\varepsilon}{2}$. Choose $N_2 \in \mathbb{N}$ such that $N_2 > N_1$ and

$$\sum_{k=0}^{N_1} |s_k - L| < \frac{\varepsilon N_2}{2}.$$

If $N > N_2$ then

$$|\sigma_N - L| \leq$$
$$\frac{1}{N+1}\sum_{k=0}^{N_1} |s_k - L| + \frac{1}{N+1}\sum_{k=N_1+1}^{N} |s_k - L|$$
$$\leq \frac{\varepsilon N_2}{2(N+1)} + \frac{\varepsilon}{2}\left(\frac{N-N_1}{N+1}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence we have shown the result.

Counter Examples

- For $x \in [0,1)$ we have $nx^n \to 0$ as $n \to \infty$. However $\int_0^1 f_n = n/(n+1) \to 1$ as $n \to \infty$.
- $\sum_{k=0}^{\infty}(-1)^k$ does not converge but its cesaro means do

$$\sigma_N = \begin{cases} \frac{N+2}{2(N+1)} & N \text{ is even} \\ \frac{1}{2} & N \text{ is odd} \end{cases}$$

Trig stuff

Angle Sum and Difference Identities

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$
$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$
$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

Double Angle Identities

$$\sin(2\theta) = 2\sin\theta \cdot \cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

Sum to Product of Two Angles

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$
$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$
$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$
$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

Product of Angels

$$\sin\theta\sin\phi = \frac{[\cos(\theta-\phi) - \cos(\theta+\phi)]}{2}$$
$$\cos\theta\cos\phi = \frac{[\cos(\theta-\phi) + \cos(\theta+\phi)]}{2}$$
$$\sin\theta\cos\phi = \frac{[\sin(\theta+\phi) + \sin(\theta-\phi)]}{2}$$
$$\cos\theta\sin\phi = \frac{[\sin(\theta+\phi) + \sin(\theta-\phi)]}{2}$$