T-Exercise 9

$L = -s(e^X - 1)$, $s > 0$, X is normally distributed with $\mu \in R$ and $\sigma > 0$

Compute $ES_\alpha(L)$ for $\alpha \in (0,1)$

First show some prerequisites which will be helpful further:

$(\Phi^{-1})'(\alpha) = \frac{1}{\Phi'(\Phi^{-1}(\alpha))}$ this can be shown as follows

$\Phi(\Phi^{-1}(\alpha)) = \alpha$

therefore $\frac{d}{d\alpha}\Phi(\Phi^{-1}(\alpha)) = \frac{d}{d\alpha}\alpha = 1 \qquad (1)$

and also by chain rule $\frac{d}{d\alpha}\Phi(\Phi^{-1}(\alpha)) = \Phi'(\Phi^{-1}(\alpha)) \cdot (\Phi^{-1})'(\alpha) \qquad (2)$

by (1) and (2) $\Phi'(\Phi^{-1}(\alpha)) \cdot (\Phi^{-1})'(\alpha) = 1 \implies (\Phi^{-1})'(\alpha) = \frac{1}{\Phi'(\Phi^{-1}(\alpha))}$

Now we use Example 1.14 from script, and show it with the rule of contraries:

$$ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_p(L)dp \overset{!}{=} s\left(1 - \frac{\exp(\mu+\frac{\sigma^2}{2})}{1-\alpha}\Phi(-\Phi^{-1}(\alpha) - \sigma)\right)$$

hence we need to show that

$$s\left((1-\alpha) - \exp(\mu + \frac{\sigma^2}{2})\Phi(-\Phi^{-1}(\alpha) - \sigma)\right)' \overset{!}{=} VaR_\alpha(L)$$

So idea is to take a derivative of $F(L)$ instead of integrating pdf.

$$s\left((1-\alpha) - \exp(\mu + \frac{\sigma^2}{2})\Phi(-\Phi^{-1}(\alpha) - \sigma)\right)' = s\left(-1 - \exp(\mu + \frac{\sigma^2}{2})\varphi(-\Phi^{-1}(\alpha) - \sigma)\underbrace{(-(\Phi^{-1})'(\alpha))}_{-\frac{1}{\Phi'(\Phi^{-1}(\alpha))}}\right)$$

where φ is pdf of standard normal distribution. Substitute φ and get

$$-s\left(1 - \exp(\mu + \frac{\sigma^2}{2})\frac{\frac{1}{\sqrt{2\pi}}\exp(-\frac{(-\Phi^{-1}(\alpha)-\sigma)^2}{2})}{\frac{1}{\sqrt{2\pi}}\exp(-\frac{(\Phi^{-1}(\alpha))^2}{2})}\right) = \underbrace{-s\left(1 - \exp(\mu - \sigma\Phi^{-1}(\alpha))\right) \overset{1.7}{=} -VaR_\alpha(L)}_{\text{see example 1.7}} \quad \square$$