T-Exercise 20

Lemma 4.8, proof:

(3) Kendall's Tau:

$\rho_\tau(X_1,X_2) = P((X_1-\tilde{X}_1)(X_2-\tilde{X}_2)>0) - P((X_1-\tilde{X}_1)(X_2-\tilde{X}_2)<0) =$
$P(X_1-\tilde{X}_1>0, X_2-\tilde{X}_2>0) + P(X_1-\tilde{X}_1<0, X_2-\tilde{X}_2<0) -$
$P(X_1-\tilde{X}_1<0, X_2-\tilde{X}_2>0) - P(X_1-\tilde{X}_1>0, X_2-\tilde{X}_2<0) =$
$\underbrace{P(X_1-\tilde{X}_1>0)P(X_2-\tilde{X}_2>0)}_{\text{since indep.}} + P(X_1-\tilde{X}_1<0)P(X_2-\tilde{X}_2<0) -$
$P(X_1-\tilde{X}_1<0)P(X_2-\tilde{X}_2>0) - P(X_1-\tilde{X}_1>0)P(X_2-\tilde{X}_2<0) \qquad (1)$

notice that: $P(X_1-\tilde{X}_1<0) = \underbrace{P(X_1>\tilde{X}_1) = P(X_1<\tilde{X}_1)}_{\text{symm., since same law}} = P(X_1-\tilde{X}_1>0)$

all other components can be done similar way, hence each component in (1) has the same prob., and they all sum up to 1, therefore probability of each is $\frac{1}{4}$
So for (1) we have $\frac{1}{4}+\frac{1}{4}-\frac{1}{4}-\frac{1}{4}=0 \quad \square$

Spearman's Rho:

$$\rho_S(X_1,X_2) = \rho_L(F_{X_1}(X_1), F_{X_2}(X_2)) = \frac{\overbrace{Cov(F_{X_1}(X_1), F_{X_2}(X_2))}^{\text{zero, due to indep.}}}{\sqrt{var(F_{X_1}(X_1))var(F_{X_2}(X_2))}} = 0$$

Absence of converse then can be shown like this. Let X, Z are independent and $P(Z=1) = P(Z=-1) = 0.5$ it follows $\rho_\tau(X, XZ) = \rho_S(X, XZ) = 0$,
however X and XZ are NOT independent $\quad \square$