// Mykhaylo Cherner // Anton Gorev

Risk Management: T-Exercise 09 Solution

By positive homogeneity and translation invariance of the $expectedshortfall$ we get: $ES_\alpha(L) = -s(ES_\alpha(e^X) - 1)$, with $L = -s(e^X - 1)$.
Since $X \sim N(\mu, \sigma)$ we can replace it by $\mu + \sigma Y$, where $Y \sim N(0,1)$ has standard normal law. Thus, we are interested in $ES_\alpha(e^{\mu+\sigma Y})$.

$$
\begin{aligned}
ES_\alpha(e^{\mu+\sigma Y}) &= \frac{1}{1-\alpha}\int_{q_\alpha(F_Y)}^{\infty} e^{(\mu+\sigma y)}\phi(y)dy \\
&= \frac{1}{1-\alpha}e^{\mu}\int_{q_\alpha(F_Y)}^{\infty} e^{(\sigma y)}\frac{1}{\sqrt{2\pi}}exp\Big(-\frac{y^2}{2}\Big)dy \\
&= \frac{1}{1-\alpha}e^{\mu}\int_{q_\alpha(F_Y)}^{\infty} \frac{1}{\sqrt{2\pi}}exp\Big(-\frac{y^2-\sigma y+\sigma^2}{2}+\frac{\sigma^2}{2}\Big)dy \\
&= \frac{1}{1-\alpha}e^{(\mu+\frac{\sigma^2}{2})}\int_{q_\alpha(F_Y)}^{\infty} \frac{1}{\sqrt{2\pi}}exp\Big(-\frac{(y-\sigma)^2}{2}\Big)dy \\
&= \frac{1}{1-\alpha}e^{(\mu+\frac{\sigma^2}{2})}\int_{q_\alpha(F_Y)}^{\infty} \phi(y+\sigma)dy \\
&= \frac{1}{1-\alpha}e^{(\mu+\frac{\sigma^2}{2})}\int_{q_\alpha(F_Y)+\sigma}^{\infty} \phi(w)dw \\
&= \frac{e^{(\mu+\frac{\sigma^2}{2})}}{1-\alpha}\Phi(-\Phi^{-1}(\alpha)-\sigma)
\end{aligned}
$$

Variable Substitution

$$
\begin{aligned}
w &= y+\sigma \\
\frac{dw}{dy} &= 1 \\
dy &= dw
\end{aligned}
$$

Bounds

$$
\begin{aligned}
y_u &= \infty \\
y_l &= q_\alpha(F_Y) \\
w &= y+\sigma \\
w_u &= \infty \\
w_l &= q_\alpha(F_Y)+\sigma
\end{aligned}
$$

Since standard distribution is symmetric, hence: $\Phi^{-1}(1-\alpha) = -\Phi^{-1}(\alpha)$

Therefore, for $ES_\alpha(L) = -s(ES_\alpha(e^X) - 1)$ we get:

$$ES_\alpha(L) = -s\Big(\frac{e^{(\mu+\frac{\sigma^2}{2})}}{1-\alpha}\Phi(-\Phi^{-1}(\alpha)-\sigma) - 1\Big)$$