T-Exercise 6

(a) Show that $VaR_\alpha(a \cdot X + b) = a \cdot VaR_\alpha(X) + b$, $b \in R$, $a \geq 0$

For $a = 0$

$VaR_\alpha(0 \cdot X + b) = VaR_\alpha(b) = b = 0 \cdot VaR_\alpha(X) + b$

For $a > 0$

$VaR_\alpha(a \cdot X + b) \stackrel{\text{def}}{=} inf\{l : l \in R,\ P(aX + b > l) \leq 1 - \alpha\} =$

$inf\{l : l \in R,\ P(X > \frac{l-b}{a}) \leq 1 - \alpha\}$

now let $l' = \frac{l-b}{a}$ hence $l = al' + b$ substitute

$VaR_\alpha(a \cdot X + b) = inf\{al' + b \in R,\ P(X < l') \geq \alpha\} =$

$a \cdot \underbrace{inf\{l' \in R,\ P(X < l') \geq \alpha\}}_{VaR_\alpha(X)} + b \quad \square$

(b) Show that $VaR_\alpha(-X) = -VaR_{1-\alpha}(X)$

By Definition 1.1 and Remark 1.2.1

$P(X > VaR_{1-\alpha}(X)) = 1 - 1 + \alpha = \alpha$

put differently

$P(-X < -VaR_{1-\alpha}(X)) = \alpha \quad \mathbf{(1)}$

now consider $VaR_\alpha(-X)$

again by remark 1.2.1

$P(-X > VaR_\alpha(-X)) = 1 - \alpha$

$P(-X < VaR_\alpha(-X)) = \alpha \quad \mathbf{(2)}$

compare (1) and (2)

$P(-X < VaR_\alpha(-X)) = P(-X < -VaR_{1-\alpha}(X))$

compare lefthand side of "under probability" expressions conclude that

$VaR_\alpha(-X) = -VaR_{1-\alpha}(X) \quad \square$