Swift-Hohenberg on a Torus

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Plan

- Overview of the Closest Point Method.
- Heat equation using closest point method.
- Swift-Hohenberg equation.

Closest Point Method

Idea: instead of discretizing surface derivatives directly, we extend the data to the embedding space so that it is constant along the normal direction, and apply normal Cartesian derivatives to it.
A simple way to obtain such extension is using the closest point: the vector connecting a point in the embedding space to its closest point on a surface is in fact normal to it.

Derivative Principles

Gradient Principle

For points $\vec{x}$ on a smooth surface S,

$$\nabla_S u(x) = \nabla(u(cp(x)))$$

because the function u(cp(x)) is constant in the normal direction and therefore only varies along the surface.

Divergence Principle

Let $\vec{v}$ be any vector field on $\mathbb{R}^d$ that is tangent at S and also tangent at all surfaces displaced by a fixed distance from S, then for points $\vec{x}$ on the surface S,

$$\nabla_S \cdot \vec{v}(x) = \nabla_S \cdot \vec{v}(x)$$

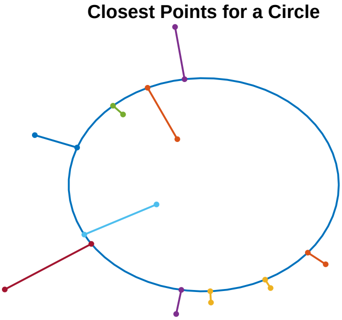
because the function u(cp(x)) is constant in the normal direction and therefore only varies along the surface.

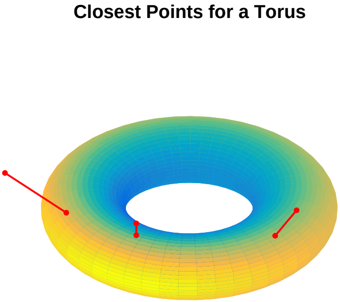
Closest Point Extension

We define the closest point extension operator E as follows:

$$Eu(\mathbf{x}) = u(cp(\mathbf{x}))$$

for all $\mathbf{x}$ in the embedding space. Note that it is linear and idempotent.





Spatial Discretization Using Closest Point Method

The (linear) spatial differential operator L_S, written in terms of surface derivatives, is discretized using the approximation of the two operators:

- Extension operator: approximated by an interpolation matrix E_h.
- Differential operator defined on the embedding space: approximated by a finite difference matrix L_h.

The original operator L_S is approximated by the composition:

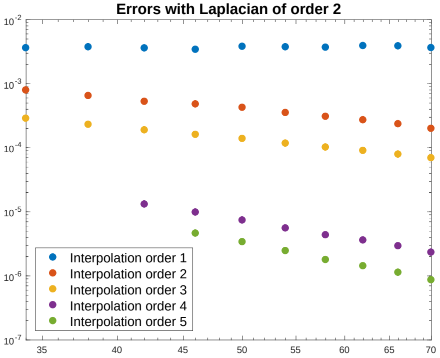
$$L_S \approx L_h \circ E_h.$$

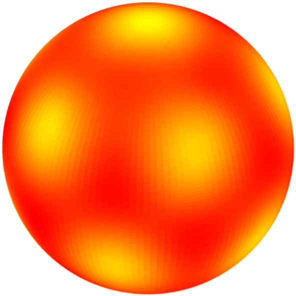
Time Stepping Using Closest Point Method

For a PDE $u_t = L_S u$, a typical time step using the CPM goes as follows:

1. Extend the data given on the surface to the embedding (computational) domain using closest point function.
2. Step forward in time on the embedding space.
3. Interpolate back to the surface.

Diffusion on a Sphere





Stabilized Diffusion Operator

The standard five-point Laplacian in combination with closest point extension:

$$\triangle u \approx \frac{1}{\Delta x^2}\big(-4u(cp(x,y)) + u(cp(x+\Delta x, y)) + u(cp(x-\Delta x, y)) \\ + u(cp(x, y+\Delta y)) + u(cp(x, y-\Delta y))\big)$$

Stabilized version:

$$\triangle u \approx \frac{1}{\Delta x^2}\big(-4u(x,y) + u(cp(x+\Delta x, y)) + u(cp(x-\Delta x, y)) \\ + u(cp(x, y+\Delta y)) + u(cp(x, y-\Delta y))\big)$$

Stabilized Diffusion Operator

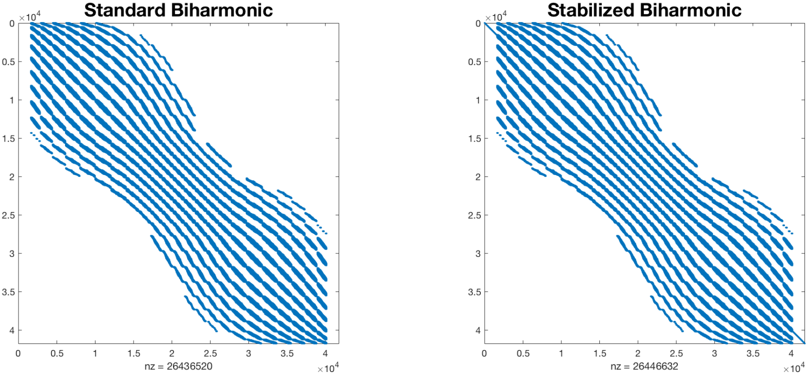
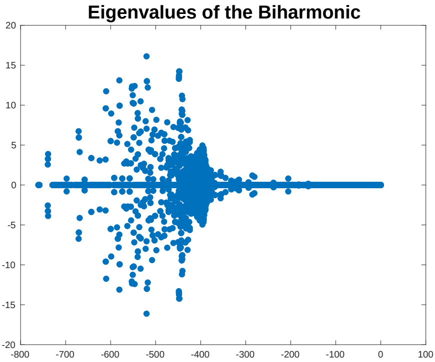
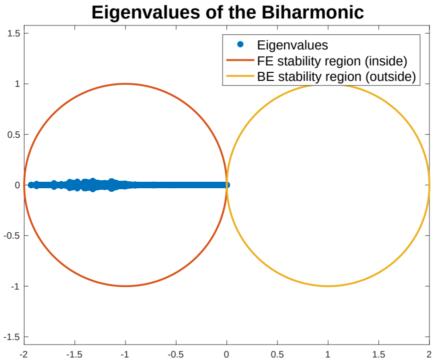


Figure: Sparsity patterns

Stability Restrictions





$$\Delta t = \frac{\Delta x^4}{36}$$

Swift-Hohenberg Equation

Finally, the equation:

$$u_t = -\triangle^2 u - \triangle u + (P-1)u + su^2 - u^3$$

This equation was derived by Swift and Hohenberg in 1977 to study thermal fluctuations on a fluid near the Rayleigh-Benard convective instability. The function u is the temperature field in a plane horizontal layer of fluid heated from below. The parameter P measures how far the temperature is above the minimum temperature required for convection: for $P < 0$, the heating is too small to cause convection, while for $P > 0$, convection occurs. The Swift-Hohenberg equation is an example of a PDE that exhibits pattern formation, including stripes, spots and spirals.

Discretization

The equation can be written as a sum of linear and non-linear terms:

$$u_t = \underbrace{-\triangle^2 u - \triangle u + (P-1)u}_{=:Lu} + \underbrace{su^2 - u^3}_{=:Nu}$$

- Linear part is very stiff (large eigenvalues), but can be inverted - treat implicitly.
- Nonlinear part is very difficult to invert - treat explicitly.

Time Stepping

Possible implicit-explicit time stepping schemes:

- IMEX Euler:

$$(I - \Delta t L)\, u^{n+1} = u^n + \Delta t N u^n$$

- Semi-implicit Backward Differentiation Formula (SBDF2)

$$(3I - 2\Delta t L)\, u^{n+1} = 4u^n + 4\Delta t N u^n - u^{n-1} - 2\Delta t N u^{n-1}$$

- Many more ...

Let's watch some videos.

Thank you for your attention!