Swift-Hohenberg on a Torus

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Summer 2017

Plan

- Overview of the Closest Point Method.
- Biharmonic heat equation on a sphere and a torus.
- Swift-Hohenberg equation.

Closest Point Method

Idea: instead of discretizing surface derivatives using parametrization, we represent them by Cartesian derivatives acting on the extension of a surface quantity to the embedding space. To achieve this, data needs to be constant in normal direction.

Gradient Principle

For points x on a smooth surface S, $\nabla_S u(x) = \nabla(u(cp(x)))$ because the function u(cp(x)) is constant in the normal direction and therefore only varies along the surface. In other words, at points x on the surface, intrinsic surface gradients $\nabla_S u(x)$ are the same as gradients of u(cp(x)).

Approximation of surface operators via Cartesian grid operators acting on normally extended data...

Closest Point Extension

To obtain the normal extension, we recall that the vector connecting a point in the embedding space to its closest point on a surface is in fact normal to it.

Stabilized Diffusion Operator

spectra of standard and stabilized operators.

Stabilized Diffusion Operator

spy plots on sphere.

Stability Restrictions

stability restrictions

Convergence on the Sphere

Swift-Hohenberg Equation

$$u_t = -\triangle^2 u - \triangle + u(P-1)u + su^2 - u^3$$

This equation was derived by Swift and Hohenberg in 1977 to study thermal fluctuations on a fluid near the Rayleigh-Benard convective instability. The function u is the temperature field in a plane horizontal layer of fluid heated from below. The parameter r measures how far the temperature is above the minimum temperature required for convection: for $r < 0$, the heating is too small to cause convection, while for $r > 0$, convection occurs. The Swift-Hohenberg equation is an example of a PDE that exhibits pattern formation, including stripes, spots and spirals.

Discretization

$$u_t = \underbrace{-\triangle^2 u - \triangle + u(P-1)u}_{=:Lu} + \underbrace{su^2 - u^3}_{=:Nu}$$

Linear part is very stiff, but can be inverted (-ish) - treat implicitly.
Nonlinear part - difficult to invert, treat explicitly.

Thank you for your attention!

Videos

References

- The Implicit Closest Point Method for the Numerical Solution of Partial Differential Equations on Surfaces. *SIAM J. SCI. COMPUT. 2009 Society for Industrial and Applied Mathematics Vol. 31, No. 6, pp. 4330-4350*
- Albert Einstein. *Zur Elektrodynamik bewegter Körper*. (German) [*On the electrodynamics of moving bodies*]. Annalen der Physik, 322(10):891?921, 1905.
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