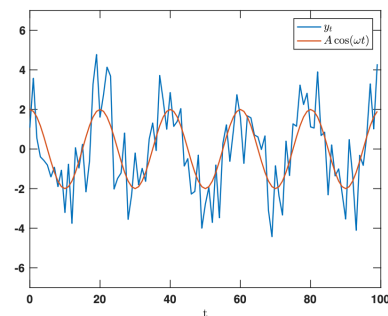


The Kalman Filter, part 2

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The Kalman filter



Example:
Consider the signal

$$y_t = A \cos(\omega t) + w_t$$

where w_t is a Gaussian white noise with variance σ_w^2 . Assuming that the frequency, ω , is known, how can one estimate the amplitude, A , using a Kalman filter?

The Kalman filter

Let the hidden state be $x_t = A$. Then,

$$x_t = x_{t-1}$$

$$y_t = \cos(\omega t)x_t + w_t$$

To simplify notation, let $c_t = \cos(\omega t)$. Thus,

$$\hat{x}_{t+1|t} = \hat{x}_{t|t}$$

$$\hat{y}_{t+1|t} = c_{t+1}\hat{x}_{t+1|t} = c_{t+1}\hat{x}_{t|t}$$

where

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t(y_t - \hat{y}_{t|t-1})$$

with

$$\mathbf{R}_{t+1|t}^{x,x} = \mathbf{R}_{t|t}^{x,x}$$

$$\mathbf{R}_{t+1|t}^{y,y} = c_{t+1}^2 \mathbf{R}_{t+1|t}^{x,x} + \sigma_w^2$$

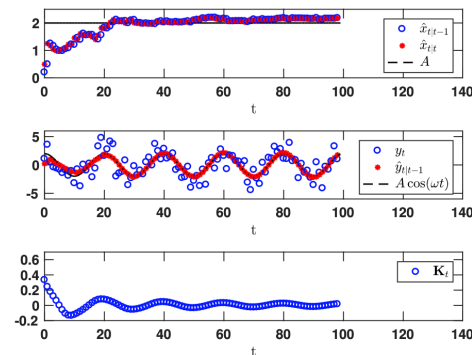
$$\mathbf{R}_{t+1|t}^{x,y} = c_{t+1} \mathbf{R}_{t+1|t}^{x,x}$$

$$\mathbf{R}_{t|t}^{x,x} = \mathbf{R}_{t|t-1}^{x,x} - K_t^2 \mathbf{R}_{t+1|t}^{y,y}$$

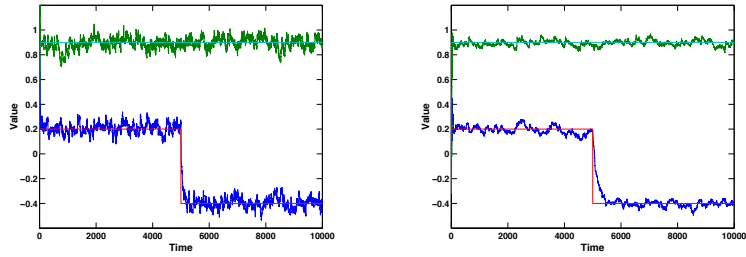
$$K_t = \frac{\mathbf{R}_{t|t-1}^{x,y}}{\mathbf{R}_{t|t-1}^{y,y}} = \frac{c_t \mathbf{R}_{t-1|t-1}^{x,x}}{c_t^2 \mathbf{R}_{t-1|t-1}^{x,x} + \sigma_w^2}$$

The Kalman filter

- 1: Set $t = 0$, initial state $\hat{x}_{t|t}$, and initial $\mathbf{R}_{t|t}^{x,x}$
- 2: **repeat**
- 3: **Predict state and observation**
 $\hat{x}_{t+1|t} = \hat{x}_{t|t}$, $\hat{y}_{t+1|t} = c_{t+1}\hat{x}_{t|t}$
- 4: **Update covariances for predictions**
 $\mathbf{R}_{t+1|t}^{x,x} = \mathbf{R}_{t|t}^{x,x}$, $\mathbf{R}_{t+1|t}^{y,y} = c_{t+1}^2 \mathbf{R}_{t|t}^{x,x} + \sigma_w^2$, $\mathbf{R}_{t+1|t}^{x,y} = c_{t+1} \mathbf{R}_{t|t}^{x,x}$
- 5: **Update Kalman gain**
 $\mathbf{K}_{t+1} = \frac{c_{t+1} \mathbf{R}_{t|t}^{x,y}}{c_{t+1}^2 \mathbf{R}_{t|t}^{x,x} + \sigma_w^2}$
- 6: **Update estimate of state**
 $\hat{x}_{t+1|t+1} = \hat{x}_{t+1|t} + K_{t+1}(y_{t+1} - \hat{y}_{t+1|t})$
- 7: $t \leftarrow t+1$
- 8: **until** end of signal



The Kalman filter



Example:

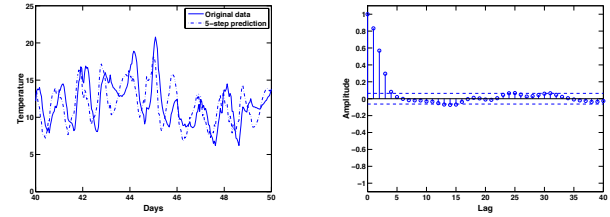
Consider an AR(2) with an unknown parameter a_1 and a_2

$$y_t + a_1 y_{t-1} + a_2 y_{t-2} = e_t$$

What should we set as states if we wish to estimate a_1 and a_2 ?

The figures show the estimates for $\mathbf{R}_\phi = 10^{-4} \mathbf{I}$ and $\mathbf{R}_\epsilon = 10^{-5} \mathbf{I}$.

The Kalman filter



Often, one is interested in predicting the process and the states k steps ahead. Such a prediction is formed as

$$\begin{aligned} \hat{\mathbf{x}}_{t+k+1|t} &= \mathbf{A} \hat{\mathbf{x}}_{t+k|t} + \mathbf{B} \hat{\mathbf{u}}_{t+k|t} \\ \mathbf{R}_{t+k+1|t}^{x,u} &= \mathbf{A} \mathbf{R}_{t+k|t}^{x,u} \mathbf{A}^T + \mathbf{B} \mathbf{R}_{t+k|t}^u \mathbf{B}^T + \mathbf{R}_\epsilon \end{aligned}$$

In the particular case when there is no input, the state prediction simplifies to

$$\hat{\mathbf{x}}_{t+k+1|t} = \mathbf{A}^k \hat{\mathbf{x}}_{t|t}$$

yielding the output prediction

$$\hat{\mathbf{y}}_{t+k|t} = \mathbf{C} \hat{\mathbf{x}}_{t+k|t} = \mathbf{C} \mathbf{A}^k \hat{\mathbf{x}}_{t|t}$$

The figures shows the 5-step prediction of the Svedala temperature example (see the book for details); note the MA($k-1$) structure of the residual.

The Kalman filter

One need to take a bit of care when predicting ARMA processes, as the required noise realisations are typically not available. For example, consider the ARMAX process

$$A(z)y_t = B(z)z^{-d}x_t + C(z)e_t$$

where

$$\begin{aligned} A(z) &= 1 + a_1 z^{-1} + \dots + a_6 z^{-6} \\ B(z) &= b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4} + b_5 z^{-5} \\ C(z) &= 1 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} \end{aligned}$$

The process may then be written on state space form as

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{A} \mathbf{x}_t + \mathbf{e}_t \\ y_t &= \mathbf{C}_{t|t-1} \mathbf{x}_{t|t-1} + w_t \end{aligned}$$

where $\mathbf{x}_{t|t-1}$ contains the model parameters using measurements up to y_{t-1} ,

$$\mathbf{C}_{t|t-1} = \begin{bmatrix} -y_{t-1} & -y_{t-2} & \dots & -y_{t-6} & e_{t-1} & e_{t-2} & e_{t-3} \\ u_t & u_{t-1} & u_{t-2} & u_{t-4} & u_{t-5} & & \end{bmatrix}$$

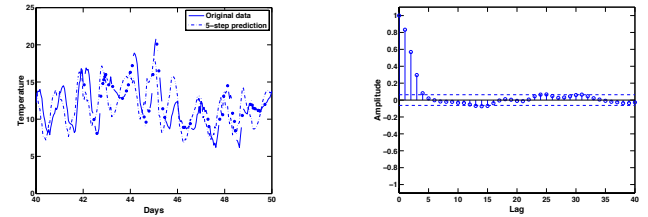
with $\mathbf{A} = \mathbf{I}$ and $w_t = e_t$. This allows for the forming of the k -step predictions

$$\begin{aligned} \hat{\mathbf{y}}_{t+1|t} &= \mathbf{C}_{t+1|t} \hat{\mathbf{x}}_{t+1|t} = \mathbf{C}_{t+1|t} \hat{\mathbf{x}}_{t|t} \\ \hat{\mathbf{y}}_{t+2|t} &= \mathbf{C}_{t+2|t} \hat{\mathbf{x}}_{t|t} \end{aligned}$$

where

$$\mathbf{C}_{t+2|t} = \begin{bmatrix} -\hat{y}_{t+1|t} & -y_t & \dots & -y_{t-4} & 0 & e_t & e_{t-1} \\ u_{t+2} & u_{t+1} & u_t & u_{t-2} & u_{t-3} & & \end{bmatrix}$$

The Kalman filter



One can also use the Kalman filter to predict missing samples; this can be done by changing the update equations to

$$\begin{aligned} \hat{\mathbf{x}}_{t|t} &= \hat{\mathbf{x}}_{t|t-1} \\ \mathbf{R}_{t|t}^{x,u} &= \mathbf{R}_{t|t-1}^{x,u} \end{aligned}$$

for the case of a sample being missing.

The figures shows the 5-step prediction of the Svedala temperature example with missing samples (marked with a star).