Predicting models with input

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Prediction of ARMAX processes

We proceed to the prediction of ARMAX processes, i.e.,

$$A(z)y_t = B(z)x_t + C(z)e_t$$

where

$$B(z) = b_d z^{-d} + b_{d+1} z^{-d-1} + \ldots + b_s z^{-s}$$

Then,

$$\begin{aligned} y_{t+k} &= \frac{C(z)}{C(z)} y_{t+k} \\ &= \frac{1}{C(z)} \Big\{ A(z)F(z) + z^{-k}G(z) \Big\} y_{t+k} \\ &= \frac{1}{C(z)} \Big\{ F(z)A(z)y_{t+k} + G(z)y_t \Big\} \end{aligned}$$

which yields

$$y_{t+k} = \frac{1}{C(z)} \Big\{ F(z) \Big[C(z)e_{t+k} + B(z)x_{t+k} \Big] + G(z)y_t \Big\}$$

Prediction of ARMAX processes

We proceed to rewrite

$$\frac{F(z)B(z)}{C(z)}x_{t+k} = \hat{F}(z)x_{t+k} + \frac{\hat{G}(z)}{C(z)}x_t$$

where the polynomials $\hat{F}(z)$ and $\hat{G}(z)$ are obtained by solving the corresponding Diophantine equation, i.e.,

$$F(z)B(z) = C(z)\hat{F}(z) + z^{-k}\hat{G}(z)$$

Thus,

$$\begin{aligned} \operatorname{ord}\left\{\hat{F}(z)\right\} &= k-1 \\ \operatorname{ord}\left\{\hat{G}(z)\right\} &= \max(q-1,\ s-1) \end{aligned}$$

where we used that $\operatorname{ord}\{F(z)B(z)\} = k-1+r$. This implies

$$\begin{aligned} y_{t+k} &= F(z)e_{t+k} + \frac{F(z)B(z)}{C(z)}x_{t+k} + \frac{G(z)}{C(z)}y_t \\ &= F(z)e_{t+k} + \hat{F}(z)x_{t+k} + \frac{\hat{G}(z)}{C(z)}x_t + \frac{G(z)}{C(z)}y_t \end{aligned}$$

Prediction of ARMAX processes

This gives the k-step prediction

$$
\begin{aligned}
\hat{y}_{t+k|t}(\mathbf{\Theta}) &= E\left\{y_{t+k}|\mathbf{\Theta}\right\} \\
&= F(z)E\{e_{t+k}|\mathbf{\Theta}\} + \hat{F}(z)E\{x_{t+k}|\mathbf{\Theta}\} + \frac{\hat{G}(z)}{C(z)}E\{x_t|\mathbf{\Theta}\} + \frac{G(z)}{C(z)}E\{y_t|\mathbf{\Theta}\} \\
&= \hat{F}(z)E\{x_{t+k}|\mathbf{\Theta}\} + \frac{\hat{G}(z)}{C(z)}x_t + \frac{G(z)}{C(z)}y_t
\end{aligned}
$$

Thus,

$$\epsilon_{t+k|t}(\mathbf{\Theta}) = F(z)e_{t+k} + \hat{F}(z)x_{t+k}$$

If e_t and x_t are independent,

$$
\begin{aligned}
V\{\epsilon_{t+k|t}(\mathbf{\Theta})\} &= V\{F(z)e_{t+k}\} + V\{\hat{F}(z)x_{t+k}|\mathbf{\Theta}\} \\
&= \sum_{\ell=0}^{k-1} f_\ell^2\sigma_e^2 + \sum_{\ell=0}^{k-1}\sum_{p=0}^{k-1}\hat{f}_\ell\hat{f}_p C\{x_{t+\ell}, x_{t+p}|\mathbf{\Theta}\}
\end{aligned}
$$

Prediction of ARMAX processes

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\begin{aligned}
\hat{y}_{t+k|t}(\boldsymbol{\Theta}) &= E\left\{y_{t+k}|\boldsymbol{\Theta}\right\} \\
&= F(z)E\{e_{t+k}|\boldsymbol{\Theta}\} + \hat{F}(z)E\{x_{t+k}|\boldsymbol{\Theta}\} + \frac{\hat{G}(z)}{C(z)}E\{x_t|\boldsymbol{\Theta}\} + \frac{G(z)}{C(z)}E\{y_t|\boldsymbol{\Theta}\} \\
&= \hat{F}(z)E\{x_{t+k}|\boldsymbol{\Theta}\} + \frac{\hat{G}(z)}{C(z)}x_t + \frac{G(z)}{C(z)}y_t
\end{aligned}
$$

Thus,

$$\epsilon_{t+k|t}(\boldsymbol{\Theta}) = F(z)e_{t+k} + \hat{F}(z)x_{t+k}$$

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\end{aligned}
$$

Example:

$$(1-0.2z^{-1})\nabla_{12}y_t = (1-0.3z^{-12})e_t + (1+0.3z^{-1}+0.4z^{-3})x_{t-4}$$

Then,

$$B(z) = z^{-4} + 0.3z^{-5} + 0.4z^{-7}$$

We obtain $F(z)$ and $G(z)$ as before, and

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B = [ 0 0 0 0 1 0.3 0 0.4 ];
BF = conv( B, F );
[Fhat,Ghat] = polydiv( BF, C, k );
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Prediction of BJ processes

When predicting Box-Jenkins processes,

$$y_t = \frac{C_1(z)}{A_1(z)} e_t + \frac{B(z)}{A_2(z)} x_{t-d}$$

we note that such processes can be rewritten as

$$A_1(z)A_2(z)y_t = A_2(z)C_1(z)e_t + A_1(z)B(z)z^{-d}x_t$$

Introduce

$$\begin{aligned} K_A(z) &= A_1(z)A_2(z) \\ K_B(z) &= A_1(z)B(z)z^{-d} \\ K_C(z) &= A_2(z)C_1(z) \end{aligned}$$

yielding the ARMAX model

$$K_A(z)y_t = K_B(z)x_t + K_C(z)e_t$$

Prediction of BJ processes

Following the steps for the ARMAX prediction,

$$y_{t+k} = \frac{1}{K_C(z)} \left\{ F(z) \Big[K_C(z) e_{t+k} + K_B(z) x_{t+k} \Big] + G(z) y_t \right\}$$

implying that

$$\frac{F(z) K_B(z)}{K_C(z)} x_{t+k} = \hat{\hat{F}}(z) x_{t+k} + \frac{\hat{\hat{G}}(z)}{K_C(z)} x_t$$

where

$$F(z) K_B(z) = K_C(z) \hat{\hat{F}}(z) + z^{-k} \hat{\hat{G}}(z)$$

with

$$\begin{aligned} \text{ord}\left\{ \hat{\hat{F}}(z) \right\} &= k - 1 \\ \text{ord}\left\{ \hat{\hat{G}}(z) \right\} &= \max(r + q - 1,\, p + s - 1) \end{aligned}$$

Prediction of BJ processes

Following the steps for the ARMAX prediction,

$$y_{t+k} = \frac{1}{K_C(z)} \left\{ F(z) \Big[K_C(z) e_{t+k} + K_B(z) x_{t+k} \Big] + G(z) y_t \right\}$$

implying that

$$\frac{F(z)K_B(z)}{K_C(z)} x_{t+k} = \hat{\hat{F}}(z) x_{t+k} + \frac{\hat{\hat{G}}(z)}{K_C(z)} x_t$$

where

$$F(z)K_B(z) = K_C(z)\hat{\hat{F}}(z) + z^{-k}\hat{\hat{G}}(z)$$

with

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Thus,

$$y_{t+k} = F(z)e_{t+k} + \hat{\hat{F}}(z)x_{t+k} + \frac{\hat{\hat{G}}(z)}{K_C(z)} x_t + \frac{G(z)}{K_C(z)} y_t$$

yielding the k-step prediction

$$\hat{y}_{t+k|t}(\boldsymbol{\theta}) = \hat{F}(z) E\{x_{t+k}|\boldsymbol{\theta}\} + \frac{\hat{G}(z)}{K_C(z)} x_t + \frac{G(z)}{K_C(z)} y_t$$