AR and ARMA Processes

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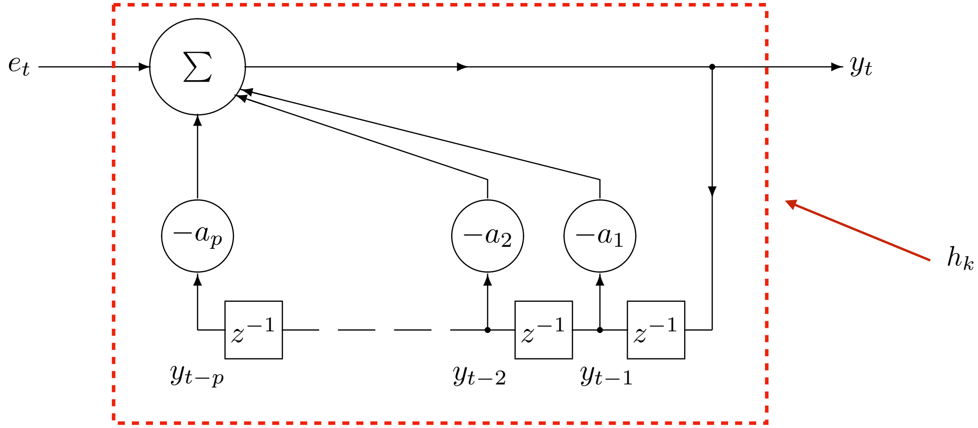
The autoregressive process



The process y_t is called an *autoregressive* (AR) process if

$$A(z)y_t = y_t + a_1 y_{t-1} + \ldots + a_p y_{t-p} = e_t$$

where $A(z)$ is a *monic* polynomial of order p (in z^{-1}), i.e.,

$$A(z) = 1 + a_1 z^{-1} + \ldots + a_p z^{-p}$$

with $a_p \neq 0$, and e_t is a zero-mean white noise process with variance σ_e^2. All zeros of the generating polynomial $A(z)$ are always within the unit circle.

The autoregressive process

Note that an AR-process is a white noise passed through a linear filter, i.e., that

$$A(z)y_t = e_t \quad \Rightarrow \quad y_t = \frac{1}{A(z)}e_t$$

Thus, the spectrum of an AR-process may be expressed as

$$\phi_y(\omega) = \frac{\sigma_e^2}{|A(\omega)|^2}$$

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The mean of an AR process is zero, as

$$E\{y_t + a_1 y_{t-1} + \ldots + a_p y_{t-p}\} = E\{e_t\} = 0$$

Thus, $m_y(1 + a_1 + \ldots + a_p) = m_y A(1) = 0$, which implies that $m_y = 0$ as all the zeros of $A(z)$ are within the unit circle, implying that $A(1) \neq 0$.

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To form the auto-covariance, post-multiply the process with y_{t-k} and take the expectation, i.e.,

$$\begin{aligned} E\{e_t y_{t-k}\} &= E\{y_t y_{t-k} + a_1 y_{t-1} y_{t-k} + \ldots + a_p y_{t-p} y_{t-k}\} \\ &= r_y(k) + a_1 r_y(k-1) + \ldots + a_p r_y(k-p) \end{aligned}$$

As e_t is uncorrelated with $y_{t-\ell}$, for $\ell > 0$, $E\{e_t y_{t-k}\} = \sigma_e^2 \delta_K(k)$, then

$$r_y(k) + a_1 r_y(k-1) + \ldots + a_p r_y(k-p) = \sigma_e^2 \delta_K(k)$$

This is the so-called Yule-Walker equations.

The autoregressive process

Example:
Consider a real-valued AR(1) process. The Yule-Walker equations then implies that

$$\begin{aligned} r_y(0) + a_1 r_y(1) &= \sigma_e^2 \\ r_y(1) + a_1 r_y(0) &= 0 \end{aligned}$$

where we have exploited that $r_y(k) = r_y(-k)$. Thus,

$$\begin{aligned} r_y(0) &= \frac{\sigma_e^2}{1 - a_1^2} \\ r_y(1) &= -a_1 r_y(0) = -a_1 \frac{\sigma_e^2}{1 - a_1^2} \end{aligned}$$

As $r_y(k) + a_1 r_y(k-1) = 0$, we may extend this to a general k as

$$r_y(k) = \left(-a_1\right)^{|k|} \frac{\sigma_e^2}{1 - a_1^2}$$

where we have again exploited the symmetry of $r_y(k)$. The PSD is

$$\phi_y(\omega) = \frac{\sigma_e^2}{\left[1 + a_1 e^{i\omega}\right]\left[1 + a_1 e^{-i\omega}\right]} = \frac{\sigma_e^2}{1 + a_1^2 + 2a_1 \cos\omega}$$

The autoregressive process

Expressed in matrix form, the Yule-Walker equations may be written as

$$\begin{bmatrix} r_y(0) & r_y(-1) & \dots & r_y(-n) \\ r_y(1) & r_y(0) & & \vdots \\ \vdots & & \ddots & r_y(-1) \\ r_y(n) & \dots & & r_y(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sigma_e^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $a_k = 0$ for $k > p$. Let

$$\boldsymbol{\theta} = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T$$

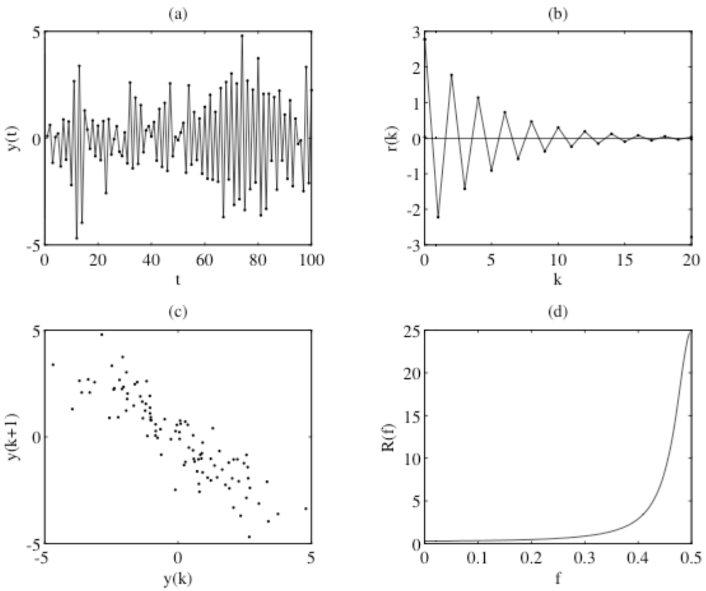
Using all but the first row then yields

$$\begin{bmatrix} r_y(1) \\ \vdots \\ r_y(n) \end{bmatrix} + \begin{bmatrix} r_y(0) & \dots & r_y(-n+1) \\ \vdots & \ddots & \vdots \\ r_y(n-1) & \dots & r_y(0) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

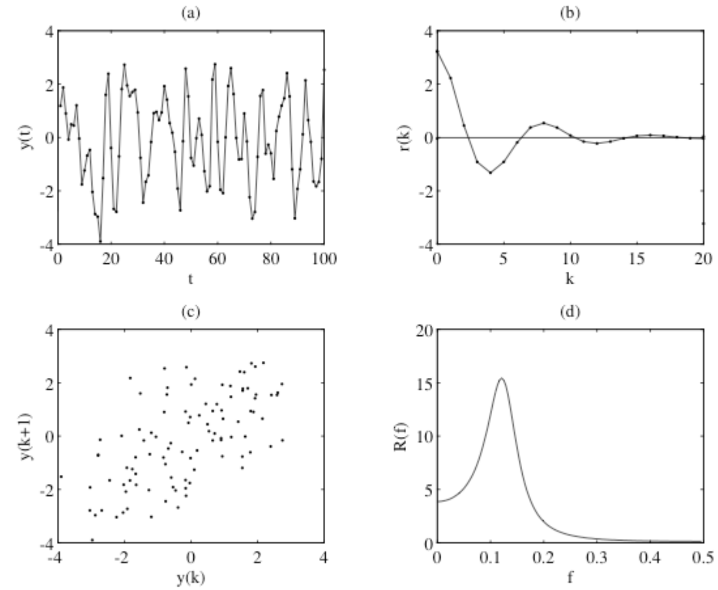
or, with obvious definitions, $\mathbf{r}_n + \mathbf{R}_n \boldsymbol{\theta} = \mathbf{0}$, implying that

$$\hat{\boldsymbol{\theta}} = -\hat{\mathbf{R}}_n^{-1} \hat{\mathbf{r}}_n$$

The autoregressive process

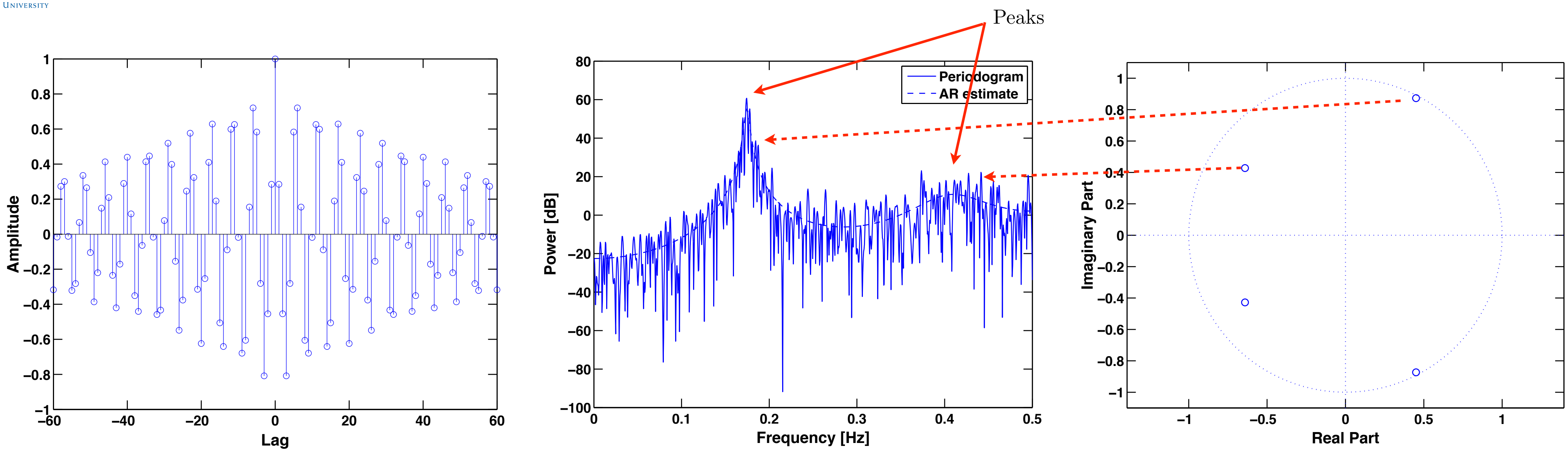


AR(1)-process $Y(t) = -0.8\,Y(t-1) + e(t)$; (a) realisation, (b) covf. func., (c) scatter-plot and (d) spectral density.



AR(2)-process $Y(t) = 1.131\,Y(t-1) - 0.64\,Y(t-2) + e(t)$;
(a) realisation, (b) covf. func., (c) scatter-plot and (d) spectral density.

The autoregressive process

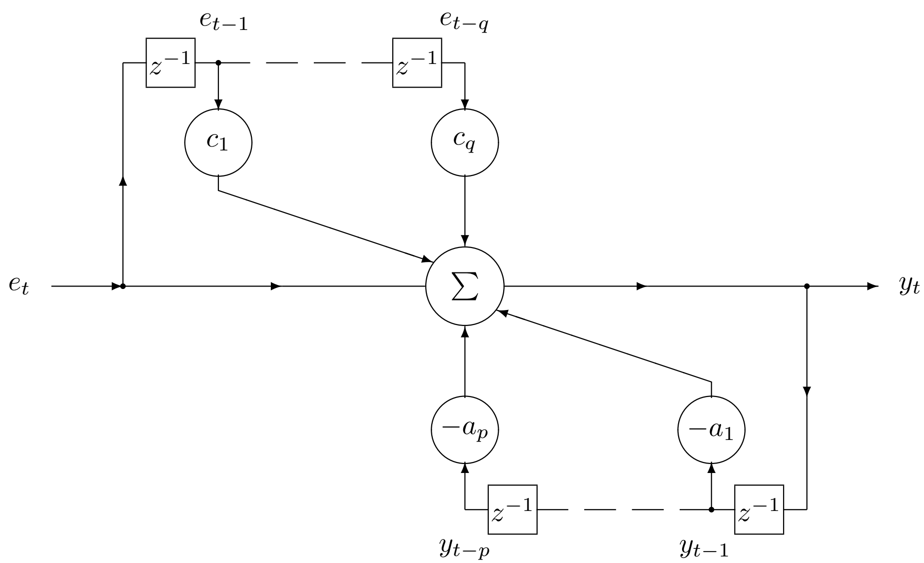


From the ACF, we are unable to determine the model order. In this case, it is an AR(4)-process, such that

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4}$$

As $\phi_y(\omega) = \sigma_e^2 \left|A(\omega)\right|^{-2}$, the roots of $A(z)$ determines where the peaks of the PSD will be located. If $y_t \in \mathbb{R}$, these will be symmetric; the PSD will thus have two peaks for the positive frequencies.

The autoregressive moving average process



An ARMA(p,q)-process contains both an AR- and an MA-part, such that

$$A(z)y_t = C(z)e_t$$

where e_t is a zero-mean white noise process with variance σ_e^2. The process is stationary if the roots of $A(z) = 0$ lie within the unit circle.

An ARMA(p,q) process will satisfy

$$m_y = 0$$

$$\phi_y(\omega) = \frac{|C(\omega)|^2}{|A(\omega)|^2}\sigma_e^2$$

$$r_y(k) + \sum_{\ell=1}^{p} a_\ell r_y(k-\ell) = 0 \quad \text{for } |k| > q$$

The autoregressive moving average process

Example:
Consider a real-valued ARMA(1,1) process, defined as

$$y_t + a_1 y_{t-1} = e_t + c_1 e_{t-1}$$

where e_t is a zero-mean white noise process. The autocovariance of y_t may then be formed by multiplying with y_{t-k} and taking the expectation, i.e.,

$$E\Big\{ y_t y_{t-k} + a_1 y_{t-1} y_{t-k} \Big\} = E\Big\{ e_t y_{t-k} + c_1 e_{t-1} y_{t-k} \Big\}$$

implying that

$$\begin{aligned}
r_y(0) + a_1 r_y(1) &= E\{e_t y_t\} + c_1 E\{e_{t-1} y_t\} \\
&= E\{e_t(-a_1 y_{t-1} + e_t + c_1 e_{t-1})\} + c_1 E\{e_{t-1} y_t\} \\
&= \sigma_e^2 + c_1 E\{e_{t-1}(-a_1 y_{t-1} + e_t + c_1 e_{t-1})\} \\
&= (1 + c_1^2 - c_1 a_1)\sigma_e^2 \\
r_y(1) + a_1 r_y(0) &= E\{e_t y_{t-1}\} + c_1 E\{e_{t-1} y_{t-1}\} = c_1 \sigma_e^2
\end{aligned}$$

and $r_y(k) = -a_1 r_y(k-1)$, for $k \geq 2$.

Inserting $r_y(1)$ from the latter equation into the former yields

$$\begin{aligned}
r_y(0) &= -a_1 \left(c_1 \sigma_e^2 - a_1 r_y(0) \right) + (1 + c_1^2 - c_1 a_1)\sigma_e^2 \\
&= \frac{1 + c_1^2 - 2c_1 a_1}{1 - a_1^2} \sigma_e^2 \\
r_y(1) &= \left(c_1 - a_1 \frac{1 + c_1^2 - 2c_1 a_1}{1 - a_1^2} \right) \sigma_e^2 = \frac{(1 - c_1 a_1)(c_1 - a_1)}{1 - a_1^2} \sigma_e^2
\end{aligned}$$

Normalizing with $r_y(0)$, the ACF is obtained as

$$\begin{aligned}
\rho_y(1) &= \frac{(1 - c_1 a_1)(c_1 - a_1)}{1 + c_1^2 - 2c_1 a_1} \\
\rho_y(k) &= (-a_1)^{k-1} \rho_y(1) \quad \text{for } k \geq 2
\end{aligned}$$

Note that $\rho_y(k)$ exhibit an exponential decay for lags larger than one. In general, for an ARMA(p,q) process, this exponential decay will start after lag $|q - p|$.