Stochastic variables

Andreas Jakobsson

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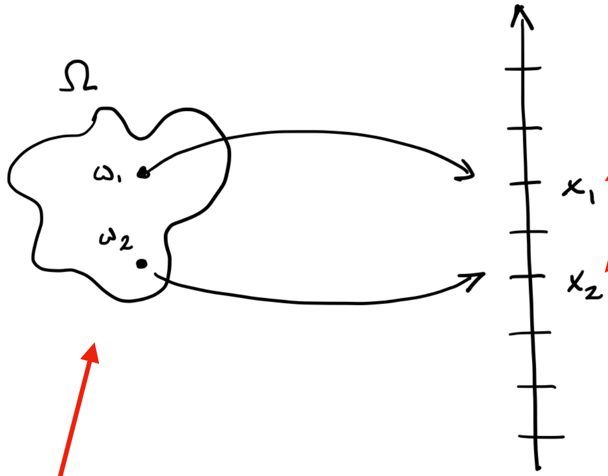
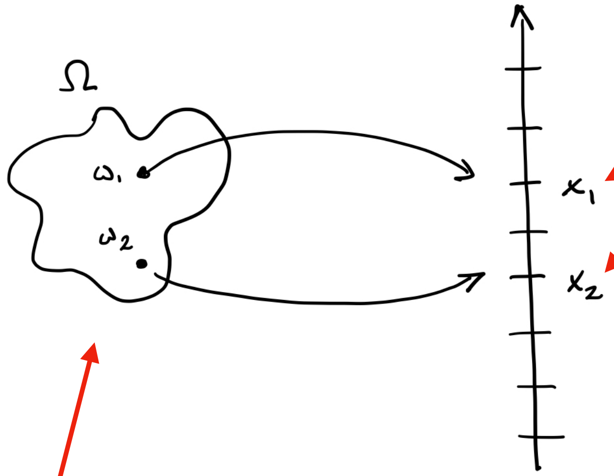


Realisations

The sample space, Ω, contains all possible outcomes

A *stochastic variable* is a mapping from an event in the sample space, Ω, to a point on the real line. This outcome is the *realisation*.

Stochastic variables



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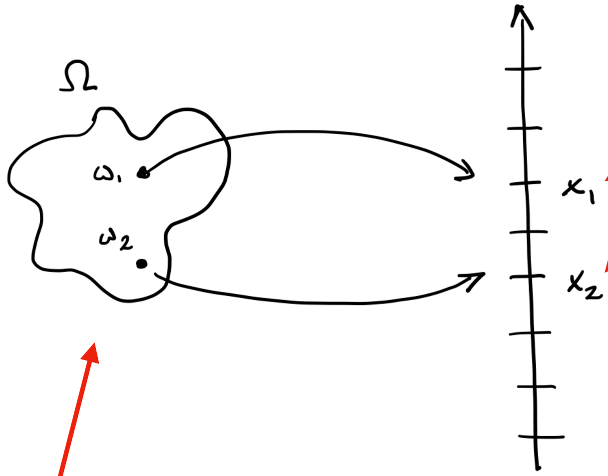
Sample space

Realisations

$\{\ 1,\ 3,\ 5,\ 1,\ 6,\ 2\ \}$

This would be an example of a one-dimensional *discrete* stochastic variable.

Stochastic variables



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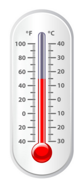
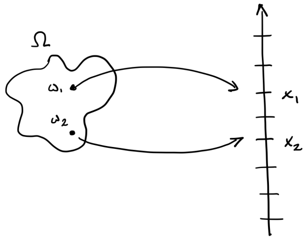


$\{\ 31.451,\ 42.327,\ \pi\ \}$

This would be an example of a one-dimensional *continuous* stochastic variable.

Stochastic variables

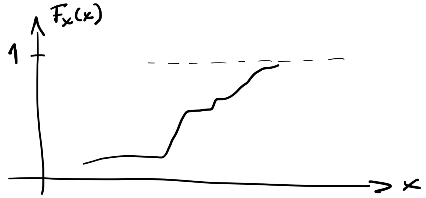


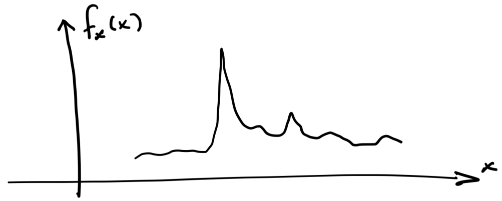
The mapping is characterised by the *probability distribution function*

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x)dx$$

where $f_X(x)$ is the *probability density function* (pdf)

$$f_X(x) = \frac{d}{dx} F_X(x)$$





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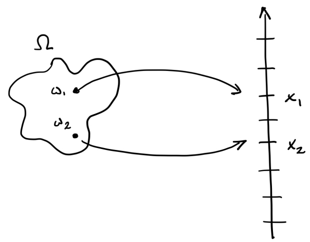

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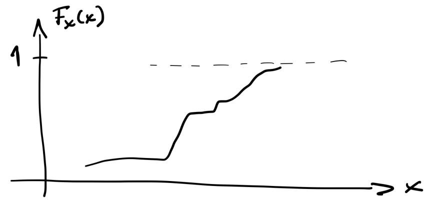

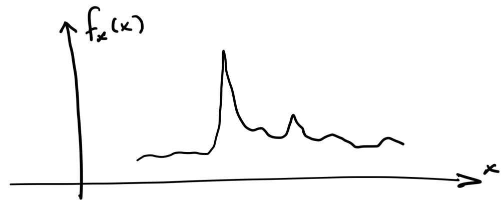

The *expected value* is defined as

$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x)dx = m_X$$

This is the value one can expect the variable to take, on average.

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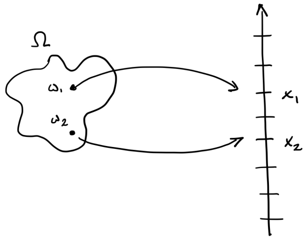

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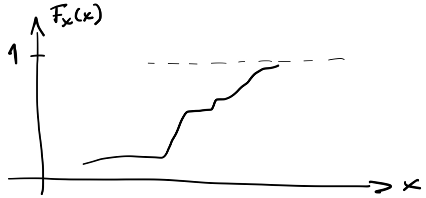

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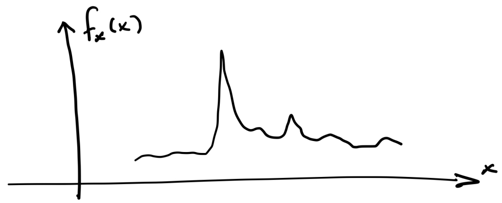

The *expected value* is defined as

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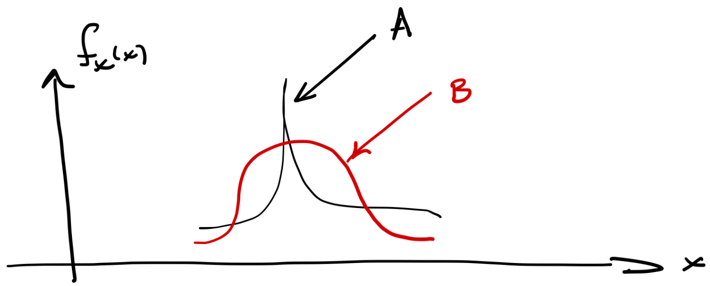
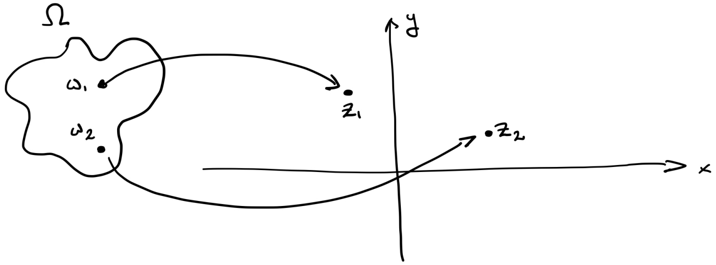
The *variance* is defined as

$$\begin{aligned} V\{X\} &= E\{(X - m_X)^2\} \\ &= E\{X^2 - 2m_X X + m_X^2\} = E\{X^2\} - 2m_X E\{X\} + m_X^2 \\ &= E\{X^2\} - m_X^2 \\ &= \int_{-\infty}^{\infty} (x - m_X)^2 f_X(x)dx \end{aligned}$$



This is a measure how much different realisations can be expected to differ from each other.

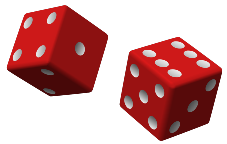
Stochastic variables



A *stochastic variable* may have more than one dimension. For example, the mapping from the sample space can be to a 2-D variable $z = \begin{bmatrix} x & y \end{bmatrix}$, making the realisation a point in a 2-D space.

Sample space

Realisations



$z_1 = \begin{bmatrix} 2 & 4 \end{bmatrix}$, $z_2 = \begin{bmatrix} 5 & 1 \end{bmatrix}$

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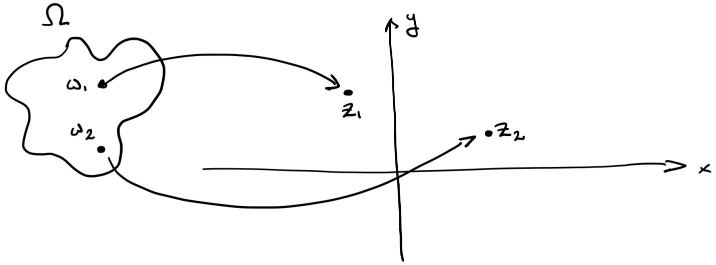

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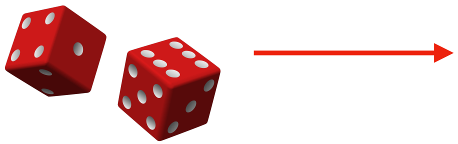


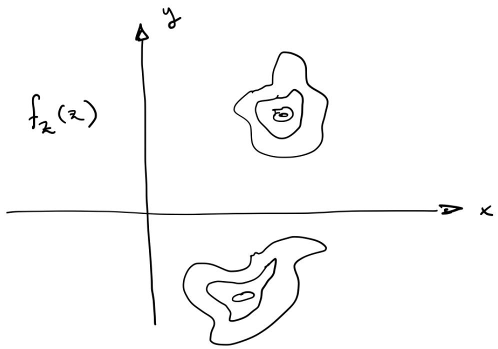
$z_1 = \begin{bmatrix} 2 & 4 \end{bmatrix}$, $z_2 = \begin{bmatrix} 5 & 1 \end{bmatrix}$

In this case, the mapping is characterised by the probability distribution function

$$F_Z(z) = F_{X,Y}(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)dxdy$$

where the pdf $f_{X,Y}(x,y)$ is

$$f_{X,Y}(x,y) = \frac{\partial}{\partial x \partial y} F_{X,Y}(x,y) = f_Z(z)$$



The pdf $f_Z(z)$ is a 3-D function

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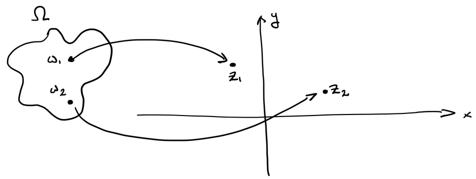

If X and Y are *statistically independent*, the pdf is separable, i.e.,

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

This is a very strong assumption. A weaker assumption is that of the variables being *uncorrelated*, implying that

$$E\{XY\} = E\{X\}E\{Y\}$$

It the variables are independent, they are also uncorrelated (show this!), but not necessarily the other way around.

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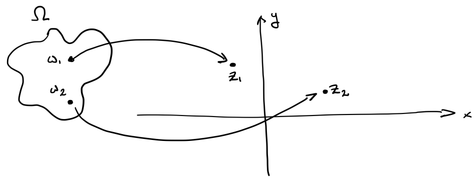

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If the variables depend on each other, this dependence can be measured via the cross-covariance function

$$r_{XY} = C\{X, Y\} = E\{(X - m_x)(Y - m_Y)^*\} = E\{XY^*\} - m_X m_Y^*$$

where $(\cdot)^*$ denotes the conjugate. Clearly, $r_{XY} = 0$ if X and Y are uncorrelated.

As the variance and cross-covariance scale depend on the scale of the stochastic variables, one instead often use the correlation coefficient

$$\rho_{XY} = \frac{C\{X, Y\}}{\sqrt{V\{X\}}\sqrt{V\{Y\}}}$$

which is bounded $0 \leq \rho_{XY} \leq 1$.