State space models

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State space models

We introduce the state space representation

$$\begin{aligned}\mathbf{x}_{t+1} &= \mathbf{A}_t\mathbf{x}_t + \mathbf{B}_t\mathbf{u}_t + \mathbf{e}_t\\ \mathbf{y}_t &= \mathbf{C}_t\mathbf{x}_t + \mathbf{w}_t\end{aligned}$$

where $\mathbf{y}_t$ is the m-dimensional measurement vector at time t, being the observed signal, whereas $\mathbf{x}_t$ is the internal n-dimensional state vector.

The matrices $\mathbf{A}_t$, $\mathbf{B}_t$, and $\mathbf{C}_t$ are known, potentially time-varying, matrices of appropriate dimensions.

The noise processess $\mathbf{e}_t$ and $\mathbf{w}_t$ detail model uncertainty and measurement noise, respectively, which are here assumed to be uncorrelated, satisfying

$$\begin{aligned}E\{\mathbf{e}_s\mathbf{e}_t^T\} &= \begin{cases}\mathbf{R}_e & \text{if } s=t\\ \mathbf{0} & \text{otherwise}\end{cases}\\ E\{\mathbf{w}_s\mathbf{w}_t^T\} &= \begin{cases}\mathbf{R}_w & \text{if } s=t\\ \mathbf{0} & \text{otherwise}\end{cases}\\ E\{\mathbf{e}_s\mathbf{w}_t^T\} &= \mathbf{0} \qquad \forall s,t\end{aligned}$$

Furthermore, we assume the system to be stable, such that $det(\lambda\mathbf{I} - \mathbf{A}) = 0$ have all its solution inside the unit circle.

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Example:
One may express an ARMA(p,q) process, on state space form using a controllable canonical form by introducing

$$d = \max(p, q+1)$$

and letting $a_\ell = 0$, for $\ell > p$, and $c_\ell = 0$, for $\ell > q$. Then,

$$\begin{aligned}\mathbf{x}_t &= \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{d-1} & -a_d\\ 1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \dots & 1 & 0\end{bmatrix}\mathbf{x}_{t-1} + \begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}e_t\\ y_t &= \begin{bmatrix}1 & c_1 & \dots & c_{d-1}\end{bmatrix}\mathbf{x}_t\end{aligned}$$

Alternatively, one may use the observable canonical form with

$$\begin{aligned}\mathbf{x}_t &= \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0\\ -a_2 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -a_{d-1} & 0 & 0 & \dots & 1\\ -a_d & 0 & 0 & \dots & 0\end{bmatrix}\mathbf{x}_{t-1} + \begin{bmatrix}1\\c_1\\c_2\\ \vdots\\c_{d-1}\end{bmatrix}e_t\\ y_t &= \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}\mathbf{x}_t\end{aligned}$$

Note that this form assumes the ARMA parameters to be known.

These are only examples of space representations; one can easily form new ones.

State space models

Example:
Consider an AR(1) with an unknown parameter a_1

$$y_t + a_1 y_{t-1} = v_t$$

We can create the state $x_t = a_1$, then

$$\begin{aligned}x_{t+1} &= 1\, x_t\\ y_t &= [\ -y_{t-1}\ ]x_t + v_t\end{aligned}$$

Thus, $\mathbf{A}_t = 1$, $\mathbf{B}_t = 0$, and $\mathbf{C}_t = [\ -y_{t-1}\ ]$.

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In case one wish to allow the estimate of a_1 to vary over time, one can then add a modelling noise, letting

$$\begin{aligned} x_{t+1} &= 1\, x_t + e_t \\ y_t &= \left[\ -y_{t-1}\ \right] x_t + v_t \end{aligned}$$

where the variance of the noise dictates how fast the parameter can change.

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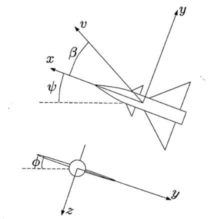
Example:
The Swedish fighter jet, Gripen, uses a state space representation with the internal states

$$\mathbf{x}_t = \left[\ v_y \quad p \quad r \quad \phi \quad \psi \quad \delta_a \quad \delta_r\ \right]^T$$

denoting the velocity in the y direction, the roll angle rate, turning angle rate, roll angle, course angle, aileron deflection and rudder deflection, respectively.



As inputs, one use the aileron deflection and rudder deflection,

$$\mathbf{u}_t = \left[\ \delta_a^c \quad \delta_r^c\ \right]^T$$

This yields the state space representation

$$\frac{\partial}{\partial t}\mathbf{x} = \begin{bmatrix} -0.292 & 8.13 & -201 & 9.77 & 0 & 12.5 & 17.1 \\ -0.152 & -2.54 & 0.561 & -0004 & 0 & 107 & 7.68 \\ 0.364 & -0.0678 & -0.481 & 0.0012 & 0 & 4.67 & -7.98 \\ 0 & 1 & 0.0401 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -20 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & -2.15 \\ -31.7 & 0.0274 \\ 0 & 1.48 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix} \mathbf{u} + \mathbf{N}\mathbf{w}$$

The output are the 4th and 5th states (roll angle, course angle).

State space models

It is not difficult to express a state space representation on the usual transfer function form.

Using the z notation yields

$$z\mathbf{x}_t = \mathbf{A}_t \mathbf{x}_t + \mathbf{B}_t \mathbf{u}_t + \mathbf{e}_t$$

implying that

$$\left(z\mathbf{I} - \mathbf{A}_t\right) \mathbf{x}_t = \mathbf{B}_t \mathbf{u}_t + \mathbf{e}_t$$

and thus

$$\mathbf{y}_t = \mathbf{C}_t \left(z\mathbf{I} - \mathbf{A}_t\right)^{-1} \mathbf{B}_t \mathbf{u}_t + \mathbf{C}_t \left(z\mathbf{I} - \mathbf{A}_t\right)^{-1} \mathbf{e}_t + \mathbf{w}_t$$

which is called the input-output or transfer function form, since the matrices

$$\begin{aligned} &\mathbf{C}_t \left(z\mathbf{I} - \mathbf{A}_t\right)^{-1} \mathbf{B}_t \\ &\mathbf{C}_t \left(z\mathbf{I} - \mathbf{A}_t\right)^{-1} \end{aligned}$$

will form the transfer functions from $\mathbf{u}_t$ and $\mathbf{e}_t$ to $\mathbf{y}_t$, respectively.