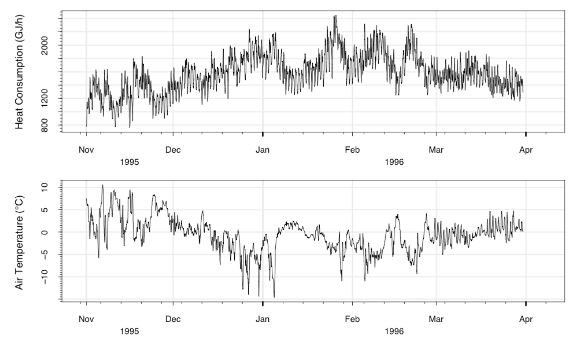
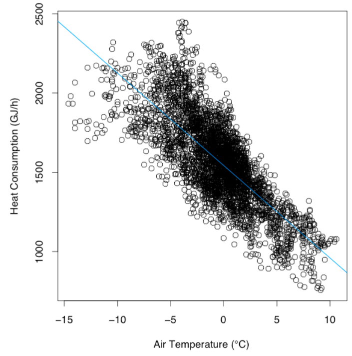
# Filtering

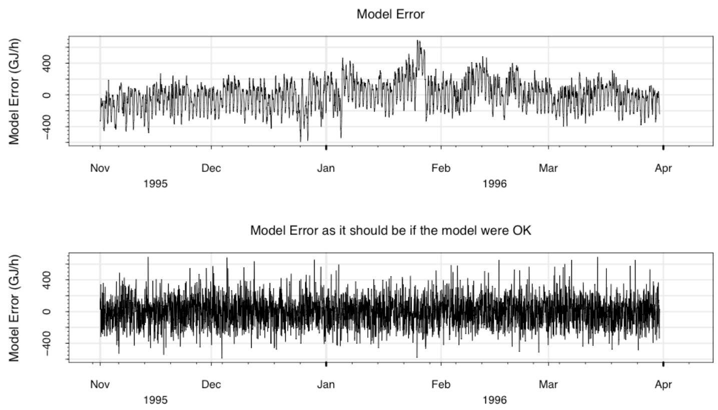
Andreas Jakobsson

## Filtering a stochastic process

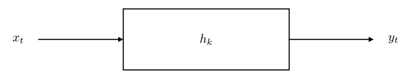



## Filtering a stochastic process



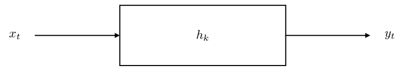

## Filtering a stochastic process




What happens if one filters a stochastic process through a *linear*, *stable*, *casual*, and *time-invariant* filter?

| | | |
|---|---|---|
| Linear: | $\alpha x_1(t) + \beta x_2(t)$ | $\Rightarrow \alpha y_1(t) + \beta y_2(t)$ |
| Time-invariant: | $x_{t-d}$ | $\Rightarrow y_{t-d}$ |

# Filtering a stochastic process




What happens if one filters a stochastic process through a *linear*, *stable*, *casual*, and *time-invariant* filter?

| | | |
|---|---|---|
| Linear: | $\alpha x_1(t) + \beta x_2(t)$ | $\Rightarrow \alpha y_1(t) + \beta y_2(t)$ |
| Time-invariant: | $x_{t-d}$ | $\Rightarrow y_{t-d}$ |

Notable, if $x_t$ is WSS / Gaussian, so is $y_t$.

Recall that such a (discrete-time) filter may be written as

$$y_t = \sum_{k=-\infty}^{\infty} h_{t-k} x_k = \sum_{k=-\infty}^{\infty} h_k x_{t-k}$$

Then,

$$m_y = E\left\{ \sum_{k=-\infty}^{\infty} h_k x_{t-k} \right\} = m_x \sum_{k=-\infty}^{\infty} h_k = m_x H(0)$$

where

$$H(\omega) = \sum_{k=-\infty}^{\infty} h_k e^{-i\omega k}$$

# Filtering a stochastic process

Similarly, the ACF may be formed as

$$r_y(k) = \sum_{m=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} h_m^* h_\ell r_x(m+k-\ell) = r_x(k) \star h_k \star h_{-k}^*$$

where $\star$ denotes the convolution operator. Expressed in the frequency domain,

$$\boxed{\phi_y(\omega) = |H(\omega)|^2 \, \phi_x(\omega)}$$

# Filtering a stochastic process

Similarly, the ACF may be formed as

$$r_y(k) = \sum_{m=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} h_m^* h_\ell r_x(m+k-\ell) = r_x(k) \star h_k \star h_{-k}^*$$

where $\star$ denotes the convolution operator. Expressed in the frequency domain,

$$\boxed{\phi_y(\omega) = |H(\omega)|^2 \, \phi_x(\omega)}$$

Defining the cross spectral density of two stationary processes, $x_t$ and $y_t$, as the DFT of the cross-covariance function, i.e.,

$$\phi_{x,y}(\omega) = \sum_{k=-\infty}^{\infty} r_{x,y}(k) e^{-i\omega k}$$

Then,

$$\phi_{x,y}(\omega) = H(\omega)\phi_x(\omega)$$

This is the so-called Wiener-Hopf equation.