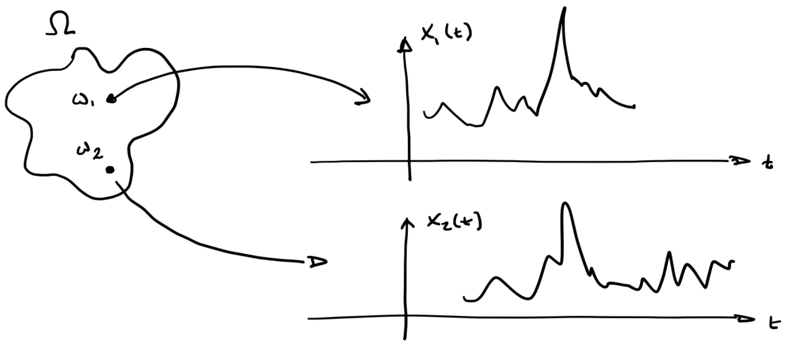
Stochastic processes

Andreas Jakobsson

Stochastic processes



A stochastic variable may have many dimensions, say N dimensions, so that

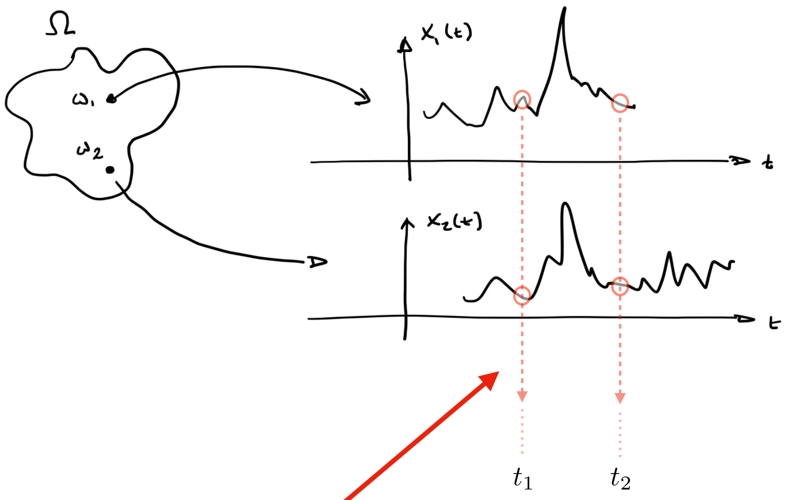
$$z = \begin{bmatrix} x_1 & \ldots & x_N \end{bmatrix}^T,$$

making the realisation a point in a N-D space. If we let N grow, making each realisation a function instead, we have a *stochastic process.*

From a practical perspective, you always only have limited amount of data, say N samples, which may be viewed as observing N samples of a stochastic process (or as an N-D stochastic variable).

As a stochastic process is a function along one (or more) variables, we often use the notation x_t, or $x(t)$, to indicate that it is a function of t. We will here use the same notation for the corresponding realisation.

Stochastic processes



A stochastic variable may have many dimensions, say N dimensions, so that

$$z = \left[\begin{array}{ccc} x_1 & \dots & x_N \end{array} \right]^T,$$

making the realisation a point in a N-D space. If we let N grow, making each realisation a function instead, we have a *stochastic process*.

From a practical perspective, you always only have limited amount of data, say N samples, which may be viewed as observing N samples of a stochastic process (or as an N-D stochastic variable).

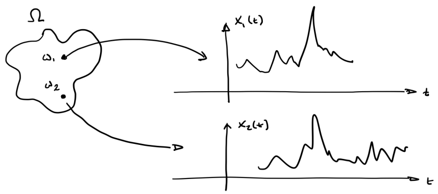
As a stochastic process is a function along one (or more) variables, we often use the notation x_t, or $x(t)$, to indicate that it is a function of t. We will here use the same notation for the corresponding realisation.

The mean function, $m_x(t_1) = E\{x_{t_1}\}$, is the mean value of the process at t_1.

The variance function, $v_x(t_1) = V\{x_{t_1}\}$, is the variance of the process at t_1.

The covariance function, $r_x(t_1, t_2) = C\{x_{t_1}, x_{t_2}\}$, is the dependence between realisations at different times.

Stochastic processes



We will restrict our attention to *wide-sense stationary* (WSS) processes. For such processes, the statistical properties do not change over time. Furthermore,

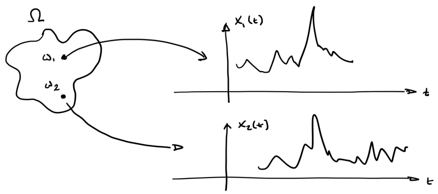
(i) The mean function is constant and finite, $m_x(t) = m_x < \infty$.

(ii) The (auto-)covariance, $C\{x_t, x_s\}$, only depends on the difference $(s - t)$ and *not* on the actual values of s and t.

(iii) The variance of the process is finite, i.e., $E\{|x_t|^2\} < \infty$.

Furthermore, we will assume that all considered processes are *ergodic*. Essentially, this implies that it is possible to estimate the characteristics of the process from a single realisation.

Stochastic processes



We will restrict our attention to *wide-sense stationary* (WSS) processes. For such processes, the statistical properties do not change over time. Furthermore,

(i) The mean function is constant and finite, $m_x(t) = m_x < \infty$.

(ii) The (auto-)covariance, $C\{x_t, x_s\}$, only depends on the difference $(s - t)$ and *not* on the actual values of s and t.

(iii) The variance of the process is finite, i.e., $E\{|x_t|^2\} < \infty$.

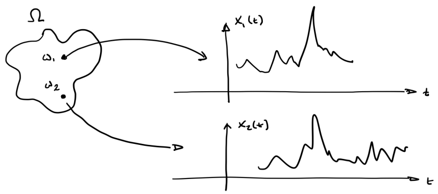
Furthermore, we will assume that all considered processes are *ergodic*. Essentially, this implies that it is possible to estimate the characteristics of the process from a single realisation.

Example:
Let $x_t = A\cos(2\pi f t + \phi)$, where ϕ is uniformly distributed on $[0, 2\pi)$ and A and ϕ are independent. Then,

$$\begin{aligned}
E\{x_t\} &= E\{A\cos(2\pi f t + \phi)\} = E\{A\}E\{\cos(2\pi f t + \phi)\} \\
&= m_A \int_{-\infty}^{\infty} \cos(2\pi f t + \phi) f_\phi(\phi) d\phi \\
&= \frac{m_A}{2\pi} \int_0^{2\pi} \cos(2\pi f t + \phi) d\phi \\
&= \frac{m_A}{2\pi} \left[\sin(2\pi f t + \phi)\right]_0^{\pi} = 0
\end{aligned}$$

The average value of this process is zero for all t. This satisfies (i). How about (ii) and (iii)?

Stochastic processes



We will restrict our attention to *wide-sense stationary* (WSS) processes. For such processes, the statistical properties do not change over time. Furthermore,

(i) The mean function is constant and finite, $m_x(t) = m_x < \infty$.

(ii) The (auto-)covariance, $C\{x_t, x_s\}$, only depends on the difference $(s - t)$ and *not* on the actual values of s and t.

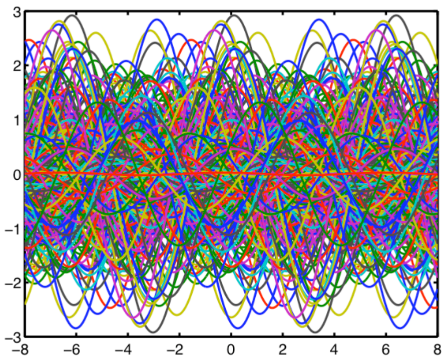
(iii) The variance of the process is finite, i.e., $E\{|x_t|^2\} < \infty$.

Furthermore, we will assume that all considered processes are *ergodic*. Essentially, this implies that it is possible to estimate the characteristics of the process from a single realisation.

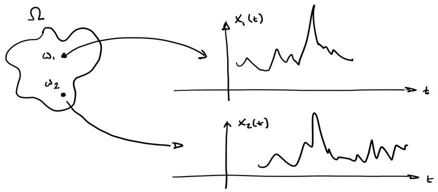
Example:
Let $x_t = A\cos(2\pi ft + \phi)$, where ϕ is uniformly distributed on $[0, 2\pi)$ and A and ϕ are independent. Then,

$$\begin{aligned} E\{x_t\} &= E\{A\cos(2\pi ft + \phi)\} = E\{A\}E\{\cos(2\pi ft + \phi)\} \\ &= m_A \int_{-\infty}^{\infty} \cos(2\pi ft + \phi) f_\phi(\phi) d\phi \\ &= \frac{m_A}{2\pi} \int_0^{2\pi} \cos(2\pi ft + \phi) d\phi \\ &= \frac{m_A}{2\pi} \left[\sin(2\pi ft + \phi)\right]_0^{\pi} = 0 \end{aligned}$$

The average value of this process is zero for all t. This satisfies (i). How about (ii) and (iii)?



Stochastic processes



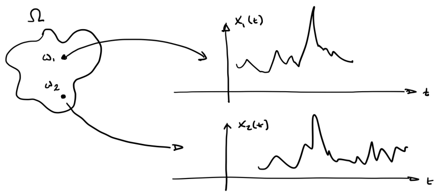
Example:
According to (ii), for a WSS process $C\{x_s, x_t\}$ should only depend on $(s-t)$. Lets check!

$$\begin{aligned}
C\{x_s, x_t\} &= E\{A^2 \cos(2\pi f s + \phi) \cos(2\pi f t + \phi)\} \\
&= E\{A^2\} E\{\cos(2\pi f s + \phi) \cos(2\pi f t + \phi)\} \\
&= \frac{E\{A^2\}}{2} E\{\cos(2\pi f (s+t) + 2\phi) + \cos(2\pi f (s-t))\} \\
&= \frac{E\{A^2\}}{4\pi} \int_0^{2\pi} \cos(2\pi f (s+t) + 2\phi) + \cos(2\pi f (s-t)) d\phi \\
&= \frac{E\{A^2\}}{4\pi} \cos(2\pi f (s-t))
\end{aligned}$$

The covariance thus only depends on $(s-t)$, with $C\{x_s, x_s\} < \infty$. This is thus a WSS process!

It is worth noting that the covariance function of a sinusoid is itself a sinusoid, with the same frequency!

Stochastic processes



The autocovariance function only depends on the difference between s and t. To simplify notation, we use $r_x(t, t-k) = r_x(k)$, i.e.,

$$r_x(k) = C\{x_t, x_{t-k}\} = E\{x_t x_{t-k}^*\} - m_x m_x^*$$

with the variance being $V\{x_t\} = r_x(0)$.

The autocovariance function of a WSS process satisfies

(i) It is conjugate symmetric, i.e., $r_x(k) = r_x^*(-k)$.

(ii) The variance is always non-negative, i.e., $r_x(0) \geq 0$.

(iii) The function takes it largest values at lag 0, i.e.,

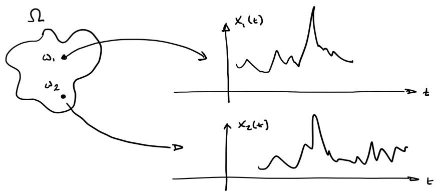
$$r_x(0) \geq |r_x(k)|, \quad \forall k$$

Example:
Returning to the sinusoidal process, $x_t = A\cos(2\pi f t + \phi)$, we note that

$$r_x(k) = C\{x_t, x_{t-k}\} = \frac{E\{A^2\}}{4\pi} \cos(2\pi f k)$$

As $\cos(\cdot)$ is symmetric, with its maximum at $k = 0$, we satisfy (i) and (iii). Also, $r_x(0) \geq 0$, so (ii) is also satisfied. If $r_x(k) = r_x(0)$ for any $k \neq 0$, the process is always periodic.

Stochastic processes



The auto-correlation function (ACF) is defined as

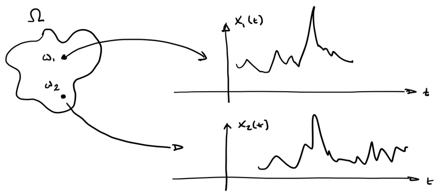
$$\rho_x(k) = \frac{r_x(k)}{r_x(0)}$$

which is thus bounded as $|\rho_x(k)| \leq 1$. Similarly, we define the cross-correlation function between x_t and x_t as

$$\rho_{x,y}(k) = \frac{r_{x,y}(k)}{\sqrt{r_x(0)}\sqrt{r_y(0)}}$$

where $r_{x,y}(k) = C\{x_t, y_{t-k}\}$. Show that $|\rho_{x,y}(k)| \leq 1$.

Stochastic processes



The auto-correlation function (ACF) is defined as

$$\rho_x(k) = \frac{r_x(k)}{r_x(0)}$$

which is thus bounded as $|\rho_x(k)| \leq 1$. Similarly, we define the cross-correlation function between x_t and x_t as

$$\rho_{x,y}(k) = \frac{r_{x,y}(k)}{\sqrt{r_x(0)}\sqrt{r_y(0)}}$$

where $r_{x,y}(k) = C\{x_t, y_{t-k}\}$. Show that $|\rho_{x,y}(k)| \leq 1$.

Recall that for two random variables x and y, it holds that

$$V\{ax + by\} = |a|^2 V\{x\} + |b|^2 V\{y\} + (a^* b + ab^*) C\{x, y\}$$

As the variance is non-negative, $V\{ax + by\} \geq 0$. Thus, if $a, b \in \mathbb{R}$,

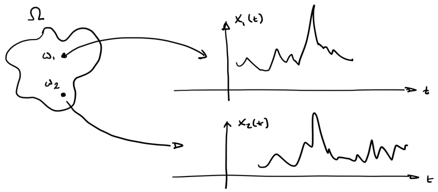
$$V\{ax_t + by_{t-k}\} = a^2 r_x(0) + b^2 r_y(0) + 2ab\, r_{x,y}(k) \geq 0$$

Let $a = 1/\sqrt{r_x(0)}$ and $b = \pm 1/\sqrt{r_y(0)}$, then

$$1 + 1 \pm 2\rho_{x,y}(k) \geq 0$$

Thus, $|\rho_{x,y}(k)| \leq 1$.

Stochastic processes



Example:
We want to determine the periodic heartbeat of a fetus in its mothers womb. As the mothers heart is also beating, we should expect our signal to be of the form

$$x_t = A_f \cos(2\pi f_f t + \phi_f) + A_m \cos(2\pi f_m t + \phi_m)$$

where A_f and f_f denote the amplitude and frequency of the fetus' heartbeat, whereas A_m and f_m are those of the mother. Clearly, $A_m \gg A_f$.

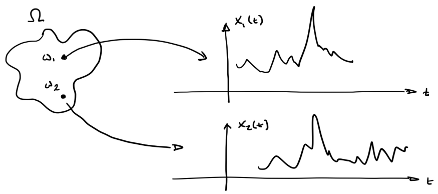
The autocovariance of x_t will thus be

$$r_x(k) = \frac{E\{A_f^2\}}{4\pi} \cos(2\pi f_f k) + \frac{E\{A_m^2\}}{4\pi} \cos(2\pi f_m k)$$

Thus, if we can estimate $r_x(k)$, we can use this to estimate the fetus' heartbeat!

In reality, the measured signal will not consist of only sinusoids; all forms of measurement are corrupted by noise. How can we deal with this?

White noise



To model various forms of measurement noise, we make use of a particular form of process, termed a *white noise*. For such a process it holds that

$$r_x(k) = \begin{cases} \sigma_x^2 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

where σ_x^2 denotes the variance of the process. The process is thus such completely uncorrelated one sample to the next!

Often, one also assumes that each value of the process has a Gaussian distribution. If such a noise is assumed to be added to the relevant signal of interest, one then speaks of a *additive Gaussian white noise* (AGWN).

Example:
Returning to the fetus data, we will assume that the measured signal is

$$y_t = x_t + w_t$$

where w_t is an AWGN with variance σ_w^2. Then,

$$r_y(k) = r_x(k) + r_w(k) = r_x(k) + \sigma_w^2 \delta_K(k)$$

with $\delta_K(k)$ denotes the Kronecker delta, i.e.,

$$\delta_K(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$