Moving Average Processes

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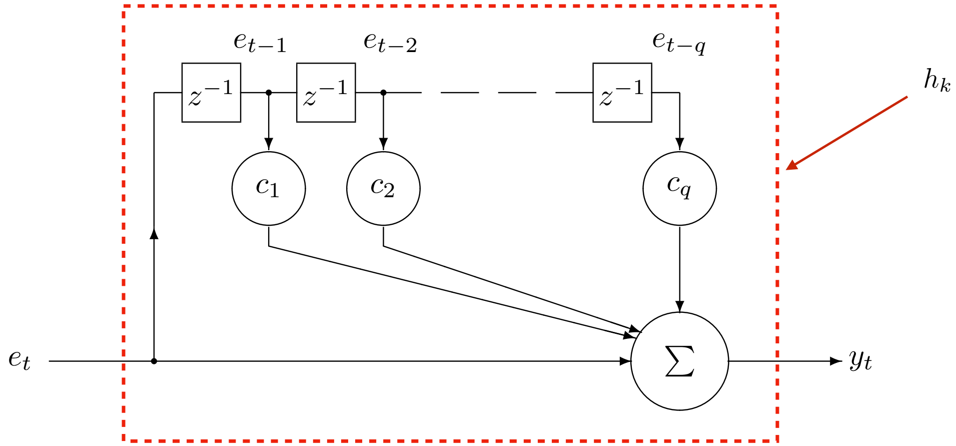
The moving average process



The process y_t is called a *moving average* (MA) process if

$$y_t = e_t + c_1 e_{t-1} + \ldots + c_q e_{t-q} = C(z) e_t$$

where $C(z)$ is a *monic* polynomial of order q (in z^{-1}), i.e.,

$$C(z) = 1 + c_1 z^{-1} + \ldots + c_q z^{-q}$$

with $c_q \neq 0$, and e_t is a zero-mean white noise process with variance σ_e^2.

The moving average process

An MA(q) process will satisfy

$$
\begin{aligned}
m_y &= E\{C(z)e_t\} = 0 \\
r_y(k) &= \begin{cases} \sigma_e^2 \left(c_k + c_1 c_{k+1} + \ldots + c_{q-k} c_q \right) & \text{if } |k| \leq q \\ 0 & \text{if } |k| > q \end{cases} \\
\phi_y(\omega) &= \sigma_e^2 \left| C(\omega) \right|^2
\end{aligned}
$$

where $C(\omega)$ indicates that $C(z)$ is evaluated at frequency ω, i.e., $z = e^{i\omega}$.

In particular, note that $r_y(k) = 0$ for $|k| > q$.

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Example:
Consider the (real-valued) MA(1) process $y_t = e_t + c_1 e_{t-1}$, i.e., $C(z) = 1 + c_1 z^{-1}$.
The auto-covariance of y_t is

$$
\begin{aligned}
r_y(0) &= \sigma_e^2(1 + c_1^2) \\
r_y(1) &= \sigma_e^2 c_1 \\
r_y(k) &= 0, \quad \text{for } |k| > 1
\end{aligned}
$$

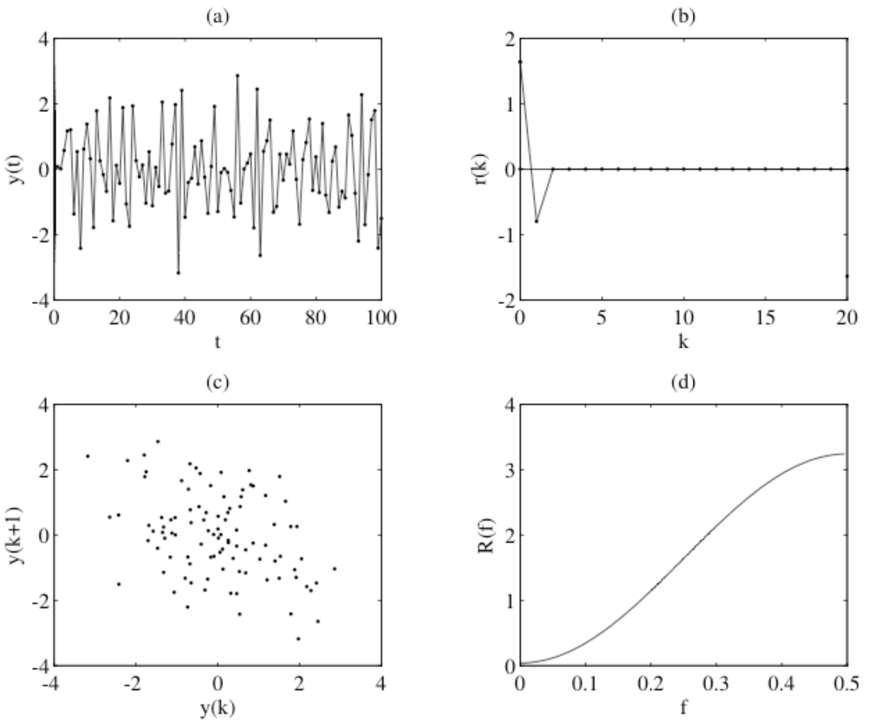
with $r_y(k) = r_y(-k)$, $\forall k$. Similarly, the PSD of y_t is

$$
\begin{aligned}
\phi_y(\omega) &= \sigma_e^2 \left|1 + c_1 e^{-i\omega}\right|^2 \\
&= \sigma_e^2 \Big(c_1 e^{i\omega} + 1 + c_1^2 + c_1 e^{-i\omega}\Big) \\
&= \sigma_e^2 \Big(1 + c_1^2 + 2c_1 \cos(\omega)\Big)
\end{aligned}
$$

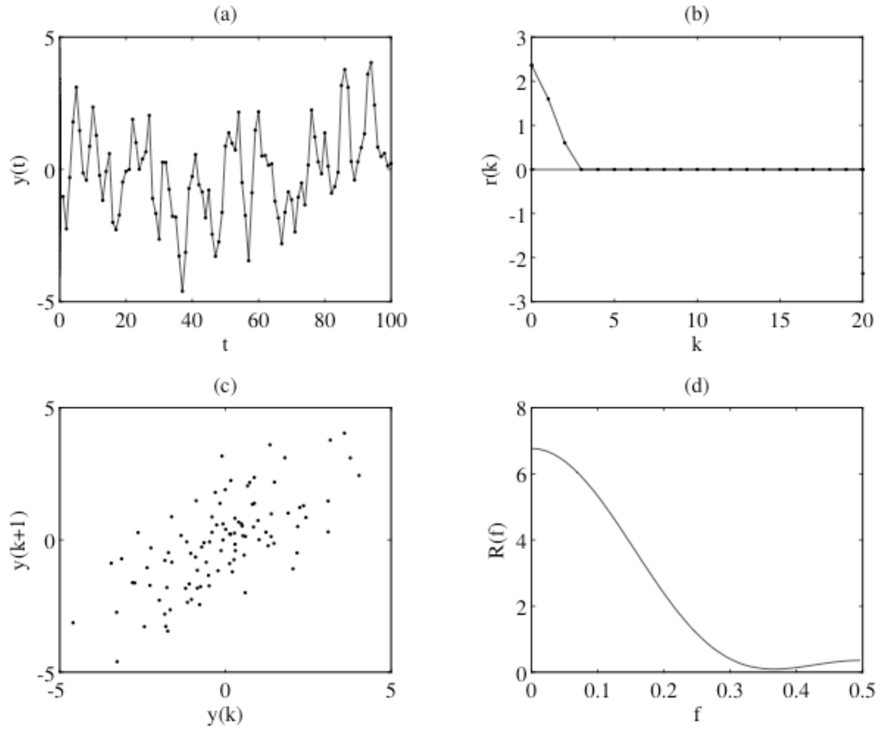
for $\omega = 2\pi f$, with $-0.5 < f \leq 0.5$

The roots of $C(z)$ will determine the locations of the nulls in $\phi_y(\omega)$.

The moving average process



MA(1)–process $Y(t) = e(t) - 0.8e(t-1)$; (a) realisation, (b) covf. func., (c) scatter-plot and (d) spectral density.



MA(2)–process $Y(t) = e(t) + e(t-1) + 0.6e(t-2)$.

The moving average process

For large N, it holds that

$$
\begin{aligned}
E\{\hat{\rho}_y(k)\} &= 0 \\
V\{\hat{\rho}_y(k)\} &= \frac{1}{N}\Big(1 + 2(\hat{\rho}_y^2(1) + \ldots + \hat{\rho}_y^2(q))\Big)
\end{aligned}
$$

for $k = q+1, q+2, \ldots$. Furthermore, $\hat{\rho}_y(k)$, for $|k| > q$, is asymptotically Normal distributed.

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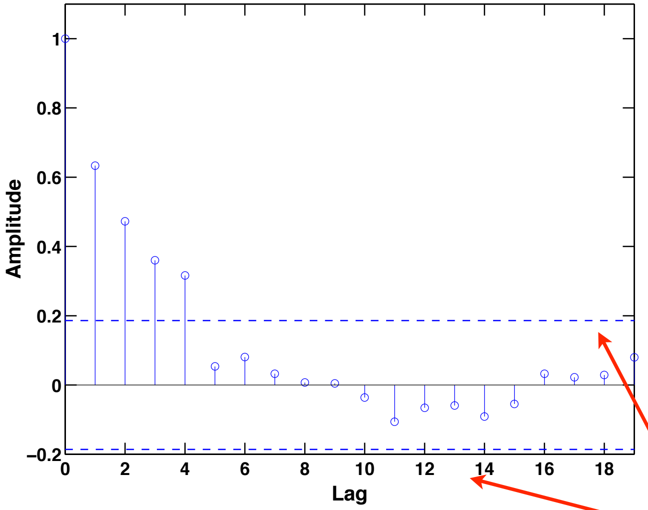
The (approximative) 95% confidence interval for an MA(q) process can be expressed as

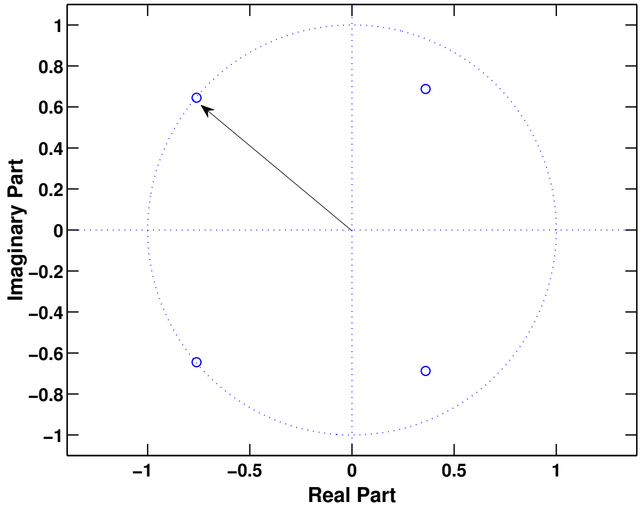
$$\hat{\rho}_e(k) \approx 0 \pm 2\sqrt{\frac{1 + 2(\hat{\rho}_y^2(1) + \ldots + \hat{\rho}_y^2(q)}{N}} \qquad \text{for } |k| \geq q+1$$

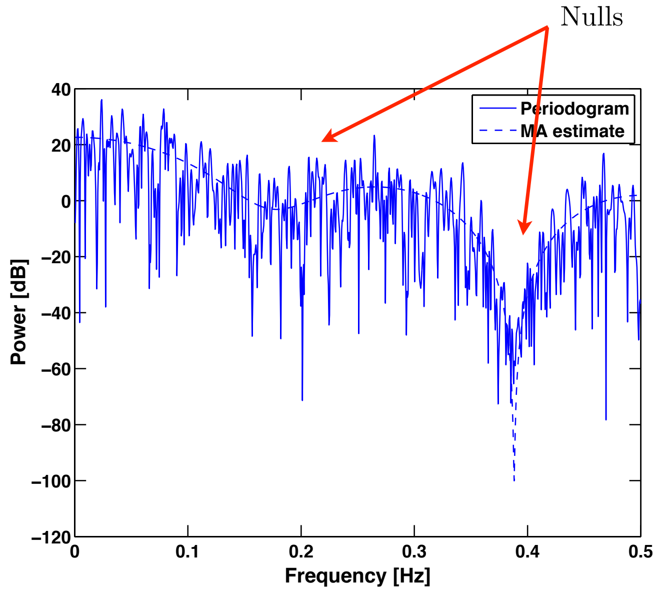
For white noise, i.e., $q = 0$, this simplifies to $\hat{\rho}_e(k) \approx 0 \pm 2/\sqrt{N}$.

Use the provided function `acf`. Remember that you can use `help acf` to learn more on how to use it.

The moving average process







$$\pm 2\sqrt{\frac{1+2(\hat{\rho}_y^2(1)+\ldots+\hat{\rho}_y^2(q)}{N}}$$

Example:
As far as we can tell, this is the ACF of an MA(4) process, i.e.,

$$C(z) = 1 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + c_4 z^{-4}$$

As $\phi_y(\omega) = \sigma_e^2 \left|C(\omega)\right|^2$, the roots of $C(z)$ determines where the nulls of the PSD will be located. If $y_t \in \mathbb{R}$, these will be symmetric; the PSD will thus have two nulls for the positive frequencies.