Time Series Analysis

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Content:

- ARMAX and Box-Jenkins models.
- Determining model orders.
- Modeling example.

The ARMAX process

One may sometimes improve the modeling of a system by considering additional relevant time series. To do so, we introduce two ways to model single-input single-output (SISO) system. The first model is the so-called ARMAX processes, given by

$$A(z)y_t = B(z)x_{t-d} + C(z)e_t$$

where x_{t-d} is the d-step delayed input signal influencing the process y_t, and

$$A(z) = 1 + a_1 z^{-1} + \ldots + a_s z^{-p}$$
$$C(z) = 1 + c_1 z^{-1} + \ldots + c_s z^{-q}$$
$$B(z) = b_0 + b_1 z^{-1} + \ldots + b_s z^{-s}$$

If $A(z) = 1$ or $C(z) = 1$, the corresponding processes are termed MAX and ARX processes, respectively.

Note that the same $A(z)$ polynomial affects both x_t and e_t.

The Box-Jenkins process

A more flexible model is the so-called Box-Jenkins (BJ) model, formed as

$$y_t = \frac{B(z)}{A_2(z)} x_{t-d} + \frac{C_1(z)}{A_1(z)} e_t$$

where

$$A_1(z) = 1 + a_1^{(1)} z^{-1} + \ldots + a_p^{(1)} z^{-p}$$
$$A_2(z) = 1 + a_1^{(2)} z^{-1} + \ldots + a_r^{(2)} z^{-r}$$

with both polynomials having all roots within the unit circle to ensure stability, and

$$C_1(z) = 1 + c_1 z^{-1} + \ldots + c_q z^{-q}$$
$$B(z) = b_0 + b_1 z^{-1} + \ldots + b_s z^{-s}$$

We will in the following focus on the BJ model, striving to identify the relevant model orders (p, q, d, r, s).

The Box-Jenkins process

To do so, we rewrite y_t as

$$y_t = \frac{B(z)z^{-d}}{A_2(z)} x_t + \tilde{e}_t = H(z)x_t + \tilde{e}_t$$

where the input, x_t, is assumed to be uncorrelated of the noise process, $\tilde{e}_t$, and

$$H(z) = \sum_{k=-\infty}^{\infty} h_k z^{-k}$$

is the transfer function between the input and the output signals, with the coefficients h_k being the impulse response weights. We assume this to be a stable, causal, and linear filter.

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If we initially ignore the noise entirely, then

$$A_2(z)H(z) = B(z)z^{-d}$$

Thus, if we knew $H(z)$, we ought to be able to determine the delay and the model orders of $A_2(z)$ and $B(z)$, i.e., (d, r, s).

Determining the BJ model orders

Writing out the resulting polynomial multiplication

$$A_2(z)H(z) = B(z)z^{-d}$$

and identifying the corresponding z^{-k} terms, imply that

$$\begin{aligned}
h_k &= 0, && k < d \\
h_k &= a_1^{(2)} h_{k-1} + \ldots + a_r^{(2)} h_{k-r} + b_0, && k = d \\
h_k &= a_1^{(2)} h_{k-1} + \ldots + a_r^{(2)} h_{k-r} - b_{k-d}, && k = d+1, d+2, \ldots, d+s \\
h_k &= a_1^{(2)} h_{k-1} + \ldots + a_r^{(2)} h_{k-r}, && k > d+s
\end{aligned}$$

Note that, if we know h_k, the first equation gives us information about d. How can we reach r and s?

Determining the BJ model orders

We assume that r only takes the values 0, 1, or 2. Then, we get three cases:

1. When $r = 0$, such that $A_2(z) = 1$, the transfer function only contains a finite number of non-zero impulse response weights, starting with $h_d = b_0$ and ending with $h_{d+s} = -b_s$.

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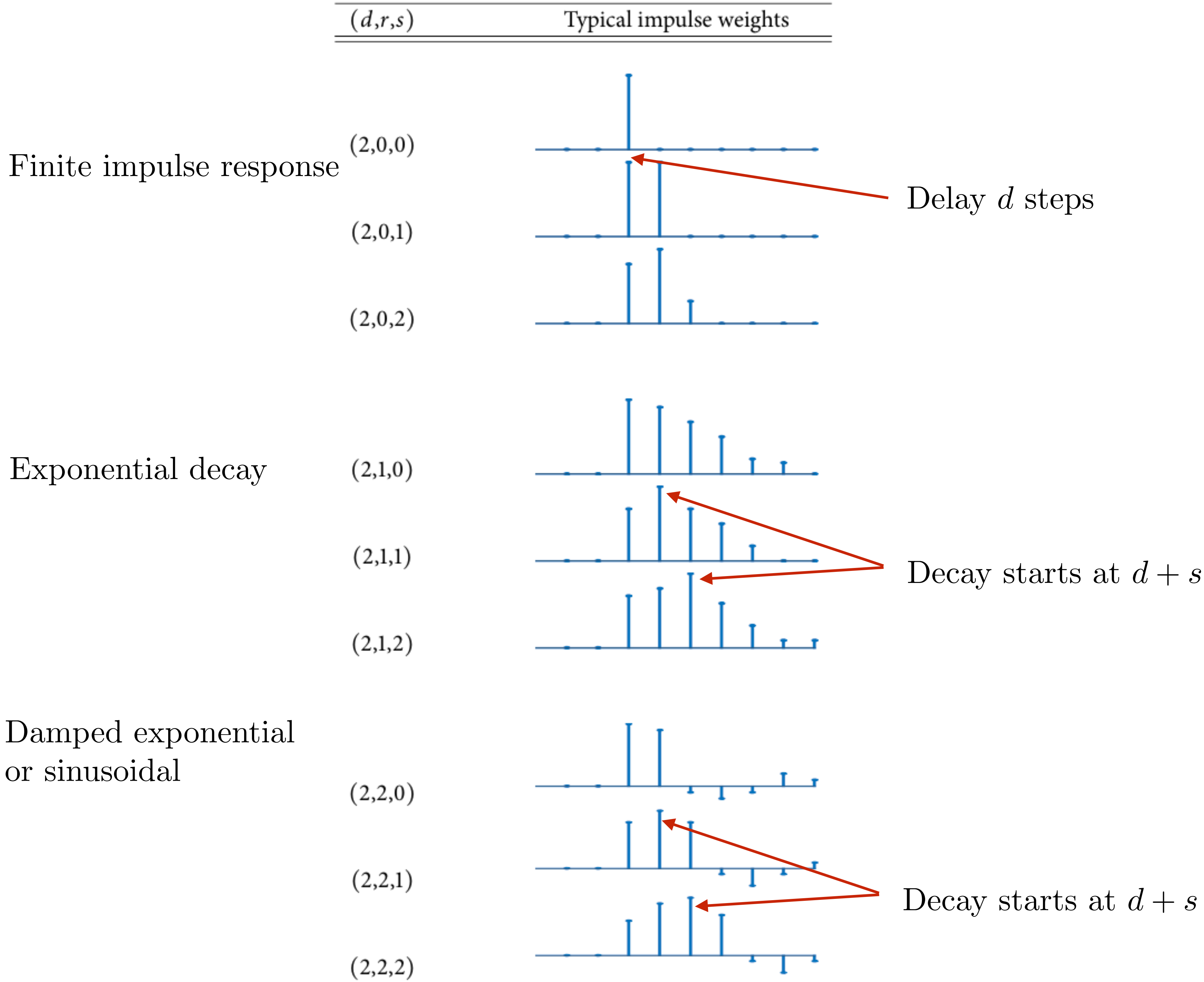
3. When $r = 2$, the impulse response starting at h_d will exhibits either a damped exponential or sinusoidal behavior; if the roots of $A_2(z)$ are real, it will follow an exponential decay, and if they are complex, it will follow a damped sinusoidal behavior. The decay will start from h_{d+s}.



(d,r,s)
Typical impulse weights
Finite impulse response
(2,0,0)
(2,0,1)
(2,0,2)
Delay d steps
Exponential decay
(2,1,0)
(2,1,1)
(2,1,2)
Decay starts at $d + s$
Damped exponential
or sinusoidal
(2,2,0)
(2,2,1)
(2,2,2)
Decay starts at $d + s$

Determining the BJ model orders

As $H(z)$ is assumed to be a causal filter,

$$y_t = h_0 x_t + h_1 x_{t-1} + \ldots + \tilde{e}_t$$

Assuming that $m_y = m_x = 0$, and that x_t is uncorrelated with $\tilde{e}_t$,

$$r_{y,x}(k) = E\{y_t x_{t-k}^*\} = h_0 r_x(k) + h_1 r_x(k-1) + \ldots$$

given that $r_{\tilde{e},x}(k) = 0$ for all k, imply that

$$\rho_{y,x}(k) = \frac{r_{y,x}(k)}{\sigma_y \sigma_x} = \frac{\sigma_x}{\sigma_y}\Big(h_0 \rho_x(k) + h_1 \rho_x(k-1) + \ldots\Big)$$

with σ_x^2 and σ_y^2 denoting the variance of x_t and y_t, respectively. If x_t is a white process,

$$\rho_{y,x}(k) = \frac{\sigma_x}{\sigma_y} h_k \rho_x(0) = \frac{\sigma_x}{\sigma_y} h_k$$

implying that

$$h_k = \frac{\sigma_y}{\sigma_x} \rho_{y,x}(k)$$

For a white input, the impulse response can thus be estimated using the cross-correlation function.

Example

Example: Consider the noise-free first-order (real-valued) system

$$y_t - a_1 y_{t-1} = x_t$$

where $|a_1| < 1$ and x_t is a zero-mean white noise with variance σ_x^2. To form the crosscorrelation, note that

$$y_t = \frac{1}{1 - a_1 z^{-1}} x_t = x_t + a_1 x_{t-1} + a_1^2 x_{t-2} + \ldots$$

and thus, using that both x_t and y_t are zero-mean,

$$r_{y,x}(k) = E\{y_t x_{t-k}\} = \begin{cases} a_1^k \sigma_x^2, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

yielding the crosscorrelation

$$\rho_{y,x}(k) = \frac{r_{y,x}(k)}{\sigma_y \sigma_x} = \begin{cases} a_1^k \sqrt{1 - a_1^2}, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

as $\sigma_y^2 = \sigma_x^2 / (1 - a_1^2)$.

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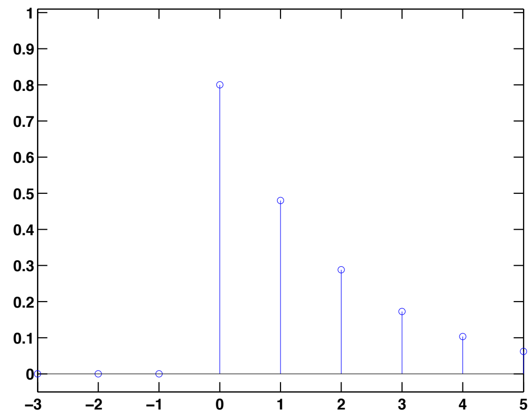

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$$\rho_{y,x}(k) = \frac{r_{y,x}(k)}{\sigma_y \sigma_x} = \begin{cases} a_1^k \sqrt{1 - a_1^2}, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

as $\sigma_y^2 = \sigma_x^2 / (1 - a_1^2)$. For example, $a_1 = 0.4$. Note that the result is just as expected given that $d = 0$ and $r = 1$.

Determining the BJ model orders

In the case when the input is not white, we cannot directly estimate h_k using $\rho_{y,x}(k)$, as

$$\rho_{y,x}(k) = \frac{r_{y,x}(k)}{\sigma_y \sigma_x} = \frac{\sigma_x}{\sigma_y}\Big(h_0 \rho_x(k) + h_1 \rho_x(k-1) + \ldots\Big)$$

To handle this, we proceed to model x_t as an ARMA process, such that

$$A_3(z) x_t = C_3(z) w_t$$

where w_t is a white process. Thus, we view the BJ process as

$$y_t = \frac{B(z) z^{-d}}{A_2(z)} \frac{C_3(z)}{A_3(z)} w_t + \frac{C_1(z)}{A_1(z)} e_t$$

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where w_t is a white process. Thus, we view the BJ process as

$$y_t = \frac{B(z) z^{-d}}{A_2(z)} \frac{C_3(z)}{A_3(z)} w_t + \frac{C_1(z)}{A_1(z)} e_t$$

Multiplying both sides with $A_3(z)/C_3(z)$ yields

$$\epsilon_t = \frac{A_3(z)}{C_3(z)} y_t = \frac{B(z) z^{-d}}{A_2(z)} w_t + \frac{A_3(z)}{C_3(z)} \frac{C_1(z)}{A_1(z)} e_t = H(z) w_t + v_t$$

where

$$v_t = \frac{A_3(z)}{C_3(z)} \frac{C_1(z)}{A_1(z)} e_t$$

Determining the BJ model orders

We thus have that

$$\epsilon_t = H(z)w_t + v_t$$

with w_t being uncorrelated with v_t. This implies that we can estimate h_k as $\rho_{\epsilon,w}(k)$. As

$$V\left\{\rho_{\epsilon,w}(k)\right\} \approx \frac{1}{N-k} \approx \frac{1}{N}$$

we may use $\pm 2/\sqrt{N}$ as our 95% confidence interval of $\rho_{\epsilon,w}(k)$.

Determining the BJ model orders

This suggests the following procedure:

1. Form an ARMA model for x_t, and construct

$$\epsilon_t = \frac{A_3(z)}{C_3(z)} y_t$$
$$w_t = \frac{A_3(z)}{C_3(z)} x_t$$

2. Estimate the impulse response, h_k, using the significant coefficients in $\rho_{\epsilon,w}(k)$. Determine suitable model orders d, r, and s.

3. Using these model orders, form initial estimates of $B(z)$ and $A_2(z)$ by treating $\tilde{e}_t$ as a white noise.

Determining the BJ model orders

Then,

4. Using this preliminary transfer function, form the residual

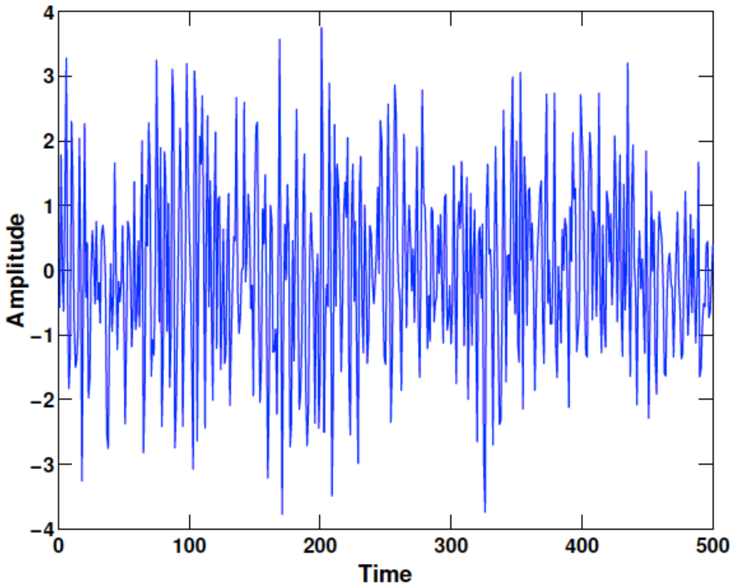
$$\tilde{e}_t = y_t - \frac{B(z)z^{-d}}{A_2(z)} x_t$$

   which is then modeled using an ARMA model, identifying the proper model orders of $A_1(z)$ and $C_1(z)$.

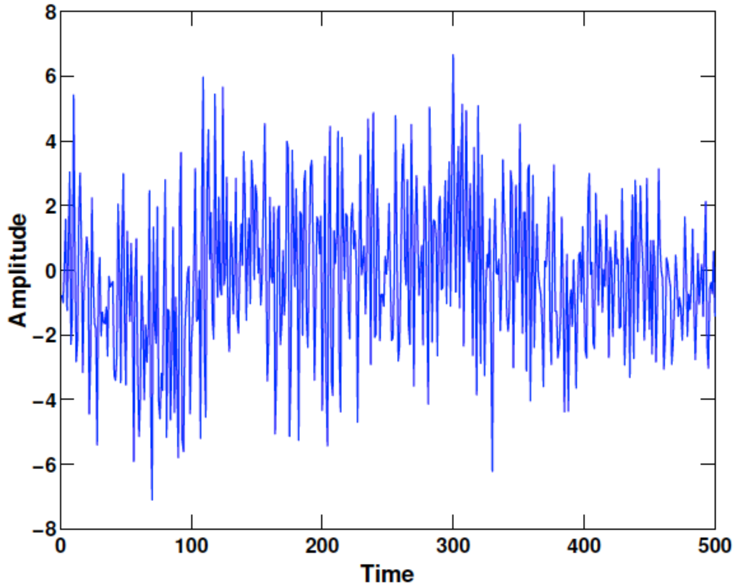
5. Estimate the coefficients for all polynomials jointly.

6. Examine the resulting model residual to determine if this is white.

Example



Input



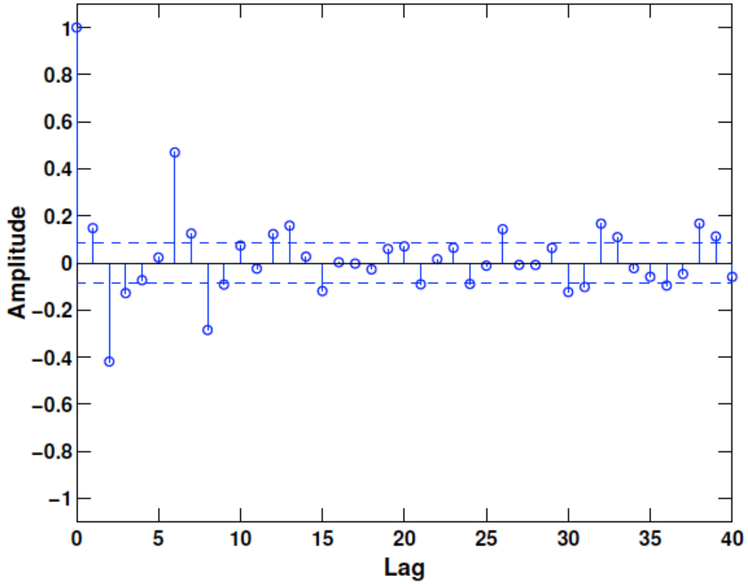
Output

Lets examine an example. We begin with forming a model for the input, x_t, such that

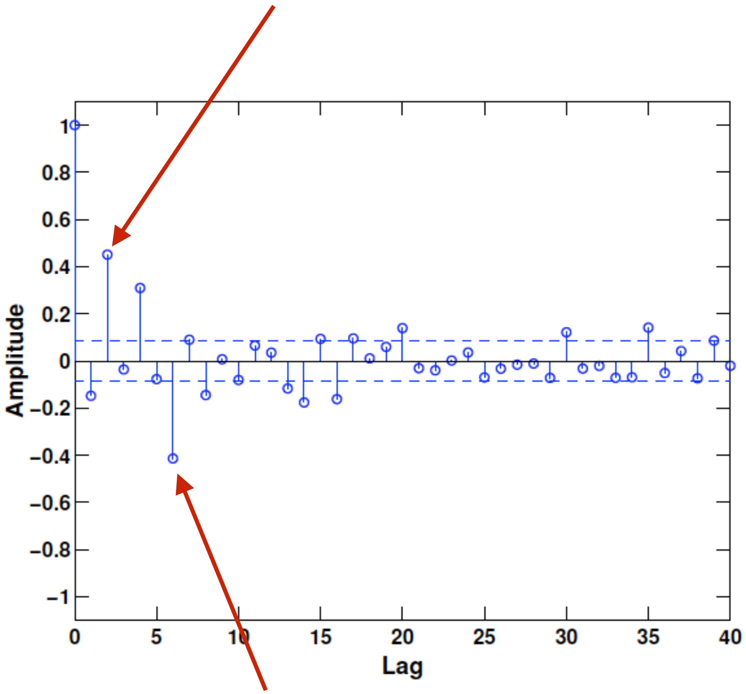
$$A_3(z)x_t = C_3(z)w_t$$

To do so, lets examine the ACF and PACF of x_t.

Example



ACF

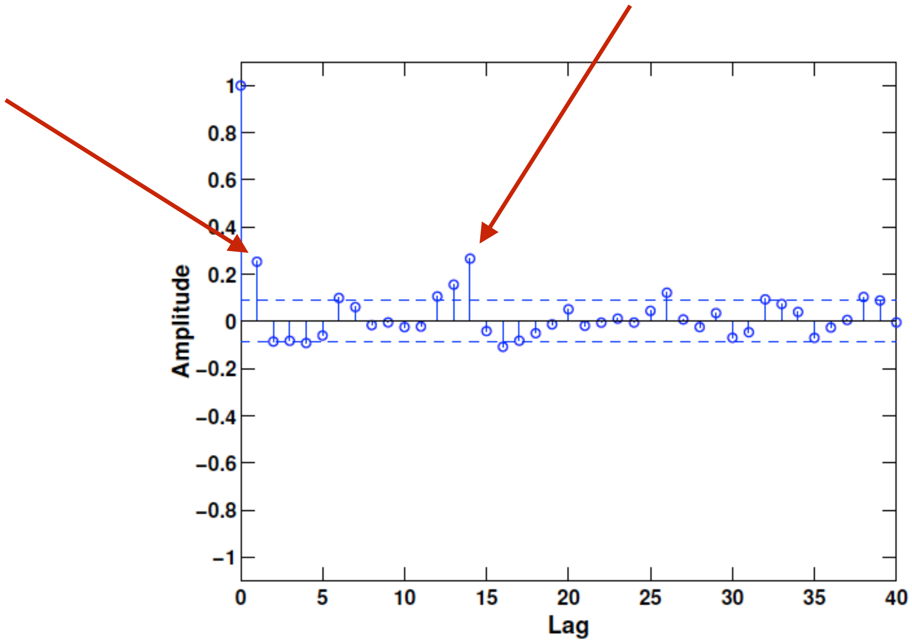


PACF

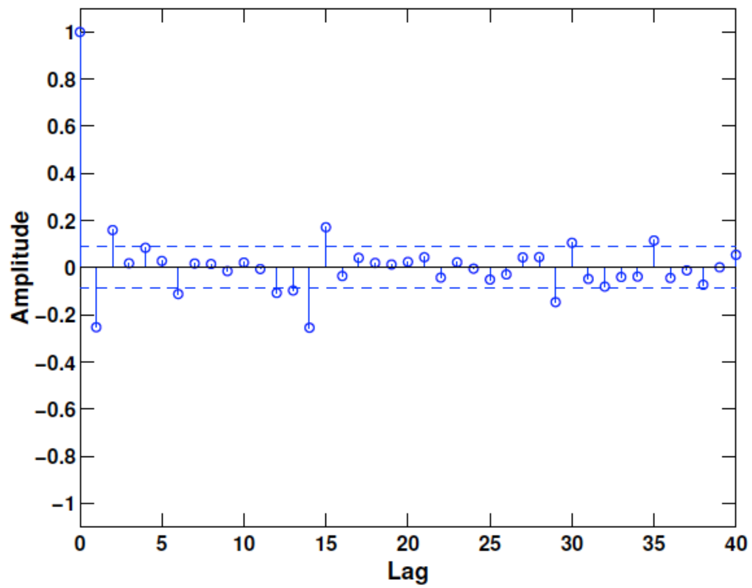
There seems to be strong dependencies for order 2 and 6, as well as, perhaps, at 4. In order to have a simple model, we begin with trying

$$A_3(z) = 1 + a_2 z^{-2} + a_6 z^{-6}$$

Example



ACF



PACF

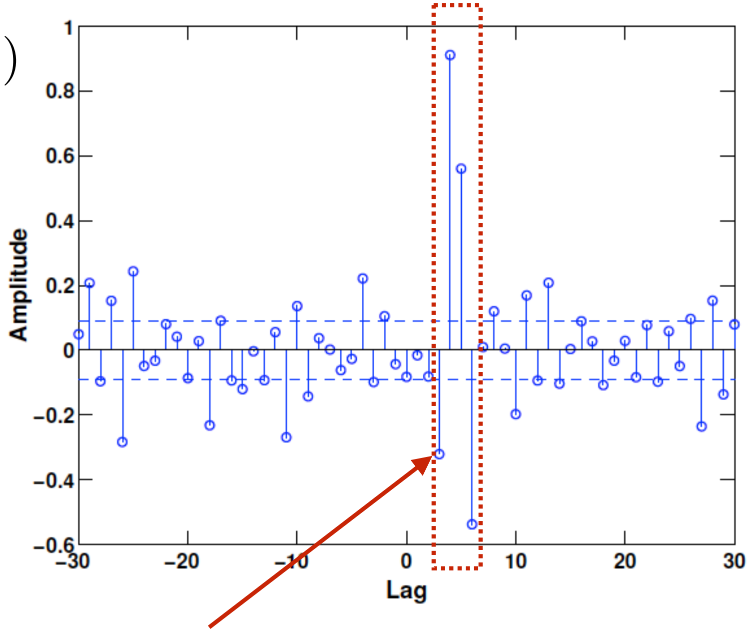
We estimate the parameters of this model and examine the ACF and PACF of the residual (above). Looking at the ACF, there seems to be strong dependencies at lag 1 and 14. We thus modify our model to

$$A_3(z) = 1 + a_2 z^{-2} + a_6 z^{-6}$$
$$C_3(z) = 1 + c_1 z^{-1} + c_{14} z^{-14}$$

We re-estimating *all* parameters and examine the residuals. These are now deemed to be white.

Example

$$\frac{\sigma_\epsilon}{\sigma_w}\rho_{\epsilon,w}(k)$$



We form the "white" input signal and the corresponding output

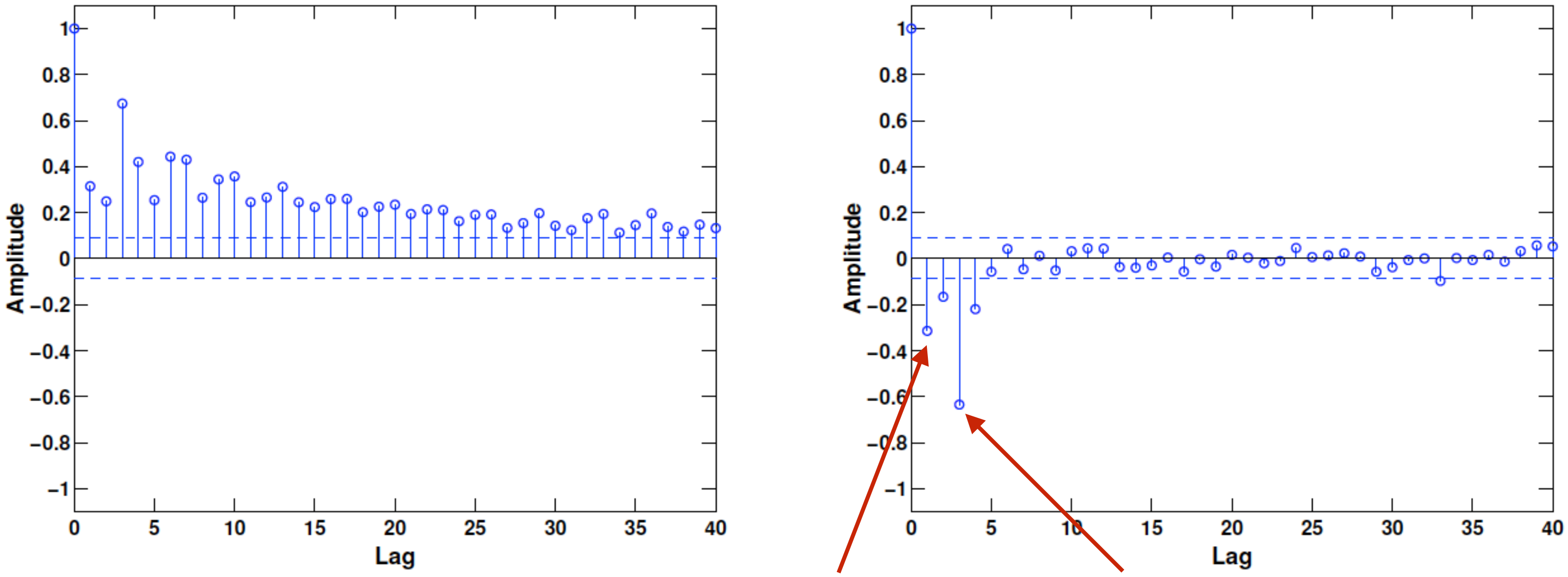
$$w_t = \frac{A_3(z)}{C_3(z)}x_t \quad \text{and} \quad \epsilon_t = \frac{A_3(z)}{C_3(z)}y_t$$

and then estimate the transfer function from w_t to ϵ_t as

$$h_k = \frac{\sigma_\epsilon}{\sigma_w}\rho_{\epsilon,w}(k)$$

The delay suggests $d = 3$. The impulse response seems to "ring", so we try $r = 2$. There seems to be 4 dominant components, i.e., $s = 3$.

Example



We pretend that the additive noise is white, and estimate the parameters detailing the model

$$y_t = \frac{B(z)z^{-d}}{A_2(z)} x_t + \tilde{e}_t$$

where $B(z)$ and $A_2(z)$ are of order s and r, respectively. We then compute the ACF and PACF of the residual $\tilde{e}_t$ (above). We form a model of the residual, beginning with using just a_1 and a_3.

Example

Examining the resulting residual suggest that we also needs c_1. This yields

$$
\begin{aligned}
A_1(z) &= 1 - 0.31z^{-1} - 0.59z^{-3} \\
C_1(z) &= 1 - 0.32z^{-1}
\end{aligned}
$$

Examining the resulting residual, we note that it is white. Using the found model, we re-estimate all the polynomials jointly, obtaining

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A_1(z) &= 1 - 0.31z^{-1} - 0.60z^{-3} \\
C_1(z) &= 1 - 0.31z^{-1} \\
A_2(z) &= 1 - 0.21z^{-1} + 0.40z^{-2} \\
B(z) &= -0.39 + 0.90z^{-1} + 0.48z^{-2}
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with $d = 3$. This process yields a white modeling residual.

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