Linear projections

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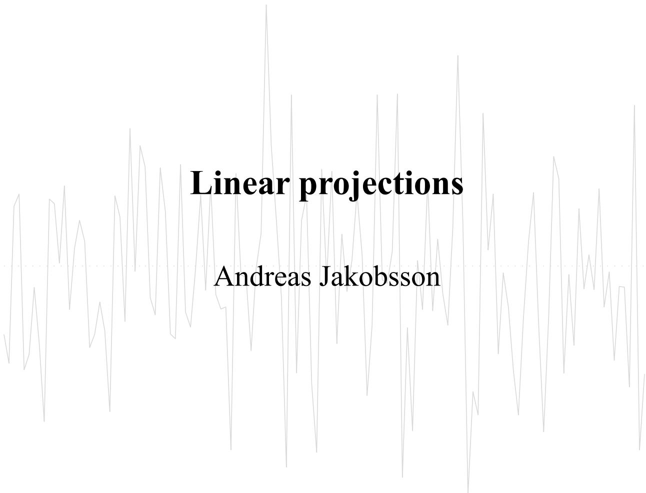

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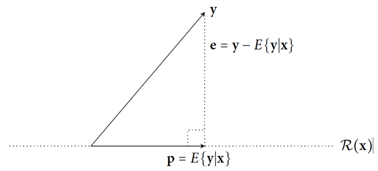


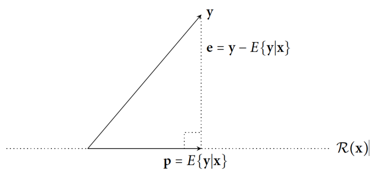
In this course, we will make use of conditional expectations to define the linear projection of one stochastic variable onto another.

The *linear projection* of $\mathbf{y}$ onto the space spanned by $\mathbf{x}$, the so-called *range space*, denoted $\mathcal{R}(\mathbf{x})$, is defined as

$$E\{\mathbf{y}|\mathbf{x}\} = \mathbf{a} + \mathbf{B}\mathbf{x}$$

where $\mathbf{a} \in \mathcal{R}(\mathbf{x})$ and $\mathbf{B}$ is a deterministic matrix of appropriate dimension.

Linear projections



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The geometrical interpretation is quite helpful. For instance, from it, we can conclude the so-called *principle of orthogonality*, stating that

$$C\Big\{\mathbf{y} - E\{\mathbf{y}|\mathbf{x}\}, \mathbf{x}\Big\} = \mathbf{0}$$

That is, the error vector $\mathbf{e} = \mathbf{y} - E\{\mathbf{y}|\mathbf{x}\}$ is uncorrelated with $\mathbf{x}$.

Linear projections

Let $\mathbf{z}$ denote the concatenated vector

$$\mathbf{z} = \begin{bmatrix} \mathbf{x}^T & \mathbf{y}^T \end{bmatrix}^T$$

having mean $E\{\mathbf{z}\} = \begin{bmatrix} \mathbf{m}_\mathbf{x}^T & \mathbf{m}_\mathbf{y}^T \end{bmatrix}^T$ and covariance matrix

$$\mathbf{R}_\mathbf{z} = \begin{bmatrix} \mathbf{R}_\mathbf{x} & \mathbf{R}_{\mathbf{x},\mathbf{y}} \\ \mathbf{R}_{\mathbf{y},\mathbf{x}} & \mathbf{R}_\mathbf{y} \end{bmatrix}$$

Then, the linear projection of $\mathbf{y}$ onto $\mathbf{x}$, can be expressed as

$$E\{\mathbf{y}|\mathbf{x}\} = \mathbf{m}_\mathbf{y} + \mathbf{R}_{\mathbf{y},\mathbf{x}}\mathbf{R}_\mathbf{x}^{-1}(\mathbf{x} - \mathbf{m}_\mathbf{x})$$

This will be the optimal linear projection, i.e., the projection that yields the minimum prediction error variance among all linear projections. Furthermore, the difference $\mathbf{e} = \mathbf{y} - E\{\mathbf{y}|\mathbf{x}\}$ will have the variance

$$V\{\mathbf{e}|\mathbf{x}\} = \mathbf{R}_\mathbf{y} - \mathbf{R}_{\mathbf{y},\mathbf{x}}\mathbf{R}_\mathbf{x}^{-1}\mathbf{R}_{\mathbf{y},\mathbf{x}}^* = E\Big\{V\{\mathbf{y}|\mathbf{x}\}\Big\}$$

If $\mathbf{x}$ and $\mathbf{y}$ are Normal distributed, then $\mathbf{e}$ and $\mathbf{x}$ are independent; otherwise, they are uncorrelated.