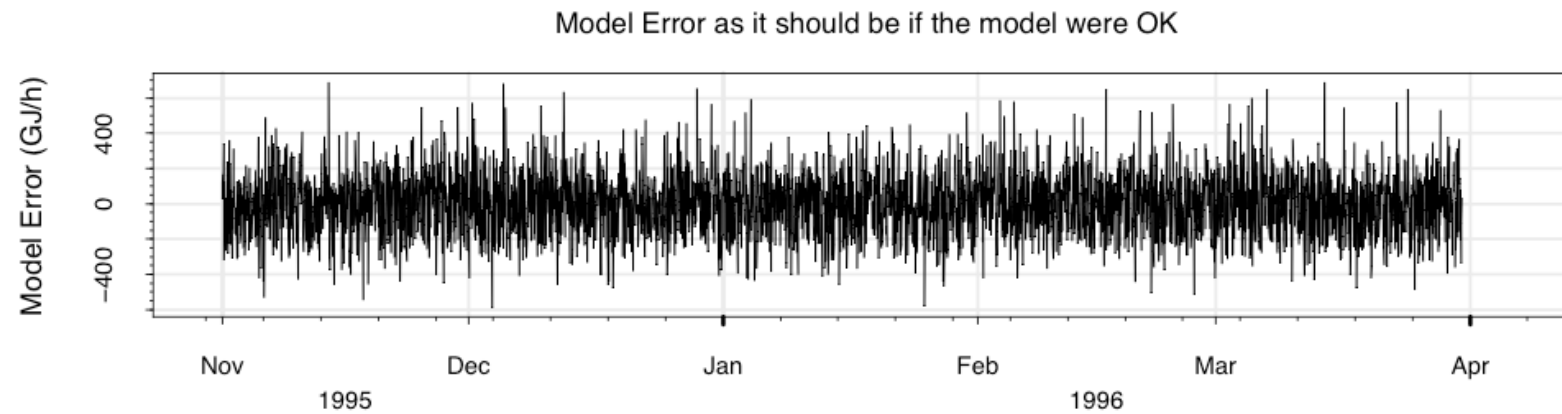# Filtering

Andreas Jakobsson
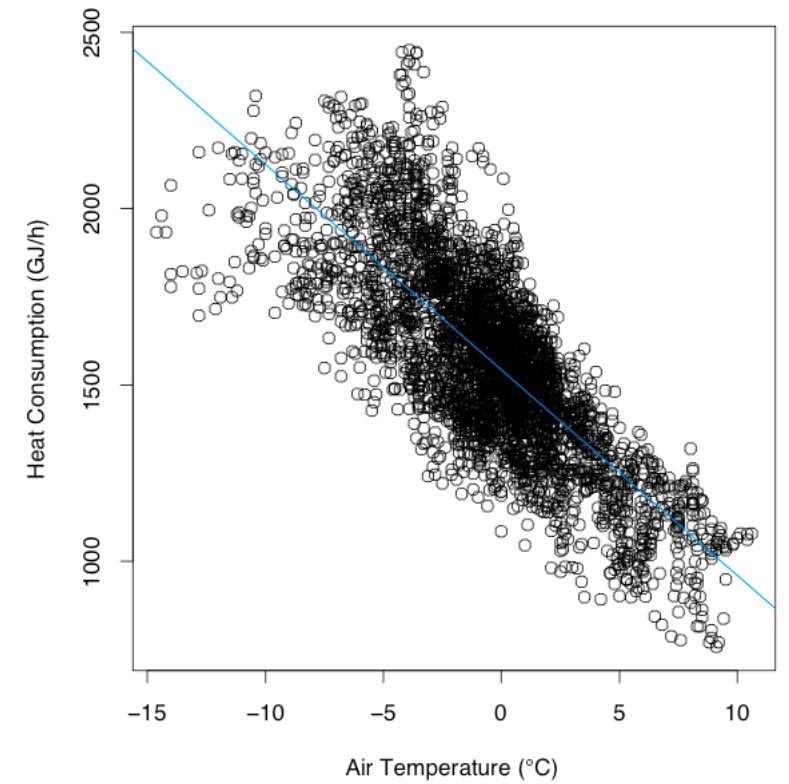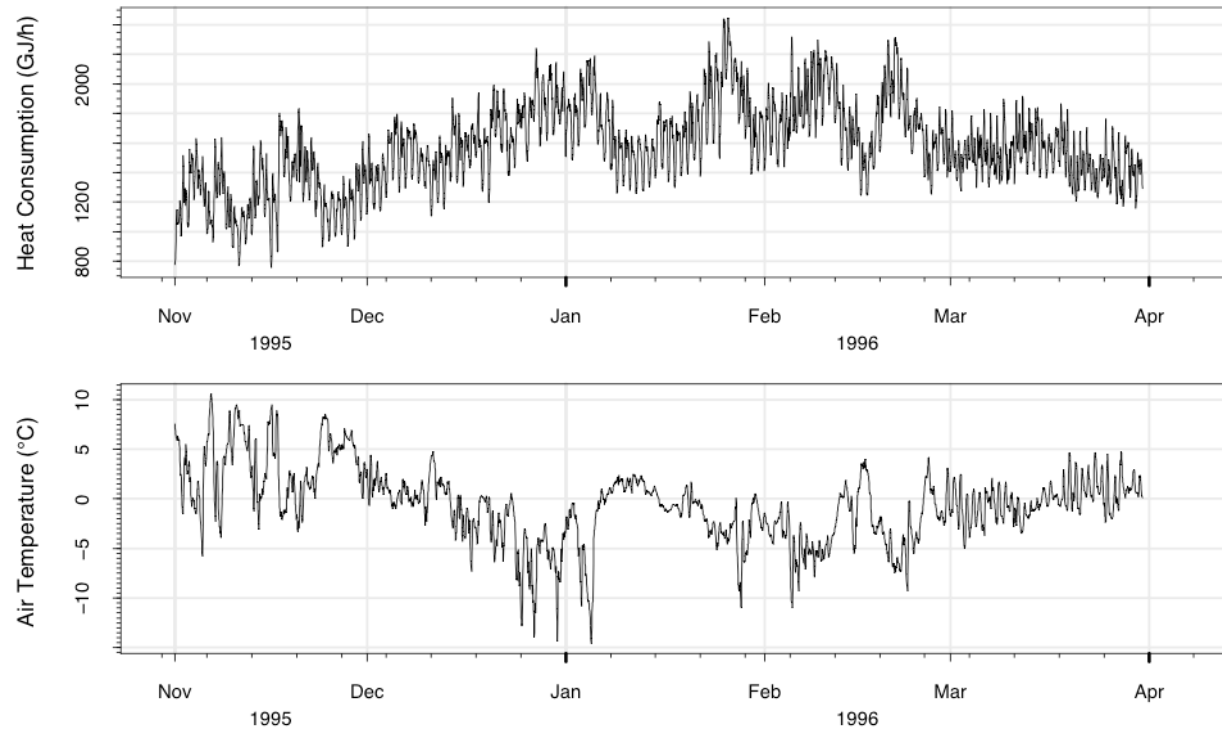
Filtering a stochastic process

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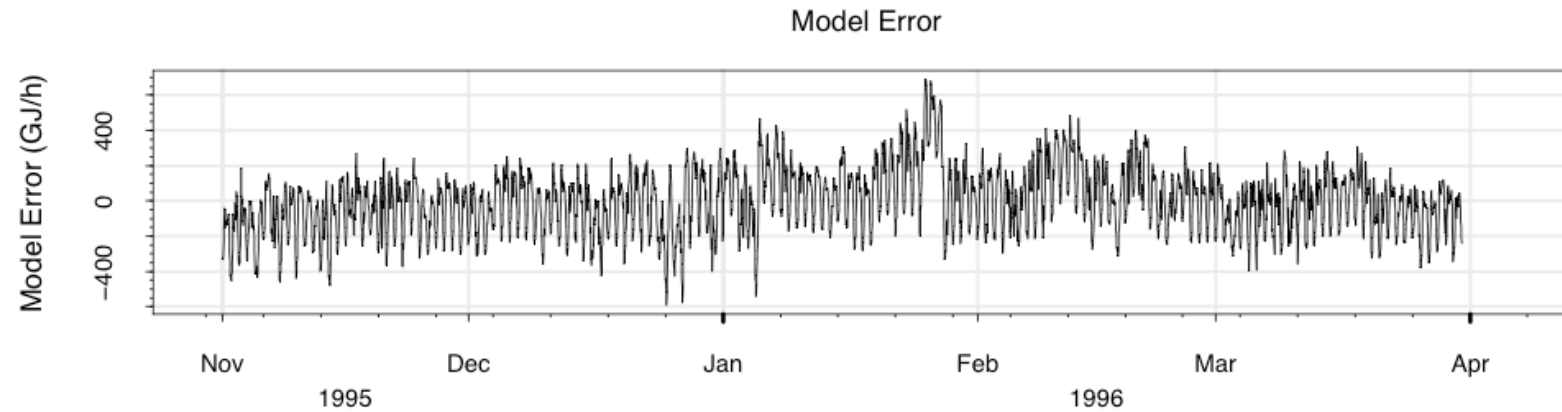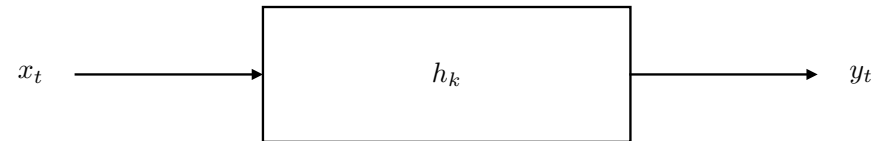
Model Error

Model Error as it should be if the model were OK

Filtering a stochastic process

x_t → h_k → y_t

What happens if one filters a stochastic process through a *linear*, *stable*, *casual*, and *time-invariant* filter?

Linear: $\qquad\qquad \alpha x_1(t) + \beta x_2(t) \quad \Rightarrow \alpha y_1(t) + \beta y_2(t)$

Time-invariant: $\quad\ x_{t-d} \qquad\qquad\qquad \Rightarrow y_{t-d}$

Filtering a stochastic process

$$x_t \longrightarrow \boxed{ h_k } \longrightarrow y_t$$
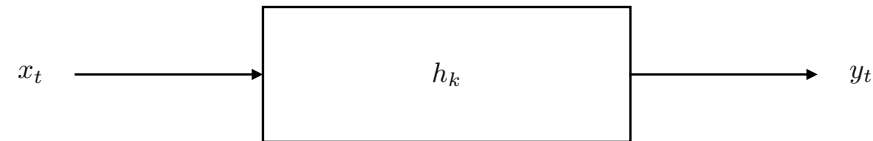
What happens if one filters a stochastic process through a *linear, stable, casual,* and *time-invariant* filter?

Linear: $\qquad \alpha x_1(t) + \beta x_2(t) \quad \Rightarrow \alpha y_1(t) + \beta y_2(t)$

Time-invariant: $\quad x_{t-d} \qquad\qquad\qquad \Rightarrow y_{t-d}$

Notable, if x_t is WSS / Gaussian, so is y_t.

Recall that such a (discrete-time) filter may be written as

$$y_t = \sum_{k=-\infty}^{\infty} h_{t-k} x_k = \sum_{k=-\infty}^{\infty} h_k x_{t-k}$$

Then,

$$m_y = E\left\{ \sum_{k=-\infty}^{\infty} h_k x_{t-k} \right\} = m_x \sum_{k=-\infty}^{\infty} h_k = m_x H(0)$$

where

$$H(\omega) = \sum_{k=-\infty}^{\infty} h_k e^{-i\omega k}$$

Filtering a stochastic process

Similarly, the ACF may be formed as

$$r_y(k) = \sum_{m=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} h_m^* h_\ell r_x(m + k - \ell) = r_x(k) \star h_k \star h_{-k}^*$$

where $\star$ denotes the convolution operator. Expressed in the frequency domain,

$$\boxed{\phi_y(\omega) = |H(\omega)|^2 \phi_x(\omega)}$$

Filtering a stochastic process

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Defining the cross spectral density of two stationary processes, x_t and y_t, as the DFT of the cross-covariance function, i.e.,

$$\phi_{x,y}(\omega) = \sum_{k=-\infty}^{\infty} r_{x,y}(k) e^{-i\omega k}$$

Then,

$$\phi_{x,y}(\omega) = H(\omega) \phi_x(\omega)$$

This is the so-called Wiener-Hopf equation.