Estimating the covariance matrix

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The covariance matrix

Consider a measurement containing N samples,

$$\mathbf{x}_N = \begin{bmatrix} x_1 & \dots & x_N \end{bmatrix}^T$$

The covariance matrix of $\mathbf{x}_N$ is

$$\mathbf{R}_{\mathbf{x}} = E\left\{\mathbf{x}_N \mathbf{x}_N^*\right\} = \begin{bmatrix} C\{x_1, x_1\} & \dots & C\{x_1, x_N\} \\ \vdots & \ddots & \vdots \\ C\{x_N, x_1\} & \dots & C\{x_N, x_N\} \end{bmatrix}$$

$$= \begin{bmatrix} r_x(0) & r_x^*(1) & r_x^*(2) & \dots & r_x^*(N) \\ r_x(1) & r_x(0) & r_x^*(1) & \dots & r_x^*(N-1) \\ r_x(2) & r_x(1) & r_x(0) & \dots & r_x^*(N-2) \\ \vdots & & \ddots & & \vdots \\ r_x(N) & r_x(N-1) & r_x(N-2) & \dots & r_x(0) \end{bmatrix}$$

where $(\cdot)^*$ denotes the conjugate. This is a *Toeplitz* structured matrix.

This structure allows for the forming of computationally efficient algorithms. Notably, one may also express the inverse of a Toeplitz matrix in closed form!

The covariance matrix

How should one proceed to estimate $\mathbf{R}_{\mathbf{x}}$ from $\mathbf{x}_N$?

This is not straight-forward, and there are several different ways to do so. Some of the more common include:

- The Toeplitz-structured estimate
- The outer-product estimate

The Toeplitz-structured estimate is formed by first estimating $\hat{r}_x(k)$, typically using the *biased* estimator, and then forming $\hat{\mathbf{R}}_{\mathbf{x}}$ using the Toeplitz structure of the matrix.

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The outer-product estimate is formed by splitting $\mathbf{x}_N$ into M subvectors of length L, such that

$$\mathbf{x}_t = \begin{bmatrix} x_t & \dots & x_{t+L-1} \end{bmatrix}^T$$

where $t = 1, \dots, M = N - L + 1$. Then, the outer-product covariance matrix estimate

$$\hat{\mathbf{R}}_{\mathbf{x}} = \frac{1}{M} \sum_{t=1}^{M} \mathbf{x}_t \mathbf{x}_t^*$$

Although the resulting $L \times L$ estimate is typically *not* a Toeplitz matrix, this is typically the preferable way to estimate $\hat{\mathbf{R}}_{\mathbf{x}}$.