DAT 600 – Linear Programming

- Linear Programming and geometrical method (slides adapted from https://cs.stanford.edu/~ermon/cs325/slides/LP.pptx)
- Simplex (slides adapted from 4.2: Maximization By The Simplex Method - Mathematics LibreTexts)

Topics in Computational Sustainability

Optimization:
Intro to Linear programming

Optimization: Mathematical Program

- Optimization problem in which the objective and constraints are given as mathematical functions and functional relationships.

Minimize $f(x_1, x_2, \ldots, x_n)$

Subject to:

$$g_1(x_1, x_2, \ldots, x_n) \;=, \;\geq, \;\leq \;\; b_1$$

$$g_2(x_1, x_2, \ldots, x_n) \;=, \;\geq, \;\leq \;\; b_2$$

$$\ldots$$

$$g_m(x_1, x_2, \ldots, x_n) \;=, \;\geq, \;\leq \;\; b_m$$

Linear Programming (LP)

- **Linear** – all the functions are linear

Ex: $f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2 x_2 + \ldots c_n x_n$

- **Programming** – does not refer to computer programming but rather "planning" - planning of activities to obtain an optimal result i.e., it reaches the specified goal best (according to the mathematical model) among all feasible alternatives.

Components of a Linear Programming Model

- A linear programming model consists of:
  - A set of decision variables
  - A (linear) objective function
  - A set of (linear) constraints

Nature Connection: Recreational Sites

Nature Connection is planning two new public recreational sites: **a forested wilderness area** and a **sightseeing and hiking park**. They own 80 hectares of forested wilderness area and 20 hectares suitable for the sightseeing and hiking park but they don't have enough resources to make the entire areas available to the public. They have a **budget of $120K** per year. They estimate a yearly management and maintenance cost of $1K per hectare for the forested wilderness area, and $4K per hectare for the sightseeing and hiking park. The expected average number of visiting hours a day per hectare are: 10 for the forest and 20 for the sightseeing and hiking park.

Question: How many hectares should Nature Connection allocate to the public sightseeing and hiking park and to the public forested wilderness area, in order to maximize the amount of recreation, (in average number of visiting hours a day for the total area to be open to the public, for both sites**) given their budget constraint?**

Steps in setting up a LP

1. Determine and label the *decision variables*.
2. Determine the objective and use the decision variables to write an expression for the *objective function*.
3. Determine the constraints - *feasible region*.
    1. Determine the *explicit constraints* and write a functional expression for each of them.
    2. Determine the *implicit constraints (e.g., nonnegativity constraints)*.

Formulation of the problem as a Linear Program

1 Decision Variables

x1 – # hectares to allocate to the public forested wilderness area
X2 – # hectares to allocate the public sightseeing and hiking park

2 Objective Function

Max 10 x1 + 20 x2

3 Constraints

$$
\begin{aligned}
x1 \quad\quad\quad\; &\leq 80 \\
x2 &\leq 20 \\
x1 + 4\,x2 &\leq 120
\end{aligned}
$$

x1 >= 0; x2 >= 0 Non-negativity constraints

Formulation of the problem as a Linear Program

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1 Decision Variables

x1 – # hectares to allocate to the public forested wilderness area

X2 – # hectares to allocate the public sightseeing and hiking park

2 Objective Function

Max 10 x1 + 20 x2

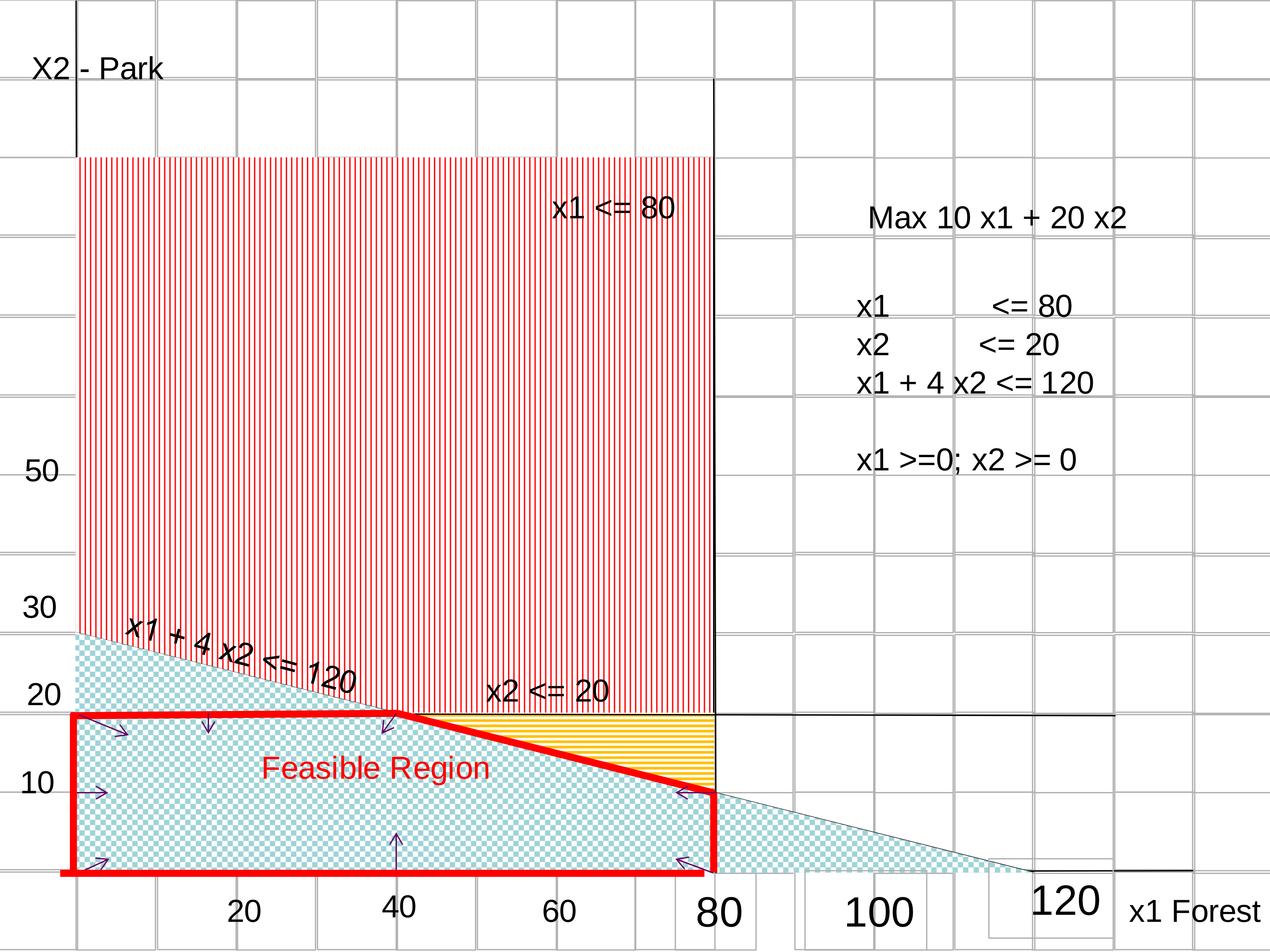
3 Constraints

x1 <= 80 Land for forest

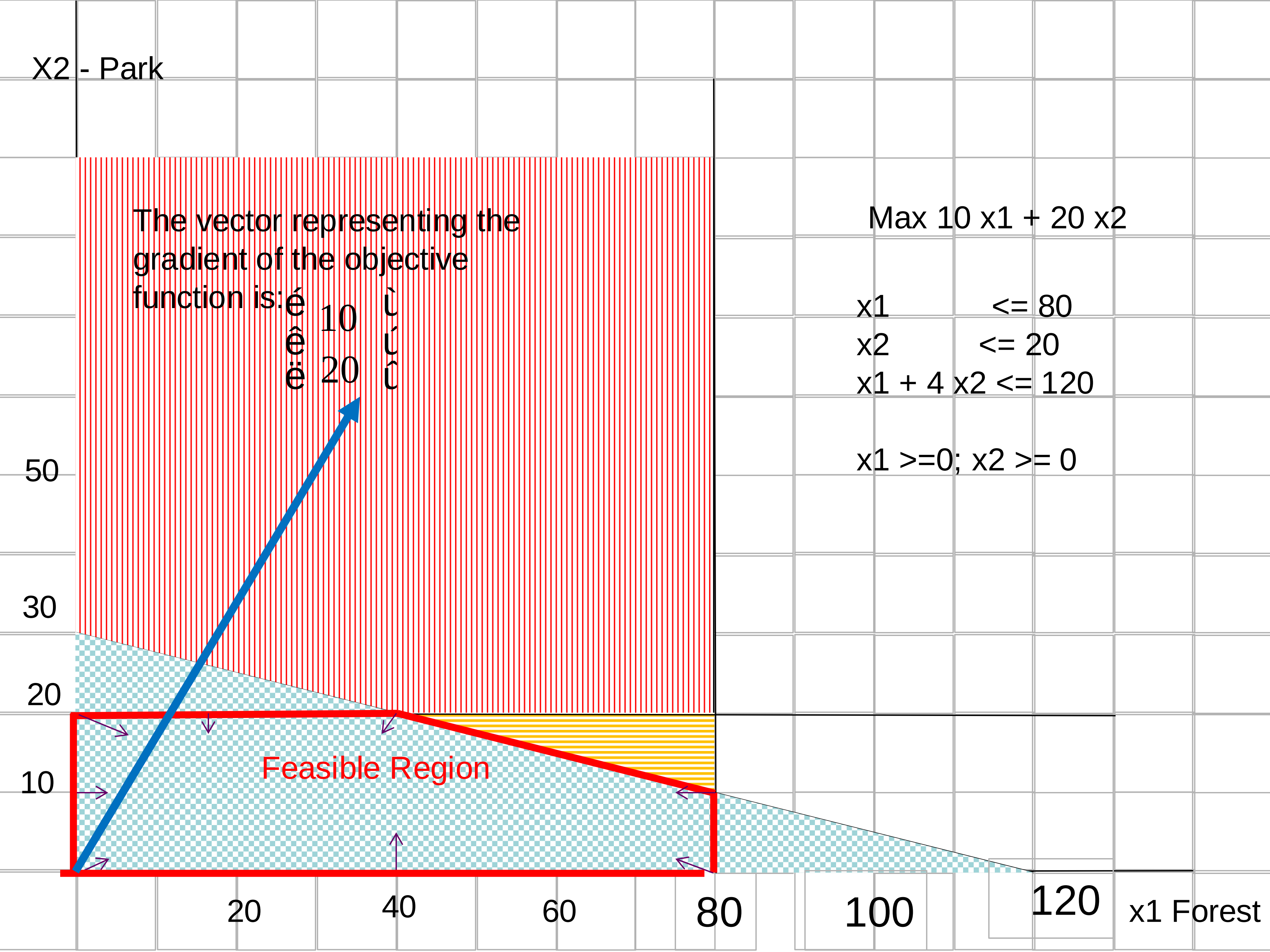
x2 <= 20 Land for Park

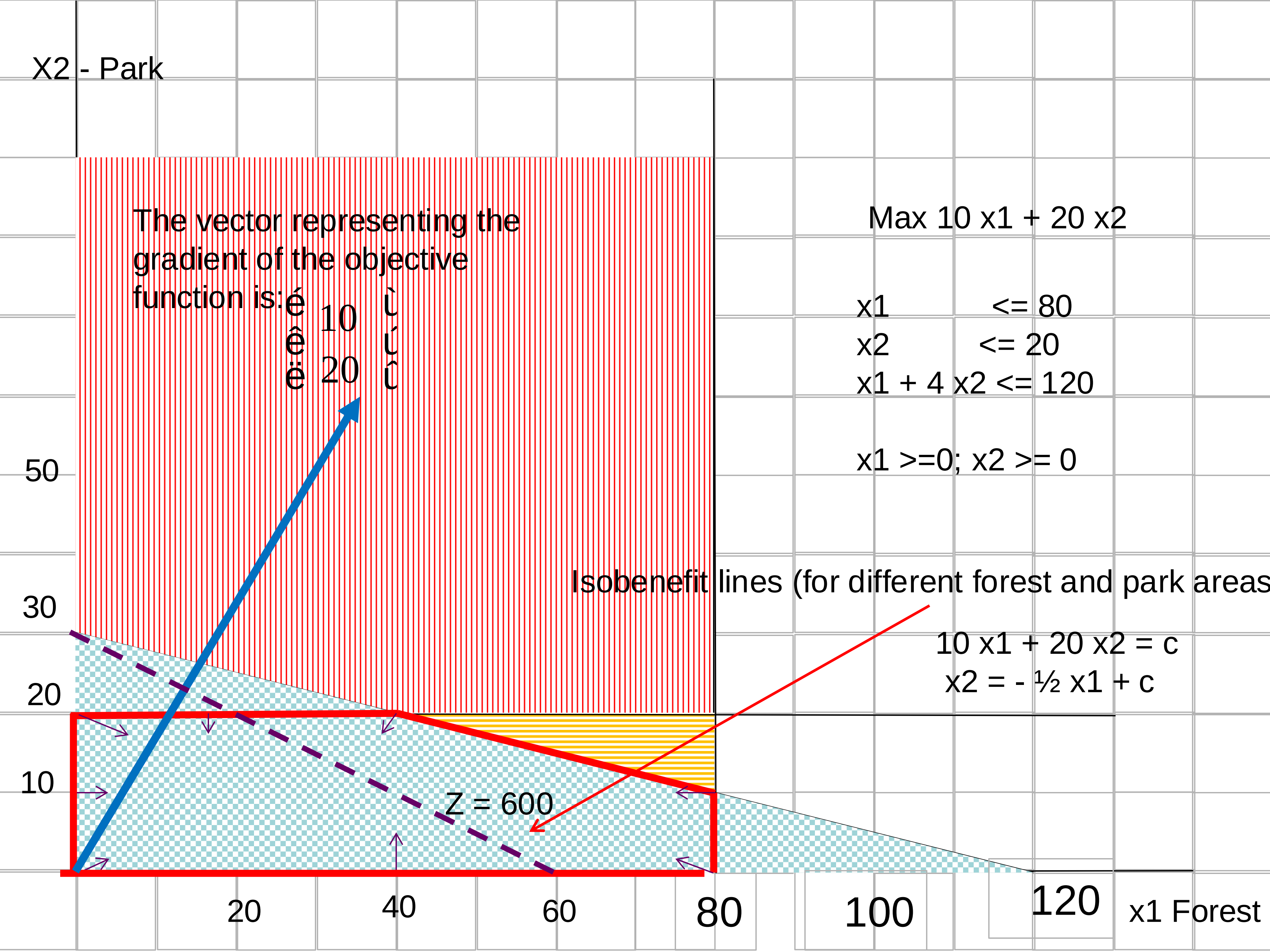
x1 + 4 x2 <= 120 Budget

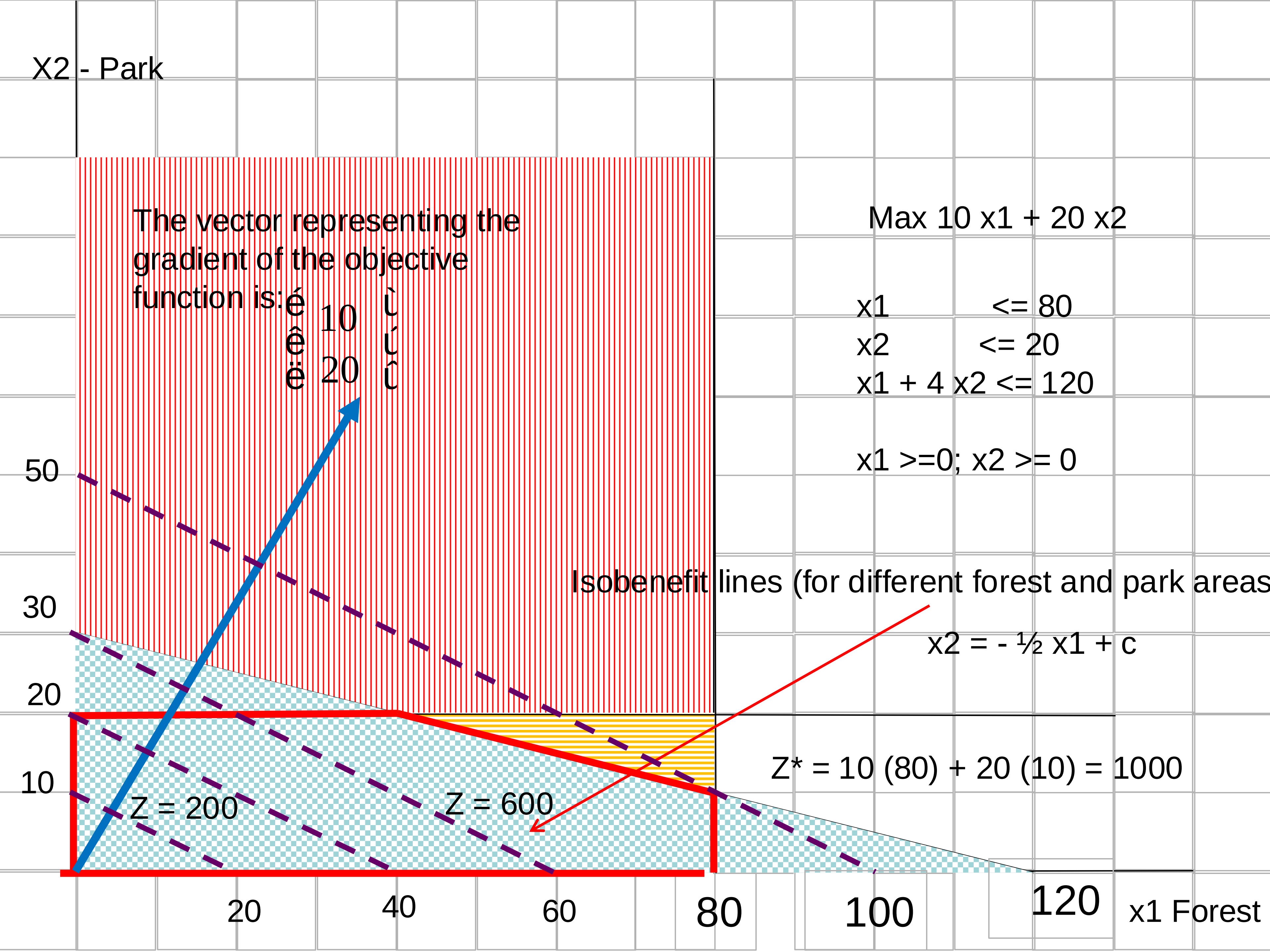
x1 >= 0; x2 >= 0 Non-negativity constraints



X2 - Park
x1 <= 80
Max 10 x1 + 20 x2
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 >=0; x2 >= 0
50
30
x1 + 4 x2 <= 120
x2 <= 20
20
Feasible Region
10
20
40
60
80
100
120
x1 Forest


X2 - Park
The vector representing the gradient of the objective function is:
10
20
Max 10 x1 + 20 x2
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 >=0; x2 >= 0
50
30
20
10
Feasible Region
20
40
60
80
100
120
x1 Forest


X2 - Park
The vector representing the
gradient of the objective
function is:é 10 ù
ê ú
ë 20 û
Max 10 x1 + 20 x2
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 >=0; x2 >= 0
Isobenefit lines (for different forest and park areas
10 x1 + 20 x2 = c
x2 = - ½ x1 + c
Z = 600
50
30
20
10
20
40
60
80
100
120
x1 Forest



X2 - Park
The vector representing the gradient of the objective function is: 10 20
Max 10 x1 + 20 x2
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 >=0; x2 >= 0
50
30
20
10
Isobenefit lines (for different forest and park areas
x2 = - ½ x1 + c
Z* = 10 (80) + 20 (10) = 1000
Z = 200
Z = 600
20
40
60
80
100
120
x1 Forest

Summary of the Graphical Method

- Draw the constraint boundary line for each constraint. Use the origin (or any point not on the line) to determine which side of the line is permitted by the constraint.

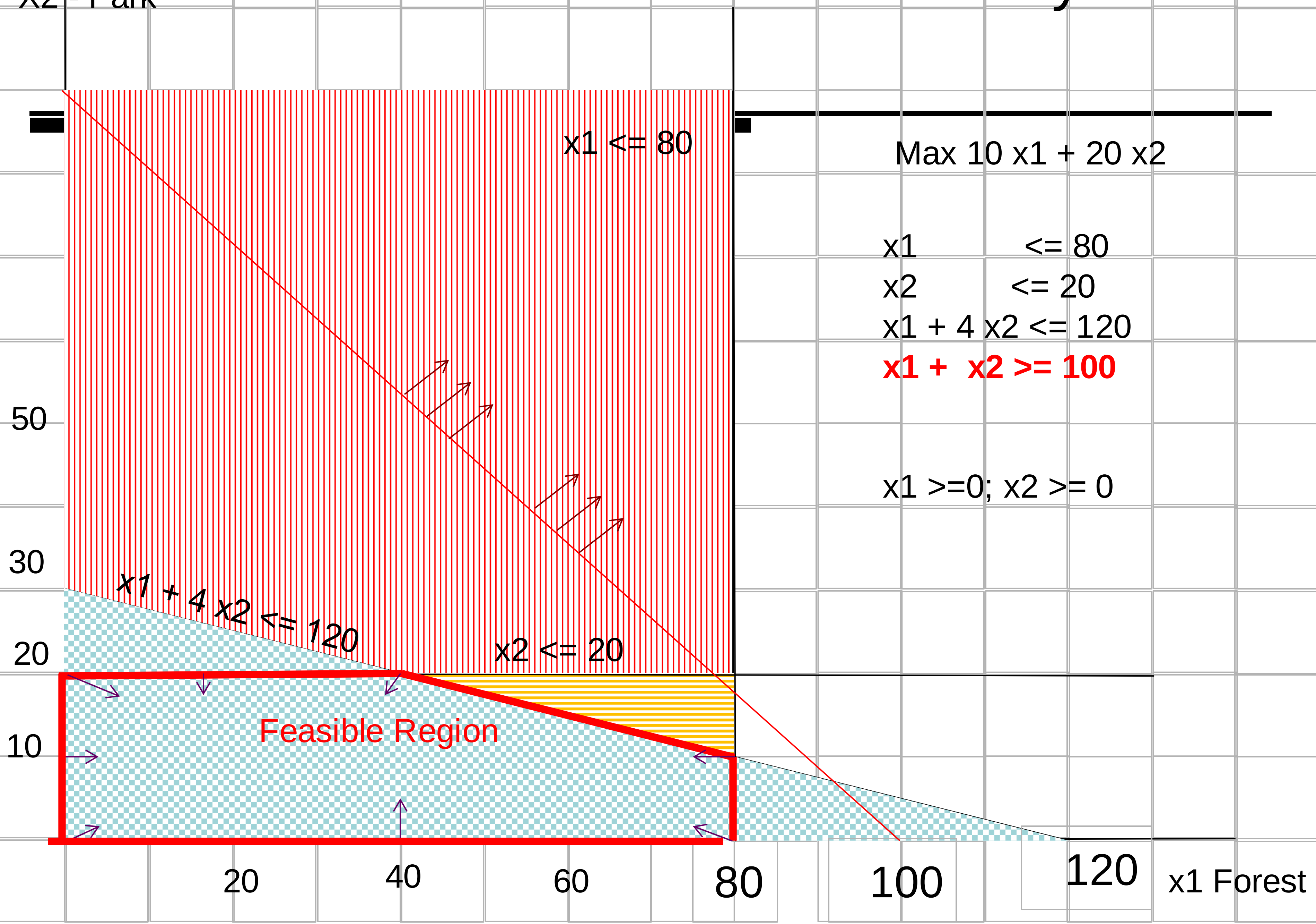
- Find the feasible region by determining where all constraints are satisfied simultaneously.

- Determine the slope of one objective function line (perpendicular to its gradient vector). All other objective function lines will have the same slope.

- Move a straight edge with this slope through the feasible region in the direction of improving values of the objective function (direction of the gradient). Stop at the last instant that the straight edge still passes through a point in the feasible region. This line is the optimal objective function line.

- A feasible point on the optimal objective function line is an optimal solution.

No Feasible Solutions – Why?

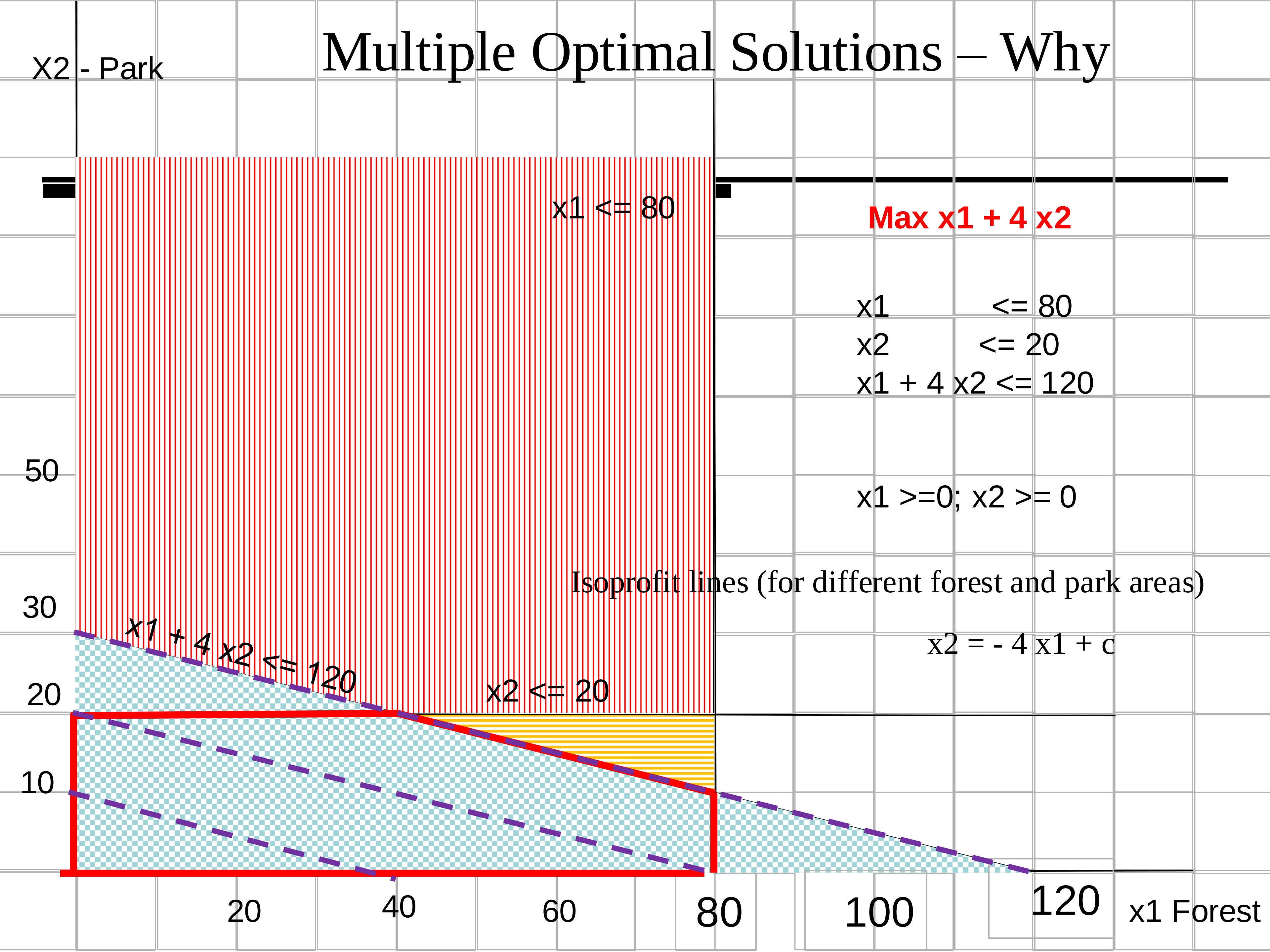


Max 10 x1 + 20 x2

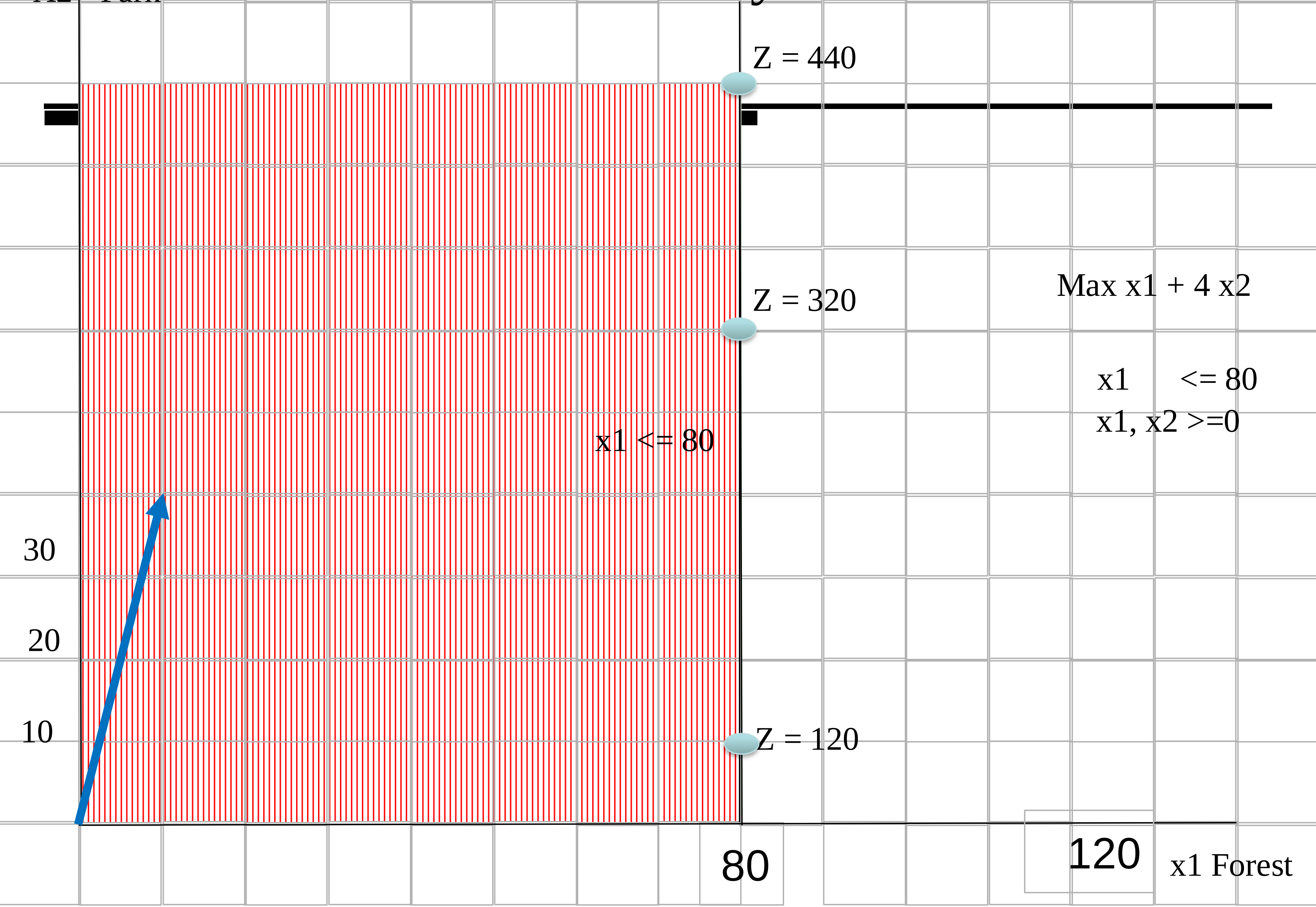
x1 <= 80
x2 <= 20
x1 + 4 x2 <= 120
x1 + x2 >= 100

x1 >=0; x2 >= 0

Multiple Optimal Solutions – Why



Unbounded – Why?



Key Categories of LP problems:

- Resource-Allocation Problems
- Cost-benefit-trade-off problems

Distribution-Network Problems

Mixed problems

Second Example: Keeping the river clean

Cost-benefit-trade-off problems

Choose the <u>mix of levels of various activities </u>to achieve <u>minimum acceptable levels for various benefits</u>

at a minimum cost.

Second Example: Keeping the River Clean

A pulp mill in Maine makes mechanical and chemical pulp, polluting the river in which it spills its spent waters. This has created several problems, leading to a change in management.

The previous owners felt that it would be too expensive to reduce pollution, so they decided to sell the pulp mill. The mill has been bought back by the employees and local businesses, who now own the mill as a cooperative. The new owners have several objectives:

1 – to keep **at least 300 people employed at the mill** (300 workers a day);

2 – to generate at least $40,000 of revenue a day

They estimate that this will be enough to pay operating expenses and yield a return that will keep the mill competitive in the long run. Within these limits, everything possible should be done to **minimize pollution**.

Both chemical and mechanical pulp require the labor of one worker for 1 day (1 workday, wd) per ton produced;

Mechanical pulp sells at $100 per ton; Chemical pulp sells at $200 per ton;

Pollution is measured by the biological oxygen demand (BOD). One ton of mechanical pulp produces 1 unit of BOD; One ton of chemical pulp produces 1.5 units of BOD.

The maximum capacity of the mill to make mechanical pulp is 300 tons per day; for chemical pulp is 200 tons per day. The two manufacturing processes are independent (I.e., the mechanical pulp line cannot be used to make chemical pulp and vice versa).

- Pollution, employment, and revenues result from the production of both types of pulp. So a natural choice for the variables is:

Decision Variables

- X1 amount of mechanical pulp produced (in tons per day, or t/d) and
- X2 amount of chemical pulp produce (in tons per day, or t/d
- Min Z = 1 X1 + 1.5 X2

Min Z =	1 X1	+	1.5 X2	
(BOD/day)	(BOD/t) (t/d)		(BOD/t) (t/d)	

Subject to:

1 X1	+	1 X2	>= 300 Workers/day
(wd/t) (t/d)		(wd/t)(t/d)	
100 X1	+	200 X2	>= 40,000 revenue/day
($/t) (t/d)	+	($/t)(t/d)	$/d
X1			<= 300 (mechanical pulp)
(t/d)			(t/d)
		X2	<= 200 (mechanical pulp)
		(t/d)	(t/d)

X1 >=0; X2 >= 0

Distribution Network Problems

Distribution-Network Problem

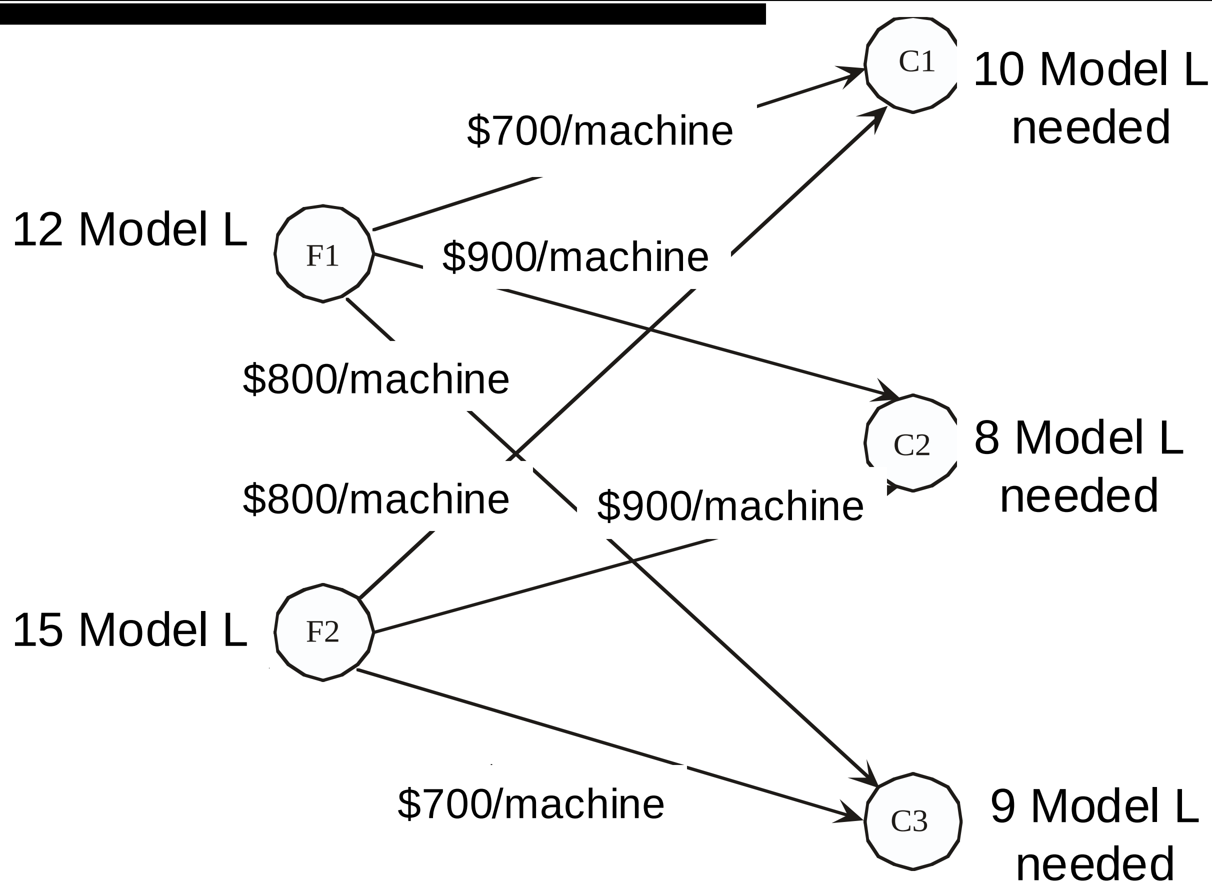
- The International Hospital Share Organization is a non-profit organization that refurbishes a variety of used equipment for hospitals of developing countries at two international factories (F1 and F2). One of its products is a large X-Ray machine.
- Orders have been received from three large communities for the X-Ray machines.

Shipping Cost for Each Machine (Model L)

From \ To	Community 1	Community 2	Community 3	Output
Factory 1	$700	$900	$800	12 X-ray machines
Factory 2	800	900	700	15 X-Ray machines
Order Size	10 X-ray machines	8 X-Ray machines	9 X-Ray machines	

Question: How many X-Ray machines (model L) should be shipped from each factory to each hospital so that shipping costs are minimized?

The Distribution Network



Question: How many machines (model L) should be shipped from each factory to each customer so that sipping costs are minimized?

- Activities – shipping lanes (not the level of production which has already been defined)
  - Level of each activity – number of machines of model L shipped through the corresponding shipping lane.

  → Best mix of shipping amounts

Example:

Requirement 1: Factory 1 must ship 12 machines
Requirement 2: Factory 2 must ship 15 machines
Requirement 3: Customer 1 must receive 10 machines
Requirement 4: Customer 2 must receive 8 machines
Requirement 5: Customer 3 must receive 9 machines

Algebraic Formulation

Let S_{ij} = Number of machines to ship from i to j (i = F1, F2; j = C1, C2, C3).

$$\text{Minimize Cost} = \$700S_{\text{F1-C1}} + \$900S_{\text{F1-C2}} + \$800S_{\text{F1-C3}} + \$800S_{\text{F2-C1}} + \$900S_{\text{F2-C2}} + \$700S_{\text{F2-C3}}$$

subject to

$$S_{\text{F1-C1}} + S_{\text{F1-C2}} + S_{\text{F1-C3}} = 12$$
$$S_{\text{F2-C1}} + S_{\text{F2-C2}} + S_{\text{F2-C3}} = 15$$
$$S_{\text{F1-C1}} + S_{\text{F2-C1}} = 10$$
$$S_{\text{F1-C2}} + S_{\text{F2-C2}} = 8$$
$$S_{\text{F1-C3}} + S_{\text{F2-C3}} = 9$$

and

$S_{ij} \geq 0$ (i = F1, F2; j = C1, C2, C3).

Algebraic Formulation

Let S_{ij} = Number of machines to ship from i to j (i = F1, F2; j = C1, C2, C3).

Minimize Cost = $\$700S_{F1\text{-}C1} + \$900S_{F1\text{-}C2} + \$800S_{F1\text{-}C3} + \$800S_{F2\text{-}C1} + \$900S_{F2\text{-}C2} + \$700S_{F2\text{-}C3}$

subject to

Factory 1: $S_{F1\text{-}C1} + S_{F1\text{-}C2} + S_{F1\text{-}C3} = 12$

Factory 2: $S_{F2\text{-}C1} + S_{F2\text{-}C2} + S_{F2\text{-}C3} = 15$

Customer 1: $S_{F1\text{-}C1} + S_{F2\text{-}C1} = 10$

Customer 2: $S_{F1\text{-}C2} + S_{F2\text{-}C2} = 8$

Customer 3: $S_{F1\text{-}C3} + S_{F2\text{-}C3} = 9$

and

$S_{ij} \geq 0$ (i = F1, F2; j = C1, C2, C3).

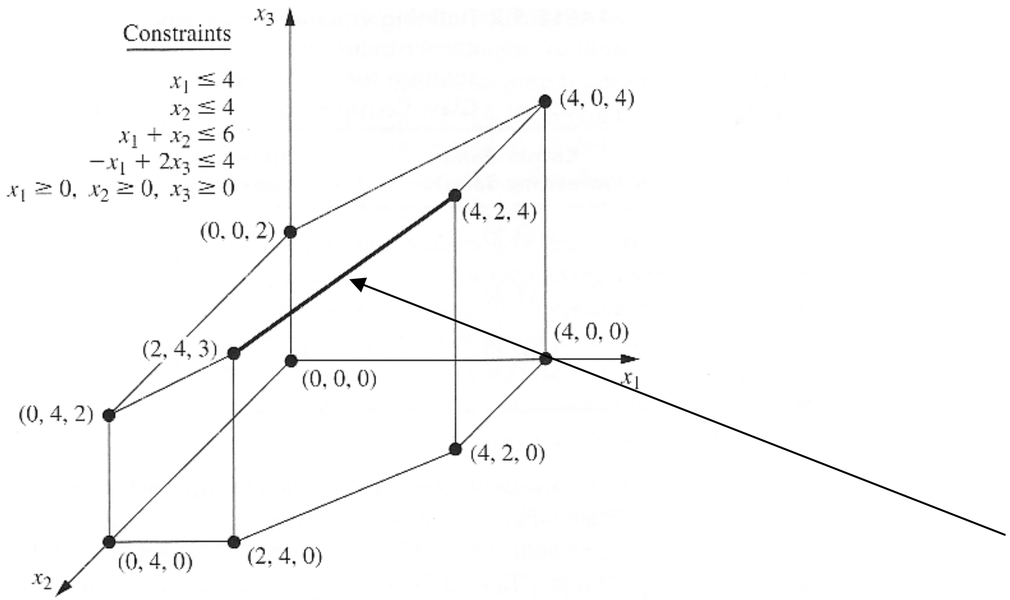
Terminology and Notation

Terminology of solutions in LP model

- Solution – not necessarily the final answer to the problem!!!
- Feasible solution – solution that satisfies all the constraints
- Infeasible solution – solution for which at least one of the constraints is violated
- Feasible region – set of all points that satisfies all the constraints (possible to have a problem without any feasible solutions)
- Binding constraint – the left-hand side and the right-hand side of the constraint are equal, I.e., constraint is satisfied in equality. Otherwise the constraint is nonbinding.
- Optimal solution – feasible solution that has the best value of the objective function.

    Largest value □ maximization problems
    Smallest value □ minimization problems
- Multiple optimal solutions, no optimal solutions, unbounded Z

3D feasible region



Simplex

Simplex

- The geometrical approach will not work for problems that have more than two variables.
- LP problems consist of literally thousands of variables and are solved by computers. We can solve these problems algebraically, but that will not be very efficient. Suppose we were given a problem with, say, 5 variables and 10 constraints. We could find all the corner points, test them for feasibility, and come up with the solution, if it exists. But the trouble is that even for a problem with so few variables, we will get more than **250** corner points, and testing each point will be very tedious.
- The simplex method was developed during the Second World War by Dr. George Dantzig. His linear programming models helped the Allied forces with transportation and scheduling problems. In 1979, a Soviet scientist named Leonid Khachian developed a method called the ellipsoid algorithm which was supposed to be revolutionary, but as it turned out it is not any better than the simplex method. In 1984, Narendra Karmarkar, a research scientist at AT&T Bell Laboratories developed Karmarkar's algorithm which has been proven to be four times faster than the simplex method for certain problems. But the simplex method still works the best for most problems.
- The simplex method uses an approach that is very efficient. It does not compute the value of the objective function at every point; instead, it begins with a corner point of the feasibility region where all the main variables are zero and then systematically moves from corner point to corner point, while improving the value of the objective function at each stage. The process continues until the optimal solution is found.

Simplex Method

1. Identify and set up a linear program in standard maximization form
2. Convert inequality constraints to equations using slack variables
3. Set up the initial simplex tableau using the objective function and slack equations
4. Find the optimal simplex tableau by performing pivoting operations.
5. Identify the optimal solution from the optimal simplex tableau

- Standard form

Maximize $c_1x_1 + c_2x_2 + \cdots c_nx_n$

subject to

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \\
x_1, x_2, \ldots, x_n &\geq 0
\end{aligned}
$$

We can rewrite this is matrix form, by setting:

- $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$,
- $\mathbf{c} = (c_1, c_2, \ldots, c_n)^\top$,
- $\mathbf{b} = (b_1, b_2, \ldots, b_m)^\top$, and
- $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$.

With those definitions we can write the LP as:

Maximize $\mathbf{c} \cdot \mathbf{x}$

subject to $\begin{cases} A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq 0 \end{cases}$

Source: https://www.matem.unam.mx/~omar/math340/std-form.html

- **Converting a LP to standard form**

  – the objective function is to be minimized

  – a constraint is lower bound

$$a_{i1}x_1 + \cdots + a_{in}x_n \geq b$$

  – there is an equality constraint

  u=*v*

  – a variable lacks a positivity constraint

$$
\begin{array}{lrcrcl}
\text{maximize} & 2x_1 & - & 3x_2 & & \\
\text{subject to} & & & & & \\
 & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & x_1 & & & \geq & 0
\end{array}
$$

*from inequalities to equalities

a.x <= b

Source: https://www.matem.unam.mx/~omar/math340/std-form.html

Simplex Method

1. **Set up the problem.** write the objective function and the inequality constraints.
2. **Convert the inequalities into equations.** add one slack variable for each inequality.
3. **Construct the initial simplex tableau.** Write the objective function as the bottom row.
4. **The most negative entry in the bottom row identifies the pivot column.**
5. **Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.** The quotients are computed by dividing the far right column by the identified column in step 4. A quotient that is a zero, or a negative number, or that has a zero in the denominator, is ignored.
6. **Perform pivoting to make all other entries in this column zero.**
7. **When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.**
8. **Read off your answers.** Get the variables using the columns with 1 and 0s. All other variables are zero. The maximum value you are looking for appears in the bottom right hand corner.

Simplex Method – problem example

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

STEP 1. Set up the problem

Let

- x_1 = The number of hours per week Niki will work at Job I and
- x_2 = The number of hours per week Niki will work at Job II.

Maximize Subject to: $Z=40x_1+30x_2$

$$x_1+x_2 \le 12$$

$$2x_1+x_2 \le 16$$

$$x_1 \ge 0; x_2 \ge 0$$

STEP 2. Convert the inequalities into equations

We add slack variables. The values of the slack variables will identify the unused amounts.

$x1+x2\le 12$ as $x1+x2+y1=12$ (we add a non-negative variable $y1$)

We rewrite the objective function
$Z=40x1+30x2$ as $-40x1-30x2+Z=0$

After adding the slack variables, our problem reads

Objective function
$-40x1-30x2+Z=0$

Subject to constraints:
$x1+x2+y1=12$
$2x1+x2+y2=16$
$x1\ge 0; x2\ge 0; y1\ge 0; y2\ge 0$

STEP 3. Construct the initial simplex tableau

- Each inequality constraint appears in its own row. (The non-negativity constraints do *not* appear as rows in the simplex tableau.)
- Write the objective function as the bottom row.
- The initial simplex tableau:

x_1	x_2	y_1	y_2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
–40	–30	0	0	1	0

- If we set x1 and x2 = 0, we g

$$\left[\begin{array}{ccc|c} y_1 & y_2 & Z & C \\ 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

- label the basic s ht of the last column as shown in the table below

x_1	x_2	y_1	y_2	Z		
1	1	1	0	0	12	y_1
2	1	0	1	0	16	y_2
–40	–30	0	0	1	0	Z

STEP 4. The most negative entry in the bottom row identifies the pivot column

- The most negative entry in the bottom row is -40; therefore the column 1 is identified.

x_1	x_2	y_1	y_2	Z		
1	1	1	0	0	12	y_1
2	1	0	1	0	16	y_2
–40	–30	0	0	1	0	Z
↑						

Question Why do we choose the most negative entry in the bottom row?
Answer The most negative entry in the bottom row represents the largest coefficient in the objective function - the coefficient whose entry will increase the value of the objective function the quickest.

The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1, x_2, x_3 etc., are zero.

It then moves from a corner point to the adjacent corner point always increasing the value of the objective function.

Since Job I pays $40 per hour as opposed to Job II which pays only $30.

STEP 5. Calculate the quotients. Identify the Pivot

The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.

Following the algorithm, in order to calculate the quotient, we divide the entries in the far right column by the entries in column 1, excluding the entry in the bottom row.

x1	x2	y1	y2	Z			
1	1	1	0	0	12	y1	$12 \div 1 = 12$
$\boxed{2}$	1	0	1	0	16	y2	$\leftarrow 16 \div 2 = 8$
–40 ↑	–30	0	0	1	0	Z	

The smallest of the two quotients, 12 and 8, is 8. Therefore row 2 is identified. The intersection of column 1 and row 2 is the entry 2, which has been highlighted. This is our pivot element.

Question Why do we find quotients, and why does the smallest quotient identify a row?
Answer To find the most strict constraint.
Question Why do we identify the pivot element?
Answer The value of the objective function is improved by changing the number of units of the variables. We may add the number of units of one variable, while throwing away the units of another. Pivoting allows us to do just that.
The variable whose units are being added is called the **entering variable,** and the variable whose units are being replaced is called the **departing variable.**

STEP 6. Perform pivoting to make all other entries in this column zero

Pivoting is a process of obtaining a 1 in the location of the pivot element, and then making all other entries zeros in that column. Make our pivot element a 1 by dividing the entire second row by 2.

x_1	x_2	y_1	y_2	Z	
1	1	1	0	0	12
$\boxed{1}$	1/2	0	1/2	0	8
–40	–30	0	0	1	0

To obtain a zero in the entry first above the pivot element, we multiply the second row by -1 and add it to row 1.

x_1	x_2	y_1	y_2	Z	
0	1/2	1	–1/2	0	4
$\boxed{1}$	1/2	0	1/2	0	8
–40	–30	0	0	1	0

To obtain a zero in the element below the pivot, we multiply the second row by 40 and add it to the last row.

x_1	x_2	y_1	y_2	Z		
0	1/2	1	–1/2	0	4	y_1
$\boxed{1}$	1/2	0	1/2	0	8	x_1
0	–10	0	20	1	320	Z

STEP 6. Perform pivoting to make all other entries in this column zero

We now determine the basic solution associated with this tableau. By arbitrarily choosing x_2=0 and y_2=0, we obtain x_1=8, y_1=4 and z=320.

x_1	x_2	y_1	y_2	Z		
0	1/2	1	–1/2	0	4	y_1
$\boxed{1}$	1/2	0	1/2	0	8	x_1
0	–10	0	20	1	320	Z

If we write the augmented matrix, whose left side is a matrix with columns that have one 1 and all other entries zeros, we get the following matrix

$$\begin{bmatrix} x_1 & y_1 & Z & | & C \\ 0 & 1 & 0 & | & 4 \\ 1 & 0 & 0 & | & 8 \\ 0 & 0 & 1 & | & 320 \end{bmatrix}$$

At this stage of the simplex, we have as solution: if Niki works 8 hours at Job I, and no hours at Job II, her profit Z will be $320.
Is this the optimal solution?

STEP 7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4

x_1	x_2	y_1	y_2	Z		
0	1/2	1	–1/2	0	4	y_1
$\boxed{1}$	1/2	0	1/2	0	8	x_1
0	–10	0	20	1	320	Z

x_1	x_2	y_1	y_2	Z			
0	$\boxed{1/2}$	1	–1/2	0	4	y_1	← 4 ÷ 1/2 = 8
1	1/2	0	1/2	0	8	x_1	8 ÷ 1/2 = 16
0	–10	0	20	1	320	Z	
	↑						

x_1	x_2	y_1	y_2	Z	
0	$\boxed{1}$	2	–1	0	8
1	1/2	0	1/2	0	8
0	–10	0	20	1	320

x_1	x_2	y_1	y_2	Z		
0	1	2	–1	0	8	x_2
1	0	–1	1	0	4	x_1
0	0	20	10	1	400	Z

We no longer have negative entries in the bottom row, therefore we are finished.

STEP 7. Question

Question Why are we finished when there are no negative entries in the bottom row?
Answer The answer lies in the bottom row. The bottom row corresponds to the equation:

$$0x_1+0x_2+20y_1+10y_2+Z=400$$

or

$$z=400-20y1-10y2$$

Since all variables are non-negative, the highest value z can ever achieve is 400, and that will happen only when y_1 and y_2 are zero.

Question

Question What if we have a minimization problem?

Answer Given that you have transformed the formulation of the problem to the standard form in its a maximization version, the method follows as stated with two changes:

- the column selection criteria is now the most ~~negative~~ **positive** entry in the bottom row;
- when there are no more ~~negative~~ **positive** entries in the bottom row, we are finished

Interesting similarities

Max Z=40x1+30x2	Min Z=-(40x1+30x2)
x1+x2≤12	-x1-x2≤-12
2x1+x2≤16	-2x1-x2≤-16
x1≥0;x2≥0	x1≥0;x2≥0

x1	x2	y1	y2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
-40	-30	0	0	1	0

x1	x2	y1	y2	Z	C
-1	-1	1	0	0	-12
-2	-1	0	1	0	-16
40	30	0	0	1	0

x1	x2	y1	y2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
-40	-30	0	0	1	0

x1	x2	y1	y2	Z	C
-1	-1	1	0	0	-12
-2	-1	0	1	0	-16
40	30	0	0	1	0

x1	x2	y1	y2	Z	C
1	1	1	0	0	12
1	0,5	0	0,5	0	8
-40	-30	0	0	1	0

x1	x2	y1	y2	Z	C
-1	-1	1	0	0	-12
1	0,5	0	-0,5	0	8
40	30	0	0	1	0

x1	x2	y1	y2	Z	C
0	0,5	1	-0,5	0	4
1	0,5	0	0,5	0	8
0	-10	0	20	1	320

x1	x2	y1	y2	Z	C
0	-0,5	1	-0,5	0	-4
1	0,5	0	-0,5	0	8
0	10	0	-20	1	-400

x1	x2	y1	y2	Z	C
0	0,5	1	-0,5	0	4
1	0,5	0	0,5	0	8
0	-10	0	20	1	320

x1	x2	y1	y2	Z	C
0	-0,5	1	-0,5	0	-4
1	0,5	0	-0,5	0	8
0	10	0	-20	1	-320

x1	x2	y1	y2	Z	C
0	1	2	1	0	***8***
1	0	1	0	0	***4***
0	0	20	10	1	***400***

x1	x2	y1	y2	Z	C
0	1	-2	1	0	**8**
1	0	1	0	0	**4**
0	0	-20	-10	1	**-400**