Chapter 2

Introduction to Number Theory

Divisibility

- We say that a nonzero b **divides** a if $a = mb$ for some m, where a, b, and m are integers
- b divides a if there is no remainder on division
- The notation $b \mid a$ is commonly used to mean b divides a
- If $b \mid a$ we say that b is a **divisor** of a

The positive divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24
$13 \mid 182$; $-5 \mid 30$; $17 \mid 289$; $-3 \mid 33$; $17 \mid 0$

Properties of Divisibility

- If $a \mid 1$, then $a = \pm 1$
- If $a \mid b$ and $b \mid a$, then $a = \pm b$
- Any $b \neq 0$ divides 0
- If $a \mid b$ and $b \mid c$, then $a \mid c$

- If $b \mid g$ and $b \mid h$, then $b \mid (mg + nh)$ for arbitrary integers m and n

11 | 66 and 66 | 198 = 11 | 198

Properties of Divisibility

- To see this last point, note that:
  - If $b \mid g$, then g is of the form $g = b * g_1$ for some integer g_1
  - If $b \mid h$, then h is of the form $h = b * h_1$ for some integer h_1
- So:
  - $mg + nh = mbg_1 + nbh_1 = b * (mg_1 + nh_1)$

    and therefore b divides $mg + nh$

> $b = 7$; $g = 14$; $h = 63$; $m = 3$; $n = 2$
> 7 | 14 and 7 | 63.
> To show 7 (3 * 14 + 2 * 63),
> we have (3 * 14 + 2 * 63) = 7(3 * 2 + 2 * 9),
> and it is obvious that 7 | (7(3 * 2 + 2 * 9)).

Division Algorithm

- Given any positive integer n and any nonnegative integer $a,$ if we divide a by n we get an integer quotient q and an integer remainder r that obey the following relationship:

$$a = qn + r \qquad\qquad 0 \leq r < n;\ q = [a/n]$$

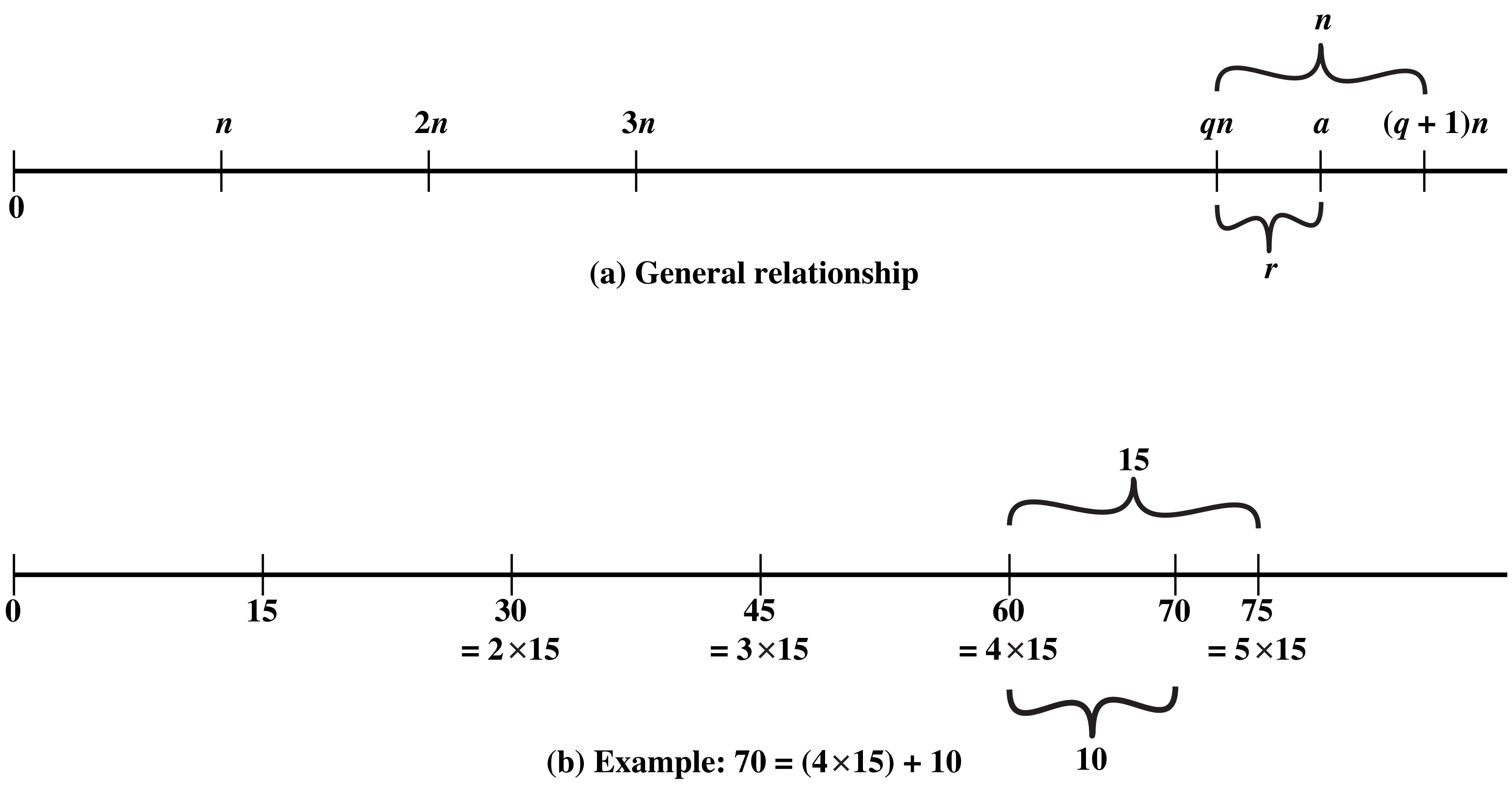


Figure 2.1 The Relationship $a = qn + r;\ 0 \le r < n$

Greatest Common Divisor (GCD)

- The greatest common divisor of a and b is the largest integer that divides both a and b
- We can use the notation gcd(a,b) to mean the **greatest common divisor** of a and b
- We also define gcd(0,0) = 0
- Positive integer c is said to be the gcd of a and b if:
  - c is a divisor of a and b
  - Any divisor of a and b is a divisor of c
- An equivalent definition is:

$$\gcd(a,b) = \max[k, \text{ such that } k \mid a \text{ and } k \mid b]$$

Modular Arithmetic

- The modulus
  - If a is an integer and n is a positive integer, we define a mod n to be the remainder when a is divided by n; the integer n is called the **modulus**
  - Thus, for any integer a:

$$a = qn + r \qquad 0 \leq r < n; \; q = [a/n]$$

$$a = [a/n] * n + (a \bmod n)$$

11 mod 7 = 4; - 11 mod 7 = 3

Modular Arithmetic

- Congruent modulo *n*
  - Two integers *a* and *b* are said to be **congruent modulo *n*** if (a mod n) = (b mod n)
  - This is written as
    $a \equiv b \pmod{n}$
  - Note that if $a \equiv 0 \pmod{n}$, then $n \mid a$

$73 \equiv 4 \pmod{23}$; $21 \equiv -9 \pmod{10}$

Properties of Congruences

- Congruences have the following properties:
  1. $a \equiv b$ (mod n) if n $(a - b)$
  2. $a \equiv b$ (mod n) implies $b \equiv a$ (mod n)
  3. $a \equiv b$ (mod n) and $b \equiv c$ (mod n) imply $a \equiv c$ (mod n)
- To demonstrate the first point, if n $(a - b)$, then $(a - b) = kn$ for some k
  - So we can write $a = b + kn$
  - Therefore, (a mod n) = (remainder when $b + kn$ is divided by n) = (remainder when b is divided by n) = (b mod n)

23 ≡ 8 (mod 5) because 23 - 8 = 15 = 5 * 3
- 11 ≡ 5 (mod 8) because - 11 - 5 = - 16 = 8 * (- 2)
81 ≡ 0 (mod 27) because 81 - 0 = 81 = 27 * 3

Modular Arithmetic

- Modular arithmetic exhibits the following properties:

  1. $[(a \bmod n) + (b \bmod n)] \bmod n = (a + b) \bmod n$
  2. $[(a \bmod n) - (b \bmod n)] \bmod n = (a - b) \bmod n$
  3. $[(a \bmod n) * (b \bmod n)] \bmod n = (a * b) \bmod n$

- We demonstrate the first property:
  - Define $(a \bmod n) = r_a$ and $(b \bmod n) = r_b$. Then we can write $a = r_a + jn$ for some integer j and $b = r_b + kn$ for some integer k
  - Then:

$$\begin{aligned}
(a + b) \bmod n &= (ra + jn + rb + kn) \bmod n \\
&= (ra + rb + (k + j)n) \bmod n \\
&= (ra + rb) \bmod n \\
&= [(a \bmod n) + (b \bmod n)] \bmod n
\end{aligned}$$

Remaining Properties:

- Examples of the three remaining properties:

11 mod 8 = 3;   15 mod 8 = 7

[(11 mod 8) + (15 mod 8)] mod 8 = 10 mod 8 = 2

(11 + 15) mod 8 = 26 mod 8 = 2

[(11 mod 8) - (15 mod 8)] mod 8 = - 4 mod 8 = 4

(11 - 15) mod 8 = - 4 mod 8 = 4

[(11 mod 8) * (15 mod 8)] mod 8 = 21 mod 8 = 5

(11 * 15) mod 8 = 165 mod 8 = 5

Exponentiation: by repeated multiplication

To find $11^7 \bmod 13$, we can proceed as follows:

$$11^2 = 121 \equiv 4 \pmod{13}$$

$$11^4 = (11^2)^2 \equiv 4^2 \equiv 3 \pmod{13}$$

$$11^7 = 11 \times 11^2 \times 11^4$$

$$11^7 \equiv 11 \times 4 \times 3 \equiv 132 \equiv 2 \pmod{13}$$

Table 2.2(a)
Arithmetic Modulo 8

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

Table 2.2(b) Multiplication Modulo 8

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Table 2.2(c)

Additive and Multiplicative Inverse Modulo 8

w	$-w$	w^{-1}
0	0	—
1	7	1
2	6	—
3	5	3
4	4	—
5	3	5
6	2	—
7	1	7

(This table can be found on page 37 in the textbook)

Table 2.3

Properties of Modular Arithmetic for Integers in Z_n

Property	Expression
Commutative Laws	$(w + x) \bmod n = (x + w) \bmod n$ $(w \times x) \bmod n = (x \times w) \bmod n$
Associative Laws	$[(w + x) + y] \bmod n = [w + (x + y)] \bmod n$ $[(w \times x) \times y] \bmod n = [w \times (x \times y)] \bmod n$
Distributive Law	$[w \times (x + y)] \bmod n = [(w \times x) + (w \times y)] \bmod n$
Identities	$(0 + w) \bmod n = w \bmod n$ $(1 \times w) \bmod n = w \bmod n$
Additive Inverse ($-w$)	For each $w \in Z_n$, there exists a z such that $w + z \equiv 0 \bmod n$

GCD

- Because we require that the greatest common divisor be positive, $\gcd(a,b) = \gcd(a,-b) = \gcd(-a,b) = \gcd(-a,-b)$
- In general, $\gcd(a,b) = \gcd(|\,a\,|, |\,b\,|)$

  gcd(60, 24) = gcd(60, - 24) = 12

- Also, because all nonzero integers divide 0, we have gcd(a,0) = | a |
- We stated that two integers *a* and *b* are relatively prime if their only common positive integer factor is 1; this is equivalent to saying that *a* and *b* are relatively prime if $\gcd(a,b) = 1$

  8 and 15 are relatively prime because the positive divisors of 8 are 1, 2, 4, and 8, and the positive divisors of 15 are 1, 3, 5, and 15. So 1 is the only integer on both lists.

Euclidean Algorithm

- One of the basic techniques of number theory
- Procedure for determining the greatest common divisor of two positive integers
- Two integers are **relatively prime** if their only common positive integer factor is 1

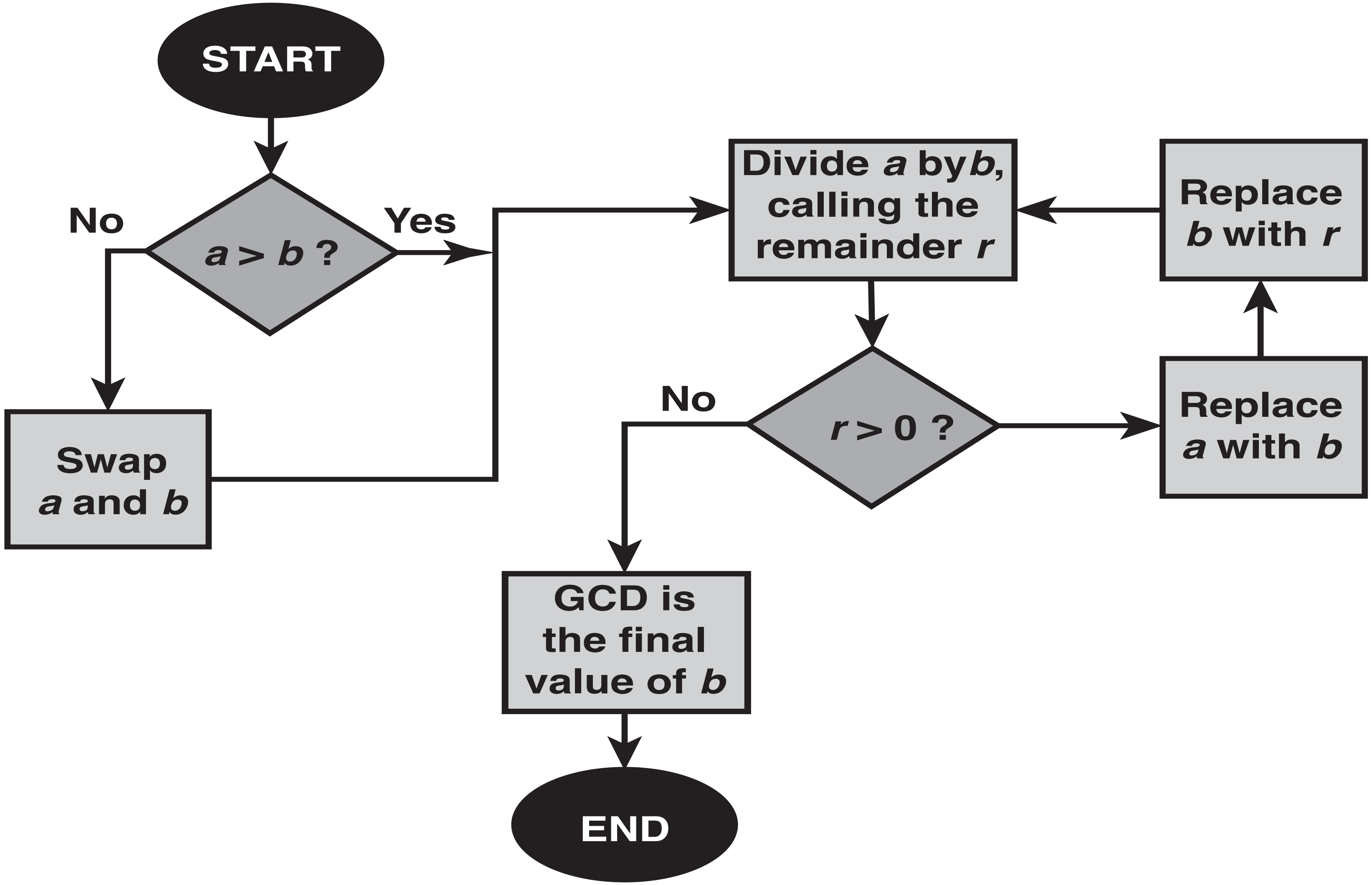


Figure 2.2 Euclidean Algorithm

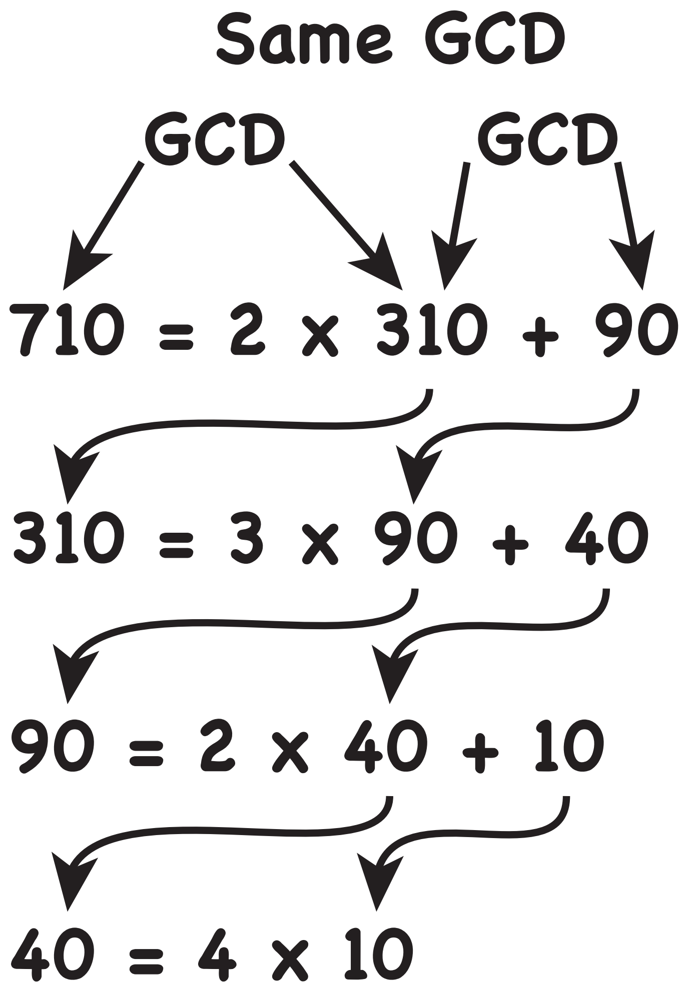


Figure 2.3 Euclidean Algorithm Example: gcd(710, 310)

To find $d = \gcd(a, b) = \gcd(1160718174, 316258250)$		
$a = q_1 b + r_1$	$1160718174 = 3 \times 316258250 + 211943424$	$d = \gcd(316258250, 211943424)$
$b = q_2 r_1 + r_2$	$316258250 = 1 \times 211943424 + 104314826$	$d = \gcd(211943424, 104314826)$
$r_1 = q_3 r_2 + r_3$	$211943424 = 2 \times 104314826 + 3313772$	$d = \gcd(104314826, 3313772)$
$r_2 = q_4 r_3 + r_4$	$104314826 = 31 \times 3313772 + 1587894$	$d = \gcd(3313772, 1587894)$
$r_3 = q_5 r_4 + r_5$	$3313772 = 2 \times 1587894 + 137984$	$d = \gcd(1587894, 137984)$
$r_4 = q_6 r_5 + r_6$	$1587894 = 11 \times 137984 + 70070$	$d = \gcd(137984, 70070)$
$r_5 = q_7 r_6 + r_7$	$137984 = 1 \times 70070 + 67914$	$d = \gcd(70070, 67914)$
$r_6 = q_8 r_7 + r_8$	$70070 = 1 \times 67914 + 2156$	$d = \gcd(67914, 2156)$
$r_7 = q_9 r_8 + r_9$	$67914 = 31 \times 2156 + 1078$	$d = \gcd(2156, 1078)$
$r_8 = q_{10} r_9 + r_{10}$	$2156 = 2 \times 1078 + 0$	$d = \gcd(1078, 0) = 1078$
Therefore, $d = \gcd(1160718174, 316258250) = 1078$		

Euclidean Algorithm	
Calculate	**Which satisfies**
$r_1 = a \bmod b$	$a = q_1 b + r_1$
$r_2 = b \bmod r_1$	$b = q_2 r_1 + r_2$
$r_3 = r_1 \bmod r_2$	$r_1 = q_3 r_2 + r_3$
$\vdots$	$\vdots$
$r_n = r_{n-2} \bmod r_{n-1}$	$r_{n-2} = q_n r_{n-1} + r_n$
$r_{n+1} = r_{n-1} \bmod r_n = 0$	$r_{n-1} = q_{n+1} r_n + 0$ $d = \gcd(a, b) = r_n$

```
Euclid(a,b)
    if (b=0) then return a;
    else return Euclid(b, a mod b);
```

Table 2.1
Euclidean Algorithm Example

Dividend	Divisor	Quotient	Remainder
$a = 1160718174$	$b = 316258250$	$q_1 = 3$	$r_1 = 211943424$
$b = 316258250$	$r_1 = 211943424$	$q_2 = 1$	$r_2 = 104314826$
$r_1 = 211943424$	$r_2 = 104314826$	$q_3 = 2$	$r_3 = 3313772$
$r_2 = 104314826$	$r_3 = 3313772$	$q_4 = 31$	$r_4 = 1587894$
$r_3 = 3313772$	$r_4 = 1587894$	$q_5 = 2$	$r_5 = 137984$
$r_4 = 1587894$	$r_5 = 137984$	$q_6 = 11$	$r_6 = 70070$
$r_5 = 137984$	$r_6 = 70070$	$q_7 = 1$	$r_7 = 67914$
$r_6 = 70070$	$r_7 = 67914$	$q_8 = 1$	$r_8 = 2156$
$r_7 = 67914$	$r_8 = 2156$	$q_9 = 31$	$r_9 = 1078$
$r_8 = 2156$	$r_9 = 1078$	$q_{10} = 2$	$r_{10} = 0$

$$ax + by = d = \gcd(a, b)$$

Extended Euclidean Algorithm			
Calculate	**Which satisfies**	**Calculate**	**Which satisfies**
$r_{-1} = a$		$x_{-1} = 1; y_{-1} = 0$	$a = ax_{-1} + by_{-1}$
$r_0 = b$		$x_0 = 0; y_0 = 1$	$b = ax_0 + by_0$
$r_1 = a \bmod b$ $q_1 = \lfloor a/b \rfloor$	$a = q_1 b + r_1$	$x_1 = x_{-1} - q_1 x_0 = 1$ $y_1 = y_{-1} - q_1 y_0 = -q_1$	$r_1 = ax_1 + by_1$
$r_2 = b \bmod r_1$ $q_2 = \lfloor b/r_1 \rfloor$	$b = q_2 r_1 + r_2$	$x_2 = x_0 - q_2 x_1$ $y_2 = y_0 - q_2 y_1$	$r_2 = ax_2 + by_2$
$r_3 = r_1 \bmod r_2$ $q_3 = \lfloor r_1/r_2 \rfloor$	$r_1 = q_3 r_2 + r_3$	$x_3 = x_1 - q_3 x_2$ $y_3 = y_1 - q_3 y_2$	$r_3 = ax_3 + by_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r_n = r_{n-2} \bmod r_{n-1}$ $q_n = \lfloor r_{n-2}/r_{n-1} \rfloor$	$r_{n-2} = q_n r_{n-1} + r_n$	$x_n = x_{n-2} - q_n x_{n-1}$ $y_n = y_{n-2} - q_n y_{n-1}$	$r_n = ax_n + by_n$
$r_{n+1} = r_{n-1} \bmod r_n = 0$ $q_{n+1} = \lfloor r_{n-1}/r_n \rfloor$	$r_{n-1} = q_{n+1} r_n + 0$		$d = \gcd(a, b) = r_n$ $x = x_n; y = y_n$

Table 2.4
Extended Euclidean Algorithm Example

$1759\,x + 550\,y = \text{gcd}(1759, 550)$

i	r_i	q_i	x_i	Y_i
–1	1759		1	0
0	550		0	1
1	109	3	1	–3
2	5	5	–5	16
3	4	21	106	–339
4	1	1	–111	355
5	0	4		

Result: $d = 1$; $x = –111$; $y = 355$

Prime Numbers

- Prime numbers only have divisors of 1 and itself
  - They cannot be written as a product of other numbers
- Prime numbers are central to number theory
- Any integer a > 1 can be factored in a unique way as

$$a = {}_{p1}{}^{a1} * p_2{}^{a2} * \ldots * p_{p1}{}^{a1}$$

where $p_1 < p_2 < \ldots < p_t$ are prime numbers and where each a_i is a positive integer

- This is known as the fundamental theorem of arithmetic

Table 2.5
Primes Under 2000

2	101	211	307	401	503	601	701	809	907	1009	1103	1201	1301	1409	1511	1601	1709	1801	1901
3	103	223	311	409	509	607	709	811	911	1013	1109	1213	1303	1423	1523	1607	1721	1811	1907
5	107	227	313	419	521	613	719	821	919	1019	1117	1217	1307	1427	1531	1609	1723	1823	1913
7	109	229	317	421	523	617	727	823	929	1021	1123	1223	1319	1429	1543	1613	1733	1831	1931
11	113	233	331	431	541	619	733	827	937	1031	1129	1229	1321	1433	1549	1619	1741	1847	1933
13	127	239	337	433	547	631	739	829	941	1033	1151	1231	1327	1439	1553	1621	1747	1861	1949
17	131	241	347	439	557	641	743	839	947	1039	1153	1237	1361	1447	1559	1627	1753	1867	1951
19	137	251	349	443	563	643	751	853	953	1049	1163	1249	1367	1451	1567	1637	1759	1871	1973
23	139	257	353	449	569	647	757	857	967	1051	1171	1259	1373	1453	1571	1657	1777	1873	1979
29	149	263	359	457	571	653	761	859	971	1061	1181	1277	1381	1459	1579	1663	1783	1877	1987
31	151	269	367	461	577	659	769	863	977	1063	1187	1279	1399	1471	1583	1667	1787	1879	1993
37	157	271	373	463	587	661	773	877	983	1069	1193	1283		1481	1597	1669	1789	1889	1997
41	163	277	379	467	593	673	787	881	991	1087		1289		1483		1693			1999
43	167	281	383	479	599	677	797	883	997	1091		1291		1487		1697			
47	173	283	389	487		683		887		1093		1297		1489		1699			
53	179	293	397	491		691				1097				1493					
59	181			499										1499					
61	191																		
67	193																		
71	197																		
73	199																		
79																			
83																			
89																			
97																			

Fermat's Theorem

- States the following:
  - If p is prime and a is a positive integer not divisible by p then

$$a^{p-1} \equiv 1 \pmod{p}$$

- An alternate form is:
  - If p is prime and a is a positive integer then

$$a^{p} \equiv a \pmod{p}$$

Table 2.6

Some Values of Euler’s Totient Function $\phi(n)$

n	$\phi(n)$	n	$\phi(n)$	n	$\phi(n)$
1	1	11	10	21	12
2	1	12	4	22	10
3	2	13	12	23	22
4	2	14	6	24	8
5	4	15	8	25	20
6	2	16	8	26	12
7	6	17	16	27	18
8	4	18	6	28	12
9	6	19	18	29	28
10	4	20	8	30	8

Euler's Theorem

- States that for every *a* and *n* that are relatively prime:

$$a^{\varnothing(n)} \equiv 1 (\bmod\ n)$$

- An alternative form is:

$$a^{\varnothing(n)+1} \equiv a (\bmod\ n)$$

Miller-Rabin Algorithm

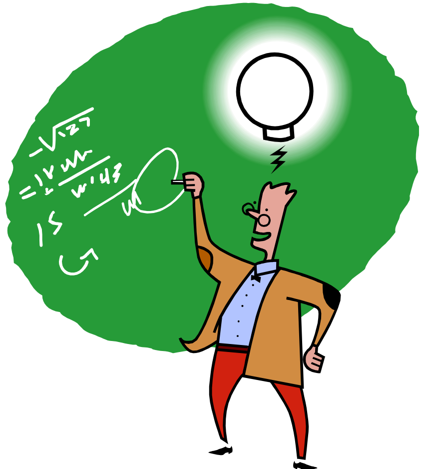
- Typically used to test a large number for primality
- Algorithm is:

TEST (n)

1. Find integers k, q, with $k > 0$, q odd, so that $(n - 1)=2^k q$;
2. Select a random integer a, $1 < a < n - 1$;
3. **if** $a^q \bmod n = 1$ **then** return (“inconclusive") ;
4. **for** $j = 0$ **to** $k - 1$ **do**
5. **if** $(a^{2^j q} \bmod n = n - 1)$ **then** return (“inconclusive") ;
6. return (“composite") ;

Deterministic Primality Algorithm

- Prior to 2002 there was no known method of efficiently proving the primality of very large numbers
- All of the algorithms in use produced a probabilistic result
- In 2002 Agrawal, Kayal, and Saxena developed an algorithm that efficiently determines whether a given large number is prime
    - Known as the AKS algorithm
    - Does not appear to be as efficient as the Miller-Rabin algorithm



Chinese Remainder Theorem (CRT)

- Believed to have been discovered by the Chinese mathematician Sun-Tsu in around 100 A.D.
- One of the most useful results of number theory
- Says it is possible to reconstruct integers in a certain range from their residues modulo a set of pairwise relatively prime moduli
- Can be stated in several ways

Provides a way to manipulate (potentially very large) numbers mod M in terms of tuples of smaller numbers

- This can be useful when M is 150 digits or more
- However, it is necessary to know beforehand the factorization of M

The 10 integers in Z_{10}, that is the integers 0 through 9, can be reconstructed from their two residues modulo 2 and 5 (the relatively prime factors of 10). Say the known residues of a decimal digit x are $r_2 = 0$ and $r_5 = 3$; that is, $x \bmod 2 = 0$ and $x \bmod 5 = 3$. Therefore, x is an even integer in Z_{10} whose remainder, on division by 5, is 3. The unique solution is $x = 8$.

Properties of Logarithms

$$y = x^{\log_x(y)}$$

$$\log_x(1) = 0$$
$$\log_x(x) = 1$$

$$\log_x(yz) = log_x(y) + \log_x(z)$$

$$\log_x(y^r) = r \times \log_x(y)$$

Discrete Logarithm

$$b \equiv a^i (\text{mod } p) \quad \text{where } 0 \leq i \leq (p - 1)$$

- This exponent i is referred to as the **discrete logarithm** of the number *b* for the base *a* (mod *p*)

Here is an example using a nonprime modulus, $n = 9$. Here $\phi(n) = 6$ and $a = 2$ is a primitive root. We compute the various powers of a and find

$$\begin{array}{ll} 2^0 = 1 & 2^4 \equiv 7 \pmod 9 \\ 2^1 = 2 & 2^5 \equiv 5 \pmod 9 \\ 2^2 = 4 & 2^6 \equiv 1 \pmod 9 \\ 2^3 = 8 & \end{array}$$

This gives us the following table of the numbers with given discrete logarithms (mod 9) for the root $a = 2$:

Logarithm	0	1	2	3	4	5
Number	1	2	4	8	7	5

To make it easy to obtain the discrete logarithms of a given number, we rearrange the table:

Number	1	2	4	5	7	8
Logarithm	0	1	2	5	4	3

Table 2.7

Powers of Integers, Modulo 19

a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	a^{15}	a^{16}	a^{17}	a^{18}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	4	8	16	13	7	14	9	18	17	15	11	3	6	12	5	10	1
3	9	8	5	15	7	2	6	18	16	10	11	14	4	12	17	13	1
4	16	7	9	17	11	6	5	1	4	16	7	9	17	11	6	5	1
5	6	11	17	9	7	16	4	1	5	6	11	17	9	7	16	4	1
6	17	7	4	5	11	9	16	1	6	17	7	4	5	11	9	16	1
7	11	1	7	11	1	7	11	1	7	11	1	7	11	1	7	11	1
8	7	18	11	12	1	8	7	18	11	12	1	8	7	18	11	12	1
9	5	7	6	16	11	4	17	1	9	5	7	6	16	11	4	17	1
10	5	12	6	3	11	15	17	18	9	14	7	13	16	8	4	2	1
11	7	1	11	7	1	11	7	1	11	7	1	11	7	1	11	7	1
12	11	18	7	8	1	12	11	18	7	8	1	12	11	18	7	8	1
13	17	12	4	14	11	10	16	18	6	2	7	15	5	8	9	3	1
14	6	8	17	10	7	3	4	18	5	13	11	2	9	12	16	15	1
15	16	12	9	2	11	13	5	18	4	3	7	10	17	8	6	14	1
16	9	11	5	4	7	17	6	1	16	9	11	5	4	7	17	6	1
17	4	11	16	6	7	5	9	1	17	4	11	16	6	7	5	9	1
18	1	18	1	18	1	18	1	18	1	18	1	18	1	18	1	18	1

Table 2.8

Tables of Discrete Logarithms, Modulo 19

(a) Discrete logarithms to the base 2, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\log_{2,19}(a)$	18	1	13	2	16	14	6	3	8	17	12	15	5	7	11	4	10	9

(b) Discrete logarithms to the base 3, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\log_{3,19}(a)$	18	7	1	14	4	8	6	3	2	11	12	15	17	13	5	10	16	9

(c) Discrete logarithms to the base 10, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\log_{10,19}(a)$	18	17	5	16	2	4	12	15	10	1	6	3	13	11	7	14	8	9

(d) Discrete logarithms to the base 13, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\log_{13,19}(a)$	18	11	17	4	14	10	12	15	16	7	6	3	1	5	13	8	2	9

(e) Discrete logarithms to the base 14, modulo 19

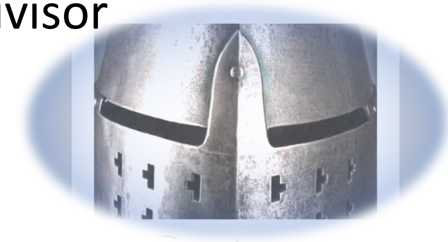
a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\log_{14,19}(a)$	18	13	7	8	10	2	6	3	14	5	12	15	11	1	17	16	4	9

(f) Discrete logarithms to the base 15, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\log_{15,19}(a)$	18	5	11	10	8	16	12	15	4	13	6	3	7	17	1	2	14	9

Summary

- Divisibility and the division algorithm
- The Euclidean algorithm
  - Greatest Common Divisor
  - Finding the Greatest Common Divisor
- Modular arithmetic
  - The modulus
  - Properties of congruences
  - Modular arithmetic operations
  - Properties of modular arithmetic
  - Euclidean algorithm revisited
  - The extended Euclidean algorithm
- Prime numbers



- Fermat’s Theorem
- Euler’s totient function
- Euler’s Theorem
- Testing for primality
  - Miller-Rabin algorithm
  - A deterministic primality algorithm
  - Distribution of primes
- The Chinese Remainder Theorem
- Discrete logarithms
  - Powers of an integer, modulo *n*
  - Logarithms for modular arithmetic
  - Calculation of discrete logarithms