

Graph Algorithms (Chapter 24)

Single-Source Shortest Paths(SSSP)

Graph Algorithms (Chapter 24)

SSSP

Goals

The shortest weighted path problem and its variants

The optimal substructure of the shortest path

Negative weights issues

Relaxation Technique

Bellman-Ford Algorithm and its time complexity

What if we have a weighted DAG

Dijkstra Algorithm

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Shortest path problem

Given a directed graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{R}$ mapping edges to weights. A path $p = \langle v_0, v_1, \dots, v_k \rangle$ has weight

$$w(p) = \sum_{i=1}^k w(v_{i-1}, v_i) \text{ (sum of weights of constituent edges)}$$

We define the *shortest-path weight* $\delta(u, v)$ from u to v by

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \xrightarrow{p} v\} & \text{if there is a path from } u \text{ to } v, \\ \infty & \text{otherwise.} \end{cases}$$

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Shortest path problem variants

Our focus

Find the a shortest path from a source $s \in V$ to each $v \in V$.

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Shortest path problem variants

Variant 1

Single-destination shortest-paths problem: Find a shortest path to a given *destination* vertex t from each vertex v . By reversing the direction of each edge in the graph, we can reduce this problem to a single-source problem.

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Shortest path problem variants

Variant 2

Single-pair shortest-path problem: Find a shortest path from u to v for given vertices u and v . If we solve the single-source problem with source vertex u , we solve this problem also. Moreover, all known algorithms for this problem have the same worst-case asymptotic running time as the best single-source algorithms.

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Shortest path problem variants

Variant 3

Find a shortest path from u to v for every pair of vertices u and v . Can be solved by running single source algorithm once from each vertex.

Can be solved faster (chap 25) but not syllabus

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Shortest path optimal substructure

Lemma 24.1 (Subpaths of shortest paths are shortest paths)

Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{R}$, let $p = \langle v_0, v_1, \dots, v_k \rangle$ be a shortest path from vertex v_0 to vertex v_k and, for any i and j such that $0 \leq i \leq j \leq k$, let $p_{ij} = \langle v_i, v_{i+1}, \dots, v_j \rangle$ be the subpath of p from vertex v_i to vertex v_j . Then, p_{ij} is a shortest path from v_i to v_j .

Proof If we decompose path p into $v_0 \xrightarrow{p_{0i}} v_i \xrightarrow{p_{ij}} v_j \xrightarrow{p_{jk}} v_k$, then we have that $w(p) = w(p_{0i}) + w(p_{ij}) + w(p_{jk})$. Now, assume that there is a path p'_{ij} from v_i to v_j with weight $w(p'_{ij}) < w(p_{ij})$. Then, $v_0 \xrightarrow{p_{0i}} v_i \xrightarrow{p'_{ij}} v_j \xrightarrow{p_{jk}} v_k$ is a path from v_0 to v_k whose weight $w(p_{0i}) + w(p'_{ij}) + w(p_{jk})$ is less than $w(p)$, which contradicts the assumption that p is a shortest path from v_0 to v_k . ■

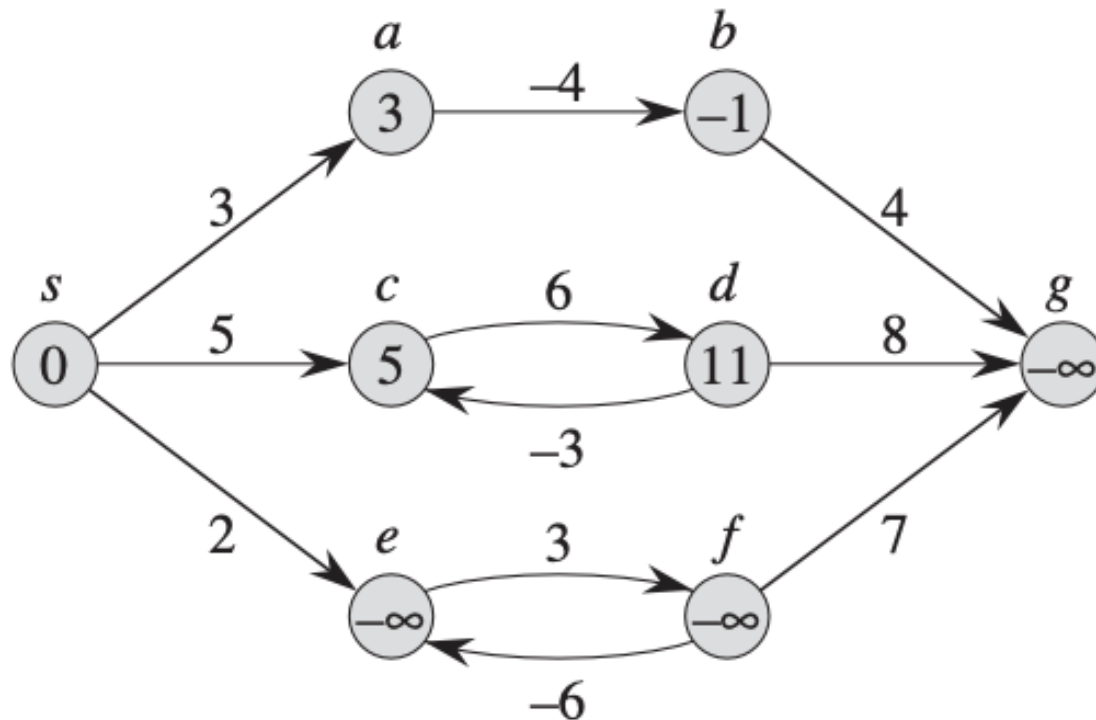
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Negative-weights and cycles

With negative weight CYCLE reachable from source s no path can be considered as the shortest path !!!

Try the shortest path $\delta(s, f)$?

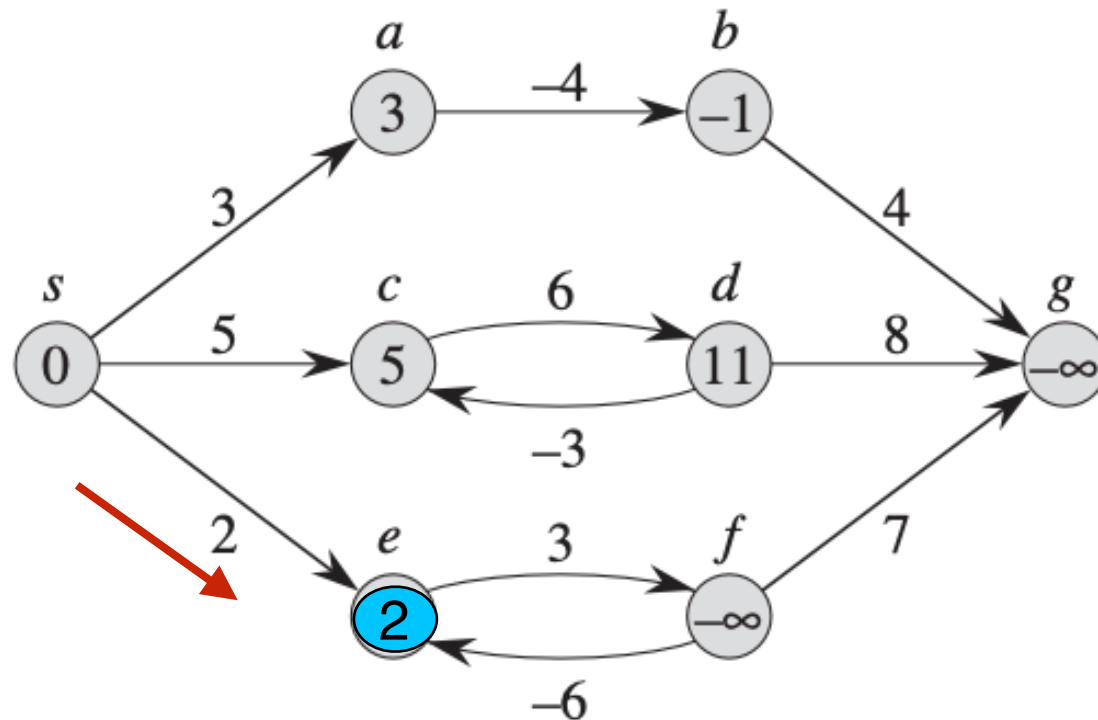


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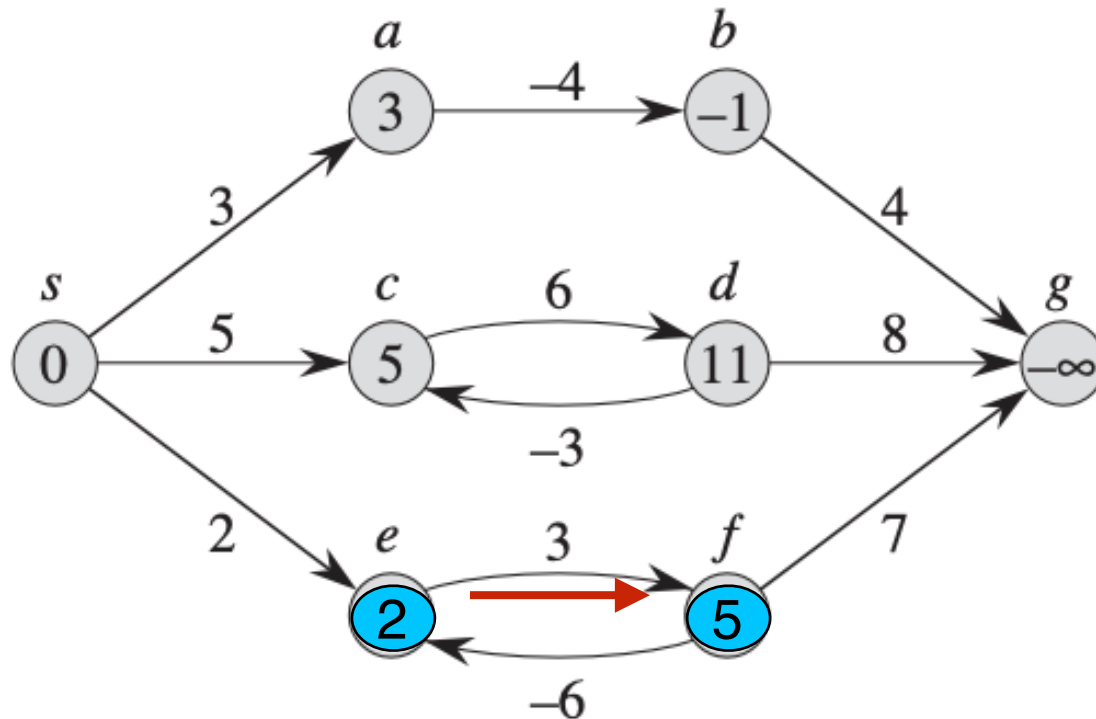


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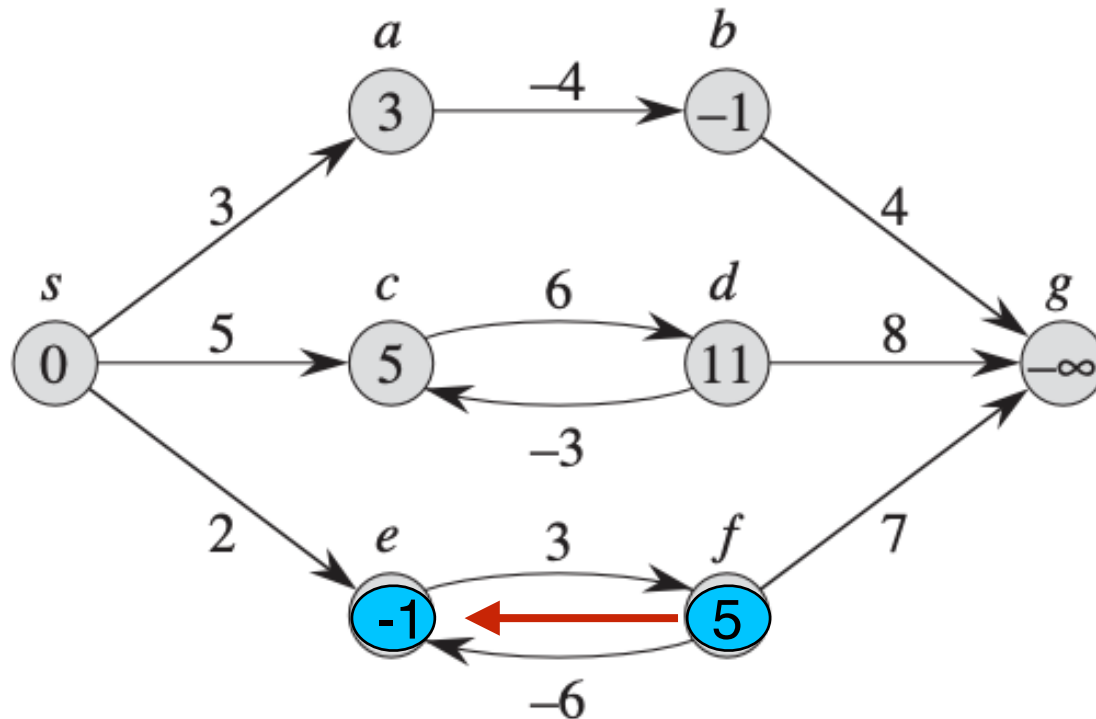


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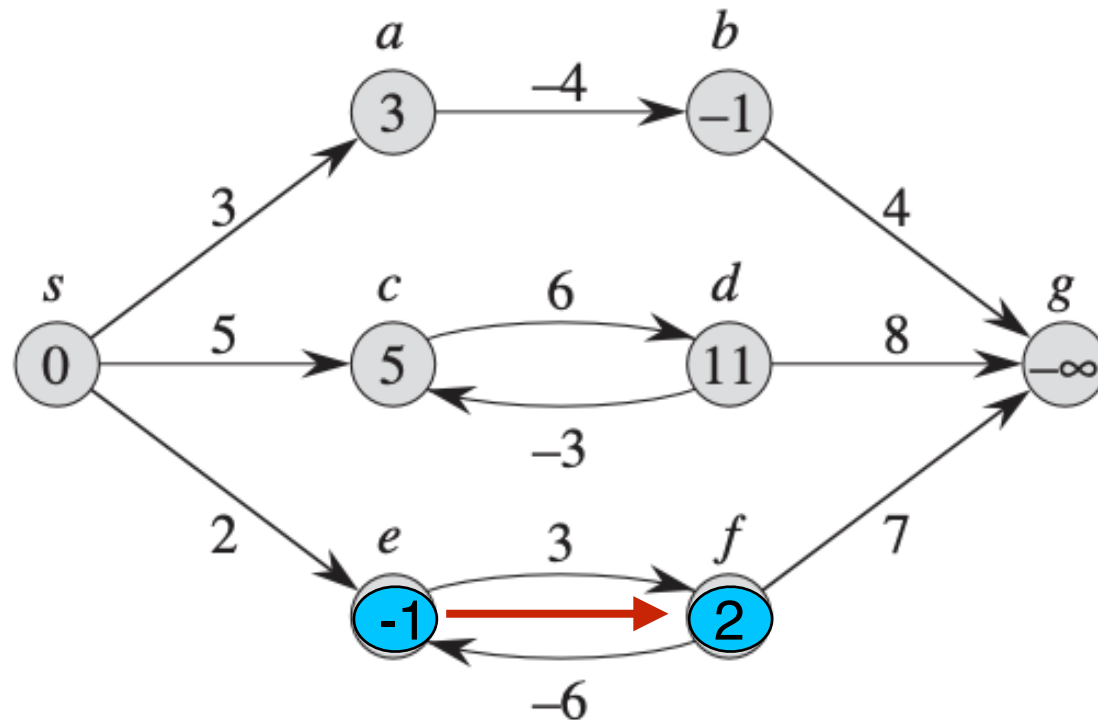


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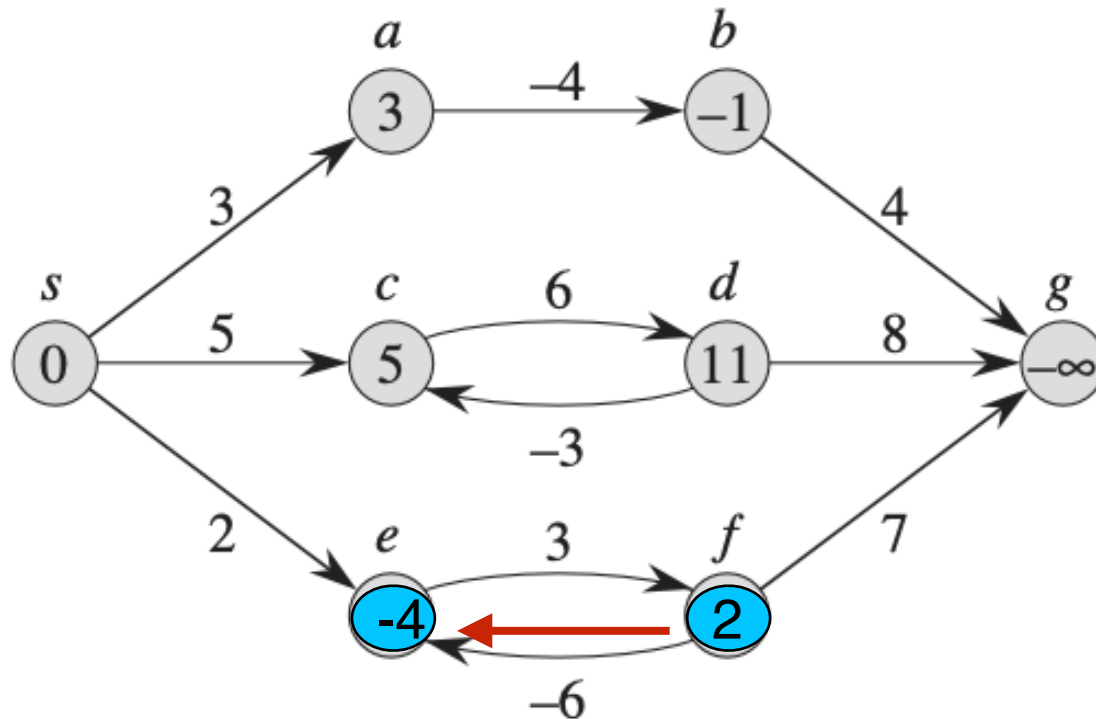


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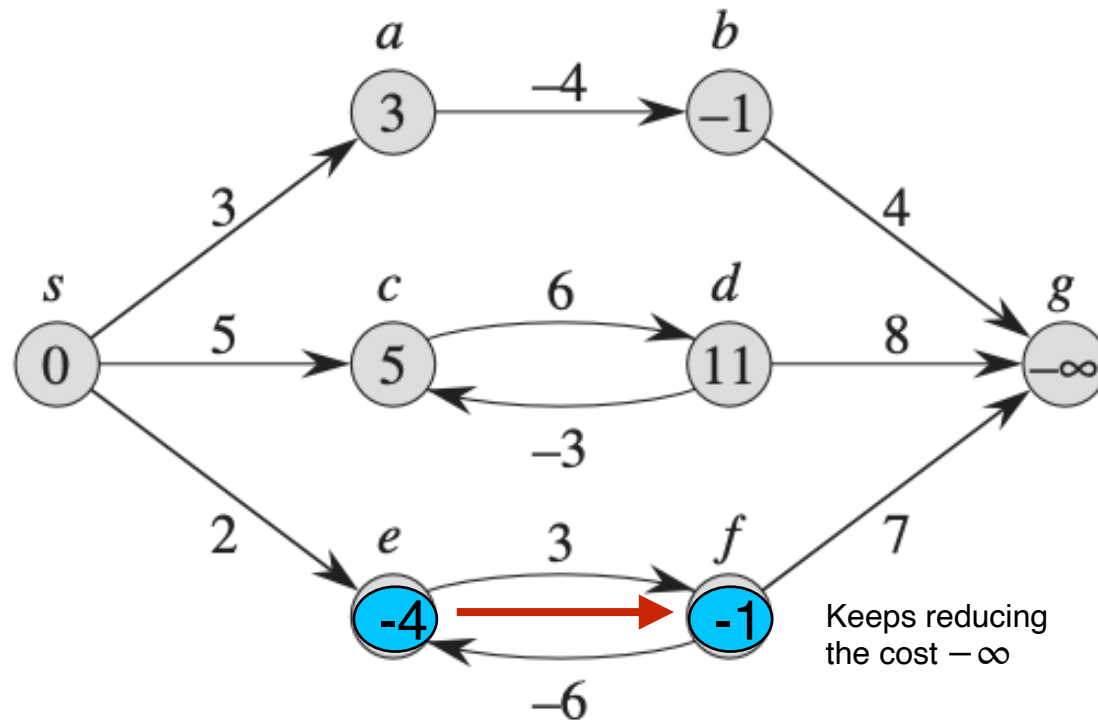


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Shortest path representation

Just like we did with BFS

We maintain for every vertex $v \in V$ a predecessor $v.\pi$ (parent) or NIL

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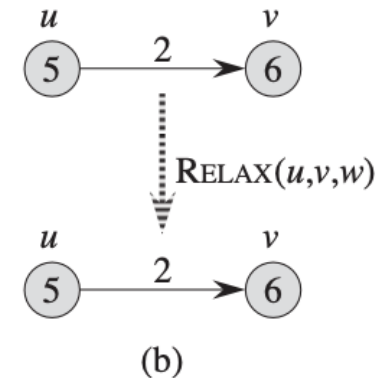
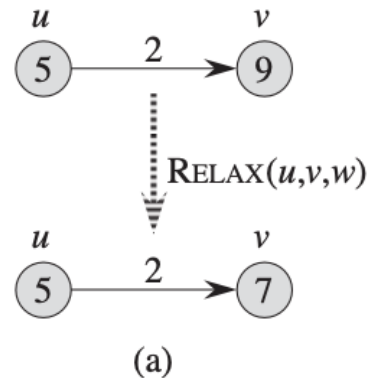
Relaxation Technique

It answers: Can I make the distance $v.d$ shorter by walking (u, v) or not.

If yes, update $v.d$ and $v.\pi$

RELAX(u, v, w)

- 1 **if** $v.d > u.d + w(u, v)$
- 2 $v.d = u.d + w(u, v)$
- 3 $v.\pi = u$



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Relaxation Properties

Triangle inequality (Lemma 24.10)

For any edge $(u, v) \in E$, we have $\delta(s, v) \leq \delta(s, u) + w(u, v)$.

Upper-bound property (Lemma 24.11)

We always have $v.d \geq \delta(s, v)$ for all vertices $v \in V$, and once $v.d$ achieves the value $\delta(s, v)$, it never changes.

No-path property (Corollary 24.12)

If there is no path from s to v , then we always have $v.d = \delta(s, v) = \infty$.

Convergence property (Lemma 24.14)

If $s \rightsquigarrow u \rightarrow v$ is a shortest path in G for some $u, v \in V$, and if $u.d = \delta(s, u)$ at any time prior to relaxing edge (u, v) , then $v.d = \delta(s, v)$ at all times afterward.

Path-relaxation property (Lemma 24.15)

If $p = \langle v_0, v_1, \dots, v_k \rangle$ is a shortest path from $s = v_0$ to v_k , and we relax the edges of p in the order $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$. This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of p .

Predecessor-subgraph property (Lemma 24.17)

Once $v.d = \delta(s, v)$ for all $v \in V$, the predecessor subgraph is a shortest-paths tree rooted at s .

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SSSP (Bellman-Ford algorithm)

Solves for positive and negative weights and returns no solution if negative cycle. If no negative cycle the result is the shortest path and their weights.

BELLMAN-FORD(G, w, s)

```
1  INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  for  $i = 1$  to  $|G.V| - 1$ 
3      for each edge  $(u, v) \in G.E$ 
4          RELAX( $u, v, w$ )
5  for each edge  $(u, v) \in G.E$ 
6      if  $v.d > u.d + w(u, v)$ 
7          return FALSE
8  return TRUE
```

Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

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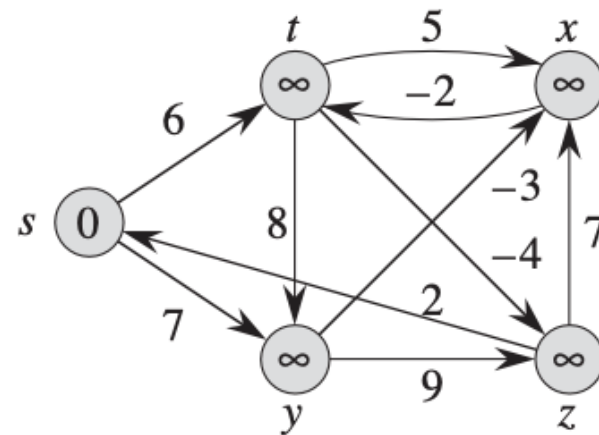
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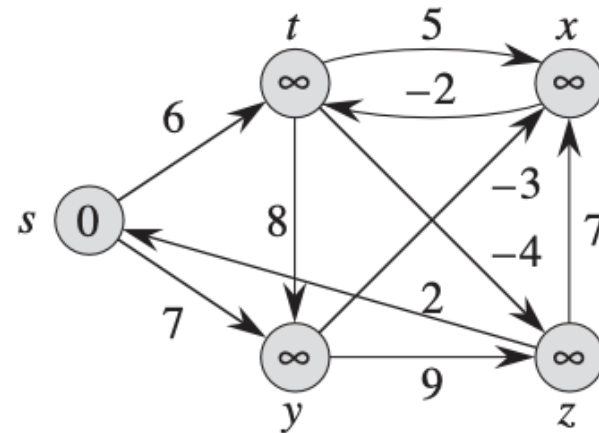
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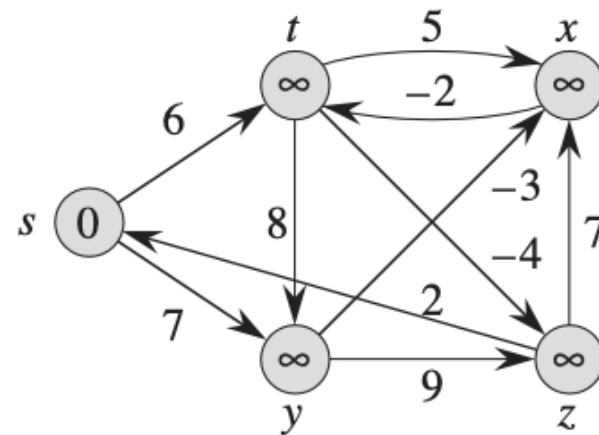
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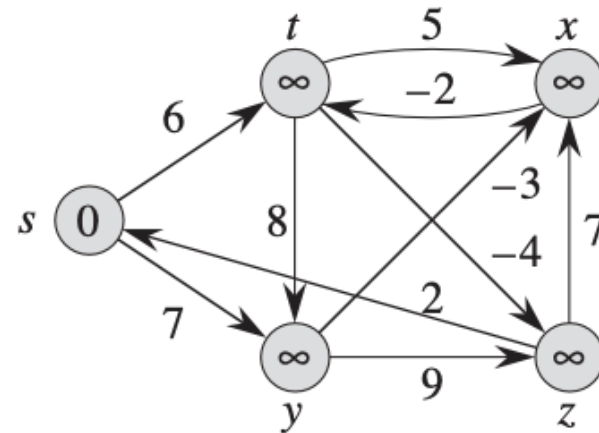
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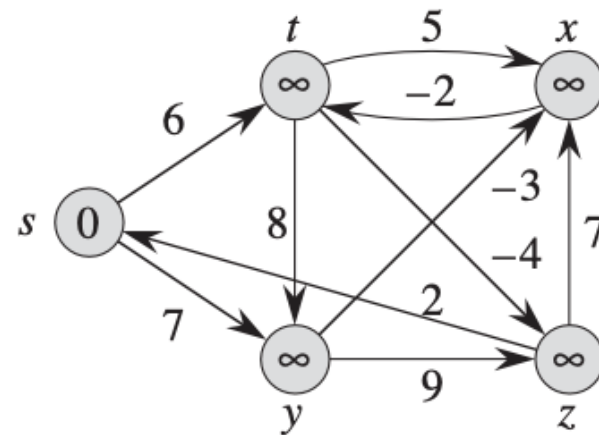
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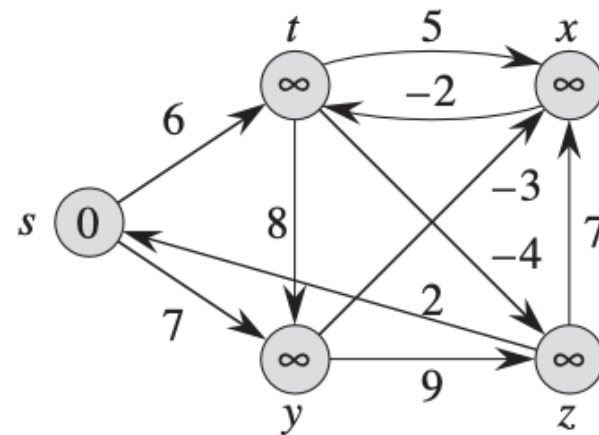
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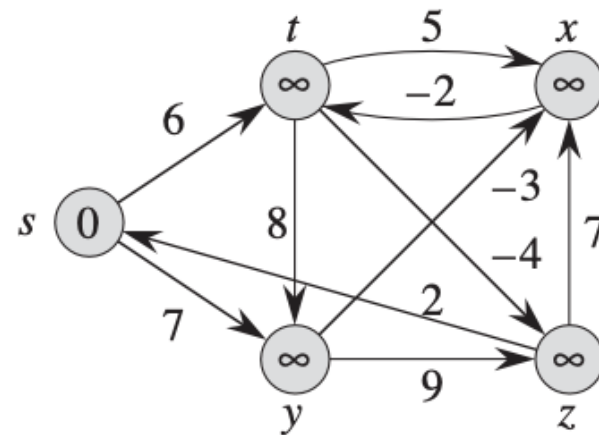
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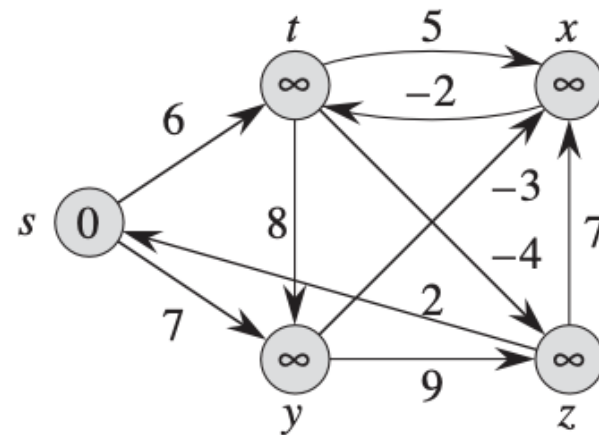
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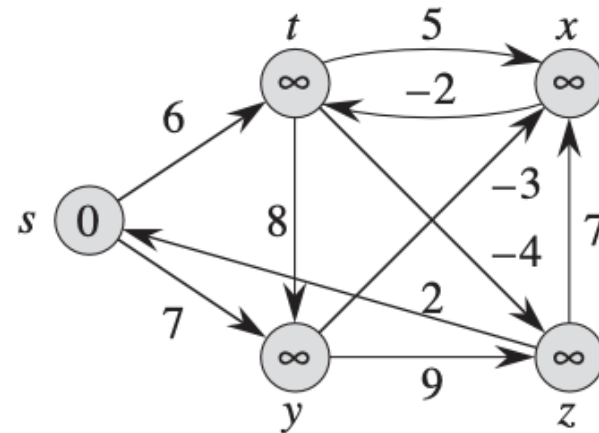
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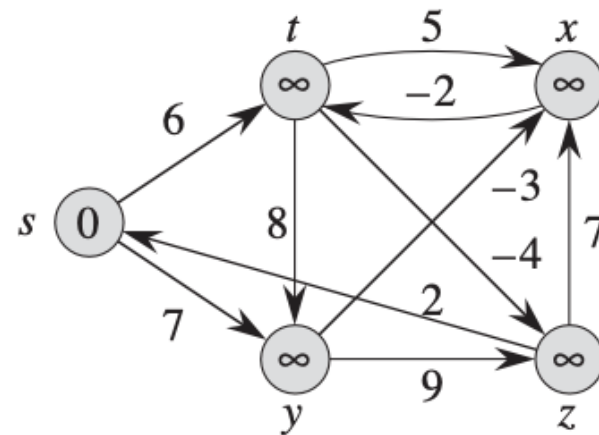
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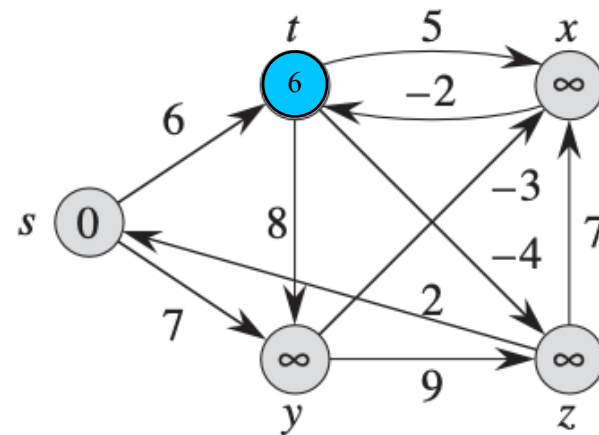
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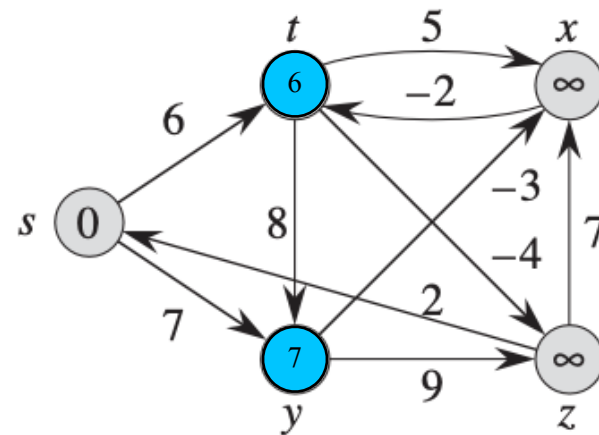
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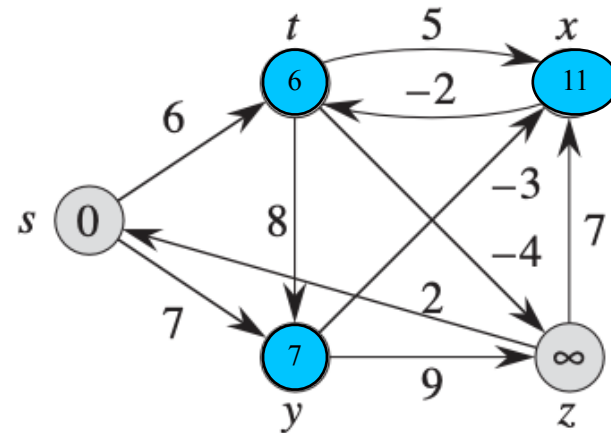
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2  for  $i = 1$  to  $|G.V| - 1$    $i = 2$ 
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SSSP (Bellman-Ford algorithm)

Example

Assume we pass the edges in the following order:

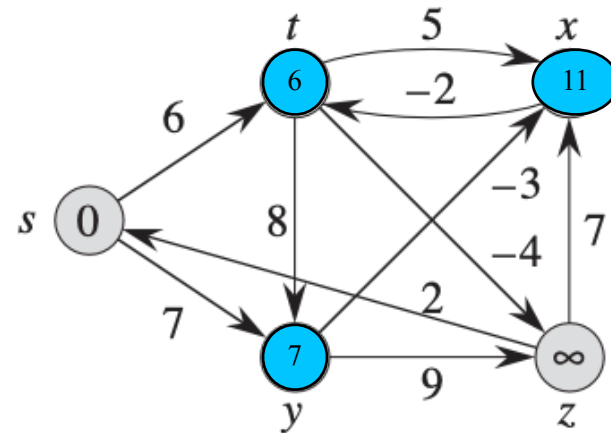
(t, x) , (t, y) , (t, z) , (x, t) , (y, x) , (y, z) , (z, x) , (z, s) , (s, t) , (s, y)

RELAX(u, v, w)

```
1  if  $v.d > u.d + w(u, v)$ 
2       $v.d = u.d + w(u, v)$ 
3       $v.\pi = u$ 
```

BELLMAN-FORD(G, w, s)

```
1  INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  for  $i = 1$  to  $|G.V| - 1$    $i = 2$ 
3      for each edge  $(u, v) \in G.E$ 
4          RELAX( $u, v, w$ )
5  for each edge  $(u, v) \in G.E$ 
6      if  $v.d > u.d + w(u, v)$ 
7          return FALSE
8  return TRUE
```



Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

Assume we pass the edges in the following order:

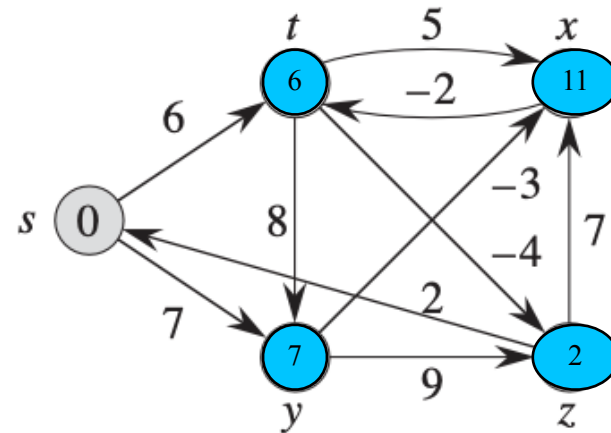
$(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)$

RELAX(u, v, w)

```
1  if  $v.d > u.d + w(u, v)$ 
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```

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Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

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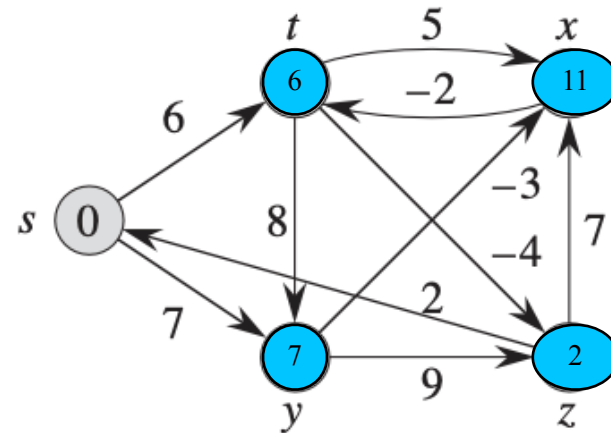
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```



Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

Assume we pass the edges in the following order:

$(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)$

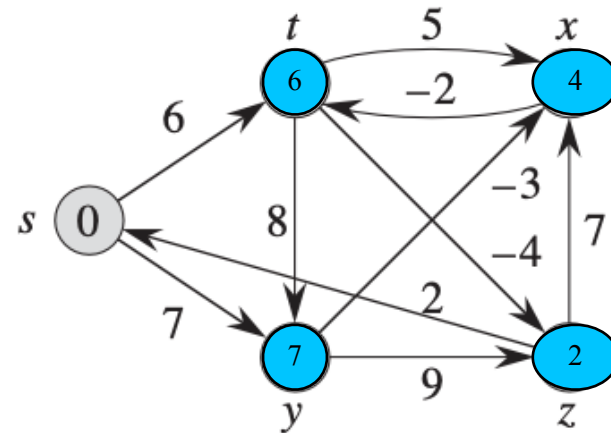
$\underline{\quad\quad\quad} \quad \underline{\quad\quad\quad} \quad \underline{\quad\quad\quad} \quad \underline{\quad\quad\quad} \quad \underline{\quad\quad\quad}$
x x x

RELAX(u, v, w)

```
1  if  $v.d > u.d + w(u, v)$ 
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BELLMAN-FORD(G, w, s)

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```



Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

Assume we pass the edges in the following order:

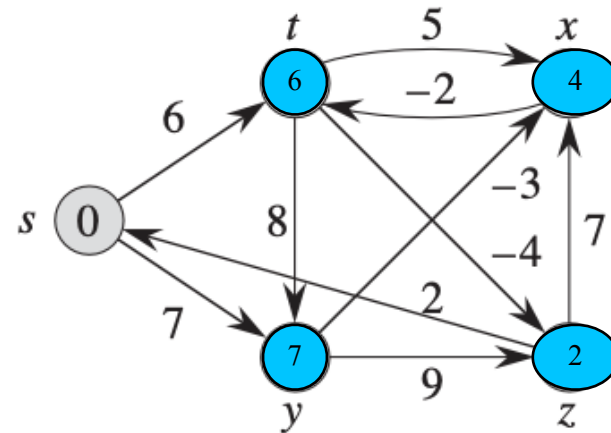
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```



Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

Assume we pass the edges in the following order:

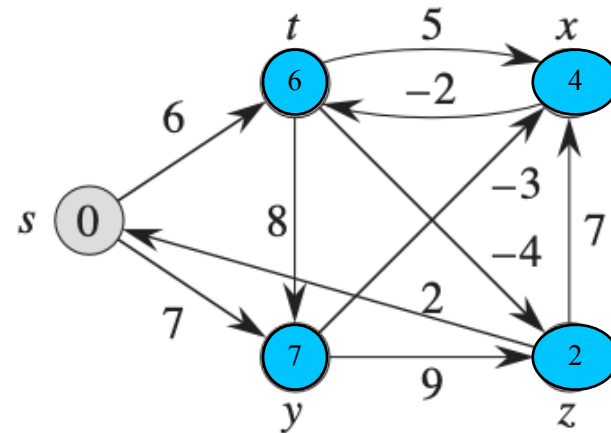
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Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

Assume we pass the edges in the following order:

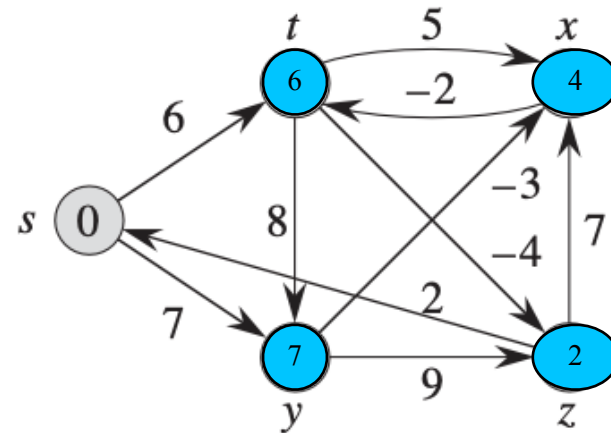
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Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

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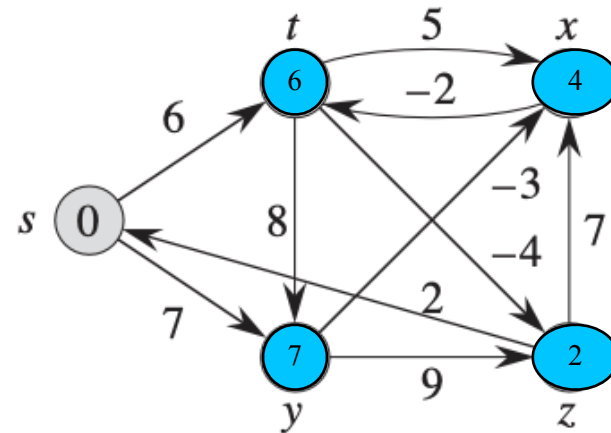
$(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)$

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Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

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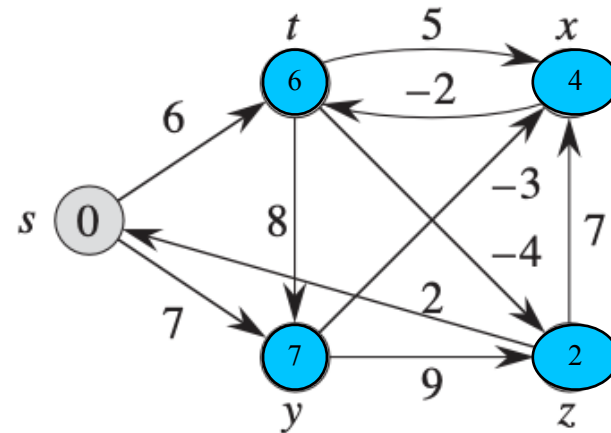
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Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

Assume we pass the edges in the following order:

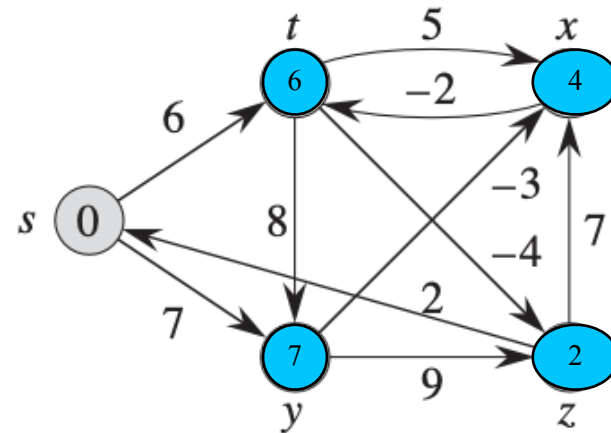
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2       $v.d = u.d + w(u, v)$ 
3       $v.\pi = u$ 
```

BELLMAN-FORD(G, w, s)

```
1  INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  for  $i = 1$  to  $|G.V| - 1$    $i = 3$ 
3      for each edge  $(u, v) \in G.E$ 
4          RELAX( $u, v, w$ )
5  for each edge  $(u, v) \in G.E$ 
6      if  $v.d > u.d + w(u, v)$ 
7          return FALSE
8  return TRUE
```



Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

Assume we pass the edges in the following order:

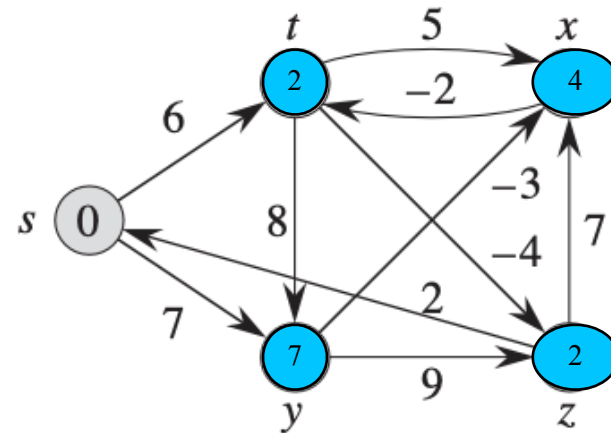
(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)

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Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

Assume we pass the edges in the following order:

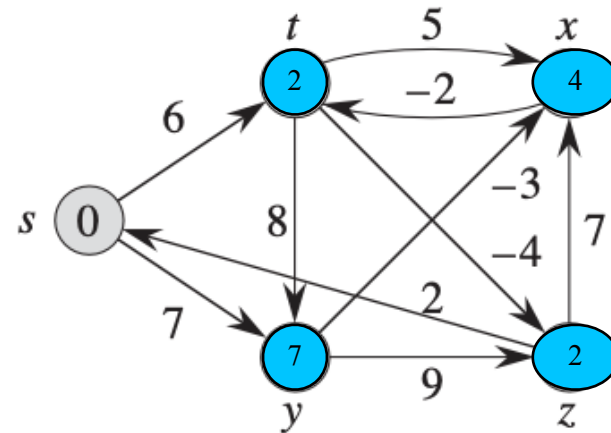
(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)

RELAX(u, v, w)

```
1  if  $v.d > u.d + w(u, v)$ 
2       $v.d = u.d + w(u, v)$ 
3       $v.\pi = u$ 
```

BELLMAN-FORD(G, w, s)

```
1  INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  for  $i = 1$  to  $|G.V| - 1$    $i = 4$ 
3      for each edge  $(u, v) \in G.E$ 
4          RELAX( $u, v, w$ )
5  for each edge  $(u, v) \in G.E$ 
6      if  $v.d > u.d + w(u, v)$ 
7          return FALSE
8  return TRUE
```



Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

Assume we pass the edges in the following order:

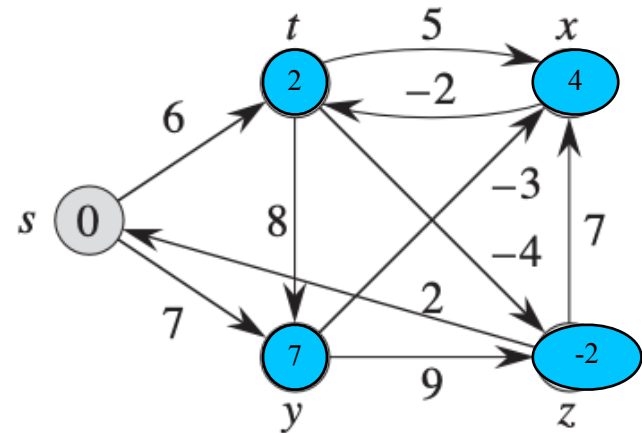
(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)

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Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Example

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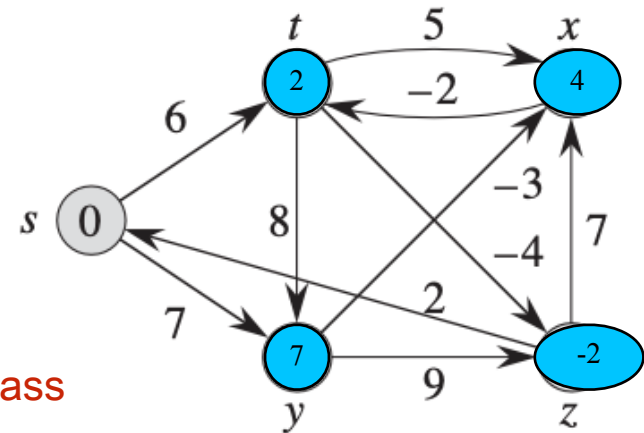
(t, x) , (t, y) , (t, z) , (x, t) , (y, x) , (y, z) , (z, x) , (z, s) , (s, t) , (s, y)

RELAX(u, v, w)

```
1  if  $v.d > u.d + w(u, v)$ 
2       $v.d = u.d + w(u, v)$ 
3       $v.\pi = u$ 
```

BELLMAN-FORD(G, w, s)

```
1  INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  for  $i = 1$  to  $|G.V| - 1$   What if  $i = 5$  one more pass
3      for each edge  $(u, v) \in G.E$ 
4          RELAX( $u, v, w$ )
5  for each edge  $(u, v) \in G.E$ 
6      if  $v.d > u.d + w(u, v)$ 
7          return FALSE
8  return TRUE
```



Nothing changes any more !

$i = 4$ is enough!

Graph Algorithms (Chapter 24.1)

SSSP (Bellman-Ford algorithm)

Runtime complexity

The Bellman-Ford algorithm runs in time $O(VE)$, since the initialization in line 1 takes $\Theta(V)$ time, each of the $|V| - 1$ passes over the edges in lines 2–4 takes $\Theta(E)$ time, and the **for** loop of lines 5–7 takes $O(E)$ time.

```
BELLMAN-FORD( $G, w, s$ )
1  INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  for  $i = 1$  to  $|G.V| - 1$ 
3      for each edge  $(u, v) \in G.E$ 
4          RELAX( $u, v, w$ )
5  for each edge  $(u, v) \in G.E$ 
6      if  $v.d > u.d + w(u, v)$ 
7          return FALSE
8  return TRUE
```

Graph Algorithms (Chapter 24.2)

SSSP (DAG)

Theorem 24.5

If a weighted, directed graph $G = (V, E)$ has source vertex s and no cycles, then at the termination of the DAG-SHORTEST-PATHS procedure, $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-paths tree.

Intuition

By relaxing the edges of a weighted dag (directed acyclic graph) $G = (V, E)$ according to a topological sort of its vertices, we can compute shortest paths from a single source in $\Theta(V + E)$ time. Shortest paths are always well defined in a dag, since even if there are negative-weight edges, no negative-weight cycles can exist.

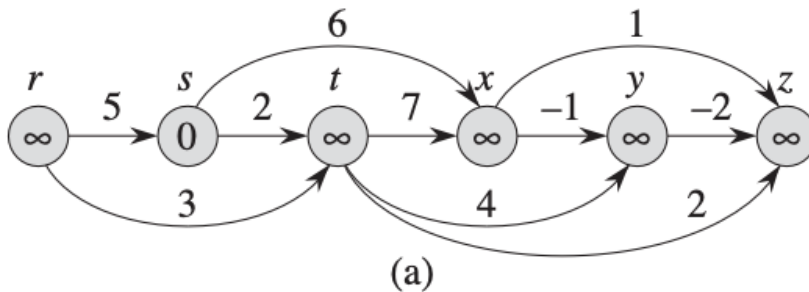
DAG-SHORTEST-PATHS(G, w, s)

- 1 topologically sort the vertices of G
- 2 INITIALIZE-SINGLE-SOURCE(G, s)
- 3 **for** each vertex u , taken in topologically sorted order
- 4 **for** each vertex $v \in G.Adj[u]$
- 5 RELAX(u, v, w)

Graph Algorithms (Chapter 24.2)

SSSP (DAG)

Example



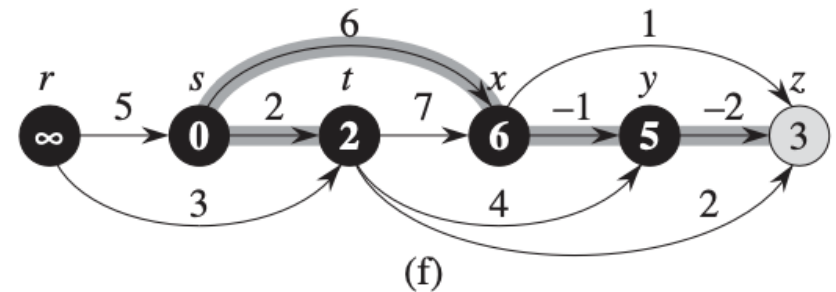
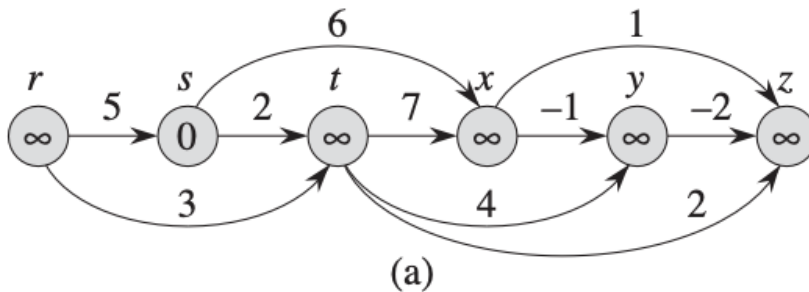
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Graph Algorithms (Chapter 24.2)

SSSP (DAG)

Example



DAG-SHORTEST-PATHS(G, w, s)

- 1 topologically sort the vertices of G
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- 4 **for** each vertex $v \in G.Adj[u]$
- 5 RELAX(u, v, w)

Graph Algorithms (Chapter 24.3)

SSSP (Dijkstra's Algorithm)

Solves SSSP on a weighted directed graph with nonnegative edge weights.

$$\forall (u, v) \in E, w(u, v) \geq 0$$

Graph Algorithms (Chapter 24.3)

SSSP (Dijkstra's Algorithm)

Intuition

Maintain a set S of vertices whose final shortest path weights from source s have already been determined. It repeatedly selects $u \in V - S$ with minimum distance estimate, adds u to S and relaxes outgoing edges from u .

DIJKSTRA(G, w, s)

```
1  INITIALIZE-SINGLE-SOURCE( $G, s$ )
2   $S = \emptyset$ 
3   $Q = G.V$ 
4  while  $Q \neq \emptyset$ 
5       $u = \text{EXTRACT-MIN}(Q)$ 
6       $S = S \cup \{u\}$ 
7      for each vertex  $v \in G.Adj[u]$ 
8          RELAX( $u, v, w$ )
```

Graph Algorithms (Chapter 24.3)

SSSP (Dijkstra's Algorithm)

Example

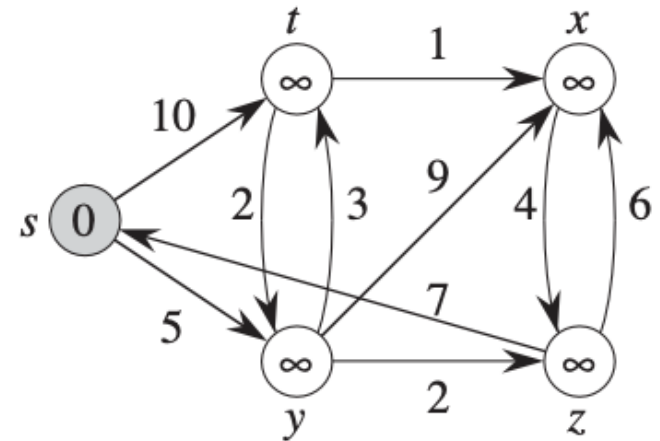
Select min distance vertex = s

$S = \{s\}$

Relax $(s, t), (s, y)$

DIJKSTRA(G, w, s)

```
1 INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  $S = \emptyset$ 
3  $Q = G.V$ 
4 while  $Q \neq \emptyset$ 
5      $u = \text{EXTRACT-MIN}(Q)$ 
6      $S = S \cup \{u\}$ 
7     for each vertex  $v \in G.Adj[u]$ 
8         RELAX( $u, v, w$ )
```



Graph Algorithms (Chapter 24.3)

SSSP (Dijkstra's Algorithm)

Example

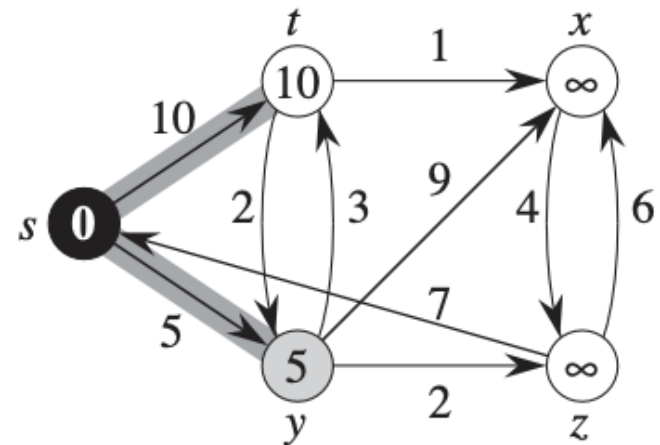
Select min distance vertex = s

$$S = \{s\}$$

Relax $(s, t), (s, y)$

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8         RELAX( $u, v, w$ )
```



Graph Algorithms (Chapter 24.3)

SSSP (Dijkstra's Algorithm)

Example

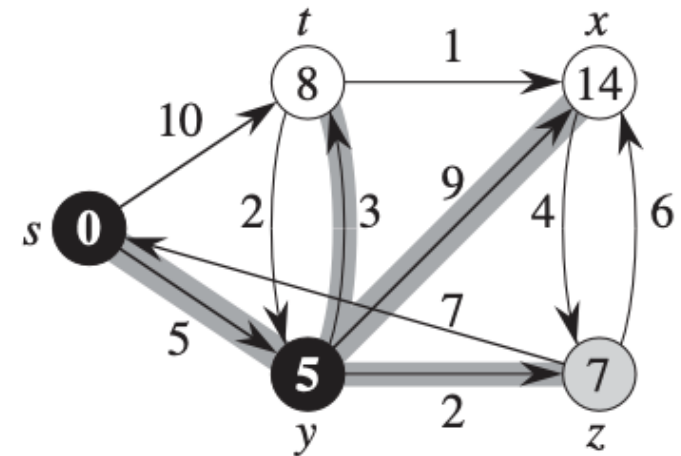
Select min distance vertex = y

$S = \{s, y\}$

Relax $(y, t), (y, x), (y, z)$

DIJKSTRA(G, w, s)

```
1 INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  $S = \emptyset$ 
3  $Q = G.V$ 
4 while  $Q \neq \emptyset$ 
5      $u = \text{EXTRACT-MIN}(Q)$ 
6      $S = S \cup \{u\}$ 
7     for each vertex  $v \in G.Adj[u]$ 
8         RELAX( $u, v, w$ )
```



Graph Algorithms (Chapter 24.3)

SSSP (Dijkstra's Algorithm)

Example

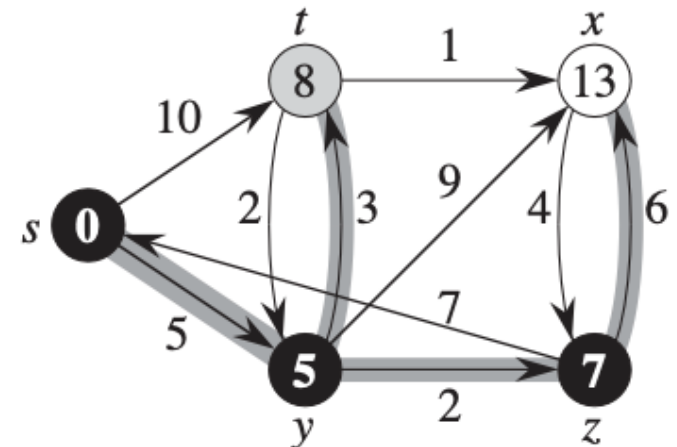
Select min distance vertex = z

$S = \{s, y, z\}$

Relax $(z, s), (z, x)$

DIJKSTRA(G, w, s)

```
1 INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  $S = \emptyset$ 
3  $Q = G.V$ 
4 while  $Q \neq \emptyset$ 
5      $u = \text{EXTRACT-MIN}(Q)$ 
6      $S = S \cup \{u\}$ 
7     for each vertex  $v \in G.Adj[u]$ 
8         RELAX( $u, v, w$ )
```



Graph Algorithms (Chapter 24.3)

SSSP (Dijkstra's Algorithm)

Example

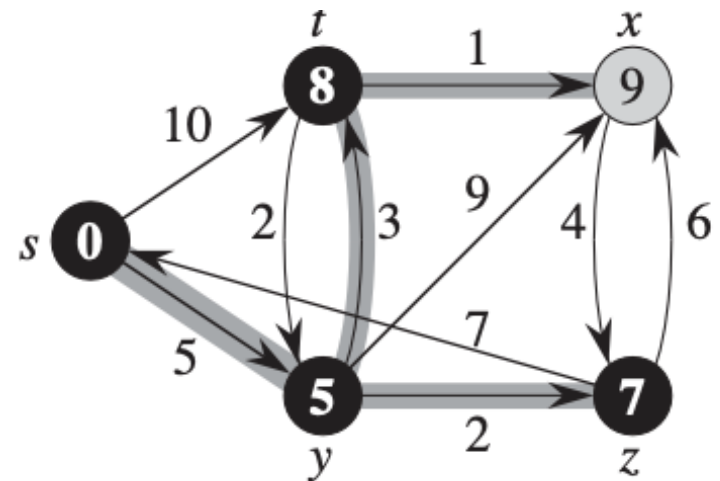
Select min distance vertex = t

$S = \{s, y, z, t\}$

Relax (t, x) , (t, y)

DIJKSTRA(G, w, s)

```
1 INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  $S = \emptyset$ 
3  $Q = G.V$ 
4 while  $Q \neq \emptyset$ 
5      $u = \text{EXTRACT-MIN}(Q)$ 
6      $S = S \cup \{u\}$ 
7     for each vertex  $v \in G.Adj[u]$ 
8         RELAX( $u, v, w$ )
```



Graph Algorithms (Chapter 24.3)

SSSP (Dijkstra's Algorithm)

Example

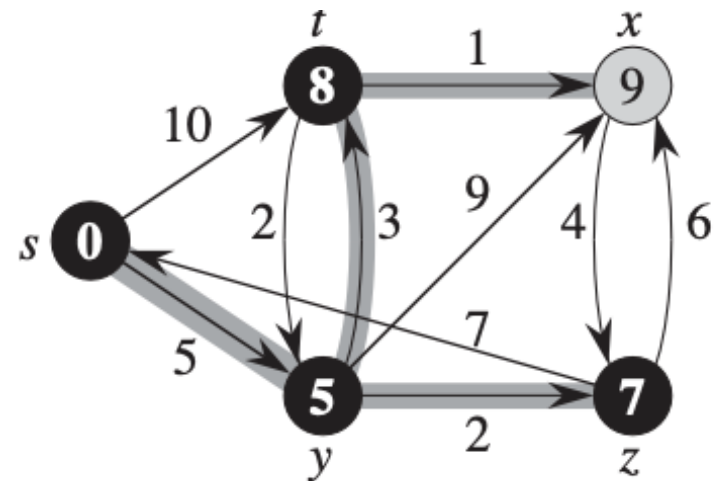
Select min distance vertex = x

$S = \{s, y, z, t, x\}$

Relax (x, z)

DIJKSTRA(G, w, s)

```
1 INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  $S = \emptyset$ 
3  $Q = G.V$ 
4 while  $Q \neq \emptyset$ 
5      $u = \text{EXTRACT-MIN}(Q)$ 
6      $S = S \cup \{u\}$ 
7     for each vertex  $v \in G.Adj[u]$ 
8         RELAX( $u, v, w$ )
```



Graph Algorithms (Chapter 24.3)

SSSP (Dijkstra's Algorithm)

Runtime complexity of Dijkstra's Algorithm
if we use min-heap for priority queue

$O(V \cdot \lg V)$ from lines 4-6

$O(E \cdot \lg V)$ from lines 7-8

$O((E + V) \cdot \lg V)$, if $E > V$ we have $O(E \cdot \lg V)$

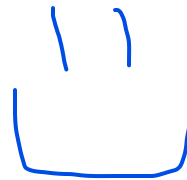
DIJKSTRA(G, w, s)

```
1 INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  $S = \emptyset$ 
3  $Q = G.V$ 
4 while  $Q \neq \emptyset$                                  $O(V)$  In total
5      $u = \text{EXTRACT-MIN}(Q)$                          $O(\lg V)$  from min-heap
6      $S = S \cup \{u\}$ 
7     for each vertex  $v \in G.Adj[u]$                  $O(E)$  In total
8         RELAX( $u, v, w$ )                             $O(\lg V)$  because for decrease key in min-heap
```

RELAX(u, v, w)

```
1 if  $v.d > u.d + w(u, v)$ 
2      $v.d = u.d + w(u, v)$ 
3      $v.\pi = u$ 
```

Summary Chapter 24



Exercices

Will come