2D Homographies

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Objective

To introduce the problem of estimating 2D projective transformations as well as some basic tools for parameter estimation.

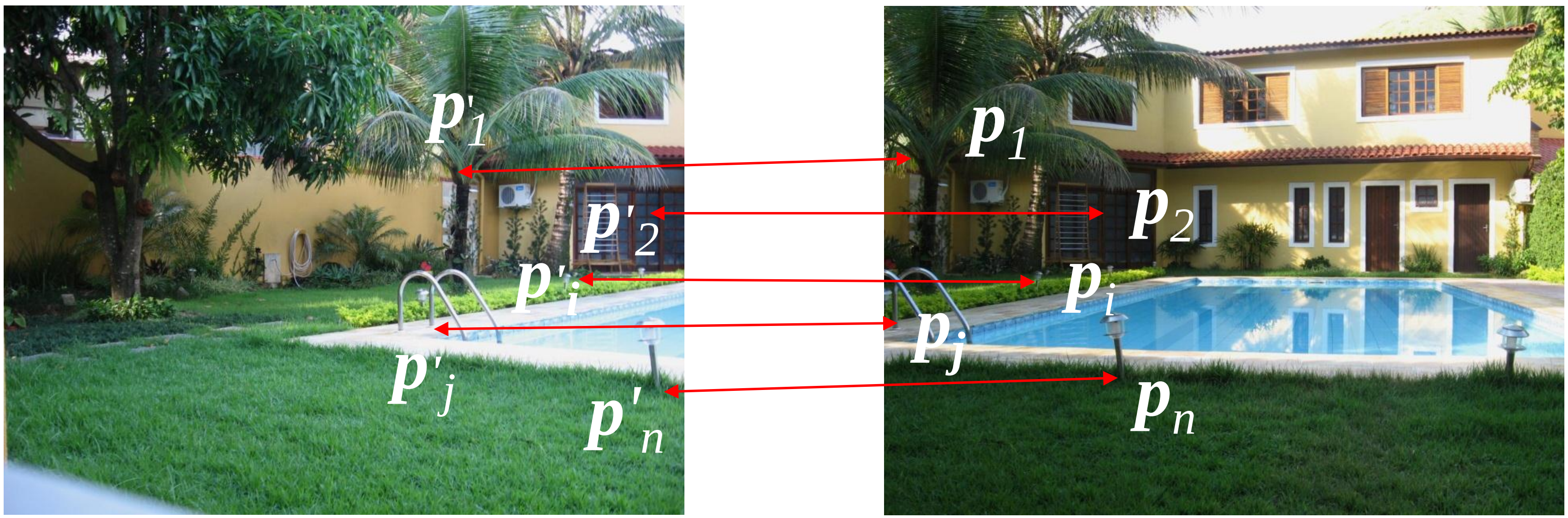
Outline

- Motivation
- Direct Linear Transformation (DLT)
- Normalization
- Robust Estimation
- Non Linear Method
- Assignment

Motivation: Panoramas

Building a Panorama

- **Step 1:** find pairs of corresponding points



Building a Panorama

- **Step 2:** Estimate the matrix $\mathcal{H}$ (the homography) such that

$$\boldsymbol{p}_i = \mathcal{H}\,\boldsymbol{p}'_i$$

where $\boldsymbol{p}_i$ and $\boldsymbol{p}'_i$ are homogeneous vectors representing corresponding points, i.e.

$$k\,(\,u_i\ v_i\ 1\,)^T = \mathcal{H}(\,u'_i\ v'_i\ 1\,)^T$$

for some $k \neq 0$ (homogeneous) and for all $1 \leq i \leq n$.

Building a Panorama

- **Step 3:** transform geometrically one image by using the matrix $\mathcal{H}$

Building a Panorama

- **Step 4:** align and blend (not here) the images

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Direct Linear Transformation

Derivation:

Let $\mathbf{h}_j^T$ be the j-th row of $\mathcal{H}$, so we may write

$$\mathcal{H} = \begin{bmatrix} \boxed{h_1 \quad h_2 \quad h_3} \\ \boxed{h_4 \quad h_5 \quad h_6} \\ \boxed{h_7 \quad h_8 \quad h_9} \end{bmatrix} \begin{matrix} \mapsto \mathbf{h}_1^T \\ \mapsto \mathbf{h}_2^T \\ \mapsto \mathbf{h}_3^T \end{matrix} \Rightarrow \mathcal{H}\boldsymbol{p}_i' = \begin{pmatrix} \mathbf{h}_1^T \boldsymbol{p}_i' \\ \mathbf{h}_2^T \boldsymbol{p}_i' \\ \mathbf{h}_3^T \boldsymbol{p}_i' \end{pmatrix}$$

Clearly $\boldsymbol{p}_i \times \mathcal{H}\boldsymbol{p}_i' = \mathbf{0}$. Thus

$$\boldsymbol{p}_i \times \mathcal{H}\boldsymbol{p}_i' = \begin{pmatrix} v_i \mathbf{h}_3^T \boldsymbol{p}_i' - \mathbf{h}_2^T \boldsymbol{p}_i' \\ \mathbf{h}_1^T \boldsymbol{p}_i' - u_i \mathbf{h}_3^T \boldsymbol{p}_i' \\ u_i \mathbf{h}_2^T \boldsymbol{p}_i' - v_i \mathbf{h}_1^T \boldsymbol{p}_i' \end{pmatrix} = \mathbf{0}$$

Direct Linear Transformation

Derivation (cont.):

After some manipulation, you obtain

$$3\left\{\overbrace{\begin{bmatrix} \mathbf{0}^T & -\boldsymbol{p}_i'^T & v_i\boldsymbol{p}_i'^T \\ \boldsymbol{p}_i'^T & \mathbf{0}^T & -u_i\boldsymbol{p}_i'^T \\ -v_i\boldsymbol{p}_i'^T & u_i\boldsymbol{p}_i'^T & \mathbf{0}^T \end{bmatrix}}^{9}\right.\begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{pmatrix} = \mathbf{0}$$

only 2 are linearly independent

where $\mathbf{h} = \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{pmatrix}$, $\qquad \mathcal{H} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$

Direct Linear Transformation

Derivation (cont.):

Each pair of points generates two equations of the form

$$\mathcal{A}_i = \begin{bmatrix} \mathbf{0}^T & -\boldsymbol{p}_i'^T & v_i \boldsymbol{p}_i'^T \\ \boldsymbol{p}_i'^T & \mathbf{0}^T & -u_i \boldsymbol{p}_i'^T \end{bmatrix} \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{pmatrix} = \mathbf{0}$$

With n pairs we get $2n$ equations on 9 unknowns!

$$\begin{bmatrix} \mathcal{A}_1 \\ \vdots \\ \mathcal{A}_n \end{bmatrix} \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{pmatrix} = \mathbf{0} \Rightarrow$$ 4 pairs of points are enough!

an homogeneous linear equation system

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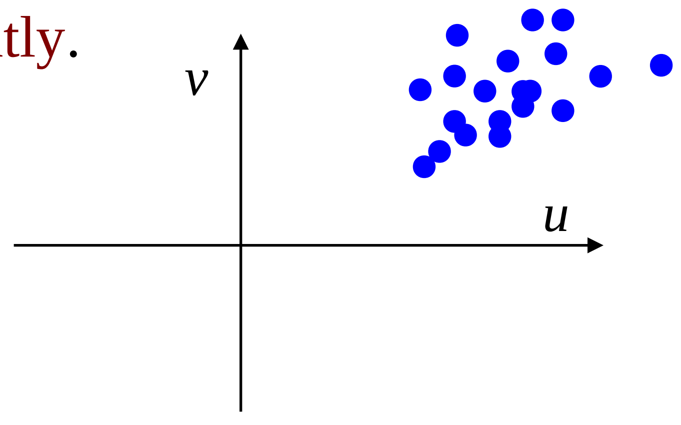
- Normalization
- Robust Estimation
- Optimal Estimation
- Assignment

Normalization

For improved estimation it is essential† for DLT to normalize the data sample, as follows

1. The points are translated so that their centroid is at the origin
2. The points are scaled so that the average distance from the origin is equal to $\sqrt{2}$.
3. This transformation is applied to each of the two images independently.



†Multiple View Geometry in computer vision, 2nd Ed., R. Hartley and A. Zisserman, 2003, Cambridge, section 4.4.4, pp.107.

Normalization

Normalization is carried out by multiplying the data vector by proper matrices $\mathbf{T}$ and $\mathbf{T}'$

$$\tilde{\boldsymbol{p}}_i = \mathbf{T}\,\boldsymbol{p}_i \quad \text{and} \quad \tilde{\boldsymbol{p}}_i' = \mathbf{T}'\,\boldsymbol{p}_i'$$

DLT is applied to $\tilde{\boldsymbol{p}}_i$ and $\tilde{\boldsymbol{p}}_i'$ to obtain $\tilde{\mathcal{H}}$, which is actually different from $\mathcal{H}$.

$$\mathbf{T}\,\boldsymbol{p}_i = \tilde{\mathcal{H}}\,\mathbf{T}'\boldsymbol{p}' \rightarrow \boldsymbol{p}_i = \underbrace{\mathbf{T}^{-1}\tilde{\mathcal{H}}\,\mathbf{T}'}_{\mathcal{H}}\boldsymbol{p}_i'$$

Normalization

The normalizing matrix $\mathbf{T}$ is given by

$$\mathbf{T} = s\begin{pmatrix} 1 & 0 & -\bar{u} \\ 0 & 1 & -\bar{v} \\ 0 & 0 & 1/s \end{pmatrix}$$

where $\bar{u}, \bar{v}$ are the mean values of u_i, v_i respectively

and $$s = \sqrt{2}\,n \Big/ \sum_{i=1}^{n} \left[(u_i - \bar{u})^2 + (v_i - \bar{v})^2 \right]^{1/2}$$

The computation of $\mathbf{T}'$ is analogous.

Normalization

- **Step-by-step**
  - Compute the normalization matrices $\mathbf{T}$ and $\mathbf{T}'$
  - Perform normalization by computing

$$\tilde{\boldsymbol{p}}_i = \mathbf{T}\ \boldsymbol{p}_i \quad \text{and} \quad \tilde{\boldsymbol{p}}_i' = \mathbf{T}'\ \boldsymbol{p}_i'$$

  - Apply DLT to $\tilde{\boldsymbol{p}}_i$ and $\tilde{\boldsymbol{p}}_i'$ and compute $\tilde{\mathcal{H}}$
  - Denormalize the result by computing

$$\mathcal{H} = \mathbf{T}^{-1}\ \tilde{\mathcal{H}}\ \mathbf{T}'$$

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Robust Estimation

RANSAC robust estimation

1. Repeat for k iterations
   a) Select a random sample of 4 pairs of corresponding points and compute $\mathcal{H}$.
   b) Calculate for each putative correspondence the "*distance*",

   $$d\,(\mathbf{p}_i, \mathcal{H}\mathbf{p}'_i) = \| \mathbf{p}_i - \mathcal{H}\mathbf{p}'_i \|$$

   from the projected to the actual point position.
   c) Determine the number of inliers consistent with $\mathcal{H}$ i.e.,

   $$d\,(\mathbf{p}_i, \mathcal{H}\mathbf{p}'_i) < t \text{ pixels.}$$

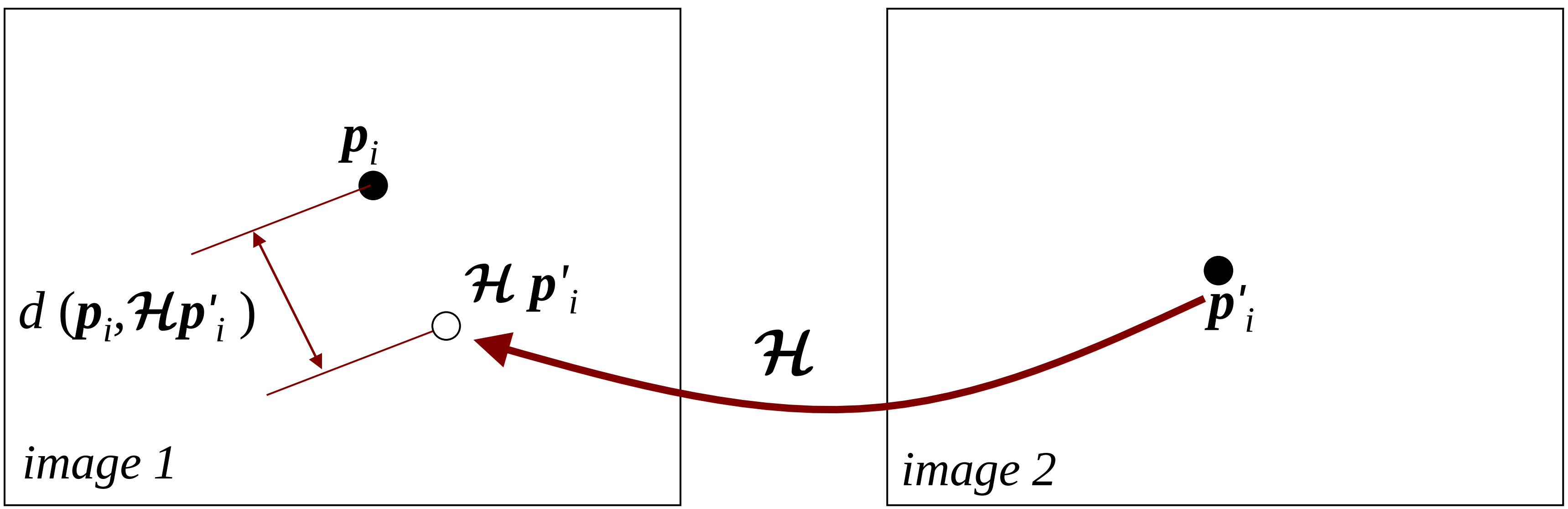
2. Choose the $\mathcal{H}$ with the largest number of inliers.
3. Recompute $\mathcal{H}$ with the inliers.

Robust Estimation

The geometric distance

It is the distance between the projected and the actual position of the corresponding points

$$d\,(\boldsymbol{p}_i, \mathcal{H}\boldsymbol{p}'_i) = \| \boldsymbol{p}_i - \mathcal{H}\boldsymbol{p}'_i \|$$



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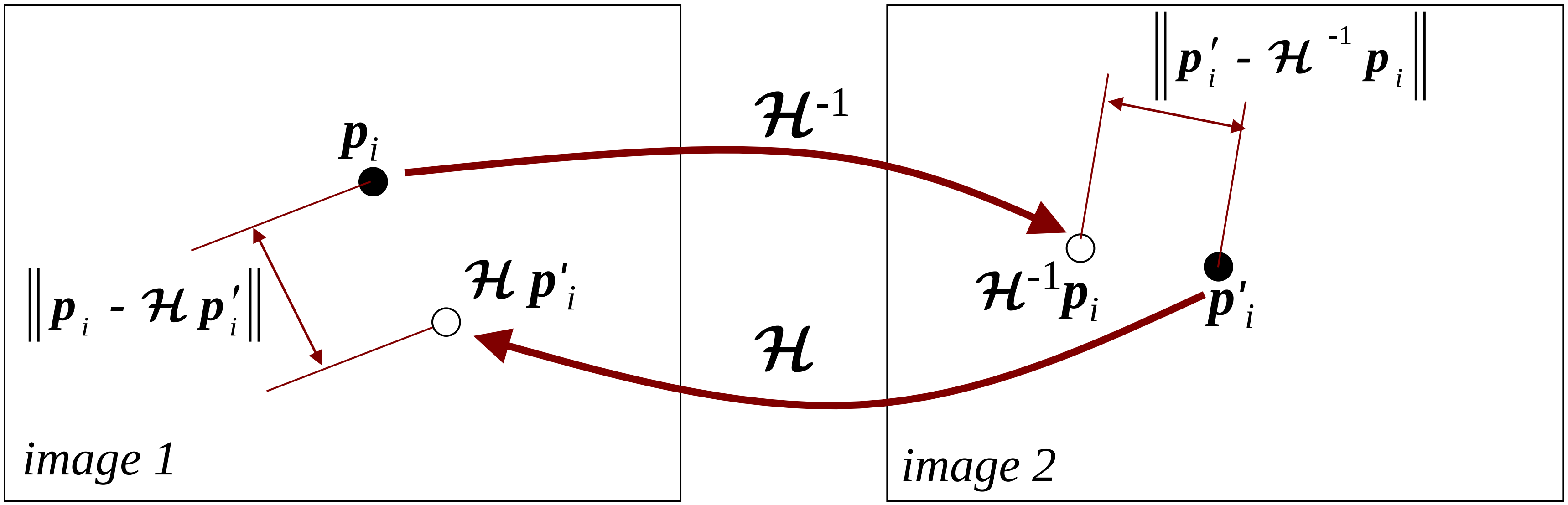
Non Linear Method

- **Reprojection error**

  Sum of the squared errors of the forward and the backward transformation, i.e.,

$$D_i = \left\| \boldsymbol{p}_i - \mathcal{H}\,\boldsymbol{p}'_i \right\|^2 + \left\| \boldsymbol{p}'_i - \mathcal{H}^{-1}\,\boldsymbol{p}_i \right\|^2$$



Non Linear Method

- **Computing $\mathcal{H}$ from the reprojection error**

  By finding a solution that minimizes the sum of reprojection errors of all *n* correspondences, i.e., if

$$D_i = \left\| \boldsymbol{p}_i - \mathcal{H}\,\boldsymbol{p}'_i \right\|^2 + \left\| \boldsymbol{p}'_i - \mathcal{H}^{-1}\,\boldsymbol{p}_i \right\|^2$$

  the homography is obtained by solving the non linear equation system below

$$\mathbf{D} = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{bmatrix} = \mathbf{0}$$

Panorama Example

- **Input Images**

- **Panorama**

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Assignment

General Formulation

i. Take a set of images of the same scene, rotating the camera around its optical center.
ii. For a pair of images collect manually a set of corresponding points (*hint:* use the function *captura_pontos*).
iii. Compute the homography $\mathcal{H}$ from these correspondences.
iv. Create a panorama from both images and from $\mathcal{H}$. (*hint:* use the function *Panorama2*).
v. Add new images by repeating steps *i* to *iv*.

Assignment

Assignment 1

1. Write a MATLAB program that implements the solution for the problem formulated in the previous slide
   i. By using DLT without normalization
   ii. By using DLT with normalization (*hint*: use the function *NormalizaPontos*)

   Compare the results obtained by each approach.

Assignment

Assignment 1 (cont.)

1. Your main task here is the development of a function that implements DLT, which may have the following help

```
%  H=DLT(u2Trans,v2Trans,uBase,vBase,)
% Computes the homography H applying the Direct Linear Transformation
% The transformation is such that
%          p          = H        p'
%   (uBase vBase 1)'=H*(u2Trans v2Trans 1)'
%
% INPUTS:
% u2Trans, v2Trans  - vectors with coordinates u and v of the image to be transformed (p')
% uBase, vBase      - vectors with coordinates u and v of the base image p
%
% OUTPUT
% H -  3x3 matrix with the Homography
%
% your name - date
```

Assignment

Assignment 2

2. Provide a new solution for the same general problem where DLT is replaced by the non linear method.

   Some hints:

   - use the MATLAB function *lsqnonlin*.
   - the result of $\mathcal{H}\,\boldsymbol{p}'_i$ is in homogeneous coordinates, i.e., it must be scaled to make the third vector component equal 1, and to obtain the actual column-row coordinates.

Assignment

Assignment 3

3. Provide a new solution for the same general problem where DLT/ the non linear method is replaced by RANSAC.

   Some hints:

   - use DLT to obtain a initial solution
   - Use the reprojection error in RANSAC step 3 (you have it from previous assignment)

Next Topic

Image Matching