A Recombining Trinomial Option Pricing Model with Time-Dependent Volatility

1 Introduction

This work develops a recombining trinomial tree framework for option pricing under time-dependent (piecewise-constant) volatility. The objective is to construct a numerically efficient and stable pricing model that supports both European and American options and can be calibrated to observed market prices. The resulting framework provides additional flexibility over constant-volatility models while retaining analytical transparency and fast computation.

2 Model Setup

2.1 Time Discretisation

Let S_0 denote the initial underlying price, r the continuously compounded risk-free rate, and T the maturity. For the sake of notational simplicity, we assume a zero dividend yield (if this was not the case, we could just replace the risk-free rate r with $b = r - q$). The time interval $[0, T]$ is divided into N equal steps of length:

$$\Delta t = \frac{T}{N}, \qquad t_i = i\Delta t.$$

2.2 Volatility Structure

The instantaneous volatility is modelled as a deterministic function of time $\sigma(t)$. For numerical implementation, it is discretised as

$$\sigma_i = \sigma(t_i), \qquad i = 0, 1, \ldots, N-1.$$

In practice, the volatility term structure is assumed to be piecewise constant:

$$\sigma(t) = \sigma_m, \quad t \in [\tau_{m-1}, \tau_m), \qquad m = 1, \ldots, M.$$

The model parameters are therefore collected in the vector

$$\Theta = (\sigma_1, \sigma_2, \ldots, \sigma_M).$$

This specification corresponds to a local volatility model in time, rather than a full spot-dependent local volatility surface.

2.3 Trinomial Price Dynamics

At each time step, the underlying price evolves according to a trinomial rule:

$$S \longrightarrow \begin{cases} Su & \text{(up)}, \\ S & \text{(middle)}, \\ Sd & \text{(down)}. \end{cases}$$

Grid Construction. To ensure the tree remains recombining (i.e., $S_{i,j} = S_0 u^j$) under time-dependent volatility, we construct a fixed spatial grid based on a reference volatility σ_{ref} (this will be the average of Θ). The up and down factors are defined as constant across all time steps:

$$u = \exp\left(\sigma_{\text{ref}}\sqrt{2\Delta t}\right), \qquad d = \frac{1}{u}.$$

A generic node price is given by

$$S_{i,j} = S_0\, u^j, \qquad j \in \{-i, -i+1, \ldots, i-1, i\}.$$

The time-dependent nature of the volatility σ_i is incorporated not by changing the grid spacing u, but by adjusting the transition probabilities $\{p_{u,i}, p_{m,i}, p_{d,i}\}$ at each step i. This approach preserves the computational efficiency of the recombining lattice while correctly capturing the local variance $\sigma_i^2 \Delta t$.

Observation: The term u can be generalized using the expression $u = \exp(\sigma_{ref}\sqrt{k\Delta t})$ where $k > 1$. Detailed derivations regarding the optimal choice of k are omitted for brevity, but $k = 2$ is specifically chosen to ensure the **positivity of transition probabilities** ($p_{u,i}, p_{m,i}, p_{d,i} \in [0, 1]$) and the **numerical stability**.

3 Risk-Neutral Probability Construction

Define the one-step multiplicative return

$$X_{i+1} = \begin{cases} u & \text{with probability } p_{u,i}, \\ 1 & \text{with probability } p_{m,i}, \\ d & \text{with probability } p_{d,i}. \end{cases}$$

Then

$$S_{i+1} = S_i X_{i+1}.$$

The probabilities are determined by matching the first two moments under the risk-neutral measure.

Normalisation

$$p_{u,i} + p_{m,i} + p_{d,i} = 1.$$

Risk-Neutral Mean

No-arbitrage requires

$$\mathbb{E}[S_{i+1} \mid S_i] = S_i e^{r\Delta t},$$

which implies

$$u p_{u,i} + p_{m,i} + d p_{d,i} = M, \qquad M := e^{r\Delta t}.$$

Local Variance

The local variance condition is imposed as

$$\mathrm{Var}[X_{i+1}] = \sigma_i^2 \Delta t.$$

Let

$$\mathbb{E}[X_{i+1}^2] = u^2 p_{u,i} + p_{m,i} + d^2 p_{d,i}.$$

Then

$$u^2 p_{u,i} + p_{m,i} + d^2 p_{d,i} = \sigma_i^2 \Delta t + M^2 =: V.$$

4 Explicit Probability Formulas

The previous conditions lead to the linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ u & 1 & d \\ u^2 & 1 & d^2 \end{pmatrix} \begin{pmatrix} p_{u,i} \\ p_{m,i} \\ p_{d,i} \end{pmatrix} = \begin{pmatrix} 1 \\ M \\ V \end{pmatrix}.$$

Since the coefficient matrix depends only on (u, d), the system admits a closed-form solution. The exact analytical probabilities are

$$p_{u,i} = \frac{V - M(d+1) + d}{(u-1)(u-d)},$$

$$p_{d,i} = \frac{V - M(u+1) + u}{(d-1)(d-u)},$$

$$p_{m,i} = 1 - p_{u,i} - p_{d,i}.$$

These expressions are used directly in the implementation for improved computational efficiency. For reasonable choices of Δt and σ_i, the resulting probabilities remain within the admissible range.

5 Option Valuation

Terminal Payoff

Let $V_{i,j}$ denote the fair value of the option at time step i and spatial node j, corresponding to the state where the underlying price is $S_{i,j}$.

At maturity $T = t_N$, we have

$$V_{N,j} = \begin{cases} \max(S_{N,j} - K, 0) & \text{for a Call,} \\ \max(K - S_{N,j}, 0) & \text{for a Put.} \end{cases} \tag{1}$$

Backward Induction

For $i = N-1, \ldots, 0$,

$$V_{i,j}^{\text{cont}} = e^{-r\Delta t} \left(p_{u,i} V_{i+1,j+1} + p_{m,i} V_{i+1,j} + p_{d,i} V_{i+1,j-1} \right).$$

For European options,

$$V_{i,j} = V_{i,j}^{\text{cont}}.$$

For American options,

$$V_{i,j} = \max\left(V_{i,j}^{\text{cont}},\ V_{i,j}^{\text{intr}}\right).$$

Where $V_{i,j}^{\text{intr}}$ represents the **intrinsic value** (the immediate exercise value) at node (i,j), calculated by applying the payoff function defined in equation (1). **The option price at inception is $V_{0,0}$.**

6 Volatility Calibration

Given market option prices $V_{\text{market}}(K_k, T_k)$, the calibration problem is formulated as

$$E(\Theta) = \sum_k w_k \left(V_{\text{model}}(K_k, T_k; \Theta) - V_{\text{market}}(K_k, T_k) \right)^2 + \lambda \sum_{m=1}^{M-1} (\sigma_{m+1} - \sigma_m)^2.$$

with calibration weights satisfying

$$\sum_k w_k = 1.$$

For simplicity, we employ uniform weights so $w_k = 1/n_{options}$ where $n_{options}$ are the number of option prices given. The calibrated volatility term structure is obtained as

$$\Theta^* = \arg\min_{\Theta} E(\Theta),$$

To find the optimal parameters Θ^*, the objective function $E(\Theta)$ is minimized numerically using the **Nelder-Mead simplex method**.

7 Conclusion

The proposed model combines a recombining trinomial lattice with time-dependent volatility, analytical probability construction, and a flexible calibration framework. This results in an efficient and robust approach for pricing and calibrating European and American options while maintaining convergence to the Black–Scholes–Merton model.