Option Prices and Pricing Theory: Combining Financial Mathematics with Statistical Modeling

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Abstract

After an overview of important developments of option pricing theory, this article describes statistical approaches to modeling the difference between the theoretical and actual prices. An empirical study is given to compare various approaches.

A cornerstone of Financial Mathematics is option pricing theory, which Ross[1] has described as "the most successful theory not only in finance, but in all of economics." A *call* (*put*) option gives the holder the right to buy (sell) the underlying asset by a certain expiration date T at a certain price K, which is called the strike price. If the option can be exercised only at T, it is termed *European* or European-style, whereas if early exercise is also allowed at any time before T, the option is termed *American* or American-style. In the seminal work of Black and Scholes,[2] a dynamic hedging argument was used in conjunction with a no-arbitrage condition to derive closed-form pricing formulas for European options written on an asset whose price S_t at time t is modeled by a geometric Brownian motion

$$S_t = S_0 \exp\{(\mu - \sigma^2/2)t + \sigma B_t\}, \tag{1}$$

where μ and σ are the mean and standard deviation of the asset's return dS_t/S_t and $\{B_t, t \geq 0\}$ is a standard Brownian motion; σ is commonly known as the asset's volatility. Merton[3] extended the Black-Scholes theory to American options, showing that optimal exercise of the option occurs when the asset price exceeds (or falls below) an exercise boundary for a call (or put) option. The Black-Scholes-Merton theory for pricing and hedging options has played a fundamental role in the development of financial derivatives; a derivative is a financial instrument having a value derived from or contingent on the values of more basic underlying variables. The European and American call and put options are "plain vanilla" products that are actively traded on many exchanges throughout the world, e.g., the Chicago Board Options Exchange that started trading options contracts in 1973. One can therefore compare the actual option prices with those given by the Black-Scholes-Merton formula that involves the price of the underlying asset and the risk-free interest rate, which are directly observable, and the volatility σ of the asset's return, which has to be estimated from past data. Discrepancies between the theoretical and actual prices suggest possible misspecification of the asset price dynamics via (1). One way to address the issue is to develop more flexible (and increasingly complex) stochastic models which incorporate stochastic volatility, stochastic interest rate, random jumps, and models incorporating both stochastic volatility and contemporaneous jumps (SVCJ) in prices and volatilities. This approach has an extensive literature and is reviewed in the next section, which is followed by a section that reviews the nonparametric approach of Hutchinson et al.[4] for European options and Broadie et al.[5] for American options.

A semiparametric approach, which combines the parametric pricing formulas given by the Black-Scholes-Merton theory with nonparametric regression applied to the discrepancies between the theoretical and observed option prices, was introduced by Lai and Wong,[6] who also performed simulation studies showing that this semiparametric approach provides good approximations to the true underlying option prices. An empirical study of its performance is given in this article, and the findings show that its improvement in predictive (out-of-sample) performance over the Black-Scholes pricing formula is limited to stable periods in which the future is similar to the past. The success of a statistical learning approach relies critically on the stationarity of the data to which it is applied. A nonparametric or semiparametric approach works best when the underlying distribution and characteristics of the data in the training period can be well extended to the data in the test period. While this is the case in typical simulation studies, the market environment may change substantially from the start of the training period till the end of the test period for an empirical study in which six-month training and test subperiods are used. We describe herein a simple modification of the semiparametric approach by using time series modeling, instead of nonparametric regression, of the discrepancies between the actual and theoretical option prices.

From Black-Scholes to SVCJ models for option pricing

Assuming that the asset price S_t follows a geometric Brownian motion, that the market has a risk-free asset with constant interest rate r, that short selling is allowed, and that continuous trading of the perfectly divisible asset can occur with no transaction costs, Black and Scholes[2] use the absence of arbitrage to show that the price of a call or put option satisfies a partial differential equation (PDE) whose solution is a function of the time $u = T - t$ to expiration and $R_t = S_t/K$:

$$P^{\mathrm{bs}}(u, R) = \begin{cases} K\{Re^{-qu}\Phi(d(u,R)) - e^{-ru}\Phi(d(u,R) - \sigma\sqrt{u})\} & \text{for call,} \\ K\{e^{-ru}\Phi(\sigma\sqrt{u} - d(u,R)) - Re^{-qu}\Phi(-d(u,R))\} & \text{for put,} \end{cases} \tag{2}$$

where q is the dividend rate paid by the underlying asset, Φ is the standard normal distribution function and

$$d(u, R) = \{\log R + (r - q)u\}/(\sigma\sqrt{u}) + \sigma\sqrt{u}/2. \tag{3}$$

Merton[3] extended the Black-Scholes theory for pricing European options to American options. Optimal exercise of the option occurs when the asset price exceeds or falls below an exercise boundary $\partial\mathcal{C}$ for a call or put option, respectively. The Black-Scholes PDE still holds in the continuation region $\mathcal{C}$ of (t, S_t) before exercise, and $\partial\mathcal{C}$ is determined by the *free boundary condition* $\partial P/\partial S = 1$ (or -1) for a call (or put) option. Unlike the explicit formula (2) for European options, there is no closed-form solution of the free-boundary PDE, and numerical methods such as finite differences are needed to compute American option prices under this theory. The free-boundary PDE can also be represented probabilistically as the value function of the optimal stopping problem

$$P^{\mathrm{a}}(u, R) = K \sup_{\tau \in \mathcal{T}_{T-u,T}} E[e^{-r(\tau-(T-u))} g(R_\tau) \,|\, R_t = R], \tag{4}$$

where $g(R) = (R-1)_+$ or $(1-R)_+$ for the call or put, with $x_+ = \max\{0, x\}$, $\mathcal{T}_{t,T}$ is the set of stopping times whose values are between t and T, and E is expectation with respect to the *risk-neutral measure* under which $\mu = r - q$ in (1).

The interest rate r in the Black-Scholes formula (2) for the price of a European option is usually taken to be the yield of a short-maturity Treasury bill at the time when the contract is initiated. The parameter in (2) that cannot be directly observed is σ. Equating (2) to the actual price of the option yields a nonlinear equation in σ whose solution is called the "implied volatility" of the underlying asset. The implied volatilities computed from call and put options with the same strike price K and time to maturity u should be equal because the put-call parity relationship (call price $-$ put price $= S_t e^{-qu} - Ke^{-ru}$) holds for both the Black-Scholes price pair in (2) and the corresponding market price pair. A call option, whose payoff function is $(S - K)_+$, is said to be *in the money*, *at the money*, or *out of the money* according to whether $S_t > K$, $S_t = K$, or $S_t < K$, respectively. Puts have the reverse terminology since the payoff function is $(K - S)_+$. According to the Black-Scholes theory, the σ in (2) is the volatility of the underlying asset and therefore does not vary with K and T . However, for some equity options, a "volatility skew" is observed (i.e., the implied volatility is a decreasing function of the strike price K). The "volatility smile" is common in foreign