

# Boltzmann Equations for the PQMSSM (v3.0)

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ABSTRACT:

## 1. General Formalism and Approximations

The general Boltzmann equation for the number distribution of a particle species can be written as[1] (assuming isotropy):

$$\frac{\partial F_i}{\partial t} - Hp \frac{\partial F_i}{\partial p} = C_i[F_i, F_j, p] \quad (1.1)$$

where  $F_i(p)$  is the number distribution of particle  $i$  as function of momentum  $p$ ,  $C$  represents a source/sink term and  $H$  is the Hubble constant:

$$H = \sqrt{\frac{8\pi}{3} \frac{\rho_T}{M_P^2}} \quad (1.2)$$

with  $M_P = 1.22 \times 10^{19}$  GeV and  $\rho_T = \sum_i \rho_i$ . The number, energy and pressure densities are given in terms of  $F_i$  as:

$$\begin{aligned} n_i(t) &= \int \frac{dp}{2\pi^2} p^2 F_i(p) \\ \rho_i(t) &= \int \frac{dp}{2\pi^2} p^2 E_i F_i(p) \\ P_i(t) &= \frac{1}{3} \int \frac{dp}{2\pi^2} \frac{p^4}{E_i} F_i(p) \end{aligned} \quad (1.3)$$

where  $m_i$  is the mass of particle  $i$  and  $E_i = \sqrt{p_i^2 + m_i^2}$ . Using Eq.(1.1), we obtain the following equations for the number and energy densities:

$$\begin{aligned} \frac{dn_i}{dt} + 3Hn_i &= \int \frac{dp}{2\pi^2} p^2 C_i \\ \frac{d\rho_i}{dt} + 3H(\rho_i + P_i) &= \int \frac{dp}{2\pi^2} p^2 E_i C_i \end{aligned} \quad (1.4)$$

The collision term,  $C_i$ , for the process  $i + f + \dots \leftrightarrow a + b + c + \dots$  is given by[2]:

$$\begin{aligned} C_i &= \frac{1}{E_i} \int \prod_{j,a} \frac{d^3 p_j}{2E_j(2\pi)^3} \frac{d^3 p_a}{2E_a(2\pi)^3} (2\pi)^4 \delta^4(p_i + p_j + \dots - p_a - p_b \dots) |\mathcal{M}|^2 \\ &\times [(1 \pm f_a)(1 \pm f_b) \dots f_i f_j \dots - f_a f_b \dots (1 \pm f_i)(1 \pm f_j) \dots] \end{aligned} \quad (1.5)$$

where the plus (minus) sign is for bosons (fermions). Below we always assume  $f_{i,j,a,\dots} \ll 1$ , so:

$$\begin{aligned} C_i &\simeq \frac{1}{E_i} \int \prod_{j,a} \frac{d^3 p_j}{2E_j(2\pi)^3} \frac{d^3 p_a}{2E_a(2\pi)^3} (2\pi)^4 \delta^4(p_i + p_j + \dots - p_a - p_b \dots) |\mathcal{M}|^2 \\ &\times [f_i f_j \dots - f_a f_b \dots] \end{aligned} \quad (1.6)$$

We will assume that  $C$  is given by:

$$C = C_{dec} + C_{prod} + C_{ann} \quad (1.7)$$

where  $C_{dec}$  contains the contributions from decays and inverse decays ( $i \leftrightarrow a+b+\dots$ ),  $C_{prod}$  contains the contributions from decay injection and inverse decay injection ( $a \leftrightarrow i+b+\dots$ ) and  $C_{ann}$  from annihilations with the thermal plasma ( $i+i \leftrightarrow a+b$ ). Below we compute each term separately, under some assumptions.

### 1.1 Annihilation Term

The annihilation term  $C_{ann}$  for the  $i + j \leftrightarrow a + b$  process is given by[1]:

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} = \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(p_i + p_j - p_a - p_b) |M|^2 [f_a f_b - f_i f_j] \quad (1.8)$$

where  $d\Pi_i = d^3p_i / ((2\pi)^3 2E_i)$ . Since we are ultimately interested in Eqs.(1.4) for the number and energy densities, we will consider the following integral:

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha = \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(p_i + p_j - p_a - p_b) |M|^2 [f_a f_b - f_i f_j] E_i^\alpha \quad (1.9)$$

where  $\alpha = 0(1)$  for the number (energy) density. Here we assume that the distributions can be approximated by<sup>1</sup>:

$$f_i \simeq \exp(-(E_i - \mu_i)/T) \quad (1.10)$$

so the annihilation term can then be written as:

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha &= -(\exp((\mu_i + \mu_j)/T) - \exp((\mu_a + \mu_b)/T)) \\ &\times \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(p_i + p_j - p_a - p_b) |M|^2 \exp(-(E_i + E_j)/T) \times E_i^\alpha \end{aligned}$$

where above we have used conservation of energy ( $E_i + E_j = E_a + E_b$ ). Assuming<sup>2</sup>:

$$\frac{n_i}{\bar{n}_i} = \exp(\mu_i/T) \quad (1.11)$$

we obtain:

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha &= -\left(\frac{n_i n_j}{\bar{n}_i \bar{n}_j} - \frac{n_a n_b}{\bar{n}_a \bar{n}_b}\right) \\ &\times \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(p_i + p_j - p_a - p_b) |M|^2 \exp(-(E_i + E_j)/T) \times E_i^\alpha \end{aligned}$$

In order to include conversion, co-annihilation and annihilation of BSM particles we must consider the cases:

- $i = j = BSM$  and  $a, b = SM, SM$ :

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha = -(n_i^2 - \bar{n}_i^2) \langle \sigma v E_i^\alpha \rangle_{ii} \quad (1.12)$$

- $i \neq j$ ,  $i, j = BSM$  and  $a, b = SM, SM$ :

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha = -(n_i n_j - \bar{n}_i \bar{n}_j) \langle \sigma v E_i^\alpha \rangle_{ij} \quad (1.13)$$

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<sup>1</sup>This approximation is only valid for particles with a thermal distribution. However, since the annihilation term is responsible for keeping the particle  $i$  in thermal equilibrium with the plasma, it is reasonable to assume a thermal distribution for  $i$  while the annihilation term is relevant.

<sup>2</sup>Note that in a more general case one can have  $T_i \neq T$  and this approximation is not valid.

- $i = j = BSM, a = b = BSM$ :

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha = - \left( n_i^2 - \frac{\bar{n}_i^2}{\bar{n}_j^2} n_j^2 \right) \langle \sigma v E_i^\alpha \rangle_{jj} \quad (1.14)$$

- $i = BSM, j = SM, a = BSM, b = SM$  and  $a \neq i$ :

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha = - \left( n_i - \frac{\bar{n}_i}{\bar{n}_j} n_j \right) \bar{n}_{SM} \langle \sigma v E_i^\alpha \rangle_{ijSM} \equiv - \left( n_i - \frac{\bar{n}_i}{\bar{n}_j} n_j \right) \tilde{\Gamma}_{ijSM} \quad (1.15)$$

where:

$$\langle \sigma v E_i^\alpha \rangle_i = \frac{1}{\bar{n}_i^2} \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(P) |M|^2 e^{-(E_i+E_j)/T} \times E_i^\alpha \quad (1.16)$$

$$\langle \sigma v E_i^\alpha \rangle_{ij} = \frac{1}{\bar{n}_i \bar{n}_j} \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(P) |M|^2 e^{-(E_i+E_j)/T} \times E_i^\alpha \quad (1.17)$$

$$\langle \sigma v E_i^\alpha \rangle_{jj} = \frac{1}{\bar{n}_j^2} \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(P) |M|^2 e^{-(E_i+E_j)/T} \times E_i^\alpha \quad (1.18)$$

$$\tilde{\Gamma}_{ijSM} = \frac{1}{\bar{n}_i} \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(P) |M|^2 e^{-(E_i+E_j)/T} \times E_i^\alpha \quad (1.19)$$

For  $\alpha = 0$ , the above equation is the well known contribution from thermal scatterings to the annihilation term. To estimate its value for  $\alpha = 1$ , we assume:

$$\langle \sigma v E \rangle \simeq \langle \sigma v \rangle \langle E_i \rangle = \langle \sigma v \rangle \frac{\rho_i}{n_i} \quad (1.20)$$

where  $\langle \rangle$  represents thermal average.

## 1.2 Decay Term

Now we derive a simplified expression for the decay (and inverse decay) term, under approximations similar to the ones used in the last section. The decay term includes the contributions from particle decay and inverse decay[2, 3]:

$$C_{dec} \simeq \frac{1}{E_i} \int \prod_a \frac{d^3 p_a}{2E_a (2\pi)^3} (2\pi)^4 \delta^4(p_i - p_a - p_b \dots) |\mathcal{M}|^2 [f_i - f_a f_b \dots] \quad (1.21)$$

As in the case of the annihilation term, we assume that the distributions for  $a, b, \dots$  can be approximated by  $f_x \simeq \exp(-(E_x - \mu_x)/T)$ , so we can write:

$$f_a f_b \dots \simeq \exp\left(\frac{\mu_a + \mu_b + \dots}{T}\right) \exp(-E_i/T) = \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \exp(-E_i/T) = \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \bar{f}_i \quad (1.22)$$

where we used conservation of energy ( $E_a + E_b + \dots = E_i$ ) and  $\bar{f}_i$  is the equilibrium distribution for the species  $i$ . Hence we can write Eq.(1.21) as:

$$\begin{aligned} C_{dec} &\simeq \left[ f_i - \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \bar{f}_i \right] \frac{1}{E_i} \int \prod_a \frac{d^3 p_a}{2E_a (2\pi)^3} (2\pi)^4 \delta^4(p_i - p_a - p_b \dots) |\mathcal{M}|^2 \\ &= \mathcal{B}_{ab\dots} \frac{\Gamma_i m_i}{E_i} \left[ f_i - \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \bar{f}_i \right] \end{aligned} \quad (1.23)$$

where  $\Gamma_i$  is the width for  $i$  and  $\mathcal{B}_{ab\dots} \equiv BR(i \rightarrow a + b + \dots)$

Once again we consider the integral:

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{dec}(p) E_i^\alpha = & -\Gamma_i \int \frac{dp}{2\pi^2} p^2 \frac{m_i}{E_i} f_i E_i^\alpha \\ & + \sum_{i \text{ decays}} \mathcal{B}_{ab\dots} \Gamma_i \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \int \frac{dp}{2\pi^2} p^2 \frac{m_i}{E_i} \bar{f}_i E_i^\alpha \end{aligned} \quad (1.24)$$

where we have included the sum over all decay channels and  $\alpha = 0(1)$  for the contribution to the number (energy) density equation. Note that both integrals are identical, except for the replacement  $f_i \rightarrow \bar{f}_i$ . The first integral in Eq.(1.24) gives:

$$-\Gamma_i \int \frac{dp}{2\pi^2} p^2 \frac{m_i}{E_i} f_i(p) E_i^\alpha = \begin{cases} -\Gamma_i m_i n_i \langle \frac{1}{E_i} \rangle, & \text{for } \alpha = 0 \\ -\Gamma_i m_i n_i, & \text{for } \alpha = 1 \end{cases} \quad (1.25)$$

where

$$\langle \frac{1}{E_i} \rangle \equiv \frac{1}{n_i} \int \frac{dp}{2\pi^2} p^2 \frac{1}{E_i} f_i(p) \quad (1.26)$$

Hence we can write Eq.(1.24) as:

$$\int \frac{dp}{2\pi^2} p^2 C_{dec}(p) E_i^\alpha = -\Gamma_i m_i \begin{cases} n_i \langle \frac{1}{E_i} \rangle - \bar{n}_i \langle \frac{1}{E_i} \rangle_{eq} \sum \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots}, & \text{for } \alpha = 0 \\ n_i - \bar{n}_i \sum \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots}, & \text{for } \alpha = 1 \end{cases} \quad (1.27)$$

For the non-equilibrium average we assume:

$$\langle \frac{1}{E_i} \rangle \simeq \frac{1}{\langle E_i \rangle} = \frac{n_i}{\rho_i} \quad (1.28)$$

which is exact in the non-relativistic limit, but it is only an approximation for the relativistic case. Although we can compute the equilibrium average ( $\langle \frac{1}{E_i} \rangle_{eq}$ ) explicitly, in order to have an exact cancellation between the decay and inverse decay terms when  $i, a$  and  $b$  are all in equilibrium, we take:

$$\langle \frac{1}{E_i} \rangle_{eq} \simeq \langle \frac{1}{E_i} \rangle = \frac{n_i}{\rho_i} \quad (1.29)$$

With the above approximations we finally obtain:

$$\int \frac{dp}{2\pi^2} p^2 C_{dec}(p) E_i^\alpha = -\Gamma_i m_i \begin{cases} \frac{n_i}{\rho_i} \left( n_i - \bar{n}_i \sum \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \right), & \text{for } \alpha = 0 \\ n_i - \bar{n}_i \sum \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots}, & \text{for } \alpha = 1 \end{cases} \quad (1.30)$$

where  $\mathcal{B}_{ab\dots} \equiv BR(i \rightarrow a + b + \dots)$ .

### 1.3 Production Term

The decay and inverse decay of other particles ( $a \rightarrow i + b + \dots$ ) can also affect the species  $i$ . The contribution from these terms we label  $C_{prod}$ , which is given by[2]:

$$C_{prod} \simeq \frac{1}{E_i} \int \frac{d^3 p_a}{2E_a (2\pi)^3} \prod_b \frac{d^3 p_b}{2E_b (2\pi)^3} (2\pi)^4 \delta^4(p_a - p_i - p_b \dots) |\mathcal{M}|^2 [f_a - f_i f_b \dots] \quad (1.31)$$

Using the same approximations of the previous section, we write:

$$f_i f_b \dots \simeq \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} e^{-E_a/T} = \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \bar{f}_a \quad (1.32)$$

Hence:

$$C_{prod} = \frac{1}{E_i} \int \frac{d^3 p_a}{2E_a (2\pi)^3} \prod_b \frac{d^3 p_b}{2E_b (2\pi)^3} (2\pi)^4 \delta^4(p_a - p_i - p_b \dots) |\mathcal{M}|^2 \left( f_a - \bar{f}_a \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \quad (1.33)$$

and

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{prod}(p) E_i^\alpha &= \int \frac{d^3 p_a}{E_a (2\pi)^3} \left( f_a - \bar{f}_a \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \\ &\times \frac{d^3 p E_i^\alpha}{2E_i (2\pi)^3} \prod_b \frac{d^3 p_b}{2E_b (2\pi)^3} (2\pi)^4 \delta^4(p_a - p_i - p_b \dots) |\mathcal{M}|^2 \end{aligned} \quad (1.34)$$

with  $\alpha = 0(1)$  for the contribution to the number (energy) density equation. For  $\alpha = 0$  we obtain:

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{prod}(p) &= \Gamma_a \mathcal{B}_i m_a \int \frac{d^3 p_a}{E_a (2\pi)^3} \left( f_a - \bar{f}_a \sum_b \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \\ &= \Gamma_a \mathcal{B}_i m_a \frac{n_a}{\rho_a} \left( n_a - \bar{n}_a \sum_b \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \end{aligned} \quad (1.35)$$

where  $\mathcal{B}_{ib\dots} \equiv BR(a \rightarrow i + b + \dots)$ ,  $\mathcal{B}_i = \sum_b \mathcal{B}_{ib\dots}$  and we have once again assumed  $\langle 1/E_a \rangle \simeq \langle 1/E_a \rangle_{eq} \simeq n_a/\rho_a$ .

For  $\alpha = 1$ , the integral in Eq.(1.34) does not take a simple form. In order to compute it, we assume:

$$E_i \simeq \frac{E_a}{2} \quad (1.36)$$

The above expression is only exact for 2-body decays and  $m_a \gg m_i, m_b$ . For the remaining cases, it is only an estimate.

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{prod}(p) E_i &\simeq \Gamma_a \mathcal{B}_i \frac{m_a}{2} \int \frac{d^3 p_a}{(2\pi)^3} \left( f_a - \bar{f}_a \sum_b \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \\ &= \Gamma_a \mathcal{B}_i \frac{m_a}{2} \left( n_a - \bar{n}_a \sum_b \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \end{aligned} \quad (1.37)$$

Combining the results for  $\alpha = 0$  and 1, we have:

$$\int \frac{dp}{2\pi^2} p^2 C_{prod}(p) E_i^\alpha = \Gamma_a \mathcal{B}_i m_a \left( n_a - \bar{n}_a \sum_b \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \begin{cases} \frac{n_a}{\rho_a}, & \text{for } \alpha = 0 \\ \frac{1}{2}, & \text{for } \alpha = 1 \end{cases} \quad (1.38)$$

## 1.4 Number and Energy Density Equations

Using the results of Eqs.(1.12)-(1.15), (1.30) and (1.38) in the Boltzmann equations for  $n_i$  and  $\rho_i$  (Eq.(1.4)), we obtain:

$$\begin{aligned} \frac{dn_i}{dt} + 3Hn_i = & (\bar{n}_i^2 - n_i^2) \langle \sigma v \rangle_{ii} + \sum_{j \neq i} (\bar{n}_i \bar{n}_j - n_i n_j) \langle \sigma v \rangle_{ij} + \sum_{j \neq i} \left( \frac{\bar{n}_i^2}{\bar{n}_j^2} n_j^2 - n_i^2 \right) \langle \sigma v \rangle_{jj} \\ & + \sum_{j \neq i} \left( \frac{\bar{n}_i}{\bar{n}_j} n_j - n_i \right) \tilde{\Gamma}_{ijSM} - \Gamma_i m_i \frac{n_i}{\rho_i} \left( n_i - \bar{n}_i \sum_{i \rightarrow \dots} \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \right) \\ & + \sum_a \Gamma_a \mathcal{B}_i m_a \frac{n_a}{\rho_a} \left( n_a - \bar{n}_a \sum_{a \rightarrow i\dots} \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) + C_i(T) \end{aligned} \quad (1.39)$$

$$\begin{aligned} \frac{d\rho_i}{dt} + 3H(\rho_i + P_i) = & \frac{\rho_i}{n_i} (\bar{n}_i^2 - n_i^2) \langle \sigma v \rangle_{ii} + \frac{\rho_i}{n_i} (\bar{n}_i \bar{n}_j - n_i n_j) \langle \sigma v \rangle_{ij} + \frac{\rho_i}{n_i} \left( \frac{\bar{n}_i^2}{\bar{n}_j^2} n_j^2 - n_i^2 \right) \langle \sigma v \rangle_{jj} \\ & + \frac{\rho_i}{n_i} \left( \frac{\bar{n}_i}{\bar{n}_j} n_j - n_i \right) \tilde{\Gamma}_{ijSM} - \sum_a \Gamma_a \mathcal{B}_i \frac{m_a}{2} \left( n_a - \bar{n}_a \sum_{a \rightarrow i\dots} \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \\ & + \tilde{C}_i(T) \frac{\rho_i}{n_i} \end{aligned} \quad (1.40)$$

where  $\mathcal{B}_{ab\dots} = BR(i \rightarrow a + b + \dots)$ ,  $\mathcal{B}_{ib\dots} = BR(a \rightarrow i + b + \dots)$ ,  $\mathcal{B}_i = \sum_b \mathcal{B}_{ib\dots}$  and we have included an extra term ( $C_i$  and  $\tilde{C}_i$ ) to allow for other possible sources for the number and energy densities. For simplicity we assume  $C_i = \tilde{C}_i$  from now on.

It is also convenient to use the above results to obtain a simpler equation for  $\rho_i/n_i$ :

$$\frac{d\rho_i/n_i}{dt} \equiv \frac{dR_i}{dt} = -3H \frac{P_i}{n_i} + \sum_a \mathcal{B}_i \frac{\Gamma_a m_a}{n_i} \left( \frac{1}{2} - \frac{n_a}{\rho_a} \frac{\rho_i}{n_i} \right) \left( n_a - \bar{n}_a \sum_{a \rightarrow i\dots} \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \quad (1.41)$$

Besides the above equations, it is useful to consider the evolution equation for entropy:

$$dS \equiv \frac{dQ^{dec}}{T} \quad (1.42)$$

where  $dQ^{dec}$  is the net energy injected from decays. With the above definition we have:

$$\begin{aligned} \dot{S} &= \frac{1}{T} \sum_i BR(i, X) \frac{d(R^3 \rho_i)^{dec}}{dt} \\ \Rightarrow \dot{S} &= \frac{R^3}{T} \sum_i BR(i, X) \Gamma_i m_i \left( n_i - \bar{n}_i \sum_{i \rightarrow \dots} \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \right) \end{aligned} \quad (1.43)$$

where  $R$  is the scale factor and  $BR(i, X)$  is the fraction of energy injected in the thermal bath from  $i$  decays.

The above expressions can be written in a more compact form if we define the following "effective thermal densities" and "effective BR":

$$\mathcal{N}_X^{th} \equiv \bar{n}_X \sum_{X \rightarrow \dots} BR(X \rightarrow 1 + 2 + \dots) \prod_k \frac{n_k}{\bar{n}_k}$$

$$\mathcal{N}_{XY}^{th} \equiv \frac{\bar{n}_X}{\mathcal{B}_{XY}^{eff}} \sum_{X \rightarrow Y + \dots} g_Y BR(X \rightarrow g_Y Y + 1 + \dots) \left( \frac{n_Y}{\bar{n}_Y} \right)^{g_Y} \prod_k \frac{n_k}{\bar{n}_k}$$

$$\mathcal{B}_{XY}^{eff} \equiv \sum_{X \rightarrow Y + \dots} g_Y BR(X \rightarrow g_Y Y + 1 + \dots)$$

where  $g_Y$  is the  $Y$  multiplicity in the final state of  $X$  decays. In addition, defining:

$$x = \ln(R/R_0), \quad N_i = \ln(n_i/s_0), \quad \text{and} \quad N_S = \ln(S/S_0) \quad (1.44)$$

we can write Eqs.(1.43), (1.39) and (1.41) as:

$$N'_S = \frac{e^{(3x-N_S)}}{HT} \sum_i BR(i, X) \Gamma_i m_i \left( n_i - \mathcal{N}_i^{th} \right) \quad (1.45)$$

$$\begin{aligned} N'_i = & -3 + \frac{n_i}{H} \left( \frac{\bar{n}_i^2}{n_i^2} - 1 \right) \langle \sigma v \rangle_{ii} + \frac{1}{H} \sum_{j \neq i} \left( \frac{\bar{n}_i}{n_i} \bar{n}_j - n_j \right) \langle \sigma v \rangle_{ij} \\ & + \frac{n_i}{H} \sum_{j \neq i} \left( \frac{\bar{n}_i^2}{n_i^2} \frac{n_j^2}{\bar{n}_j^2} - 1 \right) \langle \sigma v \rangle_{jj} + \frac{1}{H} \sum_{j \neq i} \left( \frac{\bar{n}_i}{n_i} \frac{n_j}{\bar{n}_j} - 1 \right) \tilde{\Gamma}_{ijSM} \\ & - \frac{\Gamma_i}{H} \frac{m_i}{R_i} \left( 1 - \frac{\mathcal{N}_i^{th}}{n_i} \right) + \sum_a \mathcal{B}_{ai}^{eff} \frac{\Gamma_a}{H} \frac{m_a}{R_a} \left( \frac{n_a}{n_i} - \frac{\mathcal{N}_{ai}^{th}}{n_i} \right) \end{aligned} \quad (1.46)$$

$$R'_i = -3 \frac{P_i}{n_i} + \sum_a \mathcal{B}_{ai}^{eff} \frac{\Gamma_a}{H} m_a \left( \frac{1}{2} - \frac{R_i}{R_a} \right) \left( \frac{n_a}{n_i} - \frac{\mathcal{N}_{ai}^{th}}{n_i} \right) \quad (1.47)$$

where  $' = d/dx$ .

The above equation for  $N_i$  also applies for coherent oscillating fields, if we define:

$$N_i = \ln(n_i/s_0), \quad \text{and} \quad n_i \equiv \rho_i/m_i \quad (1.48)$$

so

$$\begin{aligned} N'_i &= -3 - \frac{\Gamma_i}{H} \\ R'_i &= 0 \end{aligned} \quad (1.49)$$

where we assume that the coherent oscillating component does not couple to any of the other fields.

Collecting Eqs.(1.45)-(1.47) and (1.49) we have a closed set of first order differential equations:

- Entropy:

$$N'_S = \frac{e^{(3x-N_S)}}{HT} \sum_i BR(i, X) \Gamma_i m_i \left( n_i - \mathcal{N}_i^{th} \right) \quad (1.50)$$

- Thermal fields:

$$N'_i = -3 + \frac{n_i}{H} \left( \frac{\bar{n}_i^2}{n_i^2} - 1 \right) \langle \sigma v \rangle_{ii} + \frac{1}{H} \sum_{j \neq i} \left( \frac{\bar{n}_i}{n_i} \bar{n}_j - n_j \right) \langle \sigma v \rangle_{ij} + \frac{n_i}{H} \sum_{j \neq i} \left( \frac{\bar{n}_i^2}{n_i^2} \frac{n_j^2}{\bar{n}_j^2} - 1 \right) \langle \sigma v \rangle_{jj}$$



$$\begin{aligned}
& + \frac{1}{H} \sum_{j \neq i} \left( \frac{\bar{n}_i}{n_i} \frac{n_j}{\bar{n}_j} - 1 \right) \tilde{\Gamma}_{ijSM} - \frac{\Gamma_i}{H} \frac{m_i}{R_i} \left( 1 - \frac{\mathcal{N}_i^{th}}{n_i} \right) + \sum_a \mathcal{B}_{ai}^{eff} \frac{\Gamma_a}{H} \frac{m_a}{R_a} \left( \frac{n_a}{n_i} - \frac{\mathcal{N}_{ai}^{th}}{n_i} \right) \\
R'_i &= -3 \frac{P_i}{n_i} + \sum_a \mathcal{B}_{ai}^{eff} \frac{\Gamma_a}{H} m_a \left( \frac{1}{2} - \frac{R_i}{R_a} \right) \left( \frac{n_a}{n_i} - \frac{\mathcal{N}_{ai}^{th}}{n_i} \right)
\end{aligned} \tag{1.51}$$

- Coherent Oscillating fields:

$$\begin{aligned}
N'_i &= -3 - \frac{\Gamma_i}{H} \\
R'_i &= 0
\end{aligned} \tag{1.52}$$

As seen above, the equation for  $R_i = \rho_i/n_i$  depends on  $P_i/n_i$ . A proper evaluation of this quantity requires knowledge of the distribution  $F_i(p, t)$ . However, for relativistic (or massless) particles we have  $P_i = \rho_i/3$ , as seen from Eq.(1.3), while for particles at rest we have  $P_i = 0$ . Hence  $F_i(p, t)$  is only required to evaluate the relativistic/non-relativistic transition, which corresponds to a relatively small part of the evolution history of particle  $i$ . Nonetheless, to model this transition we approximate  $F_i$  by a thermal distribution and take  $T_i, \mu_i \ll m_i$ , where  $T_i$  is the temperature of the particle (which can be different from the thermal bath's). Under these approximations we have:

$$\begin{aligned}
\frac{P_i}{n_i} &= T_i \\
\frac{\rho_i}{n_i} &= T_i \left[ \frac{K_1(m_i/T_i)}{K_2(m_i/T_i)} \frac{m_i}{T_i} + 3 \right]
\end{aligned} \tag{1.53}$$

where  $K_{1,2}$  are the modified Bessel functions. In particular, if  $m_i/T_i \gg 1$ :

$$\frac{\rho_i}{n_i} \simeq T_i \left[ \frac{3}{2} + \frac{m_i}{T_i} + 3 \right] \Rightarrow \frac{P_i}{n_i} = T_i = \frac{2m_i}{3} \left( \frac{R_i}{m_i} - 1 \right) \tag{1.54}$$

As shown above, for a given value of  $R_i = \rho_i/n_i$ , Eq.(1.53) can be inverted to compute  $T_i$  ( $= P_i/n_i$ ):

$$\frac{P_i}{n_i} = T_i(R_i) \tag{1.55}$$

Since we are interested in the non-relativistic/relativistic transition, we can expand the above expression around  $R_i/m_i = 1$ , so  $P_i/n_i$  can be written as:

$$\frac{P_i}{n_i} = \frac{2m_i}{3} \left( \frac{R_i}{m_i} - 1 \right) + m_i \sum_{n>1} a_n \left( \frac{R_i}{m_i} - 1 \right)^n \tag{1.56}$$

where the coefficients  $a_n$  can be numerically computed from Eq.(1.53). The above approximation should be valid for  $m_i/T_i \gtrsim 1$  (or  $R_i \gtrsim m_i$ ). On the other hand, for  $m_i/T_i \ll 1$  (or  $R_i \gg m_i$ ), we have the relativistic regime, with  $P_i/n_i = R_i/3$ . Therefore we can approximate the  $P_i/n_i$  function for all values of  $R_i$  by:

$$\frac{P_i}{n_i} = \begin{cases} \frac{2m_i}{3} \left( \frac{R_i}{m_i} - 1 \right) + m_i \sum_{n>1} a_n \left( \frac{R_i}{m_i} - 1 \right)^n, & \text{for } R_i < \tilde{R} \\ \frac{R_i}{3}, & \text{for } R_i > \tilde{R} \end{cases} \tag{1.57}$$

where the coefficients  $a_n$  are given by the numerical fit of Eq.(1.53) and  $\tilde{R}$  is given by the matching of the two solutions.

Finally, to solve Eqs.(1.50)-(1.52) we need to compute  $H$  according to Eq.(1.2), which requires knowledge of the energy densities for all particles ( $\rho_i$ ) and for the thermal bath ( $\rho_R$ ). The former are directly obtained from  $N_i$  and  $R_i$ , while the latter can be computed from  $N_S$ :

$$T = \left( \frac{g_{*S}(T_R)}{g_{*S}(T)} \right)^{1/3} T_R \exp[N_S/3 - x] \Rightarrow \rho_R = \frac{\pi^2}{30} g_*(T) T^4 \quad (1.58)$$

Eqs.(1.50)-(1.52), with the auxiliary equations for  $H$  (Eq.(1.2)) and  $P_i/n_i$  (Eq.(1.57)) form a set of closed equations, which can be solved once the initial conditions for the number density ( $n_i$ ), energy density ( $\rho_i$ ) and entropy ( $S$ ) are given. For thermal fluids we assume:

$$n_i(T_R) = \begin{cases} 0 & , \text{ if } \langle \sigma v \rangle_i \bar{n}_i / H|_{T=T_R} < 10 \\ \bar{n}_i(T_R) & , \text{ if } \langle \sigma v \rangle_i \bar{n}_i / H|_{T=T_R} > 10 \end{cases} \quad (1.59)$$

$$\frac{\rho_i}{n_i}(T_R) = \frac{\bar{\rho}_i}{\bar{n}_i}(T_R) \quad (1.60)$$

where  $\bar{\rho}_i$  is the equilibrium energy density (with zero chemical potential) for the particle  $i$ . While for coherent oscillating fluids the initial condition is set at the beginning of oscillations:

$$n_i(T_i^{osc}) = \frac{\rho_i^0}{m_i(T_i^{osc})} \quad (1.61)$$

$$\frac{\rho_i}{n_i}(T_i^{osc}) = m_i \quad (1.62)$$

where  $T_i^{osc}$  is the oscillation temperature, given by  $3H(T_i^{osc}) = m_i(T_i^{osc})$  and  $\rho_i^0$  the initial energy density for oscillations.

Finally, the initial condition for the entropy  $S$  is trivially obtained, once we assume a radiation dominated universe at  $T = T_R$ :

$$S(T_R) = \frac{2\pi^2}{45} g_*(T_R) T_R^3 R_0^3 \quad (1.63)$$

## 2. Code

Here we describe how the above formalism is implemented in a numerical code for solving the coupled Boltzmann equations. In Sec.2.1 we describe how the input for specific models should be defined. Then, in Sec.2.2 we outline the procedure used to solve the Boltzmann equations and to treat some of the discrete transitions required by the formalism described above. Finally, in Sec.2.3 we describe what is the output of the code and how it can be controlled by the user.

## 2.1 Input

In order to solve the Boltzmann equations for a particular model, the user has to provide the SUBROUTINE INPUTBOLTZ(T), which, for a given (thermal bath) temperature  $T$ , fills the COMMON BLOCK:

```
COMMON/INBOLTZ/BR(NP,NP),DEGF(NP),MASS(NP),GAM(NP),  
SIGV(NP),C(NP),COHOSC(NP),TRH,NCOMPS,LABEL(NP)
```

where  $NP = 20$  and

- $NCOMPS (\leq 10)$  = the number of particles (the first component must be radiation)
- $MASS(i)$  = mass for particle  $i$  (can be temperature dependent, as in the axion case)
- $DEGF(i)$  = +-number of degrees of freedom for particle  $i$ . A plus sign should be used for bosons, while a minus should be used for fermions, i.e.  $DEGF=-2$  for neutralinos and  $DEGF=1$  for axions. The value for the  $i=1$  component (radiation) is never used, since the number of degrees of freedom in this case is given by the function  $GSTAR(T)$ .
- $GAM(i)$  = decay width for particle  $i$ , in its rest frame.
- $BR(i,j)$  = branching ratio for the decay  $i \rightarrow j + X$ , including the multiplicity factor, if the  $i$  particle decays into multiple  $j$ 's.
- $BR(i,1)$  = fraction of energy per  $i$  particle injected in the radiation fluid.
- $SIGV(i)$  = thermal averaged cross-section for the annihilation of  $i$  particles, as defined in the previous section.
- $C(i)$  = additional source term for particle  $i$ , as defined in the previous section.
- $COHOSC(i)$  = initial energy density for coherent oscillating particles. Must be zero for thermal (non-oscillating) components.
- $TRH$  = re-heat temperature.
- $LABEL(i)$  = label for particle  $i$  (optional)

## 2.2 Main Code

Once the INPUTBOLTZ subroutine is provided, the user can compute the solution for the Boltzmann equations from  $T=TRH$  to  $T=TF$ , calling:

```
CALL INPUTBOLTZ(TRH) ! (initilization)  
CALL EQSBOLTZ(TF,IOUT) ! (compute solution)
```

where, if  $\text{IOUT} > 0$ , the scale factor ( $R$ ) and energy densities as a function of  $T$  are written to  $\text{UNIT}=\text{IOUT}$ . If  $\text{TF} = 0$ , the evolution proceeds until all unstable particles have decayed and/or all coherent oscillating components have started to oscillate. Before calling EQSBOLTZ, the user must define the parameters which regulate the precision of the procedure, given by the BLOCK DEPARS:

COMMON/DEPARS/EPS,DX0,STEP,IERROR

where EPS is the relative precision for the solution  $N_i(TF)$ , DX0 is the  $x$  interval for printing the solutions in IOUT and the maximum  $\Delta x$  step and STEP is the initial  $\Delta x$  step for the evolution. Failure to solve the equations (most likely due to numerical instabilities) is indicated by  $\text{IERROR} < 0$ .

Specific components can be turned off using the COMMON BLOCK:

COMMON/SWITCHES/TURNOFF(NP)

If  $\text{TURNOFF}(I)=\text{TRUE.}$ , the  $i$ -component will not be included in the evolution of the Boltzmann equations.

The EQSBOLTZ is the main subroutine used to solve the equations, once the appropriate input has been defined. Its main steps are:

1. Set initial conditions at  $T = \text{TRH}$ : check which thermal particles are coupled/decoupled to the thermal bath and if coherent oscillating fluids are already oscillating at  $T=\text{TRH}$ . Then it sets the initial number densities and temperatures for each component, as defined in the previous section. Set  $\text{X1}=1$  and  $\text{X2}=\text{X1}+\text{DX0}$ .
2. Solve the equations between X1 and X2.
3. Check if a particle has decayed. If the particle  $i$  satisfies

$$\Gamma_i/H > 100 \quad \text{and} \quad \min_{j \neq i}(\rho_i/\rho_j) < 10^{-3} \quad (2.1)$$

the particle is neglected from here on. The decay temperature ( $T_D$ ) is defined by the sudden decay approximation:  $\Gamma_i/\gamma_i = H(T_D)$ , where  $\gamma_i$  is the boost factor ( $\gamma_i \equiv \langle E_i \rangle / m_i$ ). Although this temperature is printed out in the output, it is never used in the code.

4. Check if a particle has decoupled from the thermal bath or started to oscillate in the interval (X1,X2). Decoupling is assumed if  $\langle \sigma v \rangle_i \bar{n}_i < H/10$ , which also defines the freeze-out temperature. The oscillation temperature is given by  $3H(T_{osc}) = m_i(T_{osc})$  and defines the beginning of evolution for the oscillating components.
5. If a component has started to oscillate or if it has decoupled, loop over this interval with smaller steps until the decoupling or oscillation temperature converges ( $\Delta T_i/T_i < 0.1$ ).
6. Write temperature, scale factor and energy densities to IOUT, if  $\text{IOUT} > 0$ .
7. Set  $\text{X1}=\text{X2}$  and  $\text{X2}=\text{X1}+\text{DX0}$  and return to point 2. until  $T < \text{TF}$  (or all unstable particles have decayed and all oscillating fluids have oscillated, if  $\text{TF} \leq 0$ ).

## 2.3 Output

The standard information printed after solving the Boltzmann equations gives:

- Freeze-out temperatures ( $T_{fr}$ ) for each thermal component. As mentioned in the last section,  $T_{fr}$  is given by the decoupling condition:  $\langle\sigma v\rangle_i \bar{n}_i = H/10$ . Since the decoupling is a continuous process,  $T_{fr}$  is just an estimate for the decoupling temperature.
- Decay temperatures ( $T_D$ ) for each thermal component. Once again the decay process is continuous and  $T_D$  given in the print out is estimated by the sudden decay approximation ( $\Gamma_i/\gamma_i = H(T_D)$ ).
- Oscillation temperature ( $T_{osc}$ ).
- Entropy ratio ( $S/S_0$ ). In case of entropy injection from decays of unstable particles,  $S/S_0 > 1$ .
- Relic densities ( $\Omega_i h^2$ ) at  $T_0 = 2.725$  K. In order to consistently compute the relic densities today, we evolve  $R_i = \rho_i/n_i$  from  $TF$  to  $T_0$  assuming a trivial universe expansion:

$$R'_i = -3 \frac{P_i}{n_i} \quad (2.2)$$

Note that the result obtained above is insensitive to  $H$ , so it does not matter if there is a transition from a radiation dominated to a matter dominated (or dark energy dominated) universe between  $TF$  and  $T_0$ . Once  $R_i(T_0)$  is obtained, the relic density is given by:

$$\Omega_i h^2 = n_i(TF) \times \frac{g_{*S}(T_0) T_0^3}{g_{*S}(TF) TF^3} \times \frac{R_i(T_0)}{\rho_c/h^2} \quad (2.3)$$

- Relic densities before decay. It may be relevant to compute the relic densities of an unstable particle as it would be given if it had not decayed. In particular, this value can be used to impose BBN bounds on the decays. This quantity is computed after the particle becomes non-relativistic and well before the decay starts ( $\Gamma_i/H(T) = 1/10$ ) and is given by:

$$\tilde{\Omega}_i h^2 = \frac{\rho_i(T)}{s(T)} \times \frac{s(T_0)}{\rho_c/h^2} \quad (2.4)$$

Note that the above expression assumes a radiation dominated universe from  $T$  to  $T_0$  and should be used with caution.

- Effective number of (new) neutrinos ( $\Delta N_{eff}$ ). Since neutrinos are still coupled for  $T > 1$  MeV, this quantity is only computed below this temperature.  $\Delta N_{eff}$  is given by:

$$\Delta N_{eff}(T) = \frac{\rho_{DR}(T)}{\rho_\nu} \quad (2.5)$$

where  $\rho_{DR}$  is the total energy density of relativistic particles (excluding radiation and neutrinos) and  $\rho_\nu$  is the energy density of neutrinos after they freeze-out:

$$\rho_{DR} = \sum_{R_i/m_i > 2} \rho_i \text{ and } \rho_\nu = \frac{\pi^2}{15} \frac{7}{8} \left( \frac{4}{11} \right)^{4/3} T^4 \quad (2.6)$$

Note that  $\Delta N_{eff}$  is in general a function of temperature, since  $\rho_{DR}$  will decrease if massive particles become non-relativistic below 1 MeV.

Furthermore, if  $IOUT > 0$ , the scale factor ( $R$ ), the energy densities and  $\Delta N_{eff}$  are printed as a function of  $T$  in `UNIT=IOUT`. Also, the following quantities are stored in `COMMON BLOCKs`:

- Final relic densities and entropy ratio:

`COMMON/OUTPUT/OMEGA(NP),RS`

- Decoupling, oscillation and decay temperatures:

`COMMON/TEMPS/TDEC(NP),TOSC(NP),TDCAY(NP)`

- Relic density of unstable particles before decay ( $\tilde{\Omega}h^2$ ), temperature at the end of entropy injection (if any) and effective number of new neutrinos ( $\Delta N_{eff}$ ) *at the final temperature  $TF$* :

`COMMON/BBNINFO/UMEGA(NP),TSTAB,DNeff`

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