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Boltzmann Equations (v3.0)

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Abstract:

1. General Formalism and Approximations

The general Boltzmann equation for the number distribution of a particle species can be written as[1] (assuming isotropy):

$$\frac{\partial F_i}{\partial t} - Hp\frac{\partial F_i}{\partial p} = C_i[F_i, F_j, p] \tag{1.1}$$

where $F_i(p)$ is the number distribution of particle i as function of momentum p, C represents a source/sink term and H is the Hubble constant:

$$H = \sqrt{\frac{8\pi}{3}\frac{\rho_T}{M_P^2}} \tag{1.2}$$

with $M_P = 1.22 \times 10^{19}$ GeV and $\rho_T = \sum_i \rho_i$. The number, energy and pressure densities are given in terms of F_i as:

$$\begin{aligned} n_i(t) &= \int \frac{dp}{2\pi^2} p^2 F_i(p) \\ \rho_i(t) &= \int \frac{dp}{2\pi^2} p^2 E_i F_i(p) \\ P_i(t) &= \frac{1}{3}\int \frac{dp}{2\pi^2}\frac{p^4}{E_i} F_i(p) \end{aligned} \tag{1.3}$$

where m_i is the mass of particle i and $E_i = \sqrt{p_i^2 + m_i^2}$. Using Eq.(1.1), we obtain the following equations for the number and energy densities:

$$\begin{aligned} \frac{dn_i}{dt} + 3Hn_i &= \int \frac{dp}{2\pi^2} p^2 C_i \\ \frac{d\rho_i}{dt} + 3H(\rho_i + P_i) &= \int \frac{dp}{2\pi^2} p^2 E_i C_i \end{aligned} \tag{1.4}$$

The collision term, C_i, for the process $i + f + \ldots \leftrightarrow a + b + c + \ldots$ is given by[2]:

$$\begin{aligned} C_i = &\frac{1}{E_i}\int \prod_{j,a} \frac{d^3p_j}{2E_j(2\pi)^3}\frac{d^3p_a}{2E_a(2\pi)^3}(2\pi)^4\delta^4\left(p_i + p_j + \ldots - p_a - p_b \ldots\right)|\mathcal{M}|^2 \\ &\times \left[(1 \pm f_a)(1 \pm f_b)\ldots f_i f_j \ldots - f_a f_b \ldots (1 \pm f_i)(1 \pm f_j)\ldots\right] \end{aligned} \tag{1.5}$$

where the plus (minus) sign is for bosons (fermions). Below we always assume $f_{i,j,a,..} \ll 1$, so:

$$\begin{aligned} C_i \simeq &\frac{1}{E_i}\int \prod_{j,a} \frac{d^3p_j}{2E_j(2\pi)^3}\frac{d^3p_a}{2E_a(2\pi)^3}(2\pi)^4\delta^4\left(p_i + p_j + \ldots - p_a - p_b \ldots\right)|\mathcal{M}|^2 \\ &\times \left[f_i f_j \ldots - f_a f_b \ldots\right] \end{aligned} \tag{1.6}$$

We will assume that C is given by:

$$C = C_{dec} + C_{prod} + C_{ann} \tag{1.7}$$

where C_{dec} contains the contributions from decays and inverse decays ($i \leftrightarrow a+b+\ldots$), C_{prod} contains the contributions from decay injection and inverse decay injection ($a \leftrightarrow i+b+\ldots$) and C_{ann} from annihilations with the thermal plasma ($i + i \leftrightarrow a + b$). Below we compute each term separately, under some assumptions.

1.1 Annihilation Term

The annihilation term C_{ann} for the $i + j \leftrightarrow a + b$ process is given by[1]:

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} = \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(p_i + p_j - p_a - p_b) |M|^2 \left[f_a f_b - f_i f_j \right] \tag{1.8}$$

where $d\Pi_i = d^3 p_i / ((2\pi)^3 2E_i)$. Since we are ultimately interested in Eqs.(1.4) for the number and energy densities, we will consider the following integral:

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha = \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(p_i + p_j - p_a - p_b) |M|^2 \left[f_a f_b - f_i f_j \right] E_i^\alpha \tag{1.9}$$

where $\alpha = 0(1)$ for the number (energy) density. Here we assume that the distributions can be approximated by[1]:

$$f_i \simeq \exp(-(E_i - \mu_i)/T) \tag{1.10}$$

so the annihilation term can then be written as:

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha &= - \left(\exp((\mu_i + \mu_j)/T) - \exp((\mu_a + \mu_b)/T) \right) \\ \times \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}&(p_i + p_j - p_a - p_b) |M|^2 \exp(-(E_i + E_j)/T) \times E_i^\alpha \end{aligned}$$

where above we have used conservation of energy ($E_i + E_j = E_a + E_b$). Assuming[2]:

$$\frac{n_i}{\bar{n}_i} = \exp(\mu_i / T) \tag{1.11}$$

we obtain:

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha &= - \left(\frac{n_i n_j}{\bar{n}_i \bar{n}_j} - \frac{n_a n_b}{\bar{n}_a \bar{n}_b} \right) \\ \times \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}&(p_i + p_j - p_a - p_b) |M|^2 \exp(-(E_i + E_j)/T) \times E_i^\alpha \end{aligned}$$

In order to include convertion, co-annihilation and annihilation of BSM particles we must consider the cases:

- $i = j = BSM$ and $a, b = SM, SM$:

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha = - \left(n_i^2 - \bar{n}_i^2 \right) \langle \sigma v E_i^\alpha \rangle_{ii} \tag{1.12}$$

- $i \neq j$, $i, j = BSM$ and $a, b = SM, SM$:

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha = - \left(n_i n_j - \bar{n}_i \bar{n}_j \right) \langle \sigma v E_i^\alpha \rangle_{ij} \tag{1.13}$$

[1] This approximation is only valid for particles with a thermal distribution. However, since the annihilation term is responsible for keeping the particle i in thermal equilibrium with the plasma, it is reasonable to assume a thermal distribution for i while the annihilation term is relevant.

[2] Note that in a more general case one can have $T_i \neq T$ and this approximation is not valid.

- $i = j = BSM$, $a = b = BSM$:

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^{\alpha} = -\left(n_i^2 - \frac{\bar{n}_i^2}{\bar{n}_j^2} n_j^2\right) \langle \sigma v E_i^{\alpha} \rangle_{jj} \tag{1.14}$$

- $i = BSM$, $j = SM$, $a = BSM$, $b = SM$ and $a \neq i$:

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^{\alpha} = -\left(n_i - \frac{\bar{n}_i}{\bar{n}_j} n_j\right) \bar{n}_{SM} \langle \sigma v E_i^{\alpha} \rangle_{ijSM} \equiv -\left(n_i - \frac{\bar{n}_i}{\bar{n}_j} n_j\right) \tilde{\Gamma}_{ijSM} \tag{1.15}$$

where:

$$\begin{aligned}
\langle \sigma v E_i^{\alpha} \rangle_i &= \tfrac{1}{\bar{n}_i^2} \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(P) |M|^2 e^{-(E_i+E_j)/T} \times E_i^{\alpha} && (1.16) \\
\langle \sigma v E_i^{\alpha} \rangle_{ij} &= \tfrac{1}{\bar{n}_i \bar{n}_j} \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(P) |M|^2 e^{-(E_i+E_j)/T} \times E_i^{\alpha} && (1.17) \\
\langle \sigma v E_i^{\alpha} \rangle_{jj} &= \tfrac{1}{\bar{n}_i^2} \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(P) |M|^2 e^{-(E_i+E_j)/T} \times E_i^{\alpha} && (1.18) \\
\tilde{\Gamma}_{ijSM} &= \tfrac{1}{\bar{n}_i} \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(P) |M|^2 e^{-(E_i+E_j)/T} \times E_i^{\alpha} && (1.19)
\end{aligned}$$

For $\alpha = 0$, the above equation is the well known contribution from thermal scatterings to the annihilation term. To estimate its value for $\alpha = 1$, we assume:

$$\langle \sigma v E \rangle \simeq \langle \sigma v \rangle \langle E_i \rangle = \langle \sigma v \rangle \frac{\rho_i}{n_i} \tag{1.20}$$

where $\langle\ \rangle$ represents thermal average.

1.2 Decay Term

Now we derive a simplified expression for the decay (and inverse decay) term, under approximations similar to the ones used in the last section. The decay term includes the contributions from particle decay and inverse decay[2, 3]:

$$C_{dec} \simeq \frac{1}{E_i} \int \prod_a \frac{d^3 p_a}{2E_a (2\pi)^3} (2\pi)^4 \delta^4 (p_i - p_a - p_b \ldots) |\mathcal{M}|^2 [f_i - f_a f_b \ldots] \tag{1.21}$$

As in the case of the annihilation term, we assume that the distributions for $a, b, \ldots$ can be approximated by $f_x \simeq \exp(-(E_x - \mu_x)/T)$, so we can write:

$$f_a f_b \ldots \simeq \exp\left(\frac{\mu_a + \mu_b + \ldots}{T}\right) \exp(-E_i/T) = \frac{n_a n_b \ldots}{\bar{n}_a \bar{n}_b \ldots} \exp(-E_i/T) = \frac{n_a n_b \ldots}{\bar{n}_a \bar{n}_b \ldots} \bar{f}_i \tag{1.22}$$

where we used conservation of energy ($E_a + E_b + \ldots = E_i$) and $\bar{f}_i$ is the equilibrium distribution for the species i. Hence we can write Eq.(1.21) as:

$$\begin{aligned}
C_{dec} &\simeq \left[f_i - \frac{n_a n_b \ldots}{\bar{n}_a \bar{n}_b \ldots} \bar{f}_i\right] \frac{1}{E_i} \int \prod_a \frac{d^3 p_a}{2E_a (2\pi)^3} (2\pi)^4 \delta^4 (p_i - p_a - p_b \ldots) |\mathcal{M}|^2 \\
&= \mathcal{B}_{ab\ldots} \frac{\Gamma_i m_i}{E_i} \left[f_i - \frac{n_a n_b \ldots}{\bar{n}_a \bar{n}_b \ldots} \bar{f}_i\right]
\end{aligned} \tag{1.23}$$

where Γ_i is the width for i and $\mathcal{B}_{ab\ldots} \equiv BR(i \to a + b + \ldots)$

Once again we consider the integral:

$$\int \frac{dp}{2\pi^2} p^2 C_{dec}(p) E_i^\alpha = -\,\Gamma_i \int \frac{dp}{2\pi^2} p^2 \frac{m_i}{E_i} f_i E_i^\alpha + \sum_{i\ decays} \mathcal{B}_{ab\ldots} \Gamma_i \frac{n_a n_b \ldots}{\bar{n}_a \bar{n}_b \ldots} \int \frac{dp}{2\pi^2} p^2 \frac{m_i}{E_i} \bar{f}_i E_i^\alpha \tag{1.24}$$

where we have included the sum over all decay channels and $\alpha = 0(1)$ for the contribution to the number (energy) density equation. Note that both integrals are identical, except for the replacement $f_i \to \bar{f}_i$. The first integral in Eq.(1.24) gives:

$$-\Gamma_i \int \frac{dp}{2\pi^2} p^2 \frac{m_i}{E_i} f_i(p) E_i^\alpha = \begin{cases} -\Gamma_i m_i n_i \langle \frac{1}{E_i} \rangle \text{ , for } \alpha = 0 \\ -\Gamma_i m_i n_i \text{ , for } \alpha = 1 \end{cases} \tag{1.25}$$

where

$$\langle \frac{1}{E_i} \rangle \equiv \frac{1}{n_i} \int \frac{dp}{2\pi^2} p^2 \frac{1}{E_i} f_i(p) \tag{1.26}$$

Hence we can write Eq.(1.24) as:

$$\int \frac{dp}{2\pi^2} p^2 C_{dec}(p) E_i^\alpha = -\Gamma_i m_i \begin{cases} n_i \langle \frac{1}{E_i} \rangle - \bar{n}_i \langle \frac{1}{E_i} \rangle_{eq} \sum \mathcal{B}_{ab\ldots} \frac{n_a n_b \ldots}{\bar{n}_a \bar{n}_b \ldots} & \text{, for } \alpha = 0 \\ n_i - \bar{n}_i \sum \mathcal{B}_{ab\ldots} \frac{n_a n_b \ldots}{\bar{n}_a \bar{n}_b \ldots} & \text{, for } \alpha = 1 \end{cases} \tag{1.27}$$

For the non-equilibrium average we assume:

$$\langle \frac{1}{E_i} \rangle \simeq \frac{1}{\langle E_i \rangle} = \frac{n_i}{\rho_i} \tag{1.28}$$

which is exact in the non-relativistic limit, but it is only an approximation for the relativistic case. Although we can compute the equilibrium average ($\langle \frac{1}{E_i} \rangle_{eq}$) explicitly, in order to have an exact cancellation between the decay and inverse decay terms when i, a and b are all in equilibrium, we take:

$$\langle \frac{1}{E_i} \rangle_{eq} \simeq \langle \frac{1}{E_i} \rangle = \frac{n_i}{\rho_i} \tag{1.29}$$

With the above approximations we finally obtain:

$$\int \frac{dp}{2\pi^2} p^2 C_{dec}(p) E_i^\alpha = -\Gamma_i m_i \begin{cases} \frac{n_i}{\rho_i} \left(n_i - \bar{n}_i \sum \mathcal{B}_{ab\ldots} \frac{n_a n_b \ldots}{\bar{n}_a \bar{n}_b \ldots} \right) & \text{, for } \alpha = 0 \\ n_i - \bar{n}_i \sum \mathcal{B}_{ab\ldots} \frac{n_a n_b \ldots}{\bar{n}_a \bar{n}_b \ldots} & \text{, for } \alpha = 1 \end{cases} \tag{1.30}$$

where $\mathcal{B}_{ab\ldots} \equiv BR(i \to a + b + \ldots)$.

1.3 Production Term

The decay and inverse decay of other particles ($a \to i + b + \ldots$) can also affect the species i. The contribution from these terms we label C_{prod}, which is given by[2]:

$$C_{prod} \simeq \frac{1}{E_i} \int \frac{d^3 p_a}{2E_a (2\pi)^3} \prod_b \frac{d^3 p_b}{2E_b (2\pi)^3} (2\pi)^4 \delta^4 \left(p_a - p_i - p_b \ldots\right) |\mathcal{M}|^2 \left[f_a - f_i f_b \ldots\right] \tag{1.31}$$

Using the same approximations of the previous section, we write:

$$f_i f_b \ldots \simeq \frac{n_i n_b \ldots}{\bar{n}_i \bar{n}_b \ldots} e^{-E_a/T} = \frac{n_i n_b \ldots}{\bar{n}_i \bar{n}_b \ldots} \bar{f}_a \tag{1.32}$$

Hence:

$$C_{prod} = \frac{1}{E_i} \int \frac{d^3 p_a}{2E_a (2\pi)^3} \prod_b \frac{d^3 p_b}{2E_b (2\pi)^3} (2\pi)^4 \delta^4 \left(p_a - p_i - p_b \ldots\right) |\mathcal{M}|^2 \left(f_a - \bar{f}_a \frac{n_i n_b \ldots}{\bar{n}_i \bar{n}_b \ldots} \right) \tag{1.33}$$

and

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{prod}(p) E_i^\alpha &= \int \frac{d^3 p_a}{E_a (2\pi)^3} \left(f_a - \bar{f}_a \frac{n_i n_b \ldots}{\bar{n}_i \bar{n}_b \ldots} \right) \\ &\times \frac{d^3 p E_i^\alpha}{2E_i (2\pi)^3} \prod_b \frac{d^3 p_b}{2E_b (2\pi)^3} (2\pi)^4 \delta^4 \left(p_a - p_i - p_b \ldots\right) |\mathcal{M}|^2 \end{aligned} \tag{1.34}$$

with $\alpha = 0(1)$ for the contribution to the number (energy) density equation. For $\alpha = 0$ we obtain:

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{prod}(p) &= \Gamma_a \mathcal{B}_i m_a \int \frac{d^3 p_a}{E_a (2\pi)^3} \left(f_a - \bar{f}_a \sum_b \frac{\mathcal{B}_{ib\ldots}}{\mathcal{B}_i} \frac{n_i n_b \ldots}{\bar{n}_i \bar{n}_b \ldots} \right) \\ &= \Gamma_a \mathcal{B}_i m_a \frac{n_a}{\rho_a} \left(n_a - \bar{n}_a \sum_b \frac{\mathcal{B}_{ib\ldots}}{\mathcal{B}_i} \frac{n_i n_b \ldots}{\bar{n}_i \bar{n}_b \ldots} \right) \end{aligned} \tag{1.35}$$

where $\mathcal{B}_{ib\ldots} \equiv BR(a \to i + b + \ldots)$, $\mathcal{B}_i = \sum_b \mathcal{B}_{ib\ldots}$ and we have once again assumed $\langle 1/E_a \rangle \simeq \langle 1/E_a \rangle_{eq} \simeq n_a/\rho_a$.

For $\alpha = 1$, the integral in Eq.(1.34) does not take a simple form. In order to compute it, we assume:

$$E_i \simeq \frac{m_a^2 + m_i^2}{2m_a} \simeq \frac{E_a}{2} \left(1 + \frac{m_i^2}{m_a^2} \right) \equiv e_{inj} E_a, \tag{1.36}$$

where e_{inj} is the fraction of energy injected into the i-species. The above expression is only exact for 2-body decays and $m_a, m_i \gg m_b$. For the remaining cases, it is only an estimate.

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{prod}(p) E_i &\simeq \Gamma_a \mathcal{B}_i \frac{m_a}{2} \int \frac{d^3 p_a}{(2\pi)^3} \left(f_a - \bar{f}_a \sum_b \frac{\mathcal{B}_{ib\ldots}}{\mathcal{B}_i} \frac{n_i n_b \ldots}{\bar{n}_i \bar{n}_b \ldots} \right) \\ &= \Gamma_a \mathcal{B}_i e_{inj} m_a \left(n_a - \bar{n}_a \sum_b \frac{\mathcal{B}_{ib\ldots}}{\mathcal{B}_i} \frac{n_i n_b \ldots}{\bar{n}_i \bar{n}_b \ldots} \right) \end{aligned} \tag{1.37}$$

Combining the results for $\alpha = 0$ and 1, we have:

$$\int \frac{dp}{2\pi^2} p^2 C_{prod}(p) E_i^\alpha = \Gamma_a \mathcal{B}_i m_a \left(n_a - \bar{n}_a \sum_b \frac{\mathcal{B}_{ib\ldots}}{\mathcal{B}_i} \frac{n_i n_b \ldots}{\bar{n}_i \bar{n}_b \ldots} \right) \begin{cases} \frac{n_a}{\rho_a} & \text{, for } \alpha = 0 \\ e_{inj} & \text{, for } \alpha = 1 \end{cases} \tag{1.38}$$

1.4 Number and Energy Density Equations

Using the results of Eqs.(1.12)-(1.15), (1.30) and (1.38) in the Boltzmann equations for n_i and ρ_i (Eq.(1.4)), we obtain:

$$
\begin{aligned}
\frac{dn_i}{dt} + 3Hn_i =& \left(\bar{n}_i^2 - n_i^2\right) \langle\sigma v\rangle_{ii} + \sum_{j\neq i} \left(\bar{n}_i\bar{n}_j - n_i n_j\right) \langle\sigma v\rangle_{ij} + \sum_{j\neq i} \left(\frac{\bar{n}_i^2}{\bar{n}_j^2} n_j^2 - n_i^2\right) \langle\sigma v\rangle_{jj} \\
&+ \sum_{j\neq i} \left(\frac{\bar{n}_i}{\bar{n}_j} n_j - n_i\right) \tilde{\Gamma}_{ijSM} - \Gamma_i m_i \frac{n_i}{\rho_i} \left(n_i - \bar{n}_i \sum_{i\to\ldots} \mathcal{B}_{ab\ldots} \frac{n_a n_b \ldots}{\bar{n}_a \bar{n}_b \ldots}\right) \\
&+ \sum_a \Gamma_a \mathcal{B}_i m_a \frac{n_a}{\rho_a} \left(n_a - \bar{n}_a \sum_{a\to i\ldots} \frac{\mathcal{B}_{ib\ldots}}{\mathcal{B}_i} \frac{n_i n_b \ldots}{\bar{n}_i \bar{n}_b \ldots}\right) + C_i(T) \qquad (1.39)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\rho_i}{dt} + 3H(\rho_i + P_i) =& \frac{\rho_i}{n_i} \left(\bar{n}_i^2 - n_i^2\right) \langle\sigma v\rangle_{ii} + \frac{\rho_i}{n_i} \left(\bar{n}_i\bar{n}_j - n_i n_j\right) \langle\sigma v\rangle_{ij} + \frac{\rho_i}{n_i} \left(\frac{\bar{n}_i^2}{\bar{n}_j^2} n_j^2 - n_i^2\right) \langle\sigma v\rangle_{jj} \\
&+ \frac{\rho_i}{n_i} \left(\frac{\bar{n}_i}{\bar{n}_j} n_j - n_i\right) \tilde{\Gamma}_{ijSM} - \sum_a \Gamma_a \mathcal{B}_i e_{inj} m_a \left(n_a - \bar{n}_a \sum_{a\to i\ldots} \frac{\mathcal{B}_{ib\ldots}}{\mathcal{B}_i} \frac{n_i n_b ..}{\bar{n}_i \bar{n}_b ..}\right) \\
&+ \tilde{C}_i(T) \frac{\rho_i}{n_i} \qquad (1.40)
\end{aligned}
$$

where $\mathcal{B}_{ab\ldots} = BR(i \to a + b + \ldots)$, $\mathcal{B}_{ib\ldots} = BR(a \to i + b + \ldots)$, $\mathcal{B}_i = \sum_b \mathcal{B}_{ib\ldots}$ and we have included an extra term (C_i and $\tilde{C}_i$) to allow for other possible sources for the number and energy densities. For simplicity we assume $C_i = \tilde{C}_i$ from now on.

It is also convenient to use the above results to obtain a simpler equation for ρ_i/n_i:

$$
\frac{d\rho_i/n_i}{dt} \equiv \frac{dR_i}{dt} = -3H\frac{P_i}{n_i} + \sum_a \mathcal{B}_i \frac{\Gamma_a m_a}{n_i} \left(e_{inj} - \frac{n_a}{\rho_a}\frac{\rho_i}{n_i}\right) \left(n_a - \bar{n}_a \sum_{a\to i\ldots} \frac{\mathcal{B}_{ib\ldots}}{\mathcal{B}_i} \frac{n_i n_b ..}{\bar{n}_i \bar{n}_b ..}\right) \qquad (1.41)
$$

Besides the above equations, it is useful to consider the evolution equation for entropy:

$$
dS \equiv \frac{dQ^{dec}}{T} \qquad (1.42)
$$

where dQ^{dec} is the net energy injected from decays. With the above definition we have:

$$
\begin{aligned}
\dot{S} &= \frac{1}{T} \sum_i BR(i, X) \frac{d\left(R^3 \rho_i\right)^{dec}}{dt} \\
\Rightarrow \dot{S} &= \frac{R^3}{T} \sum_i BR(i, X) \Gamma_i m_i \left(n_i - \bar{n}_i \sum_{i\to\ldots} \mathcal{B}_{ab\ldots} \frac{n_a n_b \ldots}{\bar{n}_a \bar{n}_b \ldots}\right) \qquad (1.43)
\end{aligned}
$$

where R is the scale factor and $BR(i, X)$ is the fraction of energy injected in the thermal bath from i decays.

The above expressions can be written in a more compact form if we define the following "effective thermal densities" and "effective BR":

$$
\mathcal{N}_X^{th} \equiv \bar{n}_X \sum_{X\to\ldots} BR(X \to 1 + 2 + \ldots) \prod_k \frac{n_k}{\bar{n}_k}
$$

$$
\begin{aligned}
\mathcal{N}_{XY}^{th} &\equiv \frac{\bar{n}_X}{\mathcal{B}_{XY}^{eff}} \sum_{X \to Y + \ldots} g_Y BR(X \to g_Y Y + 1 + \ldots) \left(\frac{n_Y}{\bar{n}_Y}\right)^{g_Y} \prod_k \frac{n_k}{\bar{n}_k} \\
\mathcal{B}_{XY}^{eff} &\equiv \sum_{X \to Y + \ldots} g_Y BR(X \to g_Y Y + 1 + \ldots)
\end{aligned}
$$

where g_Y is the Y multiplicity in the final state of X decays. In addition, defining:

$$
x = \ln(R/R_0), \quad N_i = \ln(n_i/s_0), \quad \text{and} \quad N_S = \ln(S/S_0) \tag{1.44}
$$

we can write Eqs.(1.43), (1.39) and (1.41) as:

$$
N_S' = \frac{e^{(3x - N_S)}}{HT} \sum_i BR(i, X)\Gamma_i m_i \left(n_i - \mathcal{N}_i^{th}\right) \tag{1.45}
$$

$$
\begin{aligned}
N_i' &= -3 + \frac{n_i}{H}\left(\frac{\bar{n}_i^2}{n_i^2} - 1\right)\langle\sigma v\rangle_{ii} + \frac{1}{H}\sum_{j\neq i}\left(\frac{\bar{n}_i}{n_i}\bar{n}_j - n_j\right)\langle\sigma v\rangle_{ij} \\
&+ \frac{n_i}{H}\sum_{j\neq i}\left(\frac{\bar{n}_i^2}{n_i^2}\frac{n_j^2}{\bar{n}_j^2} - 1\right)\langle\sigma v\rangle_{jj} + \frac{1}{H}\sum_{j\neq i}\left(\frac{\bar{n}_i}{n_i}\frac{n_j}{\bar{n}_j} - 1\right)\tilde{\Gamma}_{ijSM} \\
&- \frac{\Gamma_i}{H}\frac{m_i}{R_i}\left(1 - \frac{\mathcal{N}_i^{th}}{n_i}\right) + \sum_a \mathcal{B}_{ai}^{eff}\frac{\Gamma_a}{H}\frac{m_a}{R_a}\left(\frac{n_a}{n_i} - \frac{\mathcal{N}_{ai}^{th}}{n_i}\right)
\end{aligned} \tag{1.46}
$$

$$
R_i' = -3\frac{P_i}{n_i} + \sum_a \mathcal{B}_{ai}^{eff}\frac{\Gamma_a}{H}m_a\left(e_{inj} - \frac{R_i}{R_a}\right)\left(\frac{n_a}{n_i} - \frac{\mathcal{N}_{ai}^{th}}{n_i}\right) \tag{1.47}
$$

where $' = d/dx$.

The above equation for N_i also applies for coherent oscillating fields, if we define:

$$
N_i = \ln(n_i/s_0), \quad \text{and} \quad n_i \equiv \rho_i/m_i \tag{1.48}
$$

so

$$
\begin{aligned}
N_i' &= -3 - \frac{\Gamma_i}{H} \\
R_i' &= 0
\end{aligned} \tag{1.49}
$$

where we assume that the coherent oscillating component does not couple to any of the other fields.

Collecting Eqs.(1.45)-(1.47) and (1.49) we have a closed set of first order differential equations:

- Entropy:

$$
N_S' = \frac{e^{(3x - N_S)}}{HT} \sum_i BR(i, X)\Gamma_i m_i \left(n_i - \mathcal{N}_i^{th}\right) \tag{1.50}
$$

- Thermal fields:

$$
N_i' = -3 + \frac{n_i}{H}\left(\frac{\bar{n}_i^2}{n_i^2} - 1\right)\langle\sigma v\rangle_{ii} + \frac{1}{H}\sum_{j\neq i}\left(\frac{\bar{n}_i}{n_i}\bar{n}_j - n_j\right)\langle\sigma v\rangle_{ij} + \frac{n_i}{H}\sum_{j\neq i}\left(\frac{\bar{n}_i^2}{n_i^2}\frac{n_j^2}{\bar{n}_j^2} - 1\right)\langle\sigma v\rangle_{jj}
$$

$$+\frac{1}{H}\sum_{j\neq i}\left(\frac{\bar{n}_i}{n_i}\frac{n_j}{\bar{n}_j}-1\right)\tilde{\Gamma}_{ijSM}-\frac{\Gamma_i}{H}\frac{m_i}{R_i}\left(1-\frac{\mathcal{N}_i^{th}}{n_i}\right)+\sum_a \mathcal{B}_{ai}^{eff}\frac{\Gamma_a}{H}\frac{m_a}{R_a}\left(\frac{n_a}{n_i}-\frac{\mathcal{N}_{ai}^{th}}{n_i}\right)$$

$$R_i' = -3\frac{P_i}{n_i}+\sum_a \mathcal{B}_{ai}^{eff}\frac{\Gamma_a}{H}m_a\left(e_{inj}-\frac{R_i}{R_a}\right)\left(\frac{n_a}{n_i}-\frac{\mathcal{N}_{ai}^{th}}{n_i}\right) \tag{1.51}$$

- Coherent Oscillating fields:

$$\begin{aligned} N_i' &= -3-\frac{\Gamma_i}{H} \\ R_i' &= 0 \end{aligned} \tag{1.52}$$

As seen above, the equation for $R_i = \rho_i/n_i$ depends on P_i/n_i. A proper evaluation of this quantity requires knowledge of the distribution $F_i(p,t)$. However, for relativistic (or massless) particles we have $P_i = \rho_i/3$, as seen from Eq.(1.3), while for particles at rest we have $P_i = 0$. Hence $F_i(p,t)$ is only required to evaluate the relativistic/non-relativistic transition, which corresponds to a relatively small part of the evolution history of particle i. Nonetheless, to model this transition we approximate F_i by a thermal distribution and take $T_i, \mu_i \ll m_i$, where T_i is the temperature of the particle (which can be different from the thermal bath's). Under these approximations we have:

$$\begin{aligned} \frac{P_i}{n_i} &= T_i \\ \frac{\rho_i}{n_i} &= T_i\left[\frac{K_1(m_i/T_i)}{K_2(m_i/T_i)}\frac{m_i}{T_i}+3\right] \end{aligned} \tag{1.53}$$

where $K_{1,2}$ are the modified Bessel functions. In particular, if $m_i/T_i \gg 1$:

$$\frac{\rho_i}{n_i} \simeq T_i\left[\frac{3}{2}+\frac{m_i}{T_i}+3\right] \Rightarrow \frac{P_i}{n_i} = T_i = \frac{2m_i}{3}\left(\frac{R_i}{m_i}-1\right) \tag{1.54}$$

As shown above, for a given value of $R_i = \rho_i/n_i$, Eq.(1.53) can be inverted to compute T_i $(= P_i/n_i)$:

$$\frac{P_i}{n_i} = T_i(R_i) \tag{1.55}$$

Since we are interested in the non-relativistic/relativistic transition, we can expand the above expression around $R_i/m_i = 1$, so P_i/n_i can be written as:

$$\frac{P_i}{n_i} = \frac{2m_i}{3}\left(\frac{R_i}{m_i}-1\right)+m_i\sum_{n>1}a_n\left(\frac{R_i}{m_i}-1\right)^n \tag{1.56}$$

where the coefficients a_n can be numerically computed from Eq.(1.53). The above approximation should be valid for $m_i/T_i \gtrsim 1$ (or $R_i \gtrsim m_i$). On the other hand, for $m_i/T_i \ll 1$ (or $R_i \gg m_i$), we have the relativistic regime, with $P_i/n_i = R_i/3$. Therefore we can approximate the P_i/n_i function for all values of R_i by:

$$\frac{P_i}{n_i} = \begin{cases} \frac{2m_i}{3}\left(\frac{R_i}{m_i}-1\right)+m_i\sum_{n>1}a_n\left(\frac{R_i}{m_i}-1\right)^n \text{ , for } R_i < \tilde{R} \\ \frac{R_i}{3} \text{ , for } R_i > \tilde{R} \end{cases} \tag{1.57}$$

where the coefficients a_n are given by the numerical fit of Eq.(1.53) and $\tilde{R}$ is given by the matching of the two solutions.

Finally, to solve Eqs.(1.50)-(1.52) we need to compute H according to Eq.(1.2), which requires knowledge of the energy densities for all particles (ρ_i) and for the thermal bath (ρ_R). The former are directly obtained from N_i and R_i, while the latter can be computed from N_S:

$$T = \left(\frac{g_{*S}(T_R)}{g_{*S}(T)}\right)^{1/3} T_R \exp[N_S/3 - x] \Rightarrow \rho_R = \frac{\pi^2}{30} g_*(T) T^4 \tag{1.58}$$

Eqs.(1.50)-(1.52), with the auxiliary equations for H (Eq.(1.2)) and P_i/n_i (Eq.(1.57)) form a set of closed equations, which can be solved once the initial conditions for the number density (n_i), energy density (ρ_i) and entropy (S) are given. For thermal fluids we assume:

$$n_i(T_R) = \begin{cases} 0 & , \text{ if } \langle\sigma v\rangle_i \bar{n}_i/H|_{T=T_R} < 10 \\ \bar{n}_i(T_R) & , \text{ if } \langle\sigma v\rangle_i \bar{n}_i/H|_{T=T_R} > 10 \end{cases} \tag{1.59}$$

$$\frac{\rho_i}{n_i}(T_R) = \frac{\bar{\rho}_i}{\bar{n}_i}(T_R) \tag{1.60}$$

where $\bar{\rho}_i$ is the equilibrium energy density (with zero chemical potential) for the particle i. While for coherent oscillating fluids the initial condition is set at the beginning of oscillations:

$$n_i(T_i^{osc}) = \frac{\rho_i^0}{m_i(T_i^{osc})} \tag{1.61}$$

$$\frac{\rho_i}{n_i}(T_i^{osc}) = m_i \tag{1.62}$$

where T_i^{osc} is the oscillation temperature, given by $3H(T_i^{osc}) = m_i(T_i^{osc})$ and ρ_i^0 the initial energy density for oscillations.

Finally, the initial condition for the entropy S is trivially obtained, once we assume a radiation dominated universe at $T = T_R$:

$$S(T_R) = \frac{2\pi^2}{45} g_*(T_R) T_R^3 R_0^3 \tag{1.63}$$

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