Cosmology
Lecture 3

The Friedmann Equation

Lecture 3 Learning Objectives

- Understand how a knowledge of $a(t)$ (i.e., the full evolution of the scale factor) enables us to determine proper distance.

- Measuring the curvature of the universe via its effect on how distant objects are perceived.

- How the Friedmann equation relates the time-dependent scale factor and curvature of a universe to its content.

Relating proper distance to scale factor

In the last two lectures, we related proper distance to the co-moving coordinate:

$$d_p(t_0) = a(t_0) \int_0^r dr = a(t_0) r = r$$

But, how do we calculate r?
As a photon travels through an expanding universe, it traverses lots of dr's. And the RW metric tells us that, if it travels along a radial path toward us:

$$a(t) dr = c dt \qquad \text{or} \qquad dr = \frac{c dt}{a(t)}$$

Integrating gives:

$$r = c \int_{t_{\rm em}}^{t_{\rm ob}} \frac{dt}{a(t)}$$

Meaning:

$$\boxed{d_p(t_0) = c \int_{t_{\rm em}}^{t_{\rm ob}} \frac{dt}{a(t)}}$$

Three key numbers

RW metric in 4D (i.e., spacetime) is:

$$ds^2 = -c^2dt^2 + a(t)^2\left[dr^2 + S_\kappa(r)^2 d\Omega^2\right]$$

where

$$S_\kappa(r) = \begin{cases} R_0 sin(r/R_0) \text{ if } \kappa > 0 & \text{+ve curvature} \\ r \text{ if } \kappa = 0 & \text{flat} \\ R_0 sinh(r/R_0) \text{ if } \kappa < 0 & \text{-ve curvature} \end{cases}$$

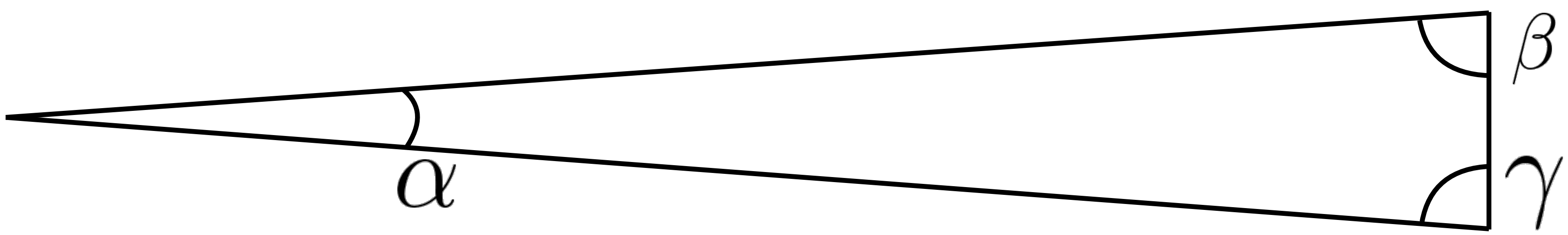
to calculate the distances between co-moving coordinates $r \;\; \theta \;\; \phi$, all we need is:

κ - sign of curvature (+1, 0, -1)

R_0 - radius of curvature

$a(t)$ - scale factor (how the Universe expands or contracts over time)

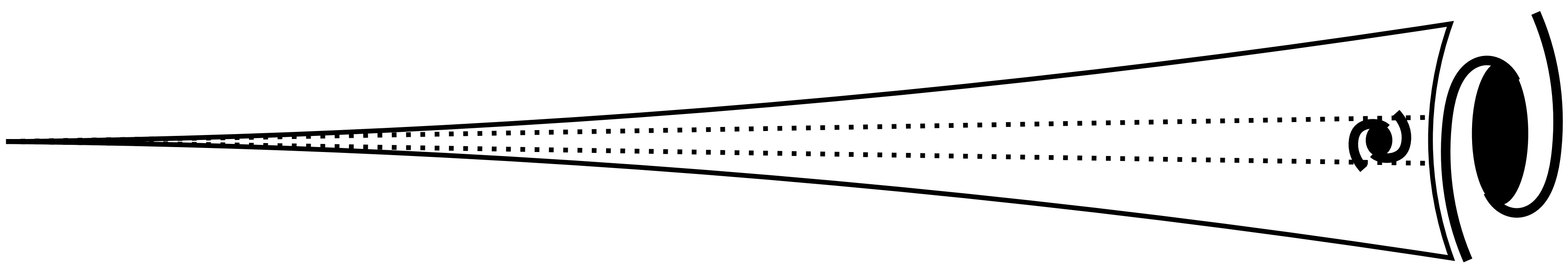
Measuring curvature



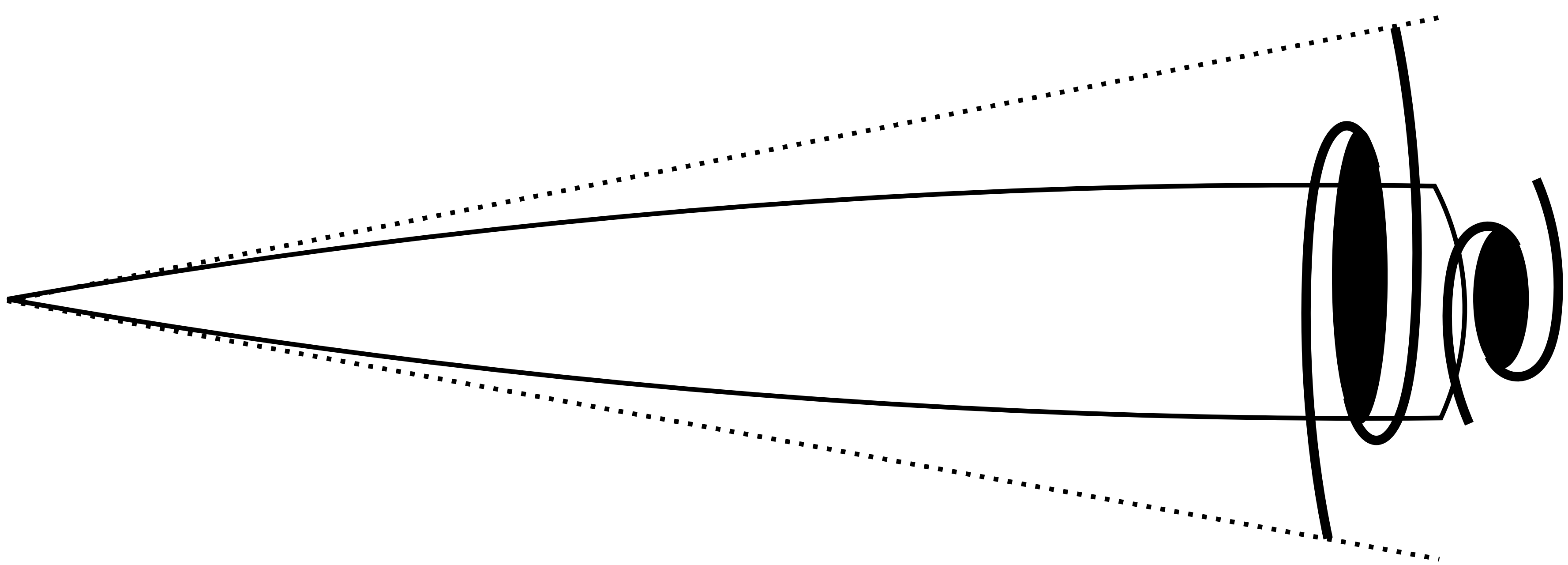
Flat

$$\alpha + \beta + \gamma = \pi + \frac{\kappa A}{R_0}$$

A = area of triangle



-ve curvature
Distant galaxies *appear* smaller than nearby ones



+ve curvature
Distant galaxies *appear* larger than nearby ones

The curvature and content of the Universe

General relativity tells us that the curvature of the Universe is explicitly linked to its energy content (where mass is energy via $E=mc^2$).

The **Field Equation** links the two. It is the G.R. equivalent to the Poisson Equation in Newtonian dynamics:

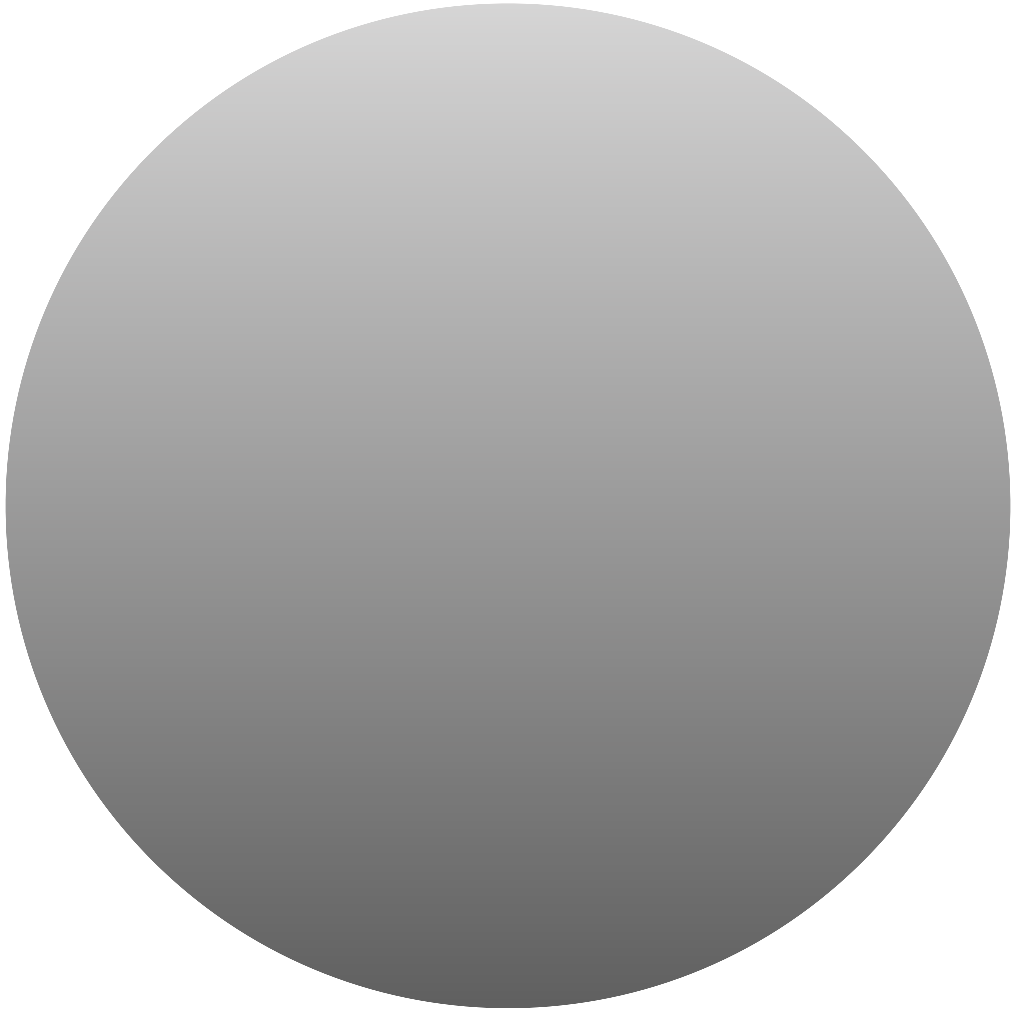
$$\nabla^2\phi = 4\pi G\rho$$

Poission equation relates gravitational potential ϕ to density ρ

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Field equation relates the curvature $G_{\mu\nu}$ to the "stress-energy" $T_{\mu\nu}$

The Universe as a perfect gas



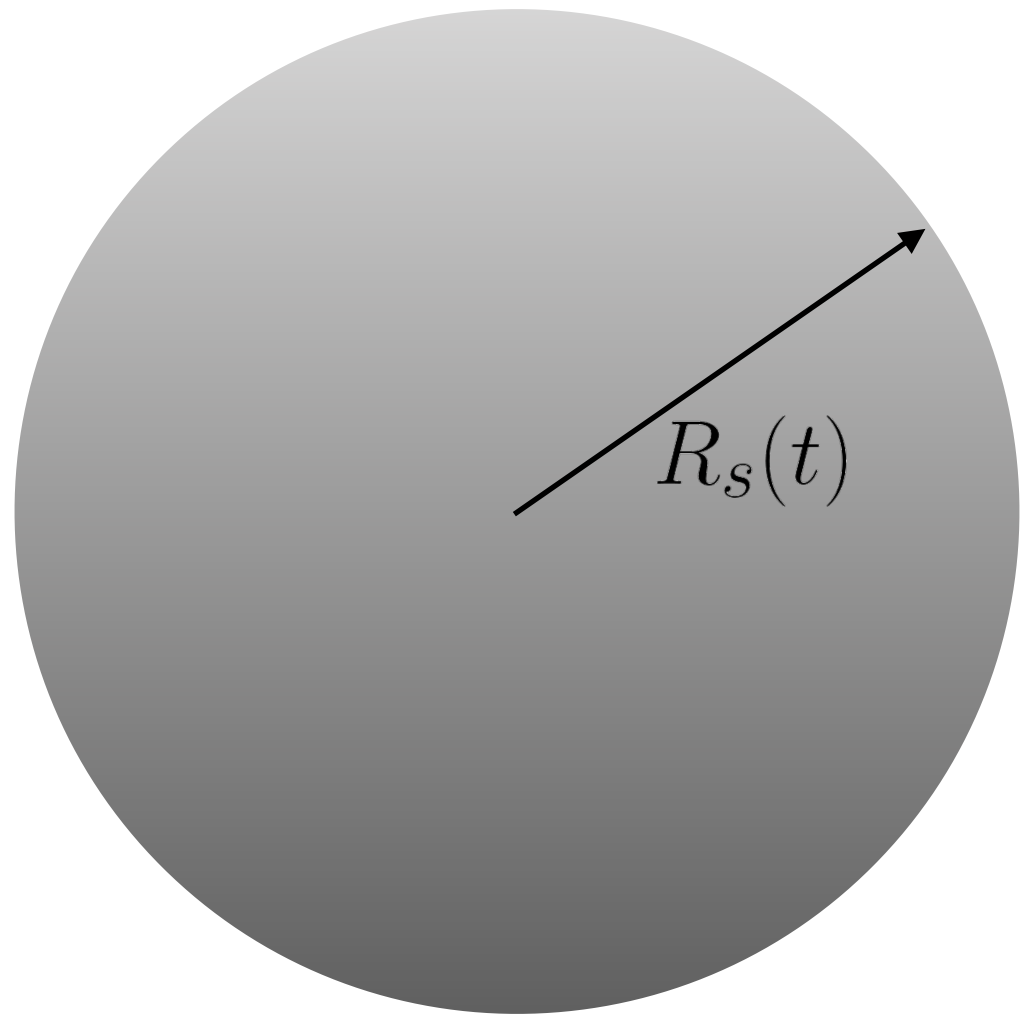
On scales large enough to say the Universe is homogeneous and isotropic, it can be approximated as being filled by a perfect gas of pressure $P(t)$ and energy density $\varepsilon(t)$

Then, $T_{\mu\nu}$ only depends on $P(t)$ and $\varepsilon(t)$

And all we need to do is relate κ, R_0 and $a(t)$ to $P(t)$ and $\varepsilon(t)$

The Newtonian Friedmann Equation

Consider a sphere of radius $R_s(t)$ and mass M_s expanding or contracting under its own gravity…



The Friedmann Equation

Newtonian:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r^2}\frac{1}{a(t)^2}$$

General relativistic:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}$$

Getting the feel of it…

- To determine the co-moving distance to a galaxy, and thus its current proper distance, we need to know how a(t) has changed throughout the time it has taken for a photon to traverse that distance.

- The curvature of the Universe affects the *perceived* sizes of distant objects. In a negatively (positively) curved universe, distant objects appear smaller (larger).

- The Friedmann Equation uses gravitational arguments to relate the curvature and expansion of the universe to its contents.

- Those contents are described in terms of pressure and energy density.