Lecture 14:
Large Scale Structure - I

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1 The dense get denser

After slide 4:

- To quantify how rapidly gravity increases the density of a region, consider a sphere of volume V, containing a spatially and time-dependent energy density, $\epsilon(r,t)$. The average density of the sphere is:
$$\bar{\epsilon}(t) = \frac{1}{V}\int_V \epsilon(r,t)d^3r \tag{1}$$
- We will also define a dimensionless density fluctuation:
$$\delta(r,t) = \frac{\epsilon(r,t) - \bar{\epsilon}(t)}{\bar{\epsilon}(t)} \tag{2}$$
- In the case of radiation and matter, the minimum possible value for $\epsilon(r,t)$ is 0 (i.e., you can't have negative mass or radiation), meaning the minimum possible value for δ is -1.
- There is no upper limit on δ (consider a black hole, which infinite density).
- Consider an *approximately* matter-only, static universe (why is it only approximately?), and add a small amount of matter within a sphere of radius R, so that the density of the sphere is $\bar{\rho}(1+\delta)$, with $\delta \ll 1$.
- The acceleration of a test mass at the sphere's surface due to the excess mass within the sphere is:
$$\ddot{R} = -\frac{G\Delta M}{R^2} = -\frac{G}{R^2}\left(\frac{4\pi}{3}R^3\bar{\rho}\delta\right) \tag{3}$$
- Giving:
$$\frac{\ddot{R}}{R} = -\frac{4\pi G\bar{\rho}}{3}\delta(t) \tag{4}$$
- We want to solve the above equation to figure out how δ changes with time, i.e., how a density fluctation increases in magnitude with time due to the effects of gravity.

- Conservation of mass tells us that:

$$M = \frac{4\pi}{3} R_0^3 \bar{\rho} = \frac{4\pi}{3} \bar{\rho}(1+\delta(t)) R(t)^3 \tag{5}$$

giving:

$$R(t) = R_0 (1+\delta(t))^{1/3} \tag{6}$$

- Since we've assumed that $\delta \ll 1$, we can use a binomial expansion (i.e., $(1+x)^\alpha \approx 1 + \alpha x$, when $|x| \ll 1$) to get:

$$R(t) \approx R_0 \left[1 - \frac{1}{3}\delta(t)\right] \tag{7}$$

- And taking the second derivative in time gives:

$$\ddot{R} \approx -\frac{1}{3} R_0 \ddot{\delta} \approx -\frac{1}{3} R_0 \ddot{\delta} \tag{8}$$

- Which, when substituted into Eq. 4 gives:

$$\ddot{\delta} = 4\pi G \bar{\rho} \delta \tag{9}$$

- Which has a solution of the form:

$$\delta(t) = A_1 e^{t/t_{\rm dyn}} + A_2 e^{-t/t_{\rm dyn}} \tag{10}$$

- And I'll leave it as an exercise to show that:

$$t_{\rm dyn} = \frac{1}{(4\pi G \bar{\rho})^{1/2}} = \left(\frac{c^2}{4\pi G \bar{\epsilon}}\right)^{1/2} \approx 9.6 \text{ hours } \left(\frac{\bar{\rho}}{1 \text{ kg m}^{-3}}\right)^{-1/2} \tag{11}$$

2 The Jeans Length

- Thankfully, pressure eventually prevents a gas cloud from collapsing completely.
- In the case of a cold, tenuous gas, there is very little pressure to support against gravitational collapse.
- As the cloud collapses, however, the pressure builds up until it begins to dominate over gravity.
- Can we calculate when this transition will occur?
- The build-up of pressure isn't trasmitted instantly throughout a gas cloud. Instead, it travels at sound speed, c_s.
- The time it takes for a pressure gradient to build up in our sphere of radius R is:

$$t_{\rm pre} \sim \frac{R}{c_s} \tag{12}$$

- where

$$c_s = c\left(\frac{dP}{d\epsilon}\right)^{1/2} = \sqrt{\omega} c \tag{13}$$

- For a cloud of gas to be stable against gravitational collapse, then the pressure must build up more quickly than the cloud can collapse, meaning $t_{\text{pre}} < t_{\text{dyn}}$.
- Comparing, then, Equations 11 and 12 means that a cloud with radius

$$R < c_s \left(\frac{c^2}{4\pi G\bar{\epsilon}}\right)^{1/2} = \lambda_J \tag{14}$$

will be stable against gravitational collapse.

- λ_J is called the Jeans length.
- Note: a more thorough evaluation of the t_{pre} gives:

$$\lambda_J = c_s \left(\frac{\pi c^2}{G\bar{\epsilon}}\right)^{1/2} \tag{15}$$

3 The Jeans length in cosmology

- In a flat universe, parts of the Jeans length equation resembles parts of the rearranged Friedmann Equation:

$$\frac{c^2}{G\epsilon} = \frac{8\pi}{3H^2} \tag{16}$$

and substituting Eqn. 16 into Eqn. 15 gives:

$$\lambda_J = 2\pi \left(\frac{2}{3}\right)^{1/2} \frac{c_s}{H} = 2\pi \left(\frac{2}{3}\right)^{1/2} \sqrt{\omega}\frac{c}{H} \tag{17}$$

- Meaning that, for a photon gas ($\omega = 1/3$):

$$\lambda_J = \frac{2\pi\sqrt{2}}{3}\frac{c}{H} \approx 3\frac{c}{H} \tag{18}$$

- Which means that any density perturbations that are smaller than a Hubble length (which is *very big*, i.e., c/H) will be stable against collapse in a radiation-only Universe.
- How is this relevant for our own Universe? Well, prior to decoupling, the energy density of photons was higher than that of *baryons*. Since pressure is mediated by baryons, not dark matter, baryons are what counts here.
- So, prior to decoupling, the Jeans length of the photon/baryon "soup" was effectively that of photons alone:

$$\lambda_J \approx 3c/H(z_{\text{dec}}) \approx 0.66 \text{ Mpc} \approx 2 \times 10^{22} \text{ m} \tag{19}$$

- Meaning everything smaller than 0.66 Mpc was impervious to gravitational collapse. At the time of decoupling, this corresponded to a mass of:

$$M_J = \rho_{\text{bary}} \left(\frac{4\pi}{3}\lambda_J^3\right) = 10^{19}\ M_\odot \tag{20}$$

which is significantly larger than the mass of a supercluster of galaxies (Note: M_J is known as the Jeans mass, and is the mass above which a cloud of gas will collapse due to gravity).

- Just *after* decoupling, however, the baryons stopped interacting with the photons, and their sound speed suddently dropped to:

$$c_s = \left(\frac{kT}{mc^2}\right)^{1/2} \approx 1.5 \times 10^{-5}c \tag{21}$$

corresponding to a 2.6×10^{-5} decrease in the Jeans length, and a resulting Jeans mass of 2×10^5 $\mathrm{M}_\odot$, wihch is small even compared to the mass of a galaxy.

- This means that the epoch of decoupling, when CMB photons stopped readily interacting with baryons, represents an important stage in the Universe's ability to form structures such as clusters and galaxies.

4 Gravitational collapse in an expanding universe

- In an expanding universe, pressure isn't the only force resisting the gravitational collapse of gas clouds.

- In such a universe, the expanding scale factor also "pulls" a gas cloud in all directions, against the inward draw of gravity.

- Or, equivalently, if the sphere is slightly overdense, it will grow slightly less rapidly than the scale factor.

- Using similar arguments to that of Section 1, we arrive at an expression that describes the growth rate of a density fluctuation as:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\bar{\rho}\delta \tag{22}$$

or,

$$\ddot{\delta} = 4\pi G\bar{\rho}\delta - 2H\dot{\delta} \tag{23}$$

- Which looks reminiscent to the equation for a damped oscillator - the $2H$ term acts to slow the growth of density perturbations.

- If we then use $\bar{\rho} = \bar{\epsilon}/c^2$, and

$$\Omega_m = \bar{\epsilon_m}/\epsilon_c = \frac{8\pi G\bar{\epsilon_m}}{3c^2H^2} \tag{24}$$

we eventually get

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2\delta = 0 \tag{25}$$

- During epochs of the universe when Ω_m is/was very small (i.e., $\Omega_m \approx 0$) will δ grow at a significant rate. For example, the solution to the above equation when the Universe was radiation dominated is:

$$\delta(t) \sim \ln(t), \tag{26}$$

and when the Universe will be strong dominated by Dark Energy, the solution will be:

$$\delta(t) \sim e^{-2Ht}, \tag{27}$$

- Meaning that structures only have time to form if a universe goes through a period of being matter-dominated, when $\delta \propto t^{2/3}$.
- Thankfully, because of Dark Matter, structures could start to form (even if from just Dark Matter) as soon as the Universe became matter-dominated at $z \sim 3440$.
- Then, once baryons and photons became decoupled, the baryons started to collapse into the dark-matter structures that had been building since $z \sim 3440$. Dark Matter therefore gave the Universe a headstart in building structure.