Cosmology
Lecture 5

Solving the Friedmann Equation
Part 2:
Solving our first universe

Lectures

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Let's briefly take stock

- The entire expansion/contraction evolution of a homogenous, isotropic universe is wholly encapsulated in the scale factor a(t).

- Knowledge of a(t) also allows us to calculate important parameters such as distances to galaxies, the age of the universe, and many other things.

- The Friedmann Equations allows us to calculate a(t) given the energy density (i.e., contents) of the universe.

- The Fluid Equation and Equations of State allows us to calculate the evolution of the energy densities of the universe.

Lecture 5 Learning Objectives

By the end of the lecture you should be able to:

- Solve the Fluid Equations so you can express $\varepsilon(t)$ as a function of scale factor and understand these in physical terms.

- Solve the Friedmann Equation for your first universe - a curved, empty universe.

- Use the resulting expression for a(t) to calculate the age of an empty universe.

- Use the resulting expression for a(t) to calculate proper distances at the time of observation and emission.

Three key equations:

We now have everything we need to solve for $a(t)$, $\varepsilon(t)$ and $P(t)$

F.E.: $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}$$

Fluid Equation: $$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$

Equation of state: $$P = \omega\varepsilon$$

Complicated by the fact that there are different types of energy in the Universe: mass energy, photon energy and dark energy.

However, these different types of energy add linearly:

$$\varepsilon = \varepsilon_{\rm m} + \varepsilon_{\rm p} + \varepsilon_{\rm d}$$
$$P = \omega_{\rm m}\varepsilon_{\rm m} + \omega_{\rm p}\varepsilon_{\rm p} + \omega_{\rm d}\varepsilon_{\rm d}$$

How does energy density change with scale factor?

Integrating the Fluid Equation gives:

$$\varepsilon(a) = \varepsilon_{i,0} a^{-3(1+\omega_i)}$$

For non-relativistic particles: $\omega_{\mathrm{m}} = 0$ so $\varepsilon_{\mathrm{m}}(a) = \varepsilon_{\mathrm{m},0} a^{-3}$

For relativistic particles: $\omega_{\mathrm{p}} = 1/3$ $\varepsilon_{\mathrm{p}}(a) = \varepsilon_{\mathrm{p},0} a^{-4}$

For dark energy: $\omega_{\mathrm{p}} = -1$ $\varepsilon_{\mathrm{d}}(a) = \varepsilon_{\mathrm{d},0} a^{-0} = \varepsilon_{\mathrm{d},0}$

Our first universe

Remember, much of cosmology is focussed on establishing how the scale factor varies with time, i.e., $a(t)$. From this, we can determine proper distances, emission times, etc.

We will now determine $a(t)$ for our first ever universe: a model universe containing nothing!

Solve:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2} \qquad \varepsilon(t) = \text{constant} = 0$$

Gives:

$$a(t) = \pm\frac{ct}{R_0} = \pm\frac{t}{t_0}$$

Where t_0 is defined as the current age of the universe by setting $a(t_0) = 1$ giving: $t_0 = R_0/c$

The result is scale factor that increases/decreases linearly with time; there are no forces acting from anything, so any initial expansion/contraction continues unchanged.

From a(t) to distances

Recall from Lecture 2: $$d_p(t_{\rm ob}) = c\int_{t_{\rm em}}^{t_{\rm ob}} \frac{dt}{a(t)}$$ i.e., current proper distance

with: $$a(t) = \frac{t}{t_0}$$ and $$(1+z) = \frac{1}{a(t_{\rm em})}$$

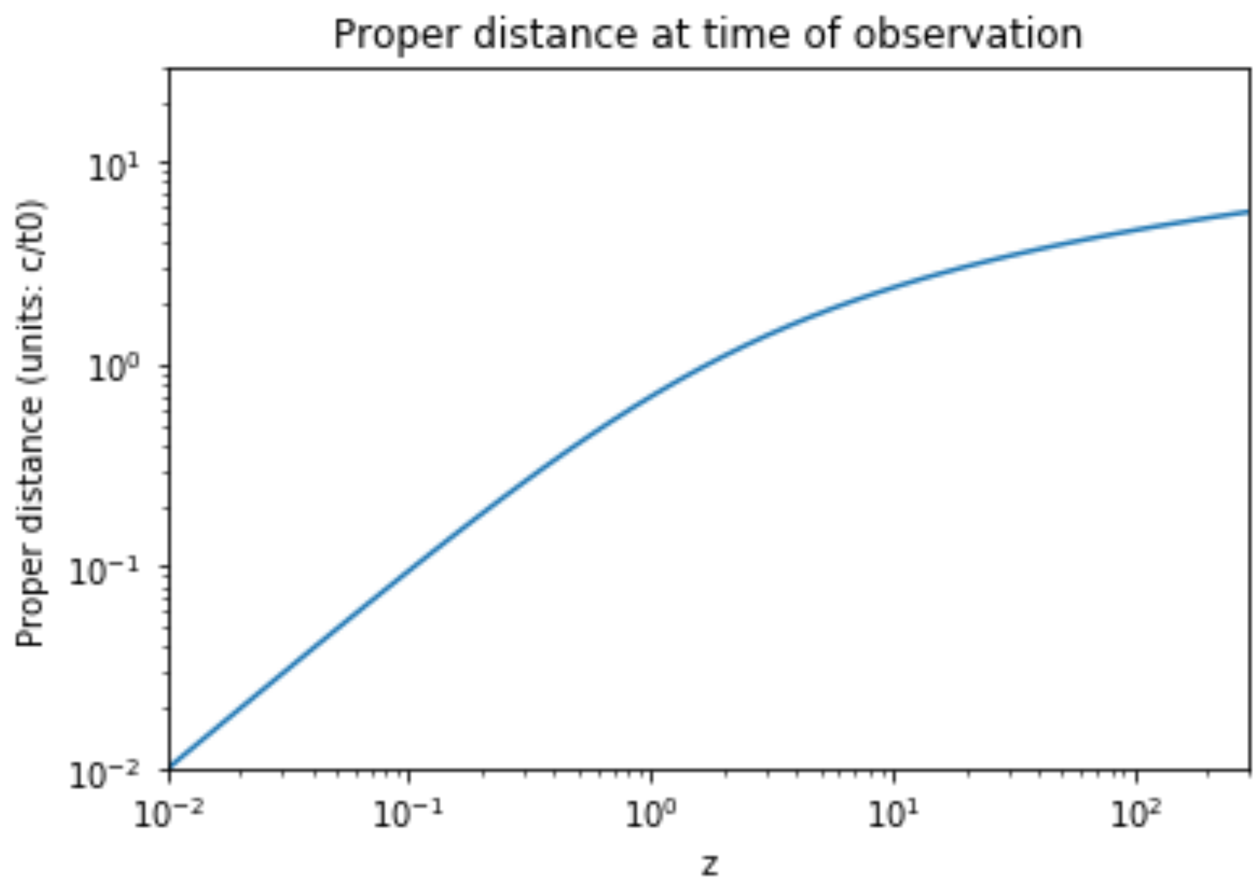
Gives: $$d_p(t_{\rm ob}) = ct_0 \ln(1+z)$$

and we assume: $$t_{\rm ob} = t_0$$ (i.e., we're observing now, at the current age of the universe)

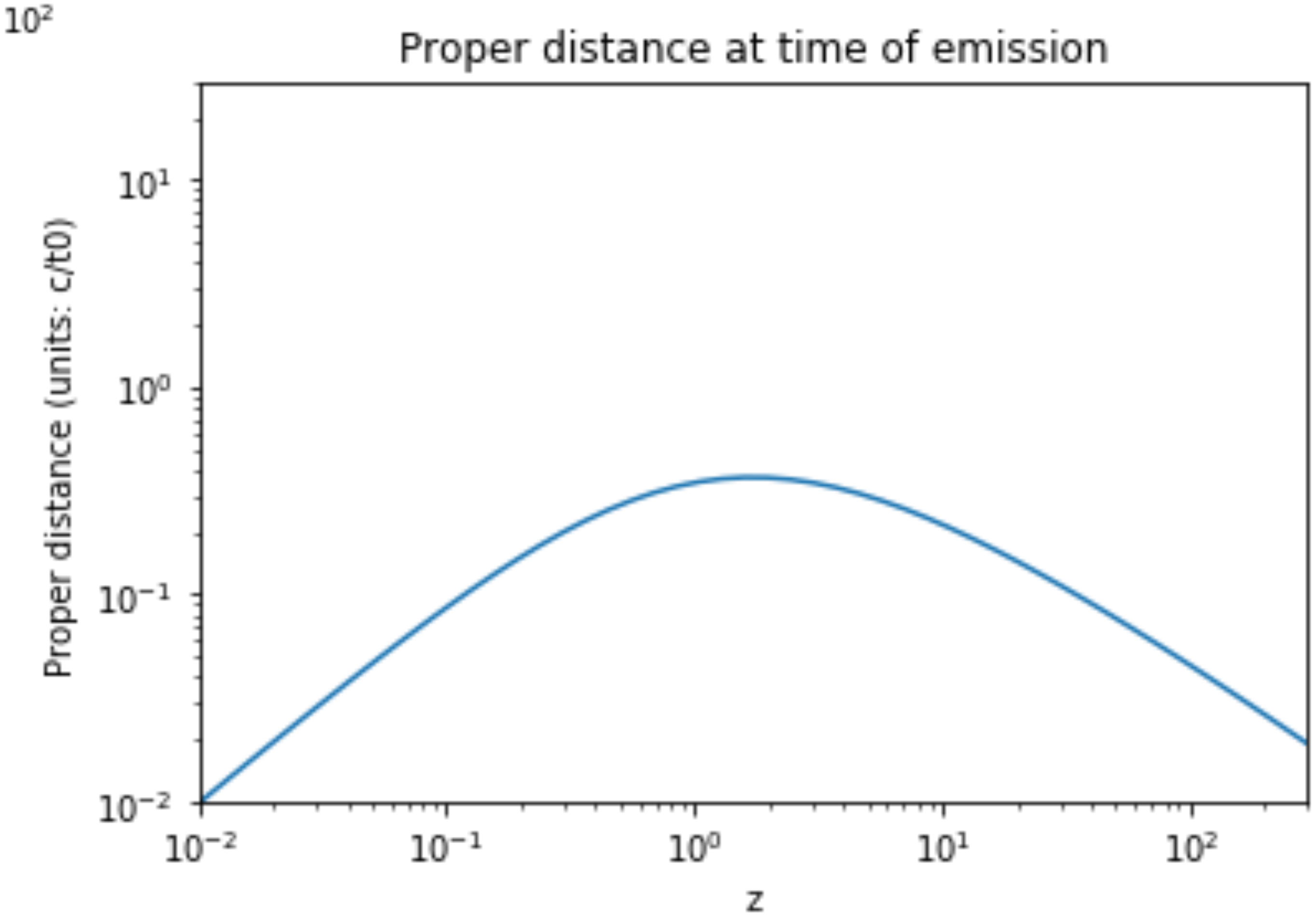
Also, since: $$\frac{a(t_{\rm ob})}{a(t_{\rm em})} = \frac{1}{(1+z)}$$

$$d_p(t_{\rm em}) = \frac{d_p(t_{\rm ob})}{1+z} = ct_0\frac{\ln(1+z)}{1+z}$$

How proper distances change with redshift



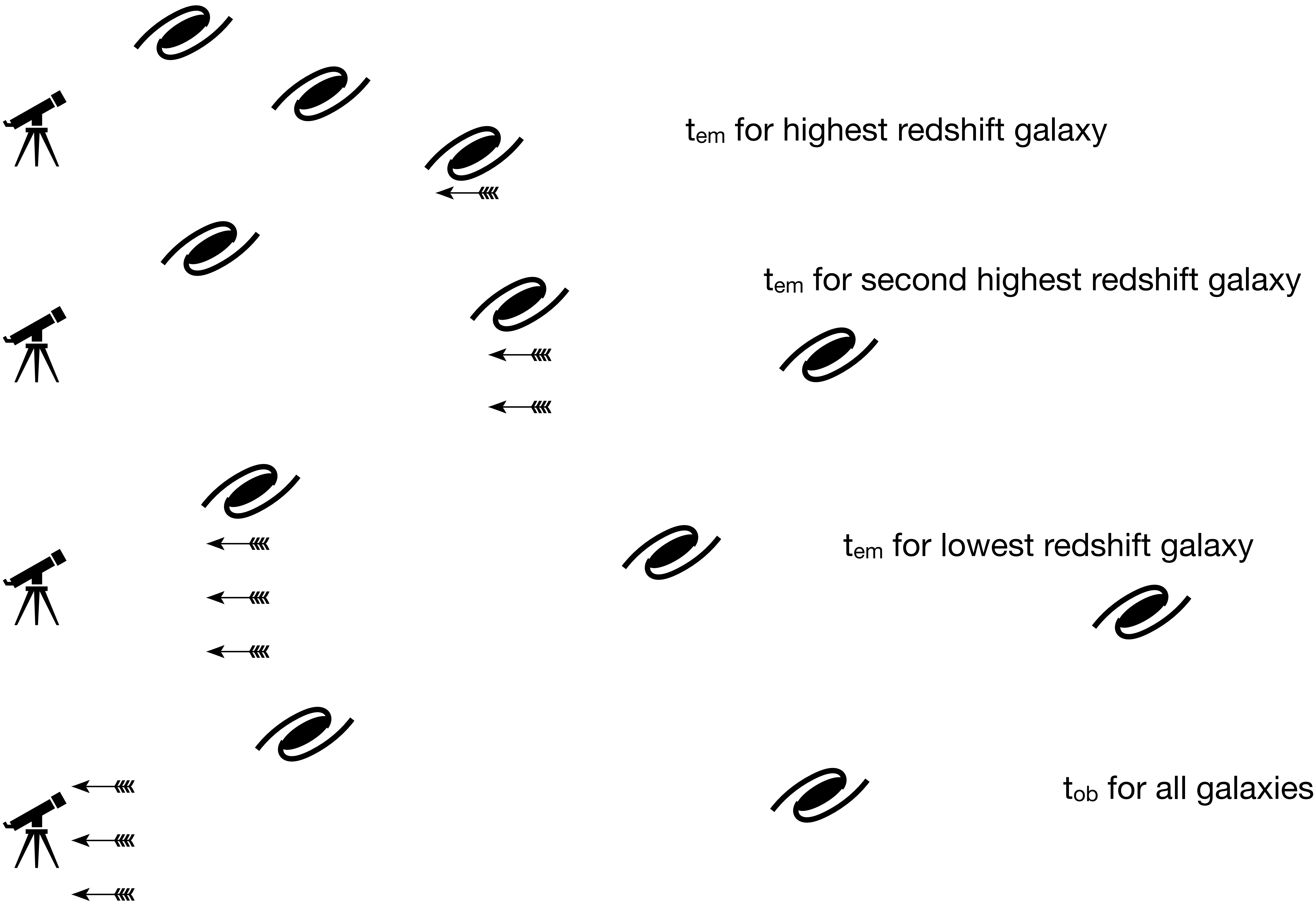
Current proper distance always increases with redshift. This increase is linear at low redshifts, but flattens at high redshifts.



Proper distance at time of emission behaves differently:
First, it increases with redshift, but then peaks around z~2 before falling again.

At high redshifts, we'd see galaxies as they were when the Universe was much smaller, and thus when the galaxies were much closer to us.

Getting the feel for it…



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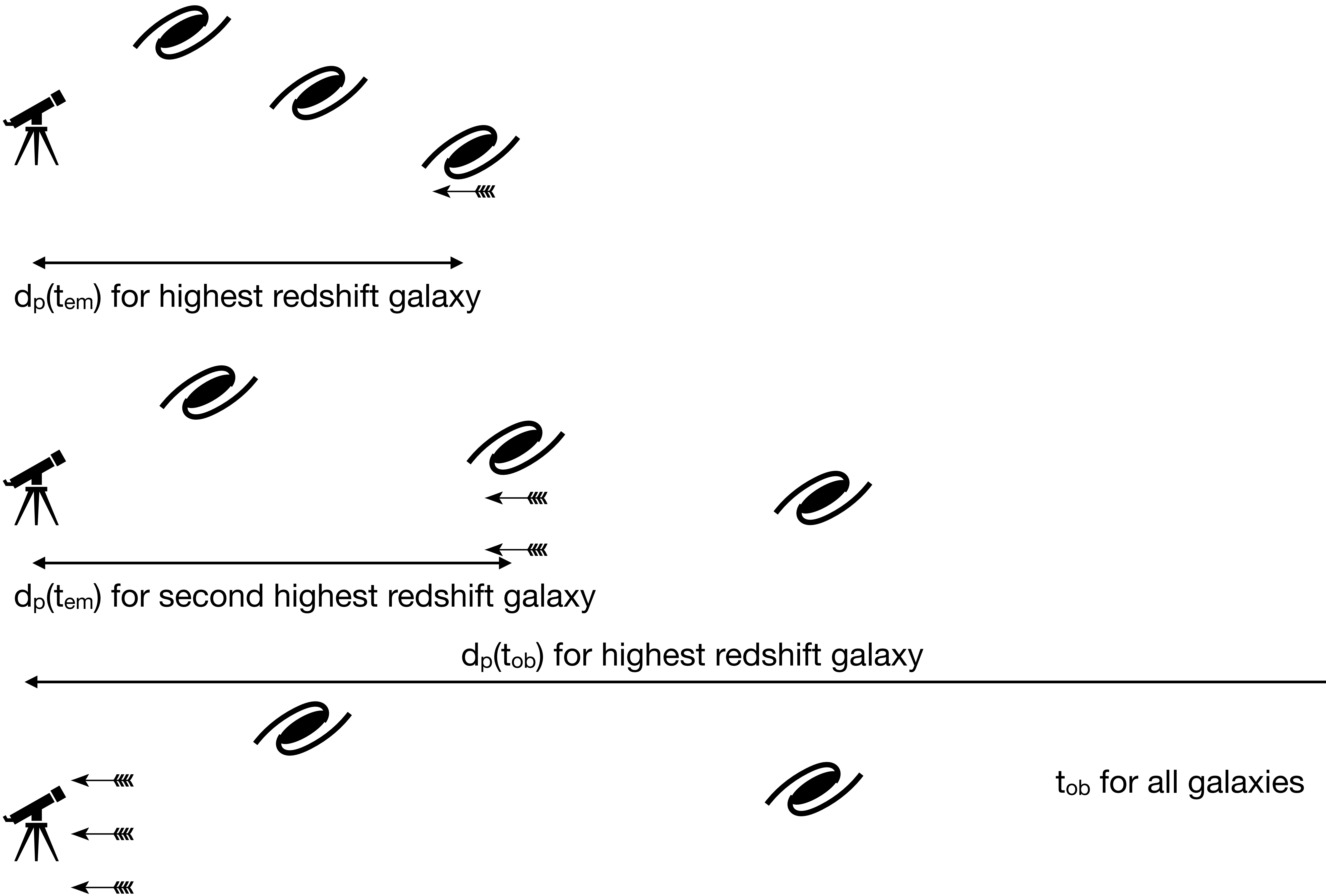

Getting the feel for it…

By integrating the Fluid Equation, we get relatively straightforward expressions for the various energy densities of the universe.

Matter density decreases by the inverse of volume, just as we'd expect.

Radiation density decreases faster with scale factor, due to the "stretching" of wavelength causing the photon energy to decrease.

Our first model universe - an empty universe - has demonstrated what we mean by "solving the Friedmann Equation".

It is, of course, a special example that's easy to solve. We'll see in the next lecture what happens when we introduce "content" to our model universes.

But, it's shown us how solving the Friedmann Equation enables us to convert redshifts (easy to measure) to distances (hard to measure).

The current distance and distance at the time of emission are different quantities related via redshift.