Cosmology
Lecture 7

The “Benchmark model” and “measurable” distances

The Benchmark Model for the real Universe

The real Universe contains $\varepsilon_{\rm m}, \varepsilon_{\rm p}, \varepsilon_{\rm d}$ and possibly $\kappa \neq 0$

In this case, the F.E. is:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\sum_i \varepsilon_i - \frac{\kappa c^2}{R_0^2 a(t)^2}$$

Which, by using $\Omega_{i,0} = \dfrac{\varepsilon_{i,0}}{\varepsilon_{c,0}}$ and the definitions in Lectures 3 and 4 becomes:

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{\rm p,0}}{a^4} + \frac{\Omega_{\rm m,0}}{a^3} + \Omega_{\rm d,0} + \frac{1-\Omega_0}{a^2}$$

multiplying by a^2, taking the square route, and using $H = \dot{a}/a$ gives:

$$\frac{da}{dt} = H_0\left(\frac{\Omega_{\rm p,0}}{a^2} + \frac{\Omega_{\rm m,0}}{a} + \Omega_{\rm d,0}a^2 + (1-\Omega_0)\right)^{1/2}$$

Solving the Benchmark Model

Solving the Benchmark model to get a(t) for the real Universe thus involves:

$$\int_0^a \frac{da}{\left(\frac{\Omega_{\mathrm{p},0}}{a^2} + \frac{\Omega_{\mathrm{m},0}}{a} + \Omega_{\mathrm{p},0} a^2 + (1 - \Omega_0)\right)^{1/2}} = H_0 \int_0^t dt = H_0 t$$

For the real Universe:

$$\Omega_{p,0} = 9 \times 10^{-5}$$
$$\Omega_{m,0} = 0.31$$
$$\Omega_{d,0} = 0.69$$
$$\Omega_0 = 1.00$$
$$H_0 = 67.7\ \mathrm{km\ s^{-1}\ Mpc}$$

But this integral can’t be solved analytically.

What cosmologists do is solve it numerically (e.g., integrate by “area under curve” methods), evaluating the LHS at various values of *a* to get the corresponding value of *t*.

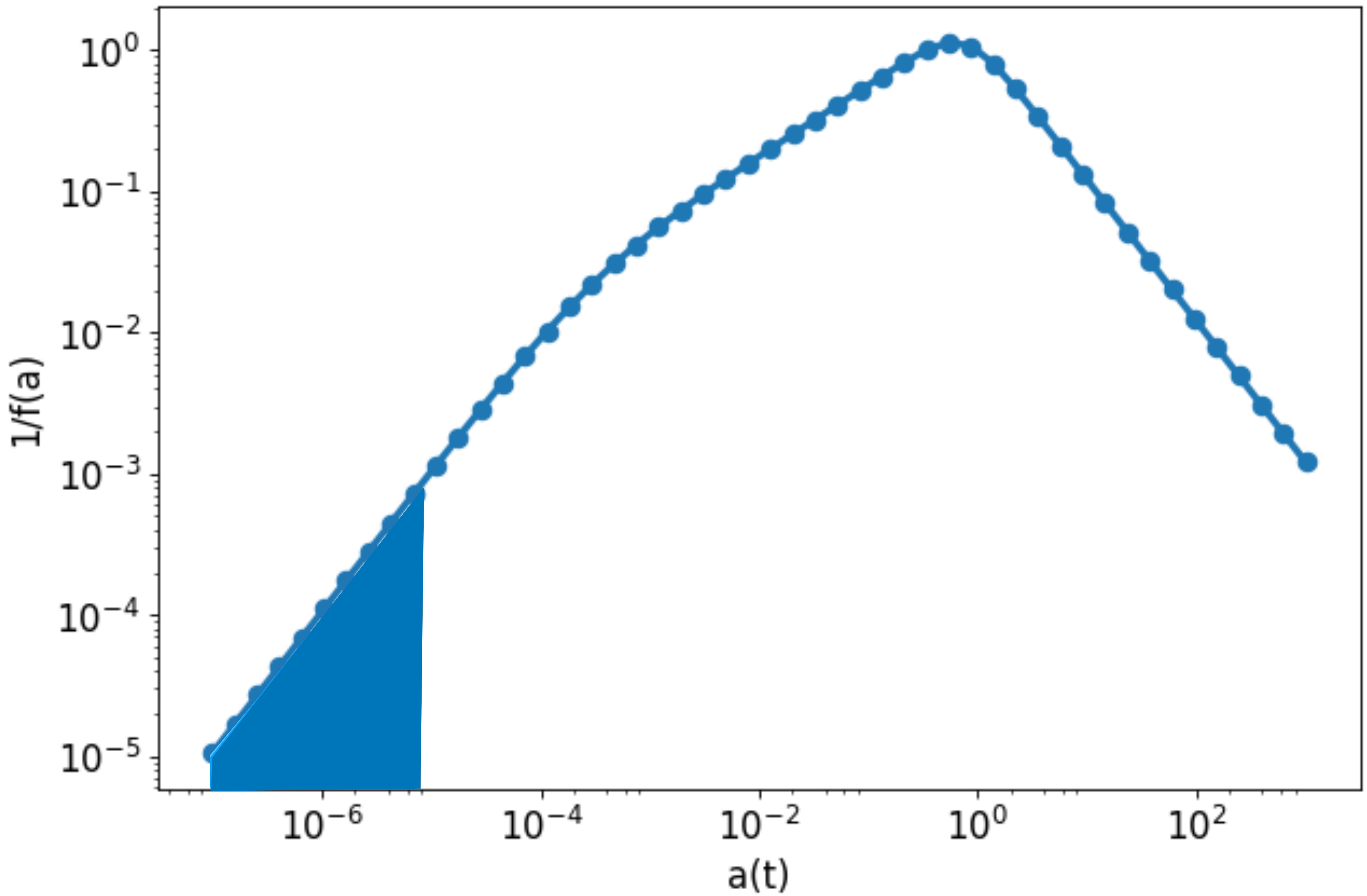
Numerically integrating the F.E.

$$\int_0^a \frac{da}{\left(\frac{\Omega_{\mathrm{p},0}}{a^2} + \frac{\Omega_{\mathrm{m},0}}{a} + \Omega_{\mathrm{d},0}a^2 + 1 - \Omega_0\right)^{1/2}} = H_0 t$$

1. Define a vector of *a* values.
2. Calculate the value of $\left(\frac{\Omega_{\mathrm{p},0}}{a^2} + \frac{\Omega_{\mathrm{m},0}}{a} + \Omega_{\mathrm{d},0}a^2 + 1 - \Omega_0\right)^{-1/2}$ at each *a* value.
3. Calculate the area under this curve between 0 and a given value of *a*.
4. This area is equal to $H_0 t$, meaning you've just calculated *t* for *a* given value of *a*.
5. Repeat steps 2-4 for each value of *a* in your vector to give *t* for every value of *a*.

We therefore get a vector of *t* values, one value for each value of *a* in our *a*-vector. We can then simply choose to plot the a vector against the t vector to get a plot of scale factor as a function of time…

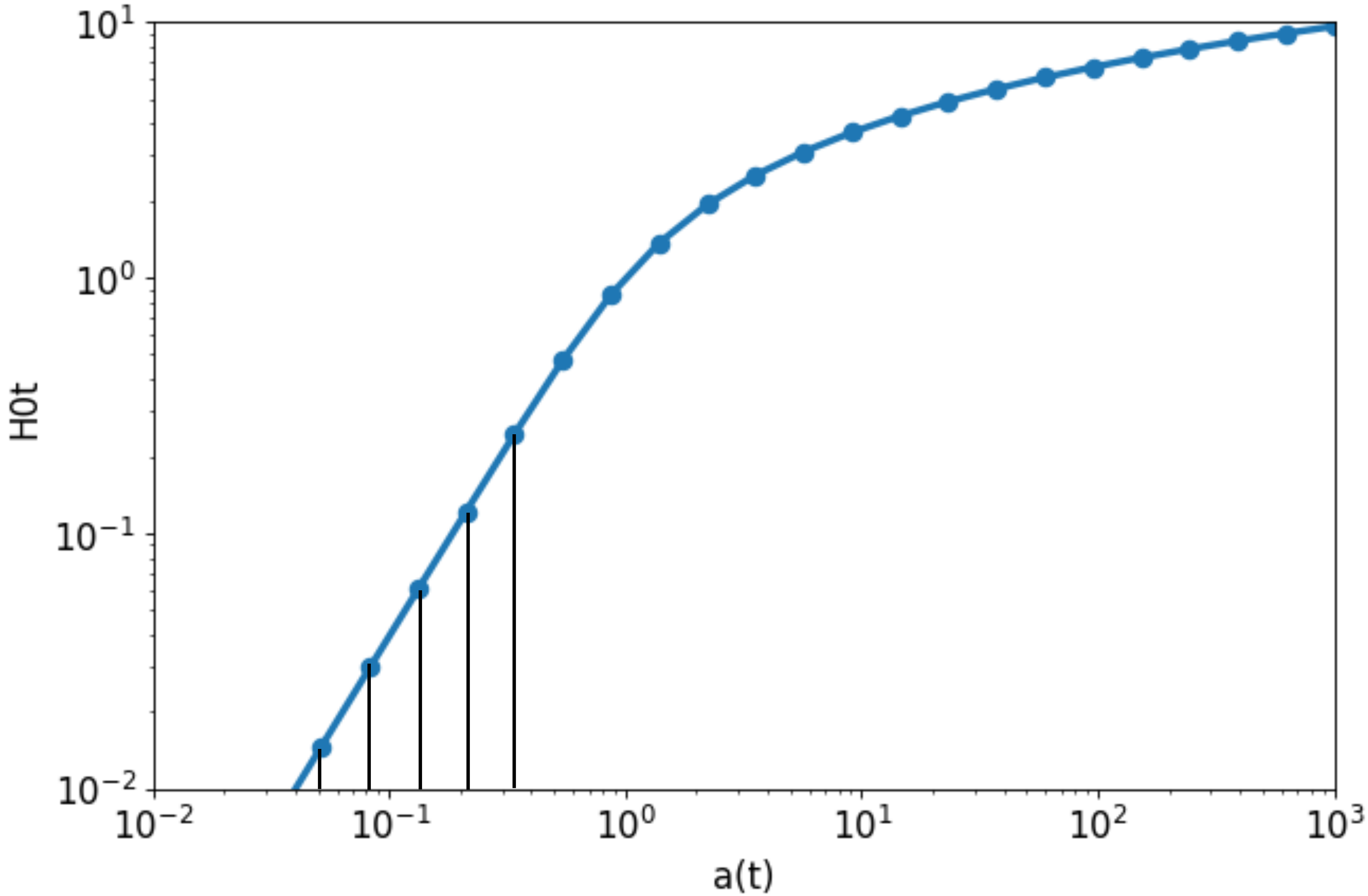
Solving the F.E. numerically



where:

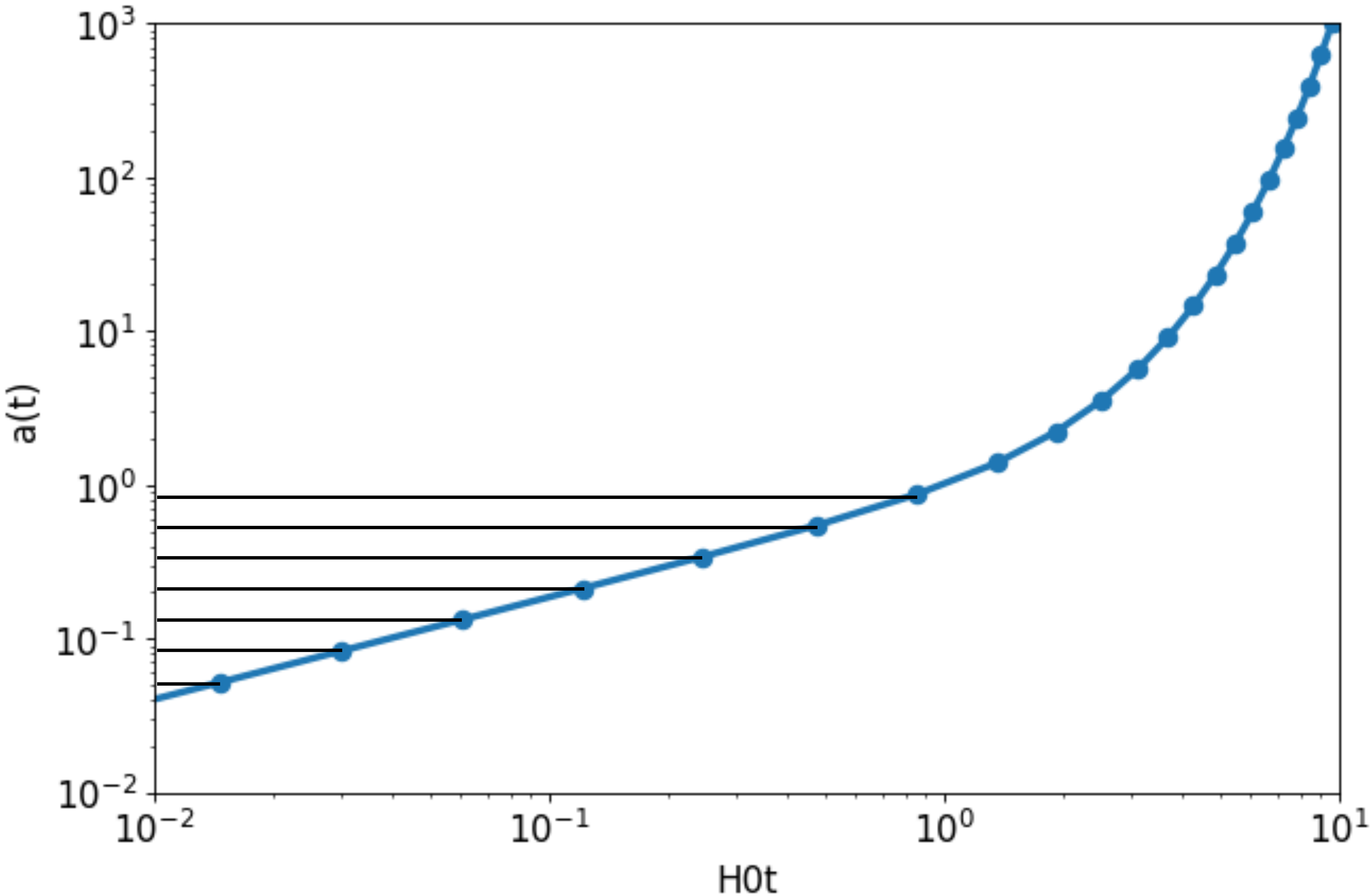
$$f(a) = \left(\frac{\Omega_{\mathrm{p},0}}{a^2} + \frac{\Omega_{\mathrm{m},0}}{a} + \Omega_{\mathrm{d},0} a^2 + 1 - \Omega_0 \right)^{1/2}$$

Solving the F.E. numerically



For each value of a, numerically integrate to calculate what value of H_0t that value of a corresponds to.

Scale factor vs. time for Benchmark Model



Then, to obtain how the scale factor changes with time, simply plot the a-vector on the y-axis, and the t-vector on the x-axis.

Proper distances within the Benchmark Model

With a numerical solution for scale factor, *a*, at different times, *t*, we are able to calculate proper distances:

$$d_p(t_0) = c \int_{t_{\rm em}}^{t_{\rm ob}} \frac{1}{a(t)} dt$$

How do we do this?

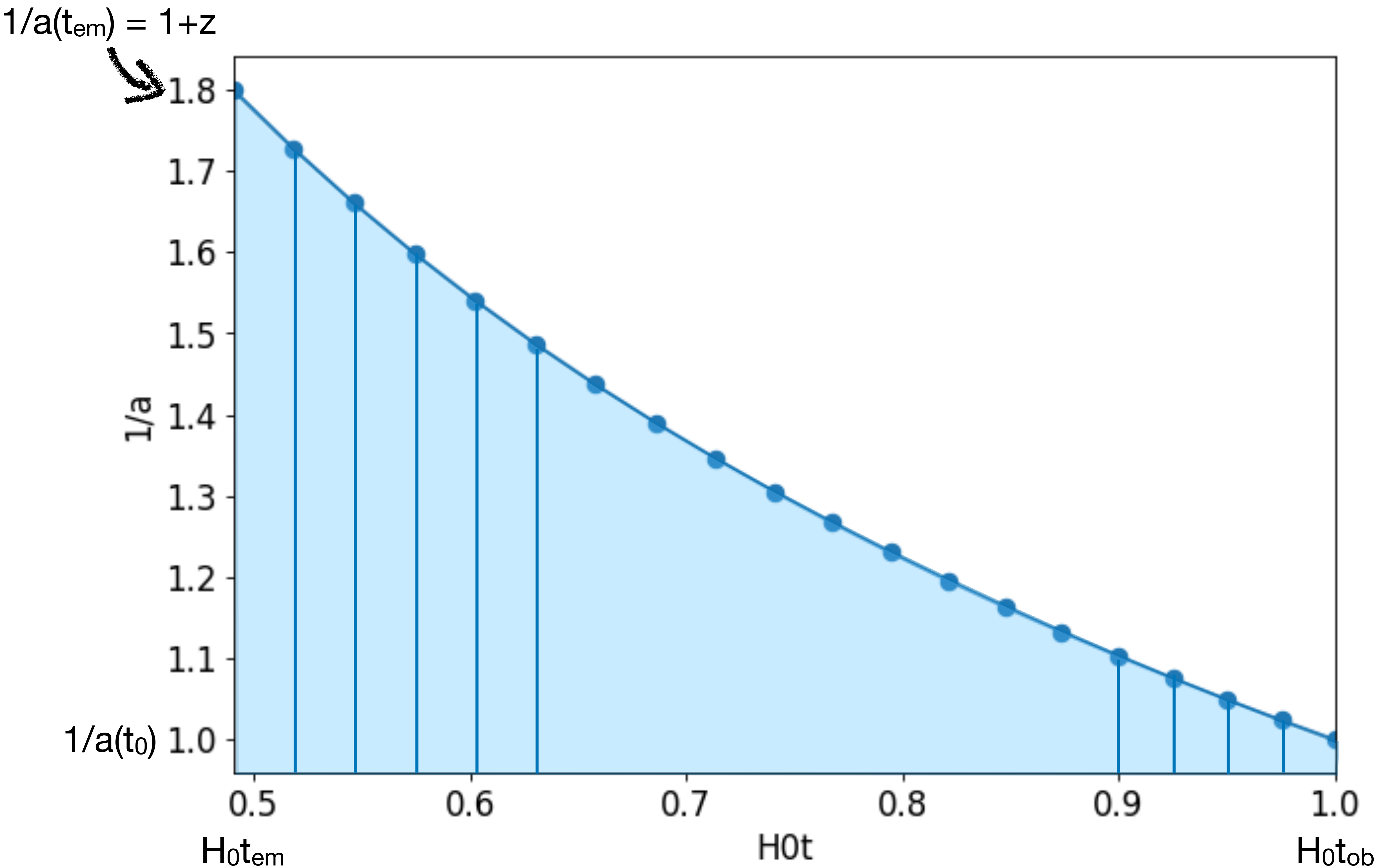
Again, we just have to calculate the area under the curve defined by 1/a(t) and between t_{em} and t_{ob}.

But, what about t_{em} and t_{ob}? Well, since:

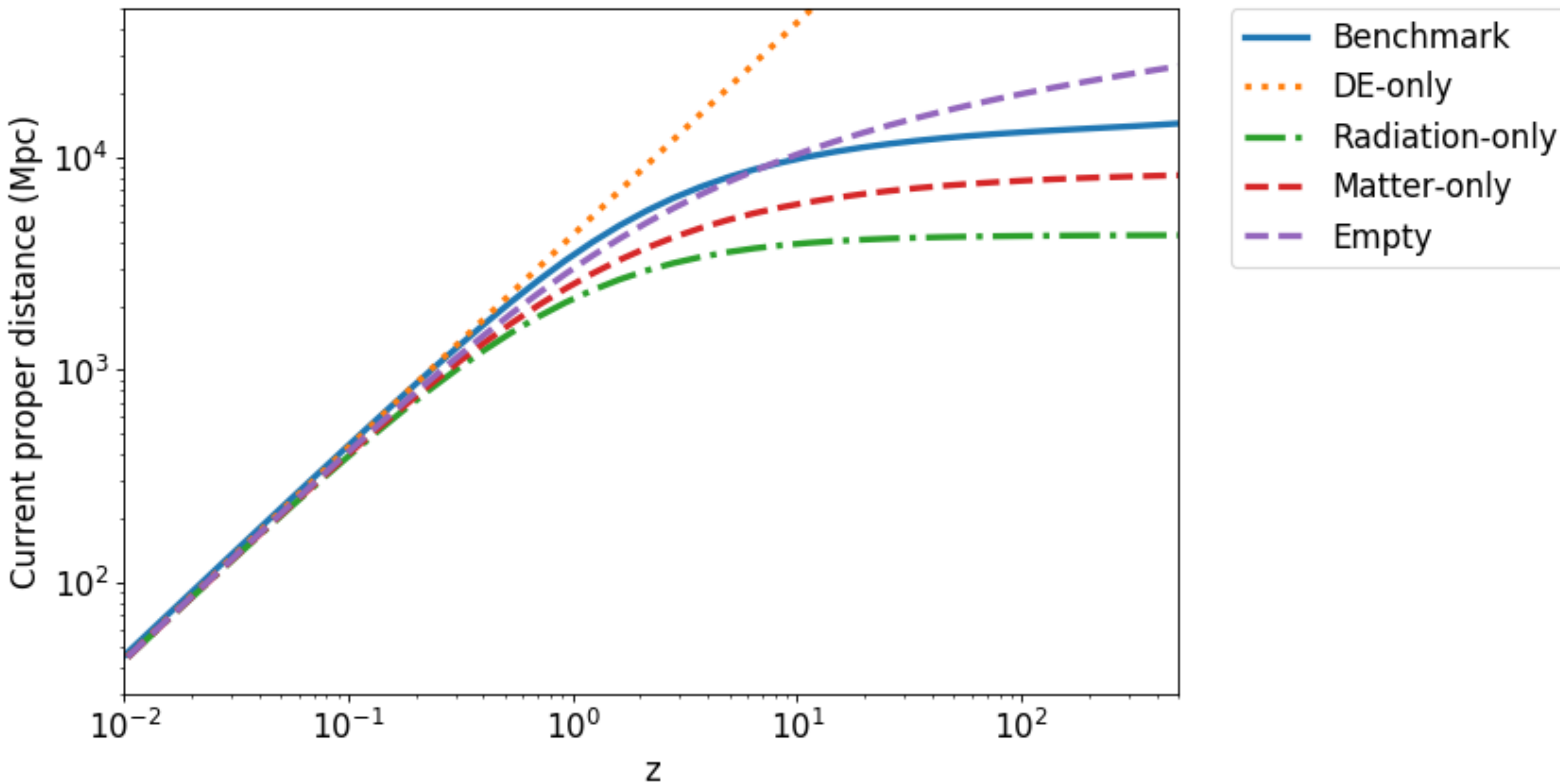
$$1 + z = \frac{1}{a(t_{\rm em})} \quad \text{and} \quad a(t_0) = 1$$

All we need to do is define our *a(t)* vector between 1/(1+z) and 1, calculate the corresponding *t* values, then sum up the area under the 1/a curve.

For z = 0.8, d_p/c given by the area under this curve:



Current proper distance vs. redshift



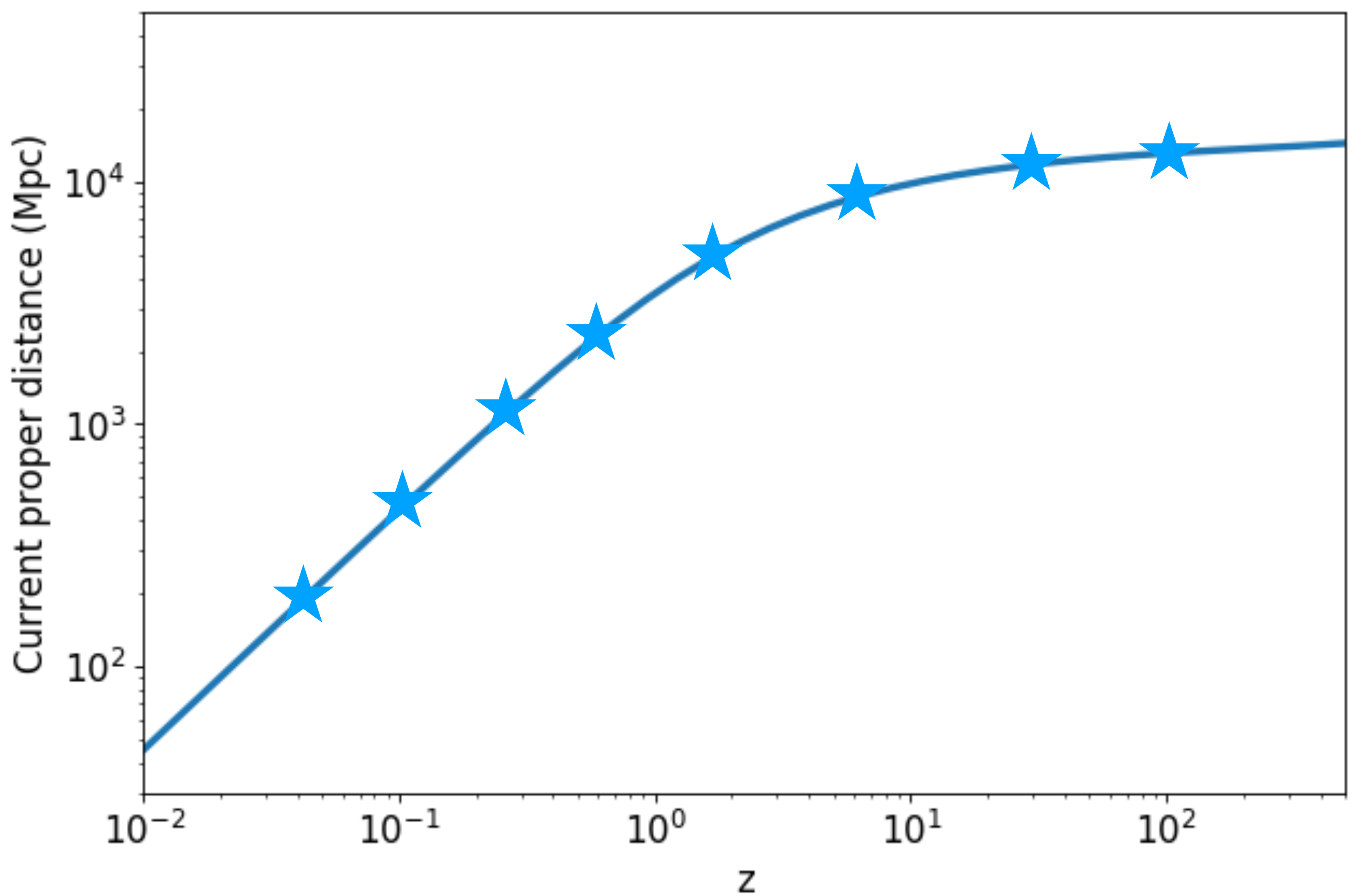
Using numerical techniques, cosmologists can calculate *a(t)* for any combination of specific energy densities…

Calculating *a(t)* for the real Universe

…but what's more important is to determine *a(t)*, and corresponding energy densities, from observations of the *real* Universe.

As we've seen, a given *a(t)* corresponds to a given relationship between redshift and proper distance.

So, work backwards: measure d_p and z for a bunch of galaxies, then find the a(t) that fits them best, which gives the corresponding energy densities:



Measurable distances

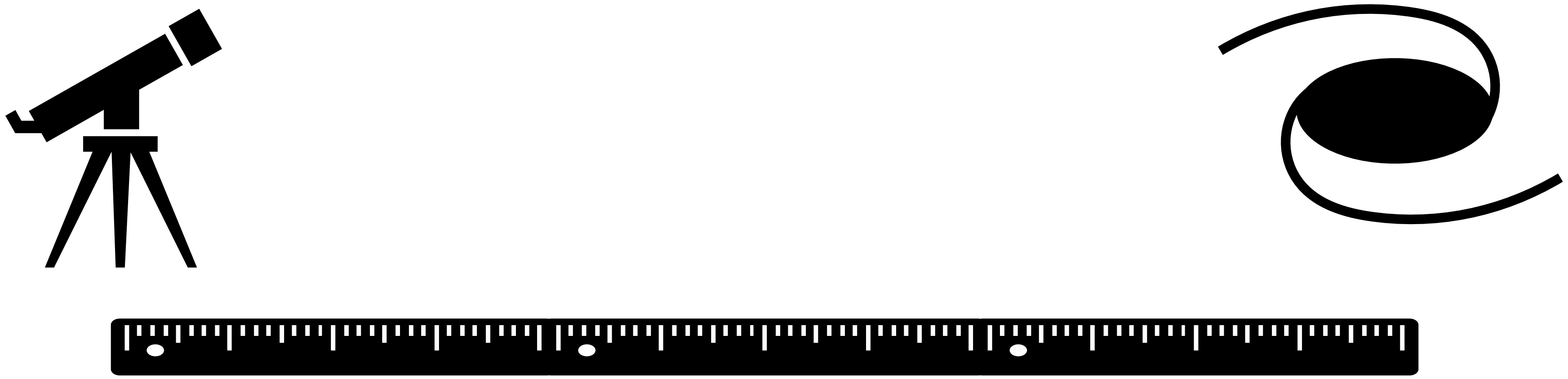
But, cosmologists clearly can't measure proper distance, since:

- They'd have to stop the Universe from expanding while we made the measurements.
- There's no measuring stick long enough to reach distant galaxies.

Thankfully, there are other ways to measure distance in the Universe:

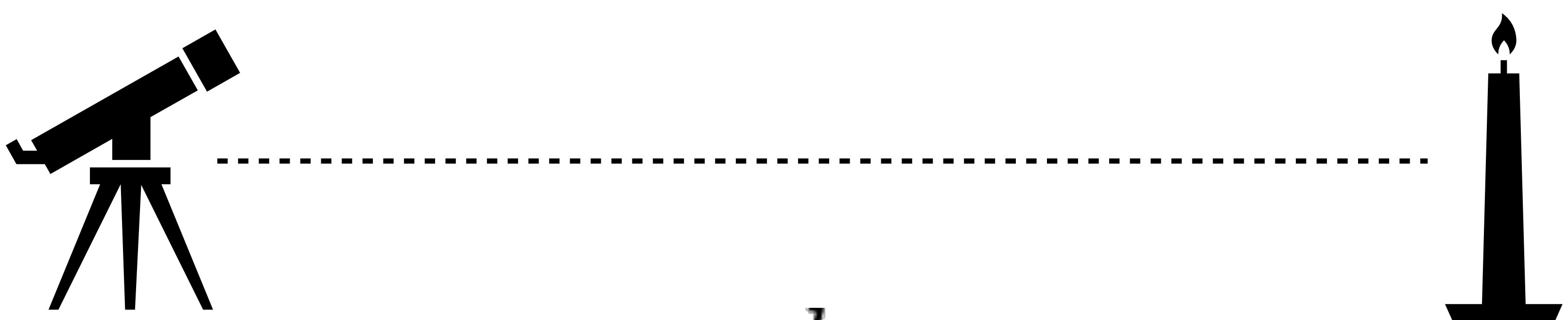
Different distances in cosmology & astronomy

Proper distance:



$$d_P$$

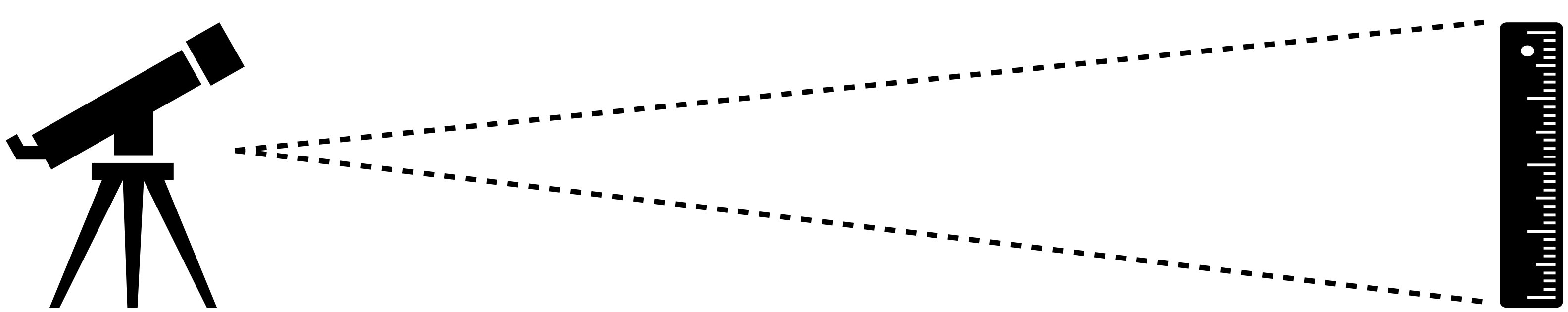
Luminosity distance:



$$d_L$$

$$\boxed{F = \frac{L}{4\pi d_L^2}}$$

Angular size distance:



$$d_A$$

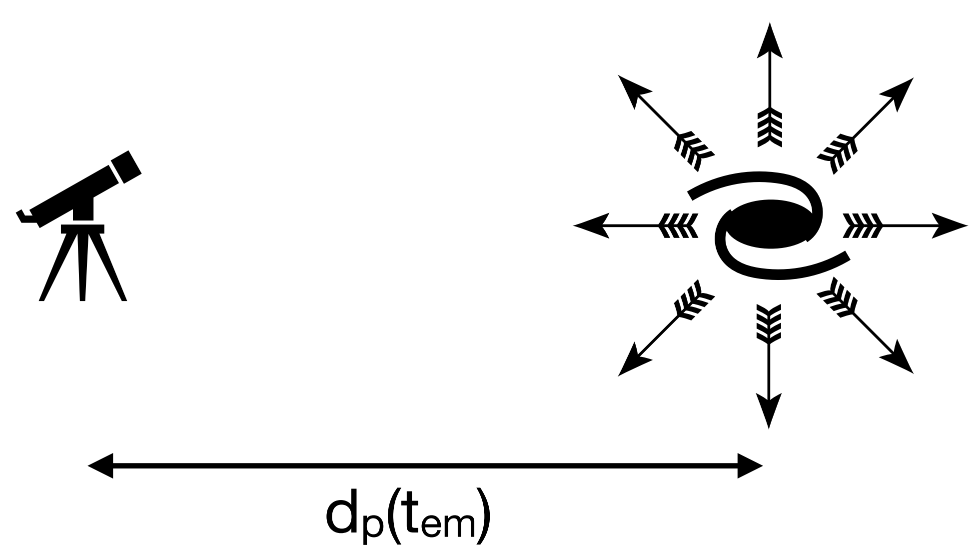
$$l = \theta d_A$$

Luminosity distance

What is flux? It is the rate of energy through a unit area:

$$F = \frac{L}{4\pi d_L^2}$$

At time $t=t_{em}$ a distant galaxy emits photons in all directions. At the time, the proper distance between us and the galaxy is $d=d_p(t_{em})$.



Luminosity distance

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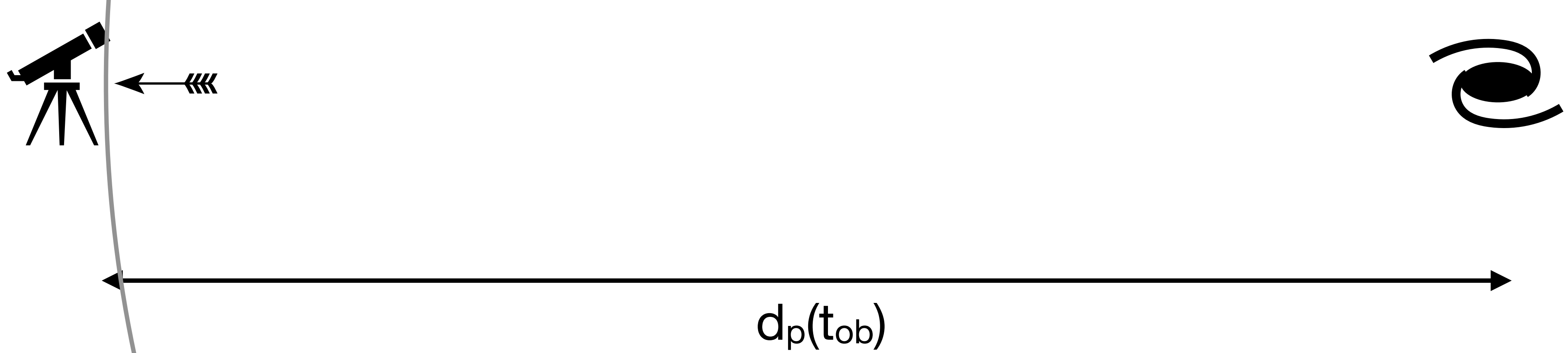
By the time those photons reach us at $t=t_{ob}$, the proper distance between us and the galaxy has increased to $d_p(t_{ob})$ - the current proper distance.

Meaning the photons are spread over an area of $4\pi d_p(t_{\rm ob})^2$ giving $F = \dfrac{L}{4\pi d_p(t_{\rm ob})^2}$

But, geometric dilution is not all we need to worry about…



Luminosity distance

What is flux? It is the rate of energy through a unit area:

$$F = \frac{L}{4\pi d_L^2}$$

As well as geometric dilution, we have to also consider the what effect the relative motions between us - as observer - and the galaxy has on flux.

While the galaxy will emit photos at a given rate, r_{em}, because of our relative motions, we will detect them at a slower rate of $r_{ob} = r_{em}/(1+z)$:

$$F = \frac{L}{4\pi d_p(t_{\rm ob})^2} \frac{1}{1+z}$$

Further, the due to the stretching of their wavelength energy of each photon will also be reduced by another factor of 1+z:

$$F = \frac{L}{4\pi d_p(t_{\rm ob})^2} \frac{1}{(1+z)} \frac{1}{(1+z)}$$

Comparing the above equation with the top equation gives:

$$\boxed{d_L = d_p(t_{\rm ob})(1+z)}$$

Luminosity distance

If you know the intrinsic luminosity of a source, and measure its flux (e.g., with a telescope) then we can calculate its Luminosity Distance:

$$d_L = \left(\frac{L}{4\pi F}\right)^{1/2}$$

and

$$d_L = d_p(t_0)(1+z)$$

So, if we know:

- the flux (easy),
- the redshift (easy), and
- the intrinsic luminosity (hard)

of a distant object, we have everything we need to fit the redshift-distance relationship, calculate a(t), and obtain cosmological parameters.

Standard Candles 1: Cepheid Variables

Standard Candles are sources whose intrinsic luminosities are known (or easily derived from observations).

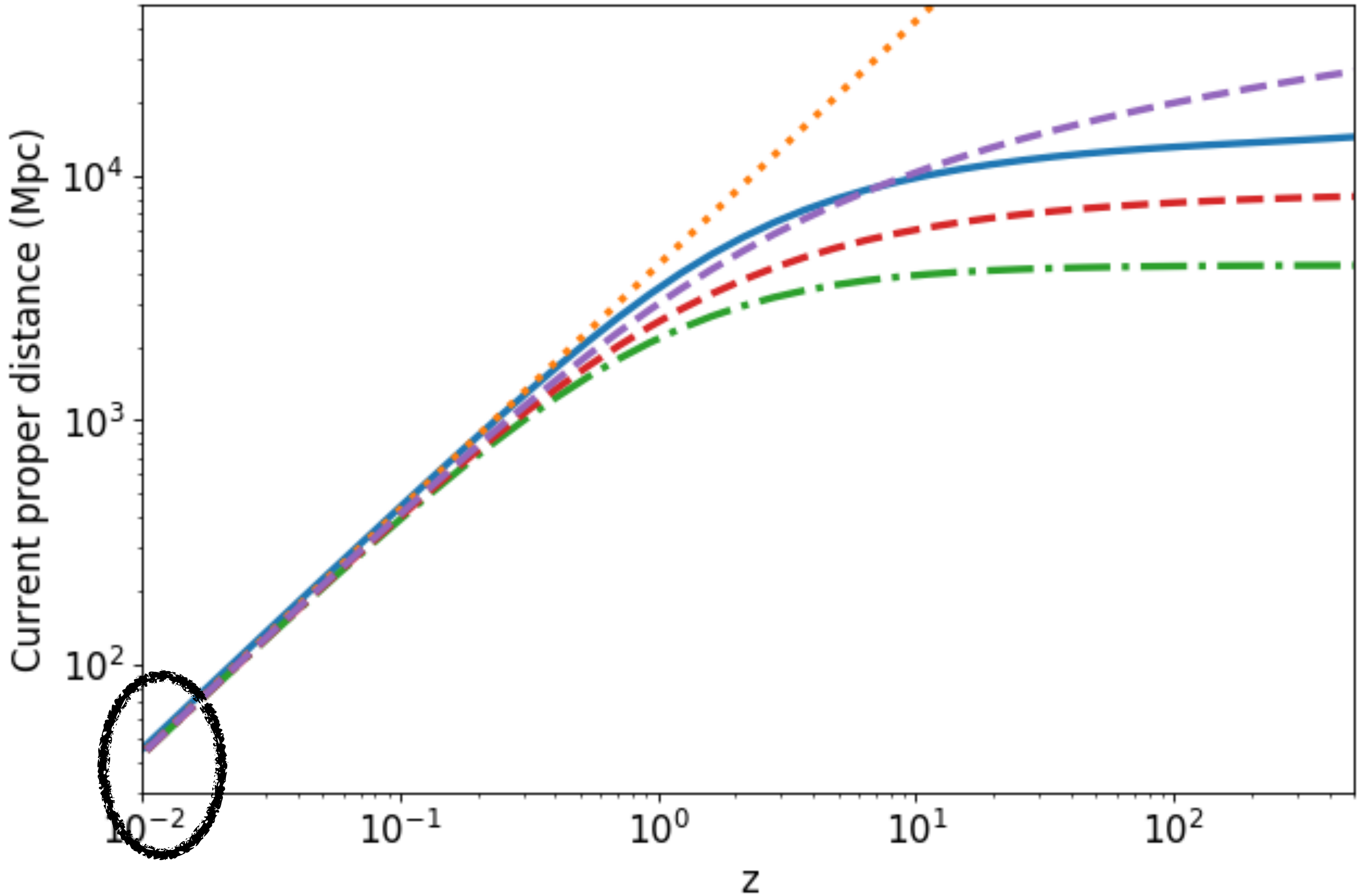
The classic Standard Candle is the *Cepheid Variable* - a star whose peak luminosity is related to the period of its variation.

Once you've measured a Cepheid's period, *P* (in days), you can easily calculate its intrinsic luminosity:

$$L_{\text{Ceph}} = 0.22 P^{-0.97} L_{\odot}$$

But, the most distant Cepheids our telescopes are able to detect are only around 30 Mpc away, corresponding to a redshift of *z* = 0.007!

Standard Candles 1: Cepheid Variables



Cepheids are fine for calculating H0, but useless for calculating a(t) since we can't observe them to high enough redshifts to use them to distinguish between models. We need something **much brighter**.

Standard Candles 2: Type 1a Supernovae

Type 1a supernovae are produced when a White Dwarf accretes mass from a binary partner.

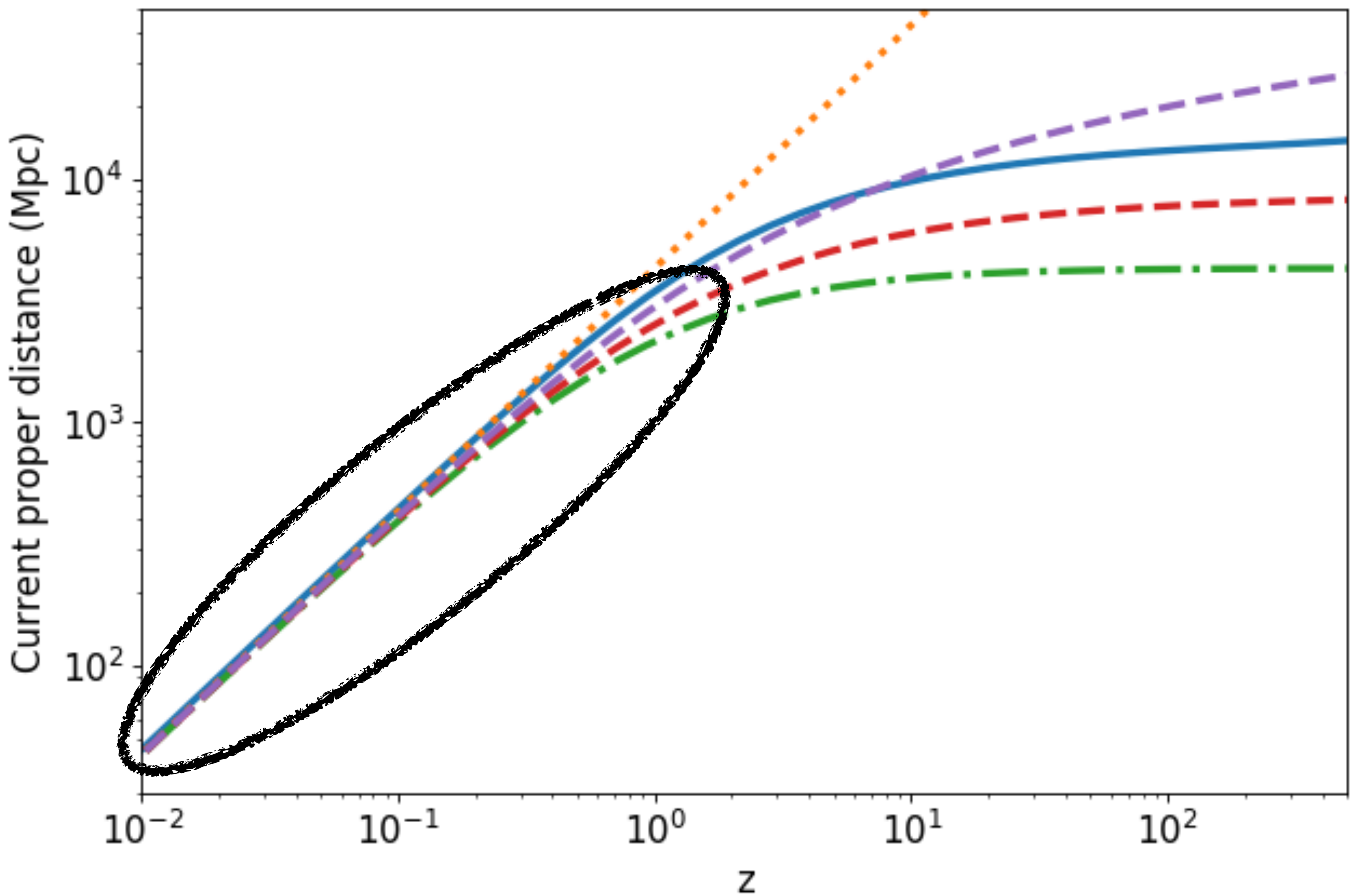
As is approaches the Chandrasekhar limit mass of 1.4MSun, it starts to undergo carbon fusion in its core.

This triggers a runaway fusion reaction which blows the star up.

Since the explosion always occurs when a White Dwarf star reaches the same given mass, its luminosity is always roughly the same.

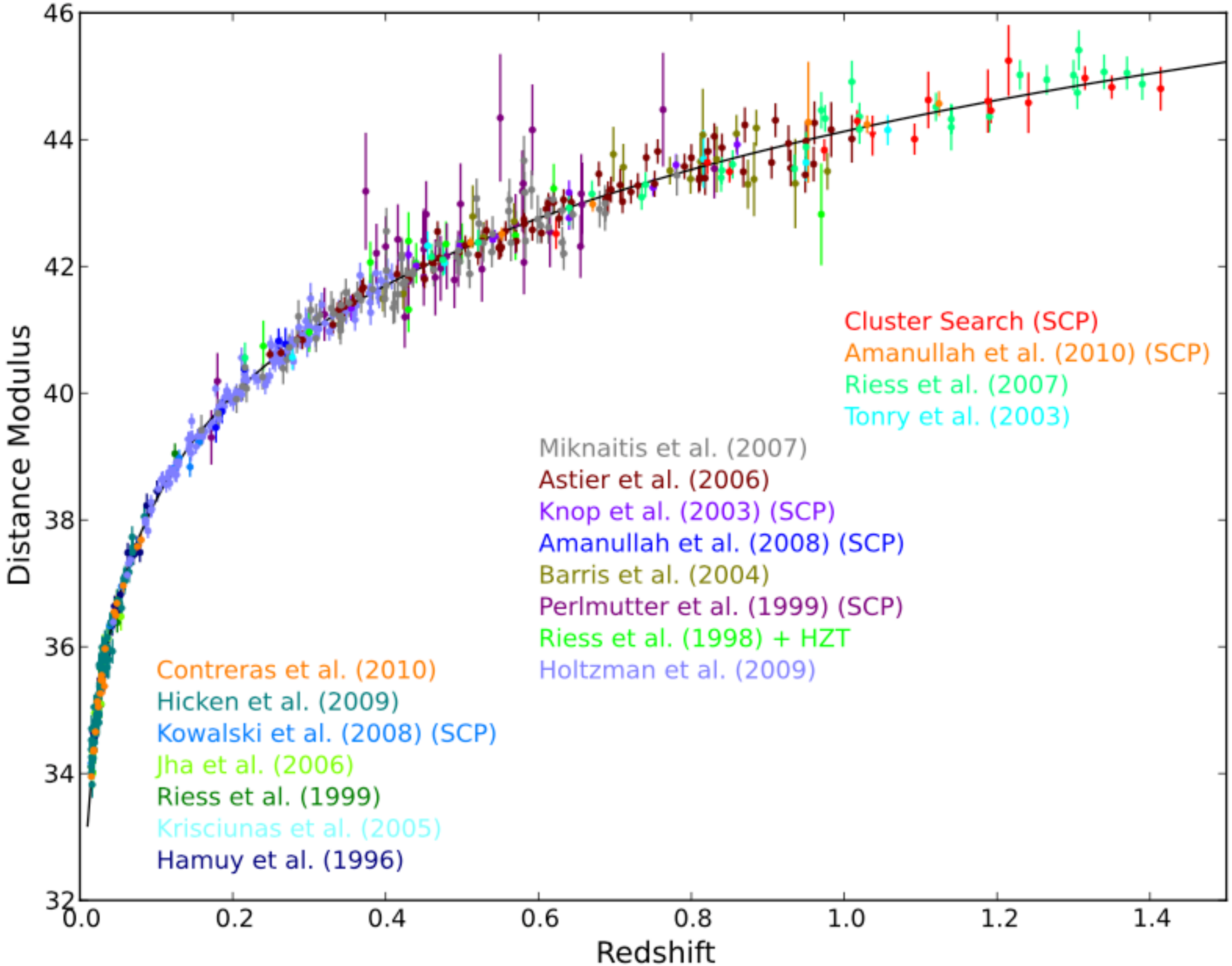
As such, it can be used as a Standard Candle.

Standard Candles 2: Type 1a Supernovae



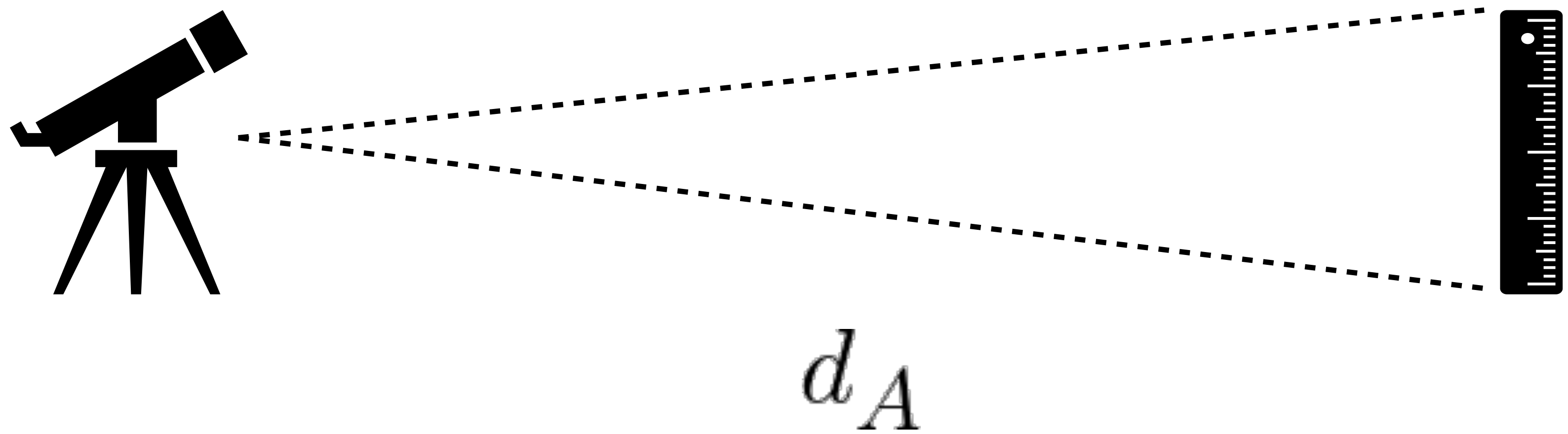
Type 1a Supernovae are about 100,000 times more luminous than the most powerful Cepheid. As such, they can be observed to much higher redshifts (the record is z~2), enabling cosmologists to determine *a(t)*.

Standard Candles 2: Type 1a supernovae



Angular size distance

Angular size distance:



$$l = \theta d_A$$

When we measure the angular size of a distant object, we are measuring the angular scale as it was at the time of emission.

As such, the angular size distance is the proper distance at time of emission:

$$\boxed{d_A = d_p(t_{\mathrm{em}}) = \frac{d_p(t_{\mathrm{ob}})}{1+z}}$$

Getting the feel of it…

We can’t solve the F.E. for the real Universe analytically.

Instead, we use computers to solve it numerically by “integrating under the curve”.

Doing so, we can determine how the scale factor evolves, and from that determine proper distances, for a given set of cosmological parameters.

However, what’s *more* important is that we can work backwards…

…from a set of observed distances and redshifts, determine the cosmological parameters of the real Universe.