Lecture 04:
Solving the Friedmann Equation: Thermodynamics and the Equation of State

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1 The critical density

- In the previous lecture, we saw (well, at least in Newtonian terms) how we could relate the expansion/contraction of the universe – parameterised as a(t) – to the energy density (ε) and curvature of the universe (κ).
- The Friedmann Equation holds true at all times throughout the universe.
- Since the RHS of the FE allows for a time-dependency via $a(t)$ and $\varepsilon(t)$, it must mean that the LHS may also be time-dependent.
- And, in general, it is (although, later we'll see one example model universe where it is not).
- As such, the Friedmann Equation tells us that the Hubble parameter is not a constant.
- But – and rather confusingly – we can write the Friedmann Equation in terms of today's parameters to obtain an expression for the Hubble constant, H_0.
- All we need to do is specify: $t = t_0$, $a(t) = a(t_0) = 1$, $\varepsilon(t) = \varepsilon(t_0) = \varepsilon_0$ and $H(t) = H(t_0) = H_0$:

$$H_0^2 = \left(\frac{\dot{a}}{a}\right)^2_{t=t_0} = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{\kappa c^2}{R_0^2} \tag{1}$$

- In a flat $\kappa = 0$ universe we know that the second term on the RHS of the F.E. is zero.
- So, we can therefore say:

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon \tag{2}$$

- Which, rearranged gives:

$$\varepsilon(t) = \frac{3c^2}{8\pi G}H(t)^2 \tag{3}$$

- This therefore represents, for a given value of H, the energy density that would force the universe to be flat.
- An energy density above that critical value would mean $\kappa = 1$ in order for the FE to balance.

- An energy density below that critical value would mean $\kappa = -1$ in order for the FE to balance.
- We therefore call this the *critical density*, ε_c, i.e.,

$$\varepsilon_c(t) = \frac{3c^2}{8\pi G} H(t)^2 \tag{4}$$

- As we'll see, it's often more convenient to think of energy densities as a fraction of this critical density (especially for the real Universe, since it appears flat):

$$\Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)} \tag{5}$$

- But, lets consider what happens if we put $\varepsilon(t) = \varepsilon_c(t)\Omega(t)$ back into the FE:

$$H(t)^2 = \frac{8\pi G}{3c^2}\Omega(t)\epsilon_c(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2} \tag{6}$$

- Substituting 4 for ε_c:

$$H(t)^2 = \frac{8\pi G}{3c^2}\Omega(t)\frac{3c^2}{8\pi G}H(t)^2 - \frac{\kappa c^2}{R_0}\frac{1}{a(t)^2} \tag{7}$$

- Cancelling, and taking the first term on the LHS to the RHS:

$$H(t)^2(1 - \Omega(t)) = -\frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2} \tag{8}$$

- Giving:

$$1 - \Omega(t) = -\frac{\kappa c^2}{R_0^2}\frac{1}{H(t)^2 a(t)^2} \tag{9}$$

- Look at all those squared values that can't change sign. And κ is time-independent.
- This means that $1 - \Omega(t)$ can *never* change sign.
- If the universe starts out with a density above the critical density, it will remain that way for ever.
- If the universe starts out with a density below the critical density, it will remain that way for ever.
- If the universe starts out with a density equal to the critical density, it will remain that way for ever.
- And today, since $a(t_0) = 1$:

$$1 - \Omega(t_0) = -\frac{\kappa c^2}{R_0^2}\frac{1}{H(t)^2} \tag{10}$$

- So, if we can measure $\Omega(t_0)$, we'll know whether the universe is open, closed or flat. If we can also measure H_0, we'll also know R_0.

2 Solving the Friedmann Equation

2.1 The Fluid Equation

- Got the FE, but we're missing some key ingredients before we can solve it to get an expression for $a(t)$ – an expression that describes the expansion or contraction of the universe.
- In particular, we're missing an expression for $\varepsilon(t)$ - how the energy density of the universe changes with time.
- To find an expression for $\varepsilon(t)$, we're going to turn to Thermodynamics.
- This seems reasonable, since Thermodynamics is all about the flow of energy.
- We'll start with the first law of thermodynamics:

$$dQ = dE + PdV \tag{11}$$

  which basically states that if there is heat flow dQ into a volume dV, either the thermal energy dE in that volume goes up, the volume increases, the pressure increases, or any combination of the three.
- In a homogeneous, isotropic universe, there is no net heat flow, otherwise there's either be a spacial place (energy flowing from one point to another) or a spacial direction (along the path of the net flow).
- And, if we also divide Eq. 11 by dt, we get:

$$0 = \dot{E} + P\dot{V} \tag{12}$$

- Let's consider a huge sphere of the universe with radius R, so big that we can ignore any clumpiness. This sphere is filled with an extremely tenuous gas with density equal to the average density of the universe (as we'll see later in the course, this is a pretty reasonable approximation, since most of the matter in the Universe is extremely tenuous).

$$V = \frac{4}{3}\pi R^3 \tag{13}$$

- We can, of course, describe R in terms of the scale factor, $R = a(t)r$:

$$V(t) = \frac{4}{3}\pi a(t)^3 r^3 \tag{14}$$

- Differentiate to get $\dot{V}$, using the chain rule to differentiate $a(t)$, i.e., :

$$\frac{d(a^3)}{dt} = \frac{d(a^3)}{da}\frac{da}{dt} = 3a^2\dot{a} \tag{15}$$

  giving:

$$\dot{V}(t) = \frac{4}{3}\pi r^3 3a(t)^2\dot{a} \tag{16}$$

- We can substitute Eq. 14 back into the above, in the form of $V(t)/a^3 = 4/3\pi r^3$ to get:

$$\dot{V}(t) = \frac{V(t)}{a(t)^3} 3a(t)^2 \dot{a} = V(t) 3 \frac{\dot{a}}{a} \tag{17}$$

- Now, we'll consider the energy part of Eq. 12. We can express the total energy content of the sphere as the energy density multiplied by the volume of the sphere:

$$E(t) = V(t)\varepsilon(t) \tag{18}$$

- Use the product rule to differentiate:

$$\frac{dE}{dt} = V(t)\frac{d\varepsilon(t)}{dt} + \varepsilon(t)\frac{dV(t)}{dt} \tag{19}$$

- And we can substitute Eq. 17 in for the final term to give:

$$\dot{E} = V(t)\left(\dot{\varepsilon} + 3\varepsilon\frac{\dot{a}}{a}\right) \tag{20}$$

- Putting Eqs. 17 & 20 into Eq. 12 gives:

$$0 = V(t)\left(\dot{\varepsilon} + 3\varepsilon\frac{\dot{a}}{a}\right) + 3PV(t)\frac{\dot{a}}{a} \tag{21}$$

- or, since $V(t) \neq 0$:

$$0 = \dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) \tag{22}$$

- Which is called the **Fluid Equation**.

2.2 Equations of State

- Great - we've created an expression for $\varepsilon(t)$ by introducing a new unknown - P!
- What we need now is an expression for P in terms of the other parameters in the Friedmann Equation.
- The easiest way to do this is via an Equation of State, which relates pressure to energy density.
- Equations of state can be really complicated, but in the tenuous, perfect gas conditions of the large-scale universe, they're really straightforward.
- Indeed, for a perfect gas they simply take the form:

$$P = \varepsilon w \tag{23}$$

- where w is a fixed constant for a given type of energy.

- For a non-relativistic perfect gas, we have the perfect gas law:

$$P = \frac{nRT}{V} = \rho RT = \frac{\rho}{\mu} kT \tag{24}$$

  where μ is the mass per particle and k is the Boltzmann constant.

- For such a gas, almost all its energy is in the form of mass, so $\varepsilon = \rho c^2$, giving:

$$P = \frac{kT}{\mu c^2} \varepsilon \tag{25}$$

- And, also for a perfect gas, the temperature and velocity are related through (from $KE = 3kT/2$:

$$3kT = \mu \langle v^2 \rangle \tag{26}$$

- So:

$$P = \frac{\langle v^2 \rangle}{3c^2} \varepsilon \tag{27}$$

- Meaning $P = w\varepsilon$, where $w = \langle v^2 \rangle / 3c^2$.
- For a non-relativistic gas, $w = \langle \frac{v^2}{3c^2} \rangle$, which is effectively 0.
- For a relativistic gas (e.g., of photons), $w = \frac{c^2}{3c^2} = 1/3$.
- And for a Dark Energy, we effectively get a negative pressure, meaning $w < -1/3$