Cosmology
Lecture 7

The “Benchmark model” and “measurable” distances

The Benchmark Model for the real Universe

The real Universe contains $\varepsilon_{\mathrm{m}}, \varepsilon_{\mathrm{p}}, \varepsilon_{\mathrm{d}}$ and possibly $\kappa \neq 0$

In this case, the F.E. is:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\sum_i \varepsilon_i - \frac{\kappa c^2}{R_0^2 a(t)^2}$$

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Which, by using $\Omega_{i,0} = \dfrac{\varepsilon_{i,0}}{\varepsilon_{c,0}}$ and the definitions in Lectures 3 and 4 becomes:

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{\rm p,0}}{a^4} + \frac{\Omega_{\rm m,0}}{a^3} + \Omega_{\rm d,0} + \frac{1-\Omega_0}{a^2}$$

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multiplying by a^2, taking the square route, and using $H = \dot{a}/a$ gives:

$$\frac{da}{dt} = H_0\left(\frac{\Omega_{\rm p,0}}{a^2} + \frac{\Omega_{\rm m,0}}{a} + \Omega_{\rm d,0}a^2 + (1-\Omega_0)\right)^{1/2}$$

Solving the Benchmark Model

Solving the Benchmark model to get a(t) for the real Universe thus involves:

$$\int_0^a \frac{da}{\left(\frac{\Omega_{\mathrm{p},0}}{a^2}+\frac{\Omega_{\mathrm{m},0}}{a}+\Omega_{\mathrm{p},0}a^2+(1-\Omega_0)\right)^{1/2}} = H_0\int_0^t dt = H_0 t$$

For the real Universe:

$$\begin{aligned}
\Omega_{p,0} &= 9\times 10^{-5}\\
\Omega_{m,0} &= 0.31\\
\Omega_{d,0} &= 0.69\\
\Omega_0 &= 1.00\\
H_0 &= 67.7\ \mathrm{km\ s^{-1}\ Mpc}
\end{aligned}$$

But this integral can’t be solved analytically.

What cosmologists do is solve it numerically (e.g., integrate by “area under curve” methods), evaluating the LHS at various values of *a* to get the corresponding value of *t*.

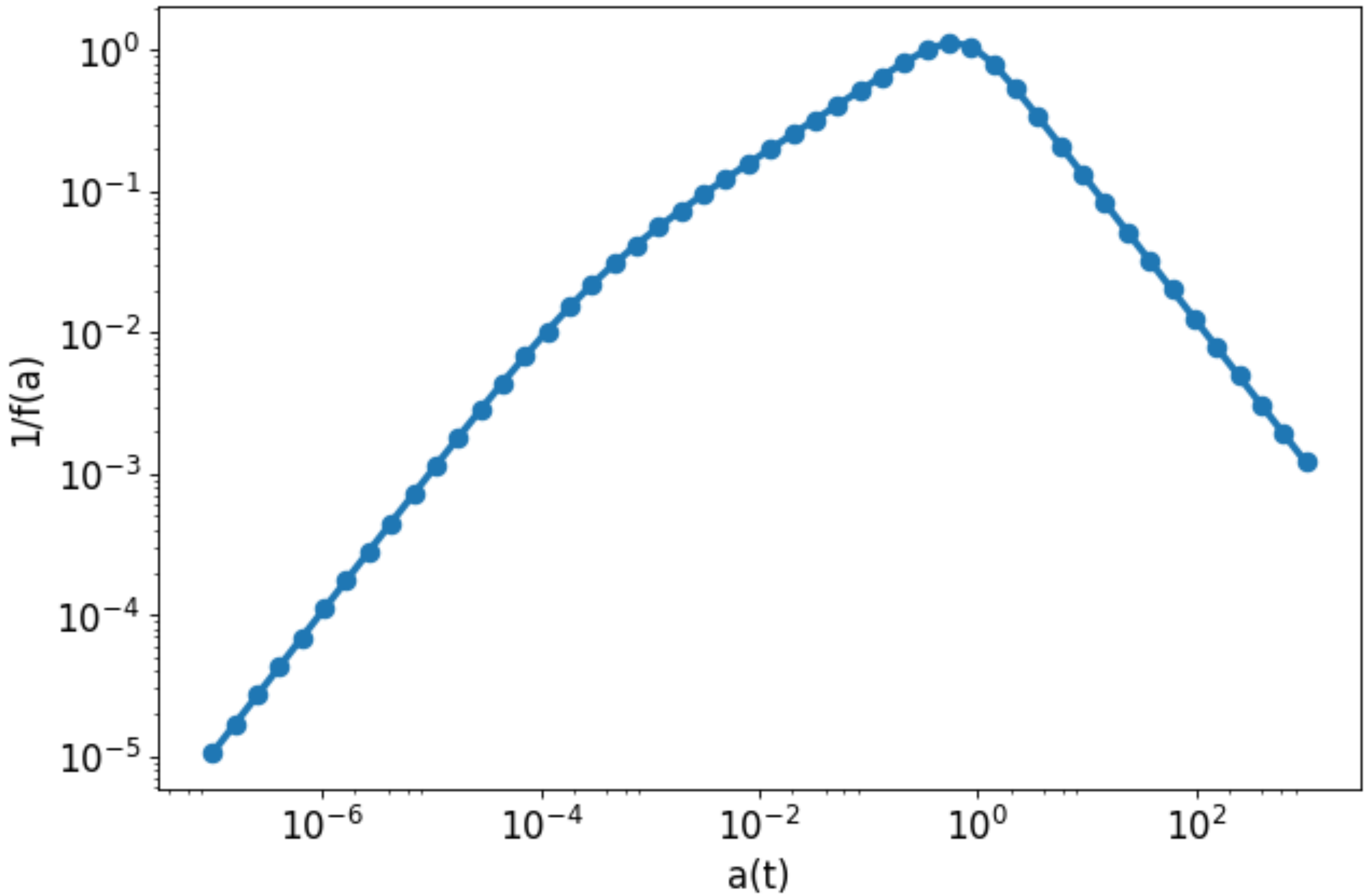
Numerically integrating the F.E.

$$\int_0^a \frac{da}{\left(\frac{\Omega_{\mathrm{p},0}}{a^2} + \frac{\Omega_{\mathrm{m},0}}{a} + \Omega_{\mathrm{d},0}a^2 + 1 - \Omega_0\right)^{1/2}} = H_0 t$$

1. Define a vector of *a* values.
2. Calculate the value of $\left(\frac{\Omega_{\mathrm{p},0}}{a^2} + \frac{\Omega_{\mathrm{m},0}}{a} + \Omega_{\mathrm{d},0}a^2 + 1 - \Omega_0\right)^{-1/2}$ at each *a* value.
3. Calculate the area under this curve between 0 and a given value of *a*.
4. This area is equal to $H_0 t$, meaning you've just calculated *t* for *a* given value of *a*.
5. Repeat steps 2-4 for each value of *a* in your vector to give *t* for every value of *a*.

We therefore get a vector of *t* values, one value for each value of *a* in our *a*-vector. We can then simply choose to plot the a vector against the t vector to get a plot of scale factor as a function of time…

Solving the F.E. numerically



where:

$$f(a) = \left(\frac{\Omega_{\mathrm{p},0}}{a^2} + \frac{\Omega_{\mathrm{m},0}}{a} + \Omega_{\mathrm{d},0} a^2 + 1 - \Omega_0 \right)^{1/2}$$

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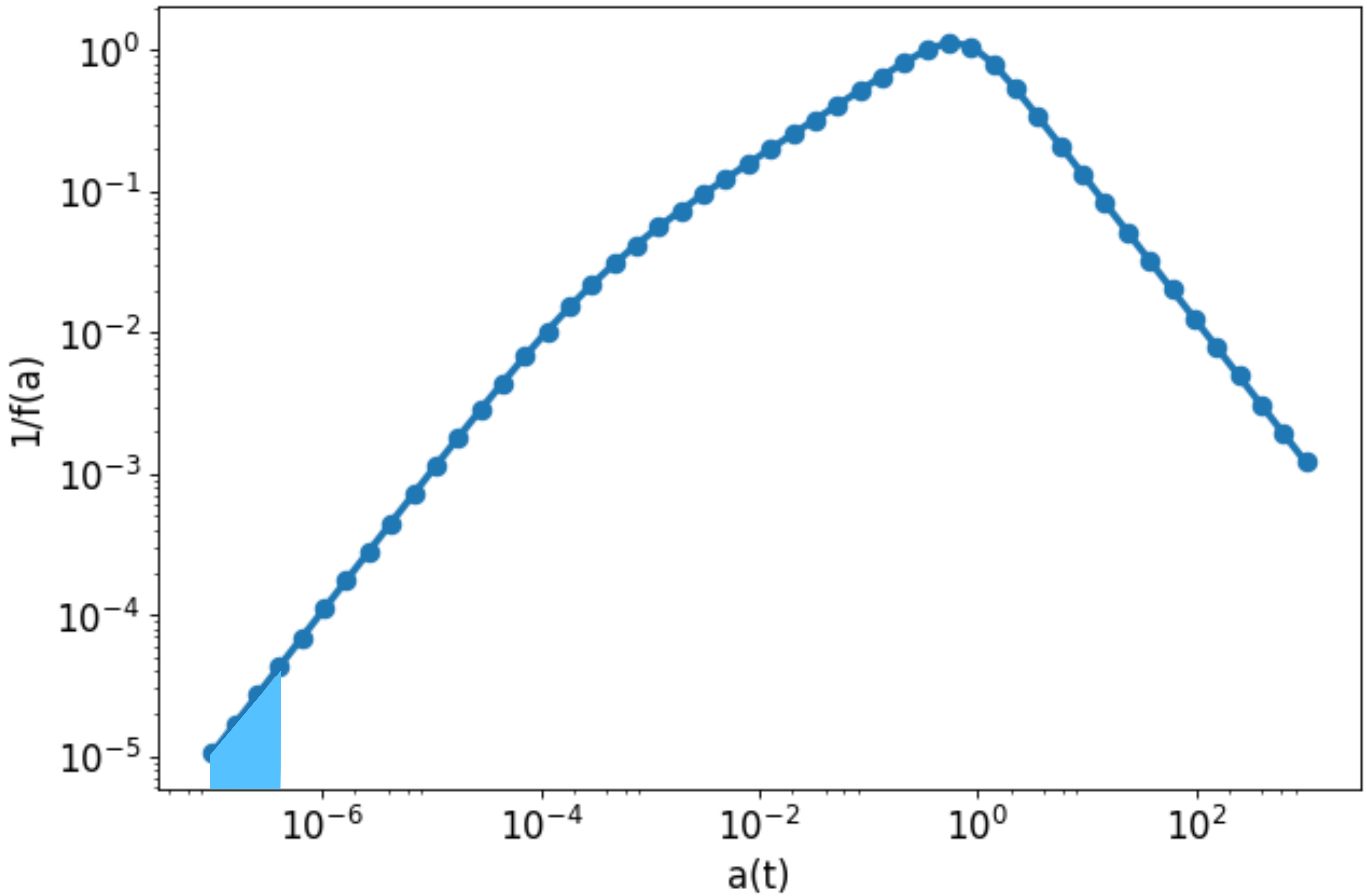

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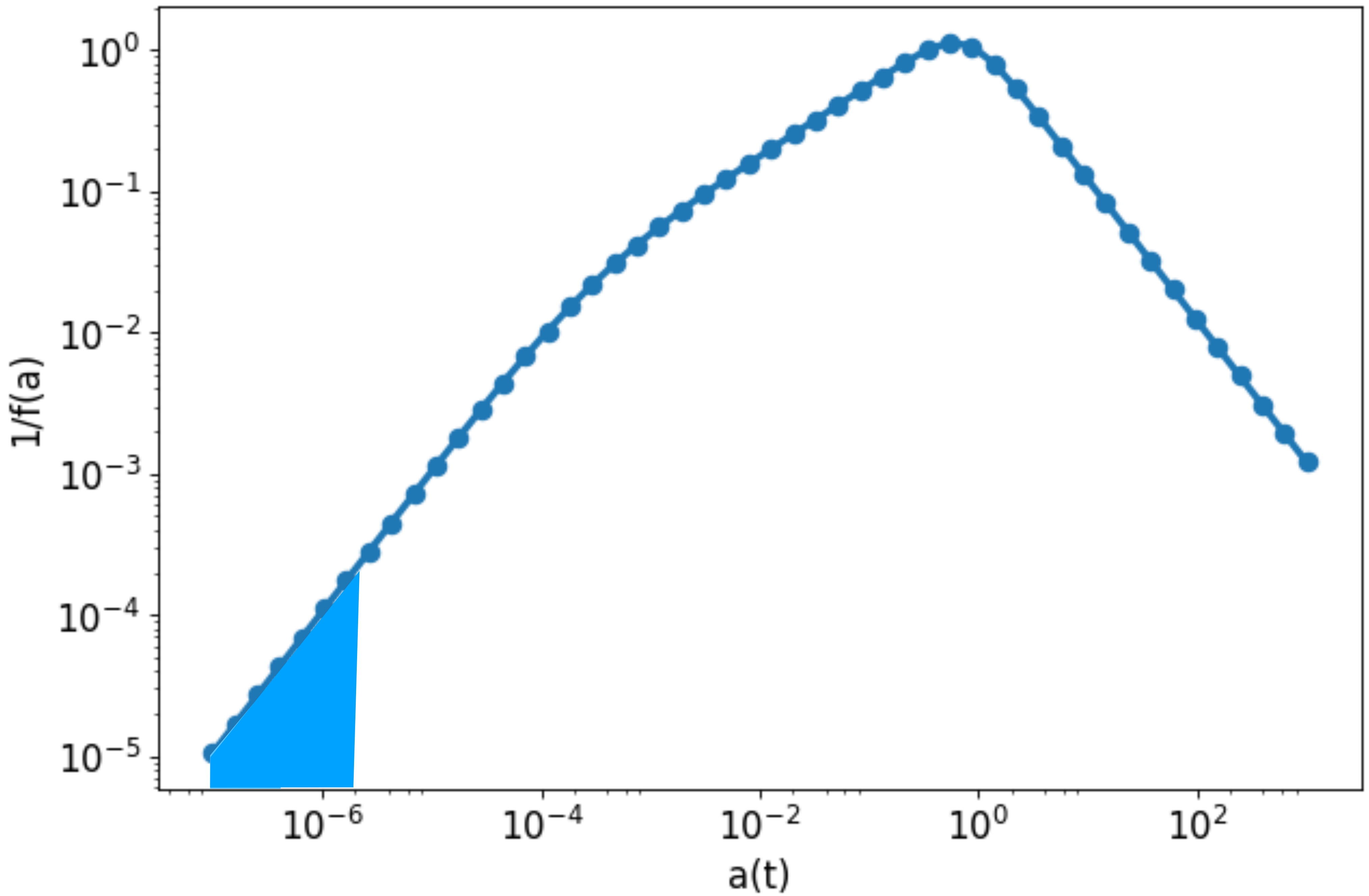

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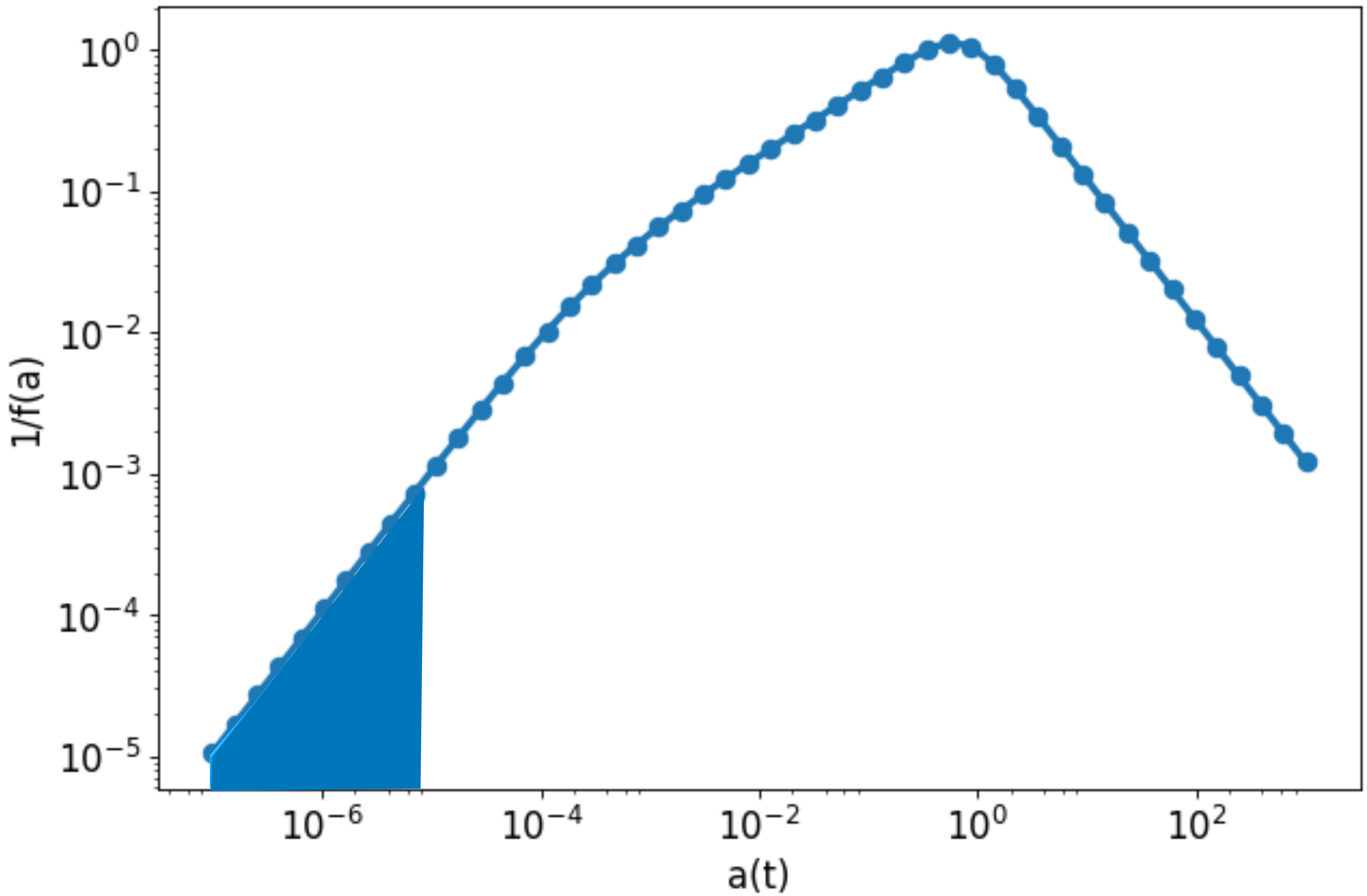

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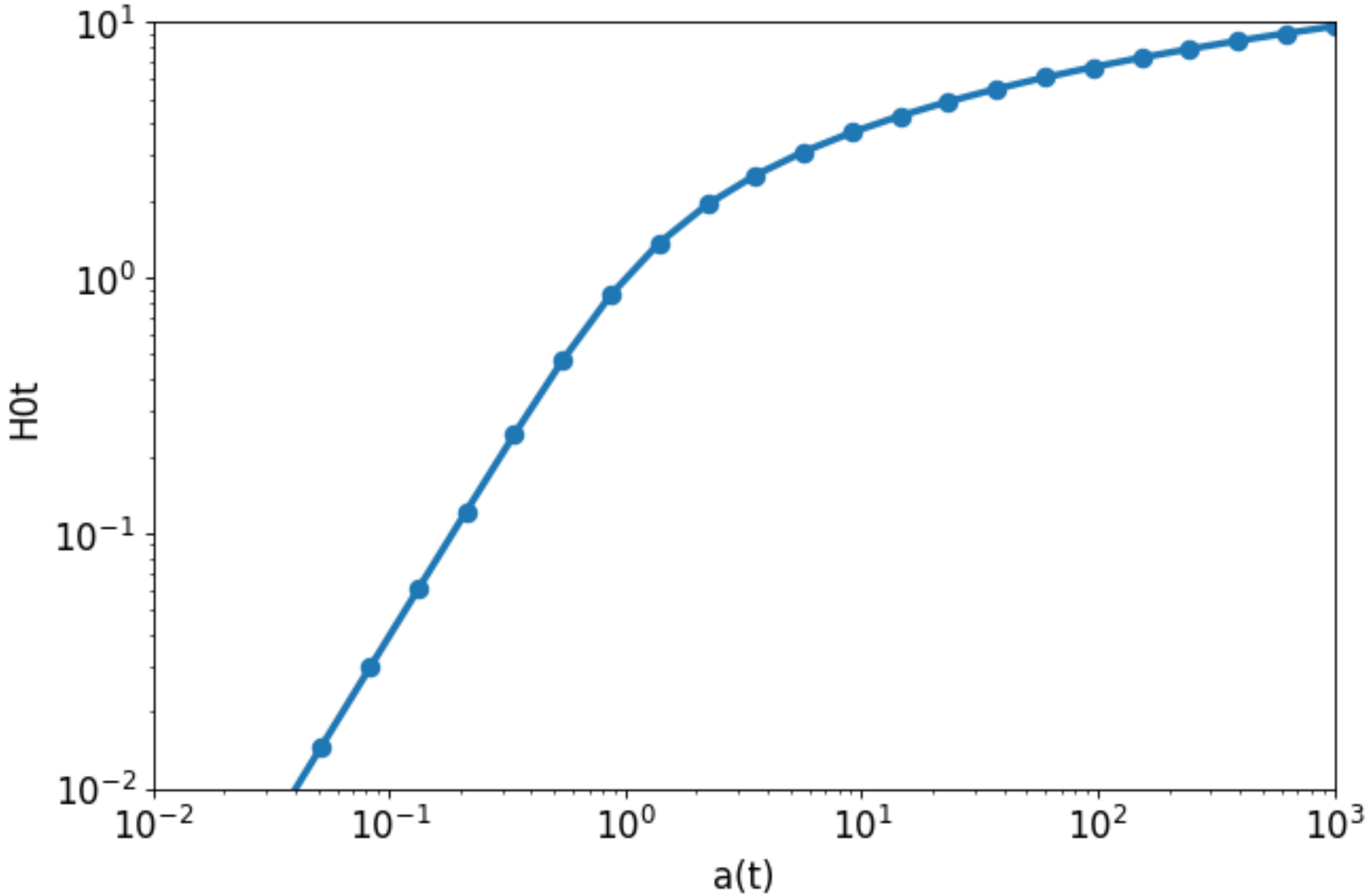
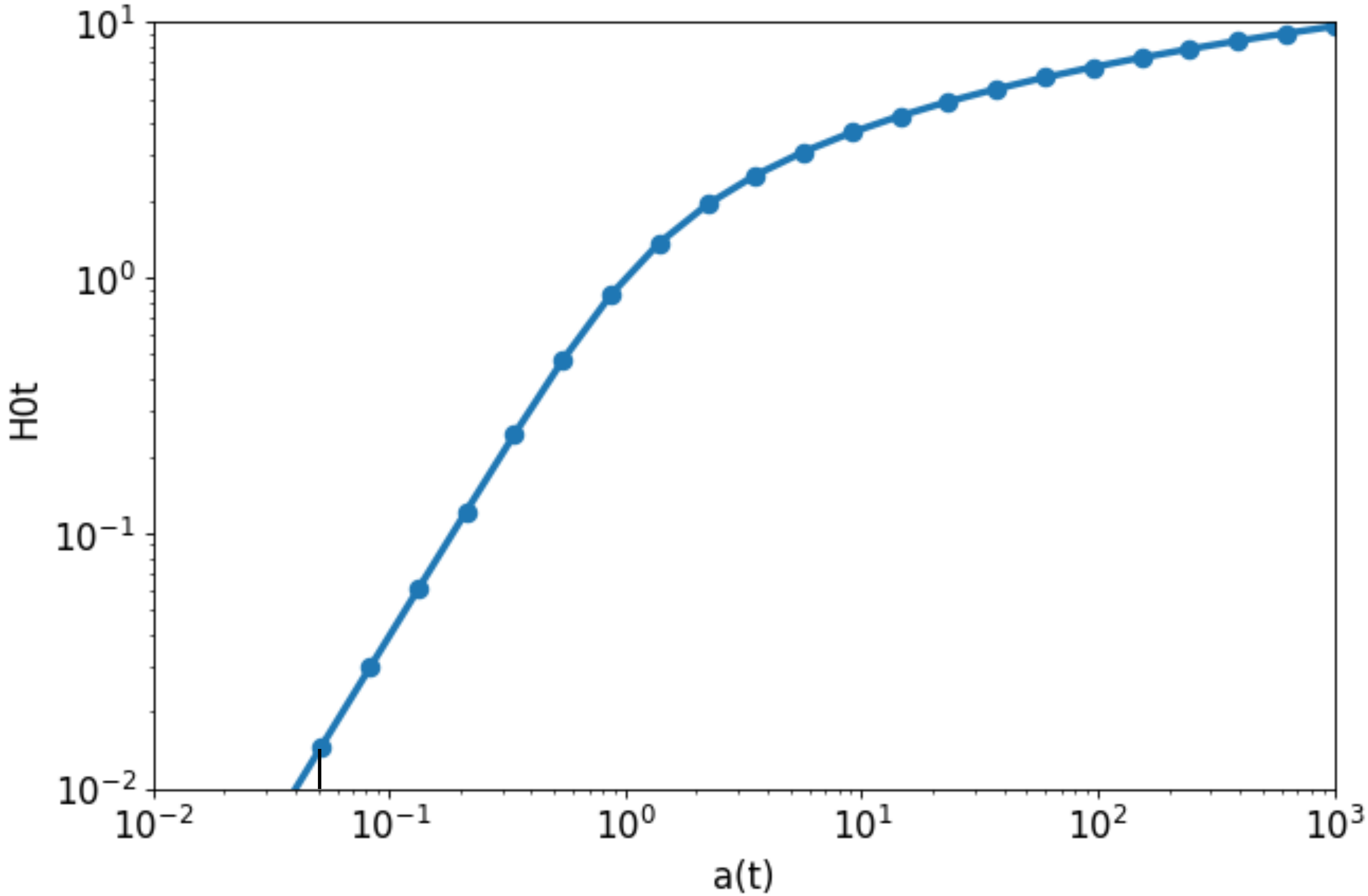


For each value of a, numerically integrate to calculate what value of H_0t that value of a corresponds to.

Solving the F.E. numerically



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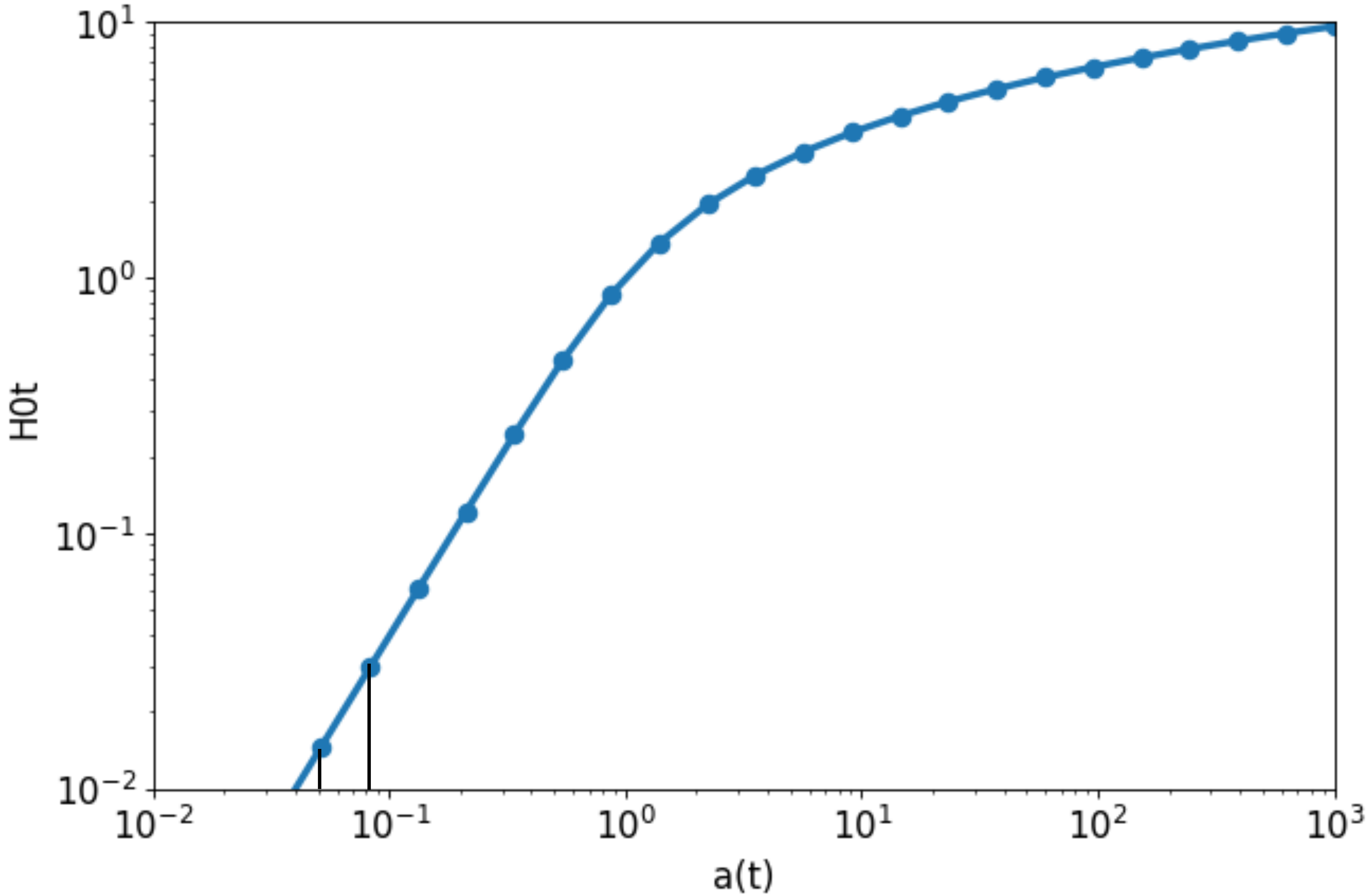

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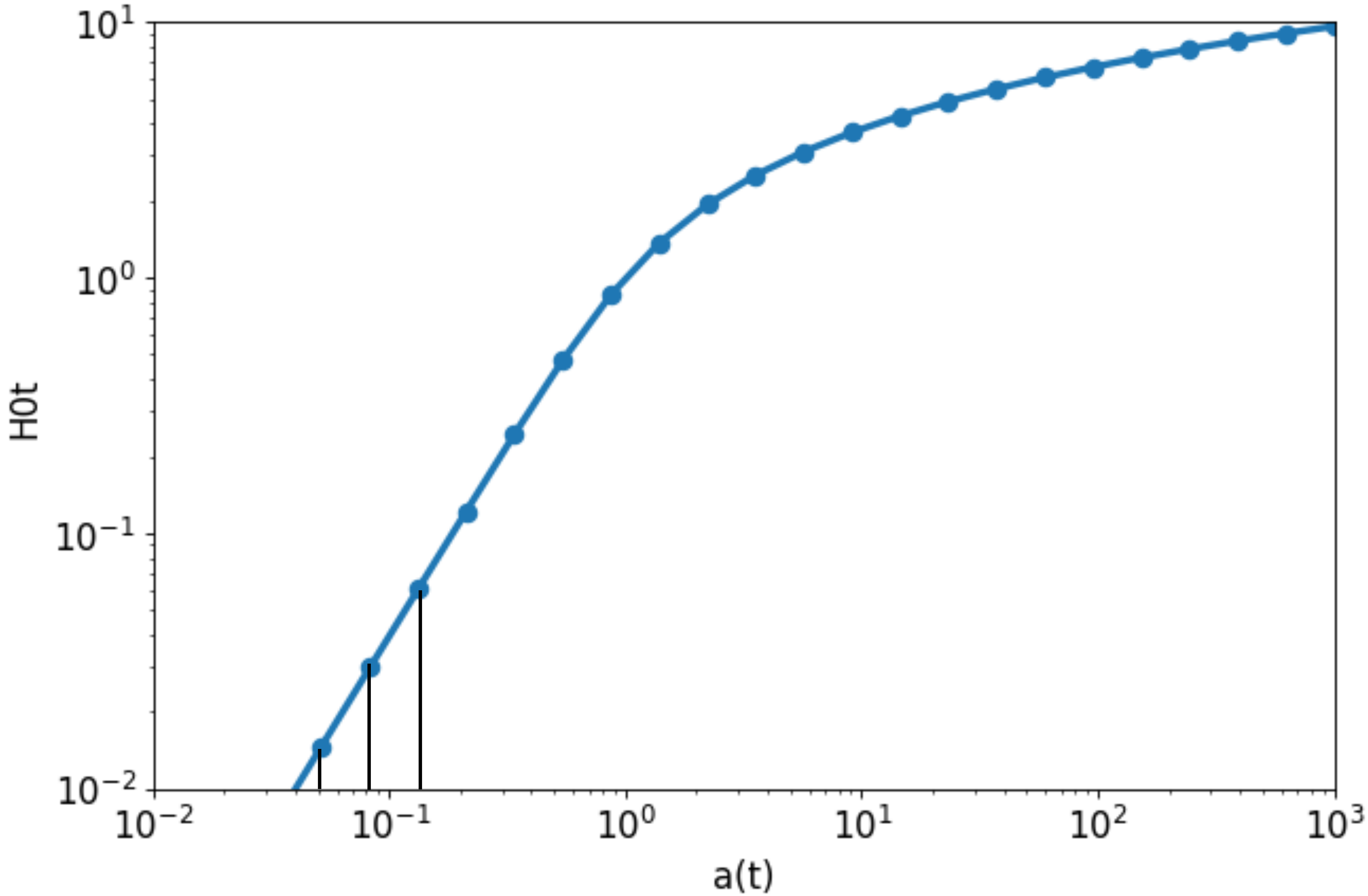

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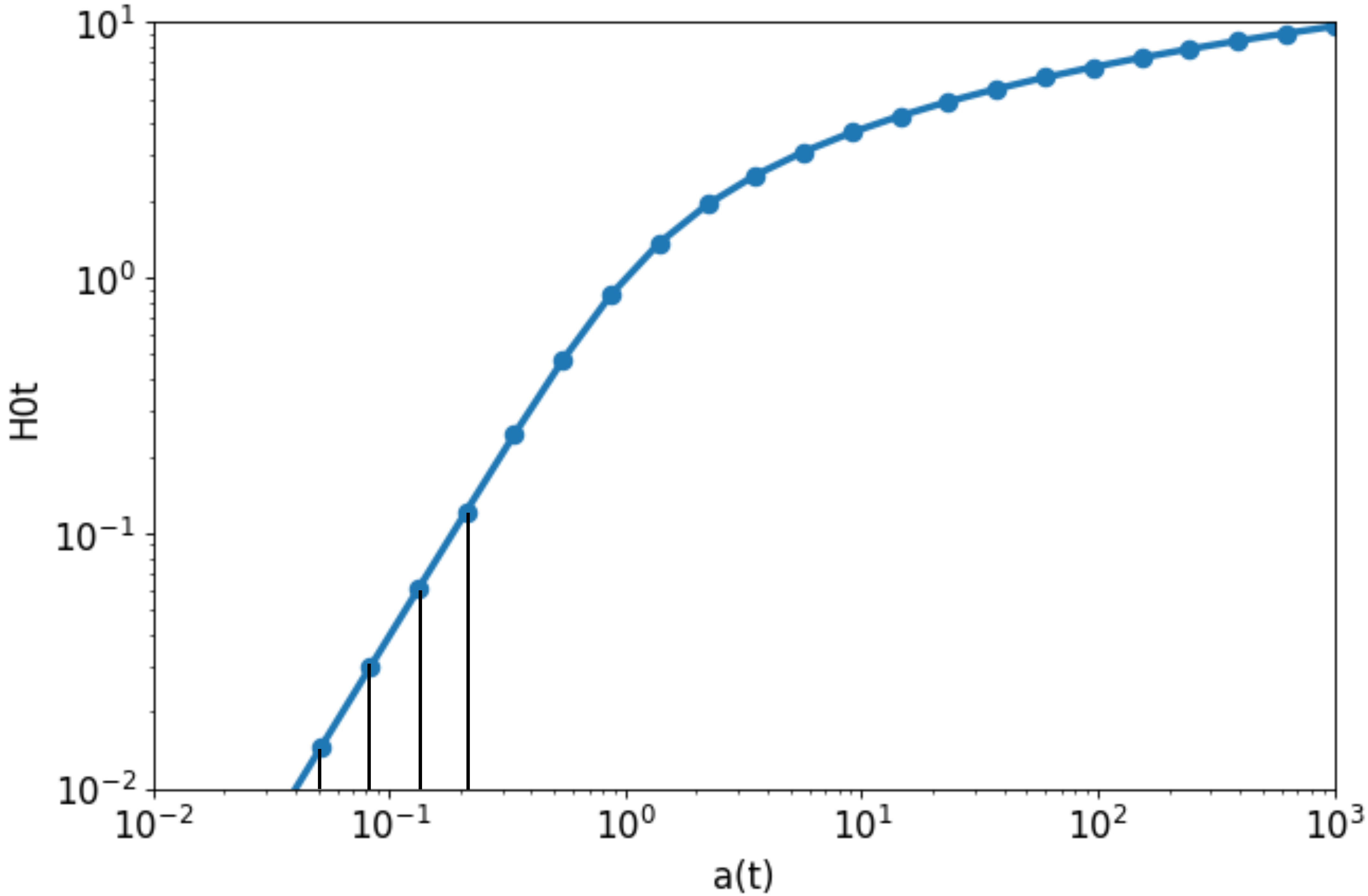

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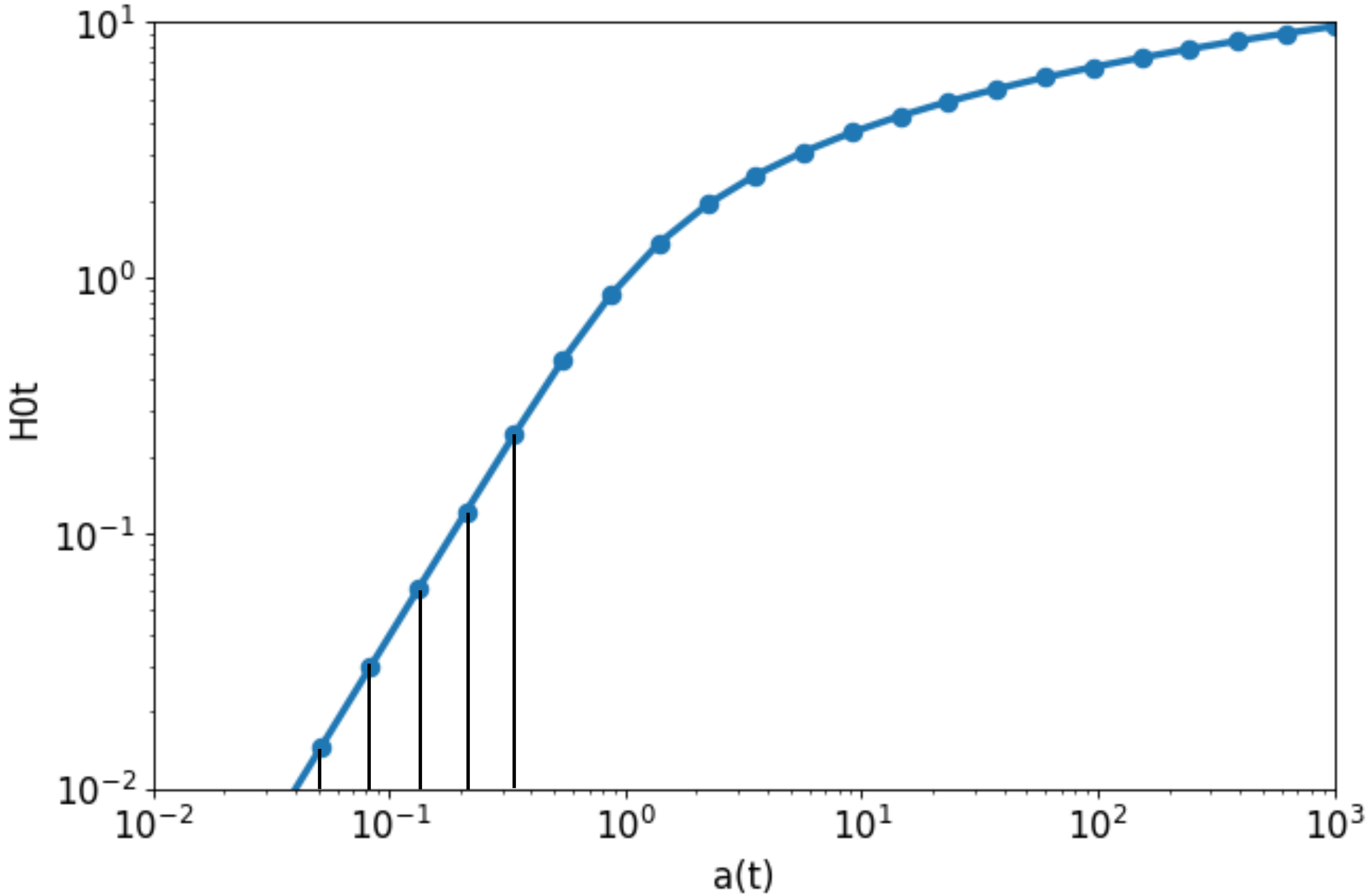

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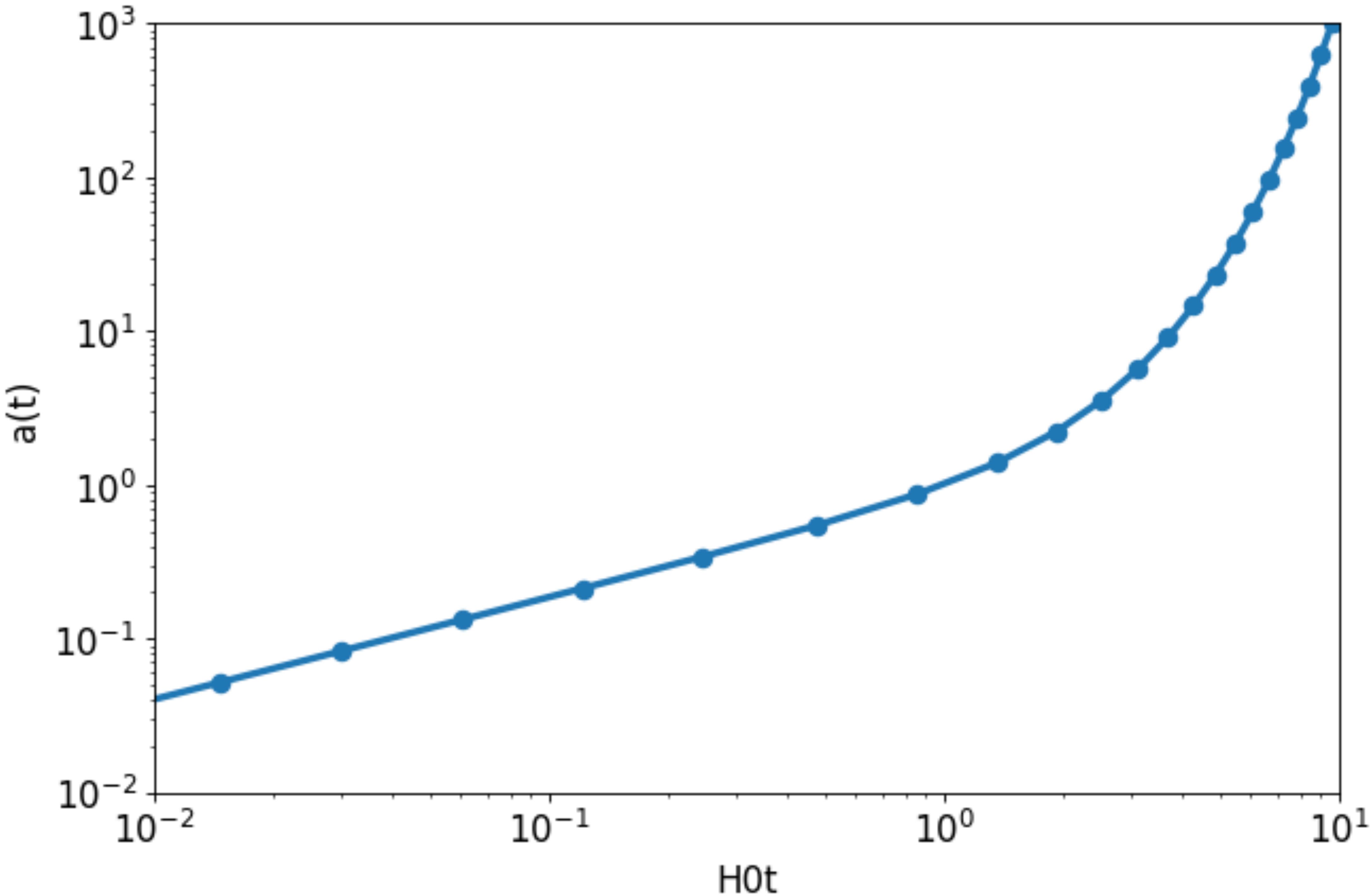
Scale factor vs. time for Benchmark Model



Then, to obtain how the scale factor changes with time, simply plot the a-vector on the y-axis, and the t-vector on the x-axis.

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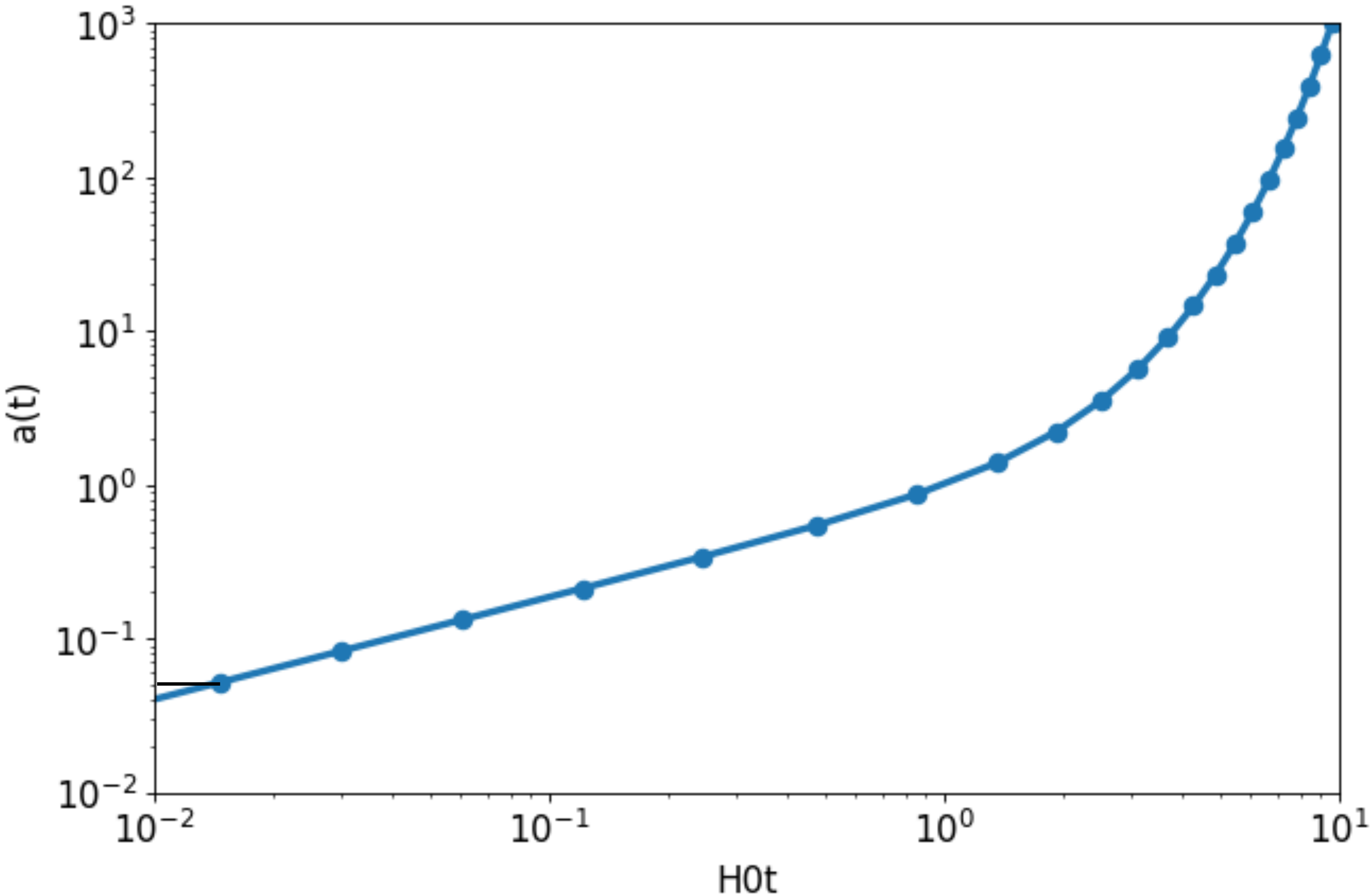

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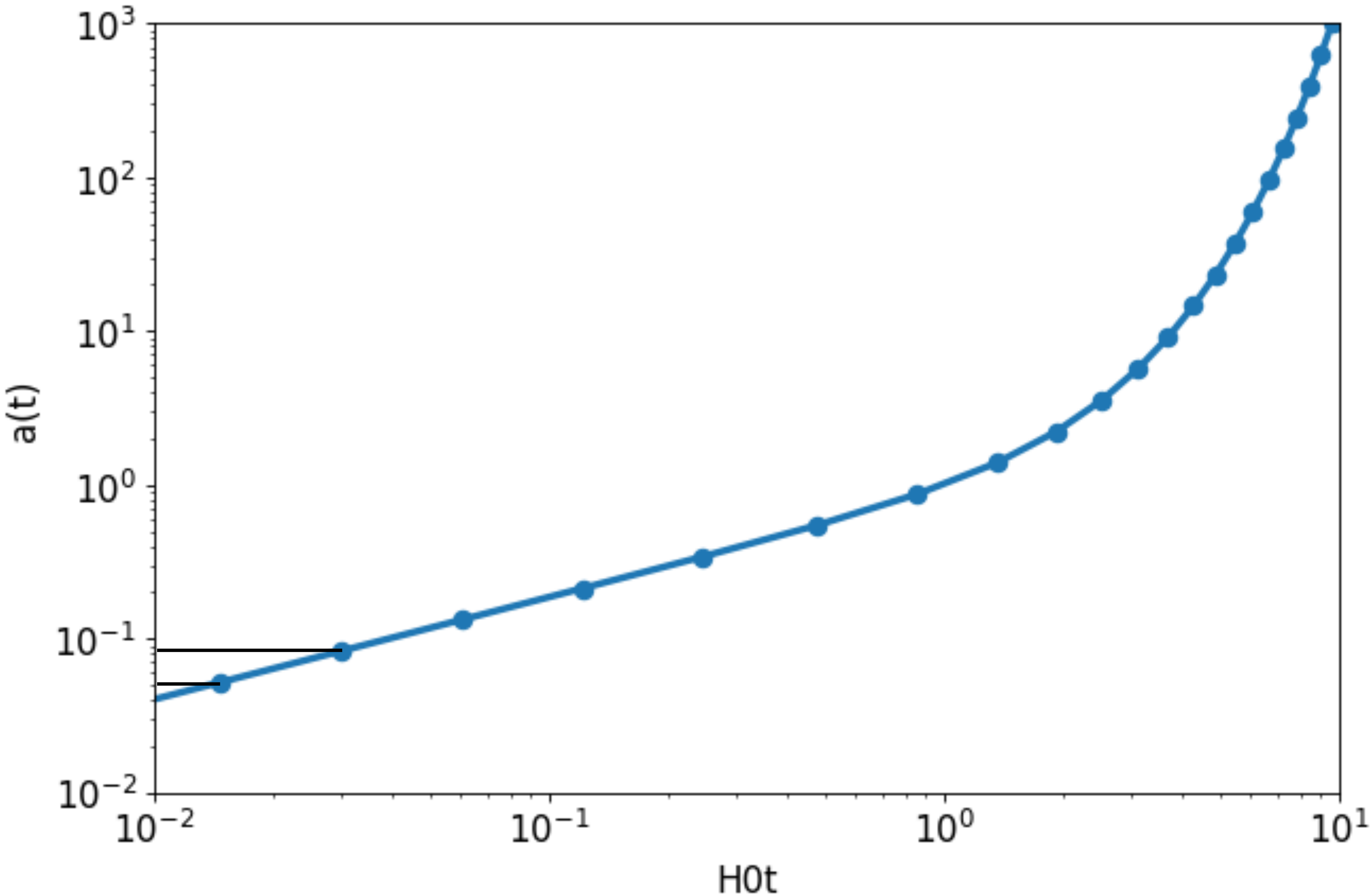

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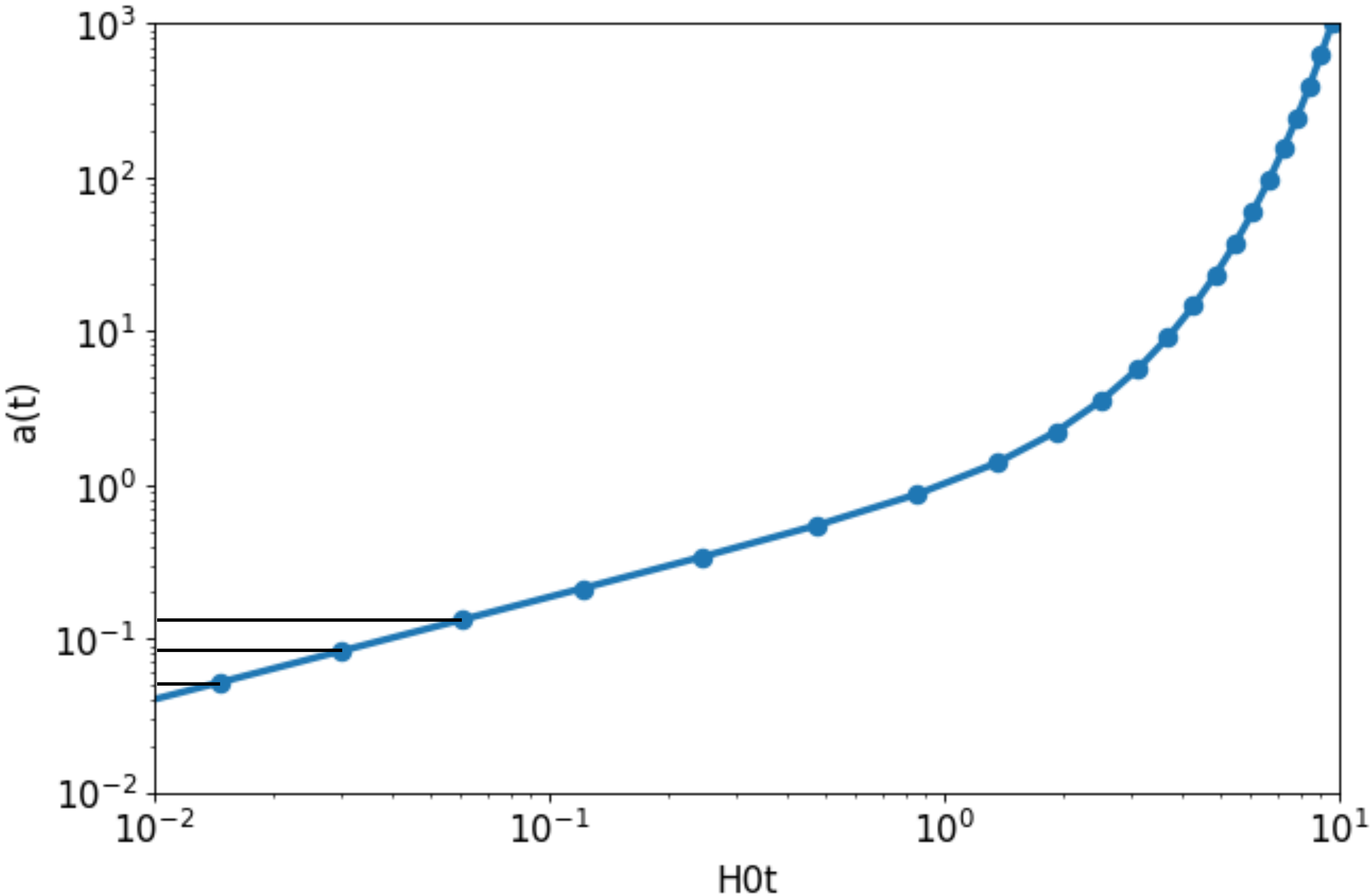

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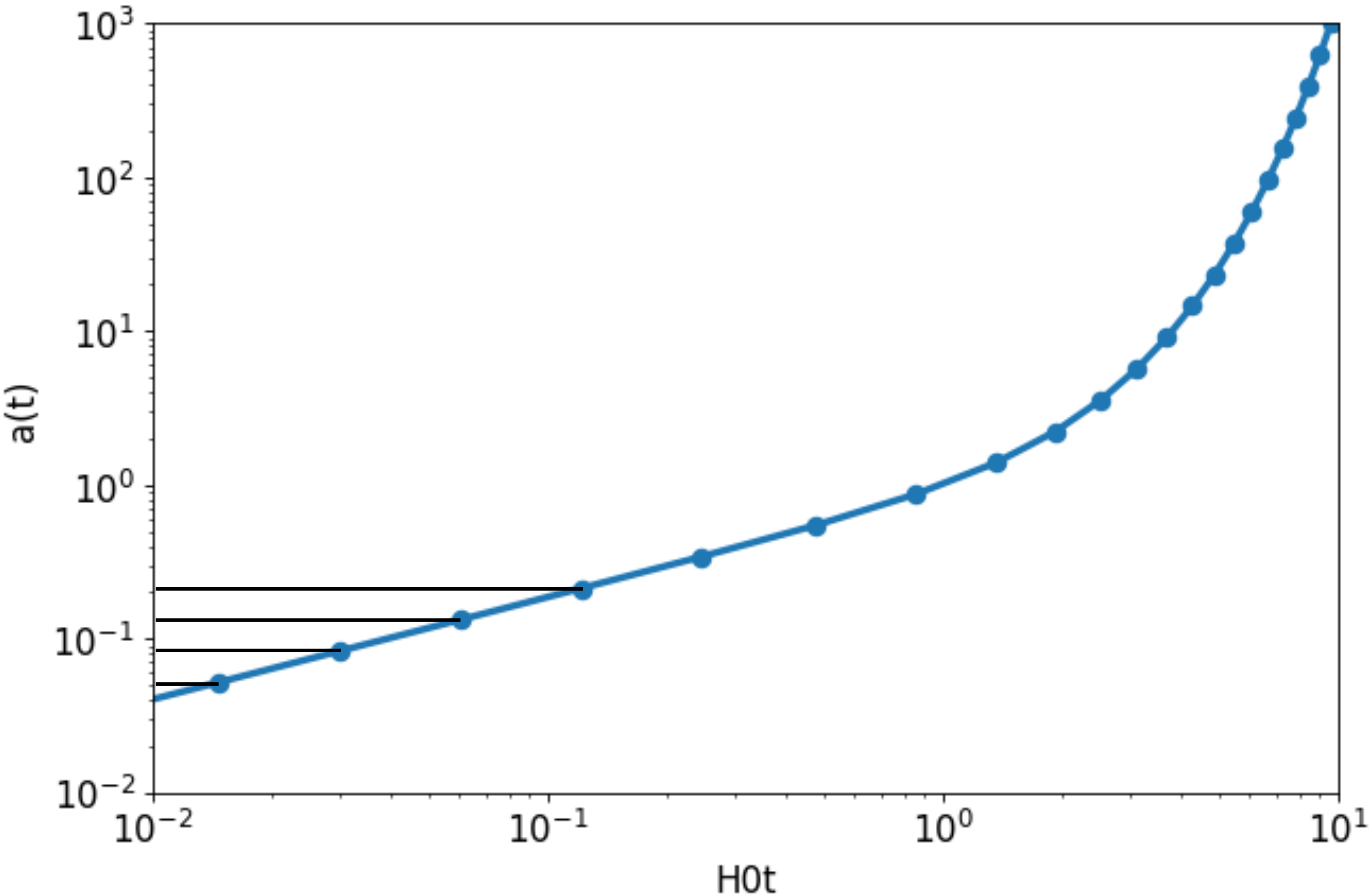

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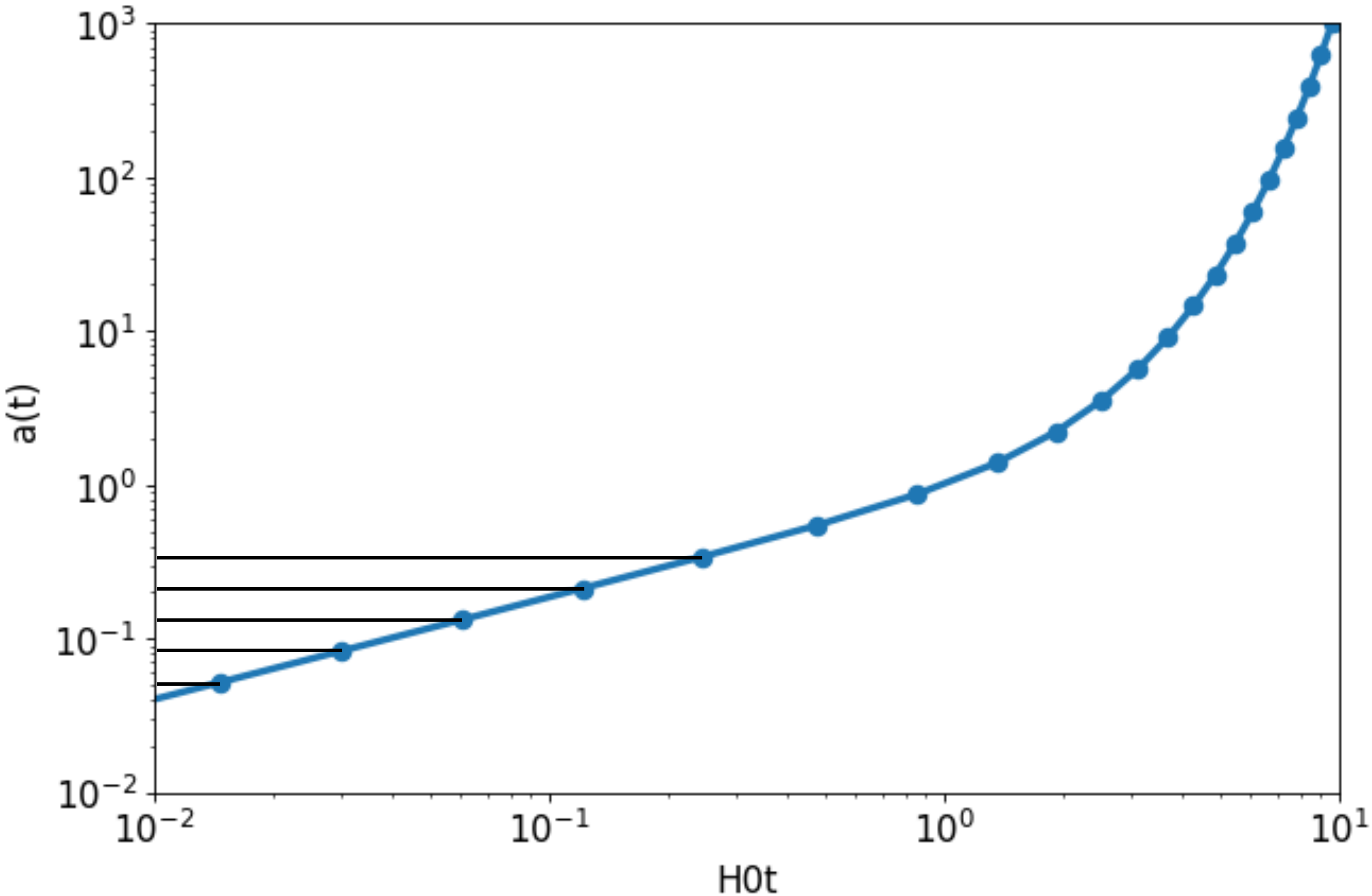

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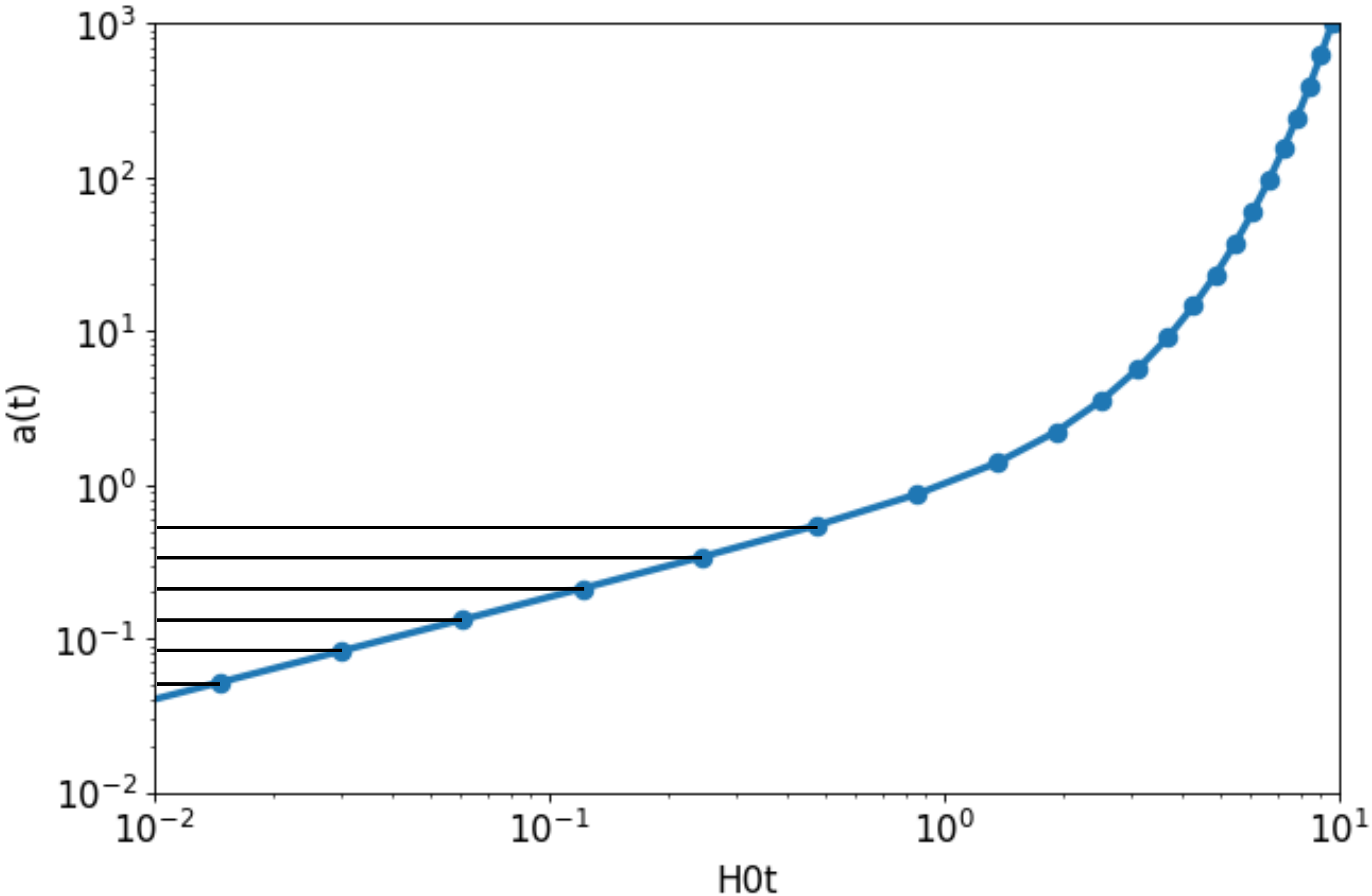

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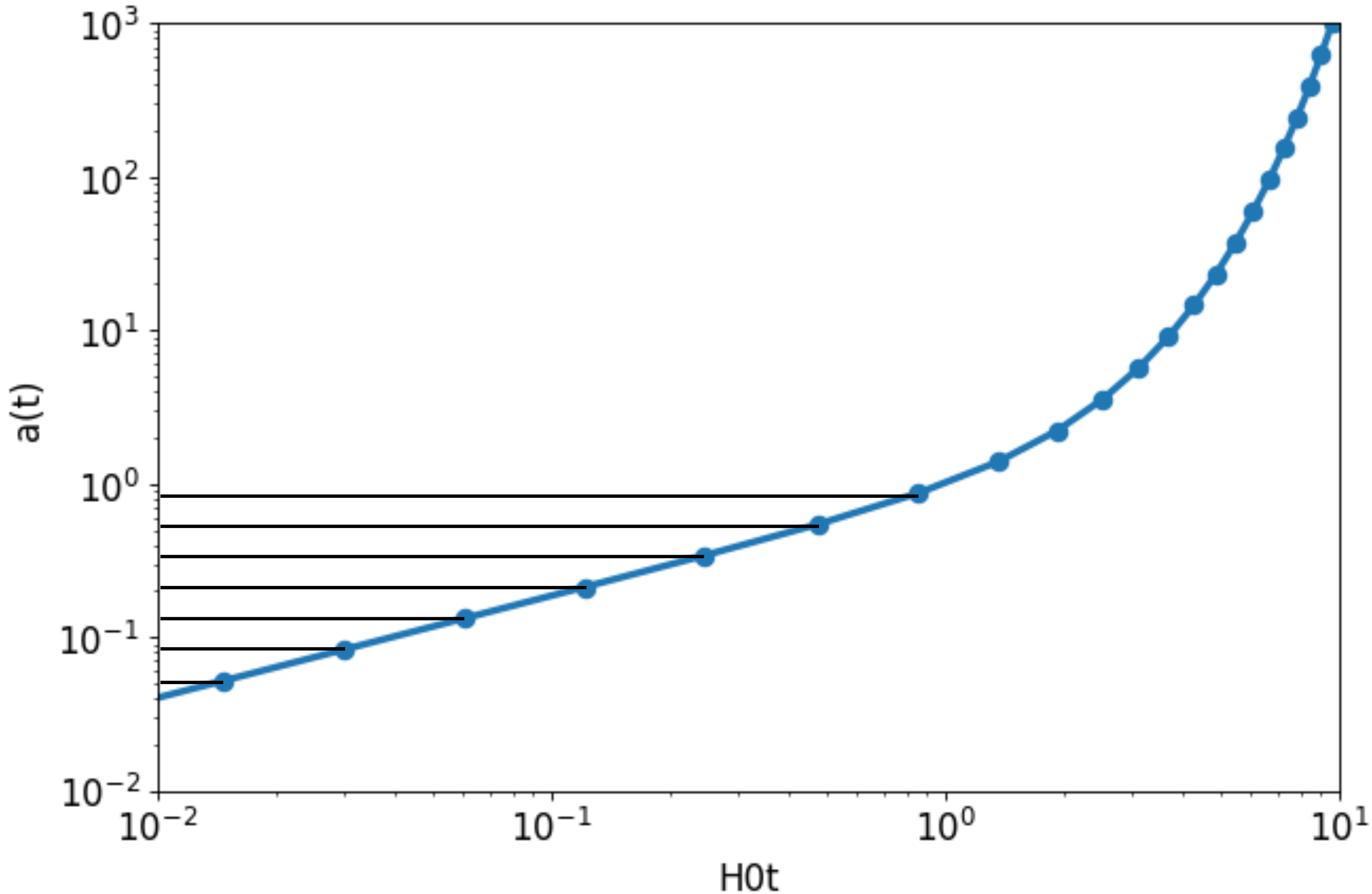

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Proper distances within the Benchmark Model

With a numerical solution for scale factor, *a*, at different times, *t*, we are able to calculate proper distances:

$$d_p(t_0) = c \int_{t_{\rm em}}^{t_{\rm ob}} \frac{1}{a(t)} dt$$

How do we do this?

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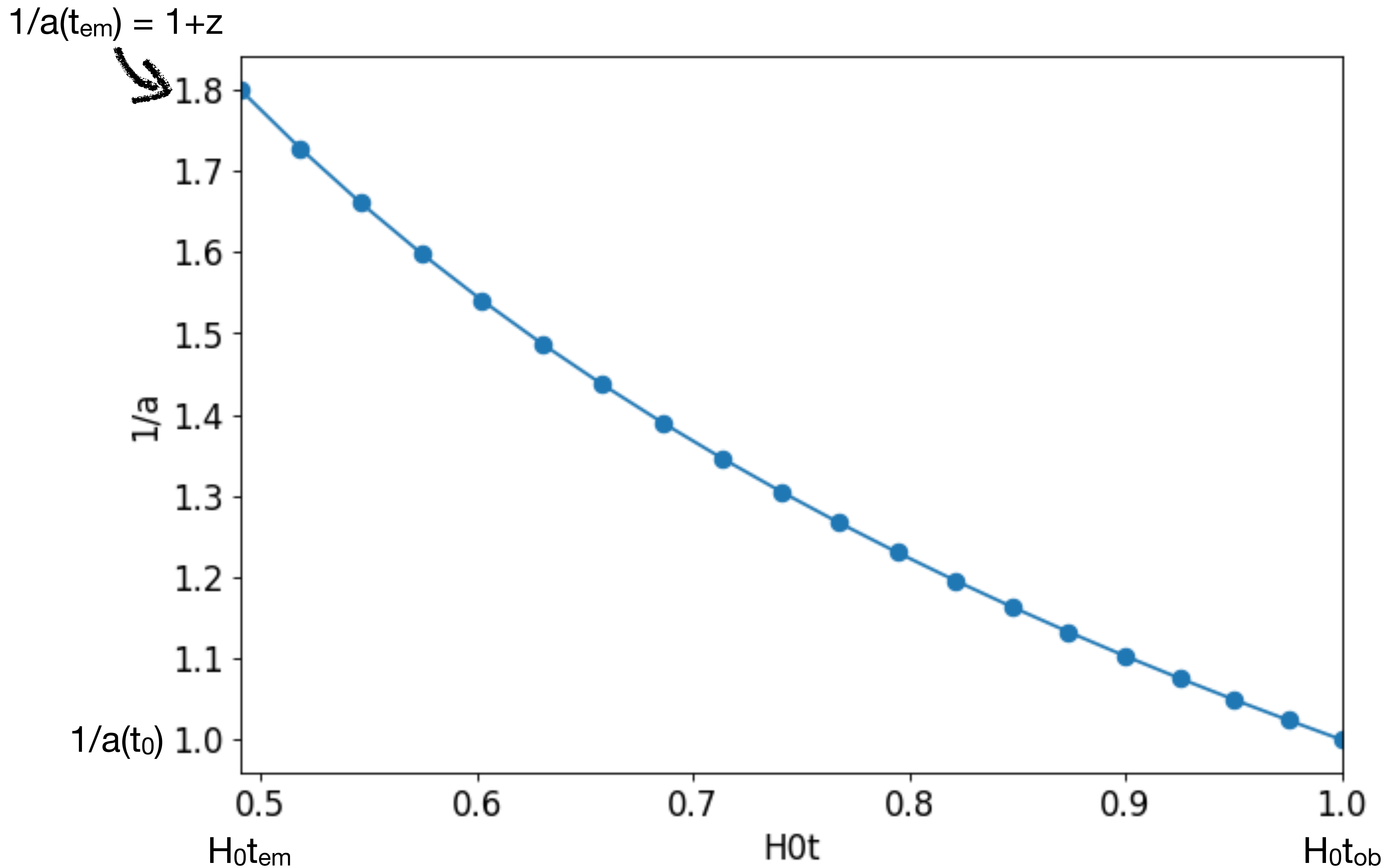
But, what about t_{em} and t_{ob}? Well, since:

$$1 + z = \frac{1}{a(t_{\rm em})} \quad \text{and} \quad a(t_0) = 1$$

All we need to do is define our *a(t)* vector between 1/(1+z) and 1, calculate the corresponding *t* values, then sum up the area under the 1/a curve between those *t* values.

For z = 0.8, d_p/c given by the area under this curve:



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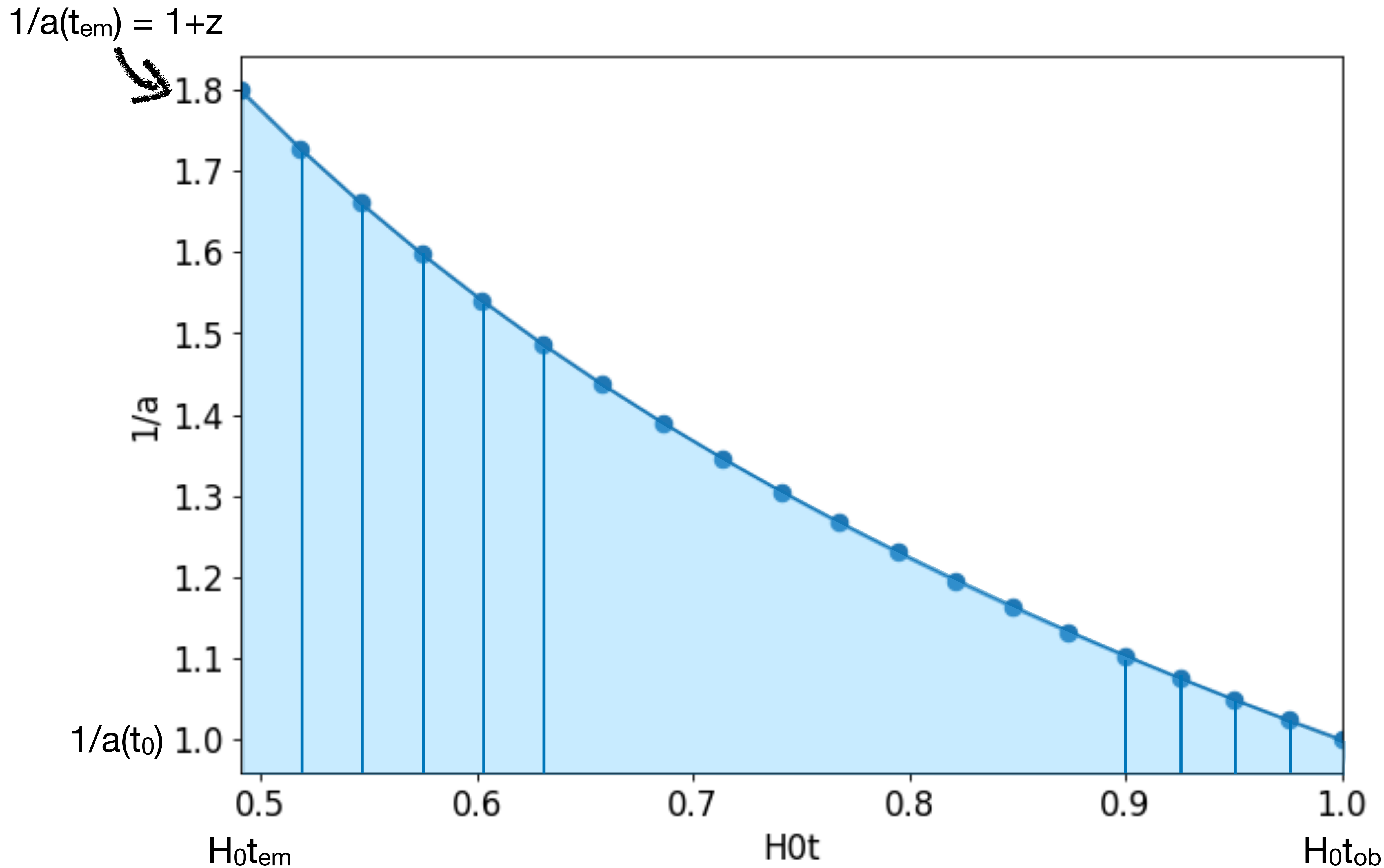
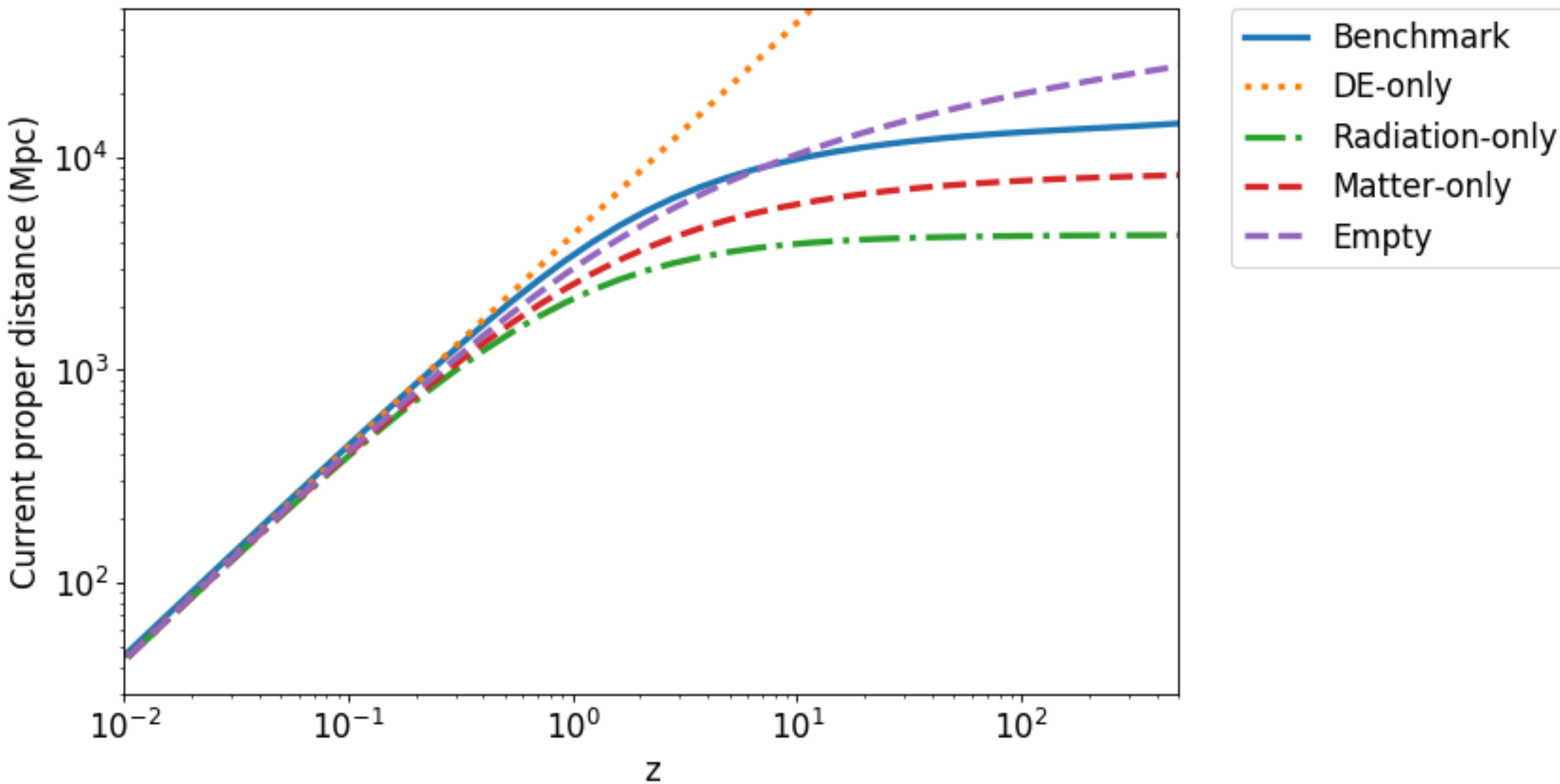


Current proper distance vs. redshift



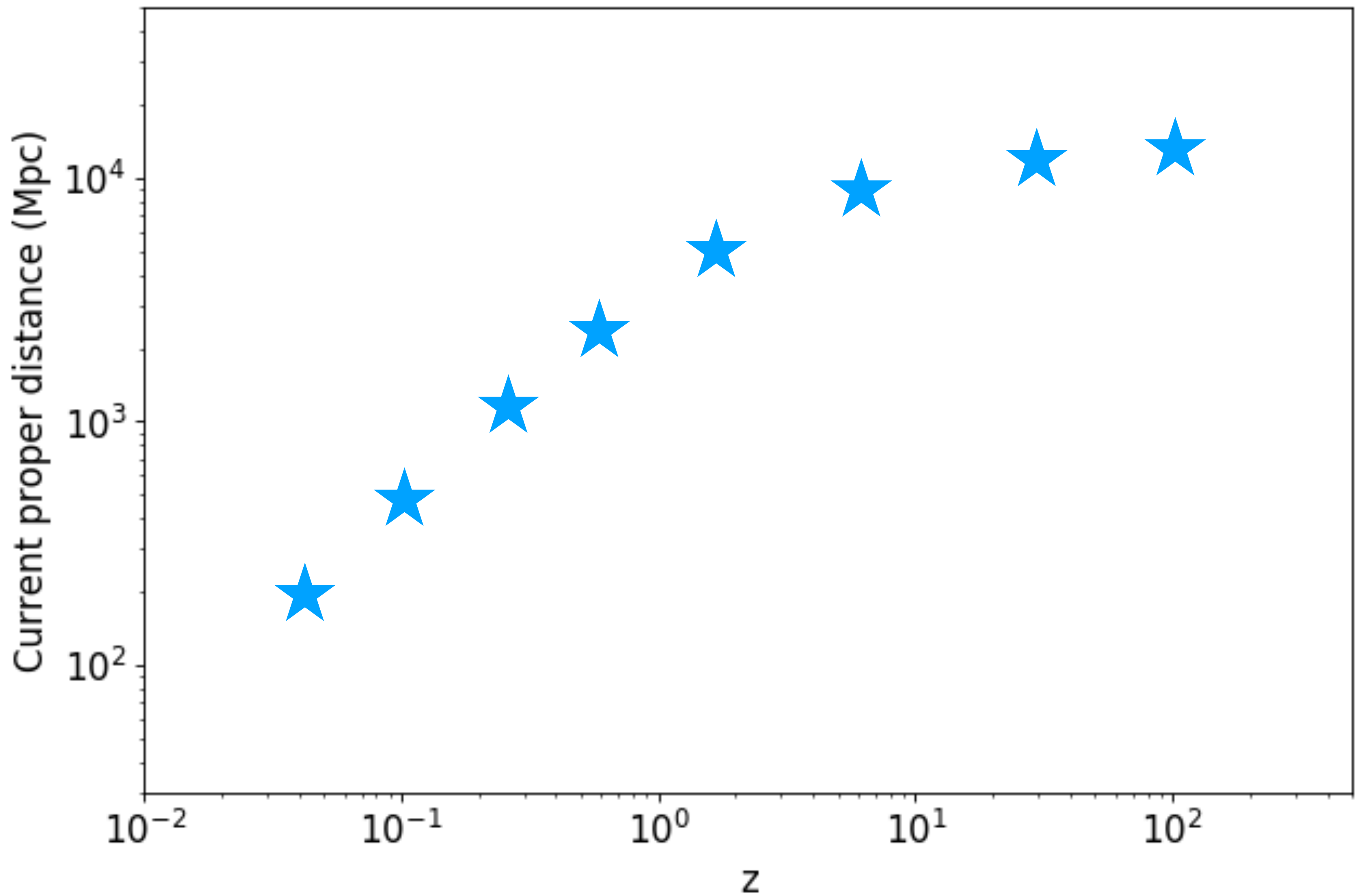
Using numerical techniques, cosmologists can calculate *a(t)* (and corresponding d_p vs. z relationships) for any combination of specific energy densities…

Calculating *a(t)* for the real Universe

…but what's more important is to determine *a(t)*, and corresponding energy densities, from observations of the *real* Universe.

As we've seen, a given *a(t)* corresponds to a given relationship between redshift and proper distance.

So, work backwards: measure d_p and z for a bunch of galaxies, then find the a(t) that fits them best, which gives the corresponding energy densities:



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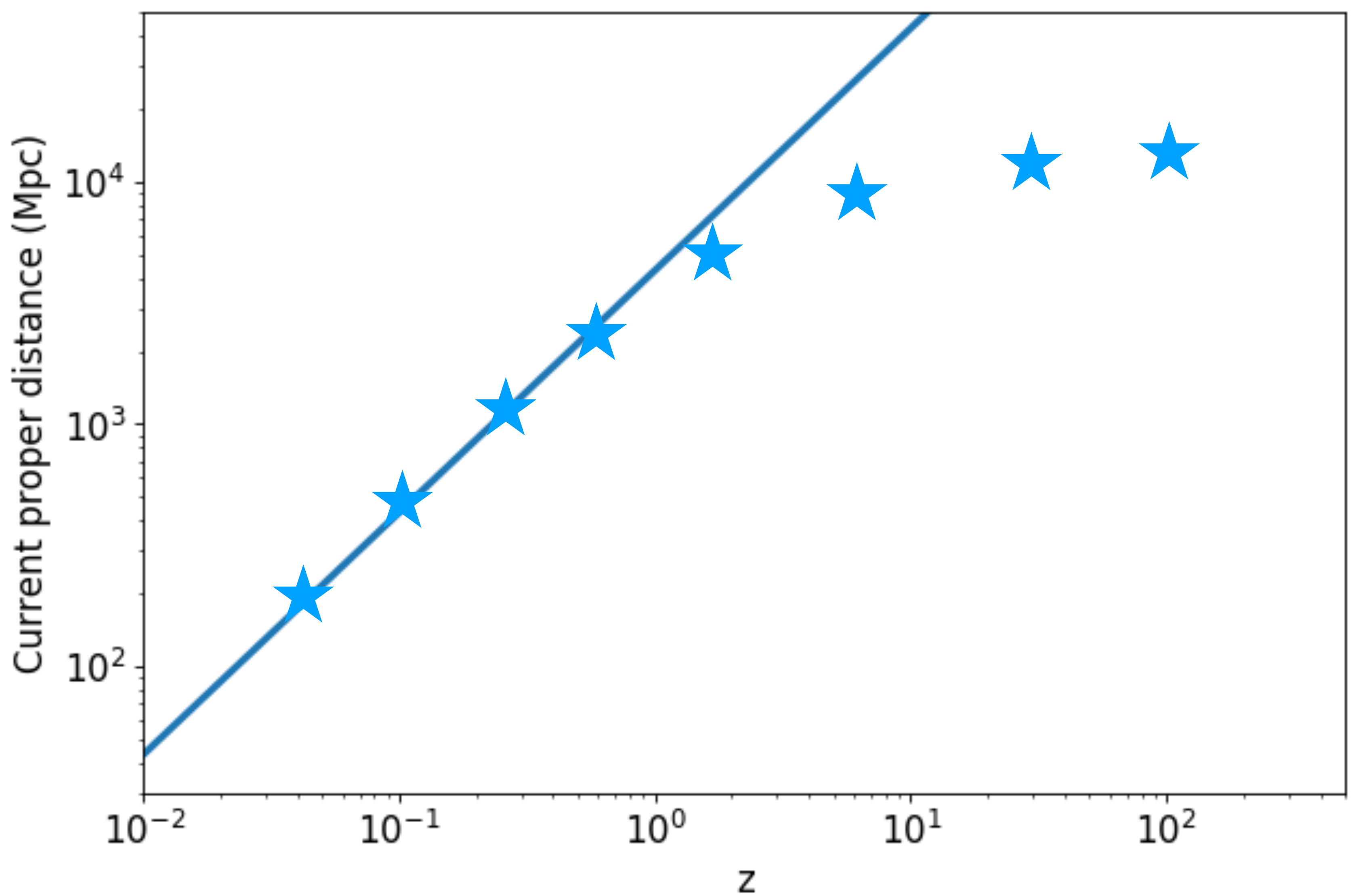

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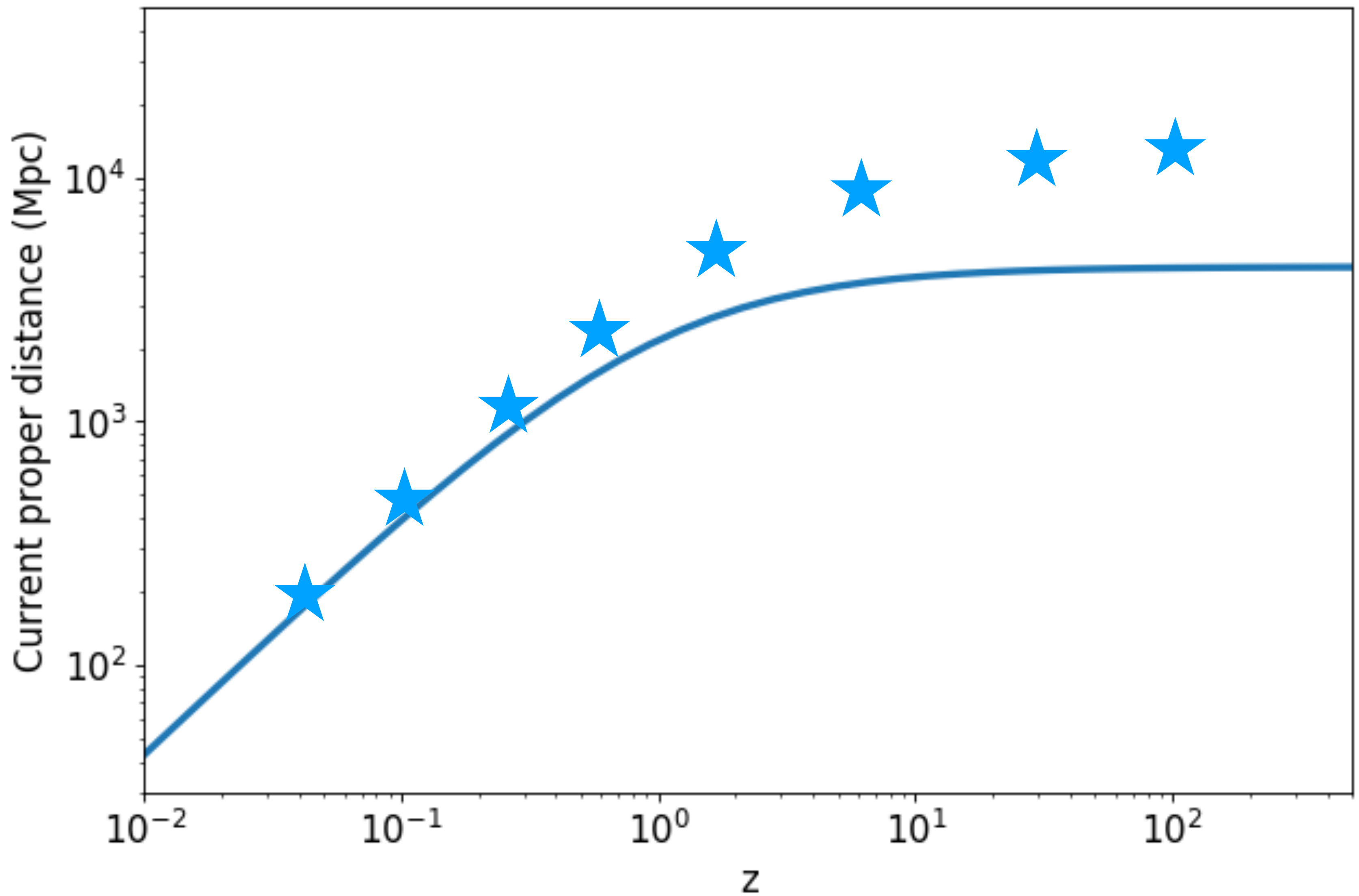

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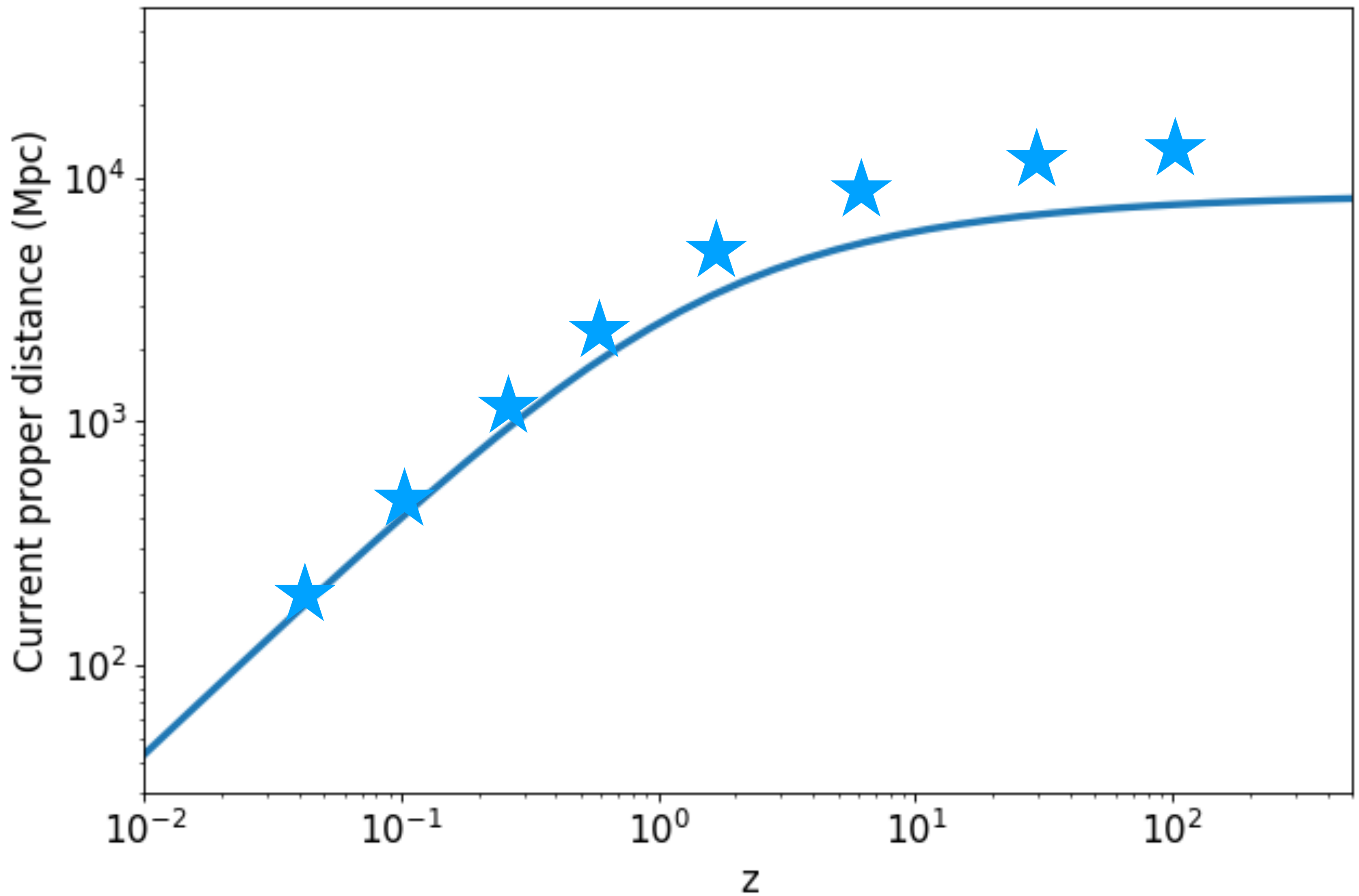

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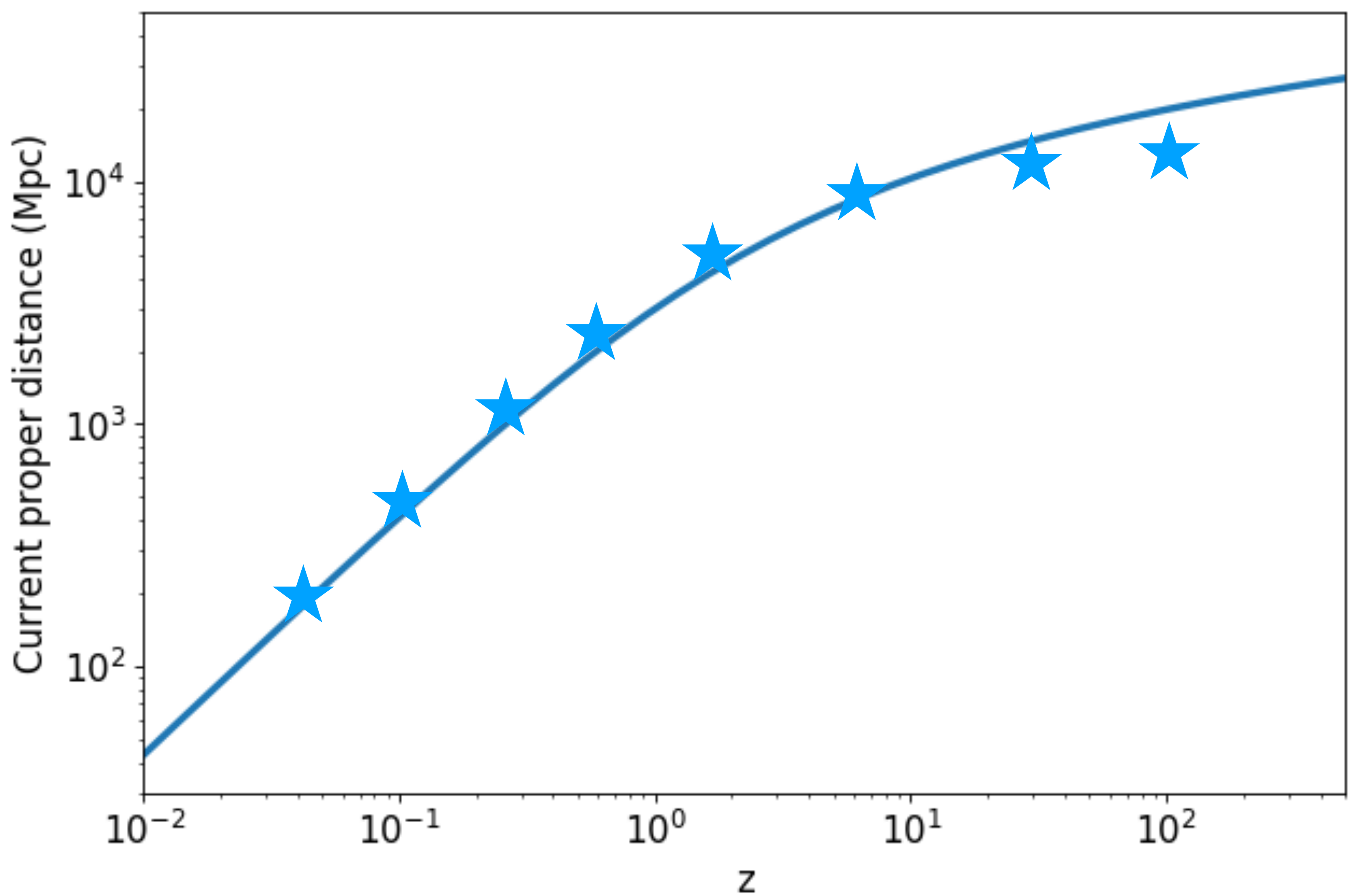

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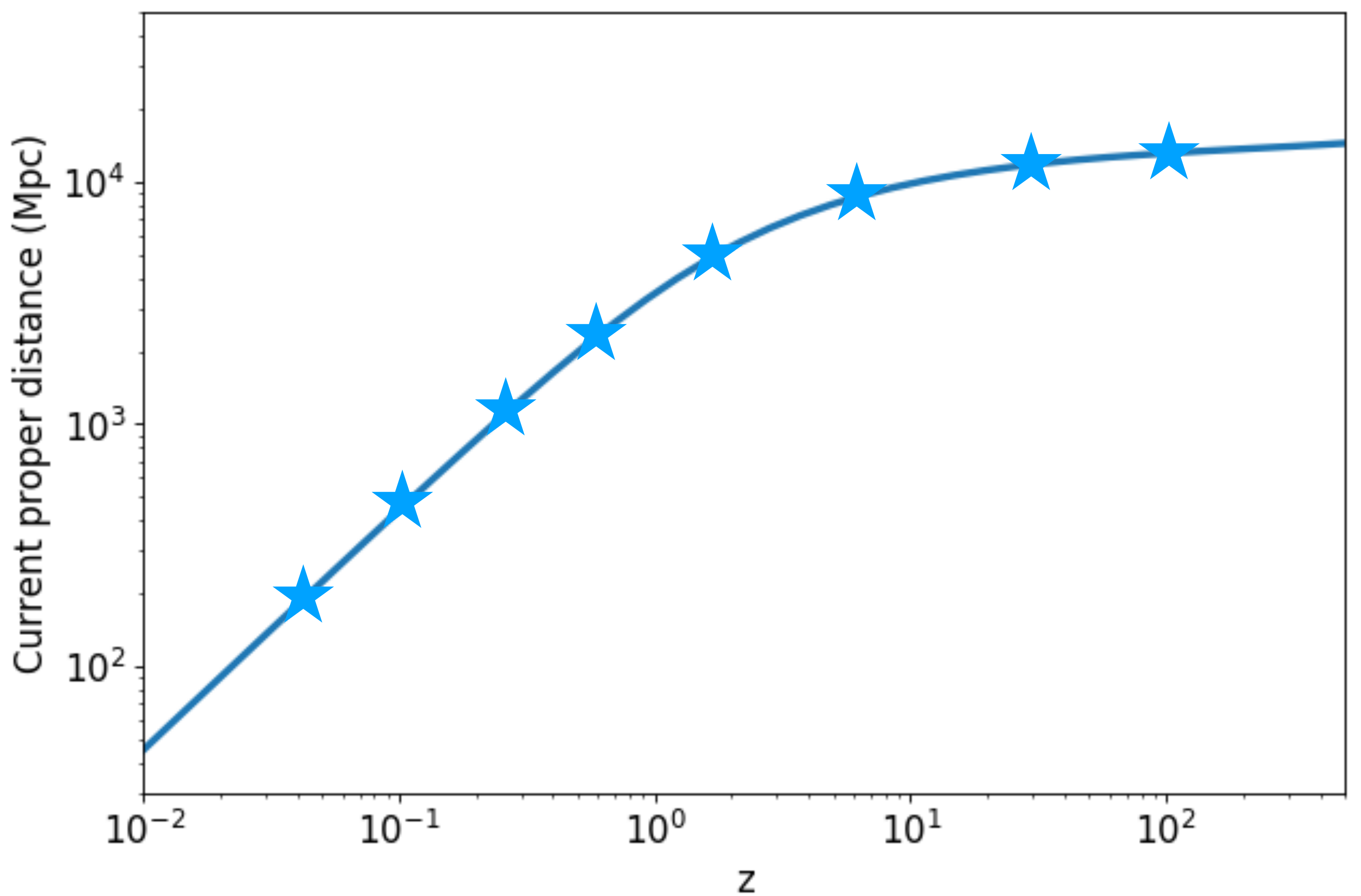


Measurable distances

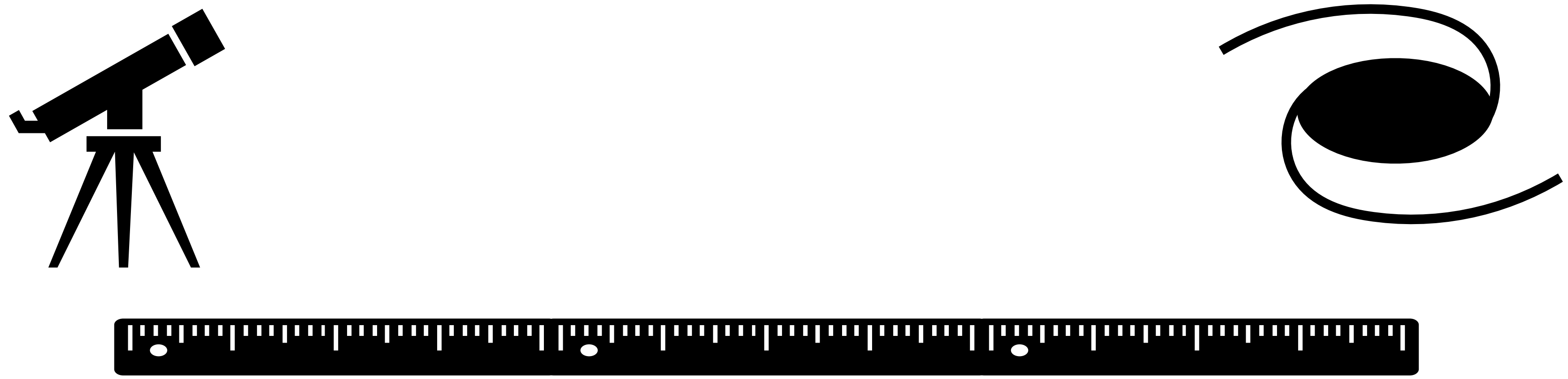
But, cosmologists clearly can’t measure proper distance, since:

- They’d have to stop the Universe from expanding while we made the measurements.
- There’s no measuring stick long enough to reach distant galaxies.

Thankfully, there are other ways to measure distance in the Universe:

Different distances in cosmology & astronomy

Proper distance:



d_P

Different distances in cosmology & astronomy

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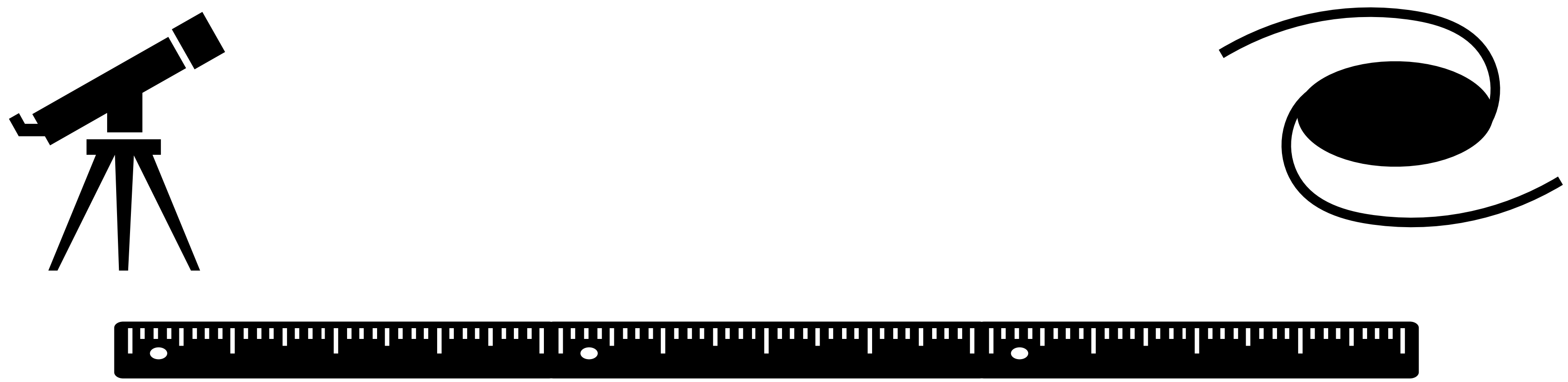

d_P

Luminosity distance:

$$F = \frac{L}{4\pi d_L^2}$$

d_L

Different distances in cosmology & astronomy

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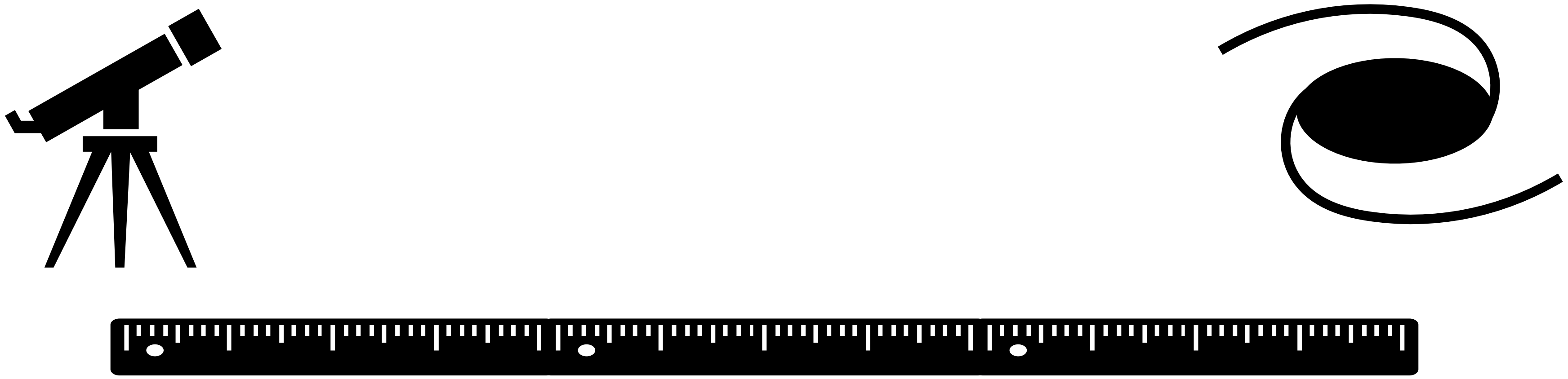

$$d_P$$

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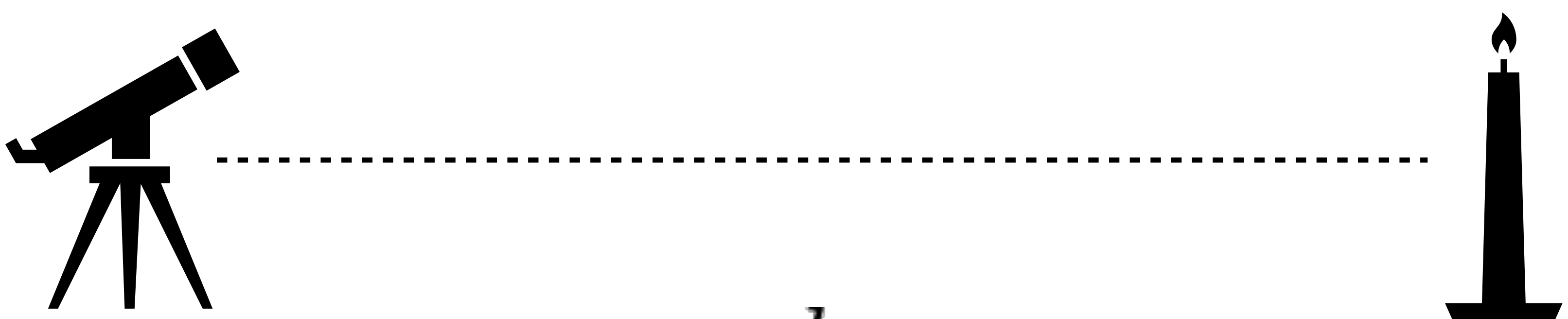

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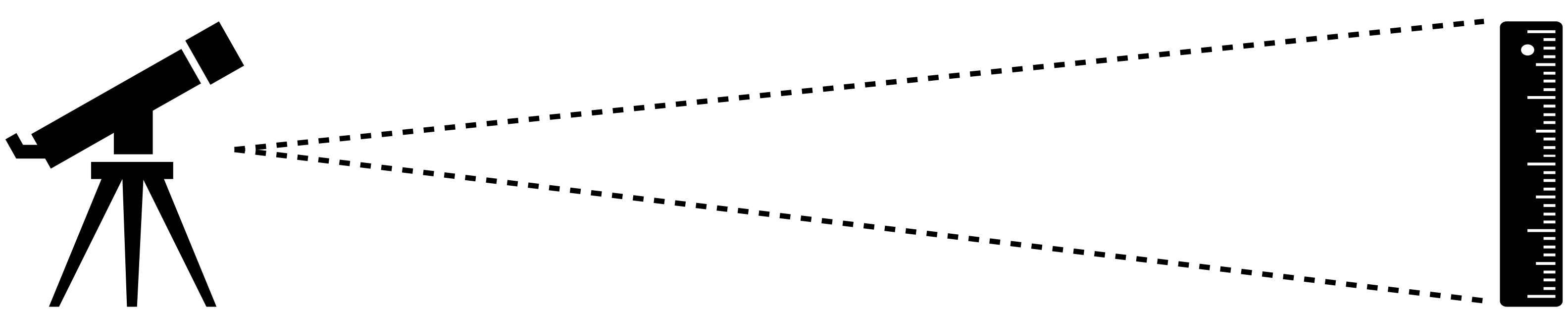
Angular size distance:



$$l = \theta d_A$$

$$d_A$$

Different distances in cosmology & astronomy

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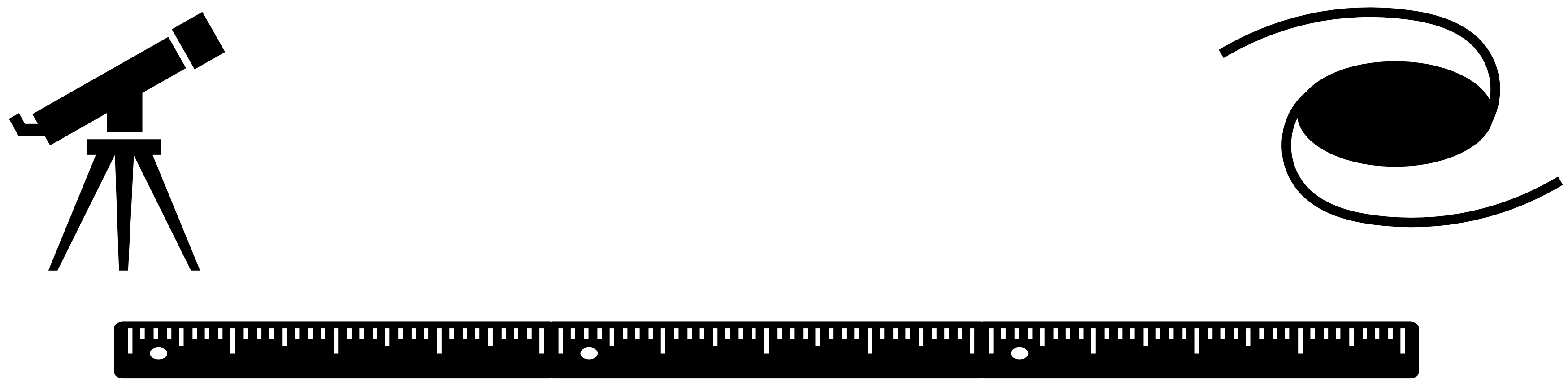

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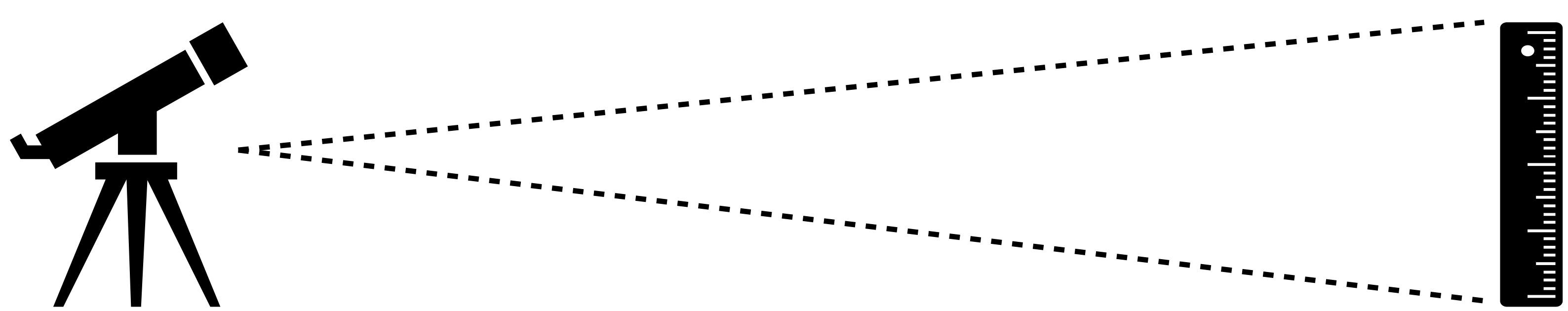

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Luminosity distance

What is flux? It is the rate of energy through a unit area:

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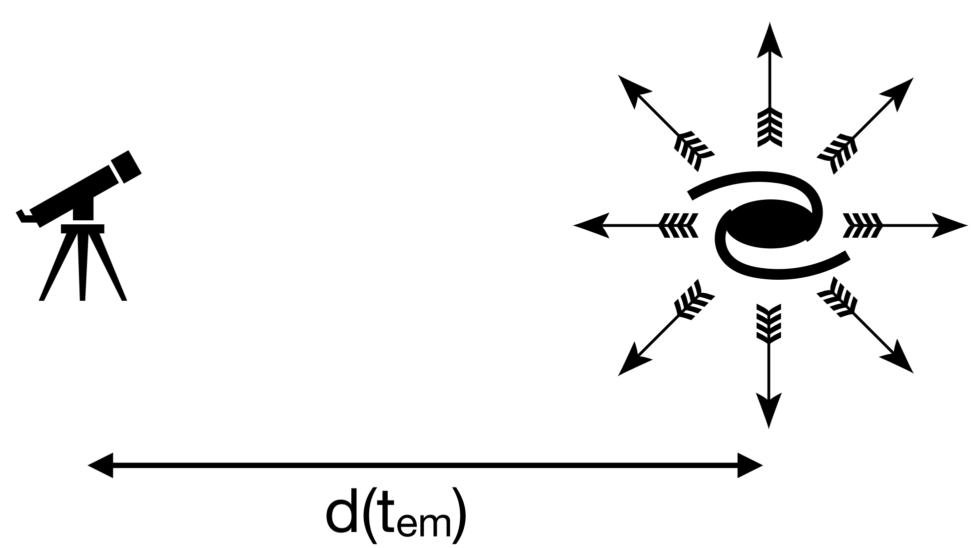
At time $t=t_{em}$ a distant galaxy emits photons in all directions. At the time, the distance between us and the galaxy is $d(t_{em})$.



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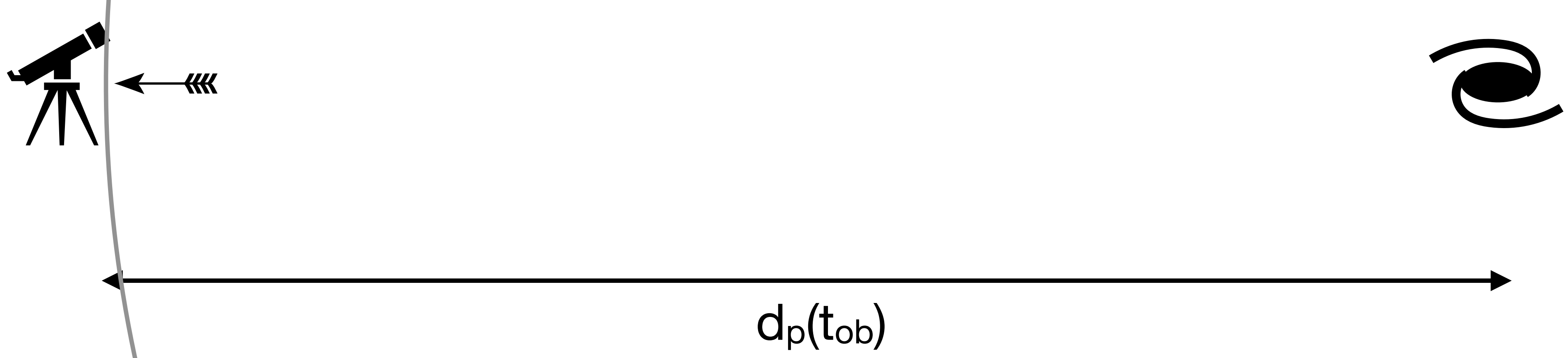

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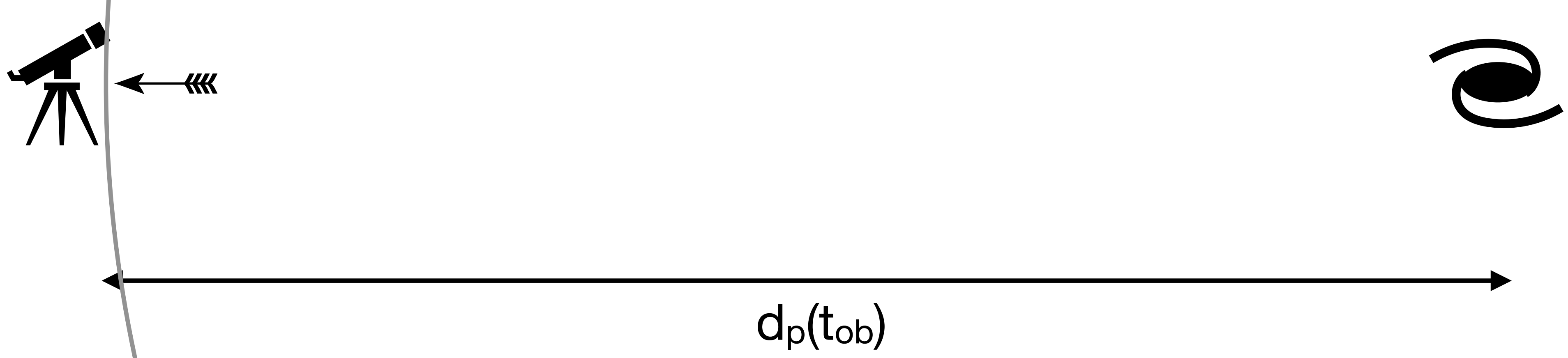
Meaning the photons are spread over an area of $4\pi d_p(t_{\rm ob})^2$ giving $F = \dfrac{L}{4\pi d_p(t_{\rm ob})^2}$



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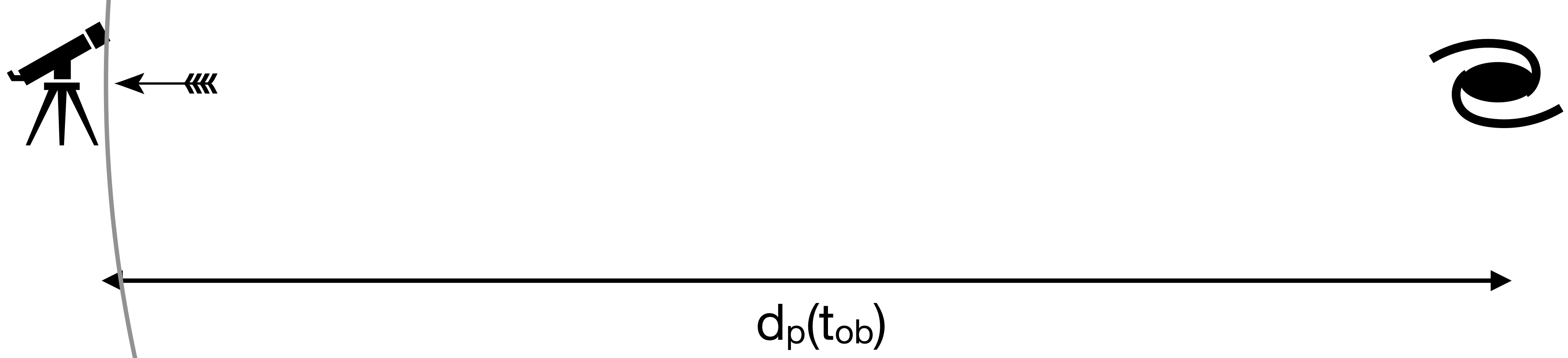
Meaning the photons are spread over an area of $4\pi d_p(t_{\rm ob})^2$ giving $F = \frac{L}{4\pi d_p(t_{\rm ob})^2}$

But, geometric dilution is not all we need to worry about…



Luminosity distance

What is flux? It is the rate of energy through a unit area:

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As well as geometric dilution, we have to also consider the what effect the relative motions between us - as observer - and the galaxy has on flux.

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Comparing the above equation with the top equation gives:

$$\boxed{d_L = d_p(t_{\rm ob})(1+z)}$$

Luminosity distance

If you know the intrinsic luminosity of a source, and measure its flux (e.g., with a telescope) then we can calculate its Luminosity Distance:

$$d_L = \left(\frac{L}{4\pi F}\right)^{1/2}$$

and

$$d_L = d_p(t_0)(1+z)$$

So, if we know:

- the flux (easy),
- the redshift (easy), and
- the intrinsic luminosity (hard)

of a distant object, we have everything we need to fit the redshift-distance relationship, calculate a(t), and obtain cosmological parameters.

Standard Candles 1: Cepheid Variables

Standard Candles are sources whose intrinsic luminosities are known (or easily derived from observations).

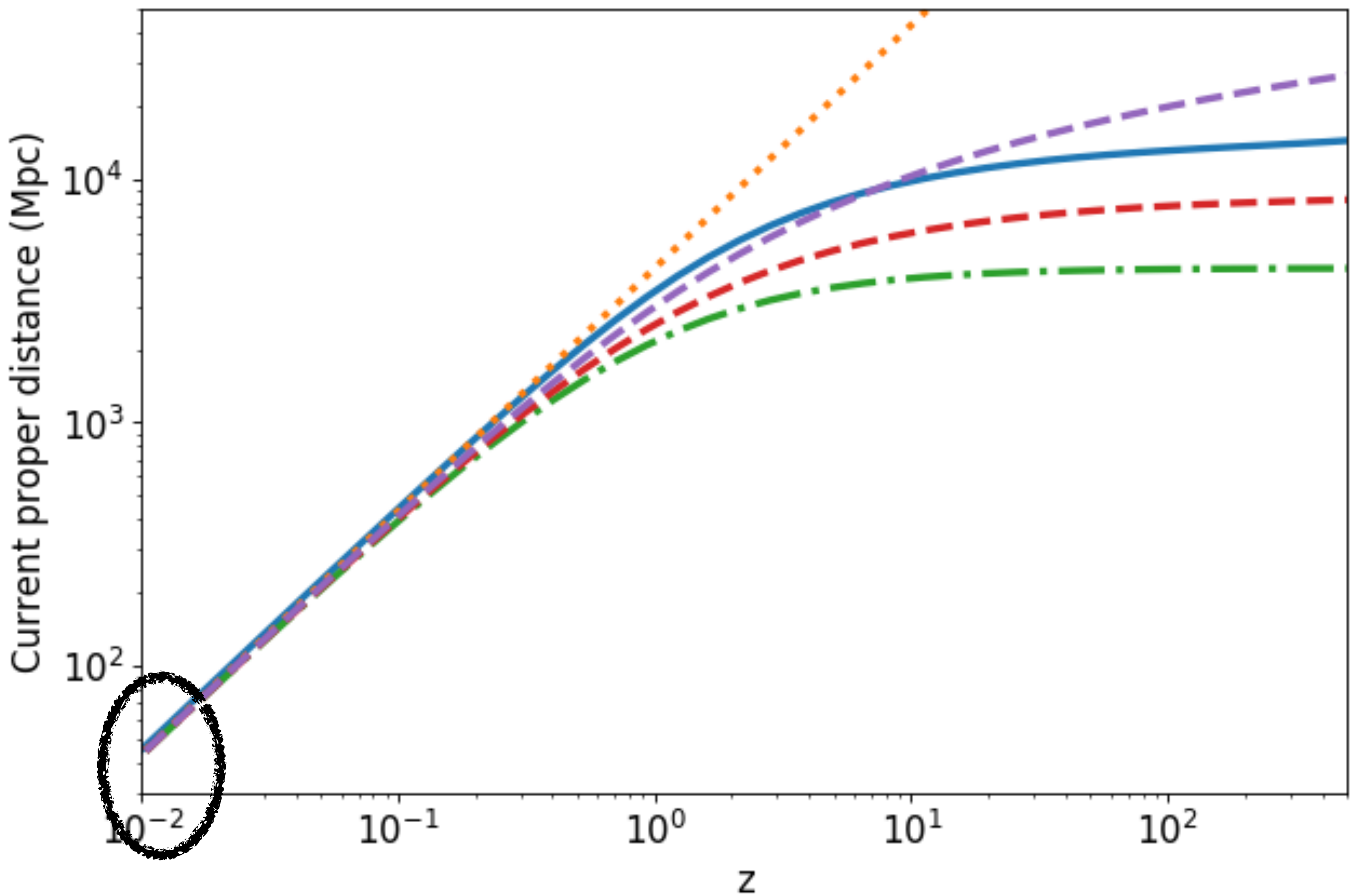
The classic Standard Candle is the *Cepheid Variable* - a star whose peak luminosity is related to the period of its variation.

Once you've measured a Cepheid's period, P (in days), you can easily calculate its intrinsic luminosity:

$$L_{\rm Ceph} = 0.22 P^{-0.97} L_{\odot}$$

But, the most distant Cepheids our telescopes are able to detect are only around 30 Mpc away, corresponding to a redshift of $z = 0.007$!

Standard Candles 1: Cepheid Variables



Cepheids are fine for calculating H0, but useless for calculating a(t) since we can't observe them to high enough redshifts to use them to distinguish between models. We need something **much brighter**.

Standard Candles 2: Type 1a Supernovae

Type 1a supernovae are produced when a White Dwarf accretes mass from a binary partner.

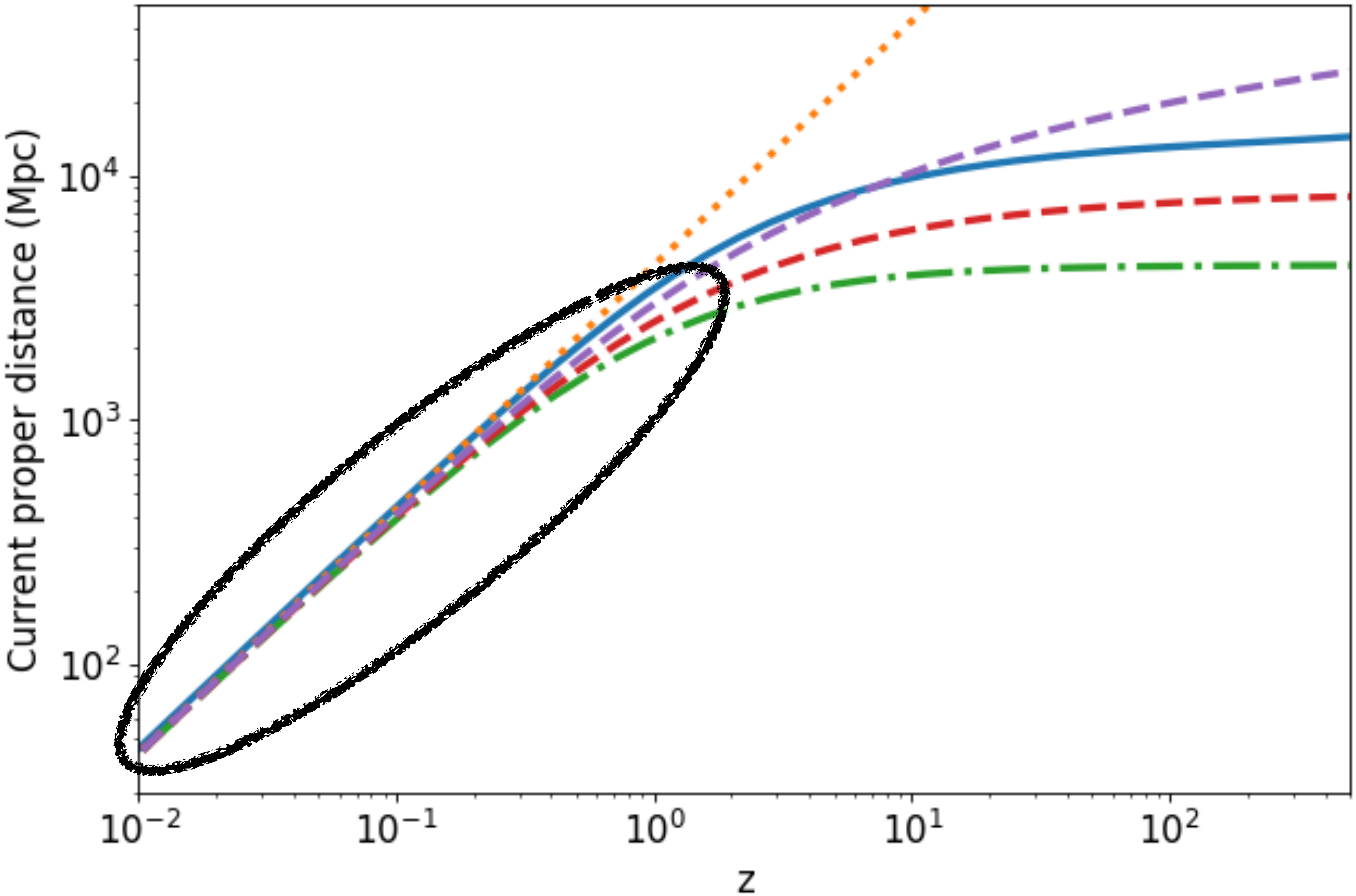
As is approaches the Chandrasekhar limit mass of 1.4MSun, it starts to undergo carbon fusion in its core.

This triggers a runaway fusion reaction which blows the star up.

Since the explosion always occurs when a White Dwarf star reaches the same given mass, its luminosity is always roughly the same.

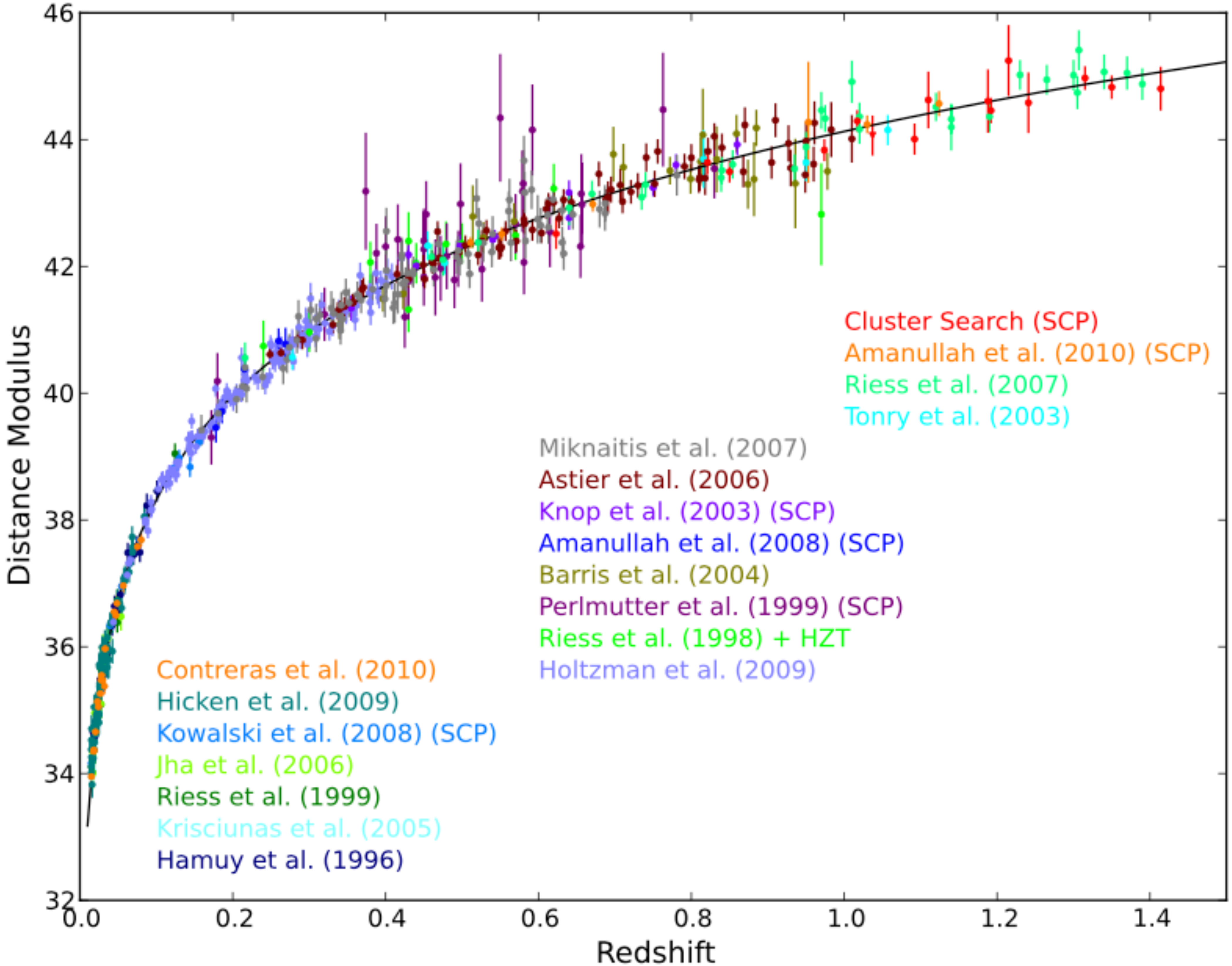
As such, it can be used as a Standard Candle.

Standard Candles 2: Type 1a Supernovae



Type 1a Supernovae are about 100,000 times more luminous than the most powerful Cepheid. As such, they can be observed to much higher redshifts (the record is z~2), enabling cosmologists to determine *a(t)*.

Standard Candles 2: Type 1a supernovae



Angular size distance

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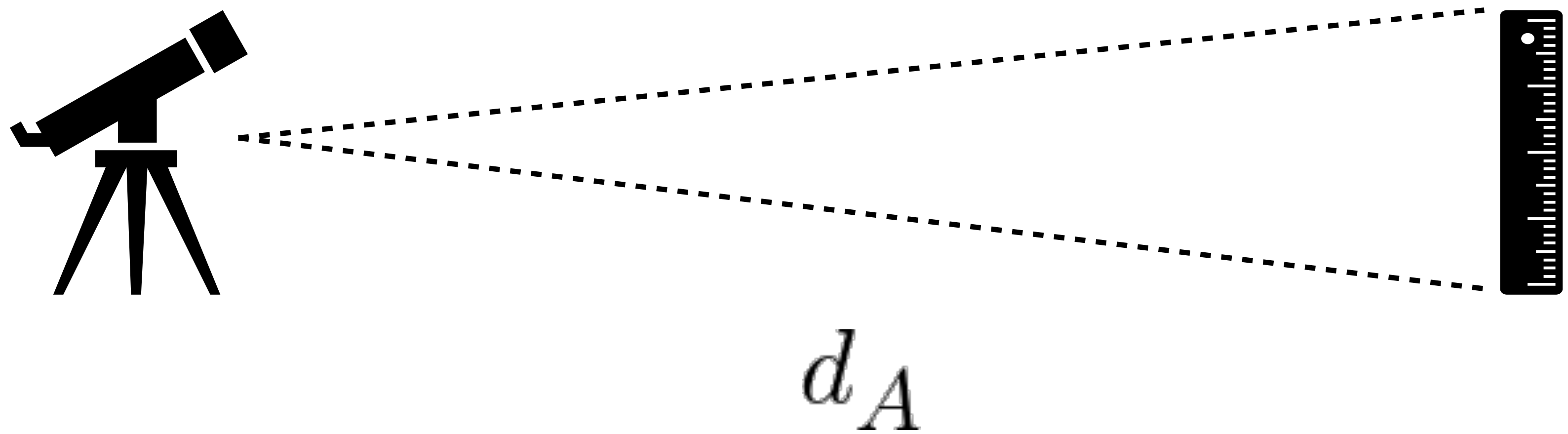

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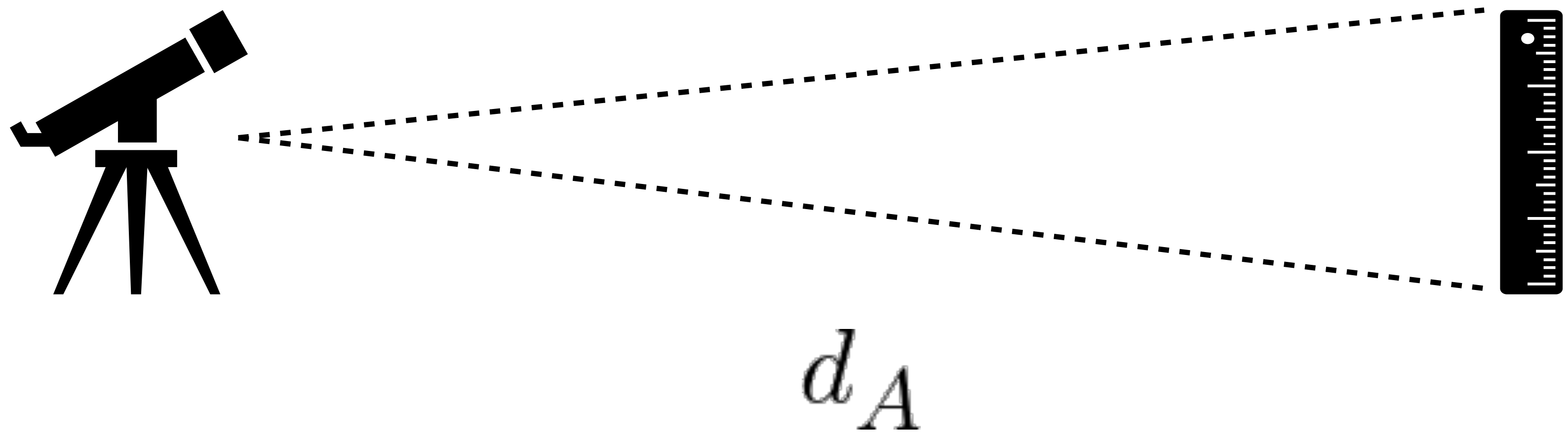

$$l = \theta d_A$$

When we measure the angular size of a distant object, we are measuring its size as it was at the time of emission.

As such, the angular size distance is the distance at time of emission:

$$\boxed{d_A = d_p(t_{\text{em}}) = \frac{d_p(t_{\text{ob}})}{1+z}}$$

Getting the feel of it…

We can't solve the F.E. for the real Universe analytically.

Instead, we use computers to solve it numerically by "integrating under the curve".

Doing so, we can determine how the scale factor evolves, and from that determine proper distances, for a given set of cosmological parameters.

However, what's *more* important is that we can work backwards…

…from a set of observed distances and redshifts, determine the cosmological parameters of the real Universe.