

# Cosmology Lecture 4

Solving the Friedmann Equation  
Part 1:  
Thermodynamics and  
The Equation of State

# Lectures

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# Key learning objectives

- Know what the critical density is, and how to derive an expression for the critical density.
- Be able to explain why a universe is *always* either positively curved, negatively curved, or flat (i.e., it can't switch the sign of its curvature over time).
- Be able to derive an equation that describes how energy density changes with time.
- The energy density of the three main types of energy:
  - Radiation;
  - Matter;
  - Dark Energy.

# The relativistic Friedmann Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2}$$

so:

$$H(t)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2}$$

we define:

$$H(t = t_0) = H_0, \quad a(t = t_0) = 1 \quad \text{and} \quad \varepsilon(t = t_0) = \varepsilon_0$$

giving:

$$H_0^2 = \frac{8\pi G}{3c^2} \varepsilon_0 - \frac{\kappa c^2}{R_0^2}$$

# The Critical Density

In a flat universe:  $\kappa = 0$

giving:

$$\varepsilon_c(t) = \frac{3c^2}{8\pi G} H(t)^2$$

if:  $\varepsilon(t) > \varepsilon_c(t)$  then  $\kappa > 0$  and universe is +vely curved

$\varepsilon(t) < \varepsilon_c(t)$  then  $\kappa < 0$  and universe is -vely curved

define:

$$\Omega(t) = \frac{\varepsilon(t)}{\varepsilon_c(t)}$$
 as the density parameter

from observations:  $0.995 < \Omega(t) < 1.005$

# The Critical Density

From

$$H(t)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2},$$

$$\varepsilon_c(t) = \frac{3c^2}{8\pi G} H(t)^2 \quad \text{and} \quad \varepsilon(t) = \Omega(t) \varepsilon_c(t)$$

get:

$$1 - \Omega(t) = -\kappa \frac{c^2}{R_0^2 a(t)^2 H(t)^2}$$

$$\Omega(t) > 1$$

If:  $\Omega(t) = 1$  then it remain so at all times

$$\Omega(t) < 1$$

# Solving the Friedmann Equation

$$H(t)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2}$$

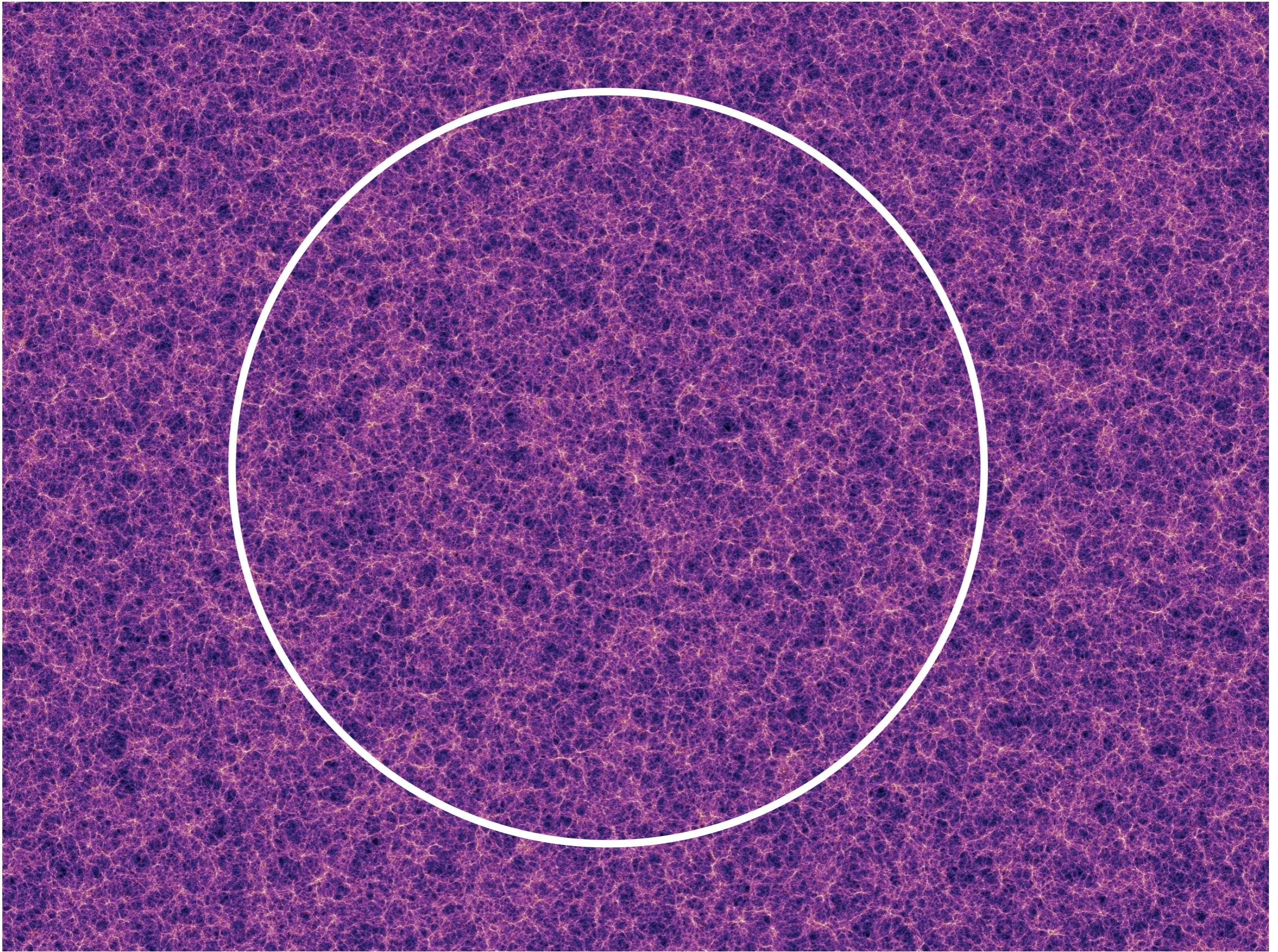
The F.E. has too many unknowns to solve for  $a(t)$  and  $\varepsilon(t)$

We need more equations...

$$dQ = dE + PdV \quad \text{First law of thermodynamics}$$

$$P = \omega \varepsilon \quad \text{Equation of state}$$

# Take a huge volume of the universe



# Solving the Friedmann Equation

From First Law of Thermodynamics, get:

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0 \quad \text{Fluid Equation}$$

which can be combined with the F.E. to get:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) \quad \text{Acceleration Equation}$$

The Equation of State simply relates  $P$  to  $\epsilon$

For a perfect “gas”:

$$P = w\epsilon \quad \text{where} \quad w = \langle v^2 \rangle / 3c^2$$

which is effectively 0 for non-relativistic (e.g., Baryonic) material, and 1/3 for photons

$w < -1/3$  results in a positive acceleration: a **“Dark Energy”**

# Getting the feel of it...

- The Friedmann Equation relates the Hubble parameter to the scale factor, curvature and energy density of the Universe.
- Using it, we can define a critical density for the Universe. If the true energy density is greater or less than this critical density, the Universe is curved.
- Current measurements are consistent with the Universe being flat; that, or the radius of curvature is very large.
- The Friedmann equation is *not* enough to determine how the scale factor and energy densities evolve over time; it's a single equation with two unknowns:  $a(t)$  and  $\varepsilon(t)$
- For that, we need to also include:
  - The First Law of Thermodynamics, which relates the scale factor to the energy density and pressure.
  - The Equation of State, which relates pressure to energy density.