Operator Automata Theory

1 Introduction

Descriptive text

2 Preliminaries

Monoid

Semiring

Semimodule

Linear transform

Norm

Automata

3 Operators

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3.1 M-Semimodules and Operator Norms

Let Σ be a finite set and M to be the monoid finitely generated by Σ:

$$M = (\Sigma, \cdot, \mathbf{1})$$

Definition 1 (Normed Monoid)**.** *A normed monoid is a monoid M with a norm $\|\cdot\| : M \to \mathbb{R}_{\geq 0}$.*

Because monoids lack addition, such norms only concern multiplication.

We may extend monoids in general to define the notion of a M-semiring. In particular, M-semirings are defined with respect to a particular monoid M, and are generated from its power set $\mathcal{P}(M)$.

Definition 2 (M-Semiring)**.** *Let M be a finitely generated monoid. A M-semiring R is a semiring such that:*

$$R = (\mathcal{P}(M), \cup, \cdot, \mathbf{0}, \mathbf{1})$$

Where semiring addition is set union $\cup$ with identity $\mathbf{0}$, and semiring multiplication $\cdot$ and identity $\mathbf{1}$ are carried over from M.

As with the case of normed monoids, we may extend this to normed M-semirings. In particular, we pay special attention to p-norms.

Definition 3 (M-Semiring p-Norm)**.** *Let M be a finitely generated and normed monoid. For $1 \leq p \leq \infty$, a p-normed M-semiring is R is equipped with a norm $\|\cdot\|_p : R \to \mathbb{R}_{\geq 0}$:*

$$\|x\|_p = \left(\sum_{a \in x} \|a\|^p \right)^{1/p}$$

The sum here utilizes the monoid norm. Observe that because M is finitely generated, each $x \in R_M$ is therefore countable, and hence so is the sum. When $p = \infty$, this is just a sup norm. Similar definitions can be found in literature [1].

Extending M-semirings, we define (M, n)-semimodules:

Definition 4 ((M, n)-Semimodule)**.** *A (M, n)-semimodule R^n is a free semimodule generated by n isomorphic copies of the M-semiring R.*

If $x \in R^n$, write x_i to denote the ith element from R in some canonical representation of R^n. Often this is just a row (horizontal) or column (vertical) vector of length n.

Again, we extend norms to (M, n)-semimodules:

Definition 5 ((M, n)-Semimodule (p, q)-Norm)**.** *Let R be a p-normed M-semiring. Let R^n be a (M, n)-semimodule and take $1 \leq p, q \leq \infty$. A (p, q)-normed (M, n)-semimodule is a semimodule with norm $\|\cdot\|_{p,q} : R^n \to \mathbb{R}_{\geq 0}$:*

$$\|x\|_{p,q} = \left(\sum_{i=1}^{n} \|x_i\|_p^q \right)^{1/q}$$

When $p = q = \infty$, these are just the sup-norm. Often we may only care about the case of $p = \infty$ and $q = 1$, which is the sup-norm over each R, and the 1-norm over the n copies of R in R^n.

With the notion of norms, we again extend such notions to linear operators mapping R^n to R^m.

Definition 6 (Linear Operator Norm)**.** *Let $\mathcal{L}$ be the space of linear operators mapping a (M, n)-semimodule R^n with norm $\|\cdot\|_{R^n}$ to a (M, m)-semimodule R^m with norm $\|\cdot\|_{R^m}$. Then define the operator norm $\|\cdot\|_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}_{\geq 0}$ as:*

$$\|T\|_{\mathcal{L}} = \inf \{c \in \mathbb{R}_{\geq 0} \, : \, \|Tx\|_{R^m} \leq c \, \|x\|_{R^n} \, , \forall x \in R^n\}$$

With the notion of a norm, we may perhaps begin to discuss weaker versions Banach spaces.

3.2 Examples

Example 1. *Consider $\Sigma = \{a, b\}$.*

4 On Automata

References

[1] Manfred Kudlek. “Iteration Lemmata for Normed Semirings (Algebraic Systems, Formal Languages and Computations)”. In: (2000).