Linear Operators on Kleene Algebras

1 Introduction

1.1 Algebraic Structures

Definition 1 (Monoid)**.** A monoid is a set S with a binary operation $\circ\colon S \times S \to S$ (multiplication) with the following axioms:

(1) $\circ$ is associative.

(2) There exists an identity element $e \in S$ such that:

$$\forall a \in S \, : e \circ a = a = a \circ e$$

Definition 2 (Semiring)**.** A semiring is a set R with two binary operations $+\colon R \times R \to R$ (addition) and $\cdot\colon R \times R \to R$ (multiplication) with the following axioms:

(1) $(R, +)$ is a commutative monoid with identity 0.

(2) $(R, \cdot)$ is a monoid with identity 1.

(3) Multiplication left and right distributes over addition.

(4) Multiplication by 0 annihilates R.

Definition 3 (Left Module Over a Semiring)**.** A left module M over a semiring R is a set with two binary operations $+\colon M \times M \to M$ (addition) and $\cdot\colon R \times M \to M$ (scalar multiplication) with the following axioms:

(1) $(M, +)$ is an abelian group.

(2) For all $s, r \in R$ and $x, y \in M$:

$$\begin{aligned} r \cdot (x + y) &= (r \cdot x) + (r \cdot y) \\ (r + s) \cdot x &= (r \cdot x) + (s \cdot x) \\ (r \cdot s) \cdot x &= r \cdot (s \cdot x) \\ 1_R \cdot x &= x \end{aligned}$$

Definition 4 (Right Module Over a Semiring)**.** A right module would be defined similarly, except all instances of scalar multiplication happen on the right side.

Definition 5 (Bimodule Over a Semiring)**.** A bimodule is a module that is both a left module and a right module with respect to scalar multiplication.

2 Linear Transformation

From linear algebra, a linear transformation (operator, map, mapping, function) is a homomorphism $T\colon V \to W$ between two modules V and W with respect to addition and scalar multiplication.