Discrete Fourier Transforms for Non-linear Real Arithmetics

General Fourier Transform

Consider the Hilbert space $\mathcal{H} = L^2([0,1])$ endowed with the orthonormal bases $\{e_n\}_{n\in\mathbb{I}}$, where we define $e_n(x) = e^{i2\pi nx}$. A Fourier transform $\mathcal{F}_n \colon \mathcal{H} \to \mathbb{C}$ can be defined as the inner product:

$$\mathcal{F}_n[f] = \langle f, e_n \rangle = \int_0^1 f(x)\overline{e_n(x)}dx = \int_0^1 f(x)e^{-i2\pi nx}dx$$

By linearity of the integral, each Fourier transform $\mathcal{F}_n$ can be seen as a linear functional. We make a stronger claim, that the Fourier transform is continuous with norm 1:

$$\|F_n\|_{\mathcal{H}^*} = \sup_{\|f\|_{\mathcal{H}}=1} \left| \int_0^1 f(x)\overline{e_n(x)}dx \right| \leq \left(\int_0^1 |f(x)|^2 dx \right)^{1/2} \left(\int_0^1 |e_n(x)|^2 dx \right)^{1/2} = \|f\|_{\mathcal{H}} \cdot \|e_n\|_H$$

The inequality is sharp when $f = e_n$. We write the partial Fourier series $S_N \colon \mathcal{H} \to \mathcal{H}$ as:

$$S_N[f] = \sum_{n=-N}^{N} \langle f, e_n \rangle = \sum_{n=-N}^{N} \int_0^1 f(x)e^{-i2\pi nx}dx$$

We omit the proof, but do note that this converges in norm for any $f \in \mathcal{H}$:

$$\lim_{N\to\infty} \|f - S_N[f]\|_{\mathcal{H}} = \lim_{N\to\infty} \left| f(x) - \sum_{n=-N}^{N} \int_0^1 f(x)e^{-i2\pi nx}dx \right| \to 0$$

Discrete Fourier Transform