Fourier Transforms for Non-linear Real Arithmetics

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0.1 The Problem(s)

We first consider the satisfiability of first-order logic formulae over non-linear real arithmetics:

$$\varphi = \bigwedge_{i=1}^{N} \left(\bigvee_{j=1}^{M} f_{i,j}(\vec{x}) \leq \varepsilon_{i,j} \right) \qquad \vec{x} \in K \subseteq \mathbb{R}^n, \;\; f_{i,j} \colon K \to \mathbb{R}, \;\; \varepsilon_{i,j} \geq 0$$

The question of satisfiability, however, is undecidable: we cannot yield algorithms that, given φ, is able to find some $\vec{x}$ if there exists a satisfying configuration. We thus seek to approximate solutions, which throws away soundness for feasibility.

However, we are interested in still another class of problems: program synthesis. Given a pre-condition P that we assume at the start of program execution, and post-condition Q, does there exists some program S that is able to transform P into Q? When P and Q are formulae over linear integer arithmetics, for instance, the corresponding program is akin to a sequence of variable assignments that correspond to a linear transformation. We never said these programs have to be fancy, did we?

0.2 Background

We provide a brief overview of the Fourier transform and the theory of linear real arithmetics.

0.2.1 The Fourier Transform

Consider the separable Hilbert space $\mathcal{H} = L^2([0,1])$ endowed with the Fourier basis $\{e_n\}_{n \in \mathbb{Z}}$, where we define $e_n(x) = e^{i2\pi nx}$. A Fourier transform $\mathcal{F}_n \colon \mathcal{H} \to \mathbb{C}$ is a linear functional that can be expressed as an inner product:

$$\mathcal{F}_n[f] = \langle f, e_n \rangle = \int_0^1 f(x)\overline{e_n(x)}dx = \int_0^1 f(x)e^{-i2\pi nx}dx$$

The partial Fourier series $S_n \colon \mathcal{H} \to \mathcal{H}$ is a (finite dimensional) compact operator:

$$S_N[f](x) = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n(x) = \sum_{n=-N}^{N} e_n(x) \int_0^1 f(y)e^{-i2\pi ny}dy$$

Note that this is an orthogonal projection. Consider some m such that $|m| > N$:

$$\langle e_m, S_N \rangle = \left\langle e_m, \sum_{n=-N}^{N} \langle f, e_n \rangle e_n \right\rangle = \sum_{n=-N}^{N} \langle f, e_n \rangle \cdot \langle e_m, e_n \rangle = \sum_{n=-N}^{N} \langle f, e_n \rangle \cdot 0 = 0$$

Furthermore, we omit the proof, but do note that this converges in norm for any $f \in \mathcal{H}$:

$$\lim_{N\to\infty} \|f - S_N[f]\|_{\mathcal{H}} = \lim_{N\to\infty} \left| f(x) - \sum_{n=-N}^{N} \int_0^1 f(x)e^{-i2\pi nx}dx \right| \to 0$$

We also remark that the class of continuous functions over compact support are dense in L^2. This motivates us to consider the problem of non-linear real arithmetics over compact (perhaps convex?) domains.

0.2.2 Multi-Dimensional Fourier Transform

We now consider the Fourier transform when we consider n dimensions. Consider the vector $\vec{N} = (N_1, \ldots, N_n) \in \mathbb{Z}^n$. For $\mathcal{H} = L^2([0,1]^n)$, we define as the orthonormal basis:

$$\phi_{\vec{N}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} e_{N_i}(x_i) = e^{i2\pi(N_1x_1 + \cdots + N_nx_n)}$$

We likewise define the Fourier transform $G_{\vec{N}} \colon H \to H$ as a linear functional in the form of an inner product, with $\vec{x} \in \mathbb{R}^n$:

$$\mathcal{G}_{\vec{N}}[f](\vec{x}) = \langle f, \phi_{\vec{N}} \rangle = \int_{[0,1]^n} f(\vec{x})\overline{\phi_{\vec{N}}(\vec{x})} d\vec{x}$$

A multi-dimensional Fourier series is thus the following operator mapping:

$$\mathcal{T}_{\vec{N}}[f](\vec{x}) = \sum_{i_1=-N_1}^{N_1} \cdots \sum_{i_n=-N_n}^{N_n} \langle f, \phi_{i_1,\ldots,i_n} \rangle \phi_{i_1,\ldots,i_n}(x)$$

I am sure a proof of convergence in L^2 norm for this exists somewhere.

0.2.3 Linear Real Arithmetics

Linear real arithmetics is a restriction of non-linear real arithmetics that we considered above. Recall the following formula:

$$\varphi = \bigwedge_{i=1}^{N} \left(\bigvee_{j=1}^{M} f_{i,j}(\vec{x}) \leq \varepsilon_{i,j} \right) \qquad \vec{x} \in K \subseteq \mathbb{R}^n, \;\; f_{i,j} \colon K \to \mathbb{R}, \;\; \varepsilon_{i,j} \geq 0$$

In **linear** real arithmetics, each inequality now has form:

$$\varphi = \bigwedge_{i=1}^{N} \left(\bigvee_{j=1}^{M} \left(\sum_{k=1}^{n} a_{i,j,k} x_k \right) \leq \varepsilon_{i,j} \right) \qquad \vec{x} = (x_1, \ldots, x_n) \in K \subseteq \mathbb{R}^n, \;\; a_{i,j,k} \in \mathbb{R}, \;\; \varepsilon_{i,j} \geq 0$$

Variants of this problem include cases when the disjunctions are unit, which is equivalent to the problem of intersection finding for halfspaces. Convex optimization problems often take such form, and have elegant theories and techniques. However, the importance here is that the theory of linear real arithmetics is decidable.

0.3 The Idea

We approach the problem of approximate satisfiability and synthesis by linearized approximations. For each $f_{i,j}$ in the boolean conjunction that makes up a non-linear real arithmetic formula, we may find Fourier series, whose coefficients are real. Such a reduction yields a problem in the domain of real arithemtics, which is decidable. We test the approach, formalize it, and perform error analysis.

0.3.1 Linearization

We rewrite a formula over non-linear real arithmetics as follows:

$$\varphi = \bigwedge_{i=1}^{N} \left(\bigvee_{j=1}^{M} T_{\vec{N}}[f_{i,j}](\vec{x}) \leq T_{\vec{N}}[\varepsilon_{i,j}](\vec{x}) \right) \qquad \vec{x} \in K \subseteq \mathbb{R}^n,\ \ \vec{N} \in \mathbb{Z}^n,\ \ f_{i,j} \colon K \to \mathbb{R},\ \ \varepsilon_{i,j} \geq 0$$

Because we have real-valued functions, we induce real-valued coefficients, and this brings the problem back into a decidable theory. We now investigate how well this may work:

Testing Stuff As a preliminary study we consider basic functions of the form:

$$f_1(x) \leq x - \frac{1}{2} \qquad f_2(x) \geq \frac{x}{2} - \frac{1}{4} \qquad x \in [0,1]$$

This set of inequalities has a solution on $x \in [1/2, 1]$. Taking their Fourier transforms:

$$\mathcal{F}_n[f_1](x) = \frac{-i\pi n + e^{i2\pi n}(1 - i\pi n) - 1}{4\pi^2 n^2} \qquad \mathcal{F}_n[f_2](x) = \frac{-i\pi n + e^{i2\pi n}(1 - i\pi n) - 1}{8\pi^2 n^2}$$

For constants, the Fourier transform yields zero. We now examine the coefficients of the Fourier series generated by f_1 and f_2:

$$
\begin{aligned}
f_1 \mapsto \quad & n=-5 \mapsto \frac{1}{10\pi}e^{\pi/2} & f_2 \mapsto \quad & n=-5 \mapsto \frac{1}{20\pi}e^{\pi/2} \\
& n=-4 \mapsto \frac{1}{8\pi}e^{\pi/2} & & n=-4 \mapsto \frac{1}{16\pi}e^{\pi/2} \\
& n=-3 \mapsto \frac{1}{6\pi}e^{\pi/2} & & n=-3 \mapsto \frac{1}{12\pi}e^{\pi/2} \\
& n=-2 \mapsto \frac{1}{4\pi}e^{\pi/2} & & n=-2 \mapsto \frac{1}{8\pi}e^{\pi/2} \\
& n=-1 \mapsto \frac{1}{2\pi}e^{\pi/2} & & n=-1 \mapsto \frac{1}{4\pi}e^{\pi/2} \\
& n=0 \mapsto \emptyset & & n=0 \mapsto \emptyset \\
& n=1 \mapsto \frac{1}{2\pi}e^{-\pi/2} & & n=1 \mapsto \frac{1}{4\pi}e^{-\pi/2} \\
& n=2 \mapsto \frac{1}{4\pi}e^{-\pi/2} & & n=2 \mapsto \frac{1}{8\pi}e^{-\pi/2} \\
& n=3 \mapsto \frac{1}{6\pi}e^{-\pi/2} & & n=3 \mapsto \frac{1}{12\pi}e^{-\pi/2} \\
& n=4 \mapsto \frac{1}{8\pi}e^{-\pi/2} & & n=4 \mapsto \frac{1}{16\pi}e^{-\pi/2} \\
& n=5 \mapsto \frac{1}{10\pi}e^{-\pi/2} & & n=5 \mapsto \frac{1}{20\pi}e^{-\pi/2}
\end{aligned} \tag{1}
$$

0.3.2 Program Synthesis

TODO: The linear methods described in Kuncak, Victor, Mayer, Mikaël and Piskac's paper is still applicable here. But really we want a more custom LRA solution that I need to read up on.

0.3.3 Error Analysis

Oh fuck.

0.3.4 Complexity Analysis

Time consuming.

0.3.5 Efficient Implementation

0.4 Remarks

References

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