

# Constrained Linear Quadratic Regular Optimization

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## 1 Introduction

We look at constrained linear quadratic regulator (LQR) optimization.

## 2 Background

Relevant background on linear systems, LQR, and matrix magic.

### 2.1 Linear Systems and the Linear Quadratic Regulator

Linear system dynamics

$$x_{t+1} = Ax_t + Bu_t, \quad (x_t)_{t=0}^\infty \in \mathbb{R}^n, \quad (u_t)_{t=0}^\infty \in \mathbb{R}^p,$$

Finite horizon linear quadratic regular (LQR) cost function

$$J = \sum_{t=0}^T x_t^\top Q x_t + u_t^\top R u_t, \quad Q \succeq 0, \quad R \succ 0,$$

which is finite when the system dynamics  $(x_t, u_t)$  tends towards the origin. Infinite-horizon LQR average-cost function

$$J = \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t^\top Q x_t + u_t^\top R u_t, \quad Q \succeq 0, \quad R \succ 0,$$

which allows for systems whose stability is not necessarily at the origin. Both versions of LQR admit an optimal feedback control sequence that is linear, i.e.,  $u = Kt$ , with respect to the algebraic Riccati equation in  $P$  where

$$P = Q + A^\top P A - A^\top P B (R + B^\top P B)^{-1} B^\top P A, \quad K = -(R + B^\top P B)^{-1} B^\top P A$$

where the solution for  $P \succ 0$ , if it exists, is unique.

## 2.2 Affine Systems

Given the affine system

$$x_{t+1} = Ax_t + Bu_t + c$$

we can reformulate it as a linear system

$$z_{t+1} = \begin{bmatrix} x_{t+1} \\ 1 \end{bmatrix} = \begin{bmatrix} A & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ 1 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t = \tilde{A}z_t + \tilde{B}u_t \quad (1)$$

## 2.3 Completing the Square

For the matrix case with variable in  $x$  from Wikipedia [2],

$$x^\top Qx + p^\top x + r = (x - h)^\top Q(x - h) + k, \quad h = -\frac{1}{2}Q^{-1}p, \quad k = r - \frac{1}{4}p^\top Q^{-1}p \quad (2)$$

where  $Q$  is assumed to be invertible.

## 3 Some Results with Affine Constraints

**Theorem 1.** *An infinite horizon average-cost LQR problem with quadratic-plus-affine terms*

$$\begin{aligned} & \text{minimize} \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t^\top Qx_t + q^\top x_t + q_0 + u_t^\top Ru_t + r^\top u_t + r_0 \\ & \text{subject to} \quad x_{t+1} = Ax_t + Bu_t, \text{ for all } t \end{aligned}$$

*admits an equivalent infinite horizon average-cost LQR problem without affine terms.*

*Proof.* Applying (2) we can write the objective as

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} (x_t - h)^\top Q(x_t - h) + (u_t - l)^\top R(u_t - l) + k,$$

with coefficients

$$h = -\frac{1}{2}Q^{-1}q, \quad l = -\frac{1}{2}R^{-1}r, \quad k = q_0 + r_0 - \frac{1}{4}(q^\top Qq + r^\top Rr)$$

Observe that  $k$  is constant with respect to the variables  $(x_t, u_t)$  and the objective is equivalently

$$k + \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} (x_t - h)^\top Q(x_t - h) + (u_t - l)^\top R(u_t - l).$$

Define the change-of-variables  $\tilde{x}_t = x_t - h$  and  $\tilde{u}_t = u_t - l$  for all  $t$ , which induces the affine system

$$\tilde{x}_{t+1} = A\tilde{x}_t + B\tilde{u}_t + (Ah + Bl - h)$$

and after the appropriate change of variables to recover a linear system as in (1),

$$z_{t+1} = \begin{bmatrix} A & (Ah + Bl - h) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_t \\ 1 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \tilde{u}_t = \tilde{A}z_t + \tilde{B}\tilde{u}_t.$$

For this, the reduced problem is then

$$\begin{aligned} \text{minimize} \quad & k + \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} z_t^\top \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} z_t + \tilde{u}_t^\top R \tilde{u}_t \\ \text{subject to} \quad & z_{t+1} = \begin{bmatrix} A & (Ah + Bl - h) \\ 0 & 1 \end{bmatrix} z_t + \begin{bmatrix} B \\ 0 \end{bmatrix} \tilde{u}_t, \text{ for all } t \end{aligned}$$

with variables in  $(z_t)_{t=0}^\infty$  and  $(u_t)_{t=0}^\infty$ . Barring the  $k$  offset in the objective, this is now equivalent to an infinite horizon average-cost LQR problem with no affine terms.  $\square$

Because quadratic-plus-affine terms can be eliminated to yield a problem with purely quadratic cost, Theorem 1 implies that quadratic systems also have an optimal feedback control sequence that is linear for the equivalent system with trajectory in  $(z_t, \tilde{u}_t)_{t=0}^\infty$ . But since  $(z_t, \tilde{u}_t)$  are linear transformations of the original trajectory  $(x_t, u_t)$ , the optimal feedback policy for the original dynamics is also linear.

**Theorem 2.** *The infinite-horizon average-cost LQR problem with affine constraints*

$$\begin{aligned} \text{minimize} \quad & \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t^\top Q x_t + u_t^\top R u_t \\ \text{subject to} \quad & x_{t+1} = Ax_t + Bu_t, \text{ for all } t \\ & Cx_t = d, \text{ for all } t \end{aligned}$$

where  $C$  is fat and full rank, admits an optimal feedback control policy that is linear if:

- There exists  $\hat{u}$  such that  $\hat{x} = A\hat{x} + B\hat{u}$ , where  $\hat{x} = C^\dagger d$ .
- $\text{ran } AN \subseteq \text{ran } B$ , where  $\text{ran } N = \ker C$ .

*Proof.* Supposing the assumptions, we can write our system dynamics as

$$\hat{x} + Nx_{t+1} = A(\hat{x} + Nx_t) + B(\hat{u} + v_t)$$

where we treat  $N$  as a projection matrix onto  $\ker C$ . However this is not quite in a standard linear systems form. Further factor the system matrices into  $A = A_1 + A_2$  and  $B = B_1 + B_2$  such that

$$\text{ran } A_1 \subseteq \text{ran } B_1 \subseteq \ker C^\perp, \quad \text{ran } A_2 \subseteq \text{ran } B_2 \subseteq \ker C,$$

Let  $y_t = Nx_t$ ; the goal of such factorization is to ensure that

$$A_1 y_t + B_1 v_t = 0 \in \ker C^\perp, \quad A_2 y_t + B_2 v_t \in \ker C$$

In effect, the  $(A_1, B_1)$  sub-system needs to eliminate action in  $\ker C^\perp$ : for which appropriate  $v_t$  can be found given  $y_t$  due to the (linear)  $\text{ran } AN \subseteq \text{ran } B$  assumption. while  $(A_2, B_2)$  is parameterized to lie purely in  $\ker C$ . Thus, we need only care about the  $(A_2, B_2)$  sub-system, for which results can be linearly translated back to the original  $(A, B)$  system. Define the following for compactness,

$$J(y_t, v_t) = \begin{bmatrix} \hat{x} \\ y_t \end{bmatrix}^\top \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \hat{x} \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ 2Q\hat{x} \end{bmatrix}^\top \begin{bmatrix} \hat{x} \\ y_t \end{bmatrix} + \begin{bmatrix} \hat{u} \\ v_t \end{bmatrix}^\top \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} \hat{u} \\ v_t \end{bmatrix} + \begin{bmatrix} 0 \\ 2R\hat{x} \end{bmatrix}^\top \begin{bmatrix} \hat{u} \\ v_t \end{bmatrix}$$

and noting that

$$x_t^\top Q x_t + u_t^\top R u_t = (\hat{x} + y_t)^\top Q (\hat{x} + y_t) + (\hat{u} + v_t)^\top R (\hat{u} + v_t) = J(y_t, v_t)$$

makes it so that the optimizatoin problem can be written as

$$\begin{aligned} & \text{minimize} \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} J(y_t, v_t) \\ & \text{subject to} \quad y_{t+1} = A_2 y_t + B_2 v_t \end{aligned}$$

with variables in  $(y_t)_{t=0}^\infty$  and  $(v_t)_{t=0}^\infty$ . Since  $J$  is a quadratic-plus-affine decomposition of the state and control costs, we know that this infinite-horizon average-cost LQR problem. As noted earlier, this system is derived from linear transformations on our original system  $(A, B)$ , and so an optimal controller linear for  $(A_2, B_2)$  also implies linearity for  $(A, B)$ .

□

## References

- [1] Stephen Boyd. “Ee363: Linear dynamical systems”. In: *Lecture. Stanford Univer., Winter Quarter* 9 (2008). URL: [https://stanford.edu/class/ee363/lectures/stoch\\_lqr.pdf](https://stanford.edu/class/ee363/lectures/stoch_lqr.pdf).
- [2] Wikipedia contributors. *Completing the square* — *Wikipedia, The Free Encyclopedia*. [Online; accessed 14-May-2020]. 2020. URL: [https://en.wikipedia.org/w/index.php?title=Completing\\_the\\_square&oldid=953923455](https://en.wikipedia.org/w/index.php?title=Completing_the_square&oldid=953923455).