

# Operator Automata Theory

## 1 Introduction

Descriptive text

## 2 Preliminaries

Monoid

Semiring

Semimodule

Linear transform

Norm

Automata

## 3 Operators

Text goes here

### 3.1 $M$ -Semimodules and Operator Norms

Let  $\Sigma$  be a finite set and  $M$  to be the free monoid finitely generated by  $\Sigma$ :

$$M = (\Sigma, \cdot, \mathbf{1})$$

**Definition 1** (Normed Monoid). *A normed monoid is a monoid  $M$  equipped with norm  $\|\cdot\|_M$ .*

Because monoids lack addition, such norms only concern multiplication.

We may extend monoids in general to define the notion of a  $M$ -semiring with respect to a monoid  $M$ . In particular,  $M$ -semirings are generated by the power set of  $M$ ,  $\mathcal{P}(M)$ .

**Definition 2** ( $M$ -Semiring). *Let  $M$  be a monoid. A  $M$ -semiring  $R_M$  is a semiring:*

$$R_M = (\mathcal{P}(M), \cup, \cdot, \mathbf{0}, \mathbf{1})$$

*Where semiring addition is set union  $\cup$  with identity  $\mathbf{0}$ , and semiring multiplication  $\cdot$  and identity  $\mathbf{1}$  are carried over from  $M$ .*

We may write  $R_M$  as just  $R$  when context is clear. Sometimes we will also just call  $M$ -semirings as semirings, because they are just special cases of semirings. As with the case of normed monoids, we may extend this to normed  $M$ -semirings:

**Definition 3** ( $M$ -Semiring  $p$ -Norm). *For  $1 \leq p \leq \infty$ , a  $p$ -normed  $M$ -semiring is  $R_M$  equipped with a norm  $\|\cdot\|_{R_M, p}$ , defined as:*

$$\forall x \in R_M : \|x\|_{R_M, p} = \left( \sum_{a \in R_M} \|a\|_M^p \right)^{1/p}$$

Observe that because  $M$  is finitely generated, each  $x \in R_M$  is therefore countable, and hence so is the sum. When  $p = \infty$ , this is just a sup norm. Similar definitions can be found in literature [1].

Extending  $M$ -semirings, we define  $(M, n)$ -semimodules:

**Definition 4** ( $(M, n)$ -Semimodule). *A  $(M, n)$ -semimodule  $R_M^n$  is a semimodule consisting of  $n$  isomorphic copies of  $R_M$ .*

If  $x \in R_M^n$ , write  $x_i$  to denote the  $i$ th element from  $R_M$  in some canonical representation of  $R_M^n$ . Often this is just a row (horizontal) or column (vertical) vector of length  $n$ .

We write  $R^n$  instead of  $R_M^n$  when  $M$  is understood from context. Again, we extend norms to  $(M, n)$ -semimodules:

**Definition 5** ( $(M, n)$ -Semimodule  $(p, q)$ -Norm). *Let  $R_M^n$  be a  $(M, n)$ -semimodule and take  $1 \leq p, q \leq \infty$ . Define the  $(M, n)$ -semimodule  $(p, q)$ -norm  $\|\cdot\|_{R_M^n, p, q}$ :*

$$\forall x \in R_M^n : \|x\|_{R_M^n, p, q} = \left( \sum_{i=1}^n \|x_i\|_{R_M, p}^q \right)^{1/q}$$

When  $p = q = \infty$ , these are just the sup-norm. Often we may only care about the case of  $p = \infty$  and  $q = 1$ , which is the sup-norm over each  $R_M$ , and the 1-norm over the  $n$  copies of  $R_M$  in  $R_M^n$ .

With the notion of norms, we again extend such notions to linear operators mapping  $R_M^n$  to  $R_M^m$ .

**Definition 6** (Linear Operator Norm). *Take  $R_M^n$  equipped with norm  $\|\cdot\|_{R_M^n, p_n, q_n}$  and  $R_M^m$  equipped with norm  $\|\cdot\|_{R_M^m, p_m, q_m}$ . Let  $\mathcal{L}$  be the space of linear operators mapping  $R_M^n$  to  $R_M^m$ , then define the  $\|\cdot\|_{\mathcal{L}}$  as:*

$$\forall T \in \mathcal{L} : \|T\|_{\mathcal{L}} = \inf \left\{ c \in \mathbb{R}_{\geq 0} : \|Tx\|_{R_M^m, p_m, q_m} \leq c \|x\|_{R_M^n, p_n, q_n}, \forall x \in R_M^n \right\}$$

## 3.2 Examples

**Example 1.** *Consider  $\Sigma = \{a, b\}$ .*

## 4 On Automata

## References

- [1] Manfred Kudlek. “Iteration Lemmata for Normed Semirings (Algebraic Systems, Formal Languages and Computations)”. In: (2000).