# Operator Automata Theory

## 1 Introduciton

The goal of operator automata theory is to view automata is to view automata as homomorphisms between linear spaces. In this sketch we develop the basic theory and preliminary results.

## 2 Preliminaries

**Definition 1** (Monoid). *A monoid* $(M, \cdot, \mathbf{1})$ *consists of a set* $M$ *for which:*

- *Monoid multiplication* $\cdot \colon M \times M \to M$ *is associative.*
- *The element* $\mathbf{1} \in M$ *is the unique identity of multiplication.*

**Definition 2** (Semiring). *A semiring* $(R, +, \cdot, \mathbf{0}, \mathbf{1})$ *consists of a set* $R$ *for which:*

- *Semiring multiplication* $\cdot \colon R \times R \to R$ *distributes over semiring addition* $+ \colon R \times R \to R$.
- $(R, +, \mathbf{0})$ *is a commutative monoid.*
- $(R, \cdot, \mathbf{1})$ *is a monoid.*
- *Multiplication by zero* $\mathbf{0}$ *annihilates* $R$.

**Definition 3** (Semimodule). *For a semiring* $R$*, a* $R$*-semimodule* $(M, +, \times, \mathbf{0})$ *is such that:*

- $(M, +, \mathbf{0})$ *is a commutative group.*
- *Scalar multiplication* $\cdot \colon R \times M \to M$ *is a semiring action on* $M$.

**Definition 4** (Field). *A field* $(K, +, \cdot, \mathbf{0}, \mathbf{1})$ *is a set* $K$ *such that:*

- *Field multiplication* $\cdot \colon K \times K \to K$ *distributes over field addition* $+ \colon K \times K \to K$.
- $(K, +, \mathbf{0})$ *is a commutative group.*

- $(K, \cdot, \mathbf{1})$ *is a commutative group.*
- *Multiplication by zero* $\mathbf{0}$ *annihilates* $K$*.*

**Definition 5** (Linear Space)**.** *For a field* $K$*, a* $K$*-linear space* $(V, +, \cdot, \mathbf{0})$ *over a field* $K$ *is a set such that:*

- $(V, +, \mathbf{0})$ *is a commutative group.*
- *Scalar multiplication* $\cdot \colon \mathbb{K} \times V \to V$ *is a field action on* $V$*.*

**Definition 6** (Linear Operator)**.** *For* $K$*-linear spaces* $V$ *and* $W$*, a linear operator* $T \colon V \to W$ *is a homomorphism.*

# 3 Operators

**(Text goes here)**

## 3.1 $M$-Semimodules and Operator Norms

Let $\Sigma$ be a finite set and $M$ to be the monoid finitely generated by $\Sigma$:

$$M = (\Sigma, \cdot, \mathbf{1})$$

**Definition 7** (Normed Monoid)**.** *A normed monoid is a monoid* $M$ *with a norm* $\|\cdot\| : M \to \mathbb{R}_{\geq 0}$*.*

The canonical norm here is length.

We may extend monoids in general to define the notion of a $M$-semiring. In particular, $M$-semirings are defined with respect to a monoid $M$, and are generated from its power set $2^M$.

**Definition 8** ($M$-Semiring)**.** *Let* $M$ *be a finitely generated monoid. A* $M$*-semiring* $R$ *is a semiring such that:*

$$R = \left(2^M, \cup, \cdot, \mathbf{0}, \mathbf{1}\right)$$

*Where semiring addition is set union* $\cup$ *with identity* $\mathbf{0}$*, and semiring multiplication* $\cdot$ *and identity* $\mathbf{1}$ *are carried over from* $M$*.*

As with the case of normed monoids, we may extend this to normed $M$-semirings. In particular, we pay special attention to $p$-norms.

**Definition 9** ($M$-Semiring $p$-Norm)**.** *Let* $M$ *be a finitely generated and normed monoid. For* $1 \leq p \leq \infty$*, a* $p$*-normed* $M$*-semiring is* $R$ *is equipped with a norm* $\|\cdot\|_p : R \to \mathbb{R}_{\geq 0}$*:*

$$\|x\|_p = \left(\sum_{a \in x} \|a\|^p\right)^{1/p}$$

The sum here utilizes the monoid norm. Observe that because $M$ is finitely generated, each $x \in R_M$ is therefore countable, and hence so is the sum. When $p = \infty$, this is just a sup norm. Similar definitions can be found in literature [1].

Extending $M$-semirings, we define $(M, n)$-semimodules:

**Definition 10** ($(M, n)$-Semimodule)**.** *A $(M, n)$-semimodule $R^n$ is a free semimodule generated by $n$ isomorphic copies of the $M$-semiring $R$.*

If $x \in R^n$, write $x_i$ to denote the $i$th element from $R$ in some canonical representation of $R^n$. Often this is just a row (horizontal) or column (vertical) vector of length $n$.

Again, we extend norms to $(M, n)$-semimodules:

**Definition 11** ($(M, n)$-Semimodule $(p, q)$-Norm)**.** *Let $R$ be a $p$-normed $M$-semiring. Let $R^n$ be a $(M, n)$-semimodule and take $1 \leq p, q \leq \infty$. A $(p, q)$-normed $(M, n)$-semimodule is a semimodule with norm $\|\cdot\|_{p,q} : R^n \to \mathbb{R}_{\geq 0}$:*

$$\|x\|_{p,q} = \left( \sum_{i=1}^{n} \|x_i\|_p^q \right)^{1/q}$$

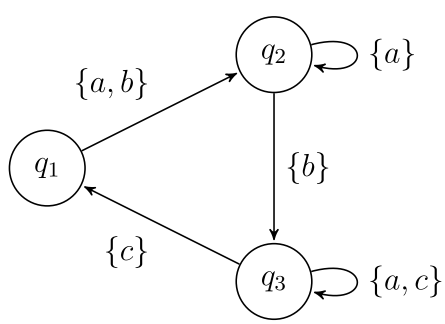
When $p = q = \infty$, these are just the sup-norm.

The goal of operator automata theory is to view automatas as linear transformations between linear spaces. To that end, we consider how to embed an automata as a linear transformation.

**Definition 12** (Something something automata)**.** *Descriptive definition text goes here.*

## 3.2 Examples

**Example 1.** *Consider $\Sigma = \{a, b, c\}$, and a finite automata below:*




*There are a few things to note in our model:*

*(1) The automata is non-deterministic, because from $q_3$, there are multiple transitions that may be taken.*

(2) *We lack the notion of a start and final state. Rather, every state is treated as both start and final in this perspective to make embedding slightly easier.*

*To embed this into our model in several steps. First, take $M$ to be the monoid generated by $\Sigma$, where monoid multiplication is taken to be string concatenation, and unit $1$ is aliased as $\varepsilon$ the empty string.*

$$M = (\Sigma, \cdot, 1)$$

*The $M$-semiring is generated using the powersets of $M$, where the unit of addition $0$ is equivalent to the empty set $\emptyset$:*

$$R = \left(2^M, \cup, \cdot, 0, 1\right)$$

*To demonstrate a better picture of how this works, we construct a transition matrix $A$ for the automata that acts on the $(M, 3)$-semimodule. Here we have $3$ because there are $3$ states. Let $A_{i,j}$ denote the transition set from state $i$ to state $j$.*

$$A = \begin{bmatrix} 0 & \{a,b\} & 0 \\ 0 & \{a\} & \{b\} \\ \{c\} & 0 & \{a,c\} \end{bmatrix}$$

*In order to perform string concatenation towards the right, transition matrices act by right-matrix multiplication. That is, if $v \in R^3$ is the initial (row) vector, then the subsequent state is $vA$.*

*To briefly demonstrate, two transitions of the matrix $A$ appears as follows:*

$$A^2 = \begin{bmatrix} 0 & \{a,b\} & 0 \\ 0 & \{a\} & \{b\} \\ \{c\} & 0 & \{a,c\} \end{bmatrix} \begin{bmatrix} 0 & \{a,b\} & 0 \\ 0 & \{a\} & \{b\} \\ \{c\} & 0 & \{a,c\} \end{bmatrix} = \begin{bmatrix} 0 & \{aa,ba\} & \{ab,bb\} \\ \{bc\} & \{aa\} & \{ab,ba,bc\} \\ \{ac,cc\} & \{ca,cb\} & \{aa,ac,ca,cc\} \end{bmatrix}$$

*In general, from graph theory, $A^k$ denotes the $k$th consecutive transition using $A$, and each entry $A^k_{i,j}$ is the set of strings that will get from state $q_i$ to $q_j$ in $k$ steps.*

# References


[1] Manfred Kudlek. "Iteration Lemmata for Normed Semirings (Algebraic Systems, Formal Languages and Computations)". In: (2000).