Discrete Fourier Transforms for Non-linear Real Arithmetics

0.1 The Problem

We first consider the satisfiability of first-order logic formulae over non-linear real arithmetics:

$$\varphi = \bigwedge_{i=1}^{N} \left(\bigvee_{j=1}^{M} f_{i,j}(\vec{x}) \leq \varepsilon_{i,j} \right) \qquad \vec{x} \in K \subseteq \mathbb{R}^n,\;\; f_{i,j} \colon K \to \mathbb{R},\;\; \varepsilon_{i,j} \geq 0$$

The quesiton of satisfiability, however, is undecidable: we cannot yield algorithms that, given φ, is able to find some $\vec{x}$ if there exists a satisfying configuration. We thus seek to approximate solutions, which throws away soundness for feasibility.

However, we are interested in still another class of problems: program synthesis. Given a pre-condition P that we assume at the start of program execution, and post-condition Q, does there exists some program S that is able to transform P into Q? When P and Q are formulae over linear integer arithmetics, for instance, the corresponding program is akin to a sequence of variable assignments that correspond to a linear transformation. We never said these programs have to be fancy, did we?

0.2 Background

0.2.1 The Fourier Transform

Consider the separable Hilbert space $\mathcal{H} = L^2([0,1])$ endowed with the Fourier basis $\{e_n\}_{n\in\mathbb{Z}}$, where we define $e_n(x) = e^{i2\pi nx}$. A Fourier transform $\mathcal{F}_n \colon \mathcal{H} \to \mathbb{R}$ is a linear functional that can be expressed as an inner product:

$$\mathcal{F}_n[f] = \langle f, e_n \rangle = \int_0^1 f(x)\overline{e_n(x)}dx = \int_0^1 f(x)e^{-i2\pi nx}dx$$

Observe that because f is real-valued, the integration yields real values. We further claim that the Fourier transform is continuous with norm 1:

$$\|F_n\|_{\mathcal{H}^*} = \sup_{\|f\|_{\mathcal{H}}=1} \left| \int_0^1 f(x)\overline{e_n(x)}dx \right| \leq \left(\int_0^1 |f(x)|^2 dx \right)^{1/2} \left(\int_0^1 |e_n(x)|^2 dx \right)^{1/2} = \|f\|_{\mathcal{H}} \cdot \|e_n\|_H$$

The inequality is sharp when $f = e_n$. We recall that the partial Fourier series $S_n \colon \mathcal{H} \to \mathcal{H}$ is a (finite dimensional) compact operator:

$$S_N[f](x) = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n(x) = \sum_{n=-N}^{N} e_n(x) \int_0^1 f(y) e^{-i2\pi ny} dy$$

Note that this is an orthogonal projection. Consider some m such that $|m| > N$:

$$\langle e_m, S_n \rangle = \left\langle e_m, \sum_{n=-N}^{N} \langle f, e_n \rangle e_n \right\rangle = \sum_{n=-N}^{N} \langle f, e_n \rangle \cdot \langle e_m, e_n \rangle = \sum_{n=-N}^{N} \langle f, e_n \rangle \cdot 0 = 0$$

Furthermore, we omit the proof, but do note that this converges in norm for any $f \in \mathcal{H}$:

$$\lim_{N\to\infty} \|f - S_N[f]\|_{\mathcal{H}} = \lim_{N\to\infty} \left| f(x) - \sum_{n=-N}^{N} \int_0^1 f(x) e^{-i2\pi nx} dx \right| \to 0$$

We also remark that the class of continuous functions over compact support are dense in L^2. This motivates us to consider the problem of non-linear real arithmetics over compact (perhaps convex?) domains.

0.2.2 Multi-Dimensional Fourier Transform

We now consider the Fourier transform when we consider n dimensions. For $\mathcal{H} = L^2([0,1]^n)$. We define as the orhonormal basis:

$$\phi_{d_1,\ldots,d_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} e_{d_i}(x_i) = e^{i2\pi(d_1x_1 + \cdots + d_nx_n)} \qquad d_1, \ldots, d_n \in \mathbb{Z}$$

We likewise define the Fourier transform $F_{i_1,\ldots,i_n} : \mathbb{R}^n \to \mathbb{R}$ as a linear functional in the form of an inner product, with $\vec{x} \in \mathbb{R}^n$:

$$\mathcal{F}_{i_1,\ldots,i_n}[f](\vec{x}) = \langle f, \phi_{i_1,\ldots,i_n} \rangle = \int_{[0,1]^n} f(\vec{x})\overline{\phi_{i_1,\ldots,i_n}(\vec{x})} d\vec{x}$$

A Fourier series for $N_1, \ldots, N_n$:

$$S_{N_1,\ldots,N_n}[f](\vec{x}) = \sum_{i_1=-N_1}^{N_1} \cdots \sum_{i_n=-N_n}^{N_n} \langle f, \phi_{i_1,\ldots,i_n} \rangle \phi_{i_1,\ldots,i_n}(\vec{x})$$

I am sure a proof of convergence in L^2 norm for this exists somewhere.

0.2.3 Discrete Fourier Transform

The discrete Fourier transform bridges the continuous and

0.3 The Problem

We may approximate the precondition as a matrix of linear inequalities over the dual space of the domain, and likewise for the codomain. This therefore becomes an LP problem with respect to the dual spaces, and all that's left is figuring out how fast the L2 norms converge