Linear Operators on Kleene Algebras

1 Introduction

1.1 Algebraic Structures

Definition 1 (Monoid). *A monoid is a set S with a binary operation $\circ\colon S \times S \to S$ (multiplication) with the following axioms:*

(1) $\circ$ is associative.

(2) There exists an identity element $e \in S$ such that for all $a \in S$:

$$e \circ a = a = a \circ e$$

Definition 2 (Commutative Monoid). *A monoid $(S, \circ)$ is commutative if $\circ$ is commutative.*

Definition 3 (Semiring). *A semiring is a set R with two binary operations $+\colon R \times R \to R$ (addition) and $\cdot\colon R \times R \to R$ (multiplication) with the following axioms:*

(1) $(R, +)$ is a commutative monoid with identity 0.

(2) $(R, \cdot)$ is a monoid with identity 1.

(3) Multiplication left and right distributes over addition.

(4) Multiplication by 0 annihilates R.

Definition 4 (Left Module Over a Semiring). *A left module M over a semiring R is a set with two binary operations $+\colon M \times M \to M$ (addition) and $\cdot\colon R \times M \to M$ (scalar multiplication) with the following axioms:*

(1) $(M, +)$ is an abelian group.

(2) For all $s, r \in R$ and $x, y \in M$:

$$\begin{aligned} r \cdot (x + y) &= (r \cdot x) + (r \cdot y) \\ (r + s) \cdot x &= (r \cdot x) + (s \cdot x) \\ (r \cdot s) \cdot x &= r \cdot (s \cdot x) \\ 1_R \cdot x &= x \end{aligned}$$

Definition 5 (Right Module Over a Semiring)**.** *A right module would be defined similarly, except all instances of scalar multiplication happen on the right side.*

Definition 6 (Bimodule Over a Semiring)**.** *A bimodule is a module that is both a left module and a right module with respect to scalar multiplication.*

If M is a module over a semiring R, we write M/R for shorthand. We implicitly assume bimodule structures, or whatever is convenient, to avoid excessive worrying over the correct left and right behavior.

Definition 7 (n-length Module)**.** *An n-length module M/R consists of n elements:*

$$M = (r_1, \ldots, r_n) \qquad\qquad r_1, \ldots, r_n \in R$$

Write M_i to also mean r_i in an abuse of notation.

Definition 8 (Linear Transformation)**.** *A linear transformation is a homomorphic map $T\colon V \to W$ between two modules V/R and W/R that preserves addition and scalar multiplication. That is, for all $r \in R$ and $x, y \in V$:*

$$\begin{aligned} T\left(0_R \cdot x\right) &= 0_W \\ T\left(r \cdot x + y\right) &= r \cdot T\left(x\right) + T\left(y\right) \end{aligned}$$

We may use matrices to represent linear transformations between n-length modules. Let $M_{n\times n}\left(R\right)$ be the set of $n \times n$ dimensional matrices with entries from semiring R.

Definition 9 (Matrix Module (Left) Multiplication)**.** *If $A \in M_{n\times n}\left(R\right)$, and M/R is some n-length module. Define left matrix-module multiplication as follows:*

$$\left(AM\right)_i = \sum_{k=1}^{n} A_{i,k} M_k$$

Here the result AM is another n-length module over semiring R. In particular, this action corresponds to left multiplication by the matrix. If instead we wish to define right multiplication:

Definition 10 (Matrix Module (Right) Multiplication)**.** *If $A \in M_{n\times n}\left(R\right)$, and M/R is some n-length module. Define right matrix-module multiplication as follows:*

$$\left(MA\right)_i = \sum_{k=1}^{n} M_k A_{k,i}$$

Typically we take the module M to have a column vector representation during left multiplication, and a row vector representation for right multiplication.

2 Kleene Algebra Mappings

Consider a Kleene algebra $\mathcal{K} = (K, +, \cdot, ^*, 0, 1)$. Recall that K forms an idempotent semiring with respect to $+$ and $\cdot$. We are primarily interested in how Kleene algebras relate to finite state machines (finite automata). In literature,

To this end, we take an n-length module M as:

Define an n-length module M as:

$$M = (k_1, k_2, \ldots, k_n) \qquad\qquad k_i \in K$$

Let M be the

2.1 Automata

In literature [1] we often have the following definitions:

Definition 11 (Alphabet)**.** *An alphabet Σ is a finite set of distinct letters (characters).*

Definition 12 (Language)**.**

References

[1] John E Savage. *Models of computation*. Vol. 136. Addison-Wesley Reading, MA, 1998.