Operator Automata Theory

1 Introduction

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3.1 M-Semimodules and Operator Norms

Let Σ be a finite set and M to be the monoid finitely generated by Σ:

$$M = (\Sigma, \cdot, \mathbf{1})$$

Definition 1 (Normed Monoid)**.** *A normed monoid is a monoid M with a norm $\|\cdot\| : M \to \mathbb{R}_{\geq 0}$.*

Because monoids lack addition, such norms only concern multiplication.

We may extend monoids in general to define the notion of a M-semiring. In particular, M-semirings are defined with respect to a particular monoid M, and are generated from its power set $\mathcal{P}(M)$.

Definition 2 (M-Semiring)**.** *Let M be a finitely generated monoid. A M-semiring R is a semiring such that:*

$$R = (\mathcal{P}(M), \cup, \cdot, \mathbf{0}, \mathbf{1})$$

Where semiring addition is set union $\cup$ with identity $\mathbf{0}$, and semiring multiplication $\cdot$ and identity $\mathbf{1}$ are carried over from M.

As with the case of normed monoids, we may extend this to normed M-semirings. In particular, we pay special attention to p-norms.

Definition 3 (M-Semiring p-Norm)**.** *Let M be a finitely generated and normed monoid. For $1 \le p \le \infty$, a p-normed M-semiring is R is equipped with a norm $\|\cdot\|_p : R \to \mathbb{R}_{\ge 0}$:*

$$\|x\|_p = \left(\sum_{a \in x} \|a\|^p \right)^{1/p}$$

The sum here utilizes the monoid norm. Observe that because M is finitely generated, each $x \in R_M$ is therefore countable, and hence so is the sum. When $p = \infty$, this is just a sup norm. Similar definitions can be found in literature [1].

Extending M-semirings, we define (M, n)-semimodules:

Definition 4 ((M, n)-Semimodule)**.** *A (M, n)-semimodule R^n is a free semimodule generated by n isomorphic copies of the M-semiring R.*

If $x \in R^n$, write x_i to denote the ith element from R in some canonical representation of R^n. Often this is just a row (horizontal) or column (vertical) vector of length n.

Again, we extend norms to (M, n)-semimodules:

Definition 5 ((M, n)-Semimodule (p, q)-Norm)**.** *Let R be a p-normed M-semiring. Let R^n be a (M, n)-semimodule and take $1 \le p, q \le \infty$. A (p, q)-normed (M, n)-semimodule is a semimodule with norm $\|\cdot\|_{p,q} : R^n \to \mathbb{R}_{\ge 0}$:*

$$\|x\|_{p,q} = \left(\sum_{i=1}^{n} \|x_i\|_p^q \right)^{1/q}$$

When $p = q = \infty$, these are just the sup-norm. Often we may only care about the case of $p = \infty$ and $q = 1$, which is the sup-norm over each R, and the 1-norm over the n copies of R in R^n.

With the notion of norms, we again extend such notions to linear operators mapping R^n to R^m.

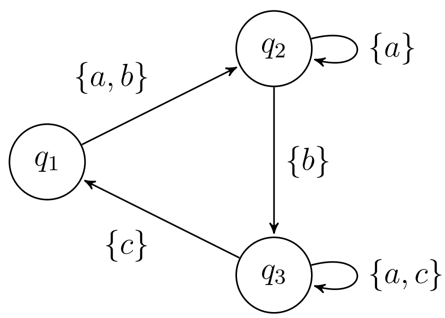
Definition 6 (Linear Operator Norm)**.** *Let $\mathcal{L}$ be the space of linear operators mapping a (M, n)-semimodule R^n with norm $\|\cdot\|_{R^n}$ to a (M, m)-semimodule R^m with norm $\|\cdot\|_{R^m}$. Then define the operator norm $\|\cdot\|_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}_{\geq 0}$ as:*

$$\|T\|_{\mathcal{L}} = \inf \left\{ c \in \mathbb{R}_{\geq 0} \; : \; \|Tx\|_{R^m} \leq c \|x\|_{R^n}, \forall x \in R^n \right\}$$

With the notion of a norm, we may perhaps begin to discuss weaker versions Banach spaces.

3.2 Examples

Example 1. *Consider $\Sigma = \{a, b, c\}$, and a finite automata below:*



There are a few things to note in our model:

(1) The automata is non-deterministic, because from q_3, there are multiple transitions that may be taken.

(2) We lack the notion of a start and final state. Rather, every state is treated as both start and final in this perspective to make embedding slightly easier.

To embed this into our model in several steps. First, take M to be the monoid generated by Σ, where monoid multiplication is taken to be string concatenation, and unit 1 is aliased as ε the empty string.

$$M = (\Sigma, \cdot, 1)$$

The M-semiring is generated using the powersets of M, where the unit of addition 0 is equivalent to the empty set $\emptyset$:

$$R = (\mathcal{P}(M), \cup, \cdot, 0, 1)$$

To demonstrate a better picture of how this works, we construct a transition matrix A for the automata that acts on the $(M, 3)$-semimodule. Here we have 3 because there are 3 states. Let $A_{i,j}$ denote the transition set from state i to state j.

$$A = \begin{bmatrix} 0 & \{a, b\} & 0 \\ 0 & \{a\} & \{b\} \\ \{c\} & 0 & \{a, c\} \end{bmatrix}$$

In order to perform string concatenation towards the right, transition matrices act by right-matrix multiplication. That is, if $v \in R^3$ is the initial (row) vector, then the subsequent state is vA.

To briefly demonstrate, two transitions of the matrix A appears as follows:

$$A^2 = \begin{bmatrix} 0 & \{a,b\} & 0 \\ 0 & \{a\} & \{b\} \\ \{c\} & 0 & \{a,c\} \end{bmatrix} \begin{bmatrix} 0 & \{a,b\} & 0 \\ 0 & \{a\} & \{b\} \\ \{c\} & 0 & \{a,c\} \end{bmatrix} = \begin{bmatrix} 0 & \{aa,ba\} & \{ab,bb\} \\ \{bc\} & \{aa\} & \{ab,ba,bc\} \\ \{ac,cc\} & \{ca,cb\} & \{aa,ac,ca,cc\} \end{bmatrix}$$

In general, from graph theory, A^k denotes the kth consecutive transition using A, and each entry $A^k_{i,j}$ is the set of strings that will get from state q_i to q_j in k steps.

4 On Automata

References

[1] Manfred Kudlek. "Iteration Lemmata for Normed Semirings (Algebraic Systems, Formal Languages and Computations)". In: (2000).