

Linear Operators on Kleene Algebras

1 Introduction

1.1 Algebraic Structures

Definition 1 (Monoid). A monoid is a set S with a binary operation $\circ: S \times S \rightarrow S$ (multiplication) with the following axioms:

- (1) \circ is associative.
- (2) There exists an identity element $e \in S$ such that for all $a \in S$:

$$e \circ a = a = a \circ e$$

Definition 2 (Commutative Monoid). A monoid (S, \circ) is commutative if \circ is commutative.

Definition 3 (Semiring). A semiring is a set R with two binary operations $+: R \times R \rightarrow R$ (addition) and $\cdot: R \times R \rightarrow R$ (multiplication) with the following axioms:

- (1) $(R, +)$ is a commutative monoid with identity 0.
- (2) (R, \cdot) is a monoid with identity 1.
- (3) Multiplication left and right distributes over addition.
- (4) Multiplication by 0 annihilates R .

Definition 4 (Left Module Over a Semiring). A left module M over a semiring R is a set with two binary operations $+: M \times M \rightarrow M$ (addition) and $\cdot: R \times M \rightarrow M$ (scalar multiplication) with the following axioms:

- (1) $(M, +)$ is an abelian group.

(2) For all $s, r \in R$ and $x, y \in M$:

$$\begin{aligned} r \cdot (x + y) &= (r \cdot x) + (r \cdot y) \\ (r + s) \cdot x &= (r \cdot x) + (s \cdot x) \\ (r \cdot s) \cdot x &= r \cdot (s \cdot x) \\ 1_R \cdot x &= x \end{aligned}$$

Definition 5 (Right Module Over a Semiring). A right module would be defined similarly, except all instances of scalar multiplication happen on the right side.

Definition 6 (Bimodule Over a Semiring). A bimodule is a module that is both a left module and a right module with respect to scalar multiplication.

If M is a module over a semiring R , we write M/R for shorthand.

Definition 7 (n -length Module). An n -length module M/R consists of n elements:

$$M = (r_1, \dots, r_n) \quad r_1, \dots, r_n \in R$$

Definition 8 (Linear Transformation). A linear transformation is a homomorphic map $T: V \rightarrow W$ between two modules V/R and W/R that preserves addition and scalar multiplication. That is, for all $r \in R$ and $x, y \in V$:

$$\begin{aligned} T(0_R \cdot x) &= 0_W \\ T(r \cdot x + y) &= r \cdot T(x) + T(y) \end{aligned}$$

We may use matrices to represent linear transformations between n -length modules. Let $M_{n \times n}(R)$ be the set of $n \times n$ dimensional matrices with entries from semiring R .

Definition 9 (Matrix Module (Left) Multiplication). If $A \in M_{n \times n}(R)$, and M/R is some n -length module. Define left matrix-module multiplication as follows:

$$(AM)_i = \sum_{k=1}^n A_{i,k} M_k$$

Here the result AM is another n -length module over semiring R . In particular, this action corresponds to left multiplication by the matrix. If instead we wish to define right multiplication:

Definition 10 (Matrix Module (Right) Multiplication). If $A \in M_{n \times n}(R)$, and M/R is some n -length module. Define right matrix-module multiplication as follows:

$$(MA)_i = \sum_{k=1}^n M_k A_{k,i}$$

Typically we take the module M to have a column vector representation during left multiplication, and a row vector representation for right multiplication.

2 Kleene Algebra Mappings

Consider a Kleene algebra $\mathcal{K} = (K, +, \cdot, *, 0, 1)$. Recall that K forms an idempotent semiring with respect to $+$ and \cdot . Define an n -length module M as:

$$M = (k_1, k_2, \dots, k_n) \quad k_i \in K$$

Let M be the