Operator Automata Theory

1 Introduction

The goal of operator automata theory is to view automata is to view automata as homomorphisms between linear spaces. In this sketch we develop the basic theory and preliminary results.

2 Preliminaries

2.1 Notation

(descriptive text here)

2.2 Algebra

Definition 1 (Monoid)**.** *A monoid* $(M, \cdot, \mathbf{1})$ *consists of a set* M *for which:*

- *Monoid multiplication* $\cdot \colon M \times M \to M$ *is associative.*
- *The element* $\mathbf{1} \in M$ *is the unique identity of multiplication.*

Definition 2 (Semiring)**.** *A semiring* $(R, +, \cdot, \mathbf{0}, \mathbf{1})$ *consists of a set* R *for which:*

- *Semiring multiplication* $\cdot \colon R \times R \to R$ *distributes over semiring addition* $+\colon R \times R \to R$.
- $(R, +, \mathbf{0})$ *is a commutative monoid.*
- $(R, \cdot, \mathbf{1})$ *is a monoid.*
- *Multiplication by zero* $\mathbf{0}$ *annihilates* R.

Definition 3 (Semimodule)**.** *For a semiring* R*, a* R*-semimodule* $(M, +, \times, \mathbf{0})$ *is such that:*

- $(M, +, \mathbf{0})$ *is a commutative group.*

- *Scalar multiplication* $\cdot\colon R \times M \to M$ *is a semiring action on* M.

Definition 4 (Group)**.** *A group* $(G, \cdot, \mathbf{1})$ *consists of a set* G *such that:*

- *Group multiplication* $\cdot\colon G \times G \to G$ *is associative.*
- $\mathbf{1}$ *is the identity under multiplication.*
- *If* $a \in G$*, then it has an inverse* $a^{-1} \in G$.

Definition 5 (Field)**.** *A field* $(K, +, \cdot, \mathbf{0}, \mathbf{1})$ *is a set* K *such that:*

- *Field multiplication* $\cdot\colon K \times K \to K$ *distributes over field addition* $+\colon K \times K \to K$.
- $(K, +, \mathbf{0})$ *is a commutative group.*
- $(K, \cdot, \mathbf{1})$ *is a commutative group.*
- *Multiplication by zero* $\mathbf{0}$ *annihilates* K.

Definition 6 (Linear Space)**.** *For a field* K*, a* K*-linear space* $(V, +, \cdot, \mathbf{0})$ *over a field* K *is a set such that:*

- $(V, +, \mathbf{0})$ *is a commutative group.*
- *Scalar multiplication* $\cdot\colon \mathbb{K} \times V \to V$ *is a field action on* V.

Definition 7 (Linear Operator)**.** *For* K*-linear spaces* V *and* W*, a linear operator* $T\colon V \to W$ *is a homomorphism.*

Definition 8 (Non-Deterministic Finite Automata)**.** *A non-deterministic finite automata (NFA)* $(\Sigma, Q, \Delta, S, F)$ *is a finite state machine that accepts or rejects strings. A NFA reads from a tape of finite tape of cells in which each cell has a letter from the finite alphabet* Σ*. At any point, the NFA is in some state* Q*, and upon reading the current tape cell's contents the transition function* $\Delta\colon \Sigma \times Q \to 2^Q$ *will transition the NFA non-deterministically to any one of the possible next states in addition to advancing to read the next cell. The set of start states* $S \subseteq Q$ *is the set of states that the NFA may begin at, while the set of final states* $F \subseteq Q$ *is the state of states that the NFA must be in in order to accept a string after reading the last letter on the tape.*

Definition 9 (Norm)**.** *A normed space* $(V, \|\cdot\|)$ *is a* K*-linear space* V *with a norm function* $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ *such that:*

- *For all* $x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$.
- *For all* $a \in K$ *and* $x \in V$, $\|ax\| = |a| \cdot \|v\|$.
- *For* $\mathbf{0} \in V$, $\|\mathbf{0}\| = 0$.

3 Operators

We leverage some of the algebraic sections previously discussed to construct representations relevant to us.

3.1 Algebraic Structures of Automata

For a NFA A, with $A = (\Sigma, Q, \Delta, S, F)$, the goal is to view A acting as a linear operator between spaces. In particular, the approach we take is to see Δ as a linear operator between spaces: this is perhaps the most obvious approach, because to begin with, Δ is already the transition function.

But the questions are then: what are the appropriate linear spaces, and what are the actions of the linear operator? One initial thought is that the transition function can, in some way, be seen as a directed acyclic graph on the states Q, where each edge is weighted by the letters in the alphabet Σ that induce the transition. The representation as an adjacency matrix does appear in literature [2]. Roughly, if M is the transition matrix, then $M_{i,j} \in 2^{\Sigma}$ denotes the states that will transition state q_i to q_j.

While such a matrix representation is useful, it is not immediately obvious how such a matrix does indeed correspond to a linear operator, especially on what linear spaces. One possible interpretation is to see the linear spaces as $|Q|$-dimensional, where each dimension of the space corresponds to one member of Q. The objects of the space is then sets of strings over Σ.

Definition 10 (String Space)**.** *Let M be the free monoid finitely generated by Σ. The string space of Σ is a semiring $\left(2^{\Sigma^\star}, \cup, \cdot, \mathbf{0}, \mathbf{1}\right)$ such that:*

- *The semiring addition is the set union $\cup\colon 2^{\Sigma^\star} \times 2^{\Sigma^\star} \to 2^{\Sigma^\star}$.*
- *The semiring multiplication is the string concatenation $\cdot\colon 2^{\Sigma^\star} \times 2^{\Sigma^\star} \to 2^{\Sigma^\star}$ such that:*

$$A \cdot B = \{a \cdot b \, : a \in A, b \in B\}$$

Here the string space is the power set of all strings generated by Σ through monoid multiplication (string concatenation). Defining the string space like this allows us to equip it with a semiring structure by viewing addition as set union. For convenience, we may write R instead of $2^{\Sigma^\star}$ when discussing the string space.

Observe that R has the structure of a 1-dimensional linear space. The big difference, however, is that the scalar elements are elements of a semiring rather than a field. Nevertheless, R is still closed under linear operations, and is therefore a linear space.

Theorem 1. *A string space R is a linear space.*

Proof. **(descriptive text here)** □

The natural extension of a 1-dimensional linear space is a n-dimensional linear space.

Definition 11 (n-String Space)**.** *For a string space R and $n \in \mathbb{Z}_+$, the n-string space R^n is then the free semimodule isomorphic to n copies of R.*

Again, we demonstrate that this is indeed a linear space.

Theorem 2. *A n-string space R^n is a linear space.*

Proof. **(descriptive text)** □

We are now ready to view

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Definition 12 (Linear String Space Operator)**.** *A linear string space operator is a linear operator $A\colon R^n \to R^m$.*

Definition 13 (Matrix Representation of Δ)**.** *For a NFA $(\Sigma, Q, \Delta, S, F)$ with string space R generated by Σ and $n = |Q|$, the matrix representation of Δ as a linear operator is a matrix $A \in M_{n\times n}(R)$, and acts by right multiplication on R^n.*

Theorem 3. *For some NFA $(\Sigma, Q, \Delta, S, F)$, the matrix representation of Δ acting by right multiplication is a linear operator.*

Proof. **(descriptive text here)** □

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Definition 14 (M-Semiring)**.** *Let Σ be a finite set, and M be the free monoid that it generates.*

Let M be a free monoid generated by

We may extend monoids in general to define the notion of a M-semiring. In particular, M-semirings are defined with respect to a monoid M, and are generated from its power set 2^M.

Definition 15 (M-Semiring)**.** *Let M be a finitely generated monoid. A M-semiring R is a semiring such that:*

$$R = \left(2^M, \cup, \cdot, \mathbf{0}, \mathbf{1}\right)$$

Where semiring addition is set union $\cup$ with identity $\mathbf{0}$, and semiring multiplication $\cdot$ and identity $\mathbf{1}$ are carried over from M.

As with the case of normed monoids, we may extend this to normed M-semirings. In particular, we pay special attention to p-norms.

Definition 16 (M-Semiring p-Norm)**.** *Let M be a finitely generated and normed monoid. For $1 \leq p \leq \infty$, a p-normed M-semiring is R is equipped with a norm $\|\cdot\|_p : R \to \mathbb{R}_{\geq 0}$:*

$$\|x\|_p = \left(\sum_{a \in x} \|a\|^p \right)^{1/p}$$

The sum here utilizes the monoid norm. Observe that because M is finitely generated, each $x \in R_M$ is therefore countable, and hence so is the sum. When $p = \infty$, this is just a sup norm. Similar definitions can be found in literature [1].

Extending M-semirings, we define (M, n)-semimodules:

Definition 17 ((M, n)-Semimodule)**.** *A (M, n)-semimodule R^n is a free semimodule generated by n isomorphic copies of the M-semiring R.*

If $x \in R^n$, write x_i to denote the ith element from R in some canonical representation of R^n. Often this is just a row (horizontal) or column (vertical) vector of length n.

Again, we extend norms to (M, n)-semimodules:

Definition 18 ((M, n)-Semimodule (p, q)-Norm)**.** *Let R be a p-normed M-semiring. Let R^n be a (M, n)-semimodule and take $1 \leq p, q \leq \infty$. A (p, q)-normed (M, n)-semimodule is a semimodule with norm $\|\cdot\|_{p,q} : R^n \to \mathbb{R}_{\geq 0}$:*

$$\|x\|_{p,q} = \left(\sum_{i=1}^{n} \|x_i\|_p^q \right)^{1/q}$$

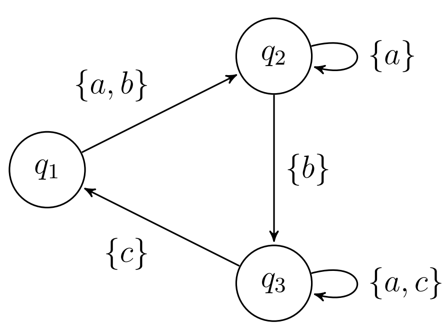
When $p = q = \infty$, these are just the sup-norm.

The goal of operator automata theory is to view automata as linear transformations between linear spaces. To that end, we consider how to embed an automata as a linear transformation.

Definition 19 (Something something automata)**.** *Descriptive definition text goes here.*

3.2 Examples

Example 1. *Consider $\Sigma = \{a, b, c\}$, and a finite automata below:*



There are a few things to note in our model:

(1) The automata is non-deterministic, because from q_3, there are multiple transitions that may be taken.

(2) We lack the notion of a start and final state. Rather, every state is treated as both start and final in this perspective to make embedding slightly easier.

To embed this into our model in several steps. First, take M to be the monoid generated by Σ, where monoid multiplication is taken to be string concatenation, and unit 1 is aliased as ε the empty string.

$$M = (\Sigma, \cdot, 1)$$

The M-semiring is generated using the power set of M, where the unit of addition 0 is equivalent to the empty set $\emptyset$:

$$R = \left(2^M, \cup, \cdot, 0, 1\right)$$

To demonstrate a better picture of how this works, we construct a transition matrix A for the automata that acts on the $(M, 3)$-semimodule. Here we have 3 because there are 3 states. Let $A_{i,j}$ denote the transition set from state i to state j.

$$A = \begin{bmatrix} 0 & \{a, b\} & 0 \\ 0 & \{a\} & \{b\} \\ \{c\} & 0 & \{a, c\} \end{bmatrix}$$

In order to perform string concatenation towards the right, transition matrices act by right-matrix multiplication. That is, if $v \in R^3$ is the initial (row) vector, then the subsequent state is vA.

To briefly demonstrate, two transitions of the matrix A appears as follows:

$$A^2 = \begin{bmatrix} 0 & \{a, b\} & 0 \\ 0 & \{a\} & \{b\} \\ \{c\} & 0 & \{a, c\} \end{bmatrix} \begin{bmatrix} 0 & \{a, b\} & 0 \\ 0 & \{a\} & \{b\} \\ \{c\} & 0 & \{a, c\} \end{bmatrix} = \begin{bmatrix} 0 & \{aa, ba\} & \{ab, bb\} \\ \{bc\} & \{aa\} & \{ab, ba, bc\} \\ \{ac, cc\} & \{ca, cb\} & \{aa, ac, ca, cc\} \end{bmatrix}$$

In general, from graph theory, A^k denotes the kth consecutive transition using A, and each entry $A^k_{i,j}$ is the set of strings that will get from state q_i to q_j in k steps.

References

[1] Manfred Kudlek. "Iteration Lemmata for Normed Semirings (Algebraic Systems, Formal Languages and Computations)". In: (2000).

[2] John E Savage. *Models of computation.* Vol. 136. Addison-Wesley Reading, MA, 1998.