Operator Automata Theory

1 Introduction

Descriptive text

2 Preliminaries

Monoid

Semiring

Semimodule

Linear transform

Norm

Automata

3 Semimodule Operators

Let Σ be a finite set and M to be the free monoid finitely generated by Σ:

$$M = (\Sigma, \cdot, \mathbf{1})$$

Definition 1 (Normed Monoid)**.** *A normed monoid is a monoid M equipped with norm $\|\cdot\|_M$.*

Because monoids lack addition, such norms only concern multiplication.

We may extend monoids in general to define the notion of a M-semiring with respect to a monoid M. In particular, M-semirings are generated by the power set of M, $\mathcal{P}(M)$.

Definition 2 (M-Semiring)**.** *A M-semiring R_M is a semiring with respect to a monoid M defined as:*

$$R_M = (\mathcal{P}(M), \cup, \cdot, \mathbf{0}, \mathbf{1})$$

Where semiring addition is set union $\cup$ with identity $\mathbf{0}$, and semiring multiplication $\cdot$ and identity $\mathbf{1}$ are carried over from M.

We may write R_M as just R when context is clear. Sometimes we will also just call M-semirings as semirings, because they are just special cases of semirings. As with the case of normed monoids, we may extend this to normed M-semirings:

Definition 3 (M-Semiring p-Norm)**.** *For $1 \leq p \leq \infty$, a normed M-semiring is a M-semiring R_M equipped with a norm $\|\cdot\|_{R_M,p}$ defined as:*

$$\forall x \in R \,:\, \|x\|_{R_M,p} = \left(\sum_{a \in R} \|a\|_M^p\right)^{1/p}$$

Observe that because M is finitely generated, each $x \in R$ is therefore countable, and hence so is the sum. When $p = \infty$, this is just a sup norm. Similar definitions can be found in literature [1].

Extending M-semirings, we define (M, n)-semimodules:

Definition 4 ((M, n)-Semimodule)**.** *A (M, n)-semimodule R_M^n is a semimodule consisting of n isomorphic copies of R_M.*

If $x \in R_M^n$, write x_i to denote the ith element from R_M in some canonical representation of R_M^n. Often this is just a row (horizontal) or column (vertical) vector of length n.

We write R^n instead of R_M^n when M is understood from context. Again, we extend norms to (M, n)-semimodules:

Definition 5 ((M, n)-Semimodule (p, q)-Norm)**.** *Let R_M^n be a (M, n)-semimodule and take $1 \leq p, q \leq \infty$. Define the (M, n)-semimodule (p, q)-norm $\|\cdot\|_{R_M^n,p,q}$:*

$$\forall x \in R_M^n \,:\, \|x\|_{R_M^n,p,q} = \left(\sum_{i=1}^{n} \|x_i\|_{R_M,p}^q\right)^{1/q}$$

When $p = q = \infty$, these are just the sup-norm. Often we care about the case of $p = \infty$ and $q = 1$, which is the sup-norm over each R_M, and the 1-norm over the n copies of R_M in R_M^n. **(Revise as needed)**

With the notion of norms, we again extend such notions to linear operators between some R_M^n and R_M^m.

Definition 6 (Linear Operator Norm)**.** *Take R_M^n equipped with norm $\|\cdot\|_{p_n,q_n}$ and R_M^m equipped with norm $\|\cdot\|_{p_m,q_m}$. Define the linear operator norm $\|\cdot\|_L$ for operators in $\mathcal{L}\left(R_M^n, R_M^m\right)$ as:*

$$\forall T \in \mathcal{L}\left(R_M^n, R_M^m\right) \ : \ \|T\|_L = \inf\left\{c \in \mathbb{R}_{\geq 0} \ : \ \|Tx\|_{p_m,q_m} \leq c \cdot \|x\|_{p_n,q_n}, \forall x \in R_M^n\right\}$$

4 On Automata

References

[1] Manfred Kudlek. "Iteration Lemmata for Normed Semirings (Algebraic Systems, Formal Languages and Computations)". In: (2000).