Constrained Linear Quadratic Regular Optimization

1   Introduction

We look at constrained linear quadratic regulator (LQR) optimization.

2   Background

Relevant background on linear systems, LQR, and matrix magic.

2.1   Linear Systems and the Linear Quadratic Regulator

Linear system dynamics

$$x_{t+1} = Ax_t + Bu_t, \qquad (x_t)_{t=0}^{\infty} \in \mathbb{R}^n, \quad (u_t)_{t=0}^{\infty} \in \mathbb{R}^p,$$

Finite horizion linear quadratic regular (LQR) cost function

$$J = \sum_{t=0}^{T} x_t^\top Q x_t + u_t^\top R u_t, \qquad Q \succeq 0, \quad R \succ 0,$$

which is finite when the system dynamics (x_t, u_t) tends towards the origin. Infinite-horizon LQR average-cost function

$$J = \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t^\top Q x_t + u_t^\top R u_t, \qquad Q \succeq 0, \quad R \succ 0,$$

which allows for systems whose stability is not necessarily at the origin. Both versions of LQR admit an optimal feedback control sequence that is linear, i.e., $u = Kt$, with respect to the algebraic Riccati equation in P where

$$P = Q + A^\top P A - A^\top P B (R + B^\top P B)^{-1} B^\top P A, \qquad K = -(R + B^\top P B)^{-1} B^\top P A$$

where the solution for $P \succ 0$, if it exists, is unique.

2.2 Affine Systems

Given the affine system

$$x_{t+1} = Ax_t + Bu_t + c$$

we can reformulate it as a linear system

$$z_{t+1} = \begin{bmatrix} x_{t+1} \\ 1 \end{bmatrix} = \begin{bmatrix} A & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ 1 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t = \tilde{A} z_t + \tilde{B} u_t \tag{1}$$

2.3 Completing the Square

For the matrix case with variable in x from Wikipedia [2],

$$x^\top Q x + p^\top x + r = (x - h)^\top Q (x - h) + k, \qquad h = -\frac{1}{2} Q^{-1} p, \qquad k = r - \frac{1}{4} p^\top Q^{-1} p \tag{2}$$

where Q is assumed to be invertible.

3 Some Results with Affine Constraints

Theorem 1. *An infinite horizon average-cost LQR problem with quadratic-plus-affine terms*

$$\begin{aligned}
&minimize \quad \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t^\top Q x_t + q^\top x_t + q_0 + u_t^\top R u_t + r^\top u_t + r_0 \\
&subject\ to \quad x_{t+1} = Ax_t + Bu_t, \ \ for\ all\ t
\end{aligned}$$

admits an equivalent infinite horizon average-cost LQR problem without affine terms.

Proof. Applying (2) we can write the objective as

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} (x_t - h)^\top Q (x_t - h) + (u_t - l)^\top R (u_t - l) + k,$$

with coefficients

$$h = -\frac{1}{2} Q^{-1} q, \qquad l = -\frac{1}{2} R^{-1} r, \qquad k = q_0 + r_0 - \frac{1}{4} \left(q^\top Q q + r^\top R r \right)$$

Observe that k is constant with respect to the variables (x_t, u_t) and the objective is equivalently

$$k + \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} (x_t - h)^\top Q (x_t - h) + (u_t - l)^\top R (u_t - l).$$

Define the change-of-variables $\tilde{x}_t = x_t - h$ and $\tilde{u}_t = u_t - l$ for all t, which induces the affine system

$$\tilde{x}_{t+1} = A\tilde{x}_t + B\tilde{u}_t + (Ah + Bl - h)$$

and after the appropriate change of variables to recover a linear system as in (1),

$$z_{t+1} = \begin{bmatrix} A & (Ah + Bl - h) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_t \\ 1 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \tilde{u}_t = \tilde{A}z_t + \tilde{B}\tilde{u}_t.$$

For this, the reduced problem is then

$$\text{minimize} \quad k + \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} z_t^\top \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} z_t + \tilde{u}_t^\top R \tilde{u}_t$$

$$\text{subject to} \quad z_{t+1} = \begin{bmatrix} A & (Ah + Bl - h) \\ 0 & 1 \end{bmatrix} z_t + \begin{bmatrix} B \\ 0 \end{bmatrix} \tilde{u}_t, \quad \text{for all } t$$

with variables in $(z_t)_{t=0}^\infty$ and $(u_t)_{t=0}^\infty$. Barring the k offset in the objective, this is now equivalent to an infinite horizion average-cost LQR problem with no affine terms. $\qquad \square$

Because quadratic-plus-affine terms can be eliminated to yield a problem with purely quadratic cost, Theorem 1 implies that quadratic systems also have an optimal feedback control sequence that is linear for the equivalent system with trajectory in $(z_t, \tilde{u}_t)_{t=0}^\infty$. But since $(z_t, \tilde{u}_t)$ are linear transformations of the original trajectory (x_t, u_t), the optimal feedback policy for the original dynamics is also linear.

4 Affine Constraints

Consider the following constrained discrete-time LQR problem

$$\text{minimize} \quad \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t^\top Q x_t + u_t^\top R u_t$$

$$\text{subject to} \quad x_{t+1} = A x_t + B u_t, \quad \forall t$$
$$C x_t = d, \quad \forall t$$

where C is fat and full-rank. Affine constraints with average costs allow us to capture the behavior of systems whose steady limiting behavior may not necessarily converge to the origin.

We know that in the unconstrained case (without the $Cx_t = d$ constraint) the optimal feedback sequence (u_t) is linear with $u_t = Kx_t$ and can be derived by solving the algebraic Riccati equation in P, as shown in [1],

$$P = Q + A^\top P A - A^\top P B (R + B^\top P B)^{-1} B^\top P A, \qquad K = -(R + B^\top P B)^{-1} B^\top P A$$

We show how to rewrite the affine constrained average-cost LQR problem. This implies that, with a few technical assumptions, the optimal policy in the affine constrained case is also linear.

4.1 Rewriting Affine Constraints

First, the constraints $Cx_t = d$ implies that we can decompose states as

$$x_t = \hat{x} + Ny_t, \qquad \hat{x} = C^\dagger d, \qquad \operatorname{ran} N = \ker C$$

where $C^\dagger$ is the Moore-Penrose Pseudoinverse of C, and although we do not fix N, we can interpret it as a projection matrix, such that $\dim y_t = \dim x_t$. In other words, the state trajectory (x_t) must lie exclusively in the affine space $\hat{x} + \ker C$. The state dynamics can then be written as

$$\hat{x} + Ny_{t+1} = A(\hat{x} + Ny_t) + Bu_t$$

Assumption 1. There exists $\hat{u}$ such that $\hat{x} = A\hat{x} + B\hat{u}$.

Under this assumption, we can decompose $u_t = \hat{u} + v_t$, such that some sub-state dynamics is now

$$Ny_{t+1} = ANy_t + Bv_t$$

Assumption 2. $\operatorname{ran}(AN) \subseteq \operatorname{ran} B$

This assumption ensures the existence of v_t that can ensure each $x_{t+1} \in \hat{x} + \ker C$: in other words, B is a sufficiently strong control matrix. We can then refactor the entire state dynamics as

$$\begin{bmatrix} \hat{x} \\ Ny_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \hat{x} \\ Ny_t \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \hat{u} \\ v_t \end{bmatrix} \tag{3}$$

and the corresponding LQR optimization problem can be written as

$$\begin{aligned} \text{minimize} \quad & \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \begin{bmatrix} \hat{x} \\ (Ny_t) \end{bmatrix}^\top \begin{bmatrix} Q & Q \\ Q & Q \end{bmatrix} \begin{bmatrix} \hat{x} \\ (Ny_t) \end{bmatrix} + \begin{bmatrix} \hat{u} \\ v_t \end{bmatrix}^\top \begin{bmatrix} R & R \\ R & R \end{bmatrix} \begin{bmatrix} \hat{u} \\ v_t \end{bmatrix} \\ \text{subject to} \quad & (3) \end{aligned}$$

with variables in (y_t) and (v_t). Since $Q \succeq 0$ and $R \succ 0$, we have $I \otimes Q \succeq 0$ and $I \otimes R \succeq 0$, so this is an average-cost LQR problem with PSD cost matrices, for which we know that the optimal control policy is linear.

References

[1] Stephen Boyd. "Ee363: Linear dynamical systems". In: *Lecture. Stanford Univer., Winter Quarter* 9 (2008). URL: https://stanford.edu/class/ee363/lectures/stoch_lqr.pdf.

[2] Wikipedia contributors. *Completing the square — Wikipedia, The Free Encyclopedia.* [Online; accessed 14-May-2020]. 2020. URL: https://en.wikipedia.org/w/index.php?title=Completing_the_square&oldid=953923455.