# Constrained Linear Quadratic Regular Optimization

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## 1 Introduction

We look at constrained linear quadratic regulator (LQR) optimization.

## 2 Background

Relevant background on linear systems, LQR, and matrix magic.

### 2.1 Linear Systems and the Linear Quadratic Regulator

Linear system dynamics

$$x_{t+1} = Ax_t + Bu_t, \qquad (x_t)_{t=0}^{\infty} \in \mathbb{R}^n, \quad (u_t)_{t=0}^{\infty} \in \mathbb{R}^p,$$

Finite horizon linear quadratic regular (LQR) cost function

$$J = \sum_{t=0}^{T} x_t^\top Q x_t + u_t^\top R u_t, \qquad Q \succeq 0, \quad R \succ 0,$$

which is finite when the system dynamics $(x_t, u_t)$ tends towards the origin. Infinite-horizon LQR average-cost function

$$J = \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t^\top Q x_t + u_t^\top R u_t, \qquad Q \succeq 0, \quad R \succ 0,$$

which allows for systems whose stability is not necessarily at the origin. Both versions of LQR admit an optimal feedback control sequence that is linear, i.e., $u = Kt$, with respect to the algebraic Riccati equation in $P$ where

$$P = Q + A^\top PA - A^\top PB(R + B^\top PB)^{-1} B^\top PA, \qquad K = -(R + B^\top PB)^{-1} B^\top PA$$

where the solution for $P \succ 0$, if it exists, is unique.

## 2.2 Affine Systems

Given the affine system

$$x_{t+1} = Ax_t + Bu_t + c$$

we can reformulate it as a linear system

$$z_{t+1} = \begin{bmatrix} x_{t+1} \\ 1 \end{bmatrix} = \begin{bmatrix} A & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ 1 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t = \tilde{A} z_t + \tilde{B} u_t \tag{1}$$

## 2.3 Completing the Square

For the matrix case with variable in $x$ from Wikipedia [1],

$$x^\top Q x + p^\top x + r = (x-h)^\top Q(x-h) + k, \qquad h = -\frac{1}{2}Q^{-1}p, \qquad k = r - \frac{1}{4}p^\top Q^{-1} p \tag{2}$$

where $Q$ is assumed to be invertible.

# 3 Some Results with Affine Constraints

**Theorem 1.** *An infinite horizon average-cost LQR problem with quadratic-plus-affine terms*

$$\begin{array}{ll} \textit{minimize} & \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t^\top Q x_t + q^\top x_t + q_0 + u_t^\top R u_t + r^\top u_t + r_0 \\ \textit{subject to} & x_{t+1} = Ax_t + Bu_t, \ \textit{for all } t \end{array}$$

*admits an equivalent infinite horizon average-cost LQR problem without affine terms.*

*Proof.* Applying (2) we can write the objective as

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} (x_t - h)^\top Q (x_t - h) + (u_t - l)^\top R (u_t - l) + k,$$

with coefficients

$$h = -\frac{1}{2}Q^{-1}q, \qquad l = -\frac{1}{2}R^{-1}r, \qquad k = q_0 + r_0 - \frac{1}{4}\left(q^\top Q q + r^\top R r\right)$$

Observe that $k$ is constant with respect to the variables $(x_t, u_t)$ and the objective is equivalently

$$k + \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} (x_t - h)^\top Q (x_t - h) + (u_t - l)^\top R (u_t - l).$$

Define the change-of-variables $\tilde{x}_t = x_t - h$ and $\tilde{u}_t = u_t - l$ for all $t$, which induces the affine system

$$\tilde{x}_{t+1} = A\tilde{x}_t + B\tilde{u}_t + (Ah + Bl - h)$$

and after the appropriate change of variables to recover a linear system as in (1),

$$z_{t+1} = \begin{bmatrix} A & (Ah + Bl - h) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_t \\ 1 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \tilde{u}_t = \tilde{A} z_t + \tilde{B} \tilde{u}_t.$$

For this, the reduced problem is then

$$\begin{array}{ll} \text{minimize} & k + \limsup\limits_{T\to\infty} \dfrac{1}{T} \displaystyle\sum_{t=0}^{T-1} z_t^\top \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} z_t + \tilde{u}_t^\top R \tilde{u}_t \\ \text{subject to} & z_{t+1} = \begin{bmatrix} A & (Ah + Bl - h) \\ 0 & 1 \end{bmatrix} z_t + \begin{bmatrix} B \\ 0 \end{bmatrix} \tilde{u}_t, \quad \text{for all } t \end{array}$$

with variables in $(z_t)_{t=0}^\infty$ and $(u_t)_{t=0}^\infty$. Barring the $k$ offset in the objective, this is now equivalent to an infinite horizon average-cost LQR problem with no affine terms. □

Because quadratic-plus-affine terms can be eliminated to yield a problem with purely quadratic cost, Theorem 1 implies that quadratic systems also have an optimal feedback control sequence that is linear for the equivalent system with trajectory in $(z_t, \tilde{u}_t)_{t=0}^\infty$. But since $(z_t, \tilde{u}_t)$ are linear transformations of the original trajectory $(x_t, u_t)$, the optimal feedback policy for the original dynamics is also linear.

**Theorem 2.** *The infinite-horizon average-cost LQR problem with affine constraints*

$$\begin{array}{ll} \textit{minimize} & \limsup\limits_{T\to\infty} \dfrac{1}{T} \displaystyle\sum_{t=0}^{T-1} x_t^\top Q x_t + u_t^\top R u_t \\ \textit{subject to} & x_{t+1} = Ax_t + Bu_t, \quad \textit{for all } t \\ & Cx_t = d, \quad \textit{for all } t \end{array}$$

*where $C$ is fat and full rank, admits an optimal feedback control policy that is linear if:*

- *There exists $\hat{u}$ such that $\hat{x} = A\hat{x} + B\hat{u}$, where $\hat{x} = C^\dagger d$.*
- $\operatorname{ran} AN \subseteq \operatorname{ran} B$, *where* $\operatorname{ran} N = \ker C$.

*Proof.* Supposing the assumptions, we can write our system dynamics as

$$\hat{x} + Nx_{t+1} = A(\hat{x} + Nx_t) + B(\hat{u} + v_t)$$

where we treat $N$ as a projection matrix onto $\ker C$. However this is not quite in a standard linear systems form. Further factor the system matrices into $A = A_1 + A_2$ and $B = B_1 + B_2$ such that

$$\operatorname{ran} A_1 \subseteq \operatorname{ran} B_1 \subseteq \ker C^\perp, \qquad \operatorname{ran} A_2 \subseteq \operatorname{ran} B_2 \subseteq \ker C,$$

Let $y_t = Nx_t$; the goal of such factorization is to ensure that

$$A_1 y_t + B_1 v_t = 0 \in \ker C^{\perp}, \qquad A_2 y_t + B_2 v_t \in \ker C$$

In effect, the $(A_1, B_1)$ sub-system needs to eliminate action in $\ker C^{\perp}$: for which appropriate $v_t$ can be found given $y_t$ due to the (linear) $\operatorname{ran} AN \subseteq \operatorname{ran} B$ assumption; while $(A_2, B_2)$ is parameterized to lie purely in $\ker C$. Thus, we need only care about the $(A_2, B_2)$ sub-system, for which results can be linearly translated back to the original $(A, B)$ system. Define the following for compactness,

$$J(y_t, v_t) = \begin{bmatrix} \hat{x} \\ y_t \end{bmatrix}^\top \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \hat{x} \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ 2Q\hat{x} \end{bmatrix}^\top \begin{bmatrix} \hat{x} \\ y_t \end{bmatrix} + \begin{bmatrix} \hat{u} \\ v_t \end{bmatrix}^\top \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} \hat{u} \\ v_t \end{bmatrix} + \begin{bmatrix} 0 \\ 2R\hat{x} \end{bmatrix}^\top \begin{bmatrix} \hat{u} \\ v_t \end{bmatrix}$$

and noting that

$$x_t^\top Q x_t + u_t^\top R u_t = (\hat{x} + y_t)^\top Q (\hat{x} + y_t) + (\hat{u} + v_t)^\top R (\hat{u} + v_t) = J(y_t, v_t)$$

makes it so that the optimization problem can be written as

$$\begin{array}{ll} \text{minimize} & \displaystyle\limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} J(y_t, v_t) \\ \text{subject to} & y_{t+1} = A_2 y_t + B_2 v_t \end{array}$$

with variables in $(y_t)_{t=0}^{\infty}$ and $(v_t)_{t=0}^{\infty}$. Since $J$ is a quadratic-plus-affine decomposition of the state and control costs, we know that this infinite-horizon average-cost LQR problem. As noted earlier, this system is derived from linear transformations on our original system $(A, B)$, and so an optimal controller linear for $(A_2, B_2)$ also implies linearity for $(A, B)$.

□

## 3.1 Quadratic Constraints

Consider the following LQR problem with quadratic constraints of form

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{t=0}^{\infty} x_t^\top Q x_t + u_t^\top R u_t \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \ \text{ for all } t \\ & x_t^\top C_x x_t \le d_x, \ \text{ for all } t \\ & u_t^\top C_u u_t \le d_u, \ \text{ for all } t \end{array} \tag{3}$$

where $C_x, C_u \succeq 0$ and $d_x, d_u > 0$.

**Theorem 3.** *Assuming strong duality holds, the optimal control policy for (3) is linear in $(x_t)$.*

*Proof.* Formulate the partial Lagrangian with constraints, omitting system dynamics,

$$L((x_t),(u_t),(\lambda_t),(\rho_t)) = \sum_{t=0}^{\infty} x_t^\top Q x_t + u_t^\top R u_t + \lambda_t(x_t^\top C_x x_t - d_x) + \rho_t(u_t^\top C_u u_t - d_u).$$

For the dual function to be defined, we require that each $Q + \lambda_t C_x \succeq 0$ and $R + \rho_t C_u \succeq 0$. Assuming strong duality holds, such optimal $(\lambda_t^\star, u_t^\star)$ co-state variables then exist, and in fact the following problem

$$\begin{aligned} \text{minimize} \quad & \sum_{t=0}^{\infty} x_t^\top (Q + \lambda_t^\star I) x_t + u_t^\top (R + \rho_t^\star I) u_t = \sum_{t=0}^{\infty} x_t^\top \tilde{Q} x_t + u_t^\top \tilde{R} u_t \\ \text{subject to} \quad & x_{t+1} = Ax_t + Bu_t, \;\; \text{for all } t \end{aligned} \tag{4}$$

where $\tilde{Q}$ and $\tilde{R}$ exist because the objective is quadratic in $(x_t, u_t)$. Consequently, problems (4) and (3) are equivalent, and so there is a stabilizing optimal controller that is linear for (4) iff there is one for (3). ◻

# References


[1] Wikipedia contributors. *Completing the square — Wikipedia, The Free Encyclopedia*. [Online; accessed 14-May-2020]. 2020. URL: https://en.wikipedia.org/w/index.php?title=Completing_the_square&oldid=953923455.