Linear Operators on Kleene Algebras

1 Introduction

1.1 Algebraic Structures

Definition 1 (Monoid)**.** *A monoid is a set S with a binary operation $\circ\colon S \times S \to S$ (multiplication) with the following axioms:*

(1) $\circ$ is associative.

(2) There exists an identity element $e \in S$ such that for all $a \in S$:

$$e \circ a = a = a \circ e$$

Definition 2 (Commutative Monoid)**.** *A monoid $(S, \circ)$ is commutative if $\circ$ is commutative.*

Definition 3 (Semiring)**.** *A semiring is a set R with two binary operations $+\colon R \times R \to R$ (addition) and $\cdot\colon R \times R \to R$ (multiplication) with the following axioms:*

(1) $(R, +)$ is a commutative monoid with identity 0.

(2) $(R, \cdot)$ is a monoid with identity 1.

(3) Multiplication left and right distributes over addition.

(4) Multiplication by 0 annihilates R.

Definition 4 (Left Module Over a Semiring)**.** *A left module M over a semiring R is a set with two binary operations $+\colon M \times M \to M$ (addition) and $\cdot\colon R \times M \to M$ (scalar multiplication) with the following axioms:*

(1) $(M, +)$ is an abelian group.

(2) For all $s, r \in R$ and $x, y \in M$:

$$r \cdot (x + y) = (r \cdot x) + (r \cdot y)$$
$$(r + s) \cdot x = (r \cdot x) + (s \cdot x)$$
$$(r \cdot s) \cdot x = r \cdot (s \cdot x)$$
$$1_R \cdot x = x$$

Definition 5 (Right Module Over a Semiring). *A right module would be defined similarly, except all instances of scalar multiplication happen on the right side.*

Definition 6 (Bimodule Over a Semiring). *A bimodule is a module that is both a left module and a right module with respect to scalar multiplication.*

If M is a module over a semiring R, we write M/R for shorthand. We implicitly assume bimodule structures, or whatever is convenient, to avoid excessive worrying over the correct left and right behavior.

Definition 7 (n-length Module). *An n-length module M/R consists of n elements:*

$$M = (r_1, \ldots, r_n) \qquad r_1, \ldots, r_n \in R$$

Write M_i to also mean r_i in an abuse of notation.

Definition 8 (Linear Transformation). *A linear transformation is a homomorphic map $T \colon V \to W$ between two modules V/R and W/R that preserves addition and scalar multiplication. That is, for all $r \in R$ and $x, y \in V$:*

$$T(0_R \cdot x) = 0_W$$
$$T(r \cdot x + y) = r \cdot T(x) + T(y)$$

We may use matrices to represent linear transformations between n-length modules. Let $M_{n \times n}(R)$ be the set of $n \times n$ dimensional matrices with entries from semiring R.

Definition 9 (Matrix Module (Left) Multiplication). *If $A \in M_{n \times n}(R)$, and M/R is some n-length module. Define left matrix-module multiplication as follows:*

$$(AM)_i = \sum_{k=1}^{n} A_{i,k} M_k$$

Here the result AM is another n-length module over semiring R. In particular, this action corresponds to left multiplication by the matrix. If instead we wish to define right multiplication:

Definition 10 (Matrix Module (Right) Multiplication). *If $A \in M_{n \times n}(R)$, and M/R is some n-length module. Define right matrix-module multiplication as follows:*

$$(MA)_i = \sum_{k=1}^{n} M_k A_{k,i}$$

Typically we take the module M to have a column vector representation during left multiplication, and a row vector representation for right multiplication.

2   Kleene Algebra Mappings

Consider a Kleene algebra $\mathcal{K} = (K, +, \cdot, {}^*, 0, 1)$. Recall that K forms an idempotent semiring with respect to $+$ and $\cdot$. We are primarily interested in how Kleene algebras relate to finite state machines (finite automata). In literature,

To this end, we take an n-length module M as:

Define an n-length module M as:

$$M = (k_1, k_2, \ldots, k_n) \qquad k_i \in K$$

Let M be the

2.1   Automata

In literature [1] we often have the following definitions:

Definition 11 (Alphabet). *An alphabet Σ is a finite set of distinct letters (characters).*

Definition 12 (Language).

References

[1]   John E Savage. *Models of computation*. Vol. 136. Addison-Wesley Reading, MA, 1998.