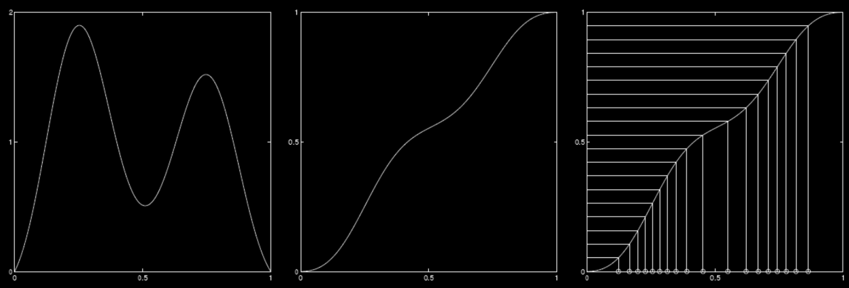
Monte Carlo Integration



70001 – Advanced Computer Graphics: Photographic Image Synthesis

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Lecture 09, Feb. 6th 2024

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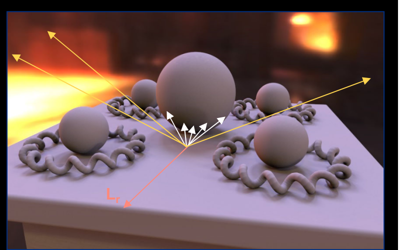
Rendering Equation [Kajiya 86]

- $L_r(x, \omega_r) = \int_\Omega f_r(x, \omega_r, \omega_i)\, L_i(x, \omega_i) \cos\theta_i\, V(\omega_i)\, d\omega_i$.
  - only discuss direct illumination integral
- Brute force solution too expensive!
  - especially under EM illumination
- Monte Carlo integration
  - stochastic approximation



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Monte Carlo Integration

- $I(f) = \int_S f(x)\, p(x)\, dx$,
  - $f(x)$ is the function defined over S
  - $p(x)$ is the probability density function of f

- Monte Carlo estimate

$$I_N(f) = \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

  - converges as $N \to \infty$
  - $X=\{x_1, x_2, \ldots , x_N\}$ are i.i.d.s drawn from $p(x)$

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Expected Value

- $I(f)$ also referred to as expected value $E(x)$
  - integral computes the average!

- $E(x)$ satisfies the following properties:
  - $E(\alpha x) = \alpha E(x)$, where α is a constant
  - $E(\sum_i x_i) = \sum_i E(x_i)$

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Variance

- $\mathrm{var}(x) = E((x - E(x))^2)$
  $= E(x^2) - E(x)^2.$
- Variance observes the following properties:
  - $\mathrm{var}(\sum_i x_i) = \sum_i \mathrm{var}(x_i)$
  - $\mathrm{var}(\alpha x) = \alpha^2 \, \mathrm{var}(x).$
- In image synthesis, variance manifests as noise!

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Variance

- $$\mathrm{var}(I_N(f)) = \mathrm{var}\left(\frac{1}{N} \sum_{i=1}^{N} f(x_i)\right) = \frac{1}{N^2} \mathrm{var}\left(\sum_{i=1}^{N} f(x_i)\right) = \frac{1}{N} \mathrm{var}(f(x_i))$$

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Monte Carlo Variance

- Variance inversely proportional to sample size N!
  - Error in estimate behaves like standard deviation
- Notion of diminishing returns
  - Image quality depends on N^2
  - Quadruple N to halve the error!
- MC integration suffers from the curse of dimensionality

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Importance Sampling

- $I(f) = \int_S f(x)\, dx$
- Unbiased MC estimate of $I(f)$

$$I(f) \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

  - $p(x_i)$ is a PDF, also called a proposal distribution

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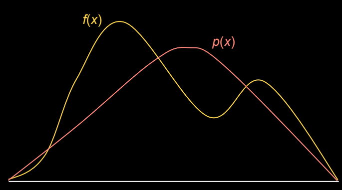
Importance Sampling

- $\frac{f}{p}$ ratio needs to have low variance!



- Importance sampling
  - choosing *p* such that *f* & *p* have similar shape
- Direct illumination integral
  - *p* chosen according to distribution of illumination
  - or BRDF

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Probability Density Function

- PDF *p* describes the relative likelihood of a certain value
  - for a random variable $x \sim p$
- A PDF has the following characteristics:
  - $p(x) \geq 0$,
  - $\int_0^1 p(x)\, dx = 1$, where $x \in [0, 1]$.

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Cumulative Density Function

- CDF C describes how to sample from a given PDF p
- $C(x) = p(X \leq x)$.
- $C(x) = \int_0^x p(x)\,dx$.
- Sampling by function inversion!
  - analytic
  - numeric

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Cumulative Density Function

- Choose a uniform variate $u_i \in [0, 1]$
- Transform u_i by C^{-1} to obtain $x_i \sim p$.
  - $x_i = C^{-1}(u_i)$.
- Since $p(x) \geq 0$, $C(x)$ is monotonically increasing
  - $C^{-1}(x)$ always exists!

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Analytic Function Inversion

- If $p(x) = 3x^2/2$, where $x \in [-1, 1]$,

  $C(x) = \int_{-1}^{1} 3x^2/2 \, dx = (x^3 + 1)/2$,

  and $C^{-1}(x) = (2y - 1)^{1/3}$, where $y = C(x)$

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Uniform Sampling a Disk

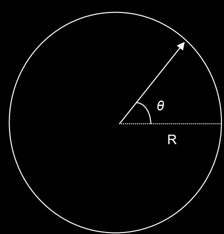
- Given a disk of radius R

  $p(r, \theta) = 1/\pi R^2$,

  $$C(r, \theta) = \int_0^{\theta} \int_0^{r} \frac{1}{\pi R^2} \, r dr d\theta$$

  $$= \theta r^2 / (2\pi R^2)$$

- $(r, \theta) = (R\sqrt{u_1}, 2\pi u_2)$



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BRDF Models

- Several BRDF models can be sampled analytically
  - cosine lobe, Gaussian lobe, GGX
- Sampling a Phong lobe

$$p(\theta, \varphi) = \frac{(n + 1) \cos^n\theta}{2\pi}$$

- $(\theta , \varphi) = (\arccos((1 - u_1)^{1/(n+1)}), 2\pi u_2)$

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Phong lobe

- $(\theta , \varphi) = (\arccos((1 - u_1)^{1/(n+1)}), 2\pi u_2)$
  - sample direction distributed about local +Z
  - need to rotate sample to be about global reflection vector ω_r!
- Half-vector parameterization $p(\omega_h)$
  - Microfacet models
  - samples generated about ω_h

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Isotropic Gaussian

- Ward BRDF $p(\omega_h) \sim \exp-(\tan^2\theta_h/\alpha^2)$ (proportionality implies CDF!)
- Sample θ_h :

  $\theta_h = \arctan(\alpha \sqrt{-\log u_1})$
- Sample φ_h :

  $\varphi_h = 2\pi u_2$

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Sampling GGX

- GGX lobe $p(\omega_h) \sim \alpha^2 / (\alpha^2 + \tan^2\theta_h)^2$
- Sample θ_h :

  $\theta_h = \arctan(\alpha\sqrt{u_1} / (\sqrt{1 - u_1}))$
- Sample φ_h :

  $\varphi_h = 2\pi u_2$

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Anisotropic Gaussian

- Ward BRDF $p(\omega_h) \sim \exp-(\tan^2 \theta_h(\cos^2\varphi_h/\alpha_x^2 + \sin^2\varphi_h/\alpha_y^2)$
- Sample θ_h :

  $\theta_h = \arctan(\sqrt{-\log u_1 / (\cos^2\varphi_h/\alpha_x^2 + \sin^2\varphi_h/\alpha_y^2)})$

- Sample φ_h :

  $\varphi_h = \arctan((\alpha_y/\alpha_x)\tan(2\pi u_2))$

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Anisotropic Phong

- Ashikhmin-Shirley $p(\omega_h) = \dfrac{\sqrt{(n_u + 1)(n_v + 1)}\cos^s\theta_h}{2\pi}$

  where $s = \cos^2\varphi_h n_u^2 + \sin^2\varphi_h n_v^2$

- Sample θ_h :

  $\theta_h = \text{acos}((1 - u_1)^{1/(s+1)})$

- Sample φ_h :

  $\varphi_h = \arctan(\sqrt{(n_u + 1)/(n_v + 1)}\tan(2\pi u_2))$

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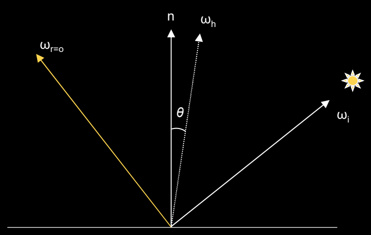
Microfacet lobe

- Analytic inversion of PDF $p(\omega_h)$
- Need for reflection of ω_r about ω_h

  $\omega_i = 2(\omega_h \cdot \omega_r)\,\omega_h - \omega_r$

  ray-tracing done for sampled ω_i



- Final step conversion of PDF $p(\omega_h)$ to $p(\omega_r)$

  $$p(\omega_r) = \frac{p(\omega_h)}{4(\omega_h \cdot \omega_i)}$$

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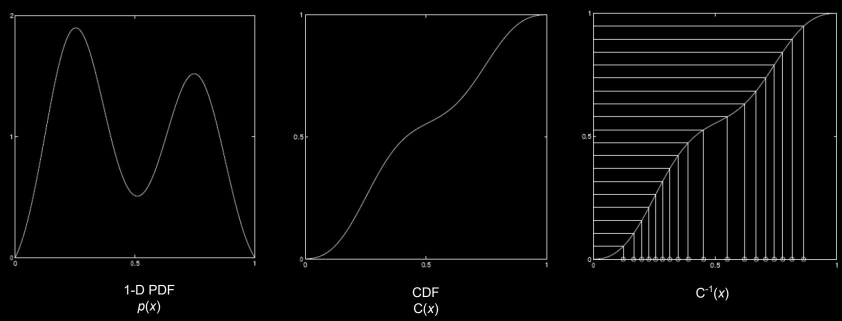
Numerical CDF Inversion



- Numerical integration of PDF
- Uniform samples on Y-axis → $x \sim p(x)$ on X-axis
- Useful for sampling from EMs!

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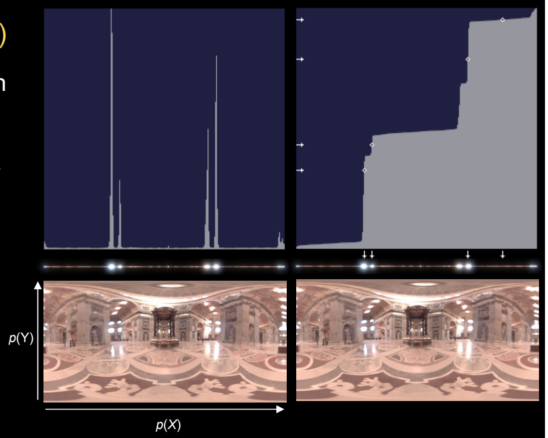
Sampling from EMs

- First sample along rows $p(Y)$
  - based on average energy in a scan line
- Then sample along columns $p(X)$ of the selected row



- Cumulative PDF

  $p(X, Y) = p(Y) \cdot p(X)$

- Need to account for $d\omega$!
  - spherical to lat-long param.
  - weighting by $\sin\theta$

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