Artificial Intelligence Assignment – 3
Exact Inference using Variable Elimination

1) P(Sick | Lecture = True, Doctor = True)

Exact Inference: We will use variable elimination[1] to compute the exact inference for the above query. The above query can be written as P(S | l, d).

$$
\begin{aligned}
P(S \mid l, d) &= \alpha P(S) \textstyle\sum_p P(p) \sum_h P(h|s,p) P(l|h) P(d|h) \\
&= \alpha P(S) \textstyle\sum_p P(p) \left[0.6 \times 0.3 \times \begin{pmatrix} 0.95 & 0.90 \\ 0.40 & 0.10 \end{pmatrix} + 0.8 \times 0.01 \times \begin{pmatrix} 0.05 & 0.10 \\ 0.60 & 0.90 \end{pmatrix}\right] \\
&= \alpha P(S) \textstyle\sum_p P(p) \begin{pmatrix} 0.1714 & 0.1628 \\ 0.0768 & 0.0252 \end{pmatrix} \\
&= \alpha P(S) \left[0.1 \times \begin{pmatrix} 0.1714 \\ 0.1628 \end{pmatrix} + 0.9 \times \begin{pmatrix} 0.0768 \\ 0.0252 \end{pmatrix}\right] \\
&= \alpha \begin{pmatrix} 0.05 \\ 0.95 \end{pmatrix} \times \begin{pmatrix} 0.08626 \\ 0.03896 \end{pmatrix} \\
&= \alpha \begin{pmatrix} 0.004313 \\ 0.037012 \end{pmatrix} \\
&\approx < 0.104, 0.896 >
\end{aligned}
$$

2) P(Sick | Doctor = False)

Exact Inference: We will use variable elimination[1] to compute the exact inference for the above query. The above query can be written as P(S = true | D = false).

$$
\begin{aligned}
P(S = true \mid D = false) &= \alpha \textstyle\sum_{P,H,L} P(S = true, P, H, L, D = true) \\
&= \alpha \textstyle\sum_{P,H,L} [P(S = true) P(P) P(H \mid S = true, P) P(L \mid H)\, P(D = false, H)] \\
&= \alpha P(S = true) \textstyle\sum_H [P(D = false|H) \sum_P P(P) P(H|S = true, P)] \\
&= \alpha P(S = true) \textstyle\sum_H P(D = false \mid H) f_P(H)
\end{aligned}
$$

Where $f_P(H = true) = (0.95)(0.1) + (0.4)(0.9) = 0.455$ and $f_P(H = false) = (0.05)(0.1) + (0.6)(0.9) = 0.545$.

Continuing,

$$
\begin{aligned}
&= \alpha 0.05(0.455 \times 0.4 + 0.545 \times 0.99) \\
&= 0.03607752\alpha
\end{aligned}
$$

Similarly, we can evaluate

$$P(S = false \mid D = false) = \alpha P(S = false) \textstyle\sum_H P(D = false \mid H) f'_P(H)$$

Where $f'_P(H = true) = (0.9)(0.1) + (0.1)(0.9) = 0.18$ and $f'_P(H = false) = (0.1)(0.1) + (0.9)(0.9) = 0.82$.

Continuing,

$$
\begin{aligned}
&= \alpha 0.95(0.18 \times 0.4 + 0.82 \times + 0.99) \\
&= 0.83961\alpha
\end{aligned}
$$

Therefore, $P(S = true \mid D = false) = \frac{0.03607752\alpha}{0.03607752\alpha + 0.83961\alpha} = 0.0411$ and $P(S = false \mid D = false) = 1 - 0.0411 = 0.9589$.

3) P(Pub | Lecture = False)

Exact Inference: We will use variable elimination[1] to compute the exact inference for the above query. The above query can be written as P(P = true | L = false).

$$P(P = true \mid L = false) = \alpha \sum_{S,H,D} P(P = true, S, H, D, L = true)$$
$$= \alpha \sum_{S,H,D}[P(P = true)P(S)P(H \mid P = true, P)P(D \mid H)\, P(L = false, H)]$$
$$= \alpha P(P = true) \sum_{H}[P(L = false|H) \sum_{S} P(S)P(H|P = true, S)]$$
$$= \alpha P(S = true) \sum_{H} P(L = false \mid H)f_S(H)$$

Where $f_S(H = true)$ = (0.95)(0.05) + (0.4)(0.95) = 0.4275 and $f_S(H = false) = (0.05)(0.05) + (0.6)(0.95) = 0.5725$.

Continuing,

$$= \alpha 0.1(0.7 \times 0.4275 + 0.2 \times 0.5725)$$
$$= 0.0445252\alpha$$

Similarly, we can evaluate

$$P(P = false \mid D = false) = \alpha P(P = false) \sum_{H} P(L = false \mid H)f'_S(H)$$

Where $f'_S(H = true)$ = (0.9)(0.05) + (0.1)(0.95) = 0.14 and $f'_S(H = false)$ = (0.1)(0.05) + (0.9)(0.95) = 0.86.

Continuing,

$$= \alpha 0.9(0.14 \times 0.7 + 0.86 \times + 0.2)$$
$$= 0.27\alpha$$

Therefore, P(P = true | D = false) = $\frac{0.0445252\alpha}{0.0445252\alpha + 0.27\alpha}$ = 0.1415 and P(S = false | D = false) = 1 – 0.1415 = 0.8585.

4) P(Pub | Lecture = False, Doctor = True)

Exact Inference: We will use variable elimination[1] to compute the exact inference for the above query. The above query can be written as P(P = true | L = false, D = true).

$$P(P = true \mid L = false, D = true) = \alpha \sum_{S,H} P(P = true, S, H, D = true, L = true)$$
$$= \alpha \sum_{S,H}[P(P = true)P(S)P(H \mid P = true, P)P(D = true \mid H)\, P(L = false, H)]$$
$$= \alpha P(P = true) \sum_{H} P(L = false \mid H)P(D = true \mid H)f_S(H)$$

Where $f_S(H = true)$ = (0.95)(0.05) + (0.4)(0.95) = 0.4275 and $f_S(H = false) = (0.05)(0.05) + (0.6)(0.95) = 0.5725$.

Continuing,

$$= \alpha 0.1(0.7 \times 0.6 \times 0.4275 + 0.2 \times 0.01 \times 0.5725)$$
$$= 0.0180695\alpha$$

Similarly, we can evaluate

$$P(P = false \mid L = false, D = true) = \alpha P(P = false) \sum_{H} P(L = false \mid H)P(D = true \mid H)f'_S(H)$$

Where $f'_S(H = \text{true}) = (0.9)(0.05) + (0.1)(0.95) = 0.14$ and $f'_S(H = \text{false}) = (0.1)(0.05) + (0.9)(0.95) = 0.86$.

Continuing,

$$= \alpha 0.9(0.7 \times 0.6 \times 0.14 + 0.2 \times 0.01 \times 0.86)$$
$$= 0.054468\alpha$$

Therefore, $P(P = \text{true} \mid L = \text{false}, D = \text{true}) = \frac{0.0180695\alpha}{0.0180695\alpha + 0.05446} = 0.2491$ and $P(P = \text{false} \mid L = \text{false}, D = \text{true}) = 1 - 0.2491 = 0.7509$.

Approximate Inference using Rejection Sampling

For computing approximate inference, we need to generate random samples, therefore using random generator we have generated 45 samples, which are listed below in the table with the random values and their topological order.

Random Values	Samples
0.85 0.97 0.14 0.11 0.83	{-S, -P, -H, +L, -D}
0.67 0.38 0.67 0.18 0.58	{-S, -P, -H, +L, -D}
0.90 0.46 0.34 0.70 0.76	{-S, -P, -H, +L, -D}
0.58 0.18 0.08 0.67 0.89	{-S, -P, +H, -L, -D}
0.42 0.35 0.72 0.37 0.56	{-S, -P, -H, +L, -D}
0.41 0.53 0.62 0.43 0.01	{-S, -P, -H, +L, -D}
0.79 0.30 0.31 0.15 0.62	{-S, -P, -H, +L, -D}
0.27 0.10 0.84 0.75 0.17	{-S, -P, -H, +L, -D}
0.49 0.18 0.25 0.43 0.87	{-S, -P, -H, +L, -D}
0.14 0.85 0.06 0.19 0.79	{-S, -P, +H, +L, -D}
0.84 0.47 0.71 0.37 0.45	{-S, -P, -H, +L, -D}
0.69 0.61 0.37 0.94 0.47	{-S, -P, -H, -L, -D}
0.97 0.87 0.85 0.75 0.97	{-S, -P, -H, +L, -D}
0.60 0.27 0.98 0.43 0.92	{-S, -P, -H, +L, -D}
0.67 0.01 0.04 0.91 0.38	{-S, +P, +H, -L, +D}
0.02 0.32 0.76 0.84 0.64	{+S, -P, -H, -L, -D}
0.79 0.87 0.38 0.54 0.83	{-S, -P, -H, +L, -D}
0.23 0.59 0.66 0.83 0.53	{-S, -P, -H, -L, -D}
0.17 0.23 0.05 0.64 0.21	{-S, -P, +H, -L, +D}
0.33 0.19 0.70 0.04 0.81	{-S, -P, -H, +L, -D}
0.89 0.97 0.96 0.39 0.85	{-S, -P, -H, +L, -D}
0.51 0.82 0.49 0.38 0.11	{-S, -P, -H, +L, -D}
0.14 0.05 0.44 0.82 0.89	{-S, +P, +H, -L, -D}
0.64 0.60 0.83 0.46 0.89	{-S, -P, -H, +L, -D}
0.00 0.43 0.37 0.41 0.17	{+S, -P, +H, -L, +D}
0.58 0.90 0.23 0.64 0.12	{-S, -P, -H, -L, -D}
0.62 0.48 0.73 0.56 0.72	{-S, -P, -H, +L, -D}
0.76 0.51 0.62 0.68 0.75	{-S, -P, -H, +L, -D}
0.49 0.01 0.78 0.91 0.70	{-S, +P, +H, -L, -D}
0.05 0.75 0.67 0.67 0.69	{-S, -P, -H, +L, -D}

0.27 0.01 0.23 0.49 0.80	{-S, +P, +H, -L, -D}
0.30 0.89 0.50 0.31 0.83	{-S, -P, -H, +L, -D}
0.88 0.23 0.93 0.01 0.51	{-S, -P, -H, +L, -D}
0.73 0.42 0.35 0.40 0.53	{-S, -P, -H, +L, -D}
0.52 0.33 0.44 0.73 0.67	{-S, -P, -H, +L, -D}
0.96 0.03 0.49 0.02 0.25	{-S, +P, +H, +L, +D}
0.24 0.51 0.38 0.56 0.44	{-S, -P, -H, +L, -D}
0.53 0.93 0.04 0.48 0.24	{-S, -P, +H, -L, +D}
0.22 0.24 0.74 0.05 0.56	{-S, -P, -H, +L, -D}
0.40 0.90 0.13 0.53 0.79	{-S, -P, -H, +L, -D}
0.88 0.08 0.11 0.16 0.24	{-S, +P, +H, +L, +D}
0.94 0.89 0.26 0.19 0.11	{-S, -P, -H, +L, -D}
0.78 0.18 0.56 0.23 0.70	{-S, -P, -H, +L, -D}
0.41 0.58 0.29 0.76 0.99	{-S, -P, -H, +L, -D}
0.01 0.11 0.02 1.00 0.55	{+S, -P, +H, -L, +D}

3) P(Pub | Lecture = False)

Approximate Inference: We will use Rejection Sampling [2] to compute the approximate inference for the above query. The above query can be written as P(P = true | L = false).

In the Rejection Sampling, we rejects/remove those samples which does not match the evidence, and for above query the evidence is L = false. So After rejecting the samples our Sample list will be as follows.

Random Values	Samples
0.58 0.18 0.08 0.67 0.89	{-S, -P, +H, -L, -D}
0.69 0.61 0.37 0.94 0.47	{-S, -P, -H, -L, -D}
0.67 0.01 0.04 0.91 0.38	{-S, +P, +H, -L, +D}
0.02 0.32 0.76 0.84 0.64	{+S, -P, -H, -L, -D}
0.23 0.59 0.66 0.83 0.53	{-S, -P, -H, -L, -D}
0.17 0.23 0.05 0.64 0.21	{-S, -P, +H, -L, +D}
0.14 0.05 0.44 0.82 0.89	{-S, +P, +H, -L, -D}
0.00 0.43 0.37 0.41 0.17	{+S, -P, +H, -L, +D}
0.58 0.90 0.23 0.64 0.12	{-S, -P, -H, -L, -D}
0.49 0.01 0.78 0.91 0.70	{-S, +P, +H, -L, -D}
0.27 0.01 0.23 0.49 0.80	{-S, +P, +H, -L, -D}
0.53 0.93 0.04 0.48 0.24	{-S, -P, +H, -L, +D}
0.01 0.11 0.02 1.00 0.55	{+S, -P, +H, -L, +D}

$$P(\text{Pub = true} \mid \text{Lecture = false}) \approx \frac{N_{PS}(\text{Pub = true, Lecture = false})}{N_{PS}(\text{Lecture = false})}$$

$$P(\text{Pub = true} \mid \text{Lecture = false}) = \frac{4}{13}$$

Similarly, P(Pub = false | Lecture = false) = 1 – P(Pub = true | Lecture = false)

$$= 1 - \frac{4}{13} = \frac{9}{13}$$

4) P(Pub | Lecture = False, Doctor = True)

Approximate Inference: We will use Rejection Sampling [2] to compute the approximate inference for the above query. The above query can be written as P(P = true | L = false, Doctor = true).

In the Rejection Sampling, we rejects/remove those samples which does not match the evidence, and for above query the evidence is L = false, Doctor = true. So After rejecting the samples our Sample list will be as follows.

Random Values	Samples
0.67 0.01 0.04 0.91 0.38	{-S, +P, +H, -L, +D}
0.17 0.23 0.05 0.64 0.21	{-S, -P, +H, -L, +D}
0.00 0.43 0.37 0.41 0.17	{+S, -P, +H, -L, +D}
0.53 0.93 0.04 0.48 0.24	{-S, -P, +H, -L, +D}
0.01 0.11 0.02 1.00 0.55	{+S, -P, +H, -L, +D}

$$P(\text{Pub = true} \mid \text{Lecture = false, Doctor = true}) \approx \frac{N_{PS}(\text{Pub = true, Lecture = false, Doctor = true})}{N_{PS}(\text{Lecture = false, Doctor = true})}$$

$$P(\text{Pub = true} \mid \text{Lecture = false, Doctor = true}) = \frac{1}{5}$$

Similarly, P(Pub = false | Lecture = false, Doctor = true) = 1 – P(Pub = true | Lecture = false, Doctor = true)

$$= 1 - \frac{1}{5} = \frac{4}{5}$$

References:

[1] Russell, Stuart, and Peter Norvig. "Artificial intelligence: a modern approach." (2002). PP: e524 – e527

[2] Russell, Stuart, and Peter Norvig. "Artificial intelligence: a modern approach." (2002). PP: e532 – e533