Probabilitati si Statistica

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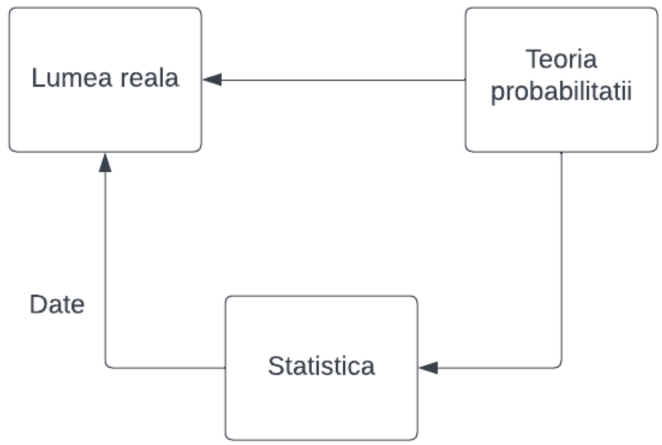
1 Cursul 1

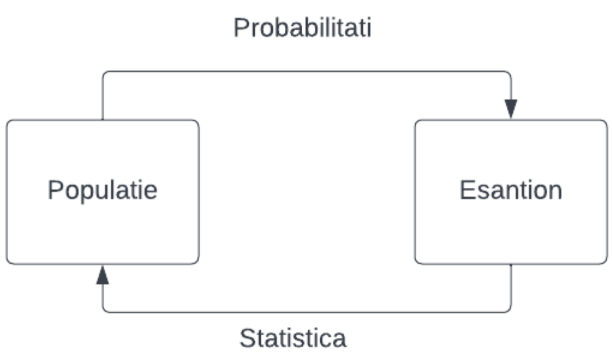
1.1 Introducere in Probabilitati si Statistica



Experiment: Urna cu bile albe si negre, proportia bilelor albe este p ϵ (0,1) necunoscut.

Probabilitati: p=0,17 ,extragem 10 bile.

Care este probabilitatea ca in cele 10 bile sa avem 4 bile de culoare alba?



Statistica: Am extras 10 bile (cu intoarcere) si 4 sunt albe.

Ce pot spune despre p?

1.2 Camp de probabilitate.Operatii cu evenimente

Experiment aleator = sir de actiuni care conduc la un rezultat necunoscut inaintea realizarii lui.
Ω = multimea evenimentelor elementare / spatiul starilor / spatiul probelor.
Ω = {H(head),T(tail)} pentru ban
Ω = {1,2,3,4,5,6} pentru zar

$\omega \epsilon \Omega$
Ω :

a) mutual exclusivitate

b) colectiv exhaustive

Dau cu banul si ma uit la vreme:

1) H si ploua

2) T si ploua

3) H si nu ploua

4) T si nu ploua

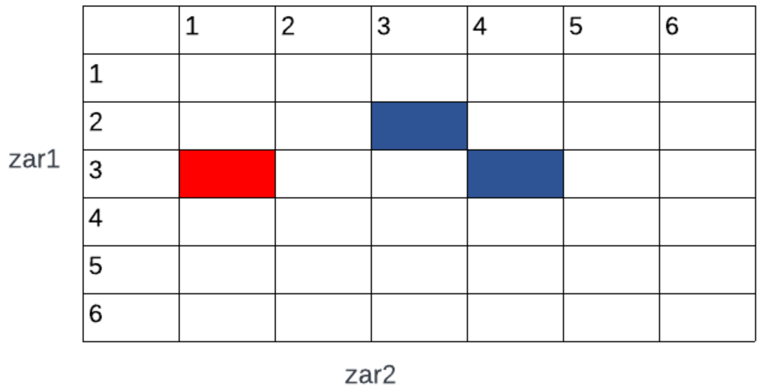
Ω = {H,T} pentru ca vremea nu influenteaza experimentul

1.2.1 Exemplul 1

Arunc cu 3 monede.
$\Omega = \{ \text{(x,y,z)} \mid \text{x,y,z} \epsilon \{H,T\} \}$

1.2.2 Exemplul 2

Arunc cu 2 zaruri.
$\Omega = \{ \text{(x,y)} \mid \text{x,y} \epsilon \{1,2,3,4,5,6\} \}$

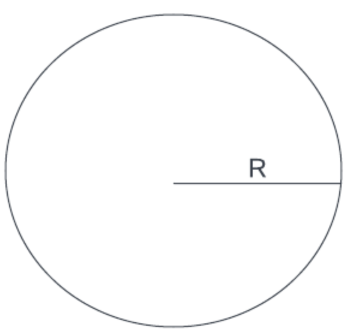


1.2.3 Exemplul 3

Ω = [0,T] , T≥0

1.2.4 Exemplul 4

$\Omega = \{(x, y) \big| x^2 + y^2 \leq R^2\}$



1.2.5 Exemplul 5

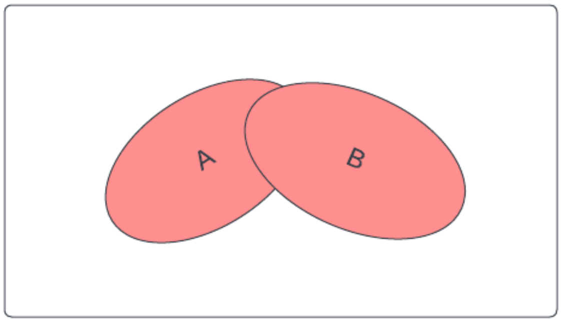
$\Omega = \{(x,y) \big| -a \leq x \leq a, -b \leq y \leq b \big| a, b > 0\}$

Definitie O submultime A $\epsilon\Omega$ se numeste eveniment.
Spunem ca evenimentul A se realizeaza daca in urma desfasurarii experimentului aleator rezulta $\omega\epsilon$A.

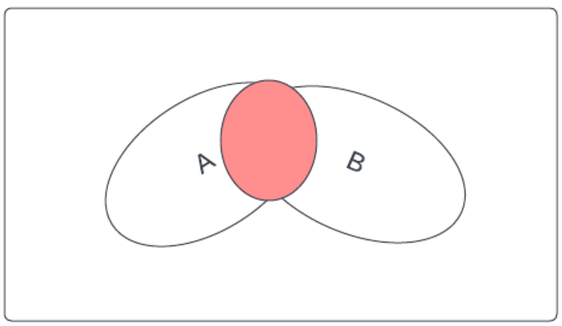
	Teoria multimii	Teoria probabilitatilor
Ω	multimea Ω	spatiul starilor/eveniment sigur
ω	un element din Ω	evenimentul elementar
$\emptyset$	multimea vida	evenimentul imposibil
A	multimea	evenimentul A
A^C	complemetara lui A in Ω	eveniment cotrar lui A
A $\cup$ B	reuniune	cel putin un eveniment din A sau B se realizeaza
A $\cap$ B	intersectie	evenimentele din A si evemenimentele din B se realizeaza simultan
A $\setminus$ B	diferenta	A se realizeaza, dar B nu
A $\triangle$ B	diferenta simetrica	sau A sau B se realizeaza, dar nu amandoua

1.3 Diagramele Venn

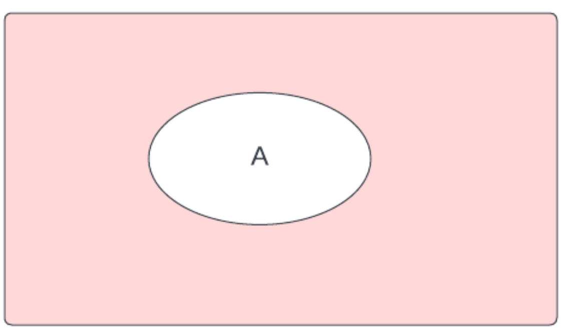
REUNIUNE



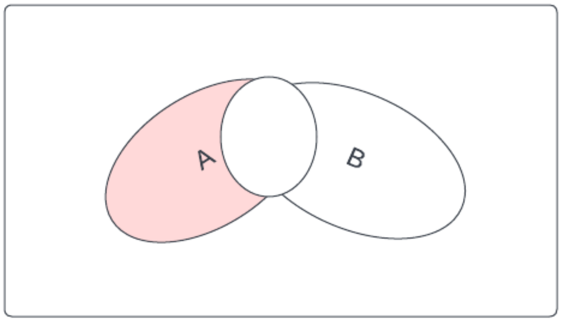
INTERSECTIE



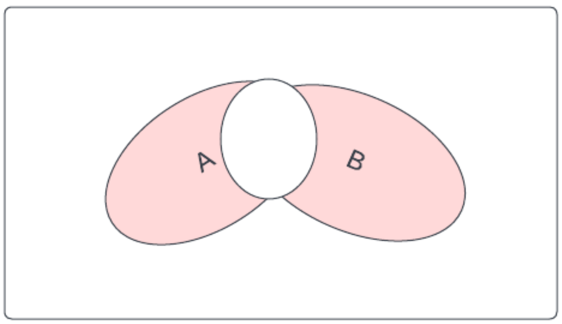
COMPLEMENTARA



A $\setminus$ B



A $\triangle$ B



Definitie Multimea evenimentelor posibile asociate experimentelor aleatoare cu spatiul starilor Ω este o submultime

$F \subseteq \mathbb{P}(\Omega)$ care verifica urmatoarele proprietati:

a) $\Omega \epsilon F$

b) daca A $\epsilon F \Rightarrow A^C \epsilon F$

c) daca A,B $\epsilon F \Rightarrow A \cup B \epsilon F$

Aruncam cu banul pana obtinem pentru prima oara H.

$\Omega = \{ 1,2,3,..\} = N^*$

A = {am obtinut pentru prima data H dupa un numar par de ori}={2,4,..}=$\bigsqcup_{n=1}^{\infty}\{2i\}$

$c^{'}$) Daca $(A_n)_n \epsilon F$ atunci $\bigsqcup_{n=1}^{\infty} A_n \epsilon F$

F care verifica a,b si $c^{'}$ se numeste σ-algebra

Ω - spatiul starilor

(Ω, F) spatiul probabilizabil (spatiul masurabil)

experiment aleator $\rightarrow (\Omega, F, \mathbb{P})$

Proprietati Avem (Ω, F) :

1) $\Omega =$ {H,T}

$F = \mathbb{P}(\Omega) = \{\emptyset, \{H\}, \{T\}, \Omega\}$

2) $\Omega =$ {1,2,3,4,5,6}

$F = \mathbb{P}(\Omega) \simeq \{0,1\}^{|\Omega|}$

2 Cursul 2

2.1 Camp de probabilitate.Operatii cu evenimente.Formule de calcul

experiment aleator $\rightarrow (\Omega, F)$

$\mathbb{P}: F \rightarrow$ [0,1]

Presupunem ca avem un experiment aleator si un eveniment A de interes. Repetam experimentul (in conditii similare) de un numar mare de ori(N).

N_A - numar de realizari ale lui A

$\frac{N_A}{N}$ - frecventa relativa de realizare a lui A

$N_A =$ {0,1,2,..,N}

$\mathbb{P}(A) \simeq \lim_{N\rightarrow\infty} \frac{N(A)}{B}$

$\frac{N(A)}{N} \epsilon [0,1]$

$\mathbb{P}(A) \epsilon [0,1]$

Daca A $= \Omega \implies N(\Omega) = N \implies \frac{N(A)}{N} = 1 \implies \mathbb{P}(A) = 1$

$\mathbb{P}(A) \epsilon [0,1]$

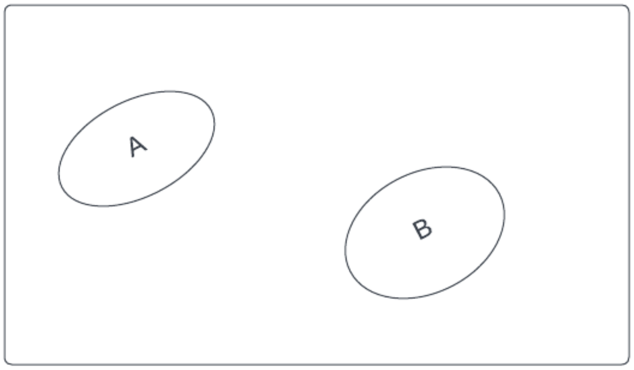
$\mathbb{P}(\Omega) = 1$

Presupunem ca A,B $\epsilon F, A \cap B = \emptyset$

$A \cup B \epsilon F$

$N(A \cup B) = N(A) + N(B)$

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ - finit aditivitate



Definitie O functie $\mathbb{P} : F \to [0,1]$ se numeste masura de probabilitate pe (Ω, F) daca verifica urmaroarele proprietati:

a) $\mathbb{P}(\Omega) = 1$

b) oricare ar fi $(A_n)_n \subseteq F$ disjuncte 2 cate 2

Experiment aleator $\to$ $(\Omega, F, \mathbb{P}(B))$ camp de probabilitate

2.1.1 Aruncatul cu banul

$\Omega = \{H, T\}$

$F = P(\Omega) = \{\emptyset, \{H\}, \{T\}, \Omega\}$

$\mathbb{P} : F \to [0,1]$

$\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\emptyset) = 0$

$\mathbb{P}\{H\} = p\, \epsilon[0,1) \Longrightarrow \mathbb{P}\{T\} = 1 - p$

Moneda echilibrata $\Longrightarrow$ p = $\frac{1}{2}$

2.1.2 Aruncatul cu zarul

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$F = P(\Omega) \Longrightarrow 2^6$ elemente

$\{0,1\}^\Omega = \{f : \Omega \to \{0,1\}\}$

$\mathbb{P} : F \to [0,1]$
$\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\emptyset) = 0$
$\mathbb{P}(\{i\}) = p_i \, \epsilon [0,1], i \epsilon \{1,2,3,4,5,6\}$

2.1.3 Proprietati

a)$\mathbb{P}(\emptyset) = 0$

$\Omega \cup \emptyset = \Omega$

$\Omega \cap \emptyset = \emptyset$

Sirul $A_n = \emptyset$

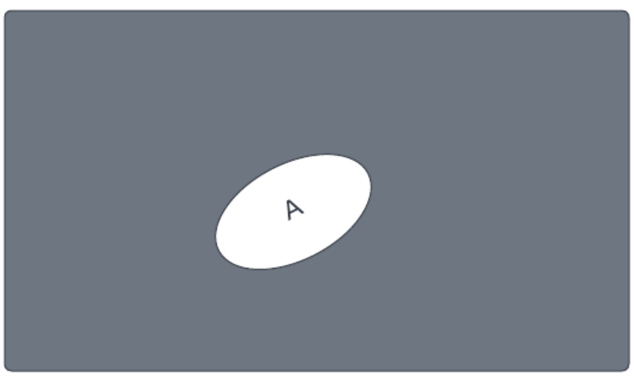
$\bigcup_n A_n = \emptyset$

Presupunem ca $\mathbb{P}(\emptyset) > 0$

$$\mathbb{P}(\emptyset) = \sum_n \mathbb{P}(\emptyset) \to \infty \implies \mathit{contradictie}$$

b) $\mathbb{P}(A_1 \cup A_2 \cup ... \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i)$, $A_1, A_2, .., A_n$ *disjuncte* 2 *cate* 2

c)$A \, \epsilon F \implies \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$



(1) $A \cap A^c = \emptyset$
(2) $A \cup A^c = \Omega$
Din (1) si (2) $\implies \mathbb{P}(A \cup A^c) = \mathbb{P}(\Omega) = 1 = \mathbb{P}(A) + \mathbb{P}(A^c)$

d) $A \subseteq B \implies \mathbb{P}(A) \leq \mathbb{P}(B)$

e) A,B ϵF, $\mathbb{P}(A \cup B) =?$ $\quad A \cup B = A \cup (B \setminus A) = \emptyset$

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) = \mathbb{P}(A) + \mathbb{P}(B \setminus (A \cap B)) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$e^{'}$) A,B,C, $\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$

f) Formula lui Poincaré

$\mathbb{P}(A_1 \cup A_2 \cup .. \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j}^{n} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k}^{n} \mathbb{P}(A_i \cap A_j \cap A_k) + ..$
$.. + (-1)^{n+1} \mathbb{P}(A_1 \cap .. \cap A_n)$

$\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$

$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$

2.2 Modelul clasic de posibilitate.Modelul lui Laplace

Fie N $\geq 1, N \epsilon \mathbb{N}$ si consideram un experiment aleator cu N rezultate posibile.

$\Omega = \{\omega_1, \omega_2, ..m\omega_n\}$

$F = \mathbb{P}(\Omega) \Longrightarrow 2^N elemente$

$\mathbb{P} : F \to [0,1]$, $\mathbb{P}(\{N_i\}) = \frac{1}{N}$, $i \epsilon \{1, 2, 3, .., N\}$ echirepartitie

Fie A ϵF

$\mathbb{P}(A) = \mathbb{P}(\bigcup_{\omega_i \epsilon A} \{\omega_i\}) = \sum_{\omega_i \epsilon A} \mathbb{P}(\{\omega_i\})$

A - multimea evenimentelor favorabile

$\mathbb{P}(A) = \frac{1}{N} \sum_{\omega_i \epsilon A} 1 = \frac{|A|}{N} = \frac{|A|}{|\Omega|} = \frac{numar\ cazuri\ favorabile}{numar\ cazuri\ posibile}$

2.2.1 Formula sumei

A,B finite disjuncte $\Longrightarrow |A \cup B| = |A| + |B|$

A,B finite oarecare $\Longrightarrow |A \cup B| = |A| + |B| - |A \cap B|$

Principiul includerii si excluderii

$A_1, A_2, .., A_n$ finita

$|A_1 \cup A_2 \cup .. \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j}^{n} |A_i \cap A_j| + \sum_{i<j<k}^{n} |A_i \cap A_j \cap A_k| +$
$.. + (-1)^{n+1} |A_1 \cap A_2 \cap .. \cap A_n|$

Aplicatie

$\varphi(n)$ - numar de numere prime cu n $\leq$ n (functia lui Euler)

$\varphi(n) = n\Pi_{p|n}(1 - \frac{1}{p})$

2.2.2 Formula produsului

A,B finite A $\times$ B $= \{(a, b)|a\epsilon A, b\epsilon B\}$

$|A \times B| = |A| \times |B|$

3 Curs 3

3.1 Schema cu revenire

Urna cu n bile 1..n si efectuam k extrageri cu revenire.

In cate moduri?(1)

Reformulare: k bile (1..k) si n nume

$(x_1, x_2, .., x_k), x_i$ -numarul urnei in care am pus bila i $\Longrightarrow n^k noduri$

$\Longrightarrow_{(1)}$ numarul de siruri de lungime k cu termeni din $\{1, 2, .., n\}$

3.2 Schema de extragere fara revenire

Urna cu n bile $\{1, 2, .., n\}$ si efectuam k extrageri fara intoarcere. In cate moduri?

Reformulare: Numarul de siruri de lungime k de termeni distincti din $\{1, 2, .., n\}$

n(n-1)(n-2)...(n-k+1) $= A_n^k = \frac{n!}{(n-k)!}$

3.2.1 Experimentul 1

Cate cuvinte puteti forma cu literele cuvantului "MATE"?

cu revenire $\Longrightarrow 4^4 = 256\, cuvinte$

fara revenire $\Longrightarrow 4! = 24\, cuvinte$

3.2.2 Experimentul 2

4 carti de matematica, 3 de fizica, 2 de istorie, 1 de geografie

Vrem sa pastram grupate cartiile pe domenii.

$\Longrightarrow 4! * (4! * \ 3! * \ 2! * \ 1!)$

3.2.3 Experimentul 3.Problema aniversarilor

n persoane

Vrem sa vedem care este probabilitatea ca cel putin 2 persoane sa se fi nascut in aceeasi zi.

$\underline{Ipoteza}:$ - anul are 365 de zile

-echirepartitia

-nu avem gemeni

Campul de probabilitate pe care lucram:

$\Omega = \{(z_1, z_2, .., z_n) | z_i \epsilon \{1, 2, .., 365\}\}$

$|\Omega| = 365^n$

$F = \mathbb{P}(\Omega)$ - multimea evenimentelor posibile

$\mathbb{P} : F \rightarrow [0, 1]$

$\mathbb{P}(\{\omega\}) = \frac{1}{365^n}$ echirepartitia

A - cel putin 2 persoane s-au nascut in aceeasi zi

$A = \{(z_1, z_2, .., z_n) \epsilon \Omega | \exists i, j, i \neq j \ a.i. \ z_i = z_j\}$

$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$

$\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$

A^c- toate cele n persoane s-au nascut in zile diferite

$\mathbb{P}(A^c) = \frac{|A^c|}{|\Omega|} = \frac{365 * 364 * .. * (365 - n + 1)}{365^n}$

$\mathbb{P}(A) = 1 - \frac{\frac{365!}{(365-n)!}}{365^n}$

Pentru n=23 $\implies \mathbb{P}(A) \approx 51\%$

3.2.4 Experimentul 4

Avem n persoane si am vrea sa formam comisii de cate 4 persoane.
Reformulare: Numarul de submultimi cu k elemente a unor submultimi cu n elemente.Ordinea nu conteaza!
$C_n^k = \frac{n!}{k!(n-k)!}$
$(x_1, x_2, .., x_k) \rightarrow \frac{n!}{(n-k)!}$
k!

Exemplul 1. 52 carti.Cate maini de 5 carti?$\rightarrow C_{52}^5$

Exemplul 2. Cate maini de 5 carti contin 2 asi, 2 popi si o dama? $\rightarrow \binom{4}{2} * \binom{4}{2} * \binom{4}{4}$

Exemplul 3. In jocul de Poker vreau sa determin probabilitatea sa obtin Fullhouse(Q,Q,3,3,3).
$\Omega = \{(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) | \omega_i \, \epsilon \, cartidejoc\}$
$|\Omega| = C_{52}^5$
$F = \mathbb{P}(\Omega)$
$\mathbb{P} : F \rightarrow [0, 1]$
$\mathbb{P}(\{\omega_1, .., \omega_5\}) \, \epsilon \, echirepartitie = \frac{1}{C_{52}^5}$
A - evenimentul prin care am obtinut Fullhouse
$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$
$|A|$-putem alege figura pentru pereche in C_{13}^1, culoarea o putem alege in C_4^2, pentru cele 3 carti avem C_{12}^1 moduri de a alege figura, iar C_4^3 pentru culoare.

$$|A| = \binom{13}{1} * \binom{4}{2} * \binom{12}{1} * \binom{4}{3}$$

Exemplul 4. Probabilitatea sa obtinem o pereche?
B- evenimentul prin care obtinem o pereche

$$\binom{13}{1} * \binom{4}{2} * \binom{12}{3} * \binom{4}{1}$$

3.3 Problema lui Newton-Pepys

a)Cel putin un 6 apare atunci cand aruncam cu 6 zaruri.

$\Omega = \{1, 2, 3, 4, 5, 6\}^6$

A - evenimentul de interes

$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = 1 - \mathbb{P}(A^c) = 1 - \frac{5^6}{6^6}$

b)Cel putin 2 valori de 6 apar cand aruncam cu 12 zaruri.

$\Omega = \{1, 2, 3, 4, 5, 6\}^{12}$

B - cel putin 2 valori de 6 in 12 zaruri

$\mathbb{P}(B) = 1 - \mathbb{P}(B^c) = 1 - \mathbb{P}(nici\ o\ valoare\ de\ 6) - \mathbb{P}(exact\ o\ valoare\ de\ 6) = 1 - \frac{5^{12}}{6^{12}} - \frac{C_{12}^{11} * 5^{11}}{6^{12}}$

c)Cel putin 3 valori de 6 apar cand aruncam cu 18 zaruri.

C - evenimentele cu cel putin 3 valori de 6 in 18 aruncari.

$\Omega = \{1, 2, 3, 4, 5, 6\}^{18}$

$\mathbb{P}(C) = 1 - \mathbb{P}(C^c)$

$\mathbb{P}(C^c) = \frac{5^{18}}{6^{18}} + \frac{\binom{18}{1} * 5^{17}}{6^{18}} + \frac{\binom{18}{2} * 5^{16}}{6^{18}}$

3.4 Partitii - coeficientul multinomial

Avem o multime cu n elemente si fie $n_1, n_2, .., n_k \epsilon \mathbb{N}$ a.i. $n_1 + n_2 + .. + n_k = n$

Consideram o partitie cu k submultimi a.i. submultimea i sa aiba n elemente.

Pentru k=2 , $n_1 + n_2 = n \Longrightarrow \binom{n}{n_1}$

Echivalent cu multimea sirurilor de lungime n cu n_1 elemente de tip 1, n_2 elemente de tip 2,.., n_k elemente de tip k.

$\binom{n}{n_1} * \binom{n-n_1}{n_2} * \binom{n-n_1-n_2}{n_3} * .. * \binom{n-n_1-n_2-..-n_{k-1}}{n_k} = \binom{n}{n_1, n_2, .., n_k}$

3.4.1 Experimentul 1

Numarul de anagrame a cuvantului "MATEMATICA". $M \to 2$
$A \to 3$
$T \to 2$
$E \to 1$
$I \to 1$
$C \to 1$
$\Longrightarrow \binom{10}{2,3,2,1,1}$

3.4.2 Experimentul 2

4 baieti si 12 fete
Profesorul formeaza in mod aleator 4 subgrupe de cate 4 studenti. Care e probabilitatea ca in fiecare subgrupa sa fie un baiat?

$\binom{16}{4,4,4,4}$

$\mathbb{P}(\omega) = \frac{1}{\binom{16}{4,4,4,4}}$
4! moduri pentru baieti

$\frac{4!*\binom{12}{3,3,3,3}}{\binom{16}{4,4,4,4}}$

3.4.3 Extragere cu revenire in care ordinea nu conteaza

In cate moduri putem plasa k bile (care nu se disting intre ele) din n urne?
Pentru n=6, k=12
$x_1 + x_2 + .. + x_n = k, x_i \epsilon \mathbb{N}^*$
$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$

4 Cursul 4

4.1 Probabilitati conditionate

4.1.1 Experimentul 1

Aruncam cu o moneda de 3 ori.
a)Care este probabilitatea sa optinem HHH?

$\Omega = \{H, T\}^3 = \{HHH, HHT, THH, THT, HTH, HTT, TTH, TTT\}$

$A = \{HHH\}$

$\mathbb{P}(A) = \frac{1}{8}$

b)Stim ca la prima aruncare am obtinut H.

$\Omega_2 = \{HHH, HHT, HTH, HTT\}$

$\mathbb{P}(A|B) = \frac{1}{4}$

B- evenimentul prin care la prima aruncare am obtinut H.

$\mathbb{P}(A|B)$ - probabilitatea realizarii lui A stiind ca B s-a realizate || probabilitatea conditionata a lui A la B

Din perspectiva frecventionala: Avem un experiment pe care il repetam de un numar N de ori. Ne intereseaza evenimentele A si B.

$$\frac{N(A\cap B)}{N(B)} = \frac{\frac{N(A\cap B)}{N}}{\frac{N(B)}{N}} \simeq \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$$

Definitie. Fie $(\Omega, F, \mathbb{P})$ un camp de probabilitate , A,B ϵF cu $\mathbb{P}(B) > 0$ atunci definim probabilitatea conditionata a lui A la evenimentul B si notam $\mathbb{P}(A|B)$ prin $\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$

$\mathbb{P}(A)$ -prior sau probabilitate a priori

$\mathbb{P}(A|B)$ -posterior sau probabilitate a posteriori

4.1.2 Exemplul 2

Carti de joc. Extragem in mod aleator 2 carti succesiv fara intoarcere.

A - prima carte de inima rosie

B - a doua carte este de inima rosie

C - a doua carte este de culoare rosie

$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\frac{13*12}{52*51}}{\frac{13}{52}} = \frac{12}{51}$

$\mathbb{P}(C|A) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(A)} = \frac{\frac{13*25}{52*51}}{\frac{13}{52}} = \frac{25}{51}$

$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\frac{13*12}{52*51}}{\frac{1}{4}} = \frac{12}{51} = \mathbb{P}(B|A)$

$\mathbb{P}(A|C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{\frac{13*25}{52*51}}{\frac{25}{52}} = \frac{25}{102} \neq \mathbb{P}(C|A)$

4.1.3 Exemplul 3

O familie are 2 copii.

a)Care este probabilitatea ca cei 2 copii sa fie de sex feminin stiind ca cel mai in varsta este fata?

b)Care este probabilitatea ca cei 2 copii sa fie de sex feminin stiind ca cel putin unul dintre ei este fata?

Ipoteza:$\{F, B\}$

$\mathbb{P}(F) = \mathbb{P}(B) = \frac{1}{2}$

sexul unui copil nu este influentat de celalalt copil

$\Omega = \{BB, BF, FB, FF\}$

$A = \{FF\}$

a) B - cel mai mic este fata $= \{FB, FF\}$

$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/4}{2/4} = 1/2$

b) C - cel putin unul este fata $= \{FB, BF, FF\}$

$\mathbb{P}(A|C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{1/4}{3/4} = 1/3$

4.1.4 Exemplul 4

Daca o aeronava apare in zona de interes scanata de un radar atunci se declanseaza o alarma cu probabilitatea de 0,99. Daca nu avem o aeronava alarma se declanseaza cu probabilitatea de 0,1. Sansa sa treaca o aeronava prin zona de interes este 0,05.
a)Care este probabilitatea ca in zona de interes sa nu avem avion si sa avem alarma?
b)Care este probabilitatea sa avem avion nedectat?

A - sa avem avion in zona de inters

B - sa se declanseze alarma

a) $\mathbb{P}(A^c \cap B) = \mathbb{P}(B|A^c) * \mathbb{P}(A^c) = \mathbb{P}(B|A^c) * (1 - \mathbb{P}(A)) = 0,1 * 0,95$
b) $\mathbb{P}(A \cap B^c) = \mathbb{P}(B^c|A) * \mathbb{P}(A) = (1 - \mathbb{P}(B|A)) * (P)(A) = 0.01 * 0.05$

Pentru formula produsului. $(\Omega, F, \mathbb{P})$ camp de probabilitate

$A_1, A_2, .., A_n \epsilon F$, $\mathbb{P}(A_1 \cap A_2 \cap ... \cap A_n) > 0$

Atunci

$\mathbb{P}(A_1 \cap A_2 \cap ... \cap A_n) = \mathbb{P}(A_1) * \mathbb{P}(A_2|A_1) * ... * \mathbb{P}(A_n|A_1 \cap A_2 \cap ... \cap A_(n-1))$

Formula probabilitatii totale

$(\Omega, F, \mathbb{P})$ camp de probabilitate o partitie a lui Ω, $\{B_1, B_2, B_3\}$ si A ϵF
$B_1, B_2, B_3 \subseteq \Omega$
$B_1 \cup B_2 \cup B_3 = \Omega$
$B_1 \cap B_2 = \emptyset$
$B_2 \cap B_3 = \emptyset$
$B_1 \cap B_3 = \emptyset$
$A = A \cap \Omega = A \cap (B_1 \cap B_2 \cap B_3) = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)$

$$\mathbb{P}(A) = \mathbb{P}(A \cap B_1) + \mathbb{P}(A \cap B_2) + \mathbb{P}(A \cap B_3) = \mathbb{P}(A|B_1) * \mathbb{P}(A|B_1) + \mathbb{P}(A \cap B_2) + \mathbb{P}(A \cap B_3) =$$
$$= \mathbb{P}(A|B_1) * \mathbb{P}(B_1) + \mathbb{P}(A|B_2) * \mathbb{P}(B_2) + \mathbb{P}(A|B_3) * \mathbb{P}(B_3)$$

Propozitie. Fie $(\Omega, F, \mathbb{P})$ un camp de probabilitate si $B_1, B_2, .., B_i \epsilon F$ o partitie pe $\Omega\, cu\, \mathbb{P}(A) > 0, i\epsilon\{1, .., n\}$.

Daca $A \epsilon F\, atunci : \mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i) * \mathbb{P}(B_i)$

4.2 Formula lui Bayes

Fie $(\Omega, F, \mathbb{P})$ camp de probabilitate $A, B\epsilon F\, cu\, \mathbb{P}(A) > 0,\ \mathbb{P}(B) > 0.$
a)$\mathbb{P}(B|A) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B)*\mathbb{P}(B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B)*\mathbb{P}(B)}{\mathbb{P}(A|B)*\mathbb{P}(B)+\mathbb{P}(A|B^c)*\mathbb{P}(B^c)}$

b) $A\epsilon F,\ B_1, .., b_n\epsilon F$ o partitie a lui $\Omega, \mathbb{P}(B_1) > 0$
$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(A|B_i)*\mathbb{P}(B_i)}{\sum_{j=1}^{n}\mathbb{P}(A|B_j)*\mathbb{P}(B_j)}$

4.2.1 Experimentul 1

Sa presupunem ca prevalenta unei boli in populatie este de 1%. Presupunem ca efectuam un test de detectie cu o acuratete de 95%.
acuratete=senzitivitatea si specificacitatea testului
$\mathbb{P}(T|D)$ = senzitivitate = rata de true positive
$\mathbb{P}(T^c|D^c)$ = specificacitatea = rata de true negative
D - pacientul este infectat
T - test pozitiv
$\mathbb{P}(T|D^c)$ = false positive
$\mathbb{P}(T^c|D)$ = false negative
Presupunem ca am efectuat testul si a iesit pozitiv . Care este probabilitatea sa avem virusul stiind ca testul este pozitiv?
$\mathbb{P}(D|T) \xlongequal{Formula\, lui\, Bayes} \frac{\mathbb{P}(T|D)*\mathbb{P}(D)}{\mathbb{P}(T)} = \frac{\mathbb{P}(T|D)*\mathbb{P}(D)}{\mathbb{P}(T|D)*\mathbb{P}(D)+\mathbb{P}(T|D^c)*\mathbb{P}(D^c)}$
$\mathbb{P}(T|D^c) = 1 - \mathbb{P}(T^c|D^c) = 0.05$
$\mathbb{P}(D|T) = \frac{0.95*0.01}{0.95*0.01+0.05*0.99} = 0.16$

Propozitie. Probabilitatea conditionata este o probabilitate.
$(\Omega, F, \mathbb{P})$ camp de probabilitate si $A\epsilon F,\ \mathbb{P}(A) > 0$
definim Q(.) $= \mathbb{P}(B|A)$
$Q(B) = \mathbb{P}(B|A)$
$(A, F \cap A)$

Q(A)=1=$\mathbb{P}(A|A) = \frac{\mathbb{P}(A \cap A)}{\mathbb{P}(A)}$
$(A_n)_n \subseteq F \cap A$ disjuncte 2 cate 2
$Q(\bigcup_n A_n) = \sum Q(A_n)$
$\mathbb{P}(\bigcup_n A_n|A) = \frac{\mathbb{P}(\bigcup_n A_n \cap A)}{\mathbb{P}(A)} = \frac{\sum \mathbb{P}(A_n \cap A)}{\mathbb{P}(A)} = \sum Q(A_n)$

4.2.2 Exemplul 2

$\Omega, F, \mathbb{P}$ camp de probabilitate
A,B,C ϵF, $\mathbb{P}(A \cap B) > 0$
$\mathbb{P}(A \cap C) > 0$
$\mathbb{P}(B \cap C) > 0$
$\mathbb{P}(A|B, C) = \frac{\mathbb{P}(B|A,C) * \mathbb{P}(A|C)}{\mathbb{P}(B|C)}$
$Q(.) = \mathbb{P}(.|C)$

5 Cursul 5

$\Omega, F, \mathbb{P},\ A\epsilon F,\ \mathbb{P}(A) > 0$
Q(B)=$\mathbb{P}(B|A), \forall\ B\epsilon F \cap A = \{F \cap A | F \epsilon F\}$

$Formula \quad lui \quad Bayes.$ $Q(.) = \mathbb{P}(.|C)$
$Q(A|B) = \frac{Q(B|A) * Q(A)}{Q(B)}$
$\mathbb{P}(A|B, C) = \frac{\mathbb{P}(B|A,C) * \mathbb{P}(A|C)}{\mathbb{P}(A|C)}$
$Q(A|B) = \mathbb{P}(A|B, C)$

5.1 Experimentul 1

Avem 2 monede (1 echilibrata si 1 trucata)
La cea trucata $\mathbb{P}(H) = 3/4$
Sansa sa alegem oricare dintre cele 2 monede este 1/2. Obtinem in urma celor 3 aruncari HHH

a) Care este probabilitatea ca moneda sa fi fost echilibrata?

b) Presupunem ca arunca pentru a 4-a oara moneda, care este probabilitatea sa fi obtinut H?

a) A - evenimentul prin care in primele 3 aruncari am obtinut HHH

B - evenimentul prin care am ales moneda echilibrata

$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)*\mathbb{P}(B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B)*\mathbb{P}(B)}{\mathbb{P}(A|B)*\mathbb{P}(B)+\mathbb{P}(A|B^c)*\mathbb{P}(B^c)} = \frac{(\frac{1}{2})^3*\frac{1}{2}}{(\frac{1}{2})^3*\frac{1}{2}+(\frac{3}{4})^3*\frac{1}{2}} = \frac{1}{1+\frac{3^3}{2^3}}$

b) C - evenimentul prin care la a 4-a aruncare am obtinut H

$\mathbb{P}(C|A) = ?$

$Q(.) = Q(.|A)$

$Q(C) = \mathbb{P}(C|A) = Q(C|B) * Q(B) + Q(C|B^c) * Q(B^c)$

$Q(B) = \mathbb{P}(B|A)$

$Q(B^c) = 1 - Q(B)$

$Q(C|B) = 1/2(\text{moneda echilibrata})$

$Q(C|B^c) = 3/4(\text{moneda trucata})$

$Q(C) = \frac{1}{2} + \frac{1}{1+\frac{3^2}{2^3}} + \frac{3}{4} * (1 - \frac{1}{1+\frac{3^3}{2^3}})$

5.2 Independenta

Doua evenimente sunt independente daca realizarea uneia nu aduce informatii suplimentare despre realizarea celeilalte. $(\Omega, F, \mathbb{P})$, $A, B \epsilon F$

$\Longrightarrow \mathbb{P}(A|B) = \mathbb{P}(B|A) \Longleftrightarrow \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \mathbb{P}(A) \Longleftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A) * \mathbb{P}(A)$

Definitie. Fie $(\Omega, F, \mathbb{P})$ un camp de probabilitate $A, B \epsilon F$. Spunema ca A si B sunt independente si notam $A \perp\!\!\!\perp B$ daca $\mathbb{P}(A \cap B) = \mathbb{P}(A) * \mathbb{P}(B)$.

Exemplu: Daca $A \perp\!\!\!\perp B$ atunci $A^c \perp\!\!\!\perp B$, $A \perp\!\!\!\perp B^c$, $A^c \perp\!\!\!\perp B^c$

5.2.1 Experimentul 1

Aruncam cu banul de 2 ori

A_1 - evenimentul prin care la prima aruncare am obtinut H

A_2 - evenimentul prin care la a 2-a aruncare am obtinut H

$\Omega = \{H, T\}^2$
$A_1 = \{HH, HT\}$
$A_2 = \{TH, HH\}$
$A_1 \cap A_2 = \{HH\}$
$\mathbb{P}(A_1 \cap A_2) = \frac{1}{4}$, $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \frac{1}{2} \Longrightarrow \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1) * \mathbb{P}(A_2)$

5.2.2 Experimentul 2

Zar cu 4 fete
Aruncam de 2 ori $\Longrightarrow \Omega = \{1, 2, 3, 4\}^2$
$A = \{primul\ zar\ are\ fata\ 1\} = \{(1, x) | x \epsilon \{1, 2, 3, 4\}\} \Longrightarrow \mathbb{P}(A) = \frac{1}{4}$
$B = \{suma\ punctelor\ este\ 5\} = \{(1, 4), (2, 3), (3, 2), (4, 1)\} \Longrightarrow \mathbb{P}(B) = \frac{1}{4}$
$C = \{minimul\ este\ 2\} \Longrightarrow \mathbb{P}(C) = \frac{5}{12}$
$D = \{maximul\ este\ 3\} \Longrightarrow \mathbb{P}(D) = \frac{3}{16}$
$A \cap B = \{(1, 4)\} \Longrightarrow \mathbb{P}(A \cap B) = \frac{1}{16} \Longrightarrow$ A si B sunt independente
$\mathbb{P}(C \cap D) = \frac{1}{16} \Longrightarrow$ C si D nu sunt independente

Definitie. Fie $(\Omega, F, \mathbb{P})$ camp de probabilitate si $A_1, A_2, .., A_n \epsilon F$.
Spunem ca $A_1, .., A_n$ sunt independente (mutual independente) daca:
$\mathbb{P}(\bigcap_{i \epsilon I} A_i) = \prod_{i \epsilon I} \mathbb{P}(A_i)$, $\forall\, I = \{1, 2, .., n\}$
Observatie. A_1, A_2, A_3 sunt independente $\Longleftrightarrow$
$\Longleftrightarrow \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_2 \cap A_3)$, $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) * \mathbb{P}(A_2) * \mathbb{P}(A_3)$
Pentru n=2 $\Longrightarrow 2^n - n - 1\ conditii = C_n^2 + C_n^3 + .. + C_n^n$

5.2.3 Experimentul 3(continuare experimentul 1)

$A_3 = cele\ 2\ sunt\ diferite = \{HT, TH\}$
$A_1 \perp\!\!\!\perp A_2$
$\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = \frac{1}{2}$
$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_2 \cap A_3) = \frac{1}{4}$
A_1, A_2, A_3 sunt independente 2 cate 2
$A_1 \cap A_2 \cap A_3 = \emptyset \Longrightarrow \mathbb{P}(A_1 \cap A_2 \cap A_3) = 0 \neq \frac{1}{8} \Longrightarrow A_1, A_2, A_3$ nu sunt independente

Definitie.$(\Omega, F, \mathbb{P})$ camp de probabilitate si $A, B, C \epsilon F$, $\mathbb{P}(C) > 0$

Spunem ca A si B sunt independente conditionat la C daca $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) * \mathbb{P}(B|C)$

$Q(.) = \mathbb{P}(.|C) \implies Q(A \cap B) = Q(A) * Q(B)$

5.2.4 Experimentul 4(continuare experiment covid)

D - $\{o\, persoana\, are\, afectiunea\}$

T - $\{testul\ a\ iesit\ pozitiv\}$

$\mathbb{P}(D) = 1\%$

acuratetea (sensibilitatea,specificacitatea) =95%

$\mathbb{P}(T|D) = \mathbb{P}(T^c|D^c) = 95\%$

$\mathbb{P}(D|T) \approx 15\%$

Sa presupunem ca persoanele mai efectueaza un test (presupunem ca rezultatele celor 2 teste sunt independente in raport cu statusul bolii respective) si testul este tot pozitiv. Care este probabilitatea sa avem COVID?

T_1 - primul test pozitiv

T_2 - al 2-lea test pozitiv

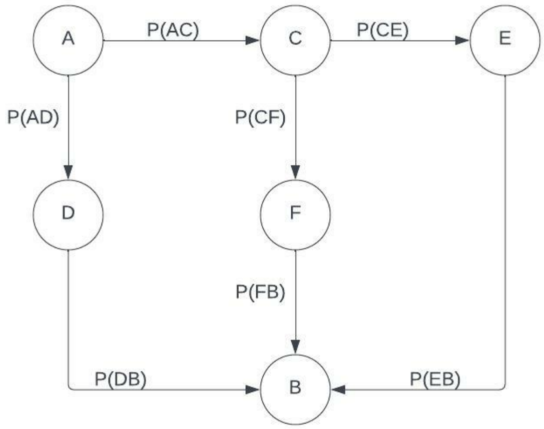
$\mathbb{P}(T_1 \cap T_2|D) = \mathbb{P}(T_1|D) * \mathbb{P}(T_2|D)$

$\mathbb{P}(T_1 \cap T_2|D^c) = \mathbb{P}(T_1|D^c) * \mathbb{P}(T_2|D^c)$

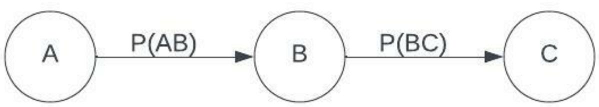
$\mathbb{P}(D|T_1 \cap T_2) = \frac{\mathbb{P}(T_1 \cap T_2|D) * \mathbb{P}(D)}{\mathbb{P}(T_1 \cap T_2)} \approx 0,78$

$\mathbb{P}(T_1 \cap T_2) = \mathbb{P}(T_1 \cap T_2|D) * \mathbb{P}(D) + \mathbb{P}(T_1 \cap T_2|D^c) * \mathbb{P}(D^c)$

5.2.5 Experimentul 5

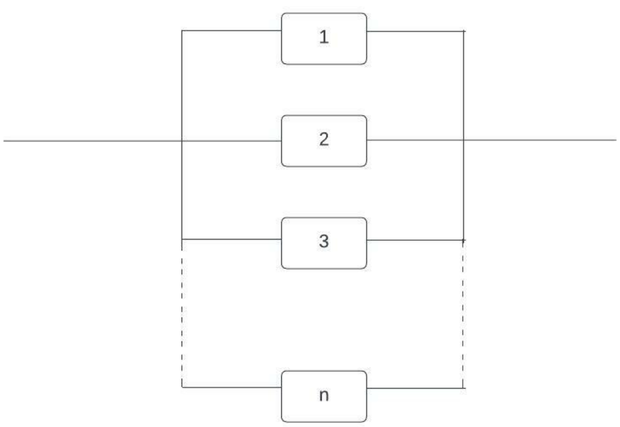


a)Subsistem serie



$\mathbb{P}_{AC} = \mathbb{P}_{AB} * \mathbb{P}_{BC}$

b)Subsistem paralel



$$\begin{aligned}\mathbb{P}(transmit\, mesaj\, sistem\, paralel) &= 1 - p(sa\, nu\, transmit) = \\ &= 1 - p(esec\, in\, nodul\, 1, .., n) = \\ &= 1 - p(esec\, in\, nodul\, 1) * p(esec\, in\, nodul\, 1) * ... * p(esec\, in\, nodul\, n) =\end{aligned}$$

$$= 1 - (1-p_1)(1-p_2)...(1-p_n))$$
$$\mathbb{P}(C \to B) = 1 - (1 - \mathbb{P}(C \to E, E \to B)) * (1 - \mathbb{P}(C \to F, F \to B)) =$$
$$= 1 - (1 - \mathbb{P}_{CE} * \mathbb{P}_{EB}) * (1 - \mathbb{P}_{CF} * \mathbb{P}_{FB})$$

5.3 Variabile aleatoare

Definitie. Fie $(\Omega, F, \mathbb{P})$ camp de probabilitate si $X : \Omega \to \mathbb{R}$ o functie reala. Spunem ca X este o variabila aleatoare daca multimea $\{\omega\epsilon\Omega | X(\omega) \leq x\}\epsilon F, \forall x\epsilon\mathbb{R}$.

5.3.1 Experiment

2 zaruri. Definim X=$\{suma\ punctelor\ de\ pe\ cele\ 2\ zaruri\}$

$3 \ \ 5 \to 8$

$X((3,5)) = 8$

X=$\{numarul\ de\ H\ din\ cele\ 2\ aruncari\}$

$\Omega = \{HH, HT, TH, TT\}$

$X(\Omega) = \{2, 1, 1, 0\}$

$X\epsilon\mathbb{R}$

Observatie.$\{X \leq x\} \subset \{\omega\epsilon\Omega | X(\omega) \leq x\}$

$\{X\epsilon A\} = \{\omega\epsilon\Omega | X(\omega)\epsilon A\} = X^{-1}(A)$

$X^{-1}(\{0\}) = \{TT\}$

$X^{-1}(\{1\}) = \{HH\}$

$X^{-1}(\{2\}) = \{HT, TH\}$

$\{X \leq x\}\epsilon F(= \mathbb{P}(\Omega))$

Daca $X < 0$, $\{X \leq x\} = \emptyset$

$X\epsilon[0, 1), \{X \leq x\} = \{TT\}$

$X\epsilon[1, 2), \{X \leq x\} = \{HT, TH\} \cup \{TT\}$

$X\epsilon[2, +\infty), \{X \leq x\} = \{HT, TH, TT, HH\} = \Omega$

5.3.2 Repartitia unei variabile aleatoare

Fie $(\Omega, F, \mathbb{P})$ camp de probabilitate si $X : \Omega \to \mathbb{R}$. Se numeste repartitia lui X probabilitatea pe $\mathbb{R}$ definite prin

$\mathbb{P}_x(A) = \mathbb{P}(X \leftarrow A) = \mathbb{P}(X^{-1}(A)) = (\mathbb{P} \circ X_{-1})(A), \forall A \, interval \, in \, \mathbb{R}$

Definitie. *Functia de repartitie. Fie* $(\Omega, F, \mathbb{P})$ *camp de probabilitate* $X : \Omega \to \mathbb{R}$
Definim functia de repartitie a lui X, $F : \mathbb{R} \to [0, 1]$ prin $F(x) = \mathbb{P}(X \leq x), \forall x \epsilon \mathbb{R}$

5.3.3 Experiment

Aruncam de 2 ori cu banul si X=numarul de H din cele doua aruncari
$F(x) = \mathbb{P}(X \leq x) = 0, x < 0$
$F(x) = \mathbb{P}(X \leq x) = \frac{1}{4}, x\epsilon[0, 1)$
$F(x) = \mathbb{P}(X \leq x) = \frac{3}{4}, x\epsilon[1, 2)$
$F(x) = \mathbb{P}(X \leq x) = 1, x\epsilon[2, +\infty)$

5.3.4 Proprietati functia de repartitie

a) F este crescatoare

$x < y \Longrightarrow F(x) \leq F(y)$

b) F este continua la dreapta

$\lim_{x \to x_0} F(x) = F(x_0)$

c)$\lim_{x \to \infty} F(x) = 0$

$\lim_{x \to \infty} F(x) = 1$

$\mathbb{P}(X = x) = \mathbb{P}(X \leq x) - \mathbb{P}(X < x) = F(x_0) - F(X-)$

6 Cursul 6

6.1 Variabile aleatoare

$X : \Omega \to \mathbb{R}$
$\{X \leq x\} \epsilon \, F, \; \forall x \epsilon \mathbb{R}$

$(\Omega, F, \mathbb{P}),\ X\ variabila\ aleatoara\ si\ \mathbb{P}_x(A) = \mathbb{P}(x\epsilon A),\ \forall A\epsilon\mathbb{R}\ interval$
$\{X\epsilon A\} = \{\omega\epsilon\mathbb{R}|x(\omega)\epsilon A\} = X^{-1}(A)$

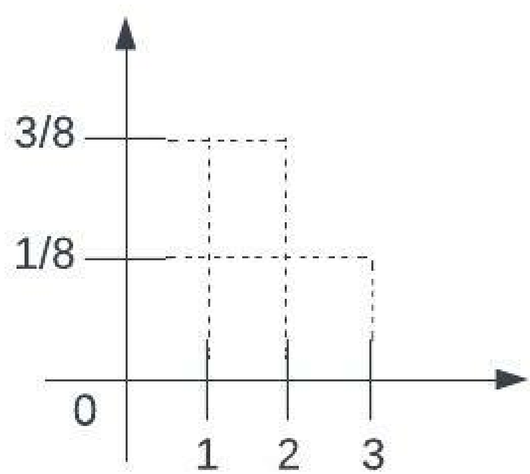
$\mathbb{P}_x(.) = (\mathbb{P} \circ X^{-1})(.)$

6.2 Functia de repartitie/cumulativa(CDF)

$F : \mathbb{R} \rightarrow [0, 1]$
$F(x) = \mathbb{P}_x((-\infty, x]) = \mathbb{P}(X \leq x),\ \forall x\epsilon\mathbb{R}$

6.2.1 Experiment

Aruncam de 3 ori cu banul.
X = numarul de H in cele 3 aruncari
Care este functia de repartitie a lui X? $\Omega = \{H, T\}^3,\ X\epsilon\{0, 1, 2, 3\}$
$\mathbb{P}(x = 0) = \mathbb{P}(TTT) = \frac{1}{8}$
$\mathbb{P}(x = 1) = \mathbb{P}(\{HTT, THT, TTH\}) = \frac{3}{8}$
$\mathbb{P}(x = 2) = \mathbb{P}(\{HHT, HHT, THH\}) = \frac{3}{8}$
$\mathbb{P}(x = 3) = \mathbb{P}(HHH) = \frac{1}{8}$



F(x)=?

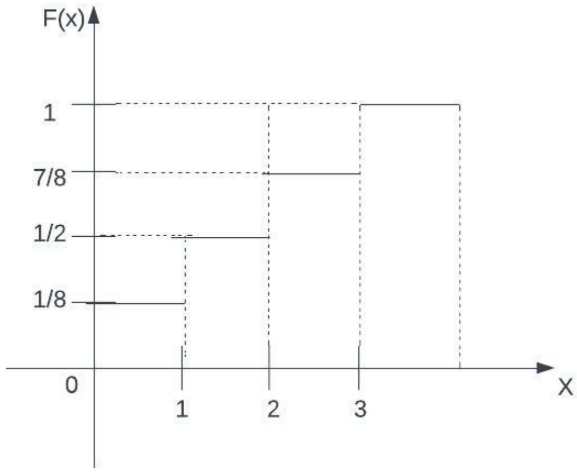
$F(x) = 0 = \mathbb{P}(\emptyset),\ x < 0$

$F(x) = \frac{1}{8},\ 0 \leq x < 1$

$F(x) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2},\ 1 \leq x < 2$

$F(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8},\ 2 \leq x < 3$

$F(x) = 1,\ 3 \leq x$



6.2.2 Proprietatile functiei de repartitie

a) F crescatoare: $\forall\, x < y \implies F(x) \leq F(y)$

b) F continua la dreapta: $\lim_{x \to x_0, x > x_0} F(x) = F(x_0)\, \forall\, x_0 \epsilon \mathbb{R}$

c) $\lim_{x \to -\infty} F(x) = 0$

$\lim_{x \to +\infty} F(x) = 1$

In plus,

d) $\mathbb{P}(X > x) = 1 - \mathbb{P}(X \leq x) = 1 - F(x)$

e) $\mathbb{P}(X < x_0) = \mathbb{P}(X \leq x_0) - \mathbb{P}(X = x_0) = \lim_{x \to x_0, x < x_0} F(x)$

f) $\mathbb{P}(X = x) = F(x) - \lim_{x \to x_0, x < x_0} F(x)$

6.3 Variabile aleatoare discrete

$X : \Omega \to \mathbb{R}\, variabila\, aleatoare\, X(\Omega) = multimea\, valorilor\, lui\, X$

$X(\Omega) \implies finita\, sau\, numarabila \longrightarrow X\, este\, variabila\, aleatoare\, discreta$

$\implies infinita\, sau\, numarabila \longrightarrow X\, este\, variabila\, continua$

X variabila aleatoare discreta, $X : \Omega \to \mathbb{R}$

$A \epsilon \mathbb{R}$

$\mathbb{P}(x \epsilon A) = ?$

$\Omega = \bigcup_{n \geq 1} X = x_n$

$\mathbb{P}(x \epsilon A) = \mathbb{P}(x \epsilon \bigcup_{x \epsilon A \cap X(\Omega)} \{x\}) = \sum_{x \epsilon A \cap X(\Omega)} \mathbb{P}(X = x)$

Definitie. Fie $(\Omega, F, \mathbb{P})$ un camp de probabilitate si $X : \Omega \to \mathbb{R}$ o variabila aleatoare distincta. Se numeste functie de masa asociata:

$$f(x) = \mathbb{P}(X = x),\ \forall x \epsilon X(\Omega),\ f : X(\Omega) \to [0, 1]$$

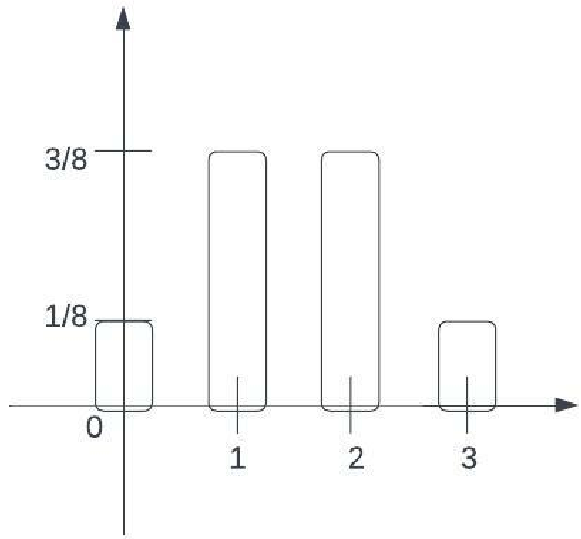
Observatie. Se mai foloseste si notatia p(x) sau $p_x(x)$.

6.3.1 Experiment

Aruncam de 3 ori cu banul , X = numarul de H din cele 3 aruncari. Determinati functia de masa a lui X.

$f(x) = \mathbb{P}(X > x)\, \forall\, x \epsilon \{0, 1, 2, 3\} = X(\Omega)$

$f(0) = \frac{1}{8},\ f(1) = \frac{3}{8},\ f(2) = \frac{3}{8},\ f(3) = \frac{1}{8}$



Observatie. $X \epsilon \{x_1, x_2, .., x_n\}$

$\mathbb{P}(X = x_i = p_i)$

$X \sim \binom{x_1\ x_2\ ..\ x_n}{p_1\ p_2\ ..\ p_n}$

6.3.2 Proprietatile functiei de masa

a) $f(x) = \mathbb{P}(X = x) \geq 0$ (*pozitiva*)

b) $\mathbb{P}(\Omega) = 1$

$\Omega = \bigcup_{x \epsilon X(\Omega)} \{X = x\} \Longrightarrow \mathbb{P}(\bigcup \{X = x\}) = 1 \Longrightarrow$

$\Longrightarrow \sum_{x \epsilon X(\Omega)} f(x) = 1$ (masa totala = 1)

Observatie.(Legatura dintre functia de masa si functia de repartitie)

$F(x) = \mathbb{P}(X \leq x) = \sum_{y \epsilon x, y \epsilon X(\Omega)} f(y)$

$f(x) = F(x) - F(x-)$

6.3.3 Exemple de variabile discrete

1) Variabila aleatoare X = c(constanta)

$f(x) = \mathbb{P}(X = x) = 1,\ x = c$

$\qquad\qquad = 0, x \neq c$

$F(x) = \mathbb{P}(X \leq x) = 0,\ x < c$

$\qquad\qquad = 1, x \geq c$

Daca x¡c $\implies \{X \leq x\} = \{\omega | X(\omega) \leq x\} = \{c \leq x\} = \emptyset$

2) Variabila aleatoare discrete Bernoulli

Avem un experiment si un eveniment A de interes. Presupunem ca $\mathbb{P}(A) = p\, \epsilon [0,1]$

$X : \Omega \to \mathbb{R}$

$X(\Omega) = 1,\ \omega \epsilon A$

$\qquad = 0,\ altfel$

$f(1) = \mathbb{P}(x = 1) = \mathbb{P}(A) = p$

$f(0) = \mathbb{P}(x = 0) = \mathbb{P}(A^c) = 1 - p$

F(x)=0, $x < 0$

=1-p, $0 \leq x < 1$

$= 1,\ x \geq 1$

$\underline{Variabila\, aleatoare\, indicator:}\ 1_A(\omega) = 1\ \ ,\omega \epsilon A$

$\qquad\qquad = 0\ \ ,\omega \notin A$

$\underline{Scrierea\, sub\, forma\, compacta\, a\, functiei\, de\, masa}$

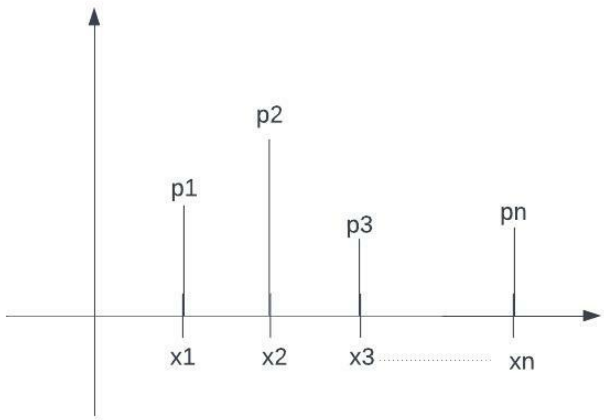
$$f(x) = p^x (1-p)^{1-x},\ x \in \{0, 1\}$$

3) $X : \Omega \to \mathbb{R}$

$X(\Omega) = \{x_1, x_2, .., x_n\}$ si sa presupunem ca $x_1 < x_2 < .. < x_n$

$\mathbb{P}(X = x_i) = p_i \in (0,1)\, , x \in \{0,1\}$

$\underline{Graficul\ functiei\ de\ masa}$



$$F(x) = \mathbb{P}(X \leq x) = \begin{cases} 0 & x \leq x_1 \\ p_1 & x_1 \leq x < x_2 \\ p_1 + p_2 & x_2 \leq x < x_3 \\ .... & .... \\ 1 & x \geq x_n \end{cases}$$

4) Variabile aleatoare de tip binomial

Presupunem ca avem un experiment aleator si A un eveniment de interes. Repetam experimentul de n ori si ne interesam la numarul de realizari ale evenimentului A.
X = numarul de realizari al evenimentului A in n repetari ale experimentului
$X \sim B(n, p)$ - variabila aleatoare repartizata binomial de parametru n si p
$X \in \{0, 1, 2, .., n\}$

$\underline{Functia\ de\ masa}$: $f(x) = \mathbb{P}(X = k) =?,\ k \in \{0, 1, .., n\}$

$$\boxed{\mathbb{P}(X = k) = C_n^k (1-p)^{n-k} * p^k}$$

Observatie. X $= y_1 + y_2 + .. + y_n$

$y_i \sim B(p)$

Experiment. Urna cu bile albe si negre: N bile, M negre.

Extragem n bile cu intoarcere.

X = numarul de bile negre dintre cele extrase

$X \sim B(n, \frac{M}{N})$

5) Variabila aleatoare repartizata hipergeometric

Avem o urna cu N bile albe si negre si M de culoare neagra. Extragem n bile fara intoarcere si ne intereseaza numarul de bile negre din cele n extrase.

X = numarul de bile negre din cele n extrase este reprezentat hipergeometric HG(n,N,M)

$X \sim HG(n, N, M),\ X \in \{0, 1, .., min\{M, n\}\}$

$\mathbb{P}(X = k) = \frac{\binom{M}{k} * \binom{N-M}{n-k}}{\binom{N}{n}}$

$\underline{Loto\ 6\ din\ 49:} \begin{cases} x_1, x_2, x_3, x_4, x_5, x_6 \\ 1\ , 17\ , 23\ , 41\ , 39\ , 5 \end{cases}$

Care este probabilitatea sa fi nimerit k=3 numere?

N=49

M=6

n=6

$\mathbb{P}(X = 3) = \frac{\binom{6}{3}\binom{43}{3}}{\binom{49}{6}}$

$\mathbb{P}(X \in \{3, 4, 5, 6\}) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) - \mathbb{P}(X = 2) \approx 0.18$

$\sum_{k=0}^{min(n,M)} \binom{M}{k}\binom{N-M}{n-k} = \binom{N}{n}$

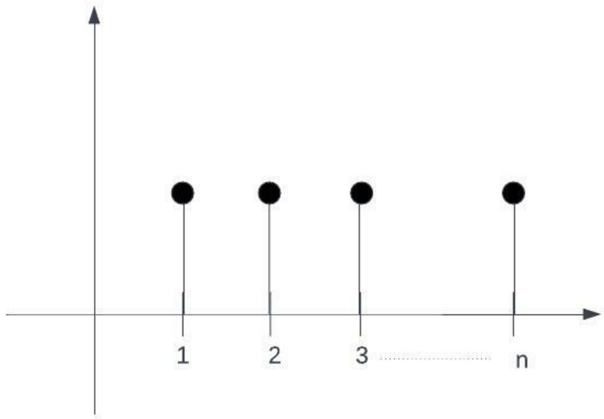
6) Uniforma pe $\{1, 2, .., n\}$ (Echirepartitia)

$X : \Omega \to \mathbb{R}$

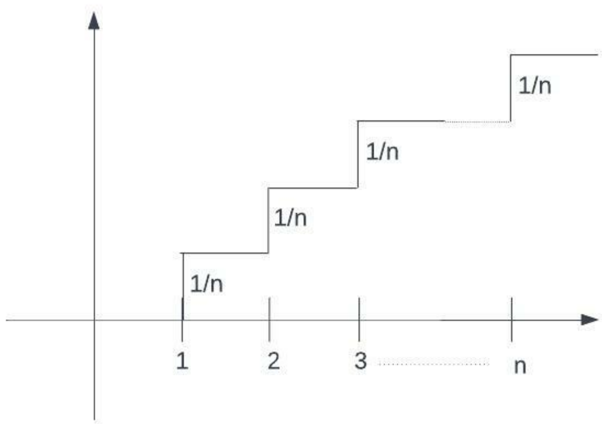
$X(\Omega) = \{1, 2, .., n\}$ (D finita)

$f(k) = \mathbb{P}(X = k) = \frac{1}{n} = \frac{1}{|D|}\ \forall k \in \{1, 2, .., n\}$

<u>*Functia de masa*</u>



<u>*Functia de repartitie*</u>



$\mathbb{P}(X \in A) = \frac{|A \cap D|}{|D|},\ \forall\ A \in \mathbb{R}$

7 Cursul 7

7.1 Experiment

O urna cu bile numerotate de la 1 la 100. Extragem 5 bile din urna (succesiv).
a) Care este repartitia variabilei aleatoare care ne da numarul bileleor ≥ 70?
b) Cum este repartitia variabilei aleatoare care ne da a 17-a extragere?
c) Care este probabilitatea ca numarul 79 sa fie extras cel putin o data?

$\underline{1.Extragere\, cu\, revenire}$
a) $X \sim B(5(numarul\ de\ extrageri), \frac{31}{100}(probabilitatea\ sa\ avem\ succes))$
b) $X_1, X_2, .., X_5 \in \{1, .., 100\}$
$X_1 \sim U(\{1, 2, .., 100\})$
$X_2 \sim U(\{1, 2, .., 100\})$
$c)\mathbb{P}(\{79\ sa\ fie\ extras\ cel\ putin\ o\ data\}) =$

$$= \mathbb{P}(\{X_1 = 79\} \cup \{X_2 = 79\} \cup \{X_3 = 79\} \cup \{X_4 = 79\} \cup \{X_5 = 79\}) =$$
$$= 1 - \mathbb{P}(\{X_1 \neq 79\} \cup \{X_2 \neq 79\} \cup \{X_3 \neq 79\} \cup \{X_4 \neq 79\} \cup \{X_5 \neq 79\}) =$$
$$= 1 - \mathbb{P}(\{X_1 \neq 79\}) * \mathbb{P}(\{X_2 \neq 79\}) * \mathbb{P}(\{X_3 \neq 79\}) * \mathbb{P}(\{X_4 \neq 79\}) * \mathbb{P}(\{X_5 \neq 79\}) =$$
$$= 1 - (\frac{99}{100})^5$$

$\underline{1.Extragere\, fara\, revenire}$
a) $Y \sim HG(5, 100, 31)$
b) $y_1, y_2, .., y_5$
$y_1 \sim U(\{1, 2, ..100\})$
$y_2 \sim U(\{1, 2, ..100\})$
$\mathbb{P}(y_2 = j) = \sum_{i=1}^{100} \mathbb{P}(y_2 = j | y_2 = i) * \mathbb{P}(y_2 = i)$
O partitie a lui $\Omega = B_1 \cap B_2 \cap .. \cap B_n$ (disjuncte 2 cate 2)
$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i) * \mathbb{P}(B_u)$
$\mathbb{P}(y_2 = j | y_2 = i) = \begin{cases} 0,\ j = i \\ \frac{1}{99},\ j \neq i \end{cases}$
$\mathbb{P}(y_2 = j) = \sum_{i=1}^{100} \mathbb{P}(y_2 = j | y_2 = i) * \mathbb{P}(y_1 = i) = 99 * \frac{1}{99} * \frac{1}{100} = \frac{1}{100}$
c) $\mathbb{P}(..) = \mathbb{P}(\{y_1 = 79\} \cup \{y_2 = 79\} \cup .. \cup \{y_5 = 79\}) = \sum_{i=1}^{5} \mathbb{P}(y_1 = 79) = \frac{5}{100}$

Repartitia geometrica si negativ binomiala

Aruncam cu o moneda in mod repetat , iar sansa de succes = p($\mathbb{P}\{H\} = p$)
X = variabila aleatoare care ne da numarul de aruncari pana obtinem pentru prima oara succes(H) incluzand primul succes
$X \in \{1, 2, 3, ..\}$
$TTH \implies x = 3$
$H \implies x = 1$
$\mathbb{P}(x = k) = (1 - p)^{k-1} * p\, , k \geq 1$
$X \sim G(p)$
$\sum_{k=1}^{\infty} (1-p)^{k-1} * p = p * \sum_{k=1}^{\infty} = \frac{p}{1-p} = \frac{p}{q} \to 1$
Daca $x \in (0, 1)$, $n \to \infty \implies \sum_{x \geq 0} x^k = \frac{1}{1-x}$

Definitie. Variabila aleatoare Z care ne da numarul de aruncari necesare pana pbtinem pentru a r-a oara succes de numeste Negativ Binomiala.

$Z \sim NB(r, p)$
$\{r, r + 1, ..\}$
$k \geq r$, $\mathbb{P}(z = k) = \binom{k-1}{r-1} * (1 - p)^(k - r) * p^r$

Variabila aleatoare de tip Poisson

Definitie. Spunem ca o variabila aleatoare este repartizata Poisson de parametru λ daca $x \in \mathbb{N}$ si $\mathbb{P}(x = k) = e^{-\lambda} * \frac{\lambda^k}{k!}$
Cand se foloseste?
$\sum_{k \geq 10} e^{-\lambda} * \frac{\lambda^k}{k!} = 1?$
$e^x = \sum_{k \geq 0} \frac{x^k}{k!}$

Functia de variabila aleatoare

$(\Omega, F, \mathbb{P})$ camp de probabilitate , X si g variabile aleatoare atunci $g \circ X$ este o variabila aleatoare.

Observatie. Daca X este discreta $\Longrightarrow g \circ X$ este o variabila aleatoare discreta.

7.2 Independenta

Doua variabile aleatoare X si Y sunt independente daca realizarea uneia nu influenteaza in niciun fel realizarea celeilalte.

Definitie. Fie $(\Omega, F, \mathbb{P})$ un camp de probabilitate si X si Y doua variabile aleatoare. Spunem ca X si Y sunt independente, $X \perp\!\!\!\perp Y$, daca evenimentele $\{X = x\}$ si $\{Y = y\}$ sunt independente $\forall x, y$

P. Fie X si Y doua variabile aleatoare (discrete). Atunci $X \perp\!\!\!\perp Y$ daca si numai daca

$$\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x) * \mathbb{P}(Y \leq y),\ \forall x, y \in \mathbb{R}$$

P. $X \perp\!\!\!\perp Y \Leftrightarrow \mathbb{P}(X \in A, Y \in B\mathbb{P}(X \in A) * \mathbb{P}(Y \in B)$,$\forall A, B \subseteq \mathbb{R}(interval)$

P. Daca X si Y variabile aleatoare astfel incat $X \perp\!\!\!\perp Y$ si g si h doua functii, atunci $g(x) \perp\!\!\!\perp h(y)$

Definitie. $X_1, X_2, .., X_n$ sunt independente daca:

$\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, .., X_n \leq x_n) = \mathbb{P}(X_1 \leq x_1) * \mathbb{P}(X_2 \leq x_2) * .. * \mathbb{P}(X_n \leq x_n),\ \forall x_1, x_2, .., x_n \in \mathbb{R}$

7.3 Media unei variabile aleatoare discrete

Repetam un experiment de N ori si ne interesam la valorile unei variabile aleatoare X de interes.

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$

1 1 1 3 3 5 8 8

$m = \frac{x_1+x_2+..+x_N}{N}$

Definitie. Fie X o variabila aleatoare discreta. Se numeste media lui X valoarea

$$\mathbb{E}[X] = \sum_x x f(x) = \sum_x x\mathbb{P}(X = x)$$

ori de cate ori $\sum_x |x|f(x) < \infty$
Daca $|x|f(x) = \infty$ atunci spunem ca X nu are medie.

8 Cursul 8

8.1 Media si momentele de ordin superior

Definitie. Fie $(\Omega, F, \mathbb{P})$ camp de probabilitate si $X : \Omega \to \mathbb{R}$ variabila aleatoare discreta, definim media variabilei aleatoare X:

$$\mathbb{E}[X] = \sum_x x * \mathbb{P}(X = x) = \sum_x x * f(x)$$

ori de cate ori $\sum |x| * f(x) < \infty$. In cazul in care seria este infinita atunci spunem ca variabila aleatoare X nu are medie.

8.1.1 Experiment 1

Aruncam cu un zar, X$\in \{1, 2, 3, 4, 5, 6\}$.
$\mathbb{P}(X = x) = \frac{1}{6}$
$\mathbb{E}[X] = \sum x * \mathbb{P}(X = x) = 1 * \frac{1}{6} + 2 * \frac{1}{6} + ... + 6 * \frac{1}{6} = 3.5$

8.1.2 Experimentul 2

$X \sim \binom{x_1\ x_2\ \dots\ x_n}{p_1\ p_2\ \dots\ p_n}$
$\mathbb{E}[X] = x_1 * p_1 + x_2 * p_2 + ... + x_n * p_n$

ex: $X \sim \binom{-1\ 0\ 1}{\frac{1}{2}\ \frac{1}{4}\ \frac{1}{4}}$
$\mathbb{E}[X] = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$

8.2 Interpretare fizica.Proprietati.

1) Daca X este constanta i.e. X=c $\Longrightarrow \mathbb{E}[X] = c$

2) Daca X > 0 atunci $\mathbb{E}[X] \geq 0$ (pozitivitate)

3) Daca X > Y atunci $\mathbb{E}[X] \geq \mathbb{E}[Y]$ (monotonie)

4) (Liniaritate) Daca X si Y variabila aleatoare discreta si a,b$\in \mathbb{R}$ atunci

$$\mathbb{E}[ax + by] = a * \mathbb{E}[x] + b * \mathbb{E}[y]$$

5) (Legatura dintre medie si probabilitati)

Fie $A \in F$ eveniment

$$\mathbb{1}_A = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

$\mathbb{P}(\mathbb{1}_A = 1) = \mathbb{P}(A)$

$\mathbb{1}_A \sim \binom{0 \quad\quad 1}{1-\mathbb{P}(A) \quad \mathbb{P}(A)}$

6) Fie X variabila aleatoare discreta si $g : \mathbb{R} \to \mathbb{R}$

Y=g(x). Atunci $\mathbb{E}[g(x)] = \sum_x g(x) * \mathbb{P}(X = x)$

7) Fie X,Y variabile aleatoare independente

$$\mathbb{E}[x * y] = \mathbb{E}[x] * \mathbb{E}[y]$$

Daca g si h sunt 2 functii atunci g(x) si h(y) sunt independente.

Definitie. Fie X o varianta aleatoare discreta. Numim momentul de ordin k ($k \geq 1$) $\mathbb{E}[(x - \mathbb{E}[x])^k]$.

Definitie. Varianta sau dispersia variantei aleatoare X este momentul centrat de ordin Ω si se noteaza cu Var(x)=$\mathbb{E}[(x - \mathbb{E}[x])^2]$.

Observatie. Arata gradul de impartire a obs fata de medie.

Experiment.

$X_1 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix}$

$X_2 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1/10 & 1/10 & 1/10 & 1/10 & 1/10 \end{pmatrix}$

$X_3 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}$

$X_4 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

$\mathbb{E}[x_1] = \mathbb{E}[x_2] = \mathbb{E}[x_3] = \mathbb{E}[x_4] = 3$

$Var(X_1) = \mathbb{E}[(x_1 - 3)^2] = \frac{(1-3)^2}{5} + \frac{(2-3)^2}{5} + \frac{(3-3)^2}{5} + \frac{(4-3)^2}{5} + \frac{(5-3)^2}{5} = 2$

$Var(X_2) = \mathbb{E}[(x_2 - 3)^2] = \frac{12}{10}$

$Var(X_3) = 4$

$Var(X_4) = 0$

8.2.1 Proprietati

1) Daca X constant $\Longrightarrow$ Var(x)=0

2) Var(x)$\geq$ 0 **!MEREU!**

3) Daca X variabila aleatoare si a$\in \mathbb{R}$ atunci Var(a+X)=Var(X)

4) Daca X variabila aleatoare si b$\in \mathbb{R}^*$ atunci Var(bX) = b^2 Var(X)

   Var(a+bX)=b^2 Var(X)

5) Var(x)=$\mathbb{E}[x^2] - \mathbb{E}[x]^2$

6) X si Y independente $\in$ Var(X+Y)=Var(X)+Var(Y)

Definitie. Fie X si Y doua variabile aleatoare . Se numeste covarianta lui X si Y:

$$\text{Cov(X,Y)}=\mathbb{E}[(x - \mathbb{E}[x])(y - \mathbb{E}[y])]$$

In general proprietatea 6 este Var(X+Y)=Var(X)+Var(Y)+2Cov(X,Y)

Definitie. Se numeste abatere standard: SD(x)=$\sqrt{Var(x)}$

σ = Varianta

σ^2 = Abatere standard

8.3 Exemple de calcul al mediei si variantei

1) X $\sim$ B(p) , X $\sim \binom{0 \quad\quad 1}{1-p \quad p}$

   $\mathbb{E}$[X]=p

   Var(X)=$\mathbb{E}[X^2] - \mathbb{E}[X]^2 = p - p^2 = p(1-p)$

2) $X \sim B(n, p)$

   $\mathbb{P}(X = k) = \binom{n}{k} * p^k * (1-p)^{n-k}$

   $\mathbb{E}[X] = \Sigma_{k=0}^{n} k * \mathbb{P}(X = k) = \Sigma_{k-0}^{n} k * \binom{n}{k} * p^k * (1-p)^{n-k}$

$\mathbb{E}[X] = \Sigma_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} * p^k * (1-p)^{n-k} = n * p$

X=$x_1 + x_2 + .. + x_n$

Var(X)=Var($X_1 + X_2 + .. + X_n$)=$\Sigma_{i=1}^{n} Var(X_i) = n * p * (1-p)$

3) Hipergiometrica HG(n,N,M)

$\mathbb{P}(X = k) = \frac{\binom{M}{k} * \binom{N-M}{n-k}}{\binom{N}{n}}$

ex: x_j la extragerea j avem bila neagra x_j=1, alba x_j=0

X=$x_1 + x_2 + .. + x_n$

extragere fara intoarcere

$\mathbb{E}[X] = \mathbb{E}[x_1] + .. + \mathbb{E}[x_n] = n * \frac{M}{N}$

$x_j \in \{0, 1\}$

$\mathbb{P}(x_j = 1) = \frac{M}{N}$

4)$X \sim Pois(\Omega)$

$\mathbb{P}(x = k) = e^{-\lambda} * \frac{\lambda^k}{k!}$

$\mathbb{E}[X] = \Sigma_{k=0}^{\infty} k * \mathbb{P}(X = k) = \Sigma k * e^{-\lambda} * \frac{\lambda^k}{k!} = \lambda$

$Var(X) = \mathbb{E}[x^2] - \mathbb{E}[x]^2$

$\mathbb{E}[x^2] = \Sigma_{k=0}^{\infty} k^2 * \mathbb{P}(x = k) = \Sigma k^2 * e^{-\lambda} * \frac{\lambda^k}{k!} = \Sigma_{k=1}^{\infty} k * e^{-\lambda} * \frac{\lambda^k}{(k-1)!}$ =$\lambda^2 + \lambda$

$\Longrightarrow$ Var(X)=λ

5) X $\sim$ Geom(p)

$X \sim \{1, 2, .., n\}$

$\mathbb{P}(x = k) = (1-p)^{k-1} * p$

$\mathbb{E}[X] = \frac{1}{p}$

8.4 Variabile aleatoare continue

Definitie. Fie (Ω,F,$\mathbb{P}$) camp de probabilitate si x:$\Omega \to \mathbb{R}$ o variabila aleatoare.

Variabila aleatoare X este continua (absolut continua) daca exista o functie f:$\mathbb{R} \to \mathbb{R}_+$ cu proprietatea ca $\mathbb{P}(x \in A) = \int_A f(x)dx, \forall A \subseteq \mathbb{R}$

Observatie. Daca A=(a,b), $\mathbb{P}(a < X < b) = \int_a^b$ f(x)dx.

Observatie. In definitia de mai sus f se numeste densitate de repartitie.

Propozitie. Daca f este densitate de repartitie, atunci

1) f ≥ 0

2) $\int_{\mathbb{R}} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 1$

Observatie. $\mathbb{P}$(x=a)=$\int_a^a f(x)dx = 0$

A=$\{a\}$

$\mathbb{P}(a < X < b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a \leq X \leq b) = \mathbb{P}(a < X \leq b)$

9 Curs 9

10 Curs 10

10.1 Repartitia normala

$\varphi = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$

$\phi(x) = \int_{-\infty}^{x} \varphi(t)dt$

$\varphi(x) = \varphi(-x)$ (simetric fata de origine)

φ densitate

1) $\varphi(x) \geq 0$

2) $\int_{-\infty}^{+\infty} \varphi(x)dx = 1$

$X \sim N(0,1)$ normala standard

$\mathbb{E}[X] = \int_{-\infty}^{+\infty} \varphi(x)dx = \int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = 0$ (functie impara)

$Var(x) = \mathbb{E}[X^2] - \underbrace{\mathbb{E}[X]^2}_{0}$

$\mathbb{E}[X^2] = \int x^2\varphi(x)dx = 1 \Longrightarrow Var(X) = 1$

Daca X$\sim N(0,1)$ atunci $\mathbb{E}[X] = 0$ si Var(X)=1

$\phi(x) = 1 - \phi(-x)$

Definitie. Spunem ca X $\sim$ N(μ, σ^2) daca admite densitate de repartitie:

$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, $x \in \mathbb{R}$

Propozitie. Daca X $\sim$ N(μ, σ^2) atunci $\exists$ Z$\sim$ N(0,1) astfel incat

X = μ + σZ

X$\sim$ $N(\underbrace{\mu}_{media}, \underbrace{\sigma^2}_{variatia})$

$\mathbb{E}[X] = \mathbb{E}[\mu + \sigma Z] = \mu + \sigma\mathbb{E}[Z] = \mu$

Z$\sim$N(2,1)

Var(X)=Var$(\mu + \sigma Z) = \sigma^2 Var(Z) = \sigma^2$

F(x)=$\mathbb{P}(X \leq x) = \mathbb{P}(\mu + \sigma Z \leq x) = \mathbb{P}(Z \leq \frac{x-\mu}{\sigma}) = \phi(\frac{x-\mu}{\sigma})$

f(x)=$\frac{d}{dx}F(x) = \frac{d}{dx}\phi(\frac{x-\mu}{\sigma}) = \varphi(\frac{x-\mu}{\sigma}) * \frac{1}{\sigma}$

$(f \circ g)' = f'(g) * g'$

Proprietatea 68-95-99,7%

Daca X$\sim$ $N(\mu, \sigma^2)$ atunci

$\mathbb{P}(|X - \mu| \leq \sigma) \simeq 68\%$

$\mathbb{P}(|X - \mu| \leq 2\sigma) \simeq 95\%$

$\mathbb{P}(|X - \mu| \leq 3\sigma) \simeq 99,7\%$

10.2 Repartitii comune, marginale si conditionate

X, Y doua variabile aleatoare $(\Omega, F, \mathbb{P})$

$\mathbb{P}((x, y) \in A \times B)$

$\mathbb{P}(X \in A)\, sau\, \mathbb{P}(Y \in B)$

$\mathbb{P}(X \in A|Y \in B)$

<u>*Cazul discret*</u>

Fie $(\Omega, F, \mathbb{P})$ camp de probabilitate si X:$\Omega \to \mathbb{R}, Y : \Omega \to \mathbb{R}$,

X(Ω) =$\{x_1, x_2, .., x_m\}$

Y(Ω) =$\{y_1, y_2, .., y_n\}$

Perechea (X,Y):$\Omega \to \mathbb{R}^2$

(X,Y)(Ω)=$\{(x_i, y_i)|i = \overline{1, m}, j = \overline{1, n}\} \to\ m * n\, valori$

<u>$Functia\, de\, masa\, a\, (X, Y)$</u>

$f_{XY}(x,y) = \mathbb{P}(X = x, Y = y), \forall x \in \{x_1, .., x_m\}, y \in \{y_1, .., y_n\}$

<u>*Proprietati*</u>

a) $f_{XY}(x,y) \geq 0, \forall x, y$

b) $\sum_{x \in X(\Omega)} \sum_{y \in Y(\Omega)} f_{XY}(x,y) = 1$

$A \subset \mathbb{R}$

$B \subset \mathbb{R}$

$\mathbb{P}((X,Y) \in A \times B) = \sum_{x \in X(\Omega)} \sum_{y \in Y(\Omega)} f_{XY}(x,y)$

$f_x(x) = \sum_y f_{xy}(x,y)$ (functia de masa a lui X, repartitia marginala a lui X)

$f_y(y) = \sum_x f_{xy}(x,y)$ (functia de masa a lui Y, repartitia marginala a lui Y)

Fie X o variabila aleatoare discreta si $A \in F, \mathbb{P}(A) > 0$

$\mathbb{P}(X = x|A) = \frac{\mathbb{P}(\{X=x\} \cap A)}{\mathbb{P}(A)}$

Daca A = $\{Y = y\}$ atunci $\mathbb{P}(X = x|Y = y) = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)} = \frac{f_{x|y}(x,y)}{f_y(y)}$ (functia de masa conditionata a lui X la Y)

10.3 Formula probabilitatii totale

$B, A_1, A_2, .., A_n \in F$

$A_1, A_2, .., A_n$ formeaza o partitie pe Ω

$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B|A_i) * \mathbb{P}(A_i)$

Daca B=x=x$\Longrightarrow \mathbb{P}(X = x) = \sum_{i=1}^{n} \mathbb{P}(X = x|A_i) * \mathbb{P}(A_i)$

$A_i = \{Y = y_i\} \Longrightarrow \mathbb{P}(X = x) = \sum_{i=1}^{n} \mathbb{P}(X = x|Y = y_i) * \mathbb{P}(Y = y_i)$

$f_x(x) = \sum_{i=1}^{n} f_{x|y}(x|y_i) f_y(y_i)$

$\mathbb{P}(A \cap B) = \mathbb{P}(A) * \mathbb{P}(B|A) = \mathbb{P}(B) * \mathbb{P}(A|B)$

A=$\{X = x\}$, $B = \{Y = y\}$

$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x) * \mathbb{P}(Y = y|X = x) = \mathbb{P}(Y = y) * \mathbb{P}(X = x|Y = y)$

$f_{x,y}(x,y) = f_x(x) * f_{y|x}(y|x) = f_y(y) * f_{x|y} f(x|y)$

10.4 Formula lui Bayes

$\mathbb{P}(X = x|Y = y) = \frac{\mathbb{P}(X=x,Y=y)}{\mathbb{P}(Y=y)} = \frac{\mathbb{P}(X=x)*\mathbb{P}(Y=y|X=x)}{\sum \mathbb{P}(X=x)\mathbb{P}(Y=y|X=x)}$
$f_{x|y}(x|y) = \frac{f_x(x)*f_{y|x}(y|x)}{\sum_{x!} f_x(x!)f_{y|x}(y|x!)}$

10.5 Media unei functii de v.a.

X$\sim X : \Omega \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$
$\mathbb{E}[g(x)] = \sum_x g(x)\mathbb{P}(X = x)$
$X, Y : \Omega \to \mathbb{R}, g : \mathbb{R}^2 \to \mathbb{R}$
$\mathbb{E}[g(x, y)] = \sum g(x, y) * \mathbb{P}(X = x, Y = y)$

Ex: X,Y
$\mathbb{E}[XY] = \sum_{x,y} xy\mathbb{P}(X = x, Y = y)$

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a) Cazul discret
X Y v.a discrete, $X \in \{x_1, .., x_n\}$ $\quad$ $Y \in \{y_1, .., y_n\}$
$f_{x,y}(x, y) = \mathbb{P}(X = x, Y = y)$ repartitie comuna
$f_x(x) = \mathbb{P}(X = x) = \sum_y f_x(x, y)$ repartitie marginala
$f_y(y) = \mathbb{P}(Y = y) = \sum_x f_{x,y}(x, y)$ repartitie marginala

12.1 Repartitia conditionata

$f_{x|y}(x|y) = \mathbb{P}(X = x|Y = y) = \frac{f_{x,y}(x,y)}{f_y(y)}$
$f_{y|x}(y|x) = \mathbb{P}(Y = y|X = x) = \frac{f_{x,y}(x,y)}{f_x(x)}$

12.2 Media conditionata

Daca X v.a discreta si $A \in Y, \mathbb{P}(A) \geq 0$ atunci am vazut ca $f_{X|A}(x) = \mathbb{P}(X = x|A)$

Media conditionata a lui X la A: $\mathbb{E}[X|A] = \sum_x x\mathbb{P}(X = x|A) = \sum_x xf_{X|A}(x)$

Daca g este o functie atunci g(x) este o v.a discreta si $\mathbb{E}[g(x)|A] = \sum_x g(x)f_{x|A}(x)$

Daca A = $\{Y = y\}$ atunci $\underline{\mathbb{E}[X|Y = y] = \sum_x xf_{x|y}(x|y)}$ (media conditionata a lui x la y)

Experiment

x/y	-1	0	2	$\sum$
1	1/18	3/18	2/18	6/18
2	2/18	0	3/18	5/18
3	0	4/18	3/18	7/18
$\sum$	3/18	7/18	8/18	1

$X \sim \begin{pmatrix} 1 & 2 & 3 \\ 6/18 & 5/18 & 7/18 \end{pmatrix}$

$Y \sim \begin{pmatrix} -1 & 0 & 2 \\ 3/18 & 7/18 & 8/18 \end{pmatrix}$

$X|Y{=}0 \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{3/18}{7/18} & 0 & \frac{4/18}{8/18} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3/7 & 0 & 4/7 \end{pmatrix}$

$\mathbb{E}[X|Y = 0] = 1 \cdot \frac{3}{7} + 2 \cdot 0 + 3 \cdot \frac{4}{7} = \frac{15}{7}$

X—Y = -1$\sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{1/18}{3/18} & \frac{2/18}{3/18} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$

$\mathbb{E}[X|Y = -1] = 1 \cdot \frac{1}{3} + 3 \cdot \frac{2}{3} + 3 \cdot 0 = \frac{5}{3}$

Analog X—Y = 2 $\sim \begin{pmatrix} 1 & 2 & 3 \\ 2/8 & 3/8 & 3/8 \end{pmatrix}$

$\mathbb{E}[X|Y = 2] = 1 \cdot \frac{2}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{3}{8} = \frac{17}{8}$

Daca X si Y sunt v.a discrete atunci $\mathbb{E}[X] = \sum \mathbb{E}[X|Y = y]y \circ \mathbb{P}(Y = y)$

$\mathbb{E}[X|Y = y] = \sum_x xf_{x|y}(x,y), f_{x|y}(x|y) = \frac{f_{x,y}(x,y)}{f_y y}$

$\sum_y \sum_x x f_{x|y}(x|y) \cdot f_y(y) = \sum_y \sum_x x f_{x,y}(x,y) = \underbrace{\sum_x x \sum_y f_{x,y}(x,y)}_{f_x(x)} = \mathbb{E}[X]$

Definitie Fie X si Y doua v.a discrete. Se numeste media conditionata a lui X la Y si se not $\mathbb{E}[X|Y]$, v.a de forma $h(y)$ pentru care $h(y) = \sum[X|Y=y], \forall y$

Experiment

Am vazut ca $\mathbb{E}[X|Y=-1] = \frac{5}{3}$

$\mathbb{E}[X|Y=0] = \frac{15}{7}$

$\mathbb{E}[X|Y=2] = \frac{17}{8}$

$\mathbb{E}[X|Y] = h(y)$ Ce valori ia aceasta v.a? $h(y) = \mathbb{E}[X|Y=y]$

$\mathbb{E}[X|Y] \sim \begin{pmatrix} 5/3 & 15/7 & 17/8 \\ \mathbb{P}(Y=-1) & \mathbb{P}(Y=0) & \mathbb{P}(Y=2) \end{pmatrix} = \begin{pmatrix} 5/3 & 15/7 & 17/8 \\ 3/18 & 7/18 & 8/18 \end{pmatrix}$

$\mathbb{E}[\mathbb{E}[X|Y]] = \frac{5}{3} \cdot \frac{3}{18} + \frac{15}{7} \cdot \frac{7}{18} + \frac{17}{8} \cdot \frac{8}{18} = \frac{5+15+17}{18} = \frac{37}{18} = \mathbb{E}[X]$

$\mathbb{E}[X] = \frac{6+10+21}{18} = \frac{37}{18}$

Media mediei conditionate este $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[x]$

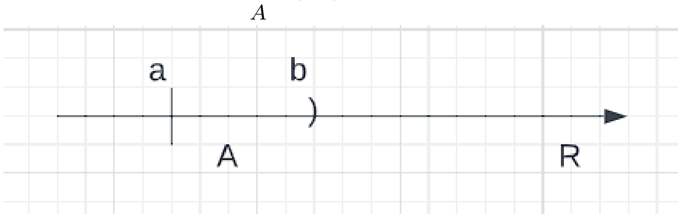
$\mathbb{E}[X|Y] \sim \begin{pmatrix} \mathbb{E}[X|Y=y] & ... & \mathbb{E}[X|Y=y_n] \\ \mathbb{P}(Y=y_1) & ... & \mathbb{P}(Y=y_n) \end{pmatrix}$

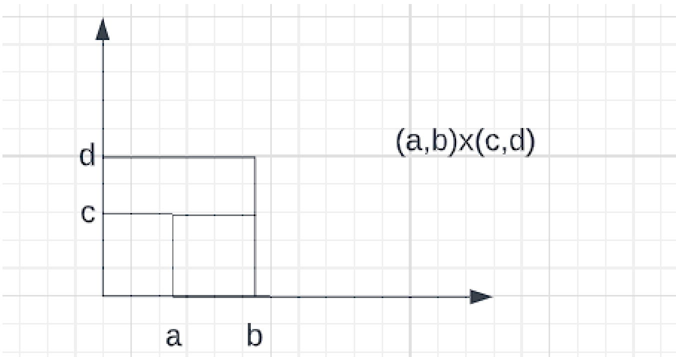
Observatie Var(X—A)=$\mathbb{E}[(X-\mathbb{E}[X|A])^2|A] = \mathbb{E}[x^2|A] - (\mathbb{E}[X|A])^2$

Var(X—Y=y)=$\mathbb{E}[X^2|Y=y] - \mathbb{E}[X|Y=y]^2$ b) Cazul v.a continue: repartitia comuna, repartitia marginala, repartitia conditionata

Definitie Fie $(\Omega, F, \mathbb{P})$ c.p si X,Y doua v.a const. Spunem ca vectorul (X,Y) formeaza o pereche de v.a continue daca exista o functie $f_{(x,y)}(x,y) \geq 0$ c prop:

$\mathbb{P}((x,y) \in A) = \int \int_A f_{(x,y)}(x,y) dxdy, \forall A \in \mathbb{R}^{\not{2}}$





Functia $f_{(x,y)}(x,y) : \mathbb{R}^2 -> \mathbb{R}$ se numeste densitatea comuna a (x,y)

Daca A=[a,b]x[c,d]

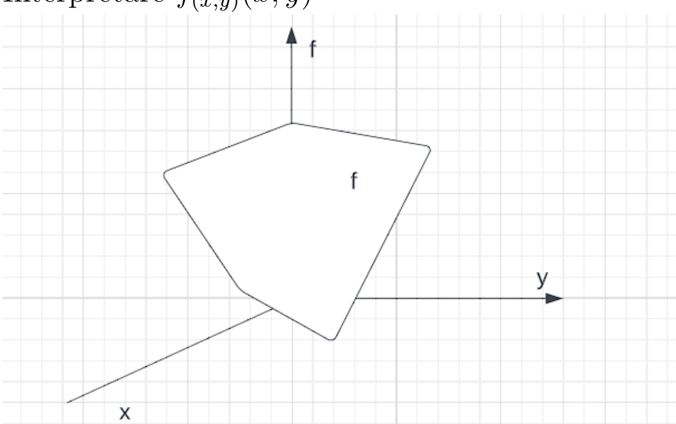
$\mathbb{P}((X,Y) \in [a,b]x[c,d]) = \mathbb{P}(a \leq x \leq b, c \leq y \leq d) = \int\limits_A \int f_(x,y)(x,y)dxdy = \int\limits_a^b \int\limits_c^d f_{(x,y)}(x,y)dydx$

Daca A=$\mathbb{R}^2$ atunci $\mathbb{P}((x,y) \in \mathbb{R}^2) = \int \int\limits_{\mathbb{R}^2} f_{(x,y)}(x,y)dxdy = 1$

$f_{(x,y)}$ este desitate $a) f_{(x,y)} \geq 0$

b)$\int \int f_{(x,y)}(x,y)dxdy = 1$

Interpretare $f_{(x,y)}(x,y)$



$\mathbb{P}(X \in (x, x+dx), y \in (y, y+dy)) = \int_x^{x+dx} \int_y^{y+dy} f_{(x,y)}(u,v)dvuv$

$dx, dy \to 0 f_{(x,y)}(x,y)dxdy$

$f_{(x,y)}(x,y) \sim \frac{\mathbb{P}X \in (x,x+dx), Y \in (y,y+dy)}{dxdy} \sim \frac{probabilitatea}{unitateadearie}$

Observatie Daca stim $f_{(x,y)}(x,y)$ atunci putem calcula price probabilitate de tipul $\mathbb{P}(X \in A, Y \in B)$

Vrem sa calculam $\mathbb{P}x \in A = \mathbb{P}(x \in A, y \in \mathbb{R}) = \int_A \int_{\mathbb{R}} f_{(x,y)}(x,y)dxdy$

Presupunem ca X si Y v.a continua cu densitatile f_x, respectiv f_y=$\mathbb{P}(X \in A) = \int_A f_x(x)$

Avem $\int_A f_x(x)dx = \int_A \int_{\mathbb{R}} f_{x,y}(x,y)dydx$

$f_x(x) = \int_{\mathbb{R}} f_{(x,y)}(x,y)dy$ desitatea marginala a lui X

Similar $f_y(y) = \int_{\mathbb{R}} f_{(x,y)}(x,y)dx$ desnitatea marginala a lui y

Cazul discret **Cazul continuu**

$f_{(x,y)}(x,y) = \mathbb{P}(X = x, Y = y)$ $f_{(x,y)}(x,y)$

Repartitie uniforma pe S$\subseteq \mathbb{R}^2$

Presupunem S$\subseteq \mathbb{R}^2 marginita$ (triunghi,dreptunghi,..)

$(X,y) \sim \mu(S) daca \exists f_{(x,y)}(x,y) \geq 0$

$f_{(x,y)}(x,y) = \begin{cases} c, (x,y) \in S \qquad 0, altfel \end{cases}$

Cum $f_{(x,y)}$ este densitate rezulta c ≥ 0 si

$\int\limits_{\mathbb{R}^2}\int f_{(x,y)x,y}(x,y)dxdy = 1 \Rightarrow \int\limits_{\mathbb{R}^2}\int c \cdot 1_s(x,y)dxdy = 1$

$c = \frac{1}{\int\int dxdy} = \frac{1}{Aria(A)}$

S=[a,b]×[c,d]

$(X,y) \sim \mu(S) daca \exists f_{(x,y)}(x,y) \geq 0$

$f_{(x,y)}(x,y) = \begin{cases} \frac{1}{\int\int_{\mathbb{R}}} \ (x,y) \in S \qquad 0, altfel \end{cases}$

Problema acului lui Buffon

O suprafata marcata cu linii paralele aflate la distanta d una fata de celelalte. Presupunem ca aruncam cu un ac de lungime l¡d. Care este probabilitatea ca acum sa intersecteze una dintre linii? $\Theta - unghiulascutitformatdeaccudreapta$

X - distanta de la mijlocul acului la cea mai apropriata dreapta

$(X,\Theta) \sim U(S), S = \{(X,\Theta)|0 \leq \Theta \leq \frac{u}{2}, 0 \leq x \leq d/2\}$

Conditia ca acul sa intersecteze o linie:

$x \leq \frac{l/2}{sin\Theta}$

Vrem sa calculam $\mathbb{P}(X \leq l/2 sin\Theta)$

Functia de repartitie(X,Y)

$\mathbb{F}(x,y)(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \int\limits_{-\infty}^{x}\int\limits_{-\infty}^{y} f_{(x,y)}(u,v)dvdu$

12.3 Repartitii conditionate

Fie $(\Omega, F, \mathbb{P})$ un c.p., X o v.a. continua si un A $\in F\, cu\, \mathbb{P}(A) \geq 0$
Definim densitatea conditionata a lui X la A, $f_{(x|A)}x$, functia $f_{x|A} \geq 0$ care verifica
$\mathbb{P}(x \in B|A) = S_B f_{x|A} x dx \forall B \subseteq \mathbb{R}$

Observatie

In locul lui A consideram ev $\{x \in A\}$ astefel incat $\mathbb{P}(X \in A) \geq 0$
$\mathbb{P}(x \in B|x \in A) = \frac{\mathbb{P}(x \in b, x \in A)}{\mathbb{P}x \in A} = \frac{\mathbb{P}(x \in A \cap B)}{\mathbb{P}(x \in A)}$

$\int\limits_{A \cap B} f dx = \int f \cdot 1_{A \cap B}(x) dx = \int\limits_{B} f(x) \cdot 1_A(x) dx$

12.4 Formula probabilitatii totale

$Fie A_1, A_2, .., A_n \in F$ care sa formeze o partitie pe $\Omega si B \in F atunci \mathbb{P} = \sum\limits_{i=1}^{n} \mathbb{P}(B|A_i) \cdot$
$\mathbb{P}A_i$

Daca $B = \{X \leq x\}$atunci X v.a cont fx densitate
$\mathbb{P}X \leq x = \sum\limits_{i=1}^{n} \mathbb{P}X \leq x|A_i \cdot \mathbb{P}A_i = \int\limits_{-\infty}^{x} f_x(t)dt = \int\limits_{-\infty}^{f_{x|A_i}} \cdot \mathbb{P}(A_i)$

Observatie

$A \in F, \mathbb{P}(A) \geq 0$ X va cont $f_x(x)$
$f_x x = f_{x|A} \mathbb{P}(A) + F_{x|A^C}(x) \cdot \mathbb{P}(A^C)$

Experiment:

Presupunem ca metroul circula la intervalul de 15 minute incepand la ora 5 am, Presupunem ca ajungem in statie in intervalul 7-7:30 in mod aleator (uniform pe acest interval). Ne propunem sa determinam rep. timpului de astepare pana la sosirea primului metrou.

Solutie: Fie Y- timpul de asteptare pana la sesiunea primului metrou. Vrem sa determinam densitatea a lui Y, f_y . Fie X timupul de sosire in statie $U[7^{10} - 7^{30}]$
$A = \{7^{10} \leq x \leq 7^{15}\}$– Urcam in metrou la 7:15
$B = \{7^{15} \leq x \leq 7^{30}\}$– Urcam in metrou la 7:30
$f_y(y) = f_{Y|A}(y) \cdot \mathbb{P}(A) + f_{y|B}(y) \cdot \mathbb{P}(B)$
$\mathbb{P}(B) = \frac{15}{20} = \frac{3}{4}$
$f_Y(y) = f_{y|A}(y) \cdot \frac{1}{4} + f_{y|B}(y) \cdot \frac{3}{4}$

$f_{Y|A}(y) = \frac{1}{5}, 0 \leq y \leq 5$
$f_{Y|B}(y) = \frac{1}{15}, 0 \leq y \leq 15$
$f_Y(y) = f_{Y|A}(y) * \mathbb{P}(A) + f_{Y|B}(Y) * \mathbb{P}(B) = \frac{1}{5} * 1_{[0,5]}(y) * \frac{1}{4} + \frac{1}{15} * 1_{[0,15]}(y) = \frac{3}{4}$
$\frac{1}{10}, 0 \leq y \leq 5$
$\frac{1}{20}, 5 \leq y \leq 15$

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13.1 Independeta v.a

$x \perp y$
$\mathbb{P}(x \in A, Y \in B) = \mathbb{P}(x \in A) \cdot \mathbb{P}(y \in B), \forall A, B \subseteq \mathbb{R}$
$A = (-\infty, x]$
$B = (-\infty, y]$
$\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x) * \mathbb{P}(Y \leq y) \forall x, y$
$\int\limits_{-\infty}^{x} \int\limits_{-\infty}^{y} f_{(x,y)}(u,v) dv du = \int\limits_{-\infty}^{x} f_x(u) du \int\limits_{-\infty}^{y} f_x(v) dv$
Derivam dupa y si x
$\frac{\partial^2}{\partial x \partial y} \int\limits_{-\infty}^{x} \int\limits_{-\infty}^{y} f_{(x,y)}(u,v) dv du = f_{(x,y)}(x,y) = f_x(x) f_y(y)$

P Fie X si Y v.a cu densitate fx si repartitia fy. Atunci x$\perp$ $f_{(x,y)}(x,y) = f_x(x) f_y(y)$

P Fie X si Y 2 va si g si h 2 fete.Daca $f_{(x|y)}(x|y) = g(x) \cdot h(y) \forall x, y \; atunci \; x \perp y$

P Daca X si Y sunt 2 v.a independente $\mathbb{E}[g(x) \cdot h(y)] = \mathbb{E}[g(x)] \cdot \mathbb{E}[h(y)]$
Formula lui Bayes
X Y v.a continua
$f_{x|y}(x,y) = f_{x|y}(x|y) f_y y = \frac{f_{x,y}(x,y)}{f_y(y)} = \frac{f_{y|x}(y|x) f_x(x)}{f_y(y)} = \frac{f_{y|x}(y|x) * f_x(x)}{\int f_{y|x}(y|x') * f_x x' * dx'}$
Experiment A Si B durata de viata pt telefonul din firma A este Exp(α_0), B Exp(α_1)
Presupunem ca primim un telefon de la A cu prob P0 si de la B cu P1 = 1-P0
Fie T durata de viata a telefonului primit
a) functia de repartitie si densitatea lui T

b) Probabilitatea ca telefonul sa fi provenit de la B stiind ca T=t

T v.a continua Fie I v.a 0, telefon produs de A; 1, telefon produs de B

$\mathbb{P}(I = 0) = p0$

$\mathbb{P}(I = 1) = p1 = 1 - p0$

T—I =0 $\sim Exp(\alpha_0)$

T—I =1 $\sim Exp(\alpha_1)$

$\mathbb{P}(T \leq t) = \mathbb{P}(T \leq t|I = 0) \cdot \mathbb{P}(I = 0) + \mathbb{P}T \leq t|I = 1 \cdot \mathbb{P}(I = 1)$

13.2 Media unei functii de v.a

X si Y doua v.a $f_{x,y}(x, y) si\ g : \mathbb{R}^2 \to \mathbb{R}$

$\mathbb{E}[g(x, y)] = \int\int g(x, y) \cdot f_{x,y}(x, y)dxdy$

In particular,

$\mathbb{E}[xy] = \int\int xyf_{x,y}(x, y)dxdy$

13.3 Media conditionata

X v.a continua si A un ev P(A)>0 $\mathbb{E}[X|A] = \int xf_{x|A}(x)dx$

Daca A=$\{Y = y\}$

$\mathbb{E}[X|Y = y] = \int xf_{x|y}(x, y)$dx

13.4 Formula probabilitatii totale

$f_x(x) = \sum \lim_{i=1}^{n} f_{x|A_i}(x) * \mathbb{P}(A_i)$

$\mathbb{E}[X] = \sum \lim_{i=1}^{n} \mathbb{E}[X|A] * \mathbb{P}(A_i)$

$\mathbb{E}[X] = \int \mathbb{E}[X|Y = y]f_y(y)dy$

<u>Definitie</u>

Fie g(y) =$\mathbb{E}[X|Y = y]$. Atunci v.a $\mathbb{E}[X|Y] = g(y)$

<u>Proprietate</u>

a) $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[x]$

$Var(X) = Var(\mathbb{E}[X|Y] + \mathbb{E}[Var(x|y)])$

N clienti

$X_1, X_2 \cdots$ sumele pe care le-au achitat

$T = X_1 + X_2 + \cdots X_\lambda$

$N(\omega_1) = 10$

$T(\omega_1) = X_1(\omega_1) + \cdots X_{10}(\omega_1)$

13.5 Covarianta si corelatie

Definitie. Fie X si Y doua v.a se numesc covariante dintre X si Y

$Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$

Convarianta este $\mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Definitie

Spunem ca X si Y sunt necorelate daca convarianta lor este 0 Daca $X \perp Y \Rightarrow$ X si Y sunt necorelate

Proprietati.// a) Cov(X,X) = Var(X)

b) Cov(X,a) =0 , a const

c) Cov(a+bx, y)=b Cov(X,Y)

d) Cov(X,Y) = Cov(Y,X)

e) Var(X+Y)=Var(X)+Var(Y)+2Cov(X,Y)

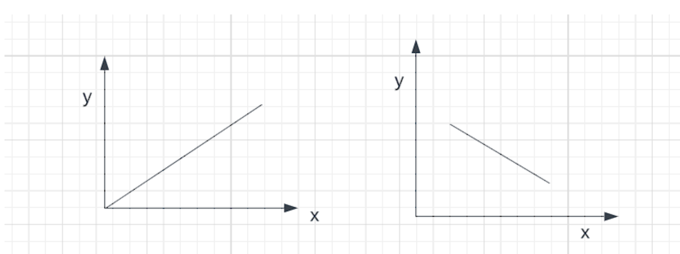
$Var(x_1 \cdots x_n) = \sum\limits_{i=1} Var(xi) + 2\sum\limits_{i<j} Cov(xi, xj)$

f) Cov(x+y,z)=Cov(x,z)+Cov(y,z)

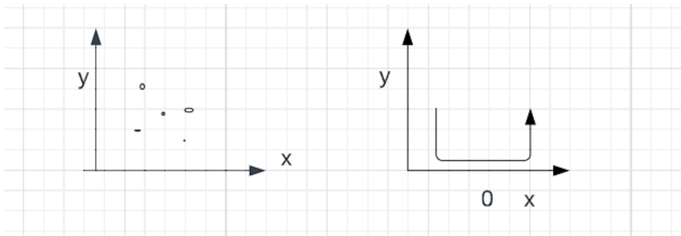
Definitie

Fie X si Y doua v.a si definim coeficientul de corelatie dintre X si Y

$\rho(X,Y) = \frac{Cov(X,Y)}{Var(X)\sqrt{Var(Y)}}$



$\rho > 0(stanga)\ \ rho < 0(dreapta)$



$\rho = 0(independent)$ $\rho = 0$

Proprietate

$\rho \in [-1, 1] Daca\ \rho = 1\ sau\ \rho = -1\ atunci$

$X = a + by\ (y = a + bx) as. \mathbb{P}(X = a + by) = 1$

13.6 Inegalitati si termeni limita

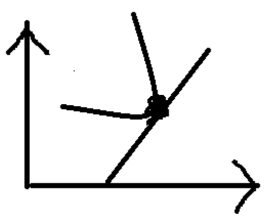
Inegalitatea Cauchy Schwartz

Fie X si Y v.a cu Var(x)$< \infty$, Atunci

$|\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[x^2]\mathbb{E}[y^2]}$

a) Functia convexa

Inegalitatea lui Jensen



Fie X v.a pozitiva. Atunci

$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$

14 Curs 14

14.1 Inegalitati

1) **Cauchy-Schwartz:** $\mathbb{E}[|XY|] \leq \sqrt{\mathbb{E}[X^2] * \mathbb{E}[Y^2]}$

2) **Jensen:**

φ convexa : $\mathbb{E}[\varphi(x)] \geq \varphi(\mathbb{E}[X])$

φ concava : $\mathbb{E}[\varphi(x)] \leq \varphi(\mathbb{E}[X])$

3) **Marcov :**

$X > 0,\ a > 0$

$\mathbb{P}(X > a) \leq \frac{\mathbb{E}[X]}{a}$

4) **Cebîșev:**

X v.a. $\mathbb{E}[X] = \mu,\ Var(x) = \sigma^2$

$\mathbb{P}((X-\mu) \geq a) \leq \frac{Var(X)}{a^2},\ a > 0$

5) **Chernoff:**

X v.a. , $a > 0,\ t > 0$ atunci $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[e^{tx}]}{e^{ta}}$

14.2 Teoreme limite

Definitie. Fie $(X_n)_{n\geq 1}$ un sir de v.a. si X o v.a peste $(\Omega, F, \mathbb{P})$.Spunem ca $(X_n)_n$ converge la X aproape sigur si notam $X_n \xrightarrow{a.s.} X$ daca :

$\mathbb{P}(lim_n X_n = X) = 1$

$A = \{\omega \in \Omega |\lim_n X_n(\omega) = X(\omega)\}$

Fie $(X_n)_n$ un sir de v.a. si X o v.a. def $(\Omega, F, \mathbb{P})$. Spunem ca X_n converge in probabilitate la X, si notam $X_n \xrightarrow{\mathbb{P}} X$ daca $\forall \epsilon > 0,\ \lim_n \mathbb{P}(|X_n - x| \geq \varepsilon) \geq 0$.

Pentru $\forall \epsilon > 0,\ \forall \delta > 0,\ \exists n_0 \in \mathbb{N}$ a.i. $n \geq n_0$

$\mathbb{P}(|X_n - x| \geq \underbrace{\varepsilon}_{acuratete}) \leq \underbrace{\delta}_{nivel\ de\ incredere}$

Definitie. Numim esantion de volum n din populatia Q , v.a. $x_1, x_2, .., x_n$ independente si identic repartizate (i.i.d)

Media esantionului : $\overline{X_n} = \frac{x_1+x_2+..+x_n}{n}$

Repartitia $x_1, x_2, .., x_n$ esantion de medie μ si varianta σ^2

$(\mathbb{E}[X_1] = \mu,\ Var(X_1) = \sigma^2)$

$\mathbb{E}[X_n] = \mathbb{E}[\frac{x_1+x_2+..+x_n}{n}] = \frac{1}{n}(\mathbb{E}[X_1] + \mathbb{E}[X_2] + .. + \mathbb{E}[X_n]) = \mu$

$Var(X_n) = Var(\frac{x_1+x_2+..+x_n}{n}) = \frac{1}{n}(Var(X_1) + .. + Var(X_n)) = \frac{\sigma^2}{n}$

14.3 Legea numerelor mari

Versiune slaba

Fie $(X_n)_n$ sir de v.a. i.i.d. cu $\mathbb{E}[X_1] = \mu$, $Var(X_1) = \sigma^2 < \infty$

Atunci $X_n \xrightarrow{\mathbb{P}} X$

Versiune tare

(X_n) v.a. i.i.d. $\mathbb{E}[|X_1|] < \infty$, $\mathbb{E}[X_1] = \mu$

Atunci $X_n \xrightarrow{a.s.} X$

<u>*Demonstratie*</u> Pentru $\varepsilon > 0$

$\mathbb{P}(|X_n - \mu| \geq \varepsilon) \xrightarrow{n\to\infty} X$

Inegalitatea lui Cebisev:

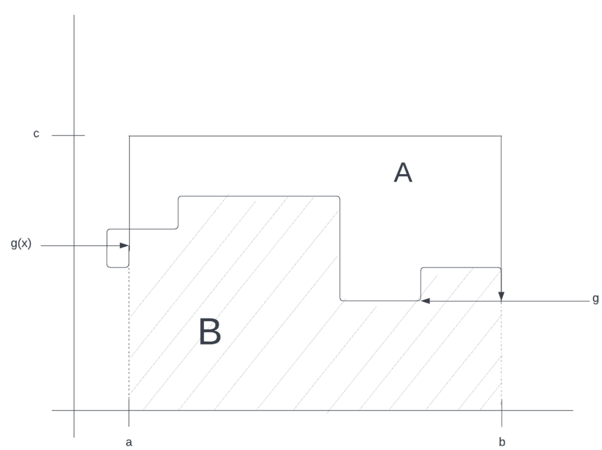
$\mathbb{P}(|X_n - \mu| \geq \varepsilon) \leq \frac{Var(X_1)}{\varepsilon^3} = \frac{\sigma^2}{\varepsilon^2} \xrightarrow{n\to\infty} 0$

Pentru un nivel de acuratete dat $\mathbb{P}(\mu - \varepsilon, \mu + \varepsilon) \to X_n \xrightarrow{n\to\infty} 1$

14.4 Integrarea Monte-Carlo

Presupunem ca avem o functie g si vrem sa calculam $\int_a^b g(x)dx$.

Presupunem ca pe $[a, b]$ avem $0 \leq g(x) \leq c$.



$A = [a, b] \times [0, c]$ -dreptunghi

$B = \{(x, y)|a \leq x \leq b, 0 \leq y \leq g(x)\}$

Generam puncte Unif(A)

$(x_1, y_1), .., (x_n, y_n) \sim U(A)$

$(x,y) \sim U(A)$ daca $f_{(x,y)}(x,y) = \{\frac{1}{Arie(A)}, (x,y) \in A\ \ , 0\, altfel\}$

Arie(a)=c(b-a)

$X \sim U([a,b])$ indep de $Y \sim U([a,c])$

$f_x(x) = \frac{1}{b-a} * \mathbb{1}_{[a,c]}(y)$

$f_{(x,y)}(x,y) = f_x(x) * f_y(y) = \frac{1}{a(b-a)} * \mathbb{1}_{[a,c]}(x,y)$

Fie $Z_i = \begin{cases} 1 & (x_i, y_i) \in B \\ 0 & altfel,\ Z_i \sim B(p) \end{cases}$

Din LNM $\Longrightarrow \sum_i = \frac{Z_1+Z_2+..+Z_n}{n} \xrightarrow{\mathbb{P}} p = \frac{Arie(B)}{c(b-a)} = \frac{\int_a^b g(x)dx}{c(b-a)}$

14.5 Teorema limita centrala(TLC)

Din LNM : $\overline{X_n} \xrightarrow{\mathbb{P}} \mathbb{E}[X_n]$

Fie $x_1, .., x_n$ i.i.d. $\mathbb{E}[x_1] = \mu,\ Var(x_1) = \sigma^2$

$\frac{X-\mathbb{E}[X]}{\sqrt{Var(X)}} = 1$

Z = $\frac{X-\mathbb{E}[X]}{\sqrt{Var(X)}} = 1$ - Z s.n. variabila normalizata

$Z_n = \frac{x_1+..x+n-\mathbb{E}[x_1+..+x_n]}{\sqrt{Var(x_1+..+x_n)}} = \frac{x_1+..+x_n-n*\mu}{\sqrt{n\sigma^2}}$ - variabila de scor

$Z_n = \sqrt{n}(\frac{\overline{X_n}-\mu}{\sigma})$

TLC

Fie $(X_n)_n$ sir de v.a. i.i.d. $\mathbb{E}[X_1] = \mu < \infty,\ Var(X_1) = \sigma^2 < \infty.$

Atunci $\lim_n \mathbb{P}(Z_n \leq x) = \phi(x)\ \forall x$

unde $Z_n = \sqrt{x} * \frac{\overline{X_n}-\mu}{\sigma} \Longrightarrow \phi(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} * e^{-t^2/2}dt$(functia de repartitie a N(0,1)

Observatie. $x_1, x_1, .., x_n$ i.i.d. , $\mathbb{E}[x_i] = \mu,\ Var(x_i) = \sigma^2,\ S_n = x_1 + x_2 + .. + x_n$

$\mathbb{P}(S_n \leq c) = \mathbb{P}(\frac{S_n-\mathbb{E}[S_n]}{Var(S_n)} \leq \frac{c-\mathbb{E}[S_n]}{\sqrt{Var(S_n)}}) = \mathbb{P}(Z_n \leq \frac{c*n\mu}{\sqrt{n\sigma^2}}) \overset{TLC}{\simeq} \phi(\frac{x-n\mu}{\sigma\sqrt{n}})$