1 Tensor-Product Basis

The tensor-product basis is

$$\phi_{(i,j)}(\alpha,\beta) = \tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta), \tag{1}$$

where $\tilde{\phi}_{(i)}(x)$ denotes the usual 1D GLL basis function at node $i \in (0,\ldots,n_p)$. For vector fields, the components of the covariant vector field are given by the tensor-product basis (1).

2 Hyperviscosity

2.1 Vector Hyperviscosity

Fourth-order vector hyperviscosity is implemented using a two stage procedure:

$$\mathbf{f} = -\mathcal{H}(1,1)\mathbf{u}^n, \tag{2}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t\mathcal{H}(\nu_d,\nu_v)\mathbf{f}. \tag{3}$$

The hyperdiffusion operator is defined implicitly via

$$\mathbf{f} = \mathcal{H}(\nu_d,\nu_v)\mathbf{u^n} \quad\Longleftrightarrow\quad \iint \mathbf{f}\cdot\boldsymbol{\phi} dA = \iint \nu_d(\nabla\cdot\boldsymbol{\phi})(\nabla\cdot\mathbf{u}^n) + \nu_v(\nabla\times\boldsymbol{\phi})^r(\nabla\times\mathbf{u}^n)_r dA, \tag{4}$$

where $dA = Jd\alpha d\beta$ and

$$(\nabla\cdot\boldsymbol{\phi}) = g^{pq}\nabla_p\phi_q = \frac{1}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\phi_\alpha + Jg^{\alpha\beta}\phi_\beta\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\phi_\alpha + Jg^{\beta\beta}\phi_\beta\right), \tag{5}$$

$$(\nabla\times\boldsymbol{\phi})^r = \epsilon^{rpq}\nabla_p\phi_q = \epsilon^{rpq}\left[\frac{\partial\phi_q}{\partial x^p} - \Gamma^k{}_{pq}\phi_k\right] = \frac{1}{J}\left[\frac{\partial\phi_\beta}{\partial\alpha} - \frac{\partial\phi_\alpha}{\partial\beta}\right]. \tag{6}$$

Here we assume that $(\nabla\cdot\mathbf{u})$ and $(\nabla\times\mathbf{u})_r$ (covariant radial component of curl) have already been computed.

2.1.1 Vector basis with zero β component

If $\phi_{(i,j)\alpha} = \tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)$ and $\phi_{(i,j)\beta} = 0$ then

$$\iint \mathbf{f}\cdot\phi dA = \iint f^\alpha\tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)dA = f^\alpha_{(i,j)}w_iw_jJ\Delta\alpha\Delta\beta \tag{7}$$

The divergent term is defined by

$$(\nabla\cdot\boldsymbol{\phi}_{(i,j)}) = \frac{1}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\phi_{(i,j)\alpha}\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\phi_{(i,j)\alpha}\right) \tag{8}$$

$$= \frac{\tilde{\phi}_{(j)}(\beta)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right) + \frac{\tilde{\phi}_{(i)}(\alpha)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right), \tag{9}$$

and

$$
\begin{aligned}
&\iint (\nabla\cdot\boldsymbol{\phi}_{(i,j)})(\nabla\cdot\mathbf{u})dA \\
&\quad= \Delta\alpha\Delta\beta\sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right)+\frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right)\right](\nabla\cdot\mathbf{u})Jw_mw_n && (10)\\
&\quad= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right)(\nabla\cdot\mathbf{u})w_m\bigg|_{\alpha=\alpha_m,\beta=\beta_j}\\
&\qquad+\Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right)(\nabla\cdot\mathbf{u})w_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} && (11)\\
&\quad= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}Jg^{\alpha\alpha}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}(\nabla\cdot\mathbf{u})w_m\bigg|_{\alpha=\alpha_m,\beta=\beta_j}\\
&\qquad+\Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}Jg^{\beta\alpha}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}(\nabla\cdot\mathbf{u})w_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} && (12)
\end{aligned}
$$

Further, the vortical term is defined by

$$(\nabla\times\boldsymbol{\phi}_{(i,j)})^r = -\frac{1}{J}\frac{\partial\phi_{(i,j)\alpha}}{\partial\beta} = -\frac{\tilde{\phi}_{(i)}}{J}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta} \tag{13}$$

and so

$$
\begin{aligned}
&\iint (\nabla\times\boldsymbol{\phi}_{(i,j)})^r(\nabla\times\mathbf{u})_r dA \\
&\quad= \Delta\alpha\Delta\beta\sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[-\frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}\right](\nabla\times\mathbf{u})_rJw_mw_n\bigg|_{\alpha=\alpha_m,\beta=\beta_n} && (14)\\
&\quad= -\Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}(\nabla\times\mathbf{u})_rw_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} && (15)
\end{aligned}
$$

Combining (7), (12) and (15) then gives

$$
\begin{aligned}
f^{\alpha}_{(i,j)} &= \frac{\nu_d}{J(\alpha_i,\beta_j)w_i}\sum_{m=0}^{n_p-1}Jg^{\alpha\alpha}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}(\nabla\cdot\mathbf{u})w_m\bigg|_{\alpha=\alpha_m,\beta=\beta_j}\\
&\quad+\frac{\nu_d}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1}Jg^{\beta\alpha}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}(\nabla\cdot\mathbf{u})w_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n}\\
&\quad-\frac{\nu_v}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}(\nabla\times\mathbf{u})_rw_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} && (16)
\end{aligned}
$$

2.1.2 Vector basis with zero α component

If $\phi_{(i,j)\alpha}=0$ and $\phi_{(i,j)\beta}=\tilde{\phi}_{(j)}(\alpha)\tilde{\phi}_{(j)}(\beta)$ then

$$\iint \mathbf{f}\cdot\phi dA = \iint f^{\beta}\tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)dA = f^{\beta}_{(i,j)}w_iw_jJ\Delta\alpha\Delta\beta \tag{17}$$

The divergent term is defined by

$$(\nabla\cdot\boldsymbol{\phi}_{(i,j)}) = \frac{1}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\phi_{(i,j)\beta}\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\phi_{(i,j)\beta}\right) \tag{18}$$

$$= \frac{\tilde{\phi}_{(j)}(\beta)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}\right) + \frac{\tilde{\phi}_{(i)}(\alpha)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}\right), \tag{19}$$

and

$$\iint(\nabla\cdot\boldsymbol{\phi}_{(i,j)})(\nabla\cdot\mathbf{u})dA$$

$$= \Delta\alpha\Delta\beta\sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}\right) + \frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}\right)\right](\nabla\cdot\mathbf{u})Jw_m w_n \tag{20}$$

$$= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}(\alpha)\right)(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}(\beta)\right)(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \tag{21}$$

$$= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}Jg^{\alpha\beta}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}Jg^{\beta\beta}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \tag{22}$$

Further, the vortical term is defined by

$$(\nabla\times\boldsymbol{\phi}_{(i,j)})^r = \frac{1}{J}\frac{\partial\phi_{(i,j)\beta}}{\partial\alpha} = \frac{\tilde{\phi}_{(j)}}{J}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha} \tag{23}$$

and so

$$\iint(\nabla\times\boldsymbol{\phi}_{(i,j)})^r(\nabla\times\mathbf{u})_r dA$$

$$= \Delta\alpha\Delta\beta\sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}\right](\nabla\times\mathbf{u})_r J w_m w_n\Big|_{\alpha=\alpha_m,\beta=\beta_n} \tag{24}$$

$$= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}(\nabla\times\mathbf{u})_r w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j} \tag{25}$$

Combining (17), (22) and (25) then gives

$$f^{\beta}_{(i,j)} = \frac{\nu_d}{J(\alpha_i,\beta_j)w_i}\sum_{m=0}^{n_p-1}Jg^{\alpha\beta}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \frac{\nu_d}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1}Jg^{\beta\beta}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n}$$

$$+ \frac{\nu_v}{J(\alpha_i,\beta_j)w_i}\sum_{m=0}^{n_p-1}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}(\nabla\times\mathbf{u})_r w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j} \tag{26}$$