1 Geometry

1.1 Terrain-Following Cartesian Geometry

For a given vertical coordinate transform $Z = Z(\alpha, \beta, \xi)$. For example, Gal-Chen and Somerville (1975) coordinates:

$$Z(\alpha, \beta, \xi) = \xi \left[z_{top} - z_s(\alpha, \beta) \right] + z_s(\alpha, \beta). \tag{1}$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial Z}{\partial \alpha} \\ \frac{\partial Z}{\partial \beta} \\ \frac{\partial Z}{\partial \xi} \end{pmatrix} \begin{pmatrix} \frac{\partial Z}{\partial \alpha} & \frac{\partial Z}{\partial \beta} & \frac{\partial Z}{\partial \xi} \end{pmatrix} \tag{2}$$

$$J = \left(\frac{\partial Z}{\partial \xi} \right) \tag{3}$$

$$g^{ij} = \begin{pmatrix} 1 & 0 & -\left(\frac{\partial Z}{\partial \xi}\right)^{-1} \left(\frac{\partial Z}{\partial \alpha}\right) \\ 0 & 1 & -\left(\frac{\partial Z}{\partial \xi}\right)^{-1} \left(\frac{\partial Z}{\partial \beta}\right) \\ -\left(\frac{\partial Z}{\partial \xi}\right)^{-1} \left(\frac{\partial Z}{\partial \alpha}\right) & -\left(\frac{\partial Z}{\partial \xi}\right)^{-1} \left(\frac{\partial Z}{\partial \beta}\right) & \left(\frac{\partial Z}{\partial \xi}\right)^{-2} \left[1 + \left(\frac{\partial Z}{\partial \alpha}\right)^2 + \left(\frac{\partial Z}{\partial \beta}\right)^2 \right] \end{pmatrix} \tag{4}$$

$$\Gamma^{\alpha}{}_{ij} = \mathbf{0}, \qquad \Gamma^{\beta}{}_{ij} = \mathbf{0}, \qquad \Gamma^{\xi}{}_{ij} = \left(\frac{\partial Z}{\partial \xi} \right)^{-1} \begin{pmatrix} \frac{\partial^2 Z}{\partial \alpha^2} & \frac{\partial^2 Z}{\partial \alpha \partial \beta} & \frac{\partial^2 Z}{\partial \alpha \partial \xi} \\ \frac{\partial^2 Z}{\partial \alpha \partial \beta} & \frac{\partial^2 Z}{\partial \beta^2} & \frac{\partial^2 Z}{\partial \xi \partial \beta} \\ \frac{\partial^2 Z}{\partial \xi \partial \alpha} & \frac{\partial^2 Z}{\partial \xi \partial \beta} & \frac{\partial^2 Z}{\partial \xi^2} \end{pmatrix} \tag{5}$$

1.2 Terrain-Following Cubed-Sphere Geometry

For a given vertical coordinate transform $r = R(\alpha, \beta, \xi)$. For example, Gal-Chen and Somerville (1975) coordinates:

$$R(\alpha, \beta, \xi) = \xi \left[z_{top} - z_s(\alpha, \beta) \right] + a + z_s(\alpha, \beta). \tag{6}$$

$$g_{ij} = \frac{a^2(1+X^2)(1+Y^2)}{\delta^4} \begin{pmatrix} 1+X^2 & -XY & 0 \\ -XY & 1+Y^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial R}{\partial \alpha} \\ \frac{\partial R}{\partial \beta} \\ \frac{\partial R}{\partial \xi} \end{pmatrix} \begin{pmatrix} \frac{\partial R}{\partial \alpha} & \frac{\partial R}{\partial \beta} & \frac{\partial R}{\partial \xi} \end{pmatrix} \tag{7}$$

$$J = \frac{1}{\delta^3} \left(\frac{\partial R}{\partial \xi} \right) a^2 (1+X^2)(1+Y^2) \tag{8}$$

$$g^{ij} = \frac{\delta^2}{a^2(1+X^2)(1+Y^2)} \begin{pmatrix} 1+Y^2 & XY & 0 \\ XY & 1+X^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \tilde{g}^{ij}, \tag{9}$$

$$\tilde{g}^{\alpha\alpha} = 0 \tag{10}$$
$$\tilde{g}^{\alpha\beta} = 0 \tag{11}$$
$$\tilde{g}^{\beta\beta} = 0 \tag{12}$$
$$\tilde{g}^{\alpha\xi} = -\frac{\delta^2}{a^2(1+X^2)(1+Y^2)}\left(\frac{\partial R}{\partial \xi}\right)^{-1}\left[(1+Y^2)\left(\frac{\partial R}{\partial \alpha}\right) + XY\left(\frac{\partial R}{\partial \beta}\right)\right] \tag{13}$$
$$\tilde{g}^{\beta\xi} = -\frac{\delta^2}{a^2(1+X^2)(1+Y^2)}\left(\frac{\partial R}{\partial \xi}\right)^{-1}\left[XY\left(\frac{\partial R}{\partial \alpha}\right) + (1+X^2)\left(\frac{\partial R}{\partial \beta}\right)\right] \tag{14}$$
$$\tilde{g}^{\xi\xi} = \left(\frac{\partial R}{\partial \xi}\right)^{-2} + \frac{\delta^2}{a^2(1+X^2)(1+Y^2)}\left(\frac{\partial R}{\partial \xi}\right)^{-2}\left[(1+Y^2)\left(\frac{\partial R}{\partial \alpha}\right)^2 + 2XY\left(\frac{\partial R}{\partial \alpha}\right)\left(\frac{\partial R}{\partial \beta}\right) + (1+X^2)\left(\frac{\partial R}{\partial \beta}\right)^2\right] \tag{15}$$

$$\Gamma^{\alpha}{}_{ij} = \begin{pmatrix} \frac{2XY^2}{\delta^2} & -\frac{Y(1+Y^2)}{\delta^2} & 0 \\ -\frac{Y(1+Y^2)}{\delta^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^{\beta}{}_{ij} = \begin{pmatrix} 0 & -\frac{X(1+X^2)}{\delta^2} & 0 \\ -\frac{X(1+X^2)}{\delta^2} & \frac{2X^2Y}{\delta^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

$$\Gamma^{\xi}{}_{ij} = \left(\frac{\partial R}{\partial \xi}\right)^{-1} \begin{pmatrix} -\frac{2XY^2}{\delta^2}\left(\frac{\partial R}{\partial \alpha}\right) + \left(\frac{\partial^2 R}{\partial \alpha^2}\right) & \frac{Y(1+Y^2)}{\delta^2}\left(\frac{\partial R}{\partial \alpha}\right) + \frac{X(1+X^2)}{\delta^2}\left(\frac{\partial R}{\partial \beta}\right) + \left(\frac{\partial^2 R}{\partial \alpha \partial \beta}\right) & \left(\frac{\partial^2 R}{\partial \alpha \partial \xi}\right) \\ \dots & -\frac{2X^2Y}{\delta^2}\left(\frac{\partial R}{\partial \beta}\right) + \left(\frac{\partial^2 R}{\partial \beta^2}\right) & \left(\frac{\partial^2 R}{\partial \beta \partial \xi}\right) \\ \dots & \dots & \left(\frac{\partial^2 R}{\partial \xi^2}\right) \end{pmatrix} \tag{17}$$

2 Hydrodynamics

The system of equations describing the hydrodynamic system in arbitrary geometry is as follows:

$$\frac{\partial u^\alpha}{\partial t} + u^\alpha \nabla_\alpha u^\alpha + u^\beta \nabla_\beta u^\alpha + u^\xi \nabla_\xi u^\alpha + \frac{1}{\rho}\nabla^\alpha p + f g^{\alpha j}\epsilon_{j\xi k}u^k = 0 \tag{18}$$
$$\frac{\partial u^\beta}{\partial t} + u^\alpha \nabla_\alpha u^\beta + u^\beta \nabla_\beta u^\beta + u^\xi \nabla_\xi u^\beta + \frac{1}{\rho}\nabla^\beta p + f g^{\beta j}\epsilon_{j\xi k}u^k = 0 \tag{19}$$
$$\frac{\partial \theta}{\partial t} + u^\alpha \frac{\partial \theta}{\partial \alpha} + u^\beta \frac{\partial \theta}{\partial \beta} = -u^\xi \frac{\partial \theta}{\partial \xi} \tag{20}$$
$$\frac{\partial u^\xi}{\partial t} + u^\alpha \nabla_\alpha u^\xi + u^\beta \nabla_\beta u^\xi + u^\xi \nabla_\xi u^\xi = -\frac{1}{\rho}\nabla^\xi p \tag{21}$$
$$\frac{\partial \rho}{\partial t} + \frac{1}{J}\frac{\partial}{\partial \alpha}(J\rho u^\alpha) + \frac{1}{J}\frac{\partial}{\partial \beta}(J\rho u^\beta) = -\frac{1}{J}\frac{\partial}{\partial \xi}(J\rho u^\xi) \tag{22}$$

Advection and geometry:

$$u^\alpha \nabla_\alpha u^k = u^\alpha \left[\frac{\partial u^k}{\partial \alpha} + \Gamma^k{}_{\alpha\alpha} u^\alpha + \Gamma^k{}_{\alpha\beta} u^\beta + \Gamma^k{}_{\alpha\xi} u^\xi \right] \tag{23}$$

$$u^\beta \nabla_\beta u^k = u^\beta \left[\frac{\partial u^k}{\partial \beta} + \Gamma^k{}_{\beta\alpha} u^\alpha + \Gamma^k{}_{\beta\beta} u^\beta + \Gamma^k{}_{\beta\xi} u^\xi \right] \tag{24}$$

$$u^\xi \nabla_\xi u^k = u^\xi \left[\frac{\partial u^k}{\partial \xi} + \Gamma^k{}_{\xi\alpha} u^\alpha + \Gamma^k{}_{\xi\beta} u^\beta + \Gamma^k{}_{\xi\xi} u^\xi \right] \tag{25}$$

Pressure gradient:

$$\nabla^k p = g^{k\alpha} \frac{\partial p}{\partial \alpha} + g^{k\beta} \frac{\partial p}{\partial \beta} + g^{k\xi} \frac{\partial p}{\partial \xi} \tag{26}$$

Coriolis Force:

$$f g^{\alpha j} \epsilon_{j\xi k} u^k = -f g^{\alpha\alpha} J u^\beta + f g^{\alpha\beta} J u^\alpha \tag{27}$$

$$f g^{\beta j} \epsilon_{j\xi k} u^k = -f g^{\beta\alpha} J u^\beta + f g^{\beta\beta} J u^\alpha \tag{28}$$

2.1 Vertical-orthogonal coordinates

Vertical-orthogonal coordinates α, β, ξ with corresponding terrain-following coordinates $\tilde{\alpha}, \tilde{\beta}, \tilde{\xi}$ with

$$\alpha = \tilde{\alpha}, \qquad \beta = \tilde{\beta}, \qquad \xi = \xi_s(\alpha, \beta) + (1 - \xi_s(\alpha, \beta))\tilde{\xi}, \tag{29}$$

$$\tilde{\alpha} = \alpha, \qquad \tilde{\beta} = \beta, \qquad \tilde{\xi} = \frac{\xi - \xi_s(\alpha, \beta)}{1 - \xi_s(\alpha, \beta)}. \tag{30}$$

$$\frac{\partial}{\partial \xi} = \frac{\partial \tilde{\xi}}{\partial \xi} \frac{\partial}{\partial \tilde{\xi}} = \frac{1}{(1 - \xi_s(\alpha, \beta))} \frac{\partial}{\partial \tilde{\xi}}, \tag{31}$$

$$\frac{\partial}{\partial \alpha} = \frac{\partial}{\partial \tilde{\alpha}} + \frac{\partial \tilde{\xi}}{\partial \alpha} \frac{\partial}{\partial \tilde{\xi}} = \frac{\partial}{\partial \tilde{\alpha}} - \frac{(1 - \xi)}{(1 - \xi_s(\alpha, \beta))^2} \left(\frac{\partial \xi_s}{\partial \alpha} \right) \frac{\partial}{\partial \tilde{\xi}}, \tag{32}$$

$$\frac{\partial}{\partial \beta} = \frac{\partial}{\partial \tilde{\beta}} + \frac{\partial \tilde{\xi}}{\partial \beta} \frac{\partial}{\partial \tilde{\xi}} = \frac{\partial}{\partial \tilde{\beta}} - \frac{(1 - \xi)}{(1 - \xi_s(\alpha, \beta))^2} \left(\frac{\partial \xi_s}{\partial \beta} \right) \frac{\partial}{\partial \tilde{\xi}}, \tag{33}$$

Define:

$$A_\alpha = \frac{\partial \tilde{\xi}}{\partial \alpha}, \qquad A_\beta = \frac{\partial \tilde{\xi}}{\partial \beta}, \qquad A_\xi = \frac{\partial \tilde{\xi}}{\partial \xi} \tag{34}$$

Hydrodynamics (assuming A_α, A_β and A_ξ are not functions of $\tilde{\xi}$):

$$\begin{aligned}
&\frac{\partial u^\alpha}{\partial t} + u^\alpha \frac{\partial u^\alpha}{\partial \tilde{\alpha}} + u^\beta \frac{\partial u^\alpha}{\partial \tilde{\beta}} + \Gamma^\alpha{}_{\alpha\alpha} u^\alpha u^\alpha + 2\ \Gamma^\alpha{}_{\alpha\beta} u^\alpha u^\beta + \Gamma^\alpha{}_{\beta\beta} u^\beta u^\beta \\
&\quad + \frac{1}{\rho}\left[g^{\alpha\alpha} \frac{\partial p}{\partial \tilde{\alpha}} + g^{\alpha\beta} \frac{\partial p}{\partial \tilde{\beta}} \right] + fJ\left[-g^{\alpha\alpha} u^\beta + g^{\alpha\beta} u^\alpha \right] \\
&\quad + \left[A_\alpha u^\alpha + A_\beta u^\beta + A_\xi u^\xi \right] \frac{\partial u^\alpha}{\partial \tilde{\xi}} + \frac{1}{\rho}\left[g^{\alpha\alpha} A_\alpha + g^{\alpha\beta} A_\beta \right] \frac{\partial p}{\partial \tilde{\xi}} = 0, \qquad (35) \\
&\frac{\partial u^\beta}{\partial t} + u^\alpha \frac{\partial u^\beta}{\partial \tilde{\alpha}} + u^\beta \frac{\partial u^\beta}{\partial \tilde{\beta}} + \Gamma^\beta{}_{\alpha\alpha} u^\alpha u^\alpha + 2\ \Gamma^\beta{}_{\alpha\beta} u^\alpha u^\beta + \Gamma^\beta{}_{\beta\beta} u^\beta u^\beta \\
&\quad + \frac{1}{\rho}\left[g^{\beta\alpha} \frac{\partial p}{\partial \tilde{\alpha}} + g^{\beta\beta} \frac{\partial p}{\partial \tilde{\beta}} \right] + fJ\left[-g^{\beta\alpha} u^\beta + g^{\beta\beta} u^\alpha \right] \\
&\quad + \left[A_\alpha u^\alpha + A_\beta u^\beta + A_\xi u^\xi \right] \frac{\partial u^\beta}{\partial \tilde{\xi}} + \frac{1}{\rho}\left[g^{\beta\alpha} A_\alpha + g^{\beta\beta} A_\beta \right] \frac{\partial p}{\partial \tilde{\xi}} = 0, \qquad (36) \\
&\frac{\partial \theta}{\partial t} + u^\alpha \frac{\partial \theta}{\partial \tilde{\alpha}} + u^\beta \frac{\partial \theta}{\partial \tilde{\beta}} = -\left[A_\alpha u^\alpha + A_\beta u^\beta + A_\xi u^\xi \right] \frac{\partial \theta}{\partial \tilde{\xi}}, \qquad (37) \\
&\frac{\partial u^\xi}{\partial t} + u^\alpha \frac{\partial u^\xi}{\partial \tilde{\alpha}} + u^\beta \frac{\partial u^\xi}{\partial \tilde{\beta}} + \left[A_\alpha u^\alpha + A_\beta u^\beta + A_\xi u^\xi \right] \frac{\partial u^\xi}{\partial \tilde{\xi}} = -\frac{1}{\rho} A_\xi \frac{\partial p}{\partial \tilde{\xi}}, \qquad (38) \\
&\frac{\partial \rho}{\partial t} + \frac{1}{J}\frac{\partial}{\partial \tilde{\alpha}}\left(J\rho u^\alpha \right) + \frac{1}{J}\frac{\partial}{\partial \tilde{\beta}}\left(J\rho u^\beta \right) = -\frac{1}{J}\frac{\partial}{\partial \tilde{\xi}}\left[J\rho\left(A_\alpha u^\alpha + A_\beta u^\beta + A_\xi u^\xi \right) \right]. \qquad (39)
\end{aligned}$$

3 Horizontal Discretization

3.1 Tensor Product Basis

The tensor-product basis is

$$\phi_{(i,j)}(\alpha, \beta) = \tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta), \qquad (40)$$

where $\tilde{\phi}_{(i)}(x)$ denotes the usual 1D GLL basis function at node $i \in (0, \ldots, n_p)$. For vector fields, the components of the covariant vector field are given by the tensor-product basis (40).

3.2 Spectral Element (Scalar Variational Form)

Consider a scalar conservation law of the form

$$\frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{F} = S, \qquad (41)$$

with $\mathbf{F} = F^\alpha \mathbf{g}_\alpha + F^\beta \mathbf{g}_\beta$. Multiplying by the tensor-product basis $\phi_{(p,q)}(\alpha, \beta)$ and integrating over the whole domain yields

$$\iint \frac{\partial \psi}{\partial t} \phi_{(i,j)} dA = -\iint (\nabla \cdot \mathbf{F}) \phi_{(i,j)} dA + \iint S \phi_{(i,j)} dA, \qquad (42)$$

where $dA = J d\alpha d\beta$. Applying integration by parts and using periodicity of the domain then leads to the weak form

$$\iint \frac{\partial \psi}{\partial t} \phi_{(i,j)} dA = \iint \nabla \phi_{(i,j)} \cdot \mathbf{F} dA + \iint S \phi_{(i,j)} dA. \qquad (43)$$

Further expanding ψ in terms of the horizontal basis functions as

$$\psi(t, \alpha, \beta) = \sum_{(s,t)} \psi_{(s,t)}(t) \phi_{(s,t)}(\alpha, \beta) \qquad (44)$$

leads to

$$\sum_{(s,t)} \frac{\partial \psi_{(s,t)}}{\partial t} \iint \phi_{(s,t)}\phi_{(i,j)} dA = \iint \nabla \phi_{(i,j)} \cdot \mathbf{F} dA + \iint S\phi_{(i,j)} dA. \tag{45}$$

Here the vertical dependence of ψ is implicit. For simplicity we now restrict our domain to a single spectral element (since the DSS procedure will later be used to account for inter-element exchange). Approximate integration is now applied,

$$\iint f(\alpha,\beta) dA = \Delta\alpha\Delta\beta \sum_{p=0}^{n_p-1}\sum_{q=0}^{n_p-1} f(\alpha_p,\beta_q) w_p w_q, \tag{46}$$

where w_i are the nodal weights of the GLL nodes on the reference element $[0,1]$. Consequently,

$$\iint \frac{\partial \psi}{\partial t}\phi_{(i,j)} dA = \Delta\alpha\Delta\beta \frac{\partial \psi_{(i,j)}}{\partial t} w_i w_j J(\alpha_i,\beta_j), \tag{47}$$

$$\iint \nabla\phi_{(i,j)} \cdot \mathbf{F} dA = \Delta\alpha\Delta\beta w_j \sum_{p=0}^{n_p-1} \frac{d\tilde{\phi}_{(i)}}{d\alpha} F^{\alpha} w_p J \bigg|_{\alpha=\alpha_p,\beta=\beta_j} + \Delta\alpha\Delta\beta w_i \sum_{q=0}^{n_p-1} \frac{d\tilde{\phi}_{(j)}}{d\beta} F^{\beta} w_q J \bigg|_{\alpha=\alpha_i,\beta=\beta_q}, \tag{48}$$

$$\iint S\phi_{(i,j)} dA = \Delta\alpha\Delta\beta S(\alpha_i,\beta_j) w_i w_j J(\alpha_i,\beta_j). \tag{49}$$

Substituting (47)-(49) into (42) gives the spectral element semi-discretization

$$\begin{aligned}\frac{\partial \psi_{(i,j)}}{\partial t} &= \frac{1}{w_i J(\alpha_i,\beta_j)} \sum_{p=0}^{n_p-1} \frac{d\tilde{\phi}_{(i)}}{d\alpha} F^{\alpha} w_p J \bigg|_{\alpha=\alpha_p,\beta=\beta_j} \\ &\quad + \frac{1}{w_j J(\alpha_i,\beta_j)} \sum_{q=0}^{n_p-1} \frac{d\tilde{\phi}_{(j)}}{d\beta} F^{\beta} w_q J \bigg|_{\alpha=\alpha_i,\beta=\beta_q} + S(\alpha_i,\beta_j)\end{aligned} \tag{50}$$

3.3 Spectral Element (Differential Form)

The spectral element method can also be derived in differential form by noting that basis functions can be interpreted as components of an interpolating polynomial. For an arbitrary function $f(\alpha,\beta)$ defined on element nodes we have

$$\frac{\partial f}{\partial \alpha}\bigg|_{\alpha=\alpha_i,\beta=\beta_j} = \sum_{p=0}^{n_p-1} \frac{d\tilde{\phi}_{(p)}}{d\alpha}\bigg|_{\alpha=\alpha_i} f(\alpha_p,\beta_j), \tag{51}$$

$$\frac{\partial f}{\partial \beta}\bigg|_{\alpha=\alpha_i,\beta=\beta_j} = \sum_{q=0}^{n_p-1} \frac{d\tilde{\phi}_{(q)}}{d\beta}\bigg|_{\beta=\beta_j} f(\alpha_i,\beta_q). \tag{52}$$

Second derivatives are defined as follows:

$$\frac{\partial^2 f}{\partial \alpha^2}\bigg|_{\alpha=\alpha_i,\beta=\beta_j} = \sum_{p=0}^{n_p-1} \frac{d\tilde{\phi}_{(p)}}{d\alpha}\bigg|_{\alpha=\alpha_i} \frac{\partial f}{\partial \alpha}(\alpha_p,\beta_j) \tag{53}$$

$$= \sum_{p=0}^{n_p-1}\sum_{q=0}^{n_p-1} \frac{d\tilde{\phi}_{(p)}}{d\alpha}\bigg|_{\alpha=\alpha_i} \frac{d\tilde{\phi}_{(q)}}{d\alpha}\bigg|_{\alpha=\alpha_p} f(\alpha_q,\beta_j) \tag{54}$$

3.4 Flux Reconstruction / Discontinuous Galerkin

4 Vertical Discretization

The following operators will be applied in the context of the vertical discretization:

Operator	Description
$\mathcal{I}_e^n$	Interpolate from nodes to edges
$\mathcal{I}_n^e$	Interpolate from edges to nodes
$\mathcal{D}_e^n$	Differentiate from nodes to edges
$\mathcal{D}_n^e$	Differentiate from edges to nodes
$\mathcal{D}_n^n$	Differentiate from nodes to nodes
$\mathcal{D}_e^e$	Differentiate from edges to edges

Exner function:

$$\Pi(\rho, \theta) = c_p \left(\frac{R \rho \theta}{p_0} \right)^{R/c_v} \tag{55}$$

Vertical discretizations:

Prognostic Variable	Operator	Choice of Staggering DG $(\rho_n \theta_n u_n^\xi)$	$(\rho_n \theta_n, u_e^\xi)$	$(\rho_n, u_e^\xi \theta_e)$
	Π_n	$\Pi_n(\rho_n, \theta_n)$	$\Pi_n(\rho_n, \theta_n)$	$\Pi_n(\rho_n, \mathcal{I}_n^e \theta_e)$
θ	$u^\xi \dfrac{\partial \theta}{\partial \xi}$	$(u_n^\xi) \mathcal{D}_n^n \theta_n$	$(\mathcal{I}_n^e u_e^\xi)(\mathcal{D}_n^n \theta)$	$(u_e^\xi)(\mathcal{D}_e^e \theta_e)$
u^ξ	$\theta \dfrac{\partial \Pi}{\partial \xi}$	$\theta_n \mathcal{D}_n^n \Pi_n$	$(\mathcal{I}_n^e \theta_n)(\mathcal{D}_e^n \Pi_n)$	$\theta_e (\mathcal{D}_e^n \Pi_n)$
ρ	$\dfrac{1}{J} \dfrac{\partial}{\partial \xi} (J \rho u^\xi)$	$\dfrac{1}{J_n} \mathcal{D}_n^n (J_n \rho_n u_n^\xi)$	$\dfrac{1}{J_n} \mathcal{D}_n^e \left[J_e (\mathcal{I}_e^n \rho_n) u_e^\xi \right]$	$\dfrac{1}{J_n} \mathcal{D}_n^e \left[J_e (\mathcal{I}_e^n \rho_n) u_e^\xi \right]$

5 Hyperviscosity

Hyperviscosity is formulated in variational form.

5.1 Fourth-Order Scalar Hyperviscosity

Fourth-order scalar hyperviscosity is implemented using a two stage procedure:

$$f = \mathcal{H}(1) \psi^n, \tag{56}$$

$$\psi^{n+1} = \psi^n - \Delta t \mathcal{H}(\nu) f. \tag{57}$$

The hyperdiffusion operator is defined implicitly via

$$f = \mathcal{H}(\nu) \psi \quad \Longleftrightarrow \quad \iint f \phi_{(i,j)} dA = \nu \iint \nabla \phi_{(i,j)} \cdot \nabla \psi dA, \tag{58}$$

where $dA = Jd\alpha d\beta$. Here

$$\iint f\phi_{(i,j)}dA = \iint f\tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)dA = f_{(i,j)}w_iw_jJ\Delta\alpha\Delta\beta, \tag{59}$$

and

$$\iint \nabla\phi_{(i,j)} \cdot \nabla\psi dA = \iint g^{pq}\nabla_p\phi\nabla_q\psi dA, \tag{60}$$

$$= \iint \frac{\partial\phi_{(i,j)}}{\partial\alpha}\left[g^{\alpha\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\alpha\beta}\frac{\partial\psi}{\partial\beta}\right] + \frac{\partial\phi_{(i,j)}}{\partial\beta}\left[g^{\beta\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\beta\beta}\frac{\partial\psi}{\partial\beta}\right]dA, \tag{61}$$

$$= \Delta\alpha\Delta\beta\sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\tilde{\phi}_{(j)}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}\left[g^{\alpha\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\alpha\beta}\frac{\partial\psi}{\partial\beta}\right]Jw_mw_n\bigg|_{\alpha=\alpha_m,\beta=\beta_n}$$

$$+ \Delta\alpha\Delta\beta\sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\tilde{\phi}_{(i)}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}\left[g^{\beta\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\beta\beta}\frac{\partial\psi}{\partial\beta}\right]Jw_mw_n\bigg|_{\alpha=\alpha_m,\beta=\beta_n} \tag{62}$$

$$= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}\left[g^{\alpha\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\alpha\beta}\frac{\partial\psi}{\partial\beta}\right]Jw_m\bigg|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}\left[g^{\beta\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\beta\beta}\frac{\partial\psi}{\partial\beta}\right]Jw_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} \tag{63}$$

In combination,

$$f_{(i,j)} = \frac{1}{w_iJ(\alpha_i,\beta_j)}\sum_{m=0}^{n_p-1}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}\left[g^{\alpha\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\alpha\beta}\frac{\partial\psi}{\partial\beta}\right]Jw_m\bigg|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \frac{1}{w_jJ(\alpha_i,\beta_j)}\sum_{n=0}^{n_p-1}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}\left[g^{\beta\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\beta\beta}\frac{\partial\psi}{\partial\beta}\right]Jw_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} \tag{64}$$

The derivatives of the scalar field ψ are obtained in the usual manner,

$$\frac{\partial\psi}{\partial\alpha}\bigg|_{\alpha=\alpha_i,\beta=\beta_j} = \sum_{m=0}^{n_p-1}\psi_{(m,j)}\frac{\partial\tilde{\phi}_{(m)}}{\partial\alpha}\bigg|_{\alpha=\alpha_m,\beta=\beta_j} \tag{65}$$

$$\frac{\partial\psi}{\partial\beta}\bigg|_{\alpha=\alpha_i,\beta=\beta_j} = \sum_{n=0}^{n_p-1}\psi_{(i,n)}\frac{\partial\tilde{\phi}_{(n)}}{\partial\beta}\bigg|_{\alpha=\alpha_i,\beta=\beta_n} \tag{66}$$

$$\tag{67}$$

5.2 Fourth-Order Vector Hyperviscosity

Fourth-order vector hyperviscosity is implemented using a two stage procedure:

$$\mathbf{f} = \mathcal{H}(1,1)\mathbf{u}^n, \tag{68}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \Delta t\mathcal{H}(\nu_d,\nu_v)\mathbf{f}. \tag{69}$$

The hyperdiffusion operator is defined implicitly via

$$\mathbf{f} = \mathcal{H}(\nu_d,\nu_v)\mathbf{u^n} \quad \Longleftrightarrow \quad \iint \mathbf{f}\cdot\boldsymbol{\phi}dA = \iint \nu_d(\nabla\cdot\boldsymbol{\phi})(\nabla\cdot\mathbf{u}^n) + \nu_v(\nabla\times\boldsymbol{\phi})^r(\nabla\times\mathbf{u}^n)_r dA, \tag{70}$$

where $dA = Jd\alpha d\beta$ and

$$(\nabla\cdot\boldsymbol{\phi}) = g^{pq}\nabla_p\phi_q = \frac{1}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\phi_\alpha + Jg^{\alpha\beta}\phi_\beta\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\phi_\alpha + Jg^{\beta\beta}\phi_\beta\right), \tag{71}$$

$$(\nabla\times\boldsymbol{\phi})^r = \epsilon^{rpq}\nabla_p\phi_q = \epsilon^{rpq}\left[\frac{\partial\phi_q}{\partial x^p} - \Gamma^k{}_{pq}\phi_k\right] = \frac{1}{J}\left[\frac{\partial\phi_\beta}{\partial\alpha} - \frac{\partial\phi_\alpha}{\partial\beta}\right]. \tag{72}$$

Here we assume that $(\nabla \cdot \mathbf{u})$ and $(\nabla \times \mathbf{u})_r$ (covariant radial component of curl) have already been computed.

5.2.1 Vector basis with zero β component

If $\phi_{(i,j)\alpha} = \tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)$ and $\phi_{(i,j)\beta} = 0$ then

$$\iint \mathbf{f} \cdot \phi dA = \iint f^{\alpha}\tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)dA = f^{\alpha}_{(i,j)}w_i w_j J\Delta\alpha\Delta\beta \tag{73}$$

The divergent term is defined by

$$(\nabla \cdot \boldsymbol{\phi}_{(i,j)}) = \frac{1}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\phi_{(i,j)\alpha}\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\phi_{(i,j)\alpha}\right) \tag{74}$$

$$= \frac{\tilde{\phi}_{(j)}(\beta)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right) + \frac{\tilde{\phi}_{(i)}(\alpha)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right), \tag{75}$$

and

$$\iint (\nabla \cdot \boldsymbol{\phi}_{(i,j)})(\nabla \cdot \mathbf{u})dA$$

$$= \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right) + \frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right)\right](\nabla \cdot \mathbf{u})Jw_m w_n \tag{76}$$

$$= \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right)(\nabla \cdot \mathbf{u})w_m\bigg|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right)(\nabla \cdot \mathbf{u})w_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} \tag{77}$$

$$= \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1} Jg^{\alpha\alpha}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla \cdot \mathbf{u})w_m\bigg|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1} Jg^{\beta\alpha}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla \cdot \mathbf{u})w_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} \tag{78}$$

Further, the vortical term is defined by

$$(\nabla \times \boldsymbol{\phi}_{(i,j)})^r = -\frac{1}{J}\frac{\partial\phi_{(i,j)\alpha}}{\partial\beta} = -\frac{\tilde{\phi}_{(i)}}{J}\frac{d\tilde{\phi}_{(j)}}{d\beta} \tag{79}$$

and so

$$\iint (\nabla \times \boldsymbol{\phi}_{(i,j)})^r(\nabla \times \mathbf{u})_r dA$$

$$= \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[-\frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{d\tilde{\phi}_{(j)}}{d\beta}\right](\nabla \times \mathbf{u})_r J w_m w_n\bigg|_{\alpha=\alpha_m,\beta=\beta_n} \tag{80}$$

$$= -\Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla \times \mathbf{u})_r w_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} \tag{81}$$

Combining (73), (78) and (81) then gives

$$f^{\alpha}_{(i,j)} = \frac{\nu_d}{J(\alpha_i,\beta_j)w_i}\sum_{m=0}^{n_p-1} Jg^{\alpha\alpha}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j} + \frac{\nu_d}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1} Jg^{\beta\alpha}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} - \frac{\nu_v}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1} \frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\times\mathbf{u})_r w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \tag{82}$$

5.2.2 Vector basis with zero α component

If $\phi_{(i,j)\alpha} = 0$ and $\phi_{(i,j)\beta} = \tilde{\phi}_{(j)}(\alpha)\tilde{\phi}_{(j)}(\beta)$ then

$$\iint \mathbf{f}\cdot\phi dA = \iint f^{\beta}\tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)dA = f^{\beta}_{(i,j)}w_iw_jJ\Delta\alpha\Delta\beta \tag{83}$$

The divergent term is defined by

$$(\nabla\cdot\boldsymbol{\phi}_{(i,j)}) = \frac{1}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\phi_{(i,j)\beta}\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\phi_{(i,j)\beta}\right) \tag{84}$$

$$= \frac{\tilde{\phi}_{(j)}(\beta)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}\right) + \frac{\tilde{\phi}_{(i)}(\alpha)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}\right), \tag{85}$$

and

$$\iint(\nabla\cdot\boldsymbol{\phi}_{(i,j)})(\nabla\cdot\mathbf{u})dA$$

$$= \Delta\alpha\Delta\beta\sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}\right) + \frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}\right)\right](\nabla\cdot\mathbf{u})Jw_mw_n \tag{86}$$

$$= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}(\alpha)\right)(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j} + \Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}(\beta)\right)(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \tag{87}$$

$$= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}Jg^{\alpha\beta}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j} + \Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}Jg^{\beta\beta}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \tag{88}$$

Further, the vortical term is defined by

$$(\nabla\times\boldsymbol{\phi}_{(i,j)})^r = \frac{1}{J}\frac{\partial\phi_{(i,j)\beta}}{\partial\alpha} = \frac{\tilde{\phi}_{(j)}}{J}\frac{d\tilde{\phi}_{(i)}}{d\alpha} \tag{89}$$

and so

$$\begin{aligned}
&\iint (\nabla \times \boldsymbol{\phi}_{(i,j)})^r (\nabla \times \mathbf{u})_r dA \\
&\quad = \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1} \sum_{n=0}^{n_p-1} \left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J} \frac{d\tilde{\phi}_{(i)}}{d\alpha} \right] (\nabla \times \mathbf{u})_r J w_m w_n \bigg|_{\alpha=\alpha_m, \beta=\beta_n} && (90) \\
&\quad = \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1} \frac{d\tilde{\phi}_{(i)}}{d\alpha} (\nabla \times \mathbf{u})_r w_m \bigg|_{\alpha=\alpha_m, \beta=\beta_j} && (91)
\end{aligned}$$

Combining (83), (88) and (91) then gives

$$\begin{aligned}
f^{\beta}_{(i,j)} = &\frac{\nu_d}{J(\alpha_i, \beta_j) w_i} \sum_{m=0}^{n_p-1} J g^{\alpha\beta} \frac{d\tilde{\phi}_{(i)}}{d\alpha} (\nabla \cdot \mathbf{u}) w_m \bigg|_{\alpha=\alpha_m, \beta=\beta_j} \\
&+ \frac{\nu_d}{J(\alpha_i, \beta_j) w_j} \sum_{n=0}^{n_p-1} J g^{\beta\beta} \frac{d\tilde{\phi}_{(j)}}{d\beta} (\nabla \cdot \mathbf{u}) w_n \bigg|_{\alpha=\alpha_i, \beta=\beta_n} \\
&+ \frac{\nu_v}{J(\alpha_i, \beta_j) w_i} \sum_{m=0}^{n_p-1} \frac{d\tilde{\phi}_{(i)}}{d\alpha} (\nabla \times \mathbf{u})_r w_m \bigg|_{\alpha=\alpha_m, \beta=\beta_j} && (92)
\end{aligned}$$