1 Geometry

1.1 Terrain-Following Cartesian Geometry

For a given vertical coordinate transform $z = z(\alpha, \beta, \xi)$. For example, Gal-Chen and Somerville (1975) coordinates:

$$z(\alpha, \beta, \xi) = \xi \left[z_{top} - z_s(\alpha, \beta) \right] + z_s(\alpha, \beta). \tag{1}$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial z}{\partial \alpha} \\ \frac{\partial z}{\partial \beta} \\ \frac{\partial z}{\partial \xi} \end{pmatrix} \begin{pmatrix} \frac{\partial z}{\partial \alpha} & \frac{\partial z}{\partial \beta} & \frac{\partial z}{\partial \xi} \end{pmatrix} \tag{2}$$

$$J = \left(\frac{\partial z}{\partial \xi} \right) \tag{3}$$

$$g^{ij} = \begin{pmatrix} 1 & 0 & -\left(\frac{\partial z}{\partial \xi}\right)^{-1} \left(\frac{\partial z}{\partial \alpha}\right) \\ 0 & 1 & -\left(\frac{\partial z}{\partial \xi}\right)^{-1} \left(\frac{\partial z}{\partial \beta}\right) \\ -\left(\frac{\partial z}{\partial \xi}\right)^{-1} \left(\frac{\partial z}{\partial \alpha}\right) & -\left(\frac{\partial z}{\partial \xi}\right)^{-1} \left(\frac{\partial z}{\partial \beta}\right) & \left(\frac{\partial z}{\partial \xi}\right)^{-2} \left[1 + \left(\frac{\partial z}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \beta}\right)^2 \right] \end{pmatrix} \tag{4}$$

$$\Gamma^{\alpha}{}_{ij} = \mathbf{0}, \qquad \Gamma^{\beta}{}_{ij} = \mathbf{0}, \qquad \Gamma^{\xi}{}_{ij} = \left(\frac{\partial z}{\partial \xi} \right)^{-1} \begin{pmatrix} \frac{\partial^2 z}{\partial \alpha^2} & \frac{\partial^2 z}{\partial \alpha \partial \beta} & \frac{\partial^2 z}{\partial \alpha \partial \xi} \\ \frac{\partial^2 z}{\partial \alpha \partial \beta} & \frac{\partial^2 z}{\partial \beta^2} & \frac{\partial^2 z}{\partial \xi \partial \beta} \\ \frac{\partial^2 z}{\partial \xi \partial \alpha} & \frac{\partial^2 z}{\partial \xi \partial \beta} & \frac{\partial^2 z}{\partial \xi^2} \end{pmatrix} \tag{5}$$

1.2 Terrain-Following Shallow-Atmosphere Cubed-Sphere Geometry

For a given vertical coordinate transform $r = r(\alpha, \beta, \xi)$. For example, Gal-Chen and Somerville (1975) coordinates:

$$r(\alpha, \beta, \xi) = \xi \left[z_{top} - z_s(\alpha, \beta) \right] + a + z_s(\alpha, \beta). \tag{6}$$

These equations are further subject to the shallow-water approximation, which removes any vertical dependence of the horizontal basis vectors. This approximation results in many instances of r being replaced with a throughout the equations of motion.

$$g_{ij} = \frac{a^2(1+X^2)(1+Y^2)}{\delta^4} \begin{pmatrix} 1+X^2 & -XY & 0 \\ -XY & 1+Y^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial r}{\partial \alpha} \\ \frac{\partial r}{\partial \beta} \\ \frac{\partial r}{\partial \xi} \end{pmatrix} \begin{pmatrix} \frac{\partial r}{\partial \alpha} & \frac{\partial r}{\partial \beta} & \frac{\partial r}{\partial \xi} \end{pmatrix} \tag{7}$$

$$J = \frac{1}{\delta^3} \left(\frac{\partial r}{\partial \xi} \right) a^2 (1+X^2)(1+Y^2) \tag{8}$$

$$g^{ij} = \frac{\delta^2}{a^2(1+X^2)(1+Y^2)} \begin{pmatrix} 1+Y^2 & XY & 0 \\ XY & 1+X^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \tilde{g}^{ij}, \tag{9}$$

$$\tilde{g}^{\alpha\alpha} = 0 \tag{10}$$
$$\tilde{g}^{\alpha\beta} = 0 \tag{11}$$
$$\tilde{g}^{\beta\beta} = 0 \tag{12}$$
$$\tilde{g}^{\alpha\xi} = -\frac{\delta^2}{a^2(1+X^2)(1+Y^2)} \left(\frac{\partial r}{\partial \xi}\right)^{-1} \left[(1+Y^2)\left(\frac{\partial r}{\partial \alpha}\right) + XY\left(\frac{\partial r}{\partial \beta}\right)\right] \tag{13}$$
$$\tilde{g}^{\beta\xi} = -\frac{\delta^2}{a^2(1+X^2)(1+Y^2)} \left(\frac{\partial r}{\partial \xi}\right)^{-1} \left[XY\left(\frac{\partial r}{\partial \alpha}\right) + (1+X^2)\left(\frac{\partial r}{\partial \beta}\right)\right] \tag{14}$$
$$\tilde{g}^{\xi\xi} = \left(\frac{\partial r}{\partial \xi}\right)^{-2} + \frac{\delta^2}{a^2(1+X^2)(1+Y^2)} \left(\frac{\partial r}{\partial \xi}\right)^{-2} \left[(1+Y^2)\left(\frac{\partial r}{\partial \alpha}\right)^2 + 2XY\left(\frac{\partial r}{\partial \alpha}\right)\left(\frac{\partial r}{\partial \beta}\right) + (1+X^2)\left(\frac{\partial r}{\partial \beta}\right)^2\right] \tag{15}$$

$$\Gamma^{\alpha}{}_{ij} = \begin{pmatrix} \frac{2XY^2}{\delta^2} & -\frac{Y(1+Y^2)}{\delta^2} & 0 \\ -\frac{Y(1+Y^2)}{\delta^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^{\beta}{}_{ij} = \begin{pmatrix} 0 & -\frac{X(1+X^2)}{\delta^2} & 0 \\ -\frac{X(1+X^2)}{\delta^2} & \frac{2X^2Y}{\delta^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

$$\Gamma^{\xi}{}_{ij} = \left(\frac{\partial r}{\partial \xi}\right)^{-1} \begin{pmatrix} -\frac{2XY^2}{\delta^2}\left(\frac{\partial r}{\partial \alpha}\right) + \left(\frac{\partial^2 r}{\partial \alpha^2}\right) & \frac{Y(1+Y^2)}{\delta^2}\left(\frac{\partial r}{\partial \alpha}\right) + \frac{X(1+X^2)}{\delta^2}\left(\frac{\partial r}{\partial \beta}\right) + \left(\frac{\partial^2 r}{\partial \alpha \partial \beta}\right) & \left(\frac{\partial^2 r}{\partial \alpha \partial \xi}\right) \\ \cdots & -\frac{2X^2Y}{\delta^2}\left(\frac{\partial r}{\partial \beta}\right) + \left(\frac{\partial^2 r}{\partial \beta^2}\right) & \left(\frac{\partial^2 r}{\partial \beta \partial \xi}\right) \\ \cdots & \cdots & \left(\frac{\partial^2 r}{\partial \xi^2}\right) \end{pmatrix} \tag{17}$$

2 Hydrodynamics

There is some flexibility in the choice of vertical velocity variable. Since the use of u^ξ as a vertical velocity variable would require perfect balance between several metric terms (and hence is more liable to set up a computational mode analogous to the one described in Klemp (2003)), we instead make use of the physical vertical velocity w. This choice further simplifies the vertical velocity evolution equation, but requires that u^ξ is recalculated when needed.

The system of equations describing the hydrodynamic system in arbitrary geometry is as follows:

$$\frac{\partial u^\alpha}{\partial t} + u^\alpha \nabla_\alpha u^\alpha + u^\beta \nabla_\beta u^\alpha + u^\xi \nabla_\xi u^\alpha + \theta \nabla^\alpha \Pi + f g^{\alpha j} \epsilon_{j\xi k} u^k = 0 \tag{18}$$

$$\frac{\partial u^\beta}{\partial t} + u^\alpha \nabla_\alpha u^\beta + u^\beta \nabla_\beta u^\beta + u^\xi \nabla_\xi u^\beta + \theta \nabla^\beta \Pi + f g^{\beta j} \epsilon_{j\xi k} u^k = 0 \tag{19}$$

$$\frac{\partial \theta}{\partial t} + u^\alpha \frac{\partial \theta}{\partial \alpha} + u^\beta \frac{\partial \theta}{\partial \beta} = -u^\xi \frac{\partial \theta}{\partial \xi} \tag{20}$$

$$\frac{\partial w}{\partial t} + u^\alpha \nabla_\alpha w + u^\beta \nabla_\beta w + u^\xi \nabla_\xi w = -\theta \left(\frac{\partial r}{\partial \xi}\right)^{-1} \frac{\partial \Pi}{\partial \xi} - g \tag{21}$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{J}\frac{\partial}{\partial \alpha}(J\rho u^\alpha) + \frac{1}{J}\frac{\partial}{\partial \beta}(J\rho u^\beta) = -\frac{1}{J}\frac{\partial}{\partial \xi}(J\rho u^\xi), \tag{22}$$

with Exner function

$$\Pi(\rho, \theta) = c_p \left(\frac{R\rho\theta}{p_0}\right)^{R/c_v} \tag{23}$$

and coordinate velocity

$$u^\xi(w, u^\alpha, u^\beta) = \left(\frac{\partial r}{\partial \xi}\right)^{-1} \left[w - \left(\frac{\partial r}{\partial \alpha}\right)^{-1} u^\alpha - \left(\frac{\partial r}{\partial \beta}\right)^{-1} u^\beta\right] \tag{24}$$

Advection and geometry:

$$u^\alpha \nabla_\alpha u^k = u^\alpha \left[\frac{\partial u^k}{\partial \alpha} + {\Gamma^k}_{\alpha\alpha} u^\alpha + {\Gamma^k}_{\alpha\beta} u^\beta + {\Gamma^k}_{\alpha\xi} u^\xi\right] \tag{25}$$

$$u^\beta \nabla_\beta u^k = u^\beta \left[\frac{\partial u^k}{\partial \beta} + {\Gamma^k}_{\beta\alpha} u^\alpha + {\Gamma^k}_{\beta\beta} u^\beta + {\Gamma^k}_{\beta\xi} u^\xi\right] \tag{26}$$

$$ \tag{27}$$

Pressure gradient:

$$\nabla^k \Pi = g^{k\alpha} \frac{\partial \Pi}{\partial \alpha} + g^{k\beta} \frac{\partial \Pi}{\partial \beta} + g^{k\xi} \frac{\partial \Pi}{\partial \xi} \tag{28}$$

Coriolis Force:

$$f g^{\alpha j} \epsilon_{j\xi k} u^k = -f g^{\alpha\alpha} J u^\beta + f g^{\alpha\beta} J u^\alpha \tag{29}$$

$$f g^{\beta j} \epsilon_{j\xi k} u^k = -f g^{\beta\alpha} J u^\beta + f g^{\beta\beta} J u^\alpha \tag{30}$$

3 Horizontal Discretization

3.1 Tensor Product Basis

The tensor-product basis is

$$\phi_{(i,j)}(\alpha, \beta) = \tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta), \tag{31}$$

where $\tilde{\phi}_{(i)}(x)$ denotes the usual 1D GLL basis function at node $i \in (0, \ldots, n_p)$. For vector fields, the components of the covariant vector field are given by the tensor-product basis (31).

3.2 Spectral Element (Scalar Variational Form)

Consider a scalar conservation law of the form

$$\frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{F} = S, \tag{32}$$

with $\mathbf{F} = F^\alpha \mathbf{g}_\alpha + F^\beta \mathbf{g}_\beta$. Multiplying by the tensor-product basis $\phi_{(p,q)}(\alpha, \beta)$ and integrating over the whole domain yields

$$\iint \frac{\partial \psi}{\partial t} \phi_{(i,j)} dA = -\iint (\nabla \cdot \mathbf{F}) \phi_{(i,j)} dA + \iint S \phi_{(i,j)} dA, \tag{33}$$

where $dA = J d\alpha d\beta$. Applying integration by parts and using periodicity of the domain then leads to the weak form

$$\iint \frac{\partial \psi}{\partial t} \phi_{(i,j)} dA = \iint \nabla \phi_{(i,j)} \cdot \mathbf{F} dA + \iint S \phi_{(i,j)} dA. \tag{34}$$

Further expanding ψ in terms of the horizontal basis functions as

$$\psi(t, \alpha, \beta) = \sum_{(s,t)} \psi_{(s,t)}(t) \phi_{(s,t)}(\alpha, \beta) \tag{35}$$

leads to

$$\sum_{(s,t)} \frac{\partial \psi_{(s,t)}}{\partial t} \iint \phi_{(s,t)} \phi_{(i,j)} dA = \iint \nabla \phi_{(i,j)} \cdot \mathbf{F} dA + \iint S \phi_{(i,j)} dA. \tag{36}$$

Here the vertical dependence of ψ is implicit. For simplicity we now restrict our domain to a single spectral element (since the DSS procedure will later be used to account for inter-element exchange). Approximate integration is now applied,

$$\iint f(\alpha, \beta) dA = \Delta\alpha \Delta\beta \sum_{p=0}^{n_p - 1} \sum_{q=0}^{n_p - 1} f(\alpha_p, \beta_q) w_p w_q, \tag{37}$$

where w_i are the nodal weights of the GLL nodes on the reference element $[0, 1]$. Consequently,

$$\iint \frac{\partial \psi}{\partial t} \phi_{(i,j)} dA = \Delta\alpha \Delta\beta \frac{\partial \psi_{(i,j)}}{\partial t} w_i w_j J(\alpha_i, \beta_j), \tag{38}$$

$$\iint \nabla \phi_{(i,j)} \cdot \mathbf{F} dA = \Delta\alpha \Delta\beta w_j \sum_{p=0}^{n_p - 1} \frac{d\tilde{\phi}_{(i)}}{d\alpha} F^\alpha w_p J \bigg|_{\alpha=\alpha_p, \beta=\beta_j} + \Delta\alpha \Delta\beta w_i \sum_{q=0}^{n_p - 1} \frac{d\tilde{\phi}_{(j)}}{d\beta} F^\beta w_q J \bigg|_{\alpha=\alpha_i, \beta=\beta_q}, \tag{39}$$

$$\iint S \phi_{(i,j)} dA = \Delta\alpha \Delta\beta S(\alpha_i, \beta_j) w_i w_j J(\alpha_i, \beta_j). \tag{40}$$

Substituting (38)-(40) into (33) gives the spectral element semi-discretization

$$\begin{aligned} \frac{\partial \psi_{(i,j)}}{\partial t} = & \frac{1}{w_i J(\alpha_i, \beta_j)} \sum_{p=0}^{n_p - 1} \frac{d\tilde{\phi}_{(i)}}{d\alpha} F^\alpha w_p J \bigg|_{\alpha=\alpha_p, \beta=\beta_j} \\ & + \frac{1}{w_j J(\alpha_i, \beta_j)} \sum_{q=0}^{n_p - 1} \frac{d\tilde{\phi}_{(j)}}{d\beta} F^\beta w_q J \bigg|_{\alpha=\alpha_i, \beta=\beta_q} + S(\alpha_i, \beta_j) \end{aligned} \tag{41}$$

3.3 Spectral Element (Differential Form)

The spectral element method can also be derived in differential form by noting that basis functions can be interpreted as components of an interpolating polynomial. For an arbitrary function $f(\alpha, \beta)$ defined on element nodes we have

$$\frac{\partial f}{\partial \alpha} \bigg|_{\alpha=\alpha_i, \beta=\beta_j} = \sum_{p=0}^{n_p - 1} \frac{d\tilde{\phi}_{(p)}}{d\alpha} \bigg|_{\alpha=\alpha_i} f(\alpha_p, \beta_j), \tag{42}$$

$$\frac{\partial f}{\partial \beta} \bigg|_{\alpha=\alpha_i, \beta=\beta_j} = \sum_{q=0}^{n_p - 1} \frac{d\tilde{\phi}_{(q)}}{d\beta} \bigg|_{\beta=\beta_j} f(\alpha_i, \beta_q). \tag{43}$$

Second derivatives are defined as follows:

$$\left.\frac{\partial^2 f}{\partial \alpha^2}\right|_{\alpha=\alpha_i,\beta=\beta_j} = \sum_{p=0}^{n_p-1} \left.\frac{d\tilde{\phi}_{(p)}}{d\alpha}\right|_{\alpha=\alpha_i} \frac{\partial f}{\partial \alpha}(\alpha_p, \beta_j) \tag{44}$$

$$= \sum_{p=0}^{n_p-1} \sum_{q=0}^{n_p-1} \left.\frac{d\tilde{\phi}_{(p)}}{d\alpha}\right|_{\alpha=\alpha_i} \left.\frac{d\tilde{\phi}_{(q)}}{d\alpha}\right|_{\alpha=\alpha_p} f(\alpha_q, \beta_j) \tag{45}$$

3.4 Flux Reconstruction / Discontinuous Galerkin

4 Vertical Discretization

The following operators will be applied in the context of the vertical discretization:

Operator	Description
$\mathcal{I}_e^n$	Interpolate from nodes to edges
$\mathcal{I}_n^e$	Interpolate from edges to nodes
$\mathcal{D}_e^n$	Differentiate from nodes to edges
$\mathcal{D}_n^e$	Differentiate from edges to nodes
$\mathcal{D}_n^n$	Differentiate from nodes to nodes
$\mathcal{D}_e^e$	Differentiate from edges to edges

Vertical discretizations:

For given variable q we denote q defined on edges by q_e and q defined on nodes by q_n.

Prognostic Variable	Operator	Choice of Staggering: DG $(\rho_n\theta_n u_n^\xi)$	$(\rho_n\theta_n, u_e^\xi)$	$(\rho_n, u_e^\xi\theta_e)$
	Π_n	$\Pi_n(\rho_n, \theta_n)$	$\Pi_n(\rho_n, \theta_n)$	$\Pi_n(\rho_n, \mathcal{I}_n^e\theta_e)$
θ	$u^\xi \frac{\partial\theta}{\partial\xi}$	$(u_n^\xi)\mathcal{D}_n^n\theta_n$	$(\mathcal{I}_n^e u_e^\xi)(\mathcal{D}_n^n\theta)$	$(u_e^\xi)(\mathcal{D}_e^e\theta_e)$
w	$\theta\frac{\partial\Pi}{\partial\xi}$	$\theta_n\mathcal{D}_n^n\Pi_n$	$(\mathcal{I}_n^e\theta_n)(\mathcal{D}_e^n\Pi_n)$	$\theta_e(\mathcal{D}_e^n\Pi_n)$
ρ	$\frac{1}{J}\frac{\partial}{\partial\xi}(J\rho u^\xi)$	$\frac{1}{J_n}\mathcal{D}_n^n(J_n\rho_n u_n^\xi)$	$\frac{1}{J_n}\mathcal{D}_n^e\left[J_e(\mathcal{I}_e^n\rho_n)u_e^\xi\right]$	$\frac{1}{J_n}\mathcal{D}_n^e\left[J_e(\mathcal{I}_e^n\rho_n)u_e^\xi\right]$

Define basis $\tilde{\phi}_{(m)}(\xi)$ as the usual 1D GLL basis function at node $m \in (0, \ldots, n_v)$ and basis $\tilde{\varphi}_{(m)}(\xi)$ as the usual 1D GL basis function at node $m \in (0, \ldots, n_v - 1)$ on the reference element $[0, 1]$. Within element i with bounds $[\xi_i, \xi_{i+1}]$ and width $\Delta\xi_i = \xi_{i+1} - \xi_i$ the continuous reconstruction over nodes takes the form

$$q_i(\xi) = \sum_{p=0}^{n_v-1} (q_n)_p \cdot \tilde{\varphi}_{(p)}\left(\frac{\xi - \xi_i}{\Delta\xi_i}\right), \tag{46}$$

and over edges

$$q_i(\xi) = \sum_{p=0}^{n_v} (q_e)_p \cdot \tilde{\phi}_{(p)}\left(\frac{\xi - \xi_i}{\Delta\xi_i}\right). \tag{47}$$

4.1 Interpolation

Interpolation is carried out in the usual manner by evaluation the continuous reconstruction (46) or (47). However, for the special case of interpolation of nodal values to a finite element edge, the reconstruction is averaged at the target point.

5 Hyperviscosity

Hyperviscosity is formulated in variational form.

5.1 Fourth-Order Scalar Hyperviscosity

Fourth-order scalar hyperviscosity is implemented using a two stage procedure:

$$f = \mathcal{H}(1)\psi^n, \tag{48}$$

$$\psi^{n+1} = \psi^n - \Delta t \mathcal{H}(\nu) f. \tag{49}$$

The hyperviscosity operator is defined implicitly via

$$f = \mathcal{H}(\nu)\psi \quad \Longleftrightarrow \quad \iint f\phi_{(i,j)} dA = \nu \iint \nabla\phi_{(i,j)} \cdot \nabla\psi dA, \tag{50}$$

where $dA = J d\alpha d\beta$. Here

$$\iint f\phi_{(i,j)} dA = \iint f\tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta) dA = f_{(i,j)} w_i w_j J \Delta\alpha\Delta\beta, \tag{51}$$

and

$$\iint \nabla\phi_{(i,j)} \cdot \nabla\psi dA = \iint g^{pq}\nabla_p\phi\nabla_q\psi dA, \tag{52}$$

$$= \iint \frac{\partial\phi_{(i,j)}}{\partial\alpha}\left[g^{\alpha\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\alpha\beta}\frac{\partial\psi}{\partial\beta}\right] + \frac{\partial\phi_{(i,j)}}{\partial\beta}\left[g^{\beta\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\beta\beta}\frac{\partial\psi}{\partial\beta}\right] dA, \tag{53}$$

$$\begin{aligned} &= \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1} \tilde{\phi}_{(j)}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}\left[g^{\alpha\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\alpha\beta}\frac{\partial\psi}{\partial\beta}\right] J w_m w_n\bigg|_{\alpha=\alpha_m,\beta=\beta_n} \\ &\quad + \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1} \tilde{\phi}_{(i)}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}\left[g^{\beta\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\beta\beta}\frac{\partial\psi}{\partial\beta}\right] J w_m w_n\bigg|_{\alpha=\alpha_m,\beta=\beta_n} \end{aligned} \tag{54}$$

$$\begin{aligned} &= \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1} \frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}\left[g^{\alpha\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\alpha\beta}\frac{\partial\psi}{\partial\beta}\right] J w_m\bigg|_{\alpha=\alpha_m,\beta=\beta_j} \\ &\quad + \Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1} \frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}\left[g^{\beta\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\beta\beta}\frac{\partial\psi}{\partial\beta}\right] J w_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} \end{aligned} \tag{55}$$

In combination,

$$\begin{aligned} f_{(i,j)} &= \frac{1}{w_i J(\alpha_i,\beta_j)} \sum_{m=0}^{n_p-1} \frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}\left[g^{\alpha\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\alpha\beta}\frac{\partial\psi}{\partial\beta}\right] J w_m\bigg|_{\alpha=\alpha_m,\beta=\beta_j} \\ &\quad + \frac{1}{w_j J(\alpha_i,\beta_j)} \sum_{n=0}^{n_p-1} \frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}\left[g^{\beta\alpha}\frac{\partial\psi}{\partial\alpha} + g^{\beta\beta}\frac{\partial\psi}{\partial\beta}\right] J w_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} \end{aligned} \tag{56}$$

The derivatives of the scalar field ψ are obtained in the usual manner,

$$\left.\frac{\partial\psi}{\partial\alpha}\right|_{\alpha=\alpha_i,\beta=\beta_j} = \sum_{m=0}^{n_p-1} \psi_{(m,j)} \left.\frac{\partial\tilde{\phi}_{(m)}}{\partial\alpha}\right|_{\alpha=\alpha_m,\beta=\beta_j} \tag{57}$$

$$\left.\frac{\partial\psi}{\partial\beta}\right|_{\alpha=\alpha_i,\beta=\beta_j} = \sum_{n=0}^{n_p-1} \psi_{(i,n)} \left.\frac{\partial\tilde{\phi}_{(n)}}{\partial\beta}\right|_{\alpha=\alpha_i,\beta=\beta_n} \tag{58}$$

$$\tag{59}$$

5.2 Fourth-Order Vector Hyperviscosity

Fourth-order vector hyperviscosity is implemented using a two stage procedure:

$$\mathbf{f} = \mathcal{H}(1,1)\mathbf{u}^n, \tag{60}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \Delta t\mathcal{H}(\nu_d,\nu_v)\mathbf{f}. \tag{61}$$

The hyperviscosity operator is defined implicitly via

$$\mathbf{f} = \mathcal{H}(\nu_d,\nu_v)\mathbf{u^n} \quad\Longleftrightarrow\quad \iint \mathbf{f}\cdot\boldsymbol{\phi}dA = \iint \nu_d(\nabla\cdot\boldsymbol{\phi})(\nabla\cdot\mathbf{u}^n) + \nu_v(\nabla\times\boldsymbol{\phi})^r(\nabla\times\mathbf{u}^n)_r dA, \tag{62}$$

where $dA = Jd\alpha d\beta$ and

$$(\nabla\cdot\boldsymbol{\phi}) = g^{pq}\nabla_p\phi_q = \frac{1}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\phi_\alpha + Jg^{\alpha\beta}\phi_\beta\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\phi_\alpha + Jg^{\beta\beta}\phi_\beta\right), \tag{63}$$

$$(\nabla\times\boldsymbol{\phi})^r = \epsilon^{rpq}\nabla_p\phi_q = \epsilon^{rpq}\left[\frac{\partial\phi_q}{\partial x^p} - \Gamma^k{}_{pq}\phi_k\right] = \frac{1}{J}\left[\frac{\partial\phi_\beta}{\partial\alpha} - \frac{\partial\phi_\alpha}{\partial\beta}\right]. \tag{64}$$

Here we assume that $(\nabla\cdot\mathbf{u})$ and $(\nabla\times\mathbf{u})_r$ (covariant radial component of curl) have already been computed.

5.2.1 Vector basis with zero β component

If $\phi_{(i,j)\alpha} = \tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)$ and $\phi_{(i,j)\beta} = 0$ then

$$\iint \mathbf{f}\cdot\phi dA = \iint f^\alpha\tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)dA = f^\alpha_{(i,j)}w_iw_jJ\Delta\alpha\Delta\beta \tag{65}$$

The divergent term is defined by

$$(\nabla\cdot\boldsymbol{\phi}_{(i,j)}) = \frac{1}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\phi_{(i,j)\alpha}\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\phi_{(i,j)\alpha}\right) \tag{66}$$

$$= \frac{\tilde{\phi}_{(j)}(\beta)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right) + \frac{\tilde{\phi}_{(i)}(\alpha)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right), \tag{67}$$

and

$$
\begin{aligned}
&\iint (\nabla\cdot\boldsymbol{\phi}_{(i,j)})(\nabla\cdot\mathbf{u})dA \\
&\quad = \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right)+\frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right)\right](\nabla\cdot\mathbf{u})Jw_m w_n \qquad (68)\\
&\quad = \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right)(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j} \\
&\qquad + \Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right)(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \qquad (69)\\
&\quad = \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1} Jg^{\alpha\alpha}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j} \\
&\qquad + \Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1} Jg^{\beta\alpha}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \qquad (70)
\end{aligned}
$$

Further, the vortical term is defined by

$$
(\nabla\times\boldsymbol{\phi}_{(i,j)})^r = -\frac{1}{J}\frac{\partial\phi_{(i,j)\alpha}}{\partial\beta} = -\frac{\tilde{\phi}_{(i)}}{J}\frac{d\tilde{\phi}_{(j)}}{d\beta} \qquad (71)
$$

and so

$$
\begin{aligned}
&\iint (\nabla\times\boldsymbol{\phi}_{(i,j)})^r(\nabla\times\mathbf{u})_r dA \\
&\quad = \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[-\frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{d\tilde{\phi}_{(j)}}{d\beta}\right](\nabla\times\mathbf{u})_r J w_m w_n\Big|_{\alpha=\alpha_m,\beta=\beta_n} \qquad (72)\\
&\quad = -\Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\times\mathbf{u})_r w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \qquad (73)
\end{aligned}
$$

Combining (65), (70) and (73) then gives

$$
\begin{aligned}
f^{\alpha}_{(i,j)} = &\frac{\nu_d}{J(\alpha_i,\beta_j)w_i}\sum_{m=0}^{n_p-1} Jg^{\alpha\alpha}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j} \\
&+\frac{\nu_d}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1} Jg^{\beta\alpha}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \\
&-\frac{\nu_v}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\times\mathbf{u})_r w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \qquad (74)
\end{aligned}
$$

5.2.2 Vector basis with zero α component

If $\phi_{(i,j)\alpha} = 0$ and $\phi_{(i,j)\beta} = \tilde{\phi}_{(j)}(\alpha)\tilde{\phi}_{(j)}(\beta)$ then

$$
\iint \mathbf{f}\cdot\phi dA = \iint f^{\beta}\tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)dA = f^{\beta}_{(i,j)}w_i w_j J\Delta\alpha\Delta\beta \qquad (75)
$$

The divergent term is defined by

$$(\nabla\cdot\boldsymbol{\phi}_{(i,j)}) = \frac{1}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\phi_{(i,j)\beta}\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\phi_{(i,j)\beta}\right) \tag{76}$$

$$= \frac{\tilde{\phi}_{(j)}(\beta)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}\right) + \frac{\tilde{\phi}_{(i)}(\alpha)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}\right), \tag{77}$$

and

$$\iint(\nabla\cdot\boldsymbol{\phi}_{(i,j)})(\nabla\cdot\mathbf{u})dA$$

$$= \Delta\alpha\Delta\beta\sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}\right) + \frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}\right)\right](\nabla\cdot\mathbf{u})Jw_m w_n \tag{78}$$

$$= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}(\alpha)\right)(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}(\beta)\right)(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \tag{79}$$

$$= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}Jg^{\alpha\beta}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}Jg^{\beta\beta}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \tag{80}$$

Further, the vortical term is defined by

$$(\nabla\times\boldsymbol{\phi}_{(i,j)})^r = \frac{1}{J}\frac{\partial\phi_{(i,j)\beta}}{\partial\alpha} = \frac{\tilde{\phi}_{(j)}}{J}\frac{d\tilde{\phi}_{(i)}}{d\alpha} \tag{81}$$

and so

$$\iint(\nabla\times\boldsymbol{\phi}_{(i,j)})^r(\nabla\times\mathbf{u})_r dA$$

$$= \Delta\alpha\Delta\beta\sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J}\frac{d\tilde{\phi}_{(i)}}{d\alpha}\right](\nabla\times\mathbf{u})_r Jw_m w_n\Big|_{\alpha=\alpha_m,\beta=\beta_n} \tag{82}$$

$$= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\times\mathbf{u})_r w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j} \tag{83}$$

Combining (75), (80) and (83) then gives

$$f^{\beta}_{(i,j)} = \frac{\nu_d}{J(\alpha_i,\beta_j)w_i}\sum_{m=0}^{n_p-1}Jg^{\alpha\beta}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \frac{\nu_d}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1}Jg^{\beta\beta}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n}$$

$$+ \frac{\nu_v}{J(\alpha_i,\beta_j)w_i}\sum_{m=0}^{n_p-1}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\times\mathbf{u})_r w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j} \tag{84}$$

5.3 Fourth-Order Scalar Hyperviscosity for DG

Fourth-order scalar hyperviscosity is implemented using a two stage procedure:

$$f = \mathcal{H}(1)\psi^n, \tag{85}$$

$$\psi^{n+1} = \psi^n - \Delta t \mathcal{H}(\nu) f. \tag{86}$$

The hyperviscosity operator is defined implicitly via

$$f = \mathcal{H}(\nu)\psi \quad \iff \quad \iint f\phi_{(i,j)} dA = \nu \left[\oint \phi_{(i,j)} \nabla\psi \cdot d\mathbf{S} - \iint \nabla\phi_{(i,j)} \cdot \nabla\psi dA \right], \tag{87}$$

where $dA = J d\alpha d\beta$. The area-integral term is handled as in section 5.1. Here

$$\iint f\phi_{(i,j)} dA = \iint f\tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta) dA = f_{(i,j)} w_i w_j J \Delta\alpha\Delta\beta. \tag{88}$$

The contour integral takes the form

$$\oint \phi_{(i,j)} \nabla\psi \cdot d\mathbf{S} = \int_R \phi_{(i,j)} \nabla\psi \cdot d\mathbf{S} + \int_T \phi_{(i,j)} \nabla\psi \cdot d\mathbf{S} + \int_L \phi_{(i,j)} \nabla\psi \cdot d\mathbf{S} + \int_B \phi_{(i,j)} \nabla\psi \cdot d\mathbf{S}, \tag{89}$$

where R, T, L and B denote the right, top, left and bottom edges, respectively. The covariant components of the face normals at the right edge are

$$N_\alpha = \frac{1}{\sqrt{g^{\alpha\alpha}}}, \qquad N_\beta = 0, \tag{90}$$

and so

$$\int_R \phi_{(i,j)} \nabla\psi \cdot d\mathbf{S} = \delta^i_{n_p-1} \sum_{n=0}^{n_p-1} \tilde{\phi}_{(j)}(\beta) \nabla^\alpha \psi N_\alpha w_n \sqrt{g_{\beta\beta}} \Delta\beta \Big|_{\alpha=\alpha_{n_p-1},\beta=\beta_n} \tag{91}$$

$$= \Delta\beta w_j \delta_{i,n_p-1} \; J\nabla^\alpha \psi \big|_{\alpha=\alpha_{n_p-1},\beta=\beta_j}, \tag{92}$$

leading to

$$f_{(i,j)} = \frac{\delta_{i,n_p-1}}{w_i \Delta\alpha} \left[g^{\alpha\alpha} \sum_{m=0}^{n_p-1} \psi \left. \frac{d\tilde{\phi}_{(m)}}{d\alpha} \right|_{\alpha=\alpha_m,\beta=\beta_j} + g^{\alpha\beta} \sum_{n=0}^{n_p-1} \psi \left. \frac{d\tilde{\phi}_{(n)}}{d\beta} \right|_{\alpha=\alpha_{n_p-1},\beta=\beta_n} \right]. \tag{93}$$

5.4 Fourth-Order Vector Hyperviscosity for DG

$$\mathbf{f} = \nu_d \nabla(\nabla\cdot\mathbf{u}) - \nu_v \nabla\times(\nabla\times\mathbf{u}). \tag{94}$$

To verify that this expression correctly separates divergence and vorticity damping,

$$\frac{\partial}{\partial t}(\nabla\cdot\mathbf{u}) = \nu_d \nabla^2(\nabla\cdot\mathbf{u}), \tag{95}$$

$$\frac{\partial}{\partial t}(\nabla\times\mathbf{u}) = -\nu_v \nabla\times(\nabla\times(\nabla\times\mathbf{u})) = \nu_v \nabla^2(\nabla\times\mathbf{u}) \tag{96}$$

For divergence damping,

$$\iint \boldsymbol{\phi}\cdot\mathbf{f} dV = \nu_d \iint \boldsymbol{\phi}\cdot\nabla(\nabla\cdot\mathbf{u}), \tag{97}$$

$$= \nu_d \left[\oint (\nabla\cdot\mathbf{u})\boldsymbol{\phi}\cdot d\mathbf{S} - \iint (\nabla\cdot\boldsymbol{\phi})(\nabla\cdot\mathbf{u}) dV \right]. \tag{98}$$

For vorticity damping,

$$\iint \boldsymbol{\phi}\cdot\mathbf{f} dV = -\nu_v \iint \boldsymbol{\phi}\cdot\nabla\times(\nabla\times\mathbf{u}) dV, \tag{99}$$

$$= -\nu_v \left[\oint (\nabla\times\mathbf{u})\times\phi\cdot d\mathbf{S} + \iint (\nabla\times\boldsymbol{\phi})\cdot(\nabla\times\mathbf{u}) dV \right] \tag{100}$$