1 Tensor-Product Basis

The tensor-product basis is

$$\phi_{(i,j)}(\alpha,\beta) = \tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta), \tag{1}$$

where $\tilde{\phi}_{(i)}(x)$ denotes the usual 1D GLL basis function at node $i \in (0,\ldots,n_p)$. For vector fields, the components of the covariant vector field are given by the tensor-product basis (1).

2 Hydrodynamics

The system of equations describing the hydrodynamic system is as follows:

$$\frac{\partial \rho}{\partial t} + \frac{1}{J}\frac{\partial}{\partial \alpha}(J\rho u^\alpha) + \frac{1}{J}\frac{\partial}{\partial \beta}(J\rho u^\beta) = -\frac{1}{J}\frac{\partial}{\partial \xi}(J\rho u^\xi), \tag{2}$$

$$\frac{\partial \mathbf{u}^\alpha}{\partial t} + u^\alpha \nabla_\alpha u^\alpha + u^\beta \nabla_\beta u^\alpha + u^\xi \nabla_\xi u^\alpha + \frac{1}{\rho}\nabla^\alpha p + \frac{f}{J}u^\beta = 0, \tag{3}$$

$$\frac{\partial \mathbf{u}^\beta}{\partial t} + u^\alpha \nabla_\alpha u^\beta + u^\beta \nabla_\beta u^\beta + u^\xi \nabla_\xi u^\beta + \frac{1}{\rho}\nabla^\beta p - \frac{f}{J}u^\alpha = 0, \tag{4}$$

2.1 Spectral Element

2.2 Flux Reconstruction / Discontinuous Galerkin

3 Hyperviscosity

3.1 Vector Hyperviscosity

Fourth-order vector hyperviscosity is implemented using a two stage procedure:

$$\mathbf{f} = \mathcal{H}(1,1)\mathbf{u}^n, \tag{5}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \Delta t \mathcal{H}(\nu_d,\nu_v)\mathbf{f}. \tag{6}$$

The hyperdiffusion operator is defined implicitly via

$$\mathbf{f} = \mathcal{H}(\nu_d,\nu_v)\mathbf{u^n} \quad \Longleftrightarrow \quad \iint \mathbf{f}\cdot\boldsymbol{\phi} dA = \iint \nu_d(\nabla\cdot\boldsymbol{\phi})(\nabla\cdot\mathbf{u}^n) + \nu_v(\nabla\times\boldsymbol{\phi})^r(\nabla\times\mathbf{u}^n)_r dA, \tag{7}$$

where $dA = J d\alpha d\beta$ and

$$(\nabla\cdot\boldsymbol{\phi}) = g^{pq}\nabla_p\phi_q = \frac{1}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\phi_\alpha + Jg^{\alpha\beta}\phi_\beta\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\phi_\alpha + Jg^{\beta\beta}\phi_\beta\right), \tag{8}$$

$$(\nabla\times\boldsymbol{\phi})^r = \epsilon^{rpq}\nabla_p\phi_q = \epsilon^{rpq}\left[\frac{\partial\phi_q}{\partial x^p} - \Gamma^k{}_{pq}\phi_k\right] = \frac{1}{J}\left[\frac{\partial\phi_\beta}{\partial\alpha} - \frac{\partial\phi_\alpha}{\partial\beta}\right]. \tag{9}$$

Here we assume that $(\nabla\cdot\mathbf{u})$ and $(\nabla\times\mathbf{u})_r$ (covariant radial component of curl) have already been computed.

3.1.1 Vector basis with zero β component

If $\phi_{(i,j)\alpha} = \tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)$ and $\phi_{(i,j)\beta} = 0$ then

$$\iint \mathbf{f} \cdot \phi dA = \iint f^{\alpha} \tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta) dA = f^{\alpha}_{(i,j)} w_i w_j J \Delta\alpha\Delta\beta \tag{10}$$

The divergent term is defined by

$$\begin{aligned}(\nabla \cdot \boldsymbol{\phi}_{(i,j)}) &= \frac{1}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\phi_{(i,j)\alpha}\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\phi_{(i,j)\alpha}\right) &(11)\\ &= \frac{\tilde{\phi}_{(j)}(\beta)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right) + \frac{\tilde{\phi}_{(i)}(\alpha)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right), &(12)\end{aligned}$$

and

$$\begin{aligned}&\iint (\nabla \cdot \boldsymbol{\phi}_{(i,j)})(\nabla \cdot \mathbf{u}) dA\\ &\quad = \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right) + \frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right)\right](\nabla\cdot\mathbf{u})Jw_m w_n &(13)\\ &\quad = \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right)(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j}\\ &\qquad + \Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1}\frac{\partial}{\partial\beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right)(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} &(14)\\ &\quad = \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1} Jg^{\alpha\alpha}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j}\\ &\qquad + \Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1} Jg^{\beta\alpha}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} &(15)\end{aligned}$$

Further, the vortical term is defined by

$$(\nabla \times \boldsymbol{\phi}_{(i,j)})^r = -\frac{1}{J}\frac{\partial\phi_{(i,j)\alpha}}{\partial\beta} = -\frac{\tilde{\phi}_{(i)}}{J}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta} \tag{16}$$

and so

$$\begin{aligned}&\iint (\nabla \times \boldsymbol{\phi}_{(i,j)})^r(\nabla\times\mathbf{u})_r dA\\ &\quad = \Delta\alpha\Delta\beta\sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[-\frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}\right](\nabla\times\mathbf{u})_r J w_m w_n\Big|_{\alpha=\alpha_m,\beta=\beta_n} &(17)\\ &\quad = -\Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}(\nabla\times\mathbf{u})_r w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} &(18)\end{aligned}$$

Combining (10), (15) and (18) then gives

$$
\begin{aligned}
f^{\alpha}_{(i,j)} = & \frac{\nu_d}{J(\alpha_i,\beta_j)w_i}\sum_{m=0}^{n_p-1} Jg^{\alpha\alpha}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}(\nabla\cdot\mathbf{u})w_m\bigg|_{\alpha=\alpha_m,\beta=\beta_j} \\
& + \frac{\nu_d}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1} Jg^{\beta\alpha}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}(\nabla\cdot\mathbf{u})w_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} \\
& - \frac{\nu_v}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1} \frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}(\nabla\times\mathbf{u})_r w_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} \qquad (19)
\end{aligned}
$$

3.1.2 Vector basis with zero α component

If $\phi_{(i,j)\alpha}=0$ and $\phi_{(i,j)\beta}=\tilde{\phi}_{(j)}(\alpha)\tilde{\phi}_{(j)}(\beta)$ then

$$
\iint \mathbf{f}\cdot\phi dA = \iint f^{\beta}\tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)dA = f^{\beta}_{(i,j)}w_iw_jJ\Delta\alpha\Delta\beta \qquad (20)
$$

The divergent term is defined by

$$
\begin{aligned}
(\nabla\cdot\boldsymbol{\phi}_{(i,j)}) &= \frac{1}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\phi_{(i,j)\beta}\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\phi_{(i,j)\beta}\right) \qquad (21) \\
&= \frac{\tilde{\phi}_{(j)}(\beta)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}\right) + \frac{\tilde{\phi}_{(i)}(\alpha)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}\right), \qquad (22)
\end{aligned}
$$

and

$$
\begin{aligned}
&\iint (\nabla\cdot\boldsymbol{\phi}_{(i,j)})(\nabla\cdot\mathbf{u})dA \\
&= \Delta\alpha\Delta\beta\sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}\right) + \frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}\right)\right](\nabla\cdot\mathbf{u})Jw_mw_n \qquad (23) \\
&= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}\frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}(\alpha)\right)(\nabla\cdot\mathbf{u})w_m\bigg|_{\alpha=\alpha_m,\beta=\beta_j} \\
&\quad + \Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}(\beta)\right)(\nabla\cdot\mathbf{u})w_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} \qquad (24) \\
&= \Delta\alpha\Delta\beta w_j\sum_{m=0}^{n_p-1}Jg^{\alpha\beta}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha}(\nabla\cdot\mathbf{u})w_m\bigg|_{\alpha=\alpha_m,\beta=\beta_j} \\
&\quad + \Delta\alpha\Delta\beta w_i\sum_{n=0}^{n_p-1}Jg^{\beta\beta}\frac{\partial\tilde{\phi}_{(j)}}{\partial\beta}(\nabla\cdot\mathbf{u})w_n\bigg|_{\alpha=\alpha_i,\beta=\beta_n} \qquad (25)
\end{aligned}
$$

Further, the vortical term is defined by

$$
(\nabla\times\boldsymbol{\phi}_{(i,j)})^r = \frac{1}{J}\frac{\partial\phi_{(i,j)\beta}}{\partial\alpha} = \frac{\tilde{\phi}_{(j)}}{J}\frac{\partial\tilde{\phi}_{(i)}}{\partial\alpha} \qquad (26)
$$

and so

$$
\begin{aligned}
&\iint (\nabla \times \boldsymbol{\phi}_{(i,j)})^r (\nabla \times \mathbf{u})_r dA \\
&\quad = \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1} \sum_{n=0}^{n_p-1} \left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J} \frac{\partial \tilde{\phi}_{(i)}}{\partial \alpha} \right] (\nabla \times \mathbf{u})_r J w_m w_n \bigg|_{\alpha=\alpha_m, \beta=\beta_n} &(27) \\
&\quad = \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1} \frac{\partial \tilde{\phi}_{(i)}}{\partial \alpha} (\nabla \times \mathbf{u})_r w_m \bigg|_{\alpha=\alpha_m, \beta=\beta_j} &(28)
\end{aligned}
$$

Combining (20), (25) and (28) then gives

$$
\begin{aligned}
f^{\beta}_{(i,j)} = &\frac{\nu_d}{J(\alpha_i, \beta_j) w_i} \sum_{m=0}^{n_p-1} J g^{\alpha\beta} \frac{\partial \tilde{\phi}_{(i)}}{\partial \alpha} (\nabla \cdot \mathbf{u}) w_m \bigg|_{\alpha=\alpha_m, \beta=\beta_j} \\
&+ \frac{\nu_d}{J(\alpha_i, \beta_j) w_j} \sum_{n=0}^{n_p-1} J g^{\beta\beta} \frac{\partial \tilde{\phi}_{(j)}}{\partial \beta} (\nabla \cdot \mathbf{u}) w_n \bigg|_{\alpha=\alpha_i, \beta=\beta_n} \\
&+ \frac{\nu_v}{J(\alpha_i, \beta_j) w_i} \sum_{m=0}^{n_p-1} \frac{\partial \tilde{\phi}_{(i)}}{\partial \alpha} (\nabla \times \mathbf{u})_r w_m \bigg|_{\alpha=\alpha_m, \beta=\beta_j} \qquad (29)
\end{aligned}
$$