1 Geometry

1.1 Terrain-Following Cartesian Geometry

For a given vertical coordinate transform $Z = Z(\alpha, \beta, \xi)$. For example, Gal-Chen and Somerville (1975) coordinates:

$$Z(\alpha, \beta, \xi) = \xi \left[z_{top} - z_s(\alpha, \beta)\right] + z_s(\alpha, \beta). \tag{1}$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial Z}{\partial \alpha} \\ \frac{\partial Z}{\partial \beta} \\ \frac{\partial Z}{\partial \xi} \end{pmatrix} \begin{pmatrix} \frac{\partial Z}{\partial \alpha} & \frac{\partial Z}{\partial \beta} & \frac{\partial Z}{\partial \xi} \end{pmatrix} \tag{2}$$

$$J = \left(\frac{\partial Z}{\partial \xi}\right) \tag{3}$$

$$g^{ij} = \begin{pmatrix} 1 & 0 & -\left(\frac{\partial Z}{\partial \xi}\right)^{-1}\left(\frac{\partial Z}{\partial \alpha}\right) \\ 0 & 1 & -\left(\frac{\partial Z}{\partial \xi}\right)^{-1}\left(\frac{\partial Z}{\partial \beta}\right) \\ -\left(\frac{\partial Z}{\partial \xi}\right)^{-1}\left(\frac{\partial Z}{\partial \alpha}\right) & -\left(\frac{\partial Z}{\partial \xi}\right)^{-1}\left(\frac{\partial Z}{\partial \beta}\right) & \left(\frac{\partial Z}{\partial \xi}\right)^{-2}\left[1 + \left(\frac{\partial Z}{\partial \alpha}\right)^2 + \left(\frac{\partial Z}{\partial \beta}\right)^2\right] \end{pmatrix} \tag{4}$$

$$\Gamma^{\alpha}{}_{ij} = \mathbf{0}, \qquad \Gamma^{\beta}{}_{ij} = \mathbf{0}, \qquad \Gamma^{\xi}{}_{ij} = \left(\frac{\partial Z}{\partial \xi}\right)^{-1} \begin{pmatrix} \frac{\partial^2 Z}{\partial \alpha^2} & \frac{\partial^2 Z}{\partial \alpha \partial \beta} & \frac{\partial^2 Z}{\partial \alpha \partial \xi} \\ \frac{\partial^2 Z}{\partial \alpha \partial \beta} & \frac{\partial^2 Z}{\partial \beta^2} & \frac{\partial^2 Z}{\partial \xi \partial \beta} \\ \frac{\partial^2 Z}{\partial \xi \partial \alpha} & \frac{\partial^2 Z}{\partial \xi \partial \beta} & \frac{\partial^2 Z}{\partial \xi^2} \end{pmatrix} \tag{5}$$

1.2 Terrain-Following Cubed-Sphere Geometry

For a given vertical coordinate transform $r = R(\alpha, \beta, \xi)$. For example, Gal-Chen and Somerville (1975) coordinates:

$$R(\alpha, \beta, \xi) = \xi \left[z_{top} - z_s(\alpha, \beta)\right] + a + z_s(\alpha, \beta). \tag{6}$$

$$g_{ij} = \frac{a^2(1+X^2)(1+Y^2)}{\delta^4} \begin{pmatrix} 1+X^2 & -XY & 0 \\ -XY & 1+Y^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial R}{\partial \alpha} \\ \frac{\partial R}{\partial \beta} \\ \frac{\partial R}{\partial \xi} \end{pmatrix} \begin{pmatrix} \frac{\partial R}{\partial \alpha} & \frac{\partial R}{\partial \beta} & \frac{\partial R}{\partial \xi} \end{pmatrix} \tag{7}$$

$$J = \frac{1}{\delta^3}\left(\frac{\partial R}{\partial \xi}\right) a^2(1+X^2)(1+Y^2) \tag{8}$$

$$g^{ij} = \frac{\delta^2}{a^2(1+X^2)(1+Y^2)} \begin{pmatrix} 1+Y^2 & XY & 0 \\ XY & 1+X^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \tilde{g}^{ij}, \tag{9}$$

$$\tilde{g}^{\alpha\alpha} = 0 \tag{10}$$
$$\tilde{g}^{\alpha\beta} = 0 \tag{11}$$
$$\tilde{g}^{\beta\beta} = 0 \tag{12}$$
$$\tilde{g}^{\alpha\xi} = -\frac{\delta^2}{a^2(1+X^2)(1+Y^2)}\left(\frac{\partial R}{\partial \xi}\right)^{-1}\left[(1+Y^2)\left(\frac{\partial R}{\partial \alpha}\right) + XY\left(\frac{\partial R}{\partial \beta}\right)\right] \tag{13}$$
$$\tilde{g}^{\beta\xi} = -\frac{\delta^2}{a^2(1+X^2)(1+Y^2)}\left(\frac{\partial R}{\partial \xi}\right)^{-1}\left[XY\left(\frac{\partial R}{\partial \alpha}\right) + (1+X^2)\left(\frac{\partial R}{\partial \beta}\right)\right] \tag{14}$$
$$\tilde{g}^{\xi\xi} = \left(\frac{\partial R}{\partial \xi}\right)^{-2} + \frac{\delta^2}{a^2(1+X^2)(1+Y^2)}\left(\frac{\partial R}{\partial \xi}\right)^{-2}\left[(1+Y^2)\left(\frac{\partial R}{\partial \alpha}\right)^2 + 2XY\left(\frac{\partial R}{\partial \alpha}\right)\left(\frac{\partial R}{\partial \beta}\right) + (1+X^2)\left(\frac{\partial R}{\partial \beta}\right)^2\right] \tag{15}$$

$$\Gamma^{\alpha}{}_{ij} = \begin{pmatrix} \frac{2XY^2}{\delta^2} & -\frac{Y(1+Y^2)}{\delta^2} & 0 \\ -\frac{Y(1+Y^2)}{\delta^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^{\beta}{}_{ij} = \begin{pmatrix} 0 & -\frac{X(1+X^2)}{\delta^2} & 0 \\ -\frac{X(1+X^2)}{\delta^2} & \frac{2X^2Y}{\delta^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

$$\Gamma^{\xi}{}_{ij} = \left(\frac{\partial R}{\partial \xi}\right)^{-1} \begin{pmatrix} -\frac{2XY^2}{\delta^2}\left(\frac{\partial R}{\partial \alpha}\right) + \left(\frac{\partial^2 R}{\partial \alpha^2}\right) & \frac{Y(1+Y^2)}{\delta^2}\left(\frac{\partial R}{\partial \alpha}\right) + \frac{X(1+X^2)}{\delta^2}\left(\frac{\partial R}{\partial \beta}\right) + \left(\frac{\partial^2 R}{\partial \alpha \partial \beta}\right) & \left(\frac{\partial^2 R}{\partial \alpha \partial \xi}\right) \\ \cdots & -\frac{2X^2Y}{\delta^2}\left(\frac{\partial R}{\partial \beta}\right) + \left(\frac{\partial^2 R}{\partial \beta^2}\right) & \left(\frac{\partial^2 R}{\partial \beta \partial \xi}\right) \\ \cdots & \cdots & \left(\frac{\partial^2 R}{\partial \xi^2}\right) \end{pmatrix} \tag{17}$$

2 Hydrodynamics

The system of equations describing the hydrodynamic system in arbitrary geometry is as follows:

$$\frac{\partial u^\alpha}{\partial t} + u^\alpha \nabla_\alpha u^\alpha + u^\beta \nabla_\beta u^\alpha + u^\xi \nabla_\xi u^\alpha + \frac{1}{\rho}\nabla^\alpha p + f g^{\alpha j}\epsilon_{j\xi k} u^k = 0 \tag{18}$$
$$\frac{\partial u^\beta}{\partial t} + u^\alpha \nabla_\alpha u^\beta + u^\beta \nabla_\beta u^\beta + u^\xi \nabla_\xi u^\beta + \frac{1}{\rho}\nabla^\beta p + f g^{\beta j}\epsilon_{j\xi k} u^k = 0 \tag{19}$$
$$\frac{\partial \theta}{\partial t} + u^\alpha \frac{\partial \theta}{\partial \alpha} + u^\beta \frac{\partial \theta}{\partial \beta} = -u^\xi \frac{\partial \theta}{\partial \xi} \tag{20}$$
$$\frac{\partial u^\xi}{\partial t} + u^\alpha \nabla_\alpha u^\xi + u^\beta \nabla_\beta u^\xi + u^\xi \nabla_\xi u^\xi = -\frac{1}{\rho}\nabla^\xi p \tag{21}$$
$$\frac{\partial \rho}{\partial t} + \frac{1}{J}\frac{\partial}{\partial \alpha}(J\rho u^\alpha) + \frac{1}{J}\frac{\partial}{\partial \beta}(J\rho u^\beta) = -\frac{1}{J}\frac{\partial}{\partial \xi}(J\rho u^\xi) \tag{22}$$

Advection and geometry:

$$u^\alpha \nabla_\alpha u^k = u^\alpha \left[\frac{\partial u^k}{\partial \alpha} + \Gamma^k{}_{\alpha\alpha} u^\alpha + \Gamma^k{}_{\alpha\beta} u^\beta + \Gamma^k{}_{\alpha\xi} u^\xi \right] \tag{23}$$

$$u^\beta \nabla_\beta u^k = u^\beta \left[\frac{\partial u^k}{\partial \beta} + \Gamma^k{}_{\beta\alpha} u^\alpha + \Gamma^k{}_{\beta\beta} u^\beta + \Gamma^k{}_{\beta\xi} u^\xi \right] \tag{24}$$

$$u^\xi \nabla_\xi u^k = u^\beta \left[\frac{\partial u^k}{\partial \xi} + \Gamma^k{}_{\xi\alpha} u^\alpha + \Gamma^k{}_{\xi\beta} u^\beta + \Gamma^k{}_{\xi\xi} u^\xi \right] \tag{25}$$

Pressure gradient:

$$\nabla^k p = g^{k\alpha} \frac{\partial p}{\partial \alpha} + g^{k\beta} \frac{\partial p}{\partial \beta} + g^{k\xi} \frac{\partial p}{\partial \xi} \tag{26}$$

Coriolis Force:

$$f g^{\alpha j} \epsilon_{j\xi k} u^k = -f g^{\alpha\alpha} J u^\beta + f g^{\alpha\beta} J u^\alpha \tag{27}$$

$$f g^{\beta j} \epsilon_{j\xi k} u^k = -f g^{\beta\alpha} J u^\beta + f g^{\beta\beta} J u^\alpha \tag{28}$$

3 Horizontal Discretization

3.1 Tensor Product Basis

The tensor-product basis is

$$\phi_{(i,j)}(\alpha,\beta) = \tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta), \tag{29}$$

where $\tilde{\phi}_{(i)}(x)$ denotes the usual 1D GLL basis function at node $i \in (0,\ldots,n_p)$. For vector fields, the components of the covariant vector field are given by the tensor-product basis (29).

3.2 Spectral Element (Scalar Variational Form)

Consider a scalar conservation law of the form

$$\frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{F} = S, \tag{30}$$

with $\mathbf{F} = F^\alpha \mathbf{g}_\alpha + F^\beta \mathbf{g}_\beta$. Multiplying by the tensor-product basis $\phi_{(p,q)}(\alpha,\beta)$ and integrating over the whole domain yields

$$\iint \frac{\partial \psi}{\partial t} \phi_{(i,j)} dA = -\iint (\nabla \cdot \mathbf{F}) \phi_{(i,j)} dA + \iint S \phi_{(i,j)} dA, \tag{31}$$

where $dA = J d\alpha d\beta$. Applying integration by parts and using periodicity of the domain then leads to the weak form

$$\iint \frac{\partial \psi}{\partial t} \phi_{(i,j)} dA = \iint \nabla \phi_{(i,j)} \cdot \mathbf{F} dA + \iint S \phi_{(i,j)} dA. \tag{32}$$

Further expanding ψ in terms of the horizontal basis functions as

$$\psi(t,\alpha,\beta) = \sum_{(s,t)} \psi_{(s,t)}(t) \phi_{(s,t)}(\alpha,\beta) \tag{33}$$

leads to

$$\sum_{(s,t)} \frac{\partial \psi_{(s,t)}}{\partial t} \iint \phi_{(s,t)} \phi_{(i,j)} dA = \iint \nabla \phi_{(i,j)} \cdot \mathbf{F} dA + \iint S \phi_{(i,j)} dA. \tag{34}$$

Here the vertical dependence of ψ is implicit. For simplicity we now restrict our domain to a single spectral element (since the DSS procedure will later be used to account for inter-element exchange). Approximate integration is now applied,

$$\iint f(\alpha,\beta)dA = \Delta\alpha\Delta\beta \sum_{p=0}^{n_p-1}\sum_{q=0}^{n_p-1} f(\alpha_p,\beta_q)w_p w_q, \tag{35}$$

where w_i are the nodal weights of the GLL nodes on the reference element $[0,1]$. Consequently,

$$\iint \frac{\partial\psi}{\partial t}\phi_{(i,j)}dA = \Delta\alpha\Delta\beta \frac{\partial\psi_{(i,j)}}{\partial t} w_i w_j J(\alpha_i,\beta_j), \tag{36}$$

$$\iint \nabla\phi_{(i,j)}\cdot \mathbf{F}dA = \Delta\alpha\Delta\beta w_j \sum_{p=0}^{n_p-1} \frac{d\tilde{\phi}_{(i)}}{d\alpha}F^\alpha w_p J\bigg|_{\alpha=\alpha_p,\beta=\beta_j} + \Delta\alpha\Delta\beta w_i \sum_{q=0}^{n_p-1} \frac{d\tilde{\phi}_{(j)}}{d\beta}F^\beta w_q J\bigg|_{\alpha=\alpha_i,\beta=\beta_q}, \tag{37}$$

$$\iint S\phi_{(i,j)}dA = \Delta\alpha\Delta\beta S(\alpha_i,\beta_j)w_i w_j J(\alpha_i,\beta_j). \tag{38}$$

Substituting (36)-(38) into (31) gives the spectral element semi-discretization

$$\begin{aligned}\frac{\partial\psi_{(i,j)}}{\partial t} &= \frac{1}{w_i J(\alpha_i,\beta_j)} \sum_{p=0}^{n_p-1} \frac{d\tilde{\phi}_{(i)}}{d\alpha}F^\alpha w_p J\bigg|_{\alpha=\alpha_p,\beta=\beta_j} \\ &\quad + \frac{1}{w_j J(\alpha_i,\beta_j)} \sum_{q=0}^{n_p-1} \frac{d\tilde{\phi}_{(j)}}{d\beta}F^\beta w_q J\bigg|_{\alpha=\alpha_i,\beta=\beta_q} + S(\alpha_i,\beta_j)\end{aligned} \tag{39}$$

3.3 Spectral Element (Differential Form)

The spectral element method can also be derived in differential form by noting that basis functions can be interpreted as components of an interpolating polynomial. For an arbitrary function $f(\alpha,\beta)$ defined on element nodes we have

$$\frac{\partial f}{\partial\alpha}\bigg|_{\alpha=\alpha_i,\beta=\beta_j} = \sum_{p=0}^{n_p-1} \frac{d\tilde{\phi}_{(p)}}{d\alpha}\bigg|_{\alpha=\alpha_i} f(\alpha_p,\beta_j), \tag{40}$$

$$\frac{\partial f}{\partial\beta}\bigg|_{\alpha=\alpha_i,\beta=\beta_j} = \sum_{q=0}^{n_p-1} \frac{d\tilde{\phi}_{(q)}}{d\beta}\bigg|_{\beta=\beta_j} f(\alpha_i,\beta_q). \tag{41}$$

3.4 Flux Reconstruction / Discontinuous Galerkin

4 Hyperviscosity

Hyperviscosity is formulated in variational form.

4.1 Fourth-Order Scalar Hyperviscosity

4.2 Fourth-Order Vector Hyperviscosity

Fourth-order vector hyperviscosity is implemented using a two stage procedure:

$$\mathbf{f} = \mathcal{H}(1,1)\mathbf{u}^n, \tag{42}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \Delta t \mathcal{H}(\nu_d, \nu_v)\mathbf{f}. \tag{43}$$

The hyperdiffusion operator is defined implicitly via

$$\mathbf{f} = \mathcal{H}(\nu_d, \nu_v)\mathbf{u^n} \quad \Longleftrightarrow \quad \iint \mathbf{f} \cdot \boldsymbol{\phi} dA = \iint \nu_d (\nabla \cdot \boldsymbol{\phi})(\nabla \cdot \mathbf{u}^n) + \nu_v (\nabla \times \boldsymbol{\phi})^r (\nabla \times \mathbf{u}^n)_r dA, \tag{44}$$

where $dA = J d\alpha d\beta$ and

$$(\nabla \cdot \boldsymbol{\phi}) = g^{pq}\nabla_p \phi_q = \frac{1}{J}\frac{\partial}{\partial \alpha}\left(Jg^{\alpha\alpha}\phi_\alpha + Jg^{\alpha\beta}\phi_\beta\right) + \frac{1}{J}\frac{\partial}{\partial \beta}\left(Jg^{\beta\alpha}\phi_\alpha + Jg^{\beta\beta}\phi_\beta\right), \tag{45}$$

$$(\nabla \times \boldsymbol{\phi})^r = \epsilon^{rpq}\nabla_p \phi_q = \epsilon^{rpq}\left[\frac{\partial \phi_q}{\partial x^p} - \Gamma^k{}_{pq}\phi_k\right] = \frac{1}{J}\left[\frac{\partial \phi_\beta}{\partial \alpha} - \frac{\partial \phi_\alpha}{\partial \beta}\right]. \tag{46}$$

Here we assume that $(\nabla \cdot \mathbf{u})$ and $(\nabla \times \mathbf{u})_r$ (covariant radial component of curl) have already been computed.

4.2.1 Vector basis with zero β component

If $\phi_{(i,j)\alpha} = \tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)$ and $\phi_{(i,j)\beta} = 0$ then

$$\iint \mathbf{f} \cdot \phi dA = \iint f^\alpha \tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta) dA = f^\alpha_{(i,j)} w_i w_j J \Delta\alpha\Delta\beta \tag{47}$$

The divergent term is defined by

$$(\nabla \cdot \boldsymbol{\phi}_{(i,j)}) = \frac{1}{J}\frac{\partial}{\partial \alpha}\left(Jg^{\alpha\alpha}\phi_{(i,j)\alpha}\right) + \frac{1}{J}\frac{\partial}{\partial \beta}\left(Jg^{\beta\alpha}\phi_{(i,j)\alpha}\right) \tag{48}$$

$$= \frac{\tilde{\phi}_{(j)}(\beta)}{J}\frac{\partial}{\partial \alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right) + \frac{\tilde{\phi}_{(i)}(\alpha)}{J}\frac{\partial}{\partial \beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right), \tag{49}$$

and

$$\iint (\nabla \cdot \boldsymbol{\phi}_{(i,j)})(\nabla \cdot \mathbf{u}) dA$$

$$= \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J}\frac{\partial}{\partial \alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right) + \frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{\partial}{\partial \beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right)\right](\nabla \cdot \mathbf{u}) J w_m w_n \tag{50}$$

$$= \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1}\frac{\partial}{\partial \alpha}\left(Jg^{\alpha\alpha}\tilde{\phi}_{(i)}(\alpha)\right)(\nabla \cdot \mathbf{u}) w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1}\frac{\partial}{\partial \beta}\left(Jg^{\beta\alpha}\tilde{\phi}_{(j)}(\beta)\right)(\nabla \cdot \mathbf{u}) w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \tag{51}$$

$$= \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1} Jg^{\alpha\alpha}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla \cdot \mathbf{u}) w_m\Bigg|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1} Jg^{\beta\alpha}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla \cdot \mathbf{u}) w_n\Bigg|_{\alpha=\alpha_i,\beta=\beta_n} \tag{52}$$

Further, the vortical term is defined by

$$(\nabla \times \boldsymbol{\phi}_{(i,j)})^r = -\frac{1}{J}\frac{\partial \phi_{(i,j)\alpha}}{\partial \beta} = -\frac{\tilde{\phi}_{(i)}}{J}\frac{d\tilde{\phi}_{(j)}}{d\beta} \tag{53}$$

and so

$$\begin{aligned}
&\iint (\nabla \times \boldsymbol{\phi}_{(i,j)})^r (\nabla \times \mathbf{u})_r dA \\
&\quad = \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1} \left[-\frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{d\tilde{\phi}_{(j)}}{d\beta}\right](\nabla \times \mathbf{u})_r J w_m w_n \Bigg|_{\alpha=\alpha_m,\beta=\beta_n} &(54)\\
&\quad = -\Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1} \frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla \times \mathbf{u})_r w_n \Bigg|_{\alpha=\alpha_i,\beta=\beta_n} &(55)
\end{aligned}$$

Combining (47), (52) and (55) then gives

$$\begin{aligned}
f^{\alpha}_{(i,j)} = &\frac{\nu_d}{J(\alpha_i,\beta_j)w_i}\sum_{m=0}^{n_p-1} J g^{\alpha\alpha}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\cdot\mathbf{u})w_m\Bigg|_{\alpha=\alpha_m,\beta=\beta_j} \\
&+ \frac{\nu_d}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1} J g^{\beta\alpha}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\cdot\mathbf{u})w_n\Bigg|_{\alpha=\alpha_i,\beta=\beta_n} \\
&- \frac{\nu_v}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1} \frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\times\mathbf{u})_r w_n\Bigg|_{\alpha=\alpha_i,\beta=\beta_n} &(56)
\end{aligned}$$

4.2.2 Vector basis with zero α component

If $\phi_{(i,j)\alpha} = 0$ and $\phi_{(i,j)\beta} = \tilde{\phi}_{(j)}(\alpha)\tilde{\phi}_{(j)}(\beta)$ then

$$\iint \mathbf{f}\cdot\phi dA = \iint f^{\beta}\tilde{\phi}_{(i)}(\alpha)\tilde{\phi}_{(j)}(\beta)dA = f^{\beta}_{(i,j)} w_i w_j J\Delta\alpha\Delta\beta \tag{57}$$

The divergent term is defined by

$$\begin{aligned}
(\nabla\cdot\boldsymbol{\phi}_{(i,j)}) &= \frac{1}{J}\frac{\partial}{\partial\alpha}\left(J g^{\alpha\beta}\phi_{(i,j)\beta}\right) + \frac{1}{J}\frac{\partial}{\partial\beta}\left(J g^{\beta\beta}\phi_{(i,j)\beta}\right) &(58)\\
&= \frac{\tilde{\phi}_{(j)}(\beta)}{J}\frac{\partial}{\partial\alpha}\left(J g^{\alpha\beta}\tilde{\phi}_{(i)}\right) + \frac{\tilde{\phi}_{(i)}(\alpha)}{J}\frac{\partial}{\partial\beta}\left(J g^{\beta\beta}\tilde{\phi}_{(j)}\right), &(59)
\end{aligned}$$

and

$$\iint (\nabla \cdot \boldsymbol{\phi}_{(i,j)})(\nabla \cdot \mathbf{u}) dA$$

$$= \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1} \left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J} \frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}\right) + \frac{\tilde{\phi}_{(i)}(\alpha_m)}{J}\frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}\right)\right](\nabla\cdot\mathbf{u})Jw_m w_n \tag{60}$$

$$= \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1} \frac{\partial}{\partial\alpha}\left(Jg^{\alpha\beta}\tilde{\phi}_{(i)}(\alpha)\right)(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1} \frac{\partial}{\partial\beta}\left(Jg^{\beta\beta}\tilde{\phi}_{(j)}(\beta)\right)(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \tag{61}$$

$$= \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1} Jg^{\alpha\beta}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \Delta\alpha\Delta\beta w_i \sum_{n=0}^{n_p-1} Jg^{\beta\beta}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n} \tag{62}$$

Further, the vortical term is defined by

$$(\nabla \times \boldsymbol{\phi}_{(i,j)})^r = \frac{1}{J}\frac{\partial \phi_{(i,j)\beta}}{\partial\alpha} = \frac{\tilde{\phi}_{(j)}}{J}\frac{d\tilde{\phi}_{(i)}}{d\alpha} \tag{63}$$

and so

$$\iint (\nabla \times \boldsymbol{\phi}_{(i,j)})^r (\nabla \times \mathbf{u})_r dA$$

$$= \Delta\alpha\Delta\beta \sum_{m=0}^{n_p-1}\sum_{n=0}^{n_p-1}\left[\frac{\tilde{\phi}_{(j)}(\beta_n)}{J}\frac{d\tilde{\phi}_{(i)}}{d\alpha}\right](\nabla\times\mathbf{u})_r J w_m w_n\Big|_{\alpha=\alpha_m,\beta=\beta_n} \tag{64}$$

$$= \Delta\alpha\Delta\beta w_j \sum_{m=0}^{n_p-1}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\times\mathbf{u})_r w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j} \tag{65}$$

Combining (57), (62) and (65) then gives

$$f^{\beta}_{(i,j)} = \frac{\nu_d}{J(\alpha_i,\beta_j)w_i}\sum_{m=0}^{n_p-1} Jg^{\alpha\beta}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\cdot\mathbf{u})w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j}$$

$$+ \frac{\nu_d}{J(\alpha_i,\beta_j)w_j}\sum_{n=0}^{n_p-1} Jg^{\beta\beta}\frac{d\tilde{\phi}_{(j)}}{d\beta}(\nabla\cdot\mathbf{u})w_n\Big|_{\alpha=\alpha_i,\beta=\beta_n}$$

$$+ \frac{\nu_v}{J(\alpha_i,\beta_j)w_i}\sum_{m=0}^{n_p-1}\frac{d\tilde{\phi}_{(i)}}{d\alpha}(\nabla\times\mathbf{u})_r w_m\Big|_{\alpha=\alpha_m,\beta=\beta_j} \tag{66}$$