ADVANCED CONTROL SYSTEMS

Manipulator Dynamics

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Outline

Kinetic and Potential Energy
of a Rigid body

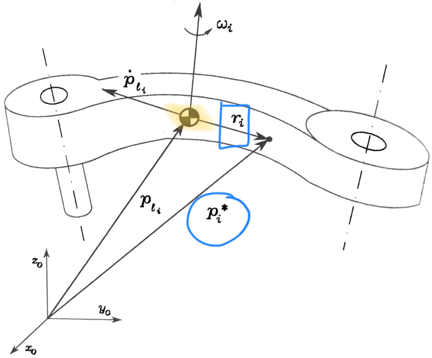
PROJECT

Equations of Motion

PROJECT

Kinetic and Potential Energy of a Rigid body

Kinematic description of Link i



- $\Sigma_0 = \{x_0, y_0, z_0\}$ base reference frame
- m_i mass of link i

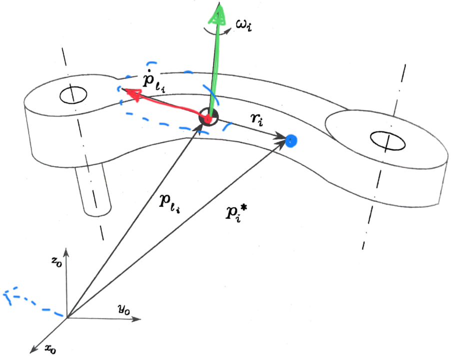
$$m_i = \int_{V_{\ell_i}} \rho dV = \int_{V_{\ell_i}} \rho(x, y, z) dxdydz$$

- p_i^* position ($\in \mathbb{R}^3$) of the generic point w.r.t. Σ_0
- p_{ℓ_i} position ($\in \mathbb{R}^3$) of the center of mass w.r.t. Σ_0

$$\boxed{p_{\ell_i} = \frac{1}{m_{\ell_i}} \int_{V_{\ell_i}} p_i^* \rho dV}$$

$$p_i^* = p_{\ell_i} + r_i \qquad r_i = p_i^* - p_{\ell_i}$$

Kinematic description of Link i



- $\dot{p}_{\ell_i}$ linear velocity ($\in \mathbb{R}^3$) of the center of mass w.r.t. Σ_0
- ω_i angular velocity ($\in \mathbb{R}^3$) of the center of mass w.r.t. Σ_0
- $\dot{p}_i^*$ linear velocity ($\in \mathbb{R}^3$) of the generic point w.r.t. Σ_0

$\omega \times =$ $S(\omega_i)$

$$
\boxed{
\begin{aligned}
\dot{p}_i^* &= \dot{p}_{\ell_i} + \omega_i \times r_i \\
&= \dot{p}_{\ell_i} + S(\omega_i) r_i
\end{aligned}
}
$$

Kinetic Energy

Assumption 1: rigid links
Assumption 2: rigid transmission

The total kinetic energy is given by the sum of the contributions relative to the motion of each link ($\mathcal{T}_{\ell_i}$) and the contributions relative to the motion of each joint motor actuator ($\mathcal{T}_{m_i}$)

$$\boxed{\mathcal{T} = \sum_{i=1}^{n} (\mathcal{T}_{\ell_i} + \mathcal{T}_{m_i})}$$

$n = \#\text{dof}$

From now on, we will consider only $\mathcal{T}_{\ell_i}$. The equations for $\mathcal{T}_{m_i}$ can be found in the textbook.

$$\boxed{\mathcal{T}_{\ell_i} = \frac{1}{2} \int_{V_{\ell_i}} \underbrace{(\dot{p}_i^*)^T \dot{p}_i^*}_{} \rho dV}$$

$\in \mathbb{R}$

$= \|\dot{p}_i^*\|^2 = \langle \dot{p}_i^*, \dot{p}_i^* \rangle$

Kinetic Energy

$$
\begin{aligned}
\mathcal{T}_{\ell_i} &= \frac{1}{2}\int_{V_{\ell_i}} (\dot p_i^*)^T \dot p_i^* \rho dV \\
&= \frac{1}{2}\int_{V_{\ell_i}} (\dot p_{\ell_i} + S(\omega_i) r_i)^T (\dot p_{\ell_i} + S(\omega_i) r_i)\rho dV \\
&= \underbrace{\frac{1}{2}\int_{V_{\ell_i}} \dot p_{\ell_i}^T \dot p_{\ell_i} \rho dV}_{\textit{Translational}} + \underbrace{\int_{V_{\ell_i}} (\dot p_{\ell_i})^T S(\omega_i) r_i \rho dV}_{\textit{Mutual}} + \underbrace{\frac{1}{2}\int_{V_{\ell_i}} r_i^T S(\omega_i)^T S(\omega_i) r_i \rho dV}_{\textit{Rotational}}
\end{aligned}
$$

where the *translational energy* is the kinetic energy of a point mass at CoM

$\left(\frac{1}{2}mv^2\right)$

$$
\underbrace{\frac{1}{2}\int_{V_{\ell_i}} \dot p_{\ell_i}^T \dot p_{\ell_i} \rho dV}_{\textit{Translational}} \overset{(\star)}{=} \dot p_{\ell_i}^T \dot p_{\ell_i} \frac{1}{2}\int_{V_{\ell_i}} \rho dV = \frac{1}{2} m_{\ell_i} \dot p_{\ell_i}^T \dot p_{\ell_i}
$$

$\|\dot p_{\ell_i}\|^2$

$(\star)$: $\dot p_{\ell_i}$ does not depend on dV

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The *mutual energy* is equal to zero

$$\underbrace{\int_{V_{\ell_i}} (\dot{p}_{\ell_i})^T S(\omega_i) r_i \rho dV}_{Mutual} = \int_{V_{\ell_i}} (\dot{p}_{\ell_i})^T S(\omega_i)(p_i^* - p_{\ell_i})\rho dV$$

$$\stackrel{(\square)}{=} (\dot{p}_{\ell_i})^T S(\omega_i) \left(\int_{V_{\ell_i}} p_i^* \rho dV - \boxed{p_{\ell_i}} \right) m_{\ell_i}$$

$$= (\dot{p}_{\ell_i})^T S(\omega_i) \left(p_{\ell_i} - p_{\ell_i} \right)$$

$$= 0$$

m_{ℓ_i} m_{ℓ_i}

$$\boxed{p_{\ell_i}} \int_{V_{\ell_i}} \rho dV$$

$$\int_{V_{\ell_i}} (p_i^* - p_{\ell_i}) \rho dV = \int_{V_{\ell_i}} p_i^* \rho dV - \int_{V_{\ell_i}} p_{\ell_i} \rho dV$$

($\square$): $\dot{p}_{\ell_i}$ and ω_i do not depend on dV

Kinetic Energy

The *rotational energy*

$$\underbrace{\frac{1}{2}\int_{V_{\ell_i}} r_i^T S(\omega_i)^T S(\omega_i) r_i \rho dV}_{Rotational} \overset{(\triangle)}{=} \frac{1}{2}\int_{V_{\ell_i}} \omega_i^T S(r_i)^T S(r_i)\omega_i \rho dV$$

symm. matrix

$$\overset{(\nabla)}{=} \frac{1}{2}\omega_i^T \left(\int_{V_{\ell_i}} S(r_i)^T S(r_i)\rho dV\right)\omega_i$$

$$\overset{(\Diamond)}{=} \frac{1}{2}\omega_i^T I_{\ell_i}\omega_i$$

A

$A^T A$

$(A^T A)^T = A^T A$

$\omega_i \times r_i = -r_i \times \omega_i = -S(r_i)\omega_i$

$(\triangle)$: $S(\omega_i) r_i = -S(r_i)\omega_i$, $\quad S(r_i) = \begin{bmatrix} 0 & -r_{iz} & r_{iy} \\ r_{iz} & 0 & -r_{ix} \\ -r_{iy} & r_{ix} & 0 \end{bmatrix}$

(∇): ω_i does not depend on dV

$(\Diamond)$: $I_{\ell_i} \triangleq \int_{V_{\ell_i}} S(r_i)^T S(r_i)\rho dV = \begin{bmatrix} I_{\ell_i\,xx} & -I_{\ell_i\,xy} & -I_{\ell_i\,xz} \\ * & I_{\ell_i\,yy} & -I_{\ell_i\,yz} \\ * & * & I_{\ell_i\,zz} \end{bmatrix}$

$-I_{\ell_i xy}$

inertia matrix

Kinetic Energy

- I_{ℓ_i} is the *inertia tensor* relative to the centre of mass of Link i expressed in the base frame Σ_0
- $I_{\ell_i} = I_{\ell_i}^T$, *symmetric matrix*
- I_{ℓ_i} depends on q, i.e. it is *configuration-dependent*

Kinetic Energy

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Σ_i

What happens to the inertia tensor when expressed w.r.t. the frame Σ_i $\boxed{(I_{\ell_i}^i)}$ attached to the Link i instead of Σ_0?

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What happens to the inertia tensor when expressed w.r.t. the frame Σ_i ($I_{\ell_i}^i$) attached to the Link i instead of Σ_0?

Since

$$\underbrace{\frac{1}{2}\omega_i^T I_{\ell_i}\omega_i}_{\Sigma_0} = \underbrace{\frac{1}{2}(\omega_i^i)^T I_{\ell_i}^i \omega_i^i}_{\Sigma_i}$$

$$\frac{1}{2}\omega_i^T I_{\ell_i}\omega_i = \frac{1}{2}(R_i\omega_i^i)^T I_{\ell_i}(R_i\omega_i^i) = \frac{1}{2}\omega_i^{iT}\boxed{R_i^T I_{\ell_i} R_i}\,\omega_i^i$$

(i.e. the product is invariant with respect to the chosen reference frame) ~~and exploiting~~ $\boxed{\omega_i^i = R_i^T\omega_i}$, we have

$$I_{\ell_i} = R_i\, I_{\ell_i}^i\, R_i^T, \qquad \boxed{(\, I_{\ell_i}^i = R_i^T\, I_{\ell_i}\, R_i\,)}$$

$R_i^{(0)}: v^i \in \Sigma_i \to v \in \Sigma_0$

$(R_i)^{-1} = R_i^T$

Kinetic Energy

- $I_{\ell_i}(q) = R_i(q)\, \boxed{I^i_{\ell_i}}\, R_i^T(q)$
- $I^i_{\ell_i}$ is constant, *configuration-independent*

Σ_i

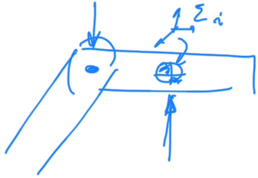
- If the axes of Link *i* frame coincide with the central axes of inertia, then the inertia cross-products are null and the inertia tensor relative to the centre of mass is a *diagonal matrix*

Kinetic Energy

- $I_{\ell_i}(q) = R_i(q)\, I_{\ell_i}^i\, R_i^T(q)$
- $I_{\ell_i}^i$ is constant, *configuration-independent*
- If the axes of Link i frame coincide with the central axes of inertia, then the inertia cross-products are null and the inertia tensor relative to the centre of mass is a *diagonal matrix*

The kinetic energy is

$$
\begin{aligned}
\mathbb{R} \ni \mathcal{T}_{\ell_i} &= \overbrace{\frac{1}{2} m_{\ell_i} \dot{p}_{\ell_i}^T \dot{p}_{\ell_i}}^{\in \mathbb{R}} + \overbrace{\frac{1}{2} \omega_i^T I_{\ell_i} \omega_i}^{\in \mathbb{R}} \\
&= \frac{1}{2} m_{\ell_i} \dot{p}_{\ell_i}^T \dot{p}_{\ell_i} + \frac{1}{2} \omega_i^T R_i\, I_{\ell_i}^i\, R_i^T \omega_i
\end{aligned}
$$



where $\dot{p}_{\ell_i}$ and ω_i are function of q (besides R_i, of course)

We actually proved the König's theorem

Theorem

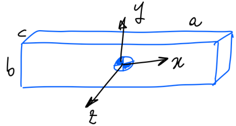
The kinetic energy of a system of particles is the sum of the kinetic energy associated to the movement of the center of mass ($\star$) and the kinetic energy associated to the movement of the particles relative to the center of mass ($\Diamond$).

$$\mathcal{T}_{\ell_i} = \underbrace{\frac{1}{2} m_{\ell_i} \dot{p}_{\ell_i}^T \dot{p}_{\ell_i}}_{(\star)} + \underbrace{\frac{1}{2} \omega_i^T R_i \, I_{\ell_i}^i \, R_i^T \omega_i}_{(\Diamond)}$$

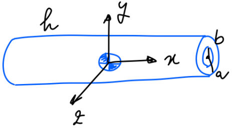
Examples of body inertia matrices

Assumptions: homogeneous body of mass m with symmetry

Body inertia matrices (w.r.t CoM): $I_C = \begin{bmatrix} I_{C,xx} & 0 & 0 \\ 0 & I_{C,yy} & 0 \\ 0 & 0 & I_{C,zz} \end{bmatrix}$



$$I_C = \begin{bmatrix} \frac{1}{12}m(b^2+c^2) & 0 & 0 \\ 0 & \frac{1}{12}m(a^2+c^2) & 0 \\ 0 & 0 & \frac{1}{12}m(a^2+b^2) \end{bmatrix}$$



$$I_C = \begin{bmatrix} \frac{1}{2}m(a^2+b^2) & 0 & 0 \\ 0 & \frac{1}{2}m\left(3(a^2+b^2)^2+h^2\right) & 0 \\ 0 & 0 & \frac{1}{2}m\left(3(a^2+b^2)^2+h^2\right) \end{bmatrix}$$

yy

zz

Parallel axis theorem (Steiner theorem)

Theorem

Let I_C be the inertia matrix with respect to a reference frame Σ_C with origin on the center of mass. The inertia I with respect to another reference frame Σ obtained translating Σ_C by the vector $r \in \mathbb{R}^3$ is given by

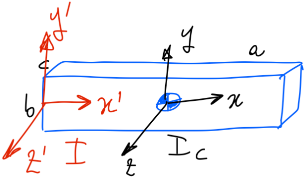
$$\boxed{I = I_C + m\underbrace{S^T(r)S(r)}_{} = I_C + m\underbrace{\left(r^T r\, I_{3\times 3} - rr^T\right)}_{}}$$

Σ $\quad$ Σ_C

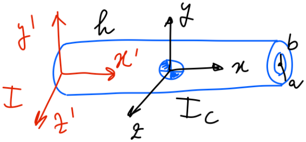
Homework. Prove

$$I_C + mS^T(r)S(r) = I_C + m\,\underbrace{\left(r^T r\, I_{3\times3} - rr^T\right)}_{\text{w.r.t. } \Sigma_C}$$

Homework.



$$I = I_C + m\left(\begin{bmatrix} -\frac{a}{2} & 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{a}{2} \\ 0 \\ 0 \end{bmatrix} I_{3\times3} - \begin{bmatrix} -\frac{a}{2} \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} -\frac{a}{2} \\ 0 \\ 0 \end{bmatrix}^T\right) = \cdots$$



$$I = I_C + m\left(\begin{bmatrix} -\frac{h}{2} & 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{h}{2} \\ 0 \\ 0 \end{bmatrix} I_{3\times3} - \begin{bmatrix} -\frac{h}{2} \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} -\frac{h}{2} \\ 0 \\ 0 \end{bmatrix}^T\right) = \cdots$$

Kinetic Energy

Given $\boxed{\mathcal{T}_{\ell_i} = \frac{1}{2} m_{\ell_i} \dot{p}_{\ell_i}^T \dot{p}_{\ell_i} + \frac{1}{2} \omega_i^T R_i I_{\ell_i}^i R_i^T \omega_i}$, how can we compute

$$\dot{p}_{\ell_i}(q) = ? \qquad \omega_i(q) = ?$$

We know that the Cartesian velocity of the EE is related to the joint velocity via the Jacobian; however, this relationship holds also for intermediate links $i = 1, \ldots, n$ (*Partial Jacobians*)

$$\dot{p}_{\ell_i} = J_P^{\ell_i}(q)\dot{q}, \qquad \omega_i = J_O^{\ell_i}(q)\dot{q}$$

R

$k > i$

$$\dot{p}_{\ell_i} = \begin{bmatrix} j_{P1}^{\ell_i} & j_{P2}^{\ell_i} & \cdots & j_{Pi}^{\ell_i} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_i \\ \dot{q}_{i+1} \\ \vdots \\ \dot{q}_n \end{bmatrix}, \qquad \omega_i = \begin{bmatrix} j_{O1}^{\ell_i} & j_{O2}^{\ell_i} & \cdots & j_{Oi}^{\ell_i} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_i \\ \dot{q}_{i+1} \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

1 2 i i+1 n

$$\dot{p}_{\ell_i} = \begin{bmatrix} j_{P1}^{\ell_i} & j_{P2}^{\ell_i} & \cdots & j_{Pi}^{\ell_i} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_i \\ \dot{q}_{i+1} \\ \vdots \\ \dot{q}_n \end{bmatrix}, \quad \omega_i = \begin{bmatrix} j_{O1}^{\ell_i} & j_{O2}^{\ell_i} & \cdots & j_{Oi}^{\ell_i} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_i \\ \dot{q}_{i+1} \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

The columns of the Jacobians are

$$\boxed{j_{Pj}^{\ell_i} = \begin{cases} z_{j-1}, & \text{prismatic joint} \\ z_{j-1} \times (p_{\ell_i} - p_{j-1}), & \text{revolute joint} \end{cases}} \qquad j_{Oj}^{\ell_i} = \begin{cases} 0, & \text{prismatic joint} \\ z_{j-1}, & \text{revolute joint} \end{cases}$$

depends on q

where

- p_{j-1} is the position vector of the origin of Frame Σ_{j-1} w.r.t. Σ_0
- z_{j-1} is the unit vector of axis z of Frame Σ_{j-1} w.r.t. Σ_0

Finally

$$\boxed{\mathcal{T}_{\ell_i} = \frac{1}{2} m_{\ell_i} \overbrace{\dot{q}^T (J_P^{\ell_i})^T J_P^{\ell_i} \dot{q}}^{\dot{q},q} + \overbrace{\frac{1}{2} \dot{q}^T (J_O^{\ell_i})^T R_i I_{\ell_i}^i R_i^T J_O^{\ell_i} \dot{q}}^{\dot{q},q}}$$

where only the blue terms ~~depends~~ depend on q

$$\mathcal{T}_{\ell_i} = \frac{1}{2} m_{\ell_i} \dot{q}^T (J_P^{\ell_i})^T J_P^{\ell_i} \dot{q} + \frac{1}{2} \dot{q}^T (J_O^{\ell_i})^T R_i I_{\ell_i}^i R_i^T J_O^{\ell_i} \dot{q}$$

The **total Kinetic Energy** is a configuration-dependent quadratic function in $\dot{q}$

$$\underline{\mathcal{T}(q, \dot{q})} = \sum_{i=1}^{n} \mathcal{T}_{\ell_i} = \frac{1}{2} \sum_{i=1}^{n} \left(m_{\ell_i} \dot{q}^T (J_P^{\ell_i})^T J_P^{\ell_i} \dot{q} + \dot{q}^T (J_O^{\ell_i})^T R_i I_{\ell_i}^i R_i^T J_O^{\ell_i} \dot{q} \right)$$

$$= \frac{1}{2} \dot{q}^T \underbrace{\left[\sum_{i=1}^{n} \left(m_{\ell_i} \underline{(J_P^{\ell_i})^T J_P^{\ell_i}} + \underline{(J_O^{\ell_i})^T R_i I_{\ell_i}^i R_i^T J_O^{\ell_i}} \right) \right]}_{B(q)} \dot{q} = \underline{\frac{1}{2} \dot{q}^T B(q) \dot{q}}$$

$A^T P A$, $P = P^T$
$\Rightarrow A^T P A$ is sym.

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$$\mathcal{T}(q,\dot{q}) = \frac{1}{2}\dot{q}^T B(q)\dot{q}$$

where $B(q) \in \mathbb{R}^{n\times n}$ is the *inertia matrix* which is

- symmetric $B(q) = B(q)^T, \ \forall q \in \mathbb{R}^n$
- positive definite $B(q) \succ 0, \ \forall q \in \mathbb{R}^n$ ($\Rightarrow$ nonsingular matrix $\forall q \in \mathbb{R}^n$)
- configuration-dependent

λ eig. of $B(q) \Rightarrow \lambda > 0$

Remarks

1. $\mathcal{T}(q,\dot{q}) \geq 0$
2. $\mathcal{T}(q,\dot{q}) = 0$ if and only if $\dot{q} = 0$

Kinetic Energy

$$\mathcal{T}(q,\dot{q}) = \frac{1}{2}\dot{q}^T B(q)\dot{q}$$

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Remarks

1. $\mathcal{T}(q,\dot{q}) \geq 0$
2. $\mathcal{T}(q,\dot{q}) = 0$ if and only if $\dot{q} = 0$

These properties are the same for selecting a candidate Lyapunov function... it is not a coincidence and it will be exploited later!

Assumption 1: rigid links
Assumption 2: rigid transmission

The total Potential energy is given by the sum of the contributions relative to each link ($\mathcal{U}_{\ell_i}$) and the contributions relative to each joint motor actuator ($\mathcal{U}_{m_i}$)

$$\boxed{\mathcal{U} = \sum_{i=1}^{n} (\mathcal{U}_{\ell_i} + \mathcal{U}_{m_i})}$$

From now on, we will consider only $\mathcal{U}_{\ell_i}$. The equations for $\mathcal{U}_{m_i}$ can be found in the textbook.
Without elastic components, the potential energy is only due to the gravitational forces

$$\mathcal{U}_{\ell_i} = -\int_{V_{\ell_i}} g_0^T p_i^* \rho dV = -m_{\ell_i} g_0^T p_{\ell_i}, \qquad \boxed{\mathcal{U} = -\sum_{i=1}^{n} m_{\ell_i} g_0^T p_{\ell_i}}$$

z

where g_0 is the gravity acceleration vector in the base frame Σ_0 ($\mathbf{g}_0 = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$)

Potential Energy

$$\mathcal{U} = \sum_{i=1}^{n} \mathcal{U}_i = -\sum_{i=1}^{n} m_{\ell_i} g_0^T p_{\ell_i}$$

q

The position of the center of mass of the link i w.r.t. the base reference frame Σ_0, p_{ℓ_i}, can be expressed w.r.t. the reference frame Σ_i attached to the link, $\boxed{p^i_{\ell_i}}$, by

$$\underbrace{\begin{pmatrix} p_{\ell_i} \\ 1 \end{pmatrix}}_{\Sigma_0} = \underbrace{T_1^0(q_1)\, T_2^1(q_2) \cdots T_i^{i-1}(q_i)} \underbrace{\begin{pmatrix} p^i_{\ell_i} \\ 1 \end{pmatrix}}_{\Sigma_i}$$

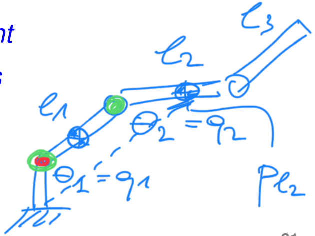
does not depend on q

where $T_j^{j-1}(q_j)$ are the homogeneous transformation matrices.

- *The coordinate of the CoM with respect to Σ_i, $p^i_{\ell_i}$, is constant*
- $\mathcal{U}_i = \mathcal{U}_i(q_1, q_2, \ldots, q_i)$ *for open kinematic chain manipulators*

p_{ℓ_1} is a function of q_1,

p_{ℓ_2} is a function of q_1, q_2, ...



"link" causality


Project

To do

- Compute the kinetic energy
- Compute the potential energy

Nicola Marchiotti
Fabio Castellini

Equations of Motion

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The Lagrangian is given by

$$\mathcal{L}(q,\dot{q}) = \mathcal{T}(q,\dot{q}) - \mathcal{U}(q) = \boxed{\frac{1}{2}\dot{q}^T B(q)\dot{q}} + \boxed{\sum_{i=1}^{n} m_{\ell_i} g_0^T p_{\ell_i}}$$

$n = \#\text{DoF}$

$f(q_1, \dots, q_i)$

$q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix}$

We have to solve

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial q}\right)^T = \tau}$$

where τ_i is the generalized force performing work on the q_i generalized coordinate. τ_i is non-conservative.

Let's compute all the derivatives one by one

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)^T = B(q)\dot{q}, \in \mathbb{R}^n \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)^T = B(q)\ddot{q} + \dot{B}(q)\dot{q}$$

$$\left(\frac{\partial \mathcal{L}}{\partial q}\right)^T = \frac{1}{2}\left(\frac{\partial}{\partial q}\dot{q}^T B(q)\dot{q}\right)^T - \left(\frac{\partial \mathcal{U}}{\partial q}\right)^T$$

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$$B(q)\ddot{q} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_n \\ \vdots \\ \ddot{q}_n \end{bmatrix}$$

i-th row of $B(q)$

$$B(q)\ddot{q} + \dot{B}(q)\dot{q} - \frac{1}{2}\left(\frac{\partial}{\partial q}\dot{q}^T B(q)\dot{q}\right)^T + \left(\frac{\partial \mathcal{U}}{\partial q}\right)^T = \tau$$

For the $i-th$ DOF, we have

$$\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n} \frac{db_{ij}(q)}{dt}\dot{q}_j - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial b_{jk}(q)}{\partial q_i}\dot{q}_k\dot{q}_j - \sum_{j=1}^{n} m_{\ell_j} g_0^T \frac{\partial p_{\ell_j}}{\partial q_i} = \tau_i$$

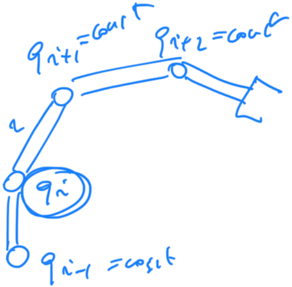
and finally

$$\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \underbrace{\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial b_{ij}(q)}{\partial q_k}\dot{q}_k\dot{q}_j - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial b_{jk}(q)}{\partial q_i}\dot{q}_k\dot{q}_j}_{\triangleq \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}(q)\dot{q}_k\dot{q}_j} \underbrace{- \sum_{j=1}^{n} m_{\ell_j} g_0^T j_{P_i}^{\ell_j}(q)}_{\triangleq g_i(q)} = \tau_i$$

$$\frac{d b_{ij}(q)}{dt} = \frac{\partial b_{ij}(q)}{\partial q_k}\frac{dq_k}{dt} = \frac{\partial b_{ij}(q)}{\partial q_k}\dot{q}_k$$

$$\boxed{\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}(q)\dot{q}_k\dot{q}_j + g_i(q) = \tau_i,} \qquad i = 1, \ldots, n$$

- $\boxed{b_{ii}(q)}$ is the moment of inertia ~~at~~ around Joint i axis when the other joints are blocked ($q_j = const, \forall j \neq i$)
- $\boxed{b_{ii}(q) = b_{ii} > 0}$
- $\boxed{b_{ij}}$ effects of acceleration of Joint j on Joint i



$$\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}(q)\dot{q}_k\dot{q}_j + g_i(q) = \tau_i, \qquad i = 1, \ldots, n$$

- $b_{ii}(q)$ is the moment of inertia at Joint i axis when the other joints are blocked ($q_j = const, \forall j \neq i$)
- $b_{ii}(q) = b_{ii} > 0$
- b_{ij} effects of acceleration of Joint j on Joint i

$j=k$

- $h_{ijj}\dot{q}_j^2$ is the centrifugal effect induced on Joint i by velocity of Joint j, ($h_{iii} \equiv 0, \forall i$)

$j \neq k$

- $h_{ijk}\dot{q}_j\dot{q}_k$ is the Coriolis effect induced on Joint i by velocities of Joints j and k

Equations of Motion

$$\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}(q)\dot{q}_k\dot{q}_j + g_i(q) = \tau_i, \qquad i = 1, \ldots, n$$

- $b_{ii}(q)$ is the moment of inertia at Joint i axis when the other joints are blocked ($q_j = const, \forall j \neq i$)
- $b_{ii}(q) = b_{ii} > 0$
- b_{ij} effects of acceleration of Joint j on Joint i
- $h_{ijj}\dot{q}_j^2$ is the centrifugal effect induced on Joint i by velocity of Joint j, ($h_{iii} \equiv 0, \forall i$)
- $h_{ijk}\dot{q}_j\dot{q}_k$ is the Coriolis effect induced on Joint i by velocities of Joints j and k
- g_i is the moment generated at Joint i axis of the manipulator by gravity.

Equations of Motion

$$\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}(q)\dot{q}_k\dot{q}_j + g_i(q) = \tau_i, \qquad i = 1, \ldots, n$$

Second-order non-linear diff. equations

- linear terms in acceleration $\ddot{q}$
- quadratic terms in velocity $\dot{q}$
- nonlinear terms in position q

mainly trigonometric functions (if there are revolute joints)

Equations of Motion

The equations of motion

$$\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \boxed{\sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}(q)\dot{q}_k\dot{q}_j} + g_i(q) = \tau_i, \qquad i = 1, \ldots, n$$

can be rewritten as

$$\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \boxed{\sum_{j=1}^{n} c_{ij}(q,\dot{q})\dot{q}_j} + g_i(q) = \tau_i, \qquad i = 1, \ldots, n$$

$\square \in \mathbb{R}$

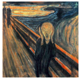
Equations of Motion

The equations of motion

$$\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n} \boxed{\sum_{k=1}^{n} h_{ijk}(q)\dot{q}_k}\,\dot{q}_j + g_i(q) = \tau_i, \qquad i = 1, \ldots, n$$

can be rewritten as

$$\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n} \boxed{c_{ij}(q, \dot{q})}\,\dot{q}_j + g_i(q) = \tau_i, \qquad i = 1, \ldots, n$$



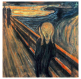
The choice of $\{c_{ij}\}$ is not unique!

The equations of motion

$$\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}(q)\dot{q}_k\dot{q}_j + g_i(q) = \tau_i, \qquad i = 1, \dots, n$$

can be rewritten as

$$\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n} c_{ij}(q,\dot{q})\dot{q}_j + g_i(q) = \tau_i, \qquad i = 1, \dots, n$$



The choice of $\{c_{ij}\}$ is not unique!

However, there is a clever choice: *Christoffel symbols of the first type*

$$
\begin{aligned}
\sum_{j=1}^{n} c_{ij}(q)\dot{q}_j &= \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}\dot{q}_k\dot{q}_j = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial b_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial b_{jk}}{\partial q_i}\right)\dot{q}_k\dot{q}_j \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial b_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)\dot{q}_k\dot{q}_j \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n}\boxed{\underbrace{\frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)}_{\triangleq c_{ijk}}}\dot{q}_k\dot{q}_j \\
&= \sum_{j=1}^{n}\boxed{\underbrace{\sum_{k=1}^{n} c_{ijk}\dot{q}_k}_{\triangleq c_{ij}}}\dot{q}_j
\end{aligned}
$$

Property: $\boxed{c_{ijk} = c_{ikj}}$

Equations of Motion

The n equations of motion

$$\boxed{\sum_{j=1}^{n} b_{ij}(q)\ddot{q}_j + \sum_{j=1}^{n} c_{ij}(q,\dot{q})\dot{q}_k\dot{q}_j + g_i(q) = \tau_i,} \qquad i = 1,\dots,n$$

can be written more compactly as

$$\boxed{B}\underset{\ddot q}{\boxed{}} + \boxed{C(q,\dot q)}\underset{\dot q}{\boxed{}} + \underset{g(q)}{\boxed{}} = \underset{\tau}{\boxed{}}$$

$$\boxed{B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau}$$

If we have to take into account friction (*viscous friction* $F_v\dot{q}$, *Coulomb friction* $F_s\text{sign}(\dot{q})$)

$F_v > 0$ pos. def. matrix
$F_s > 0$ " " "

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F_v\dot{q} + F_s\text{sign}(\dot{q}) + g(q) = \tau$$

If the end-effector interacts with the environment via the *external wrench* h_e we end up with

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F_v\dot{q} + F_s\text{sign}(\dot{q}) + g(q) = \tau - J^T(q)h_e$$

Set of n nonlinear second-order differential equations.


Project

PROJECT – Assignment # 3

To do

- equations of motion (dynamic model)

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Matteo Meneghetti