ADVANCED CONTROL SYSTEMS

Manipulator Dynamics

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Outline

Complements

Linearization

Operational Space Dynamic Model

PROJECT

Complements

Assumption: only the spinning rotor velocity is taken into account in the rotational part of the kinetic energy.

This means that the kinetic energy of the i-th motor (usually located on the link $i-1$) is given by

$$\begin{aligned} T_{m_i} = \frac{1}{2} I_{m_i} \dot{\theta}_{m_i}^2 &= \frac{1}{2} I_{m_i} n_i^2 \dot{q}_i^2 && [n_i \text{ is the gear ratio}] \\ &= \frac{1}{2} B_{m_i} \dot{q}_i^2 && [B_{m_i} > 0] \end{aligned}$$

The total kinetic energy for the n rotors is

$$T_m(\dot{q}) = \sum_{i=1}^{n} T_{m_i}(\dot{q}_i) = \sum_{i=1}^{n} \frac{1}{2} B_{m_i} \dot{q}_i^2 = \frac{1}{2} \dot{q}^T B_m \dot{q}, \quad \text{with } B_m := \begin{bmatrix} B_{m_1} & 0 & & \\ 0 & B_{m_2} & \ddots & \\ & \ddots & \ddots & 0 \\ & & 0 & B_{m_n} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

with $B_m \succ 0$ by construction and constant.

Rotor Inertia

The dynamic model

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F_v\dot{q} + F_s\mathrm{sign}(\dot{q}) + g(q) = \tau - J^T(q)h_e$$

can be easily integrated with the contribution of B_m as

$$(B(q) + B_m)\,\ddot{q} + C(q,\dot{q})\dot{q} + F_v\dot{q} + F_s\mathrm{sign}(\dot{q}) + g(q) = \tau - J^T(q)h_e$$

since the rotor kinetic energy does not play any role in $C(q,\dot{q})$ because B_m is constant.

τ_i is the torque *after* the gear box of the i-th motor.

$$\tau_i = n_i\tau_{m_i}$$

The matrices in the dynamic model

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F_v\dot{q} + F_s\text{sign}(\dot{q}) + g(q) = \tau - J^T(q)h_e$$

satisfy the following properties

- the element $B_{nn}(q)$ is always constant: $B_{nn}(q) = b_{nn}$
- $k_0 \leq \|B(q)\| \leq k_1 + k_2\|q\| + k_3\|q\|^2$
- $\|C(q,\dot{q})\| \leq (k_4 + k_5\|q\|)\,\|\dot{q}\|$
- $\|g(q)\| \leq k_6 + k_7\|q\|$

If the robot has only revolute joints or bounded prismatic joints ($q_i = d_i \in [d_{i,min}, d_{i,max}]$)

- $k_0 \leq \|B(q)\| \leq k_1$
- $\|C(q,\dot{q})\| \leq k_4\|\dot{q}\|$
- $\|g(q)\| \leq k_6$

Elastic joints

Why *elasticity*?

- Motion transmissions by belts, cables, harmonic drives, etc have *intrinsic* flexibility
- In collaborative robotics / physical Human-Robot interaction the compliance of the robot is increase by inserting *elastic elements* (e.g. Serial Elastic Actuators SEA)

  $\rightarrow$ increase safety

  $\rightarrow$ increase energy efficiency

Why *joint* elasticity?

- flexibility is modeled as *concentrated at the joints* to make the analysis easier
- flexibility is constrained to small deformation (i.e. *linear elastic regime*)

  $\rightarrow$ a stiffness coefficient k_i for the i-th joint

With elastic elements at each joint, *2n generalized coordinates* are needed to model the manipulator dynamics:

- n before the elastic elements, i.e. at the motors' side after the gear box $\theta \in \mathbb{R}^n$ (i.e. $\theta_i = \frac{\theta_{m,i}}{n_i}$)
- n after the elastic elements, i.e. at the links' side $q \in \mathbb{R}^n$

It is necessary to add the *elastic potential energy* U_e

Let $U_{e,i}$ be the elastic potential energy of the i-th joint

$$U_{e,i} = \frac{1}{2}k_i\left(q_i - \theta_i\right)^2 = \frac{1}{2}k_i\left(q_i - \frac{\theta_{m,i}}{n_i}\right)^2, \qquad k_i > 0$$

then ($K \succ 0$)

$$U_e = \sum_{i=1}^{n} U_{e,i} = \sum_{i=1}^{n} \frac{1}{2}k_i\left(q_i - \theta_i\right)^2 = \frac{1}{2}(q-\theta)^T K (q-\theta), \text{ where } K := \begin{bmatrix} k_1 & 0 & & \\ 0 & k_2 & \ddots & \\ & \ddots & \ddots & 0 \\ & & 0 & k_n \end{bmatrix} \in \mathbb{R}^{n\times n}$$

The dynamic model obtained solving the Euler equations with the motor kinematic energy (i.e. the inertia matrix B_m) and the elastic potential energy (i.e. the stiffness matrix K) is

$$\begin{aligned} B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + K(q-\theta) &= 0 \\ B_m\ddot{\theta} + K(\theta - q) &= \tau \end{aligned}$$

This is a system of $2n$ second-order differential equations on (q,θ).

If it is necessary to take into account external torques performing work on q and friction effects, the previous equations become

$$\begin{aligned} B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) + K(q-\theta) &= -J^T(q)h_e \\ B_m\ddot{\theta} + F_m\dot{\theta} + K(\theta - q) &= \tau \end{aligned}$$

Linearization

Linearization at an equilibrium point q_e

The Lagrangian model

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

can be written in the *state-space representation* $\dot{x}(t) = f(x,u)$ by defining the state vector x as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleq \begin{bmatrix} q \\ \dot{q} \end{bmatrix},$$

and re-arranging the implicit differential equations as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} x_2 \\ -B^{-1}(x_1)\left(C(x_1,x_2)x_2 + g(x_1)\right) \end{bmatrix} + \begin{bmatrix} 0 \\ B^{-1}(x_1)u \end{bmatrix} \\ &= f(x_1,x_2,u) \end{aligned}$$

We want to derive the *linear dynamic approximation* of the robot which is valid around a given equilibrium point q_e.

Since q_e is constant, then $\dot{q}_e = 0$. The equilibrium state vector is

$$x_e = \begin{bmatrix} x_{e1} \\ x_{e2} \end{bmatrix} = \begin{bmatrix} q_e \\ 0 \end{bmatrix},$$

Moreover, $(q, \dot{q}) = (q_e, 0) \Rightarrow \ddot{q} = 0$.

The nonlinear model evaluated in $(q_e, 0)$

$$B(q_e)\,0 + C(q_e, 0)\,0 + g(q_e) = u_e$$

gives the corresponding ‘equilibrium’ command

$$u_e = g(q_e)$$

We now consider the variations around the equilibrium point $(q_e, 0)$ and the command u_e

$$
\begin{aligned}
q &= q_e + \delta_q \\
\dot{q} &= \dot{q}_e + \dot{\delta}_q = \dot{\delta}_q \\
\ddot{q} &= \ddot{q}_e + \ddot{\delta}_q = \ddot{\delta}_q \\
u &= u_e + \delta_u
\end{aligned}
$$

we have (around the equilibrium point)

$$
\begin{aligned}
B(q)\ddot{q} &\simeq B(q_e)\ddot{q}_e + \left.\frac{\partial(B(q)\ddot{q})}{\partial q}\right|_{\substack{q=q_e\\ \dot{q}=0\\ \ddot{q}=0\\ u=u_e}} \delta_q + \left.\frac{\partial(B(q)\ddot{q})}{\partial \dot{q}}\right|_{\substack{q=q_e\\ \dot{q}=0\\ \ddot{q}=0\\ u=u_e}} \dot{\delta}_q + \left.\frac{\partial(B(q)\ddot{q})}{\partial \ddot{q}}\right|_{\substack{q=q_e\\ \dot{q}=0\\ \ddot{q}=0\\ u=u_e}} \ddot{\delta}_q \\
&\simeq B(q_e)\ddot{\delta}_q
\end{aligned}
$$

$$
\begin{aligned}
C(q,\dot q)\dot q &\simeq C(q_e,\dot q_e)\dot q_e + \left.\frac{\partial (C(q,\dot q)\dot q)}{\partial q}\right|_{\substack{q=q_e\\ \dot q=0\\ \ddot q=0\\ u=u_e}} \delta_q + \left.\frac{\partial (C(q,\dot q)\dot q)}{\partial \dot q}\right|_{\substack{q=q_e\\ \dot q=0\\ \ddot q=0\\ u=u_e}} \dot\delta_q + \left.\frac{\partial (C(q,\dot q)\dot q)}{\partial \ddot q}\right|_{\substack{q\\ \dot q\\ \ddot q\\ u}} \\
&\simeq 0 + o(\delta_q, \dot\delta_q)
\end{aligned}
$$

since $\|C(q,\dot q)\| \leq (k_4 + k_5\|q\|)\,\|\dot q\|$, i.e. $(C(q,\dot q)\dot q)$ is quadratic w.r.t. $\dot q$.
$o(\delta_q, \dot\delta_q)$ contains second or higher order infinitesimal terms.

$$
\begin{aligned}
C(q,\dot q)\dot q &\simeq C(q_e,\dot q_e)\dot q_e + \left.\frac{\partial (C(q,\dot q)\dot q)}{\partial q}\right|_{\substack{q=q_e\\ \dot q=0\\ \ddot q=0\\ u=u_e}} \delta_q + \left.\frac{\partial (C(q,\dot q)\dot q)}{\partial \dot q}\right|_{\substack{q=q_e\\ \dot q=0\\ \ddot q=0\\ u=u_e}} \dot\delta_q + \left.\frac{\partial (C(q,\dot q)\dot q)}{\partial \ddot q}\right|_{\substack{q\\ \dot q\\ \ddot q\\ u}} \\
&\simeq 0 + o(\delta_q, \dot\delta_q)
\end{aligned}
$$

since $\|C(q,\dot q)\| \leq (k_4 + k_5\|q\|)\,\|\dot q\|$, i.e. $(C(q,\dot q)\dot q)$ is quadratic w.r.t. $\dot q$.
$o(\delta_q, \dot\delta_q)$ contains second or higher order infinitesimal terms.

$$
\begin{aligned}
g(q) &\simeq g(q_e) + \left.\frac{\partial g(q)}{\partial q}\right|_{\substack{q=q_e\\ \dot q=0\\ \ddot q=0\\ u=u_e}} \delta_q + \left.\frac{\partial g(q)}{\partial \dot q}\right|_{\substack{q=q_e\\ \dot q=0\\ \ddot q=0\\ u=u_e}} \dot\delta_q + \left.\frac{\partial g(q)}{\partial \ddot q}\right|_{\substack{q=q_e\\ \dot q=0\\ \ddot q=0\\ u=u_e}} \ddot\delta_q \\
&\simeq g(q_e) + G(q_e)\delta_q
\end{aligned}
$$

where $G(q)$ is the Jacobian matrix $\frac{\partial g(q)}{\partial q}$. (Remark $\|g(q)\| \leq k_6 + k_7\|q\|$)

The overall ODE around the equilibrium point is

$$B(q_e)\ddot{\delta}_q + g(q_e) + G(q_e)\delta_q + o(\delta_q, \dot{\delta}_q) = u_e + \delta_u$$

and finally

$$B(q_e)\ddot{\delta}_q + G(q_e)\delta_q = \delta_u$$

Let $\delta_x = \begin{bmatrix} \delta_q \\ \dot{\delta}_q \end{bmatrix}$ be the state vector. The state space model around the equilibrium point is

$$\dot{\delta}_x = \underbrace{\begin{bmatrix} 0 & I \\ -B^{-1}(q_e)G(q_e) & 0 \end{bmatrix}}_{\text{`}A\text{' matrix}} \delta_x + \underbrace{\begin{bmatrix} 0 \\ B^{-1}(q_e) \end{bmatrix}}_{\text{`}B\text{' matrix}} \delta_u$$

The matrices 'A' and 'B' are constant.

Linearization along a trajectory $q_d(t)$

It makes more sense to linearize the model around an *equilibrium trajectory* $q_d(t)$.

From $q_d(t)$, we can compute $\dot{q}_d(t)$ and $\ddot{q}_d(t)$ and so the *nominal command* $u_d(t)$

$$B(q_d)\ddot{q}_d + C(q_d, \dot{q}_d)\dot{q}_d + g(q_d) = u_d$$

We now consider the variations around the nominal trajectory

$$\begin{aligned} q &= q_d + \delta_q \\ \dot{q} &= \dot{q}_d + \dot{\delta}_q \\ \ddot{q} &= \ddot{q}_d + \ddot{\delta}_q \\ u &= u_d + \delta_u \end{aligned}$$

i.e.

$$B(q_d + \delta_q)(\ddot{q}_d + \ddot{\delta}_q) + C(q_d + \delta_q, \dot{q}_d + \dot{\delta}_q)(\dot{q}_d + \dot{\delta}_q) + g(q_d + \delta_q) = u_d + \delta_u$$

Let's now compute the first order approximation.

Inertia matrix

$$B(q_d + \delta_q) \simeq B(q_d) + \sum_{i=1}^{n} \left. \frac{\partial B_i(q_d)}{\partial q} \right|_{q=q_d} e_i^T \delta_q$$

Coriolis and centrifugal term

$$C(q_d + \delta_q, \dot{q}_d + \dot{\delta}_q)(\dot{q}_d + \dot{\delta}_q) = c(q_d + \delta_q, \dot{q}_d + \dot{\delta}_q) \simeq c(q_d, \dot{q}_d) + \left. \frac{\partial c(q, \dot{q})}{\partial q} \right|_{\substack{q = q_d \\ \dot{q} = q_d}} \delta_q + \left. \frac{\partial c(q, \dot{q})}{\partial \dot{q}} \right|_{\substack{q = q_d \\ \dot{q} = q_d}} \dot{\delta}_q$$

and so

$$C(q_d + \delta_q, \dot{q}_d + \dot{\delta}_q)(\dot{q}_d + \dot{\delta}_q) \simeq C(q_d, \dot{q}_d)\dot{q}_d + C_q(q_d, \dot{q}_d)\delta_q + C_{\dot{q}}(q_d, \dot{q}_d)\dot{\delta}_q$$

Gravity term

$$g(q_d + \delta_q) \simeq g(q_d) + G(q_d)\delta_q$$

Cancelling out the terms related to the nominal trajectory and the higher order terms

$$\left(B(q_d) + \sum_{i=1}^{n} \frac{\partial B_i(q_d)}{\partial q}\bigg|_{q=q_d} e_i^T \delta_q\right)(\ddot{q}_d + \ddot{\delta}_q) + C(q_d, \dot{q}_d)\dot{q}_d + C_q(q_d, \dot{q}_d)\delta_q + C_{\dot{q}}(q_d, \dot{q}_d)\dot{\delta}_q + \\ + g(q_d) + G(q_d)\delta_q = u_d + \delta_u$$

we end up with the following dynamic model on the variations

$$B(q_d)\ddot{\delta}_q + \sum_{i=1}^{n} \frac{\partial B_i(q_d)}{\partial q}\bigg|_{q=q_d} \ddot{q}_d e_i^T \delta_q + C_q(q_d, \dot{q}_d)\delta_q + C_{\dot{q}}(q_d, \dot{q}_d)\dot{\delta}_q + G(q_d)\delta_q = \delta_u$$

By defining the “damping”-like term

$$D(q_d, \dot{q}_d, \ddot{q}_d) \triangleq \sum_{i=1}^{n} \frac{\partial B_i(q_d)}{\partial q}\bigg|_{q=q_d} \ddot{q}_d e_i^T + C_q(q_d, \dot{q}_d) + G(q_d)$$

We finally get

$$B(q_d)\ddot{\delta}_q + C_{\dot{q}}(q_d, \dot{q}_d)\dot{\delta}_q + D(q_d, \dot{q}_d, \ddot{q}_d)\delta_q = \delta_u$$

The equivalent state-space model around the nominal trajectory is

$$\dot{\delta}_x = \underbrace{\begin{bmatrix} 0 & I \\ -B^{-1}(q_d)D(q_d, \dot{q}_d, \ddot{q}_d) & -B^{-1}(q_d)C_{\dot{q}}(q_d, \dot{q}_d) \end{bmatrix}}_{\text{`}A\text{' matrix}} \delta_x + \underbrace{\begin{bmatrix} 0 \\ B^{-1}(q_d) \end{bmatrix}}_{\text{`}B\text{' matrix}} \delta_u$$

The matrices '$A(t)$' and '$B(t)$' are *time-varying*.

We have a time-varying linear system.

Operational Space Dynamic Model

Operational Space Dynamic Model

We end up with the following model for the manipulator having the generalized torque as input (τ) and the generalized coordinates as output ($\dot{q}, \ddot{q}$)

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau - J^T(q)h_e$$

How does this model change if we use the Cartesian coordinate $x = \begin{bmatrix} p \\ \phi \end{bmatrix}$ of the end-effector?

The determination of the dynamic model with Lagrange formulation using operational space variables allows a complete description of the system motion only in the case of a *nonredundant manipulator*, i.e.

$$\text{size}(x) = \text{size}(q),$$

otherwise internal motions could occur.

Torques at the end effector h corresponding to joint torques τ

$$\tau = J^T(q)h$$

Relationship between q and x: direct kinematics

$$x = \kappa(q)$$

Relationship between $\dot{q}$ and $\dot{x}$: analytical Jacobian. $J = T_A(\phi)J_A$, $T_A(\phi) = \begin{bmatrix} I & 0 \\ 0 & T(\phi) \end{bmatrix}$

$$\dot{x} = J_A(q)\dot{q}$$

Relationship between $\ddot{q}$ and $\ddot{x}$

$$\ddot{x} = J_A(q)\ddot{q} + \dot{J}_A(q,\dot{q})\dot{q}$$

We end up with

$$B_A(x)\ddot{x} + C_A(x,\dot{x})\dot{x} + g_A(x) = u - u_e$$

where

$$\begin{aligned}
B_A(x) &\overset{(*)}{=} (J_A B^{-1} J_A^T)^{-1} \\
C_A(x,\dot{x})\dot{x} &= B_A J_A B^{-1} C\dot{q} - B_A \dot{J}_A \dot{q} \\
g_A(x) &= B_A J_A B^{-1} g \\
u &= T_A^T(x) h \\
u_e &= T_A^T(x) h_e
\end{aligned}$$

$(*)$ Assumption: B_A nonsingular, i.e. J_A full rank (no kinematic and representation singularities)

For non-redundant manipulator in a nonsingular configuration, we have the simplified expression

$$\begin{aligned} B_A(x) &= J_A^{-T} B J_A^{-1} \\ C_A \dot{x} &= J_A^{-T} C \dot{q} - B_A \dot{J}_A \dot{q} \\ g_A(x) &= J_A^{-T} g \\ u &= T_A^T(x) h \\ u_e &= T_A^T(x) h_e \end{aligned}$$

$(*)$ Assumption: B_A nonsingular, i.e. J_A full rank (no kinematic and representation singularities)


Project

PROJECT – Assignment # 5

To do

- Compute the dynamic model in the operational space