ADVANCED CONTROL SYSTEMS

Manipulator Dynamics

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Outline

Dynamic model of robotic manipulators

Mathematical models

System theory studies the stability of systems like

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ y(t) = h(t, x(t), u(t)) \end{cases} \qquad \Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t)) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

and provides tools to design controllers able to

- stabilize the closed-loop system, and
- guarantee performance specifications

Σ is just a *mathematical representation* of the *real* plant

Mathematical models

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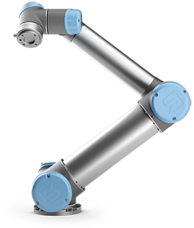
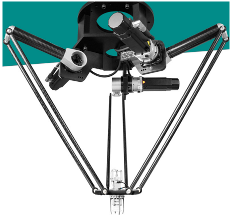
Σ is just a *mathematical representation* of the *real* plant

"All models are wrong, but some are useful." –George E.P. Box

Mathematical models

How can I compute a model for a robotic manipulator?





Mathematical models

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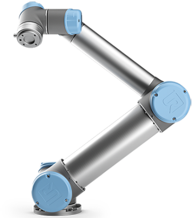
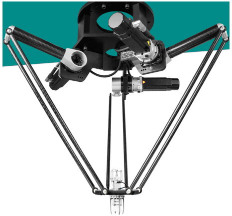
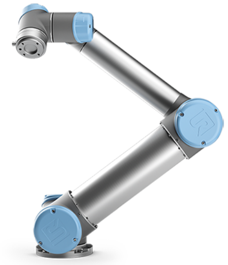




Why do I need a model?

Dynamics



The dynamical model mapping the generalized forces u into the generalized coordinates $q, \dot{q}$ will take the expression

$$\boxed{B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = u}$$

$q \in \mathbb{R}^n$

n # doF

where

- $B(q)$ is the inertia tensor
- $\boxed{C(q,\dot{q})\dot{q}}$ is the Coriolis and centrifugal term
- $F\dot{q}$ is the friction
- $g(q)$ is the gravity term

The dynamical model is a *set of second-order nonlinear differential equations*

$$\mathcal{F}(q,\dot{q},\ddot{q}) = u$$

$\Rightarrow B(x_1)\dot{x}_2 + C(x_1,x_2)x_2 + Fx_2 + g(x_1) = u$

MIMO

If $B(q)$ is nonsingular, the dynamical model

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = u$$

can be written in the *state-space representation* $\dot{x}(t) = f(t,x,u)$ by defined the state vector x as

$q \in \mathbb{R}^n$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleq \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \in \mathbb{R}^{2n}$$

S-function (Simulink)

and re-arranging the implicit differential equations as

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -B^{-1}(x_1)\left(C(x_1,x_2)x_2 + Fx_2 + g(x_1)\right) \end{bmatrix} + \begin{bmatrix} 0 \\ B^{-1}(x_1)u \end{bmatrix}}$$

This formulation is useful to model the robot in Simulink

Assumption: B is nonsingular

$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$

Another choice for the state vector is

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \triangleq \begin{bmatrix} q \\ B(q)\dot{q} \end{bmatrix},$$

where the second half of $\bar{x}$ is the *generalized momentum*.

The state space model is

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \textbf{[Exercise]}$$

Dynamics – Exercise

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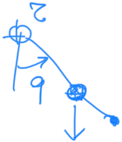


↓ gravity

Exercise. Let

$$\boxed{I\ddot{q} + F\dot{q} + \underbrace{mgd\sin q} = \tau}$$

be the model of 1-Dof link under gravity, where I is the inertia, m is the mass, d is the distance of the center of mass to the pivoting point, q is the angle with respect to the vertical axis (the same of the gravity q).

Write the state space model corresponding to the ODE.

$$B(q)\ddot{q} + \underline{C(q,\dot{q})\dot{q}} + F\dot{q} + g(q) = \tau$$

Why the dynamic model is important

Derivation of the dynamic model of a manipulator plays an important role for

- simulation of motion,
- analysis of manipulator structures,
- design of control algorithms,
- design of prototype arms, joints, transmissions and actuators

The dynamic model provides the relationship between the joint actuator torques $u(t)$ and the robot motion, i.e. configuration $q(t)$.

We will start with the model in the joint space (i.e. q), later in operational space (i.e. x, *do not confuse this x with the state vector*)

1. Lagrange formulation (related to mechanical energy)
2. Newton-Euler formulation (recursive formulation)

Why the dynamic model is important

When a model is available it is possible to address the following problems:

- direct dynamics: $u \mapsto (q, \dot{q}, \ddot{q})$
- inverse dynamics: $(q, \dot{q}, \ddot{q}) \mapsto u$
- dynamic parameter identification

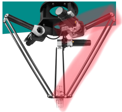
- motion control
- force control

- motion planning ($\rightarrow$ *Robotics, Vision and Control* course)
- visual servoing ($\rightarrow$ *Robotics, Vision and Control* course)
- Human-Robot Interaction / Cooperative Robots / Teleoperation ($\rightarrow$ *Physical Human-Robot Interaction* course)

Dynamics taxonomy

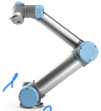
Serial Robots (open kin chain)

Parallel robots (closed kin chain)



Rigid-Joint–Rigid-Link Manipulators

4 5 6

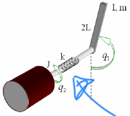


3

1

2

Flexible-Joint–Rigid-Link Manipulators



electric elements

Under-actuated robots

4

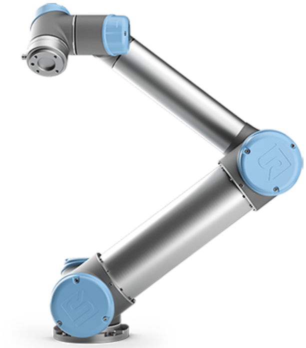
6

Over-actuated robots

7

6

Direct dynamics



inputs

$$\tau(t) = \begin{bmatrix} \tau_1(t) \\ \tau_2(t) \\ \vdots \\ \tau_n(t) \end{bmatrix}, \; t \in [0, T]$$

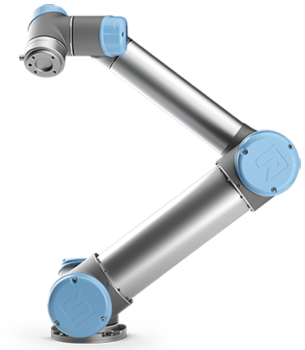
time-series

outputs

$$q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix}, \; t \in [0, T]$$

motion

initial conditions:
$q(0), \dot{q}(0)$ at $t = 0$



desired trajectory

$$q_d(t),\, \dot{q}_d(t)\; \ddot{q}_d(t)$$
$$t \in [0, T]$$

required commands

$$\tau_d(t)$$
$$t \in [0, T]$$

Here we assume that the desired trajectory $q_d(t)$ is known. ($\rightarrow$ *Robotics, Vision and Control* course)

Lagrange Formulation

Let $q \in \mathbb{R}^n$ be the generalized coordinates for a n-DOF manipulator (e.g. joint variables), $\mathcal{T}(q, \dot{q})$ be the kinetic energy, and $\mathcal{U}(q)$ be the potential energy

The Lagrangian of the mechanical system is

$$\boxed{\mathcal{L}(q, \dot{q}) = \mathcal{T}(q, \dot{q}) - \mathcal{U}(q)}$$

The Lagrange equations are

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial q}\right)^T = \tau}$$

or, equivalently

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \qquad i = 1, \ldots, n$$

where τ_i are the generalized forces (external or dissipative).

The Lagrange equations establish the relations existing between the generalized forces applied to the manipulator and the joint positions, velocities and accelerations

The contributions to the generalized forces are given by the nonconservative forces, i.e., the joint actuator torques, the joint friction torques, as well as the joint torques induced by end-effector forces at the contact with the environment

$$\begin{gathered}\text{Mechanical structure + parameters}\\ \downarrow \\ \mathcal{T}(q,\dot q),\ \mathcal{U}(q)\\ \downarrow \\ \text{Dynamics by solving Lagrange equations}\end{gathered}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial q}\right)^T = \tau$$

symbolic toolbox

Why is the Lagrangian $\mathcal{L}(q, \dot{q}) = \mathcal{T}(q, \dot{q}) - \mathcal{U}(q)$ like that?

Lagrangian function and Lagrange equation

Why is the Lagrangian $\mathcal{L}(q,\dot{q}) = \mathcal{T}(q,\dot{q}) - \mathcal{U}(q)$ like that?

Why are the Lagrange equations $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial q}\right)^T = \tau$ like that?

$$q = \in \mathbb{R}^n \qquad \frac{\partial f(x)}{\partial x} = \nabla_x f(x) = \in \mathbb{R}^{1\times n}$$

$$\tau = \in \mathbb{R}^m$$

Lagrangian function and Lagrange equation

Why is the Lagrangian $\mathcal{L}(q, \dot{q}) = \mathcal{T}(q, \dot{q}) - \mathcal{U}(q)$ like that?

Why are the Lagrange equations $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial q}\right)^T = \tau$ like that?

The answer is at the core of Geometric mechanics. We will just provide the intuition by studying the motion of the simplest mechanical system.

Newtonian mechanics

A *point mass* is an indealized zero-dimension object completely described by its position $q \in \mathbb{R}$ and its mass $m \in \mathbb{R}^+$.

The *Newton's law* for the motion of a point mass is

$$\boxed{m\ddot{q} = F}$$

where F is the total force acting on the point mass.

The *kinematic energy* is

$$\boxed{\mathcal{T}(\dot{q}) = \frac{1}{2}m\dot{q}^2}$$

Newtonian mechanics

A *Newtonian potential system* is

$$\boxed{m\ddot{q} = -\frac{\partial \mathcal{U}}{\partial q}}$$

where $\mathcal{U}(q)$ is the potential energy (real-valued function).

The *total energy* of a Newtonian potential system is

$$\boxed{\mathcal{E} \triangleq \mathcal{T} + \mathcal{U}}$$

Theorem (Conservation of energy)

In any Newtonian potential system, total energy is conserved.

Theorem

Every Newtonian potential system

$$m\ddot{q} = -\frac{\partial \mathcal{U}}{\partial q}$$

is equivalent to the Euler-Lagrange equation

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \left(\frac{\partial \mathcal{L}}{\partial q}\right) = 0}$$

for the Lagrangian

$$\boxed{\mathcal{L}(q, \dot{q}) = \mathcal{T}(\dot{q}) - \mathcal{U}(q).}$$

The domain of $\mathcal{L}$ is called the (velocity) phase space.

Proof.

Since $\mathcal{L}(q,\dot{q}) = \mathcal{T}(\dot{q}) - \mathcal{U}(q)$, we have

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{T}}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\Big(\underbrace{\frac{1}{2}m\dot{q}^2}_{\mathcal{T}}\Big) = m\dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{d\, m\dot{q}}{dt} = m\ddot{q}$$

m is constant

$$\frac{\partial \mathcal{L}}{\partial q} = -\frac{\partial \mathcal{U}}{\partial q}.$$

Finally

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \left(\frac{\partial \mathcal{L}}{\partial q}\right) = 0 \iff \boxed{m\ddot{q} = -\frac{\partial \mathcal{U}}{\partial q}}$$

□

The energy function for a Lagrangian $\mathcal{L}(q, \dot{q})$ is

$$\boxed{\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \dot{q}} \cdot \dot{q} - \mathcal{L}}$$

$\mathcal{E} = T + U$

$\mathcal{L} = T - U$

Theorem

In any Lagrangian system, the energy function is conserved.

Proof.

$$\boxed{\frac{d\mathcal{E}}{dt} = \ldots = 0}$$

$\mathcal{E} =$ const.

□

⚠ if there are external or friction forces, the Euler-Lagrange equations are equal to their sum $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \left(\frac{\partial \mathcal{L}}{\partial q}\right) = \tau$

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Is the Lagrangian mechanics “just” a ri-formulation of the Newtonian mechanics?

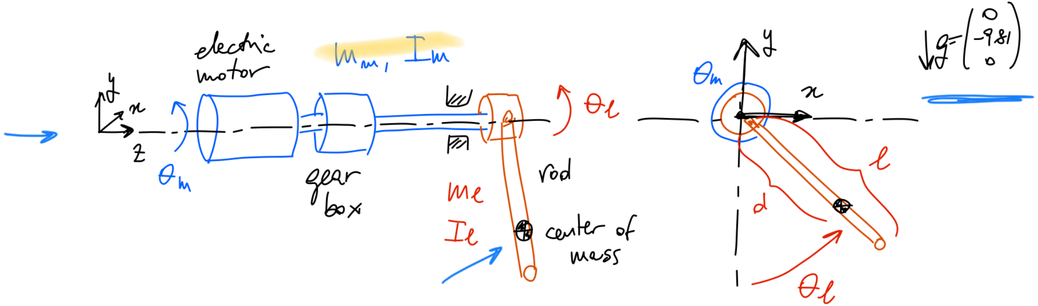
Absolutely not!

It is strongly related to many important results in Physics and Mathematics

- Variation principle
- Hamilton’s principle of stationary action
- Principle of Least Action
- D’Alambert principle
- Principle of Virtual works

and, in our context, it allows as to manage holonomic constraints (e.g. two links of a robotic manipulator connected by a joint)

Example: 1 DoF



Relations between positions and torques

$$\boxed{\theta_m = n\theta_\ell,} \quad \boxed{\dot\theta_m = n\dot\theta_\ell,} \quad \boxed{\tau_m = \frac{1}{n}\tau_\ell} \quad \boxed{\tau_\ell = n\tau_m}$$

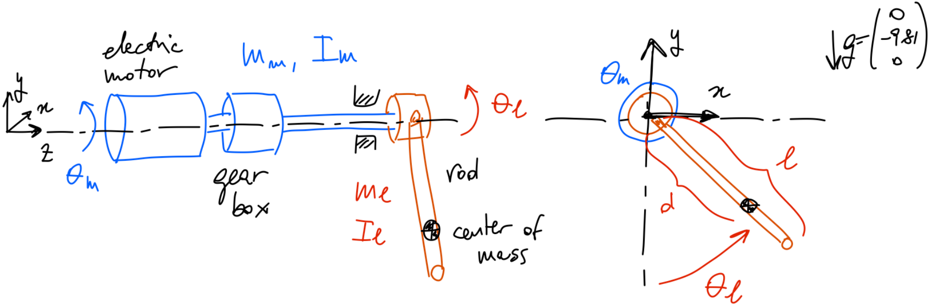
(*) (**)

Kinematic energy of the motor

$$\mathcal{T}_m = \frac{1}{2}I_m\dot\theta_m^2 = \frac{1}{2}I_m n^2\dot\theta_\ell^2$$

$\dot\theta_m \tau_m = \dot\theta_\ell \tau_\ell$

coming in power



Kinematic energy of the link

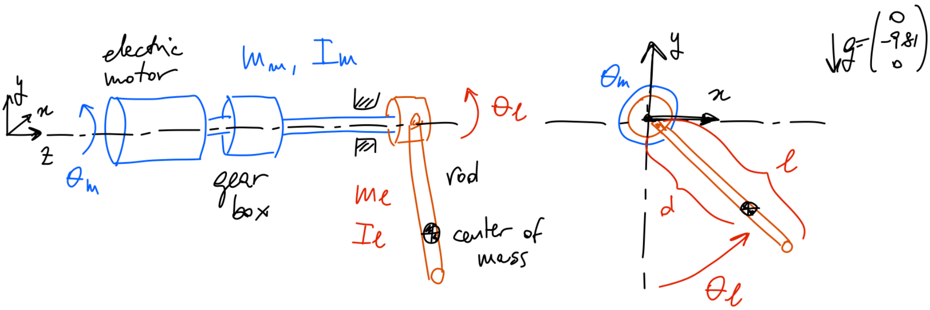
Steiner! theorem

$$\mathcal{T}_\ell = \frac{1}{2}\underbrace{(I_\ell + \boxed{m_\ell d^2})}\dot\theta_\ell^2 = \frac{1}{2}\bar{I}_\ell\dot\theta_\ell^2$$

Total Kinematic energy $(q = \theta_\ell)$

$q = \theta_\ell$

$$\mathcal{T} = \mathcal{T}_m + \mathcal{T}_\ell = \frac{1}{2}I_m n^2\dot\theta_\ell^2 + \frac{1}{2}\bar{I}_\ell\dot\theta_\ell^2 = \frac{1}{2}\left(I_m n^2 + \frac{1}{2}\bar{I}_\ell\right)\dot\theta_\ell^2 = \frac{1}{2}I\,\dot\theta_\ell^2 = \frac{1}{2}I\,\dot q^2$$



Potential energy ($q = \theta_\ell$)

$$\mathcal{U} = \mathcal{U}_0 - \underbrace{mgd\cos\theta_l}_{\ell} = \mathcal{U}_0 - m_\ell gd\cos q$$

m_ℓ

Lagrangian function

$$\boxed{\mathcal{L} = \mathcal{T} - \mathcal{U} = \frac{1}{2}I\dot{q}^2 - \mathcal{U}_0 + m_\ell gd\cos q}$$

$$\mathcal{L}(q,\dot q) = \mathcal{T}(\dot q) - \mathcal{U}(q) = \frac{1}{2} I \dot q^2 - \mathcal{U}_0 + mgd\cos(q)$$

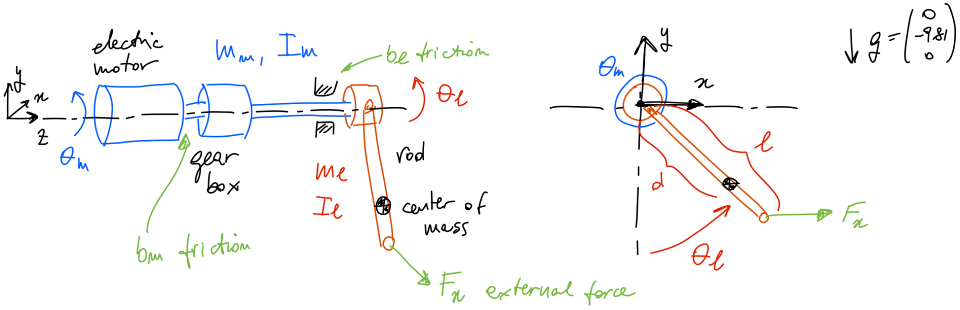
$q = \theta_\ell$

Terms within the Lagrange equation $\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q}\right) - \left(\frac{\partial \mathcal{L}}{\partial q}\right) = \tau,}$ where we set $\tau = \tau_\ell$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot q} &= I\dot q \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q}\right) &= I\ddot q \\ \frac{\partial \mathcal{L}}{\partial q} &= -mgd\sin(q). \end{aligned}$$

Finally

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q}\right) - \left(\frac{\partial \mathcal{L}}{\partial q}\right) = \tau \iff \boxed{I\ddot q + mgd\sin(q) = n\tau_m}$$



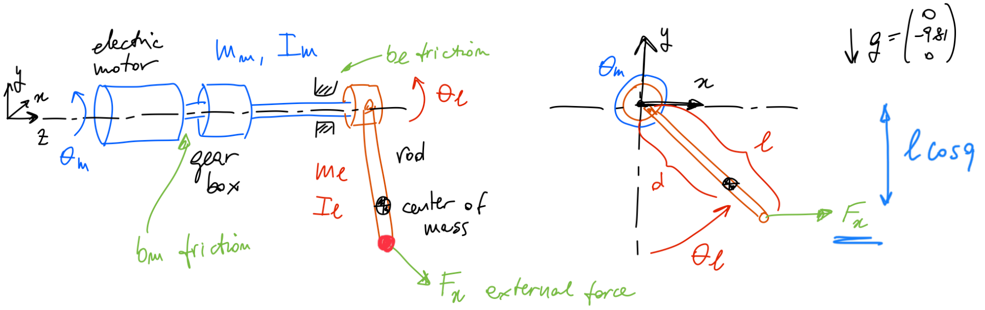
What happen with frictions ($b_m\dot{\theta}_m$, $b_\ell\dot{\theta}_\ell$)?

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \left(\frac{\partial \mathcal{L}}{\partial q}\right) = \tau - b_m n\dot{q} - b_\ell\dot{q}$$

$$\boxed{I\ddot{q} + (b_m n + b_\ell)\dot{q} + mgd\sin(q) = n\tau_m}$$

control command

Example 1



What happen with an external force along the x axix F_x?

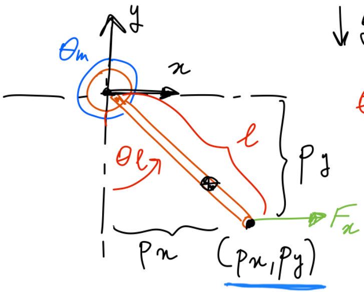
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \left(\frac{\partial \mathcal{L}}{\partial q}\right) = \tau - b_m n\dot{q} - b_\ell \dot{q} \boxed{+ \ell\cos(q)F_x}$$

$$I\ddot{q} + (b_m n + b_\ell)\dot{q} + mgd\sin(q) = n\tau_m + \ell\cos(q)F_x$$

control command

interaction force

Example 1



$\downarrow g = \begin{pmatrix} 0 \\ -9.81 \\ 0 \end{pmatrix}$

$\theta_\ell = q$

$q = q(t)$

$\dot{p}_x = \dfrac{dp_x}{dt}$

$$\begin{aligned}
p_x &= \ell \sin(q) \\
p_y &= -\ell \cos(q) \\
\dot{p}_x &= \ell \cos(q)\, \dot{q} \\
\dot{p}_y &= \ell \sin(q)\, \dot{q}
\end{aligned}$$

$$\boxed{\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = \underbrace{\begin{bmatrix} \ell \cos(q) \\ \ell \sin(q) \end{bmatrix}}_{J(q)} \dot{q},}$$

$J(q)$ Jacobian

$\begin{bmatrix} \ell \cos q \\ \ell \sin q \end{bmatrix}$

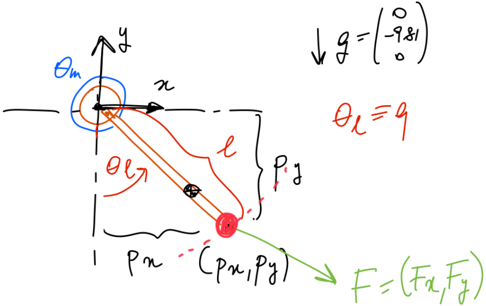
Relationship between external forces at the end-effector F_e and joint torques τ_e

$$\boxed{\tau_e = \boxed{J^T(q)} F_e}$$

i.e. equivalent joint torque τ_e due to the force F_e applied at the tip (p_x, p_y). In our case

$$F_e = \begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} F_x \\ 0 \end{bmatrix} \quad \rightarrow \quad \tau_e = \ell \cos(q) F_x$$

Example 1



$$
\begin{aligned}
p_x &= \ell \sin(q) \\
p_y &= -\ell \cos(q) \\
\dot p_x &= \ell \cos(q)\, \dot q \\
\dot p_y &= \ell \sin(q)\, \dot q \\
\begin{bmatrix} \dot p_x \\ \dot p_y \end{bmatrix} &= \underbrace{\begin{bmatrix} \ell \cos(q) \\ -\ell \sin(q) \end{bmatrix}}_{J(q)} \dot q, \qquad J(q) \text{ Jacobian}
\end{aligned}
$$

$$B(q)\ddot q + C(q,\dot q)\dot q + F\dot q + g(q) = \tau - J^T(q)h$$

With $F_e = \begin{bmatrix} F_x \\ F_y \end{bmatrix}$ we will have

$$\boxed{I\ddot q + (b_m n + b_\ell)\dot q + mgd\sin(q) = n\tau_m + \begin{bmatrix} \ell\cos(q) & -\ell\sin(q) \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}}$$

where the non-conservative torques are indicated in blue.

Example 1

Exercise. Re-write the dynamical model with $q = \theta_m$

Euler-Lagrange equations with constraints

Theorem

Every constrained Newtonian potential system

$$m\ddot{q}_i = -\frac{\partial \mathcal{U}}{\partial q_i} + C_i, \qquad i = 1, \dots, N$$

with constraints $f_j(q) = c_j$, $j = 1, \dots, k$ *and and constraint forces* C_i *satisfying*

$$\boxed{(C_1, \dots, C_N) = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j(q)}{\partial q}}$$

is equivalent to the Euler-Lagrange equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \left(\frac{\partial \mathcal{L}}{\partial q}\right) = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j(q)}{\partial q}} \quad (*)$$

for the Lagrangian $\mathcal{L}(q, \dot{q}) = \mathcal{T}(\dot{q}) - \mathcal{U}(q)$.

Constraints

Remark.
The Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \left(\frac{\partial \mathcal{L}}{\partial q}\right) = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j(q)}{\partial q}$$

for the Lagrangian $\mathcal{L}(q, \dot{q}) = \mathcal{T}(\dot{q}) - \mathcal{U}(q)$ are equivalent to the the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \bar{\mathcal{L}}}{\partial \dot{q}}\right) - \left(\frac{\partial \bar{\mathcal{L}}}{\partial q}\right) = 0 \qquad (**)$$

for the Lagrangian $\boxed{\bar{\mathcal{L}}(q, \dot{q}) = \mathcal{L}(q, \dot{q}) + \sum_{j=1}^{k} \lambda_j f_j(q).}$

Hint.

$$\boxed{\frac{\partial \bar{\mathcal{L}}}{\partial q} = \frac{\partial \mathcal{L}}{\partial q} + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j(q)}{\partial q}}$$

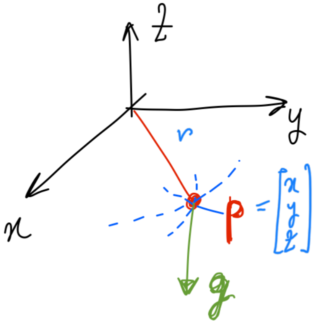
Constraints

The numbers λ_j are called *Lagrange multipliers*. To find explicit equations of motion in the variables q_i and $\dot{q}_i$ only, the Lagrange multipliers must be eliminated, which can be difficult.

The problem is much easier if the constraints have simple relationships with the coordinates $\rightarrow$ changes of coordinates

Example: Spherical pendolum

Example 2: Spherical pendolum



Spherical pendolum: a point mass of mass m, suspended from the origin by a massless rigid rod of length r

Constant gravitational force: $\boldsymbol{g} = -g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Constraint:

$f(p) = c$

$$f(x, y, z) = r^2, \qquad f(x, y, z) \triangleq x^2 + y^2 + z^2$$

Configuration space:

$$\boxed{Q = \{\boldsymbol{p} = (x, y, z) \in \mathbb{R}^3 | \, \|\boldsymbol{p}\| = r\}}$$

i.e. the sphere of radius r

$Q \neq \mathbb{R}^3$

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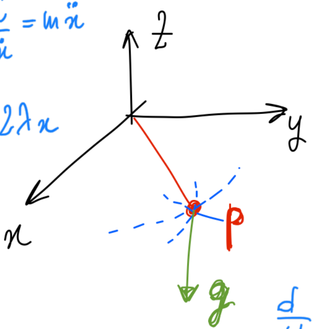
$$\frac{\partial \bar{\mathcal{L}}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\frac{\partial \bar{\mathcal{L}}}{\partial \dot{x}} = m\ddot{x}$$

$$\frac{\partial \bar{\mathcal{L}}}{\partial x} = 2\lambda x$$



Unconstrained Lagrangian: $\mathcal{L} = T - U$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

Constrained Lagrangian: $f(p) = c$

$$\boxed{\bar{\mathcal{L}} = \mathcal{L} - \lambda f}$$

$$\stackrel{(*)}{=} \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz - \lambda(x^2 + y^2 + z^2)$$

The motion is determined by the *Euler-Lagrange equations for $\bar{L}$*

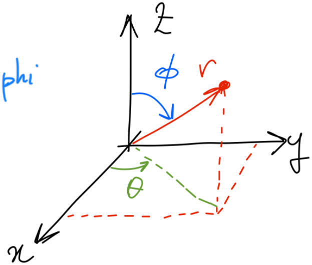
$$\frac{d}{dt}\frac{\partial \bar{\mathcal{L}}}{\partial \dot{q}} - \frac{\partial \bar{\mathcal{L}}}{\partial q} = 0$$

$$\begin{aligned} m\ddot{x} &= 2\lambda x \\ m\ddot{y} &= 2\lambda y \\ m\ddot{z} &= 2\lambda z + mg \end{aligned}$$

with the *constraint equation* $x^2 + y^2 + z^2 = r^2$

$(*)$: *The change of the sign in front of λ is just to make the following equations better looking.*

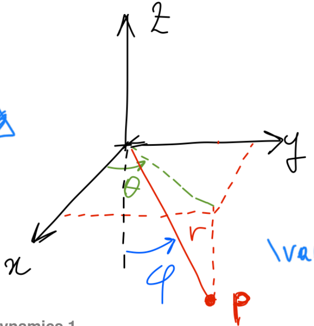
\phi



The EL equations are easier to solve in spherical coordinates

Cartesian coordinates $(x, y, z) \leftrightarrow (r, \phi, \theta)$ Spherical coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin\phi\cos\theta \\ r\sin\phi\sin\theta \\ r\cos\phi \end{bmatrix}$$



\varphi

Cartesian coordinates $(x, y, z) \leftrightarrow (r, \varphi, \theta)$ Spherical coordinates

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin\varphi\cos\theta \\ r\sin\varphi\sin\theta \\ -r\cos\varphi \end{bmatrix}}$$

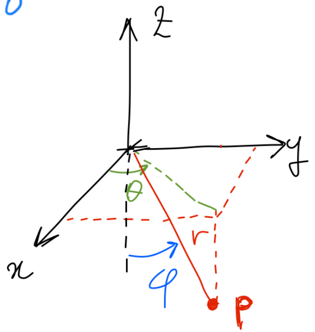
Constraint:

$f(r, \varphi, \theta) = r^2,$ (r is constant here!)

$x^2+y^2+z^2=r^2$

$r = const$

$\dot{r} = 0$



Constrained Lagrangian:

$$\bar{\mathcal{L}} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2 + r^2 \sin^2\varphi\,\dot{\theta}^2) + mgr\cos\varphi - \lambda r^2$$

Euler-Lagrange equations for $\bar{\mathcal{L}}$

$$
\begin{aligned}
0 \equiv \boxed{\frac{d}{dt}(m\dot{r})} &= \overbrace{mr\dot{\varphi}^2 + mr\sin^2(\varphi)\,\dot{\theta}^2 + mg\cos\varphi - 2\lambda r}^{\frac{\partial \bar{\mathcal{L}}}{\partial r}} \quad (1)\\
\frac{d}{dt}(mr^2\dot{\varphi}) &= \overbrace{mr^2\sin(\varphi)\cos(\varphi)\dot{\theta}^2 - mgr\sin\varphi}^{\frac{\partial \bar{\mathcal{L}}}{\partial \varphi}} \quad (2)\\
\frac{d}{dt}(mr^2\sin^2\varphi\,\dot{\theta}) &= \overbrace{0}^{\frac{\partial \bar{\mathcal{L}}}{\partial \theta}} \quad (3)
\end{aligned}
$$

$\lambda(t)$

Since r is constant, in the first equation we have $\frac{d}{dt}(m\dot{r}) = 0$ and so such equation is irrelevant for determining the motion $(\theta(t), \varphi(t))$. Such dynamics is given by the second and third equations from where the Lagrange multiplier λ disappeared.

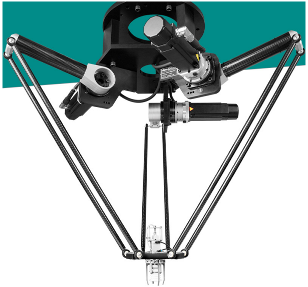
Exercise. Solve the EL equations (1)–(3) with Matlab

Exercise. Compute λ as a function of $\theta(t)$ and $\varphi(t)$ solutions of (2) and (3)

Exercise. Obtain the dynamic model of the Example 1 using the Lagrange multiplier

Euler-Lagrange equations with constraints

Remark. The Euler-Lagrange equations with constraints are needed to derive the dynamic model of parallel robots, like the Delta robot.




Project

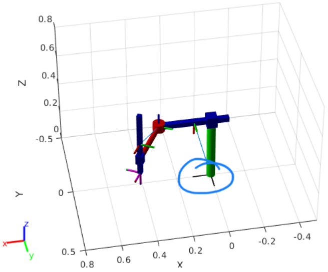
PROJECT

A different 3DoF robot for each of you (example `PRP.urdf`)

This activity is a *personal work*: it is your exam !!! NO FOCUS GROUP

I give you the robot "shape", you should choose the numerical parameters for the simulations

Example `PRP.urdf`



The reference frames in the Matlab plot of the URDF ARE NOT the frames related to the DH convention

PROJECT – Assignment # 1

Simone Cremasco
Michele Sandrini

To do

- DH table
- direct kinematics
- inverse kinematics
- Jacobians (geometric and analytical)

By hand, and cross-checking with Robotics toolbox