Controllabilità

$$\textbf{Gramiano di Controllabilità :}\quad \boldsymbol{\Gamma}(t) = \int_0^t e^{\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{B}^\top e^{\boldsymbol{A}^\top\tau}d\tau \quad \text{non singolare } (\det \neq 0)$$

$$\textbf{Matrice di Controllabilità :}\quad \mathcal{T} = \begin{bmatrix}\boldsymbol{B} & \boldsymbol{AB} & \boldsymbol{A}^2\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B}\end{bmatrix} \quad \text{rank}(\mathcal{T}) = n$$

$$\mathcal{T}^{-1} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \vdots & \iddots & 1 & 0 \\ \alpha_3 & \vdots & \alpha_{n-1} & \iddots & 0 & 0 \\ \vdots & \alpha_{n-1} & 1 & \cdots & 0 & 0 \\ \alpha_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{P} = \mathcal{T}^{-1}\mathcal{T}$$

$$\boldsymbol{A}_c = \boldsymbol{P}^{-1}\boldsymbol{AP} \quad \boldsymbol{B}_c = \boldsymbol{P}^{-1}\boldsymbol{B}$$

$$\boldsymbol{C}_c = \boldsymbol{CP} \quad \boldsymbol{D}_c = \boldsymbol{D}$$

$$\boldsymbol{A}_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{-\alpha_0}{\alpha_n} & \frac{-\alpha_1}{\alpha_n} & \frac{-\alpha_2}{\alpha_n} & \cdots & \frac{-\alpha_{n-1}}{\alpha_n} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{B}_c$$

$$\boldsymbol{C}_c = \begin{bmatrix}\beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots & \beta_{n-1}\end{bmatrix} \begin{bmatrix}\beta_n\end{bmatrix} = \boldsymbol{D}_c$$

$$\begin{aligned} W(s) &= \frac{\beta_m s^m + \beta_{m-1}s^{m-1} + \cdots + \beta_1 s + \beta_0}{\alpha_n s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0} \\ &= \boldsymbol{C}(s\mathbb{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} \end{aligned}$$

$$\begin{aligned} P(s) &= \det(s\mathbb{I} - \boldsymbol{A}) \quad \to \quad \alpha \\ P_c(s) &= (s - \overline{\lambda}_1)\cdots(s - \overline{\lambda}_n) \quad \to \quad \overline{\alpha} \end{aligned}$$

$$\boldsymbol{K}_c = \begin{bmatrix}\overline{\alpha}_0 - \alpha_0 & \overline{\alpha}_1 - \alpha_1 & \cdots & \overline{\alpha}_{n-1} - \alpha_{n-1}\end{bmatrix}$$
$$\boldsymbol{K} = \boldsymbol{K}_c\boldsymbol{P}^{-1}$$

Osservabilità

$$\textbf{Gramiano di Osservabilità :}\quad \boldsymbol{O}(t) = \int_0^t e^{\boldsymbol{A}^\top\tau}\boldsymbol{C}^\top\boldsymbol{C}e^{\boldsymbol{A}\tau}d\tau \quad \text{non singolare } (\det \neq 0)$$

$$\textbf{Matrice di Osservabilità :}\quad \mathcal{O} = \begin{bmatrix}\boldsymbol{C} & \boldsymbol{CA} & \boldsymbol{CA}^2 & \cdots & \boldsymbol{CA}^{n-1}\end{bmatrix}^\top \text{rank}(\mathcal{O}) = n$$

$$\mathcal{O}^{-1} = \mathcal{T}^{-1}$$

$$\boldsymbol{A}_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0/\alpha_n \\ 1 & 0 & \cdots & 0 & -\alpha_1/\alpha_n \\ 0 & 1 & \cdots & 0 & -\alpha_2/\alpha_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1}/\alpha_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \end{bmatrix} = \boldsymbol{B}_o$$

$$\boldsymbol{C}_o = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1\end{bmatrix} \begin{bmatrix}\beta_n\end{bmatrix} = \boldsymbol{D}_o$$

Osservatore di Luenberger:

$$\boldsymbol{P}_o = \mathcal{O}^{-1}\mathcal{O} \qquad \boldsymbol{L}_o = \begin{bmatrix} \overline{\alpha}_0 - \alpha_0 \\ \overline{\alpha}_1 - \alpha_1 \\ \vdots \\ \overline{\alpha}_{n-1} - \alpha_{n-1} \end{bmatrix}$$

$$\boldsymbol{A}_o = \boldsymbol{A} - \boldsymbol{L}_o \quad \boldsymbol{B}_o = \begin{bmatrix}\boldsymbol{B} & \boldsymbol{L}_o\end{bmatrix} \quad \boldsymbol{C}_o = \boldsymbol{C} \quad \boldsymbol{D}_o = \boldsymbol{D}$$

$$\boldsymbol{L} = \boldsymbol{P}_o^{-1}\boldsymbol{L}_o \quad \overline{\boldsymbol{u}}(t) = \begin{bmatrix}\boldsymbol{u}(t) \\ \boldsymbol{y}()\end{bmatrix}$$

Sylvester per Matrice di transizione di stato

$$e^{\boldsymbol{A}t} = \sum_{i=0}^{n-1}\beta_i(t)\boldsymbol{A}^i = \beta_0(t)\boldsymbol{I} + \beta_1(t)\boldsymbol{A} + \cdots + \beta_{n-1}(t)\boldsymbol{A}^{n-1}$$

$\mu_a = 1$

$$\begin{cases} \beta_0(t) + \lambda_1\beta_1(t) + \lambda_1^2\beta_2(t) + \cdots + \lambda_1^{n-1}\beta_{n-1}(t) = e^{\lambda_1 t} \\ \beta_0(t) + \lambda_2\beta_1(t) + \lambda_2^2\beta_2(t) + \cdots + \lambda_2^{n-1}\beta_{n-1}(t) = e^{\lambda_2 t} \\ \quad\vdots \qquad \vdots \\ \beta_0(t) + \lambda_n\beta_1(t) + \lambda_n^2\beta_2(t) + \cdots + \lambda_n^{n-1}\beta_{n-1}(t) = e^{\lambda_n t} \end{cases}$$

$\mu_a > 1$

$$\begin{cases} \beta_0(t) + \lambda\beta_1(t) + \cdots + \lambda^{n-1}\beta_{n-1}(t) = e^{\lambda t} \\ \dfrac{d}{d\lambda}\left(\beta_0(t) + \lambda\beta_1(t) + \cdots + \lambda^{n-1}\beta_{n-1}(t)\right) = \dfrac{d}{d\lambda}e^{\lambda t} \\ \qquad\vdots \qquad\qquad \vdots \\ \dfrac{d^{\nu-1}}{d\lambda^{\nu-1}}\left(\beta_0(t) + \lambda\beta_1(t) + \cdots + \lambda^{n-1}\beta_{n-1}(t)\right) = \dfrac{d^{\nu-1}}{d\lambda^{\nu-1}}e^{\lambda t} \end{cases}$$

Autovalori complessi $\lambda, \lambda' = \alpha \pm j\omega$:

$$\begin{cases} \beta_0(t) + \text{Re}(\lambda)\beta_1(t) + \text{Re}(\lambda^2)\beta_2(t) + \cdots + \text{Re}(\lambda^{n-1})\beta_{n-1}(t) = e^{\lambda_1 t}\cos(\omega t) \\ \text{Im}(\lambda)\beta_1(t) + \text{Im}(\lambda)^2\beta_2(t) + \cdots + \text{Im}(\lambda^{n-1})\beta_{n-1}(t) = e^{\lambda t}\sin(\omega t) \end{cases}$$

dove $\text{Re}(\lambda) = \alpha$ e $\text{Im}(\lambda) = \omega$.