Dynamic Random Graphs and Unstructured P2P Networks, Analysis of Two Models Inspired by the Bitcoin Network

Abstract

Inspired by the work of Becchetti-Clementi-Natale-Pasquale-Trevisan, we propose two dynamic random graphs that are a dynamical extension of their work. We empirically show that these models stabilizes on a set of bounded degree expander graphs in $\mathcal{O}(\log n)$ rounds and that information spreading protocols, such the classical Flooding protocol, under certain conditions, correctly terminate in $\mathcal{O}(\log n)$ rounds even when the graphs have bad expanding properties. However, we point out that under extremely dynamical settings of the graphs, classical distributed protocols designed for static networks become unusable on the dynamic ones.

The formal analysis of these two models is complicated by the complex dependencies that arise between edges and between choices made in different rounds, also, for the first model, by its non time-reversibility, that prevents us from explicitly calculating the steady-state distribution. However, we provide the steady-state distributions of the first model, that is $1/\boldsymbol{E}_x(\tau_x^+), x \in \Omega$, and for the second model $\boldsymbol{\pi} \sim Poisson(\lambda/q)$. We also show that, under certain conditions, structural results obtained by Pandurangan-Raghavan-Upfal can be extended to our second model.

Our dynamic random graphs captures some properties of classical P2P Networks, in particular how agents select servers in Bitcoin blockchain protocols when new agents joins the network and connections or agents in the network may disappear.

1. Introduction

Bitcoin [1] is a cryptocurrency invented in 2008 by an unknown person or group of people using the name Satoshi Nakamoto. It is fully dematerialized and unlike common legal tender currencies, it is not issued and recognized by any governmental or financial authority. With the name bitcoin you can refer to the entire system or unit of currency. In general is convention to use the term "Bitcoin" for referring to the system and "bitcoin" for the coin. In recent years there has been a lot of talk about Bitcoin in fact. in a short time it has from a system used by few users for "playful" or illegal purposes, such as trafficking of weapons or drugs in the Deep Web, to a system widely used to make simple purchases in online stores such Etsy, buy tickets, book hotels through CheapAir and even pay university fees. In fact, the number of universities that allow students to pay using Bitcoin is continuously increasing, some among many: ESMT in Berlin, the University of Nicosia and the King's College.
This change is largely due to the increase in bitcoin market capitalization achieved in 2017, with an increase in value of approximately 770 billion dollars, putting the system at the centre of the worldwide media and financial attention, triggering a cryptocurrencies boom.

In addition to its high "monetary value" this great interest in Bitcoin is due to its reliability, transparency and security.
Much of the trust in Bitcoin comes from the fact that it requires no trust. Bitcoin is completely open source and decentralized. This means that anyone has access to the entire source code at any time. Any developer in the world can therefore verify exactly how Bitcoin works. All transactions and bitcoins issued can be consulted transparently in real time by anyone.All payments can be made without reliance on third parties and the entire system is protected by cryptographic algorithms heavily subjected to peer review such as those used for online banking. No organization or individual can control Bitcoin and the system remains secure even if not all of its users can be considered reliable. Communication takes place via a peer-to-peer network, where there is no central server, but all users of the network contribute equally to its correct functioning.

Given the huge success it has received, Bitcoin has been under the scientific spotlight for several years. In fact, efforts are being made to formalize the system and to analyze its properties in order to have a deep understanding of the system and its limits. A factor that slows down the progress in this direction is the strong orientation towards system security, which means that a lot of information regarding the network cannot be obtained. For example, it is possible to know the number of active users in the network and the country from which they connect, but it is not possible to know the connections between them. It would be very interesting to know the real structure of the Bitcoin P2P network in order to observe its evolution over time, the structure, influential nodes and other interesting properties.

Knowing the topology of the Bitcoin network is a "hot topic" that affects several research areas. A very active one it is cyber security where different techniques have been proposed to try to discover and analyze the topology of the real network, in order to study its robustness to cyber attacks [2],[3],[4]. In fact, knowledge of its structure could expose it to attacks aimed at isolating agents or even partitioning the network. In the most recent article [4] a Bitcoin network inference technique based on the use of orphan transactions was proposed. The results of the above technique are very interesting. Unfortunately, the analyses by Delgado et al. were carried out on a Bitcoin test network and not on the real one, since this technique could slow down user transactions and therefore damage the Bitcoin network itself.

In a recent article [5], a random graph model inspired by the generating process of the P2P Bitcoin network was proposed: a distributed algorithm takes a Δ-regular graph and two integers d and c as input and returns a bounded degree graph with good expansion properties in $\mathcal{O}(\log n)$ rounds. This result is very interesting, as it provides indications on the "quality"; of the topology of the P2P Bitcoin Network. In the "Future Works" section of the article, in order to make the model as realistic as possible, the authors express a strong interest in studying the behaviour of this algorithm in a dynamic scenario, where nodes and edges enter or leave the network.

In this paper we propose two natural extensions of the model proposed by Becchetti et al.
In the second section we provide all the preliminaries that you need in order to read properly this paper, the formal description of the two models and a collection of the obtained results.

In the third section we prove two theorems for the formal analysis of the first dynamic random graph and we show that the Markov Chain associated with the first model non-reversible.

In the fourth section we describe the results obtained from the simulations of the first model and we show that the process has a drift towards a class of states that have expansion properties very similar to random d-regular graphs.

In the fifth section we provide the formal analysis of the second model and we show that the process can be reduced to a Markovian Queue.

Finally, in the sixth section we describe the results from the simulations of the second model observing the expansion properties and we point out some problems of classical distributed algorithms applied on dynamic random graphs.

2. Preliminaries and main result

For an undirected graph $\mathcal{G}(V,E)$, the *volume* of a subset of nodes $U \subseteq V$, is $vol(U) = \sum_{v \in U} d_u$. Consider two subsets $U, X \subseteq V$, we define $e(U,X)$ as the number of edges in $\mathcal{G}$ with one end-point in U and the other in X.

Definition 2.1. *A graph $\mathcal{G}(V,E)$ is an ϵ-expander if, for every subset $U \subset V$ with $|U| \leq \frac{n}{2}$, the number $e(U, V \setminus U)$ of edges in the cut $(U, V \setminus U)$ is at least $\epsilon \cdot vol(U)$.*

We also give the spectral definition of Expander Graph:

Definition 2.2. *A d-regular graph $\mathcal{G}$ with n vertices, is a (n,d,α)-expander if $|\lambda_2(\mathcal{G})|, |\lambda_n(\mathcal{G})| \leq \alpha d$, where :*

$$d = \lambda_1(\mathcal{G}) \geq \lambda_2(\mathcal{G}) \geq \cdots \geq \lambda_n(\mathcal{G})$$

is the spectrum of the adjacency matrix of the graph $\mathcal{G}$.

In the next sections, we analyze the behaviour of Edge Dynamic Graph and Vertex Dynamic Graph which have been informally decribed in the introduction, a more formal description is given below.

***Edge Dynamic** ($\mathcal{G}$,d,c,p)*

At each time step t (round), $\forall u \in V$ evolves with the following rules:

1. Let $N_u^{start} = \{v \in V : u \sim v\}$ the set of neighbours of u at the beggining of the round, if $|N_u^{start}| < d$, u chooses a set of vertices $X \subseteq V \setminus N_u^{start}$ such that $|X| + |N_u^{start}| = d$ u.a.r. and creates the edges $u \sim v$, $v \in X$.
2. Let $N_u^{end} = \{v \in V : u \sim v\}$ the set of neighbours of u at the end of the round ,if $|N_u^{end}| > c \cdot d$ the vertex u chooses a set $Y \subseteq N_u^{end}$ such that $|N_u^{end}| - |Y| = c \cdot d$ u.a.r. and deletes the edges $u \sim v$, $v \in Y$.
3. Each edge $u \sim v$ in the graph falls with probability p.

Vertex Dynamic *($\mathcal{G}$,d,c,λ, q)*

At each time step t (round):

1. $N(t)$ vertices enter in the graph, where $N(t)$ is a Poisson process with intensity parameter λ.
2. Let $N_u^{start} = \{v \in V : u \sim v\}$ the set of neighbours of u at the beggining of the round, if $|N_u^{start}| < d$, u chooses a set of vertices $X \subseteq V \backslash N_u^{start}$ such that $|X| + |N_u^{start}| = d$ u.a.r. and creates the edges $u \sim v$, $v \in X$.
3. Let $N_u^{end} = \{v \in V : u \sim v\}$ the set of neighbours of u at the end of the round ,if $|N_u^{end}| > c \cdot d$ the vertex u chooses a set $Y \subseteq N_u^{end}$ such that $|N_u^{end}| - |Y| = c \cdot d$ u.a.r. and deletes the edges $u \sim v$, $v \in Y$.
4. Each vertex leave $\mathcal{G}$ with probability q.

We next define a class of almost-regular graphs in which, for appropriate parameter values, the models seem to stabilize and how we can measure the expansion properties fo these dynamic random graphs.

Definition 2.3. *A graph $\mathcal{G}(V,E)$ is (d,cd)-regular if the degree d_u of any node $u \in V$ is such that $d_u \in \{d, \dots cd\}\}$.*

The expansion properties we measure for the Edge Dynamic and Vertex Dynamic models are related to the mixing time of a simple Random Walk on the evolving graph. Let A to be the adjacency matrix of $\mathcal{G}_t$, we define $P = \frac{1}{d}A$ as the normalized adjacency matrix. P also defines a transition matrix of a random walk on the graph $\mathcal{G}_t$. We know that P is real, symmetric and doubly stochastic. Then the spectrum of P is: $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1$, $\max\{|\lambda_2|, |\lambda_n|\} \leq \alpha$ and the *stationary distribution* of the random walk on $\mathcal{G}_t$ is the uniform distribution $\boldsymbol{u}P = P\boldsymbol{u} = \boldsymbol{u}$ (because P is symmetric). In our case we have that $\boldsymbol{v}P = \boldsymbol{v}$ where $\boldsymbol{v}(i) = \frac{d_i}{2|E|}, \forall i \in V$.

Theorem 2.1. *[6] Let $\mathcal{G}_t$ be an (n,d,α)-expander with normalized adjacency matrix P. Then for each probability vector $\boldsymbol{p}$ and $t \in \mathbb{N}$:*

$$\left\| P^t \boldsymbol{p} - \boldsymbol{u} \right\|_1 \leq \sqrt{n} \cdot \alpha^t$$

We know that

$$\left\| \boldsymbol{\mu} - \boldsymbol{\nu} \right\|_{TV} = \frac{1}{2} \left\| \boldsymbol{\mu} - \boldsymbol{\nu} \right\|_1$$

so we can give a bound of the *spectral expansion* of a graph $\mathcal{G}$ in terms of *mixing time* $\tau(\epsilon)$ of the random walk on that graph. Where $\tau(\epsilon) = \min_t \{d(t) \leq \epsilon\}$ and $d(t) = \max_i \left\| P^t(i, \dot{)} - \boldsymbol{\nu} \right\|$. We can define the *relaxation time* as $t_{rel} = \frac{1}{\gamma}$ and bound $\tau(\epsilon)$ using the following Theorem:

Theorem 2.2. *[7] For each aperiodic irriducible and reversible Random Walk P:*

$$(t_{rel} - 1) \log\left(\frac{1}{2\epsilon}\right) \leq \tau(\epsilon) \leq t_{rel} \log\left(\frac{1}{2\epsilon\sqrt{\boldsymbol{\pi_{min}}}}\right)$$

This Theorem infromally tells us that

$$\gamma \text{ small} \iff \text{High Mixing Time} \iff \text{Poor Expansion properties}$$

Finally, we define the distributed protocol known as Flooding or more commonly as Broadcast[8]. Given an undirected network, initially, only one vertex has a message that he wants to transmit to the rest of nodes. To do this, he sends a copy of this message to all its neighbors. A vertex that receives the message for the first time forwards it to all its neighbors, and so on. If a vertex get the message but he already have received, does nothing. The protocol ends when all the network has the new message.

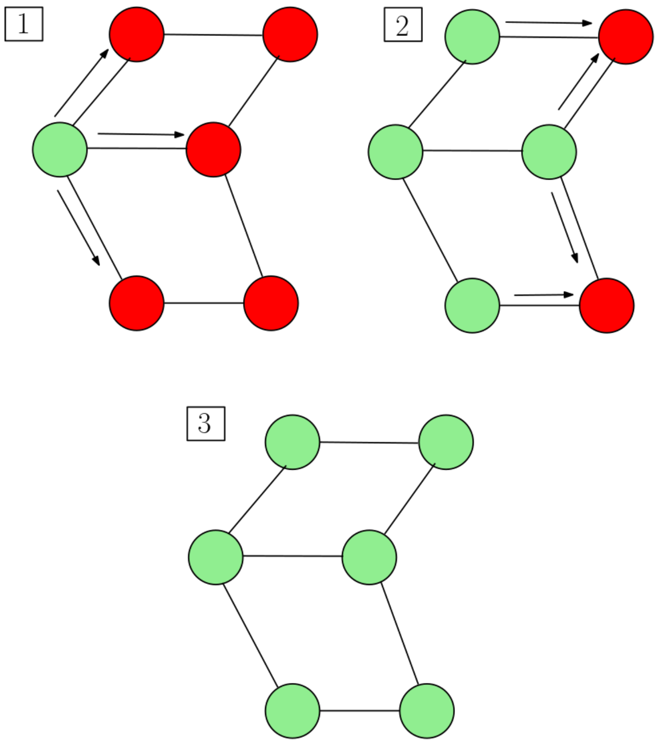

Figure 1: Example of the Flooding protocol

It is clear that a good P2P network (also a general network) must guarantee the termination of the protocol,.That is, it must ensure that when a node announces, for example, a new transaction, in a few time steps, the whole network has become aware of it. For both Dynamic Random Graphs we analyzed the flooding time by simulating the information spreading process on the evolving network, studying the average termination time and the average number of informed nodes at each time step.

Our main results can be formally summarized as follows.

Edge Dynamic

Theorem 2.3. *The Edge Dynamic model has a unique stationary distribution* $\boldsymbol{v}(x) = \frac{1}{\boldsymbol{E}_x(\tau_x^+)}$[1].

Theorem 2.4. *The Edge Dynamic model is not Time Reversible.*

The following are observations made from the results of the experiments.

Observation 2.1. *For appropriate values of* p *and for* $d \geq 3$, $c > 1$ *the Edge Dynamic model stabilizes on average in an expander configuration in* $\mathcal{O}(\log n)$ *rounds.*

Observation 2.2. *The Edge Dynamic model seems to have a drift in a subset of expander configurations.*

Observation 2.3. *The Flooding Protocol executed on the Edge Dynamic model, for* $d \geq 3, c > 1$ *and* $0 \leq p \leq 0.7$ *on average ends in* $\mathcal{O}(\log n)$ *rounds.*

Vertex Dynamic

Theorem 2.5. *The Vertex Dynamic model has the following stationary distribution*

$$\boldsymbol{v}(i) = \frac{\left(\frac{\lambda}{q}\right)^i}{i!} e^{-\frac{\lambda}{q}}, \quad i \geq 0$$

Theorem 2.6. *Assuming to have an exponential time of permanence Exp(μ) all Network Size results obtained in [9] are also valid for our model.*

The following are observations made from the results of the experiments.

Observation 2.4. *After reaching* $\frac{\lambda}{q}$ *vertices in the network, for appropriate values of* q, λ *and for* $d \geq 3$, $c > 1$ *the Vertex Dynamic model stabilizes on average in an expander configuration.*

Observation 2.5. *The Flooding Protocol executed on the Vertex Dynamic model for* $d \geq 3, c > 1$ *and for appropriate values of* q *on average ends in* $\mathcal{O}(\log n)$ *rounds.*

[1] Where $x \in \Omega$ is a state of the stocastic process (a configuration of the Dynamic Graph), τ_x^+ is the first return time in the state x and $\boldsymbol{E}_x(\tau_x^+)$ in the mean first return time in x.

3. Proof of Theorems 2.3 and 2.4

In order to analyze and proove Theorem 2.3 and 2.4 we define a discrete time Markov Chain representing the stochastic model as follows.

Let the state space Ω be the set of all possibile configuration of the graph, and $n = |V|$ the number of verices of the graph, we have that

$$|\Omega| = \sum_{i=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{i} = 2^{\binom{n}{2}}$$

Without loss of generality let us assume that $\{0\} \in \Omega$ is the starting configuration of the model $\mathcal{G}(V, \{\emptyset\})$ at time $t = 0$ and so the inital density of the chain is $\boldsymbol{\pi}_0(0) = 1$.
Let $P \in \mathbb{R}^{|\Omega| \times |\Omega|}$ the transition matrix of the Markov Chain, it is clear that $P(x, y) > 0, x, y \in \Omega$ if and only if it is possible to "jump" from the configuration $\mathcal{G}_x$ to the configuration $\mathcal{G}_y$ in one step ($\mathcal{G}_x \rightsquigarrow \mathcal{G}_y$), formally:

$$P(x, y) = \begin{cases} 0 < p_{x,y} \leq 1 & \text{If } \mathcal{G}_x \rightsquigarrow \mathcal{G}_y. \\ 0 & \text{Otherwise.} \end{cases}, \quad \forall x, y \in \Omega$$

We can easily observe that assuming to have a zero probability of edge failure, $p = 0$, we obtain the RAES model [5], and so a Dynamic Random Graphs that converges *w.h.p.* in $\mathcal{O}(\log n)$ rounds to an ϵ- expander configuration.

Assuming $p \neq 0$ and $\boldsymbol{\pi}_0(0) = 1$ we get a subset $\mathcal{D} \subseteq \Omega$ with the following property:

$$P^t(x, y) = 0,\ x \in \Omega \backslash \mathcal{D},\ y \in \mathcal{D},\ t \geq 0$$

namely, there is a set of states that can never be reached starting from state $\{0\}$.
So let us consider the restricted Markov Chain with state space $\Omega' = \Omega \backslash \mathcal{D}$. Let R be the transition matrix of the restricted chain, we can easily observe that $\forall x, y \in \Omega', \exists t > 0 :\ R^t(x, y) > 0$, hence R is aperiodic, irreducible and positive-recurrent. From this observation follows the proof of Theorem 2.3 because R has unique stationary distribution $\boldsymbol{v}$ and $\boldsymbol{v}(x) = \frac{1}{\boldsymbol{E}_x(\tau_x^+)}$, $x \in R$.

We have just shown that the restricted chain has unique stationary distribution. After this result the following question arises:" Can the stationary distribution be easily calculated? ". To answer this question we have to analyze the reversibility property of the restricted chain R. Because for Markov Chains with exponential state space is not possible to directly calculate $\boldsymbol{v}P = \boldsymbol{v}$, but if the chain is time-reversible we can guess $\boldsymbol{v}$ and solve instead the time reversibility equations (that are easier to solve) and quickly check if our guess is right.

Reversibility Analysis

If a Markov Chain has a probability distribution $\boldsymbol{v}$ on Ω such that satisfies the detailed balance equations:

$$\boldsymbol{v}(x)P(x,y) = \boldsymbol{v}(y)P(y,x), \;\; \forall x, y \in \Omega$$

the following proposition holds

Proposition 1. *Let P be the transition matrix of a Markov Chain with state space* Ω*. Each probability distribution* $\boldsymbol{v}$ *that satisfy the detailed balance equations is a steady-state distribution for P.*

Proof.

$$\sum_{y\in\Omega} \boldsymbol{v}(y)P(x,y) = \sum_{y\in\Omega} \boldsymbol{v}(x)P(x,y) = \boldsymbol{v}(x)$$

■

As a first step, we have to check if the dynamic is Time-Reversible, to do this we rely on the Kolmogorov Criterion.

Theorem 3.1. *[10] An irreducible, positive recurrent, aperiodic Markov chain with transition matrix P is reversible if and only if its stationary Markov chain satisfies*

$$P(i_1,i_2)\cdot P(i_2,i_3)\cdots P(i_{n-1},i_n)\cdot P(i_n,i_1) = P(i_1,i_n)\cdot P(i_n,i_{n-1})\cdots P(i_3,i_2)\cdot P(i_2,i_1)\cdot P(i_1,i_n) \quad (1)$$

For each finite sequence of states $i_1, i_2 \ldots i_n \in \Omega$

However we have obtained the following non-reversibility result

Theorem 3.2. *The dynamic R^2 is not Time Reversible*

Proof. Let us consider the following four-state cycle:

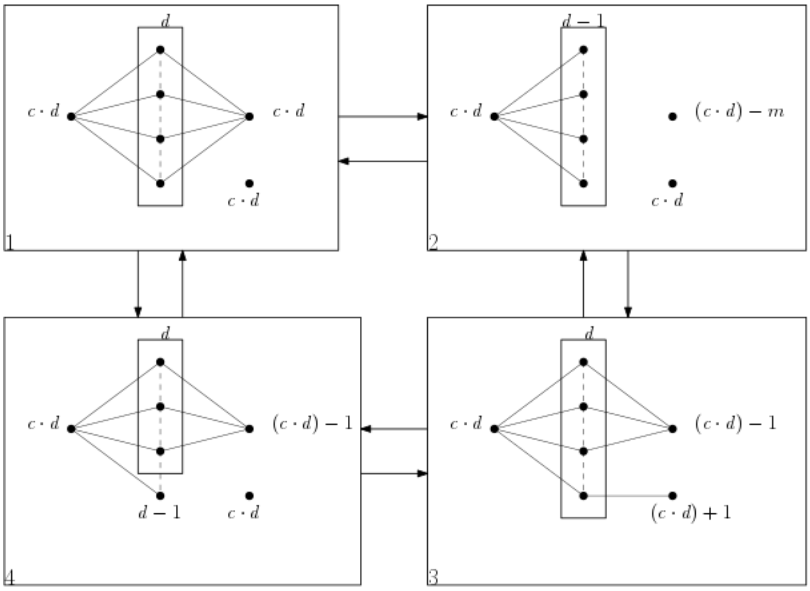

Figure 2: Four-State cycle that does not satisfy the Kolmogorov Criterion

Without loss of generality, let us assume the following conditions: to start from the state number 1, every vertex of the graph $\mathcal{G}$ has degree between d and $c \cdot d$ and that p is the falling probability of an edge.

We define the descriptions of the transitions with the relative jumping probabilities from one state to another.

Transition probabilities from 1 to 2 and 2 to 1:
Let m be the vertices at the center of the Figure 2, each of these m vertices has exactly d neighbors, and also assume that $(c \cdot d) - m \geq c \cdot d$. Starting from 1, the only event that allows the chain to jump in the state 2 ($P(1,2)$) is the following:

"Only the m edges connecting the central vertices to the right one falls."

Let K be the total number of edges of the graph in the state 1, we can define

$$P(1,2) = p^m (1-p)^{K-m}$$

The only way to jump from state 2 to 1 is:

"The central vertices chooses u.a.r. the right vertex and connect to it, also no edge falls."

[2] R is the restricted dynamic on $\Omega' = \Omega \backslash \mathcal{D}$.

Let $\left(\frac{(n-1)-(d-1)}{n-1}\right)^m = a^m$, then $P(2,1) = a^m(1-p)^K$.

Transition probabilities from 2 to 3 and 3 to 2:
$P(2,3)$ can be described by the following event:

"$m-1$ vertices connects to the right vertex, one to the bottom right one and no edge falls."

So we have that $P(2,3) = a^m(1-p)^K$, observe that in 3 the bottom right vertex has degree $(c \cdot d)+1$. For $P(3,2)$ we have that:

"The bottom right vertex, chooses u.a.r. and deletes the central vertex from his neighborhood and also $m-1$ edges fall."

So $P(3,2) = \frac{1}{(c\cdot d)+1} p^{m-1}(1-p)^{K-m-1}$.

Transition probabilities from 3 to 4 and 4 to 3:
$P(3,4)$ can be defined as follows:

"The bottom right vertex chooses u.a.r. and deletes the central vertex from his neighborhood and no edge falls."

It follows that $P(3,4) = \frac{1}{(c\cdot d)+1}(1-p)^{K-1}$ and clearly $P(4,3) = a(1-p)^K$.

Transition probabilities from 4 to 1 and 1 to 4:
Finally $P(4,1)$ and $P(1,4)$ are the followings: $P(4,1) = a(1-p)^k$ and $P(1,4) = p(1-p)^{K-1}$.

Now we can verify if this four state cycle satisfies the Kolmogorov Criterion. We have that

$P(1,2) \cdot P(2,3) \cdot P(3,4) \cdot P(4,1) = p^m(1-p)^{K-m} a^m (1-p)^K \frac{1}{(c\cdot d)+1}(1-p)^{K-1} a(1-p)^K$
$= p^m(1-p)^{4K-m-1} a^{m+1} \frac{1}{(c\cdot d)+1}$

$P(1,4) \cdot P(4,3) \cdot P(3,2) \cdot P(2,1) = p(1-p)^{K-1} a(1-p)^K \frac{1}{(c\cdot d)+1} p^{m-1}(1-p)^{K-m-1} a^m(1-p)^K$
$= p^m(1-p)^{4K-m-2} a^{m+1} \frac{1}{(c\cdot d)+1}$

So we have that

$$P(1,2) \cdot P(2,3) \cdot P(3,4) \cdot P(4,1) \neq P(1,4) \cdot P(4,3) \cdot P(3,2) \cdot P(2,1)$$

Then this four state cycle does not satisfy the Kolmogorov Criterion, so the dynamic R is not Time Reversible.

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4. Edge Dynamic simulations

In this section we describe the results obtained from the simulations by studying different model properties, such as: expansion, convergence time, degree, volume and flooding[8] time. Each experiment was repeated for m times, so the results are given on average. Specifically, the dynamic random graph $\mathcal{G}(n,d,c,p)$ has been simulated $m = 10$ times by varying the parameters.

The expansion properties of $\mathcal{G}$ were measured using the spectral gap of the transition matrix P_t of the random walk performed on the dynamic graph at each time step $t \geq 0$ and the convergence time was established using the following heuristic:

> **Spectral Convergence:**
> At generic time $t \geq 0$, given the spectral gaps $\gamma_{t-\log n}, \gamma_{(t-\log n)+1}, \dots, \gamma_t$ of the latest $\log n$ executions of the model, if they differ only by an ε value then the dynamic random graph is converged in a stationary configuration with spectral gap value in the range $[\gamma_t - \varepsilon, \gamma_t + \varepsilon]$.

So after the convergence, we can assume that the expansion properties of the stationary configuration are defined by γ_t. Since giving the average spectral gap of the m simulations is not meaningful to understand how the model evolves, we define $\gamma_{gap}^{(max)}$ and $\gamma_{gap}^{(min)}$ respectively as the maximum and minimum value of the convergence spectral gaps.

The flooding time was measured as follows: after the spectral convergence, the flooding protocol was simulated by observing the average stopping time. Below are the experiments results.

For the following experiments, small d (2, 3, 4) and c (1.5, 3) values were used so that the resulting dynamic graph can be considered a good expanders (sparse and with good connectivity properties).

Convergence time and expansion

p=0.0	Best Expansion d = 4, c = 1.5				Worst Expansion d = 2, c = 1.5			
	Range of the Spectral Gap		Convergence Time		Range of the Spectral Gap		Convergence Time	
Nodes	γ_{gap}^{min}	γ_{gap}^{max}	**Mean**	**Standard Deviation**	γ_{gap}^{min}	γ_{gap}^{max}	**Mean**	**Standard Deviation**
64	0.20	0.27	8	1	0.01	0.05	7	1
256	0.18	0.21	10	1	0.02	0.03	9	1
512	0.18	0.21	10	1	0.01	0.02	10	1
1024	0.18	0.20	11	1	0.01	0.02	12	1

Table 1: Best and Worst Expansion scores for $p = 0.0$.

Table 1 shows the best and worst expansion results for different values of n, d, c with $p = 0.0$ (assuming this falling probability value the model is the one proposed by Becchetti et al [5]). As can be seen from the table, for $d = 2$ and $c = 1.5$ we obtained a graph with spectral gap $\gamma = 1 - \lambda_2 \simeq 0$ (having $\lambda_2 \simeq 1$). This result tells us that the resulting graph from convergence is likely bipartite or not connected and so is not an Expander graph. Conversely, for $d = 4$ and $c = 1.5$ we obtained a spectral gap between 0.2 and 0.3 which suggests an expansion quality increase of the graph. More precisely, these expansion values are very close to those of a 4-regular graph, suggesting that the graph is a good expander. This results empirically confirms the theoretical ones obtained in [5]. Introducing a non-zero failure probability of the edges, the expansion quality of the dynamic random graph tends to deteriorate. Specifically, we analyzed the behaviour of the spectral gap before and after such random falls. We observed that for values of $0 < p \leq 0.1$ the model converges on average in $\mathcal{O}(\log n)$ time steps with a low standard deviation σ of 1. For the spectral expansion, the converged dynamic random graph, before and after the edges disappearance, maintains a spectral gap between 0.2 and 0.3. This result suggests that the graph maintains a good expansion property even with 10% faults occurring. In terms of the Markov Chain R (defined on $\Omega' = \Omega \backslash \mathcal{D}$) described before these results tells us that, for $0 < p \leq 0.1$ the chain is stationary on a subset of $E \subset \Omega'$ composed by expander configurations, so the graph can be considered a good dynamic expander. In the following table you can see what has just been said.

Spectral Expansion												
p	**0.002**				**0.005**				**0.1**			
d = 4 **c = 1.5**	**Before**		**After**		**Before**		**After**		**Before**		**After**	
Nodes	$\gamma_{gap}^{(min)}$	$\gamma_{gap}^{(max)}$	$\gamma_{gap}^{(min)}$	$\gamma_{gap}^{(max)}$	$\gamma_{gap}^{(min)}$	$\gamma_{gap}^{(max)}$	$\gamma_{gap}^{(min)}$	$\gamma_{gap}^{(max)}$	$\gamma_{gap}^{(min)}$	$\gamma_{gap}^{(max)}$	$\gamma_{gap}^{(min)}$	$\gamma_{gap}^{(max)}$
64	0.20	0.28	0.20	0.28	0.21	0.24	0.18	0.21	0.18	0.25	0.17	0.21
256	0.17	0.21	0.17	0.21	0.17	0.18	0.16	0.18	0.18	0.21	0.16	0.20
512	0.19	0.20	0.19	0.20	0.17	0.18	0.16	0.17	0.18	0.21	0.17	0.19
1024	0.18	0.20	0.18	0.20	0.17	0.18	0.16	0.17	0.18	0.21	0.17	0.19

Table 2: Spectral Expansion before and after the random falls for $0 < p \leq 0.1$, $d = 4$ and $c = 1.5$

Spectral Expansion before random falls **d = 4, c = 1.5**												
p	**0.2**		**0.3**		**0.4**		**0.5**		**0.6**		**0.7**	
Nodes	$\gamma_{gap}^{(min)}$	$\gamma_{gap}^{(max)}$	$\gamma_{gap}^{(min)}$	$\gamma_{gap}^{(max)}$	$\gamma_{gap}^{(min)}$	$\gamma_{gap}^{(max)}$	$\gamma_{gap}^{(min)}$	$\gamma_{gap}^{(max)}$	$\gamma_{gap}^{(min)}$	$\gamma_{gap}^{(max)}$	$\gamma_{gap}^{(min)}$	$\gamma_{gap}^{(max)}$
64	0	0.25	0.16	0.22	0.13	0.20	0.01	0.19	0.07	0.15	0	0.10
256	0	0.19	0.15	0.18	0.13	0.16	0	0.13	0	0.10	0	0.18
512	0	0.19	0	0.16	0	0.15	0	0.12	0	0.10	0	0.28
1024	0	0.18	0	0.16	0.13	0.16	0	0.13	0	0.10	0	0.18

Table 3: Spectral Expansion before the random falls for $0.2 \leq p \leq 0.7$, $d = 4$ and $c = 1.5$

Varying p from 0.2 to 0.7 we observed that the expansion quality of the graph deteriorates as p increases, but from Table 3 an interesting property of the model may also be noticed, that is, to have a spectral gap between 0 and 0.3 before the random falling phase. This means that the dynamic random graph, after the spectral convergence, at each time step, before the random falling phase, has really good expansion properties as those in Table 1. In terms of the Markov chain R, this result tells us that as p increases in the range $[0.2, 0.7]$, the chain, at each time step, after the falling phase, leave the subset $E \subset \Omega'$ and enters a set of non expander states. However, these $N \subset \Omega'$ such that $N \cap E = \emptyset$ encodes configurations that are close to the expanders ones, because the chain only needs one R.A.E.S. step to come back in such E, we refer to this set of states as nearly expanders configurations. Such result suggests that the chain drifts towards a specific class of states. Specifically it tells us that the model, for $p \in (0, 0.7]$ has a drift to the expander configurations, and so that the dynamic graph tends to be a dynamic expander [3]. Regarding the convergence time, from the experiments, we observed that, on average, for $p \in [0, 0.7]$,the spectral gap takes $\mathcal{O}(\log n)$ rounds to stabilize with low standard deviation σ[4].

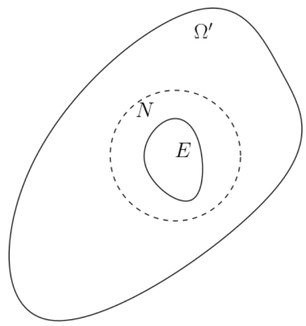

Figure 3: State space Ω' of the Markov Chain R.

[3]This result lead us to Observation 2.2

[4]This final observation together with those of Table 1, 2 lead us to Observation 2.1

We also investigated the structure of the graph after the spectral convergence 4, observing the average number of (d, cd)-regular vertices before and after the random falling phase as p increases.

We observed that, before the random falling process, the graph, on average, assumes a (d, cd)-regular configuration with low standard deviation σ, confirming expanding-drift result shown in Table 2, 3.

Regarding the structure after the random falling phase (that is the structure of the dynamic random graph at each time step), as we expected, we observed that: as p increases, the average number of (d, cd)-regular nodes decreases (this final result is shown in Table 4).

Structural Properties d = 4, c = 1.5				
Nodes	**64**	**256**	**512**	**1024**
p	**Average (d,cd)-regular**	**Average (d,cd)-regular**	**Average (d,cd)-regular**	**Average (d,cd)-regular**
0	61	243	485	970
0.002	63	254	509	1019
0.1	47	145	460	922
0.2	32	130	263	820
0.3	20	83	166	334
0.4	11	45	90	182
0.5	4	19	40	80
0.6	1	6	13	26
0.7	0	1	3	6

Table 4: Structural properties for $p \in [0, 0.7]$, $d = 4$ and $c = 1.5$

Flooding time analysis

Flooding Time d = 4, c = 1.5								
Nodes	**64**		**256**		**512**		**1024**	
p	**Average Flooding Time**	**Std Flooding Time**	**Average Flooding Time**	**Std Flooding Time**	**Average Flooding Time**	**Std Flooding Time**	**Average Flooding Time**	**Std Flooding Time**
0	**4**	0	**5**	1	**6**	0	**7**	0
0.002	4	0	5	1	7	2	7	1
0.1	4	1	5	2	7	2	8	1
0.2	5	1	6	1	7	1	9	1
0.3	6	1	7	1	9	1	9	1
0.4	7	1	9	1	10	1	11	1
0.5	8	2	11	1	13	1	14	1
0.6	11	2	16	1	23	3	19	2
0.7	14	3	20	3	24	0	25	2

Table 5: Average Flooding Time for $p \in [0, 0.7], d = 4$ and $c = 1.5$

In Table 5 are shown the results of simulations for $d = 4$ and $c = 1.5$, that gives us the best expanding dynamic random graph. From the data we can easily observe that the Flooding protocol, on average, terminates in $\mathcal{O}(\log n)$ rounds for values of $p \in [0, 0.7]$, this means that the information sent via broadcast by a single node to the whole network, propagates completely in $\mathcal{O}(\log n)$ rounds even if at any time step 70% of the edges randomly disappear[5]. This results suggests that the Edge Dynamic graph faithfully represent a p2p network in which the number of nodes is fixed because from the previous observations we have that the dynamic graph tends to be a good dynamic expander and guarantees the diffusion of information throughout the network in logarithmic time.

[5]This lead us to Observation 2.3.

5. Proof of Theorems 2.5 and 2.6

In order to prove the Theorem 2.5 it is sufficient to note that the model can be interpreted as the following Markovian Queue[11]

> ***M/G/∞***
> At each time step $t \geq 0$:
>
> 1. $N(t)$ vertices enter in the graph, where $N(t)$ is a Poisson process with intensity parameter λ.
> 2. Each vertex leave $\mathcal{G}$ with probability q.

It is known from [12], that the stationary distribution of this process is $\boldsymbol{v} \sim Poisson(\frac{\lambda}{q})$ [6], that is:

$$\boldsymbol{v}(i) = \frac{\left(\frac{\lambda}{q}\right)^i}{i!} e^{-\frac{\lambda}{q}},\ i \geq 0$$

Also, from this result, we can state that the average number of vertices in the graph is $\frac{\lambda}{q}$. If we slightly change the model assuming that each vertex in the graph has an exponential life time $Exp(\mu)$, our analysis boils down to that proposed in [9]. Below, we quote one of the theorems that can be extended to our model:

Theorem 5.1. *[9] Let $n = \frac{\lambda}{\mu}$ and $\mathcal{G}(V_t, E_t)$ the dynamic random graph at time t, then*

- *For each $t = \Omega(n)$, $|V_t| = \Theta(n)$ w.h.p.*
- *If $\frac{t}{n} \to \infty$, then $|V_t| = n + o(n)$ w.h.p.*

This Theorem tells us that, the network, after a time proportional to the average number of nodes in the network, contains $\frac{\lambda}{\mu}$ vertices with high probability. The statement provide us a starting point to formulate a convergence criterion for the model simulation.

[6] Proving the Theorem 2.5.

6. Vertex Dynamic simulations

In this section, as in Section 4, we describe the model simulations results. Each experiment was repeated for k times, so the results are given on average. $\mathcal{G}(\lambda, q, d, c)$ has been simulated for $k = 30$ times, observing the same properties measured in Section 4.

Using the results of Section 5 we defined the following model convergence criterion:

Almost Regular Convergence:
Wait for the model to get a stationary configuration of $\frac{\lambda}{q}$ vertices, then wait for 90% of the vertices being (d, cd)-regular.

In this scenario, the flooding time was measured as follows: after the Almost Regular Convergence, the flooding protocol was simulated by observing the same parameters observed in Section 4.

Preliminary analysis and a priori considerations

By performing the experiments we noticed that for particular values of q the flooding protocol fails.That is, all the informed nodes leave the network, blocking the information spreading process or at each round most of the informed ones leaves the graph heavily slowing down the diffusion process. It has been observed that for values of q higher than 0.1 the broadcast protocol fails or is too slow. Therefore the experiments that will be reported below will be those for which the information spreading process ends correctly. The main reason is that the network must be able to ensure that every message sent by any agent spread completely (and faster) on the whole network and if we have a dynamic graph that does not guarantee this property, then it does not model a real P2P network. We also observed that for $q \geq 0.1$ the Vertex Dynamic doesn't reach the percentage of (d, cd)-regular nodes required by the convergence criterion. In fact, for values of $q \geq 0.1$ we only have that at most 70% of the nodes satisfy this semi-regular property, so for the sake of clarity we will show only the results for $q < 0.1$. Moreover, many experiments have been carried out by varying λ and q, more specifically for $\lambda \in \{1, 2, 10, 20, 50\}$ and $q < 0.1$ such that $\frac{\lambda}{q} \in \{20, 50, 100, 200, 400, 500, 1000\}$. Finally, since we have a large amount of data with very similar results, we will show a sample of the results instead of showing them all.

Structural properties and Flooding time

d	c	Average (d,cd)-regular	Std (d,cd)-regular	Average |V|	Std |V|	Average time disconnected	Std time disconnected	Average Volume	Std Volume	Average Deg	Std Deg	Average Flooding Time	Std Flooding Time
2	1.5	92	4	95	5	0.2	0.0	228	12	3	0.2	14	2
2	2	93	5	95	5	0.1	0.1	257	14	4	0.2	10	1
2	3	94	5	95	5	0.1	0.1	279	15	6	0.2	8	1
3	1.5	91	5	96	5	0.1	0.1	326	18	5	0.2	8	2
3	2	93	4	96	4	0.1	0.1	383	21	6	0.3	7	1
3	3	92	5	96	5	0.1	0.1	401	25	8	0.3	6	1
4	1.5	91	5	96	5	0.1	0.0	457	24	6	0.4	6	1
4	2	91	4	96	4	0.1	0.1	500	24	8	0.4	6	1
4	3	92	4	97	4	0.1	0.1	524	27	11	0.4	5	1

Table 6: Results for $\frac{\lambda}{q} = 100$

d	c	Average (d,cd)-regular	Std (d,cd)-regular	Average |V|	Std |V|	Average time disconnected	Std time disconnected	Average Volume	Std Volume	Average Deg	Std Deg	Average Flooding Time	Std Flooding Time
2	1.5	964	33	967	33	0.14	0.01	2369	83	3	0.16	21	2
2	2	964	33	966	33	0.06	0.06	2630	93	4	0.18	15	1.5
2	3	965	33	965	33	0.03	0.03	2836	100	6	0.19	12	1
3	1.5	960	33	965	33	0.12	0.02	3348	115	5	0.23	11	1
3	2	963	33	966	33	0.04	0.04	3892	134	6	0.27	9	0.5
3	3	962	33	965	33	0.00	0.00	4101	143	8	0.29	8	0.5
4	1.5	960	33	965	33	0.10	0.10	4694	163	6	0.32	8	0.8
4	2	961	33	966	33	0.05	0.05	5122	181	8	0.36	**7**	05
4	3	962	33	966	33	0.01	0.01	5302	184	11	0.37	**6**	0.4

Table 7: Results for $\frac{\lambda}{q} = 1000$

Table 6, 7 show the results of a sample of the model simulations. From them we can initially observe that for $d = 4$, on average, the dynamic graph have a logarithmic diameter and so the average flooding time is $\mathcal{O}(\log n)$. For others values of d we observed that the flooding time slightly deteriorates. As a first hypothesis, to justify this deterioration, we thought that the random dynamic graph on average was disconnected (and therefore that it was not a good dynamic expander). To analyze this phenomenon, we observed the worst simulation and we found out that the graph, for extremely small values of d, was often disconnected. These results are shown below, in Figure 4. Conversely, to analyze the structure of the graph in optimal settings we observed the best simulation in terms of flooding time and (d, cd)-regularity of the graph (Figure 5, 6).

Plottings for $\frac{\lambda}{q} = 100$

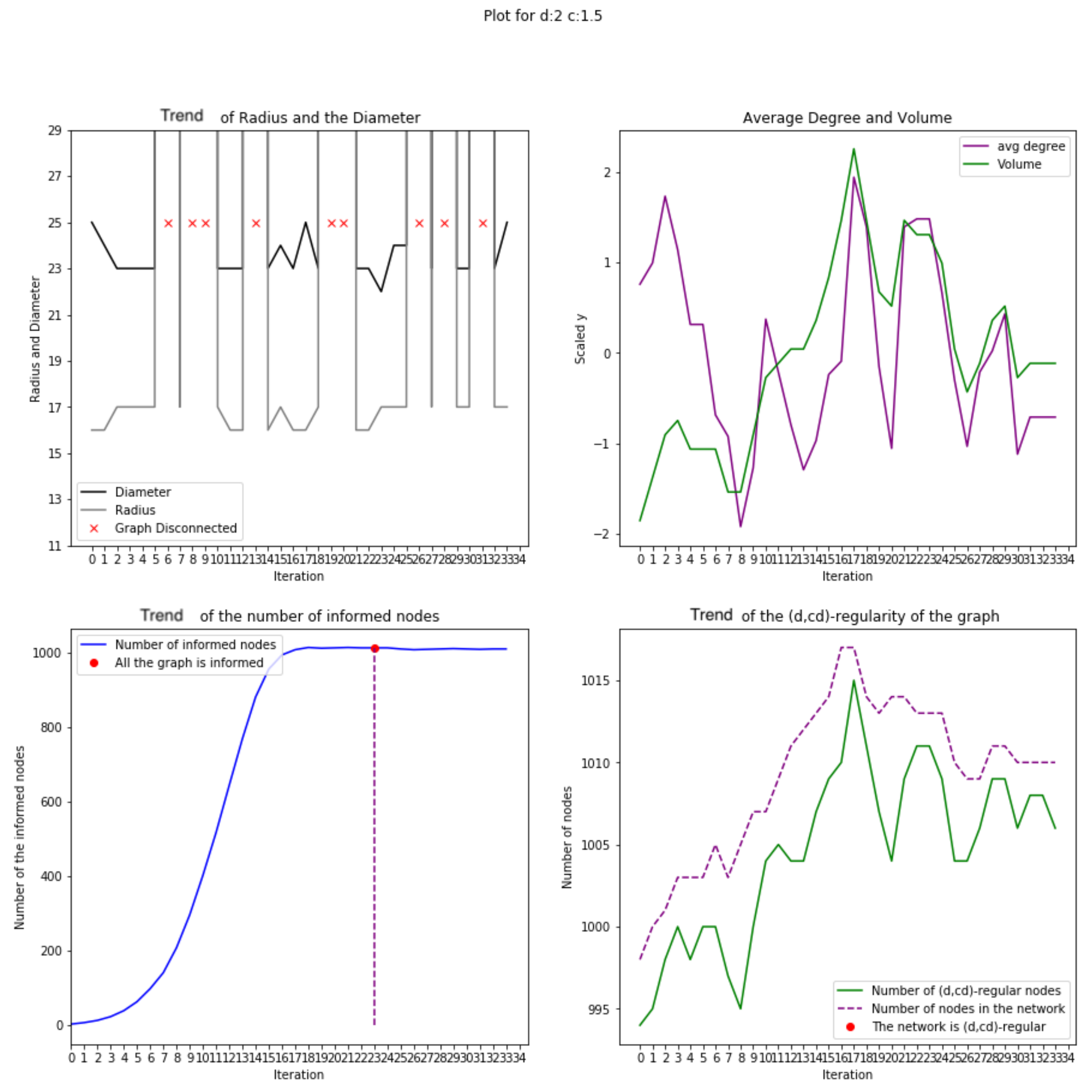

Figure 4: **Top left**: trend of the diameter and radius of the graph over time. **Top right**: relationship between average degree and volume of the graph over time. **Bottom left**: number of nodes informed by the flooding protocol at each of time step. **Bottom right**: number of nodes (d, cd)-regular and the total number of vertices in the network for each time step.

Figure 4 shows the worst results (in terms of flooding time) for $\frac{\lambda}{q} = 100$, notice that we have really small values of d and c, so implies that the dynamic graph is disconnected for most of the simulation and when is not, its diameter is high (between 23 and 25). From the figure we can observe that the information spreading process correctly ends in 23 time steps. We can also see that about 90% of the network is (d, cd)-regular but there is no time step where all nodes have this property.

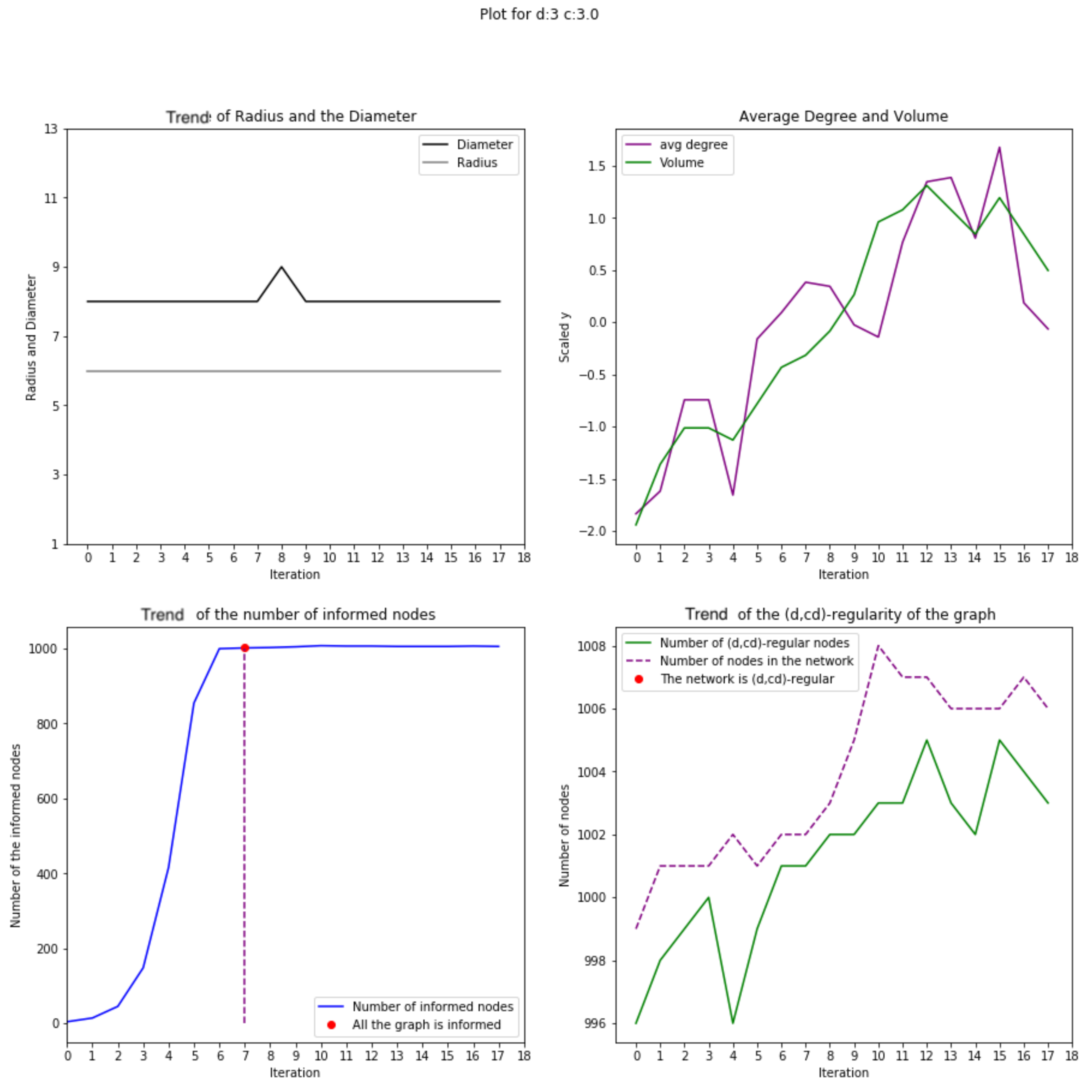

Figure 5: **Top left**: trend of the diameter and radius of the graph over time. **Top right**: relationship between average degree and volume of the graph over time. **Bottom left**: number of nodes informed by the flooding protocol at each of time step. **Bottom right**: number of nodes (d, cd)-regular and the total number of vertices in the network for each time step.

Figure 5 shows the best Flooding simulation on the dynamic random graph. From the bottom left plot we can observe that the protocol correctly ends in 7 time steps, that is logarithmic in the expected size of the network $\frac{\lambda}{q} = 100$. Furthermore from the top left plot we can observe that the network stabilizes on a logarithmic diameter, supporting the Flooding time result.

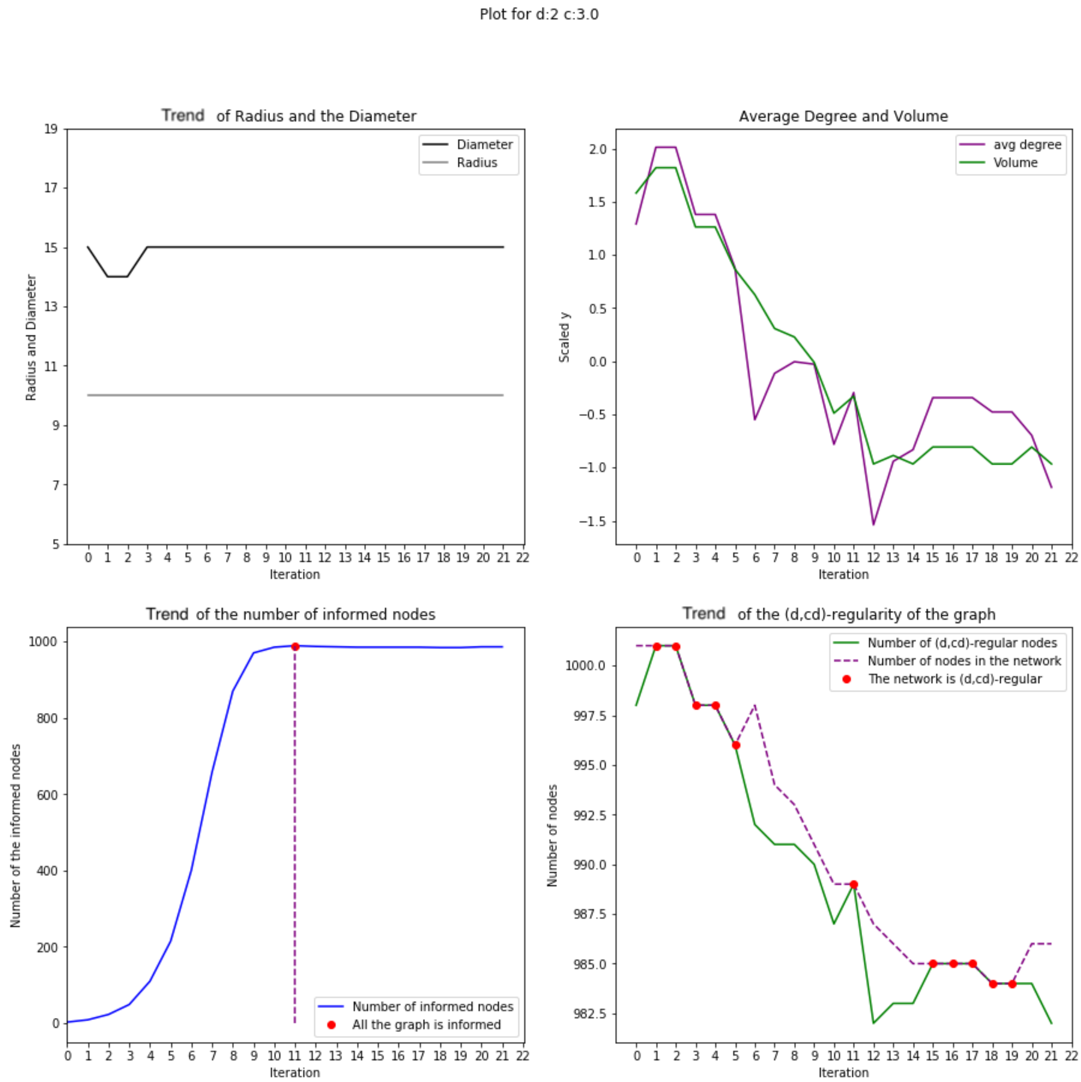

Figure 6: **Top left**: trend of the diameter and radius of the graph over time. **Top right**: relationship between average degree and volume of the graph over time. **Bottom left**: number of nodes informed by the flooding protocol at each of time step. **Bottom right**: number of nodes (d, cd)-regular and the total number of vertices in the network for each time step.

Figure 6 shows the best (d, cd)-regular simulation, as we can see from the plottings, having an almost regular structure with $d = 2, c = 3$ does not imply a good Flooding time (as in Figure 5). That is because this dynamic random graph has worst expansion properties than the one in Figure 5.

Plottings for $\frac{\lambda}{q} = 1000$

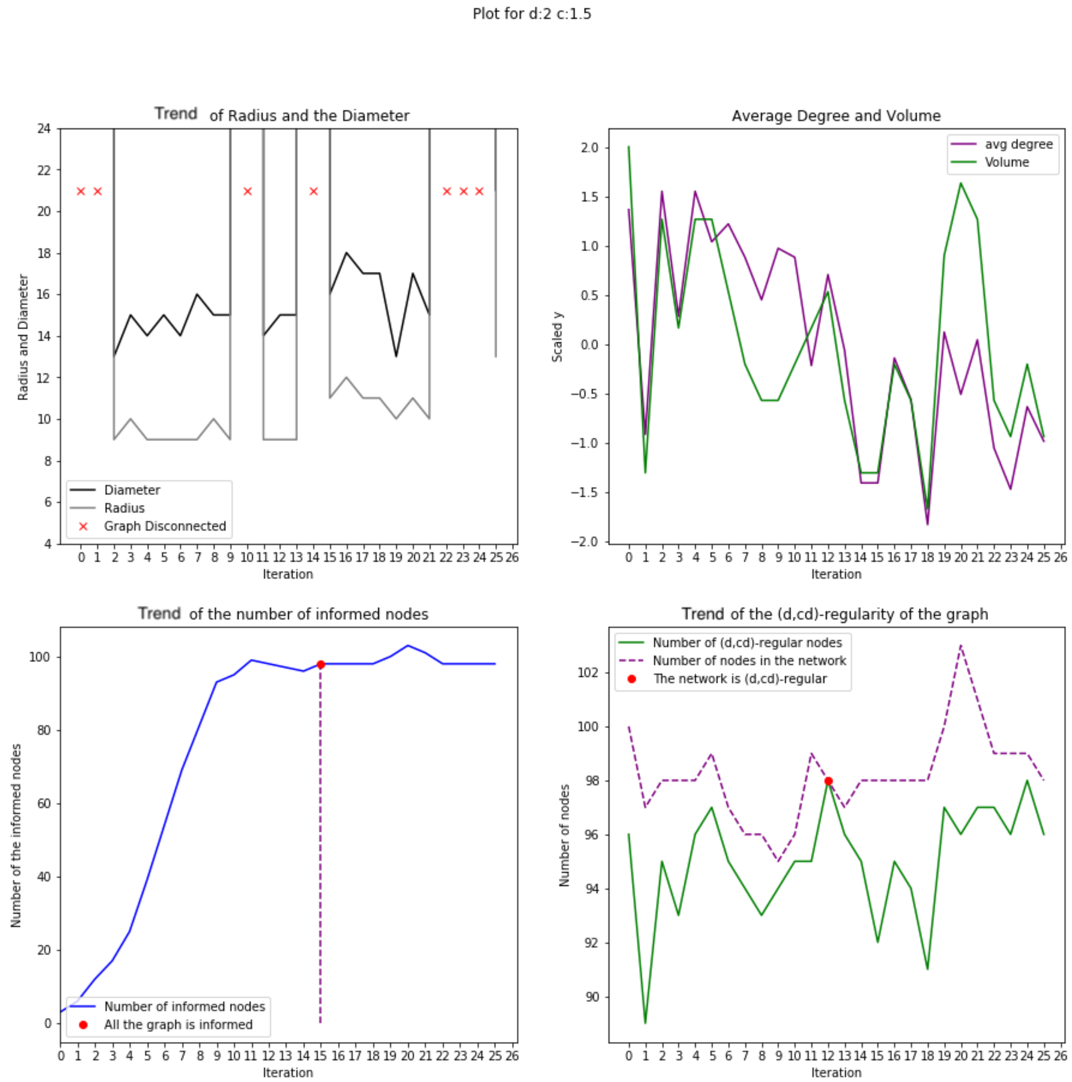

Figure 7: **Top left**: trend of the diameter and radius of the graph over time. **Top right**: relationship between average degree and volume of the graph over time. **Bottom left**: number of nodes informed by the flooding protocol at each of time step. **Bottom right**: number of nodes (d, cd)-regular and the total number of vertices in the network for each time step.

Figure 7 shows the worst results (in terms of flooding time) for $\frac{\lambda}{q} = 1000$, notice that we have really small values of d and c, so implies that the dynamic graph is disconnected for most of the simulation. From the figure we can observe that the information spreading process correctly ends in 15 time steps. We can also see that about 90% of the network is (d, cd)-regular and there is only one time step in which all nodes have this property.

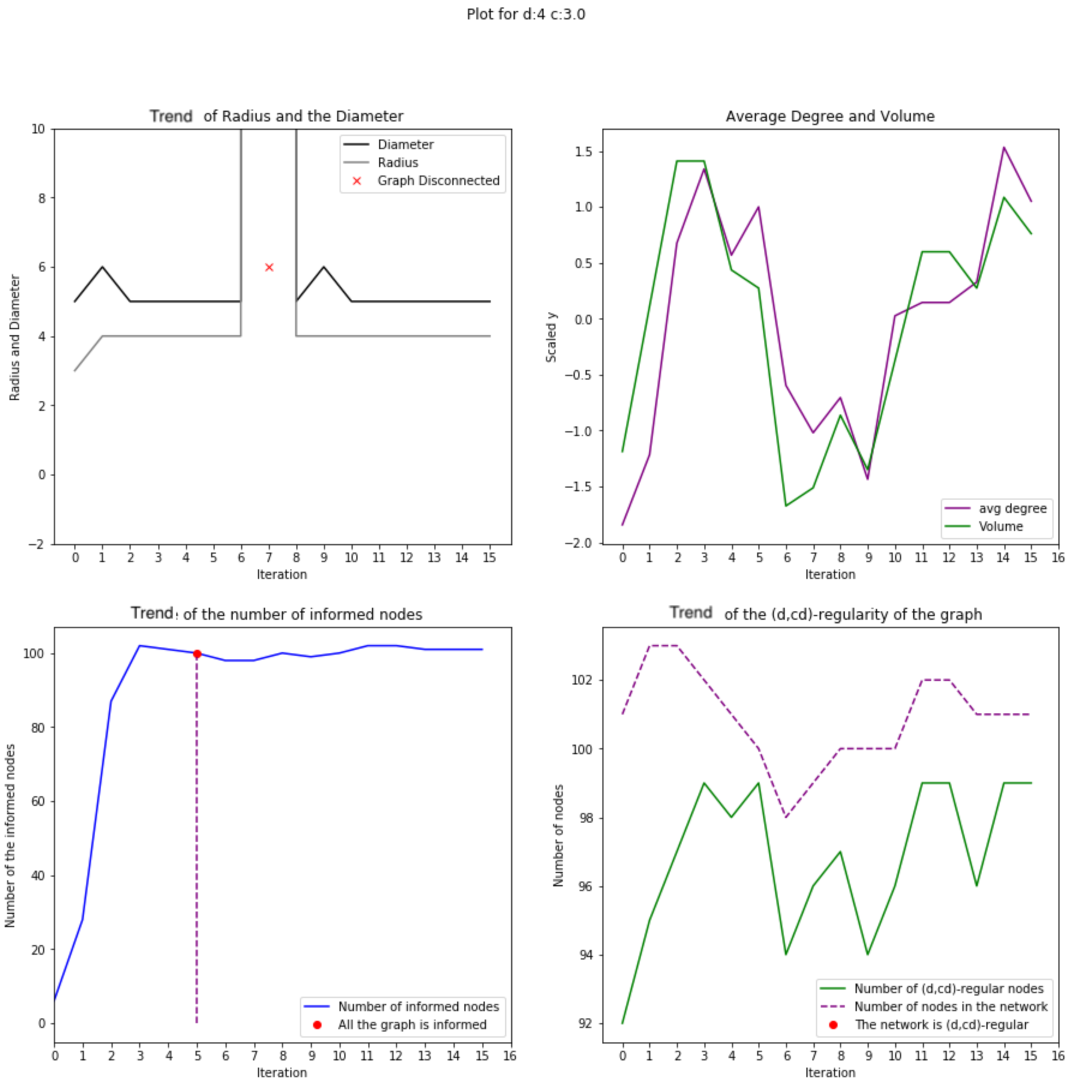

Figure 8: **Top left**: trend of the diameter and radius of the graph over time. **Top right**: relationship between average degree and volume of the graph over time. **Bottom left**: number of nodes informed by the flooding protocol at each of time step. **Bottom right**: number of nodes (d, cd)-regular and the total number of vertices in the network for each time step.

Figure 8 shows the best Flooding simulation on the dynamic random graph. From the bottom left plot we can observe that the protocol correctly ends in 5 time steps, that is logarithmic in the expected size of the network $\frac{\lambda}{q} = 1000$. Furthermore from the top left plot we can observe that the network, on average, stabilizes on a logarithmic diameter, supporting the Flooding time result.

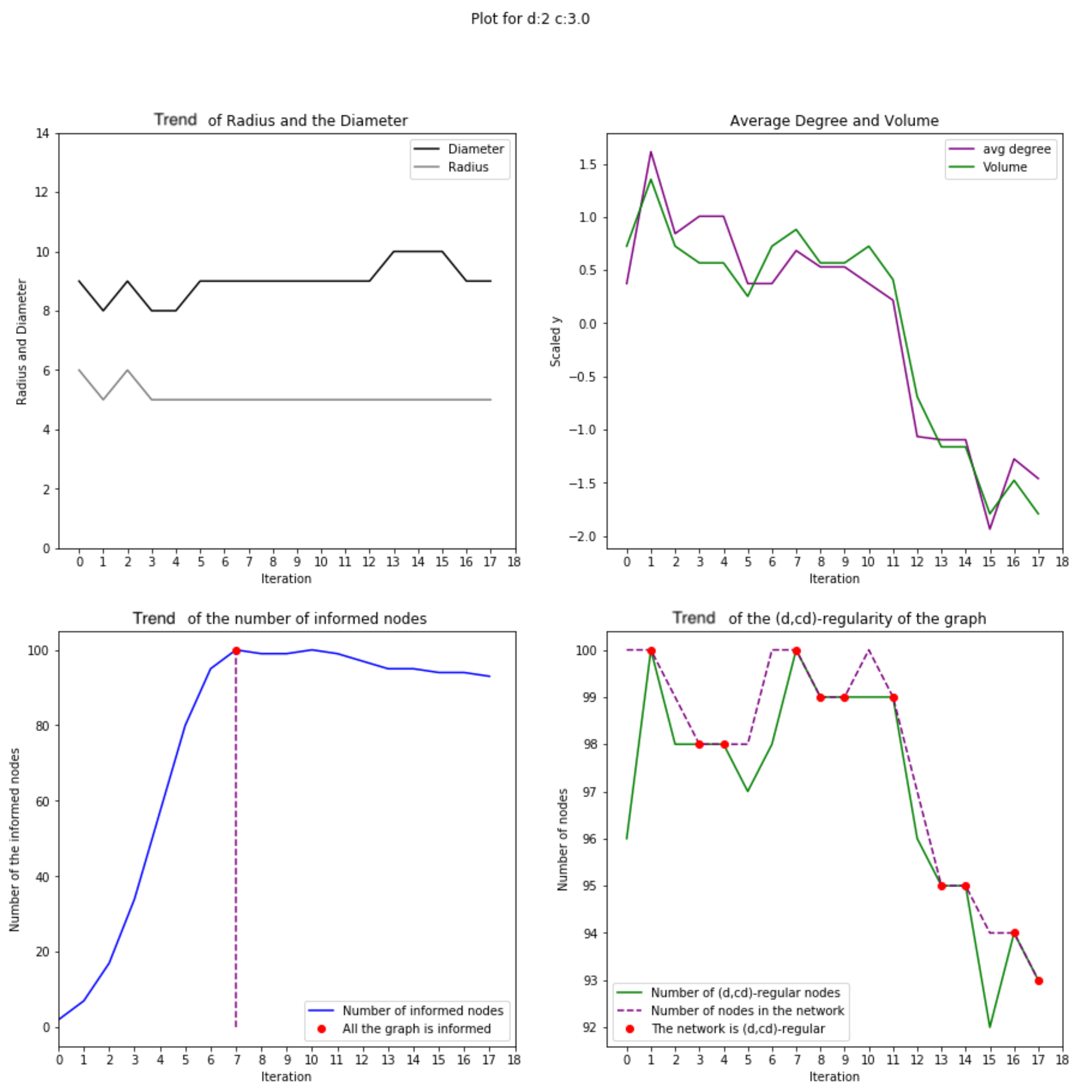

Figure 9: **Top left**: trend of the diameter and radius of the graph over time. **Top right**: relationship between average degree and volume of the graph over time. **Bottom left**: number of nodes informed by the flooding protocol at each of time step. **Bottom right**: number of nodes (d, cd)-regular and the total number of vertices in the network for each time step.

Figure 9 shows the best (d, cd)-regular simulation, as we can see from the plottings, having an almost regular structure with $d = 2, c = 3$ does not imply a good Flooding time (as in Figure 8). That is because this dynamic random graph has worst expansion properties than the one in Figure 8.

From the data we also observed that for fixed q and increasing values of λ the Vertex Dynamic model, as the intensity parameter of the Poisson process increases, takes longer to converge. Clearly, for $d = 2$ (and increasing λ with fixed q) we obtained the worst Flooding performances. That is the dynamic graph does not have a logarithmic diameter, and the arrival of new agents in the network worsens its structure. In the following table is shown the relation between the average flooding time and the average number of informed nodes at each time step.

		$\lambda = 10, q = 0.02$			$\lambda = 10, q = 0.01$			$\lambda = 20, q = 0.02$		
d	**c**	**Average Flooding Time**	**Std Flooding Time**	**Average Informed Nodes at Each Time Step**	**Average Flooding Time**	**Std Flooding Time**	**Average Informed Nodes at Each Time Step**	**Average Flooding Time**	**Std Flooding Time**	**Average Informed Nodes at Each Time Step**
2	1.5	56	16	7	46	10	17	237	73	4
2	2	25	5	14	23	5	30	50	13	16
2	3	17	3	18	15	3	39	25	6	27
3	1.5	37	13	10	33	8	22	143	51	6
3	2	16	3	19	13	2	44	29	7	24
3	3	12	3	22	11	2	49	18	5	24
4	1.5	18	5	16	15	3	41	41	11	18
4	2	14	4	21	11	2	49	21	5	31
4	3	12	3	22	10	1	51	17	4	35

Table 8: Table of the relation between the average flooding time and the average number of informed nodes at each time step.

From Table 8 we can easily observe that the optimum average flooding time is for $d = 3, 4$ that is because for these parameter values the graphs assumes $\mathcal{O}(\log n)$ diameter.

Expansion properties and Flooding time

As it concern the expansion quality of the Vertex Dynamic Random Graph, we observed (as the Edge Dynamic) that for $d \in 2,3$ the spectral gap assumes nearly zero values ad for $d = 4$ assumes a maximum value of 0.3. Therefore we obtained that our best configurations assumes spectral properties very similar to those of 4-regular random graphs. We indeed obtained that for these configurations the Flooding protocol has the best termination time.The following table shows the trend, over all the m simulations, of the average number of informed nodes at each time step. We can easily observe that for $d = 4$, and $d = 3, c \in 2,3$ we have a lot of new informed nodes at each time step. This means that for these values we have, on average, better expanding configurations.

Average Informed Nodes at Each Time Step

		$\lambda = 1$		$\lambda = 10$		$\lambda = 20$
d	**c**	$q = 0.001$	$q = 0.01$	$q = 0.01$	$q = 0.02$	$q = 0.02$
2	1.5	32	4	17	7	4
2	2	40	5	30	14	16
2	3	45	5	39	18	27
3	1.5	47	5	23	10	7
3	2	54	6	44	19	25
3	3	56	6	49	22	35
4	1.5	56	6	41	16	19
4	2	60	6	49	21	31
4	3	60	7	51	22	35

Table 9: Average number of informed nodes at each time step

We have that the proposed model, for $d = 4$ and $c \in 1.5, 2, 3$ has a spectral gap similar to that of d-regular graphs and maintains, on average, a logarithmic diameter (ensuring a logarithmic flooding termination time).

7. Conclusions

We proposed two Dynamic Random Graphs inspired by the generation process of the Bitcoin P2P Network. More precisely we extended the work proposed by Becchetti et al. [5] defining two dynamic versions of their model. For the first one we have showed that it has a unique steady-state distribution, that is the mean of first return time on a generic configuration $\boldsymbol{\nu}(x) = \frac{1}{\boldsymbol{E}_x(\tau_x^+)}$, $x \in \Omega$, subsequently we have found that the process is not time reversible. Therefore it is difficult to formally find or calculate that steady-state distribution. In order to face this problem we simulated the model and studied his behaviour finding that, for appropriate parameter values, after $\mathcal{O}(\log n)$ rounds it assumes semi-regular topologies with an expansion quality very similar to that of d-regular random graphs. This result tells us that the network has strong connection properties and that it guarantees the connectivity of the nodes even in the presence of faults. Furthermore, we have shown that the model guarantees the rapid diffusion of new information, also in catastrophic scenarios in which most of the interconnections between the agents crashes. This means that new transactions always spreads successfully along the network.

For the second model we have shown that can be reduced to a Markovian Queue $M\backslash G\backslash\infty$ and that, the vertices have a Poissonian steady-state distribution with expected value defined by the ratio between the intensity parameter and the life parameter of the nodes. We also have showed that if we assume to have an exponential distributed lifetime of the vertices in the network, we can extend the results proposed by G. Pandurangan et al. in [9]. Thanks to the simulations we observed that the flooding protocol fails in highly-dynamic settings, in the sense that the nodes remain in the network for short time intervals. This result suggest us that we need to design ad hoc algorithms for these dynamic networks. We have showed that the graph, for appropriate parameter values, was a dynamic expander and the Flooding protocol (for appropriate lifetime parameter values) correctly terminated in $\mathcal{O}(\log n)$ rounds even when the graph turns out to be a bad expander. This model appears to be promising for the correct representation of the Bitcoin P2P Network as it guarantees good connectivity properties and captures the dynamic aspect of a P2P network in which the nodes join and leave the network.

These two models are a novel representation of P2P networks which allow us to study their evolution over time and their properties in more detail than the classic techniques based on static random graphs. Therefore these could be a very interesting new research topics to be explored in distributed algorithm and graph theory.

8. Future Works

For the Edge Dynamic Random Graph it is interesting to continue the theoretical analysis and give a bound of the mixing time (in an approximate way), and formally calculate the drift towards the class of states $E \subset \Omega'$ of (d, cd)-regular graphs.

For the Vertex Dynamic Random Graph would be a good idea to analyze in depth the model and formally bound the diameter, the convergence and the flooding time. Furthermore, it would be interesting to study and define new distributed protocols for dynamic networks, as we have seen that classical distributed algorithms do not work properly on highly dynamic topologies. Finally, could be interesting investigating different settings of this model, for example, in which each vertex has no more a fixed number of neighbours to connect with, but instead a random one following a probability distribution i.e. Poisson.

References

[1] S. Nakamoto, Bitcoin: A peer-to-peer electronic cash system - white paper (2008).

[2] A. Miller, J. Litton, A. Pachulski, N. Gupta, D. Levin, N. Spring, B. Bhattacharjee, Discovering bitcoin's public topology and influential nodes, et al (2015).

[3] T. Neudecker, P. Andelfinger, H. Hartenstein, Timing analysis for inferring the topology of the bitcoin peer-to-peer network, in: 2016 Intl IEEE Conferences on Ubiquitous Intelligence & Computing, Advanced and Trusted Computing, Scalable Computing and Communications, Cloud and Big Data Computing, Internet of People, and Smart World Congress (UIC/ATC/ScalCom/CBDCom/IoP/SmartWorld), IEEE, 2016, pp. 358–367.

[4] S. Delgado-Segura, S. Bakshi, C. Pérez-Solà, J. Litton, A. Pachulski, A. Miller, B. Bhattacharjee, Txprobe: Discovering bitcoin's network topology using orphan transactions, in: International Conference on Financial Cryptography and Data Security, Springer, 2019, pp. 550–566.

[5] L. Becchetti, A. Clementi, E. Natale, F. Pasquale, L. Trevisan, Finding a bounded-degree expander inside a dense one, in: Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SIAM, 2020, pp. 1320–1336.

[6] S. Hoory, N. Linial, A. Wigderson, Expander graphs and their applications, Bulletin of the American Mathematical Society 43 (4) (2006) 439–561.

[7] D. A. Levin, Y. Peres, Markov chains and mixing times, Vol. 107, American Mathematical Soc., 2017.

[8] N. Santoro, Design and Analysis of Distributed Algorithms (Wiley Series on Parallel and Distributed Computing), Wiley-Interscience, USA, 2006.

[9] G. Pandurangan, P. Raghavan, E. Upfal, Building low-diameter peer-to-peer networks, Selected Areas in Communications, IEEE Journal on 21 (2003) 995 – 1002. `doi:10.1109/JSAC.2003.814666`.

[10] F. P. Kelly, Reversibility and stochastic networks, Cambridge University Press, 2011.

[11] H. C. Tijms, A first course in stochastic models, John Wiley and sons, 2003.

[12] G. F. Newell, The $m/g/\infty$ queue, SIAM Journal on Applied Mathematics 14 (1) (1966) 86–88. `arXiv:https://doi.org/10.1137/0114007`, `doi:10.1137/0114007`.
URL `https://doi.org/10.1137/0114007`