# Assignment 3 – Inverse Problems

## A simple trans-dimensional inversion

In this assignment we shall carry out a very simple trans-dimensional inversion step-by-step. Based on the data and other assumptions we are going to choose between two solutions. One solution is a 1-parameter model, and the other solution is a 2-parameter model. The observed data are contaminated by noise with a known distribution.

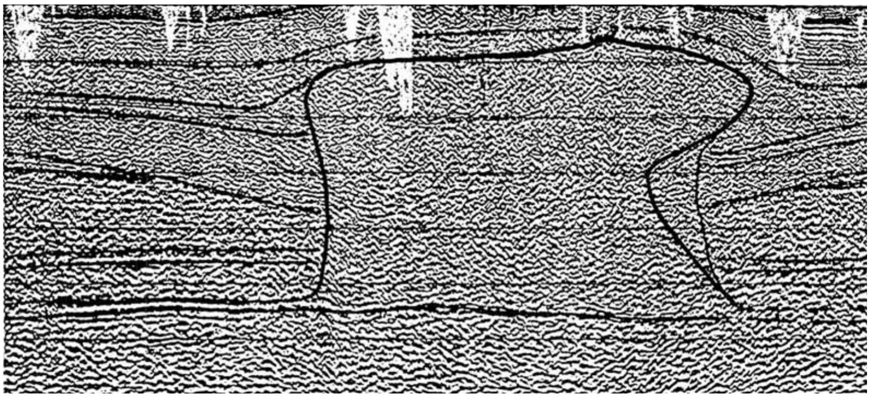

Figure 1: Seismic reflection profile with an outline of the Mors salt dome in Northern Denmark (Larsen and Baumann, DGF Bulletin, p.151-155, 1982)

The seismic reflection profile in figure 1 suggests (together with other data) that the subsurface consists of layered sediments surrounding a near-cylindrical salt structure. Such structures are formed by deep seated salt, originally situated in a horizontal layer, that has moved upward and has penetrated the overlying sediments. The driving force of this movement is buoyancy due to lower density of the salt, compared to the surrounding sediments.

### The problem

We will investigate and compare two simple models of this salt structure, using measurements of the gravity at the surface, and in a borehole. We assume that the structure is a perfect vertical cylinder with a lower density than the surrounding sediments. The term *density contrast* is used to denote the difference between the cylinder density and the surrounding density. The depth to the bottom of the salt structure is from the seismic data estimated to be 5400 m, and the top is at 1000 m. The radius of our cylindrical model is 4700 m.

We will compare two cylindrical models (see figure 2):

1. A homogeneous cylinder of height 4400 m with (unknown) density $\rho^{(1)}$
2. A stack of two homogeneous cylinders with a total height of 4400 m with (unknown) densities $\rho_1^{(2)}$ and $\rho_2^{(2)}$. Based on geological considerations we assume that the height of the upper sub-cylinders is 3000 m, and the height of the lower sub-cylinders is 1400 m.

We consider the two models (with 1 and 2 parameters, respectively) to be *a priori* equally acceptable.

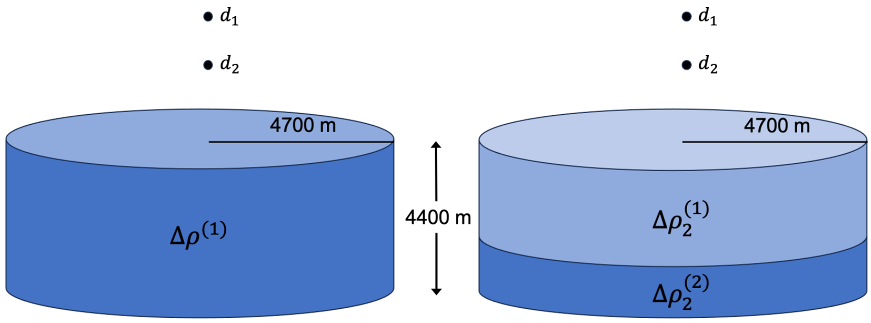


Figure 2: The two cylinder models to be compared.

Assume that we measure the gravity at points $P_1$ and $P_2$ on the symmetry axis at height 1000 m above the top surface (which is at the surface), and at height 500 m (in a borehole), respectively. We obtain the following deviation in gravity acceleration from the normal value (the field from a hypothetic, homogeneous Earth) :

$$d_1 = -0.00021453 \; m/s^2 \qquad d_2 = -0.00024499 \; m/s^2. \tag{1}$$

The uncertainty of both data is $\pm$ 0.00001 $m/s^2$.

We will now attempt an inversion of this data to obtain density contrasts, using the two different cylindrical models. To do this, we need the forward relation between data and model parameters.

It can be shown that the gravity acceleration above a homogeneous vertical cylinder of height $H$ and radius $R$, at a point $P$ on its symmetry axis is:

$$\Delta g = 2\pi G \, \Delta\rho \, ((b_{top} - h_{top}) - ((b_{bot} - h_{bot})) \tag{2}$$

where $G$ is the Gravitational constant, $\Delta\rho$ is the density contrast, $h_{top}$ and $h_{bot}$ are the vertical distances from $P$ to the top and the bottom of the cylinder, respectively, and $b_{top}$ and $b_{bot}$ are the distances from $P$ to a point on the circumference of the top surface and the bottom surface of the cylinder, respectively.

1. Assuming that the priors of the density contrasts $\Delta\rho^{(1)}$, $\Delta\rho_1^{(2)}$ and $\Delta\rho_2^{(2)}$ are all constant in the interval $\{-500, 0\}$ $kg/m^3$, and that the noise on each data value is uniformly distributed in the interval $\{-\sigma, \sigma\}$, where $\sigma = 0.00001$ $m/s^2$, write down the joint prior $p_{prior}(\mathbf{d}, \mathbf{m}, k)$ for the trans-dimensional problem (where $\mathbf{m}$ contains the unknown density parameter(s), and $k$ is the number of parameters/layers).

2. Replacing the data with the forward functions of the model parameters (for each $k$), write down the (unnormalized) joint posterior $p_{post}(\mathbf{d}, \mathbf{m}, k)$ in the $\mathbf{m}, k$ space.

3. Integrate $p_{post}(\mathbf{d}, \mathbf{m}, k)$ over $\mathbf{m}$ for each $k$ to compute the marginal probability distribution $p(k)$. Which model has the highest posterior probability?

**Hint**

To solve this assignment, you can seek inspiration in the slides 'Bayesian-Model-Selection.pdf' from Week 6, in the section: *A Tomograpic Example* (slide 19-25).

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