# Analytical derivation of the conoscopic interference pattern of a birefringent uniaxial crystal.

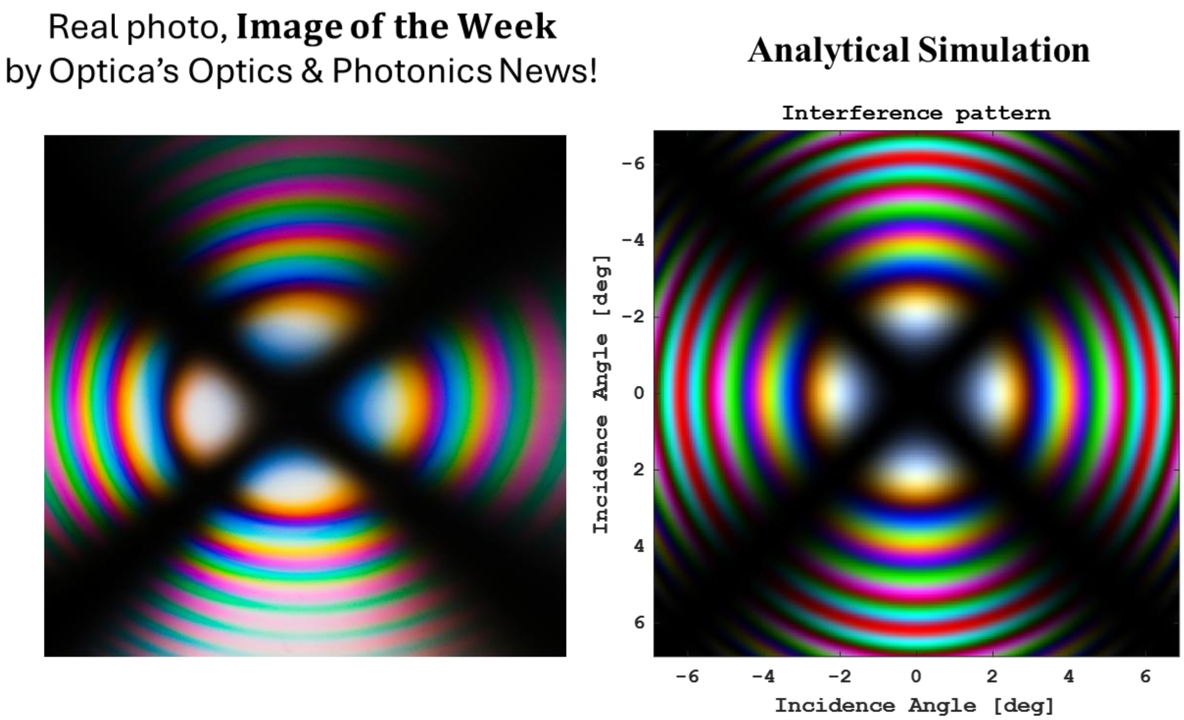


*Figure 1 Comparison between a real photo of the interference pattern of a birefringent crystal with the optical axis perpendicular to the crystal front face and the numerical simulation using the analytical derivation of the interference pattern.*

In a uniaxial birefringent crystal, there is a single direction known as the optical axis that represents a center of symmetry for the material. Namely, rotating the material around this axis does not change its optical behavior. Each direction with a specific angle with respect to this axis is optically equivalent. No matter its polarization state, the light propagating parallel to the optic axis is governed by a refractive index $n_o$ known as the ordinary refractive index. If we consider any other propagation direction, we can always decompose the incoming light polarization into two orthogonal polarizations. One linear polarization that is perpendicular to the optic axis, which is called the ordinary ray and experiences a refractive index value $n_o$, and another polarization propagating in the same direction will be oriented partly in the direction of the optic axis. This polarization is called the extraordinary ray and is governed by a different, direction-dependent refractive index.

The propagation of the ordinary ray is simply described by $n_o$ as if there were no birefringence involved. The extraordinary ray, as its name suggests, propagates differently from any wave in an isotropic optical material. This beam has an effective refractive index that has a value between $n_o$ and $n_e$. Its power flow is not in the same direction as the wave normal. Notably, while the power flow does not respect Snell's law and behaves extraordinarily, the direction of the wave normal follows it.

To compute the interference pattern, we first need to calculate the optical path difference between the ordinary and extraordinary rays as a function of the angle of incidence. We can define it as $\theta$ the angle between the optical axis and the extraordinary ray wave normal. The extraordinary beam experiences an effective refractive index $\bar{n}$ given by:

$$\frac{1}{\bar{n}^2} = \frac{sin(\theta)^2}{n_e^2} + \frac{cos(\theta)^2}{n_o^2} \tag{0.1}$$

Or equivalently:

$$\frac{1}{\bar{n}^2} = \frac{1}{n_e^2} - \left(\frac{1}{n_e^2} - \frac{1}{n_o^2}\right) cos(\theta)^2 \tag{0.2}$$

If we consider a light beam impinging on a crystal at an angle $i$ as in Figure 2. We can apply Snell's law to find the angles of refraction $r$, and $\bar{r}$ for the ordinary. and extraordinary wave normals, respectively. The Snell's equations for these wave normals are:

$$sin(i) = n_o sin(r) \tag{0.3}$$

$$sin(i) = \bar{n} sin(\bar{r}) \tag{0.4}$$

With the help of these equations, we can evaluate the path difference between the extraordinary and ordinary waves when they propagate at an angle $i$ into a plate with an optical axis parallel to the crystal surface, as shown in Figure 2. Following Figure 2 we can calculate the path difference as $\Delta P = P_e - P_o = AC - (AB + BD)$ where $AC = \frac{t}{cos(\bar{r})}$,

$AB = \frac{t}{cos(r)}$, $BC = t(tan(\bar{r}) - tan(r))$, and $BD = BCsin(i)$

Therefore $\Delta P = \frac{t}{cos(\bar{r})}\bar{n} - \frac{t}{cos(r)}n_o - t\left[tan(\bar{r}) - tan(r)\right]sin(i)$ and using (0.3 and 0.4) we obtain $\Delta P = \frac{t}{cos(\bar{r})}\frac{sin(i)}{sin(\bar{r})} - \frac{t}{cos(r)}\frac{sin(i)}{sin(r)} - t\left[tan(\bar{r}) - tan(r)\right]sin(i)$ which is equivalent to:

$$\Delta P = t sin(i)[cot(\bar{r}) - cot(r)] \tag{0.5}$$

From which the term $cot(r)$ can be evaluated by considering that:

$cot(r)^2 = \frac{cos(r)^2}{sin(r)^2} = \frac{1 - sin(r)^2}{sin(r)^2} = \frac{1}{sin(r)^2} - 1$ and using 0.3 we obtain $cot(r)^2 = \frac{n_o^2}{sin(i)^2} - 1$ leading to:

$$cot(r) = \sqrt{\frac{n_o^2 - sin(i)^2}{sin(i)^2}} = \frac{n_o}{sin(i)}\sqrt{1 - \frac{sin(i)^2}{n_o^2}} \tag{0.6}$$

The term $cot(\bar{r})$ can be written as:

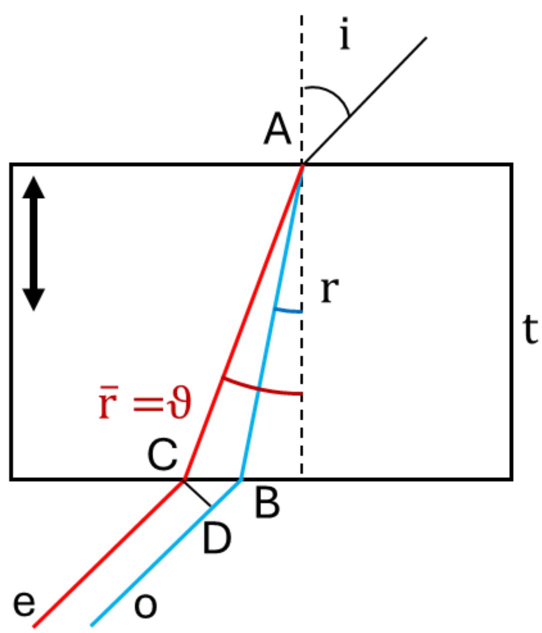


*Figure 2 Scheme of propagation for a ray of light incident with an angle i onto a birefringent crystal with the optical axis oriented as the double arrow in the figure. Inside the crystal, the light is split into two rays at different angles. After the crystal, the ordinary and extraordinary rays travel collinearly with the same angle as the incident ray, but having a lateral shear CD*

$$\cot(\bar{r}) = \frac{cos(\bar{r})^2}{sin(\bar{r})^2} = \frac{1 - sin(\bar{r})^2}{sin(\bar{r})^2}. \tag{0.7}$$

We can evaluate the term $sin(\bar{r})^2$ from 0.4 $sin(\bar{r})^2 = \frac{1}{\bar{n}^2} sin(i)^2$ where we can use 0.2 to evaluate the term $\frac{1}{\bar{n}^2}$ leading to:

$$sin(\bar{r})^2 = \left[\frac{1}{n_e^2} - \left(\frac{1}{n_e^2} - \frac{1}{n_o^2}\right) cos(\theta)^2\right] sin(i)^2 \tag{0.8}$$

We can evaluate the term $cos(\theta)^2$ of the last equation by considering Figure 1, which describes the scheme of propagation in an uniaxial birefringent crystal with the optical axis perpendicular to the crystal entrance face. In this case, the $\theta = \bar{r}$ and substituting this value in equation 0.8 we can solve for $sin(\bar{r})^2$ obtaining:

$$sin(\bar{r})^2 = \frac{\frac{1}{n_o^2} sin(i)^2}{\left[1 - \left(\frac{1}{n_e^2} - \frac{1}{n_o^2}\right) sin(i)^2\right]} \tag{0.9}$$

We can insert this expression into 1.7, leading to:

$$cot(\bar{r})^2 = \frac{1 - \left(\frac{1}{n_e^2} - \frac{1}{n_o^2}\right) sin(i)^2 - \frac{1}{n_o^2} sin(i)^2}{\frac{1}{n_o^2} sin(i)^2} \tag{0.10}$$

Finally, we can write the path difference as:

$$\Delta P = t\left[\sqrt{\frac{1 - \left(\frac{1}{n_e^2} - \frac{1}{n_o^2}\right) sin(i)^2 - \frac{1}{n_o^2} sin(i)^2}{\frac{1}{n_o^2}}} - n_o \sqrt{1 - \frac{sin(i)^2}{n_o^2}}\right] \tag{0.11}$$

Now we can compute the interference equation:

$$I = E_{OP}^2 + E_{EP}^2 + 2E_{OP}E_{EP} \cos\frac{2\pi}{\lambda} \Delta P \tag{0.12}$$

Where $E_{OP}$ and $E_{EP}$ are the projections of the ordinary and extraordinary wave electric fields along the 2$^{nd}$ polarizer transmission direction.

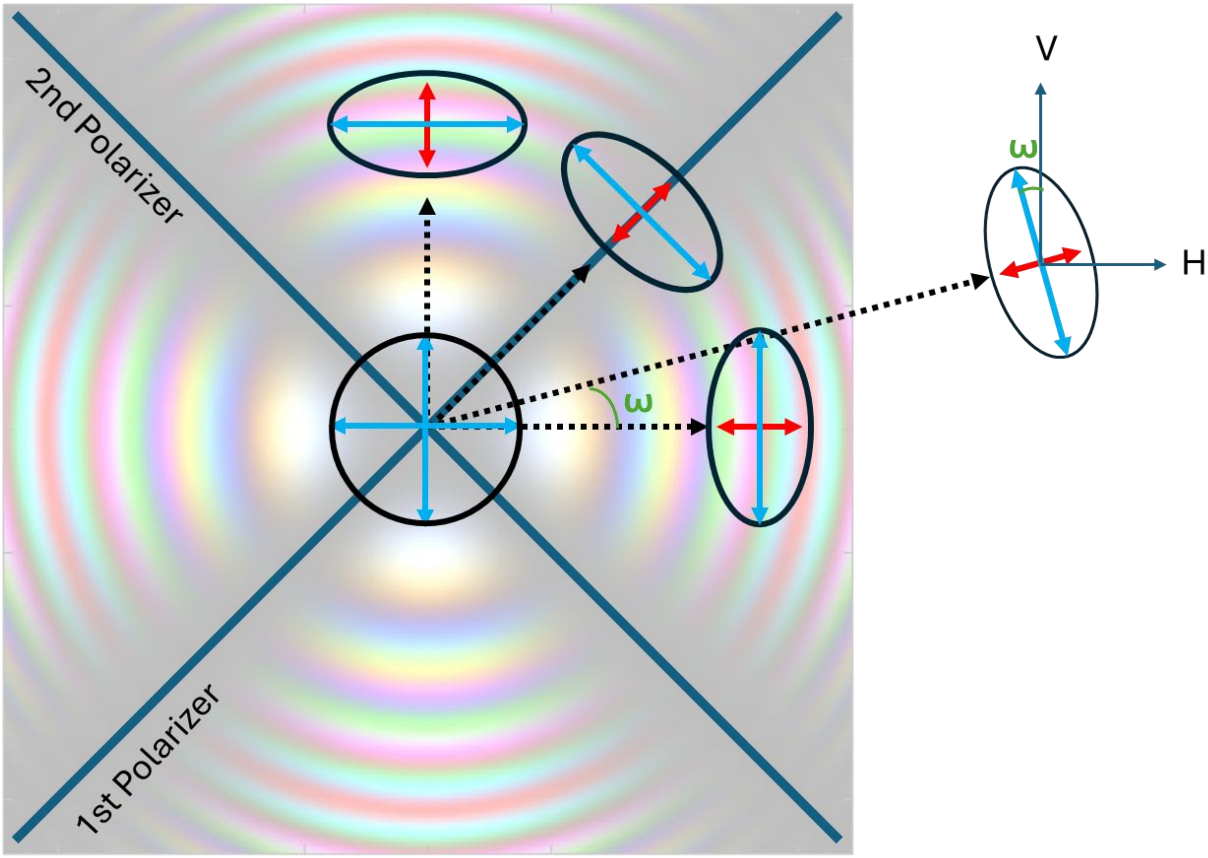


*Figure 3 Scheme of the shape of a cross-section through the cristal's optical refractive index ellipsoid in 3D that would be seen at each position corresponding to a defined incidence angle i and a defined angle ω with respect to the horizontal axis.*

Considering Figure 3, for every ω from [0;2π], the vertical V and horizontal H polarization components created by the 1st polarizer are mixed into the ordinary and extraordinary wave electric field as:

$$E = V\ \sin\omega + H\ \cos\omega$$

$$O = V\ \cos\omega - H\ \sin\omega$$

And the projections of these electric fields onto the 2nd polarizer transmission directions are:

$$E_{EP} = -\cos\left(\frac{\pi}{4} + \omega\right)(V\ \sin\omega + H\ \cos\omega) = \left(\frac{\sqrt[2]{2}}{2}\cos\omega - \frac{\sqrt[2]{2}}{2}\sin\omega\right)(V\ \sin\omega + H\ \cos\omega))$$

$$E_{OP} = \cos\left(\frac{\pi}{4} - \omega\right)(V\ \cos\omega - H\ \sin\omega) = \left(\frac{\sqrt[2]{2}}{2}\cos\omega + \frac{\sqrt[2]{2}}{2}\sin\omega\right)(V\ \cos\omega - H\ \sin\omega)$$

Now, considering the first polarizer at 45° with respect to the vertical direction, we have H=V; therefore, we obtain:

$$E_{EP} = -\frac{\sqrt[2]{2}}{2}\mathrm{H}\cos^2\omega + \frac{\sqrt[2]{2}}{2}\,V\sin^2\omega$$

$$E_{OP} = \frac{\sqrt[2]{2}}{2}\mathrm{V}\cos^2\omega - \frac{\sqrt[2]{2}}{2}\,H\sin^2\omega$$

So now we can insert $E_{OP}$ and $E_{EP}$ into the interference equation 0.12, obtaining:

$$I = \frac{1}{2}(H^2 + V^2)cos^4\omega + \frac{1}{2}(H^2 + V^2)sin^4\omega - 2HVcos^2\omega sin^2\omega + 2 * \left[\frac{1}{2}(H^2 + V^2)cos^2\omega sin^2\omega - \frac{1}{2}(HV)(cos^4\omega + sin^4\omega)\right] * \cos\frac{2\pi}{\lambda}\Delta P$$

Plotting the result numerically, we obtain what is shown in Figure 1, where the simulation closely represents the reality captured with a photographic camera. You can access the full code at this repository: https://github.com/Antonio-Nireos/Birefringence

Antonio Perri, PhD, NIREOS CTO

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