Conditional Distributions Review - Fall 2024

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Introduction

This document is a mini recap for understanding conditional distributions and marginalization techniques in probability theory. The goal is to clarify the concepts of conditional densities and the roles of variables and constants, helping students differentiate between evaluating functions and computing densities.

1. Overview of Notation

- **Function of a Variable**: Notation like $f(x)$ means that x is the variable. Here, $f(x)$ changes when x changes.
- **Function Evaluated at a Constant**: Notation like $f(x = a)$ or simply $f(a)$ means that x is set to a specific value a, resulting in a fixed number.

Example of Function vs. Evaluation:

- **Function**: $f(x) = x^2$
- **Evaluation**: $f(3) = 3^2 = 9$

This distinction is essential for conditional densities, where we often use variables and constants in different contexts.

2. Conditional Density for Random Variables

Suppose we have three random variables X, Y, and Z with a joint density $f_{X,Y,Z}(x, y, z)$.

Goal

Find the conditional density of one variable (or group) given the others.

- **Conditional Density Notation**: $f_{A|B=b}(a)$ tells us:
  - A is the variable of interest (the one we're focusing on).
  - $B = b$ fixes the value of B to b.
  - $f_{A|B=b}(a)$ is then the density of A assuming B is fixed at b.

3. Examples

Example 1: Single Variable A, Two Variables B

Define Groups:

- Let $A = X$ (single variable).
- Let $B = (Y, Z)$ (two variables).

Goal: Compute $f_{X|Y=y,Z=z}(x)$, the conditional density of X given $Y = y$ and $Z = z$.

1. **Joint Density** $f_{X,Y,Z}(x, y, z)$: This is a function of the three variables x, y, and z.
2. **Marginal Density of B (i.e., Y and Z)**: Integrate out X using a dummy variable u to avoid confusion:
$$f_{Y,Z}(y, z) = \int_{-\infty}^{\infty} f_{X,Y,Z}(u, y, z)\, du$$
   Tip: Since you're looking for the marginal of Y and Z, integrate out everything that's not Y or Z. In this case, integrate over X.
3. **Conditional Density Formula**:
$$f_{X|Y=y,Z=z}(x) = \frac{f_{X,Y,Z}(x, y, z)}{f_{Y,Z}(y, z)}$$

Example 2: Two Variables A, Single Variable B

Define Groups:

- Let $A = (X, Y)$ (two variables).
- Let $B = Z$ (single variable).

Goal: Compute $f_{X,Y|Z=z}(x, y)$, the conditional density of (X, Y) given $Z = z$.

1. **Joint Density** $f_{X,Y,Z}(x, y, z)$: This function now has three variables: x, y, and z.
2. **Marginal Density of B (i.e., Z)**: Integrate out X and Y using dummy variables u and v:
$$f_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y,Z}(u, v, z)\, du\, dv$$
Tip: Since you want the marginal of Z only, integrate out everything that's not Z, in this case, both X and Y.
3. **Conditional Density Formula**:
$$f_{X,Y|Z=z}(x, y) = \frac{f_{X,Y,Z}(x, y, z)}{f_Z(z)}$$

4. General Tips for Marginalizing

- If you want the marginal density of one variable (e.g., X): Integrate out everything that's not X.
$$f_X(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y,Z}(x, y, z)\, dy\, dz$$
- If you want the marginal density of two variables (e.g., X and Y): Integrate out everything that's not X or Y.
$$f_{X,Y}(x, y) = \int_{-\infty}^{\infty} f_{X,Y,Z}(x, y, z)\, dz$$

5. General Formula Recap

- Conditional Density of A Given $B = b$:

$$f_{A|B=b}(a) = \frac{f_{A,B}(a,b)}{f_B(b)}$$

- Steps to Remember:
  - Identify which variable is A (the one you are conditioning on) and which is B (the one fixed at a constant).
  - Joint Density $f_{A,B}(a,b)$: Evaluated with A as the variable and B set to the constant b.
  - Marginal Density of B: Integrate over A using a dummy variable to get $f_B(b)$ and evaluate at b.

6. Key Takeaways

- Variable vs. Constant: Remember, $f(x)$ is a function of x, but $f(a)$ or $f(x = a)$ is evaluated at $x = a$ and is now a number.
- Dummy Variables in Integrals: Use different letters in integrals (e.g., u, v) to represent that we are integrating over those variables, leaving the rest as fixed values.
- Conditional Density: Think of $f_{A|B=b}(a)$ as "the density of A when B is fixed at b."

Exercise

Consider a joint density function given by $f_{X,Y,Z}(x,y,z) = c(x+2y+3z)$, where $x, y, z \in [0,1]$.

Questions

1. Find the value of c that makes $f_{X,Y,Z}(x,y,z)$ a valid probability density function over the range of x, y, z.

2. Find the marginal densities for X, Y, and Z separately:
   - $f_X(x)$
   - $f_Y(y)$
   - $f_Z(z)$

3. Find the marginal joint densities:
   - $f_{X,Y}(x,y)$
   - $f_{X,Z}(x,z)$
   - $f_{Y,Z}(y,z)$

4. Find the conditional density $f_{X|Y=y,Z=z}(x)$:
   - Then evaluate the conditional density at specific values:
     (a) $f_{X|Y=0.1,Z=0.2}(x)$
     (b) $f_{X|Y=0.5,Z=0.5}(x)$

5. Calculate $\Pr(X > 0.1|Y = 0.5, Z = 0.7)$, using the conditional density $f_{X|Y=0.5,Z=0.7}(x)$.