1.3 Counting and combinatorial methods

1. A box contains 100 balls, of which r are red. Suppose that the balls are drawn from the box one at a time, at random, without replacement. Determine

   (a) the probability that the first ball drawn will be red;

   1 2 3 49 50 51 98 99 100

   ___ × ___ × ___ . . . ___ × ___ × ___ . . . ___ × ___ × ___

   [redacted] $\frac{r}{100}$

   (b) the probability that the 50th ball drawn will be red; and

   ___ × ___ × ___ . . . ___ × ___ × ___ . . . ___ × ___ × ___

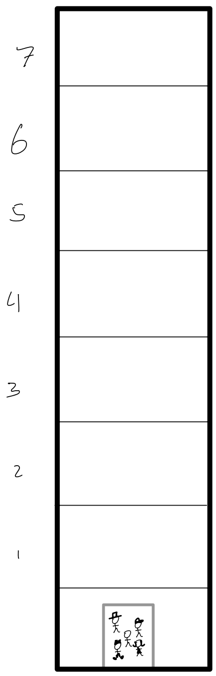
   [redacted] $\frac{r}{100}$

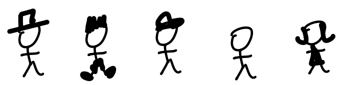
   (c) the probability that the last ball drawn will be red.

   ___ × ___ × ___ . . . ___ × ___ × ___ . . . ___ × ___ × ___

   [redacted] $\frac{r}{100}$

2. An elevator in a building starts with five passengers and stops at seven floors. If every passenger is equally likely to get off at each floor and all the passengers leave independently of each other, what is the probability that no two passengers will get off at the same floor?





___ × ___ × ___ × ___ × ___

$_7P_5$

7^5

[redacted]

ABBccDC

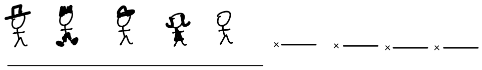
$\frac{6!}{2!3!}$

3. Seats assignment

$$nPr = \frac{n!}{(n-r)!}$$

$5P5$

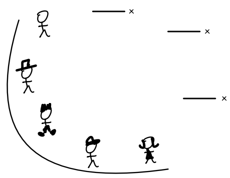
(a) If k people are seated in a random manner in a **row** containing n seats $(n > k)$, what is the probability that the people will occupy k adjacent seats in the row?



$$\frac{k!(n-k+1)}{\frac{n!}{(n-k)!}}$$

[redacted]

(b) If k people are seated in a random manner in a **circle** containing n seats $(n > k)$, what is the probability that the people will occupy k adjacent seats in the circle?



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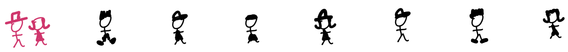
(c) If n people are seated in a random manner in a **row** containing n seats, what is the probability that two particular people A and B will be seated next to each other?

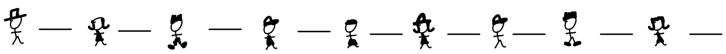


$\underline{n}\ \underline{n-1}\ \underline{n-2}\ \_\ \underline{\cdots}\ \_\ \underline{1} = n!$

$$\frac{2!(n-1)!}{n!}$$

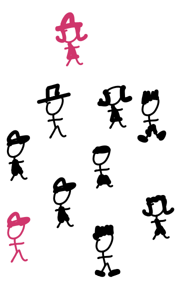
$$\frac{2!(n-1)!}{n!} = \frac{2}{n}$$

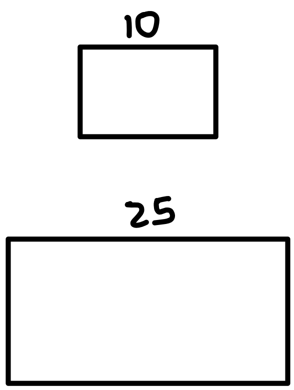
(d) If n people are seated in a random manner in a **row** containing $2n$ seats, what is the probability that no two people will occupy adjacent seats?



[redacted]

4. Suppose that 35 people are divided in a random manner into two teams in such a way that one team contains 10 people and the other team contains 25 people. What is the probability that two particular people A and B will be on the same team?

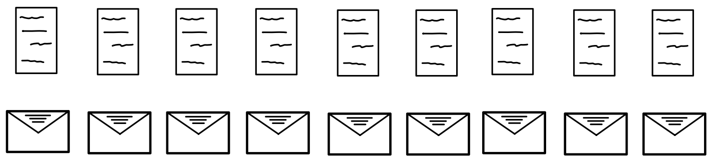




$$\frac{\binom{33}{8} + \binom{33}{22}}{35}$$

[redacted]

5. If n letters are placed at random in n envelopes, what is the probability that exactly $n-1$ letters will be placed in the correct envelopes?



Assume one letter per envelop :

0

Asume several letters per envelop:

$n \cdot \underbrace{\frac{1}{n} \cdot \frac{1}{n} \cdots \frac{1}{n}}_{n-1} \cdot \frac{n-1}{n}$

$\boxed{\text{For you}}$

$n \, \frac{n-1}{n^n}$

6. (1 extra credit) Distribute n cookies to k kids. Each kid receives at least two cookies. For convenience, we assume $n \geq 2k$. How many possible arrangements are there?

Hint: Distribute the remaining $n-2k$

$\boxed{\text{For you}}$

1.4 Conditional probability

1. A box contains three cards. One card is red on both sides, one card is green on both sides, and one card is red on one side and green on the other. One card is selected from the box at random, and the color on one side is observed. If this side is green, what is the probability that the other side of the card is also green?

R R G
G R G

$$Pr(\text{both sides } G \mid \text{one side } G) = \frac{Pr(\text{both sides } G, \text{ one side } G)}{Pr(\text{one side } G)}$$

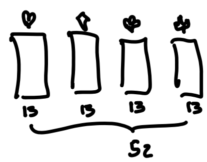
$$= \frac{Pr(\text{both sides } G)}{Pr(\text{one side } G)}$$

$$Pr(\text{both sides } G) = 1/3$$

$$Pr(\text{one side } G) = Pr(\text{one side } G, \text{ one side } R) + Pr(\text{one side } G, \text{ one side } G)$$

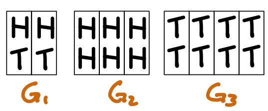
$$= 1/3 + 1/3$$

$$= 2/3$$



$$\boxed{Pr(\text{both sides } G \mid \text{one side } G) = \frac{1}{2}}$$

2. A box contains three coins with a head on each side, four coins with a tail on each side, and two fair coins. If one of these nine coins is selected at random and tossed once, what is the probability that a head will be obtained?



$Pr(\text{Getting } H)$

LTP $A_1, \ldots, A_n$

$$P(B) = \sum P(B \cap A_i) = \sum P(B \mid A_i) \cdot P(A_i)$$

MR $P(C \cap D) = P(C \mid D) P(D)$

$$P(\text{Getting a heart type}, \text{Getting a heart type}) = P(\text{Getting a heart type} \mid \text{Getting a heart type}) \times P(\text{Getting a heart type})$$

$\frac{12}{51}$

$\frac{13}{52}$

$$Pr(\text{Get } H) = \sum_{i=1}^{3} Pr(\text{Get } H, \text{ Select coin from group } i\text{-th})$$

$$= \sum_{i=1}^{3} Pr(\text{Get } H \mid \text{Select coin from group } i\text{-th}) Pr(\text{Select coin from group } i\text{-th})$$

$$= \frac{1}{2} \cdot \frac{2}{9} + 1 \cdot \frac{3}{9} + 0 \cdot \frac{4}{9}$$

$$= \frac{4}{9}$$

BT $P(A_j \mid B) = \dfrac{P(B \mid A_j) P(A_j)}{\sum_{i=1}^{n} P(B \mid A_i) P(A_i)}$

3. Suppose that a box contains five coins and that for each coin there is a different probability that a head will be obtained when the coin is tossed. Let p_i denote the probability of a head when the i^{th} coin is tossed ($i = 1, ..., 5$), and suppose that $p_1 = 0$, $p_2 = 1/4$, $p_3 = 1/2$, $p_4 = 3/4$, and $p_5 = 1$.

(a) In the case of tossing once of a randomly selected coin, if a $\boxed{\text{head is obtained}}$ what is the posterior probability that the ith coin was selected ($i = 1, ..., 5$)?

$$\Pr\left(\text{The } i\text{-th coin is selected} \mid \text{Head is obtained}\right) \overset{BT}{=} \frac{\Pr\left(\text{Head is obtained} \mid \text{The } i\text{-th coin is selected}\right)\Pr\left(\text{The } i\text{-th coin is selected}\right)}{\sum_{k=1}^{5} \Pr\left(\text{Head is obtained} \mid \text{The } k\text{-th coin is selected}\right)\Pr\left(\text{The } k\text{-th coin is selected}\right)}$$

$$= \frac{\Pr\left(\text{Head is obtained} \mid \text{The } i\text{-th coin is selected}\right)\underbrace{(1/5)}_{\text{Prior}}}{\sum_{k=1}^{5} \Pr\left(\text{Head is obtained} \mid \text{The } k\text{-th coin is selected}\right)(1/5)}$$

$$= \frac{\Pr\left(\text{Head is obtained} \mid \text{The } i\text{-th coin is selected}\right)}{\sum_{k=1}^{5} \Pr\left(\text{Head is obtained} \mid \text{The } k\text{-th coin is selected}\right)}$$

$$= \frac{2}{5}\Pr\left(\text{Head is obtained} \mid \text{The } i\text{-th coin is selected}\right)$$

$$= \begin{cases} 0 & i=1 \\ 1/10 & i=2 \\ 1/5 & i=3 \\ 3/10 & i=4 \\ 2/5 & i=5 \end{cases}$$

k	$\Pr\left(\text{Head is obtained} \mid \text{The } k\text{-th coin is selected}\right)$
1	0
2	1/4
3	1/2
4	3/4
5	1

(b) What would be the probability of getting another head if the same coin were tossed once more?

$$\Pr\left(\text{Head is obtained in the 2nd toss} \mid \text{Head is obtained in the 1st toss}\right) = \frac{\Pr\left(\text{Head is obtained in the 2nd toss}, \text{Head is obtained in the 1st toss}\right)}{\Pr\left(\text{Head is obtained in the 1st toss}\right)}$$

$$\overset{LTP}{=} \frac{\sum_{k=1}^{5} \Pr\left(\text{Head is obtained in the 2nd toss}, \text{The } k\text{-th coin is selected}, \text{Head is obtained in the 1st toss}\right)}{\Pr\left(\text{Head is obtained in the 1st toss}\right)}$$

$$= \frac{\sum_{k=1}^{5} \Pr\left(\text{Head is obtained in the 2nd toss} \mid \text{The } k\text{-th coin is selected}, \text{Head is obtained in the 1st toss}\right)\Pr\left(\text{Head is obtained in the 1st toss} \mid \text{The } k\text{-th coin is selected}\right)\Pr\left(\text{The } k\text{-th coin is selected}\right)}{\Pr\left(\text{Head is obtained in the 1st toss}\right)}$$

$$= \sum_{k=1}^{5} \Pr\left(\text{Head is obtained in the 2nd toss} \mid \text{The } k\text{-th coin is selected}, \text{Head is obtained in the 1st toss}\right)\frac{\Pr\left(\text{Head is obtained in the 1st toss} \mid \text{The } k\text{-th coin is selected}\right)\Pr\left(\text{The } k\text{-th coin is selected}\right)}{\Pr\left(\text{Head is obtained in the 1st toss}\right)}$$

$$= \sum_{k=1}^{5} \Pr\left(\text{Head is obtained in the 2nd toss} \mid \text{The } k\text{-th coin is selected}, \text{Head is obtained in the 1st toss}\right)\Pr\left(\text{The } k\text{-th coin is selected} \mid \text{Head is obtained}\right)$$

why?

$$= \sum_{k=1}^{5} Pr\left(\begin{array}{c}\text{Head is}\\ \text{obtained}\\ \text{in the 2nd toss}\end{array} \middle| \begin{array}{c}\text{The k-th coin}\\ \text{is selected}\end{array}\right) Pr\left(\begin{array}{c}\text{The k-th coin}\\ \text{is selected}\end{array} \middle| \begin{array}{c}\text{Head is}\\ \text{obtained}\end{array}\right)$$

$$= 0 \cdot 0 + \frac{1}{10} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{2} + \frac{3}{10} \cdot \frac{3}{4} + \frac{2}{5} \cdot 1$$

(c) If the first toss of the chosen coin resulted in a tail, and the same coin were tossed again, what would be the probability of getting a head on the second toss?

I. $Pr\left(\begin{array}{c}\text{The i-th coin}\\ \text{is selected}\end{array} \middle| \begin{array}{c}\text{Tail is}\\ \text{obtained}\end{array}\right) = \frac{2}{5} Pr\left(\begin{array}{c}\text{Tail is}\\ \text{obtained}\end{array} \middle| \begin{array}{c}\text{The i-th coin}\\ \text{is selected}\end{array}\right)$

II. $Pr\left(\begin{array}{c}\text{Head is}\\ \text{obtained}\\ \text{in the 2nd toss}\end{array} \middle| \begin{array}{c}\text{Tail is}\\ \text{obtained}\\ \text{in the 1st toss}\end{array}\right) = \sum_{k=1}^{5} Pr\left(\begin{array}{c}\text{Head is}\\ \text{obtained}\\ \text{in the 2nd toss}\end{array} \middle| \begin{array}{c}\text{The k-th coin}\\ \text{is selected}\end{array}\right) Pr\left(\begin{array}{c}\text{The i-th coin}\\ \text{is selected}\end{array} \middle| \begin{array}{c}\text{Tail is}\\ \text{obtained}\end{array}\right)$

d. Suppose that one coin is selected at random from the box and is tossed repeatedly until a head is obtained.

a) If the first head is obtained on the fourth toss, what is the posterior probability that the i^{th} coin was selected $(i = 1, ..., 5)$? Hint: You got 1 head, 3 tails

b) If we continue to toss the same coin until another head is obtained, what is the probability that exactly three additional tosses will be required? Hint: use LTP to condition in all possible coins

For you