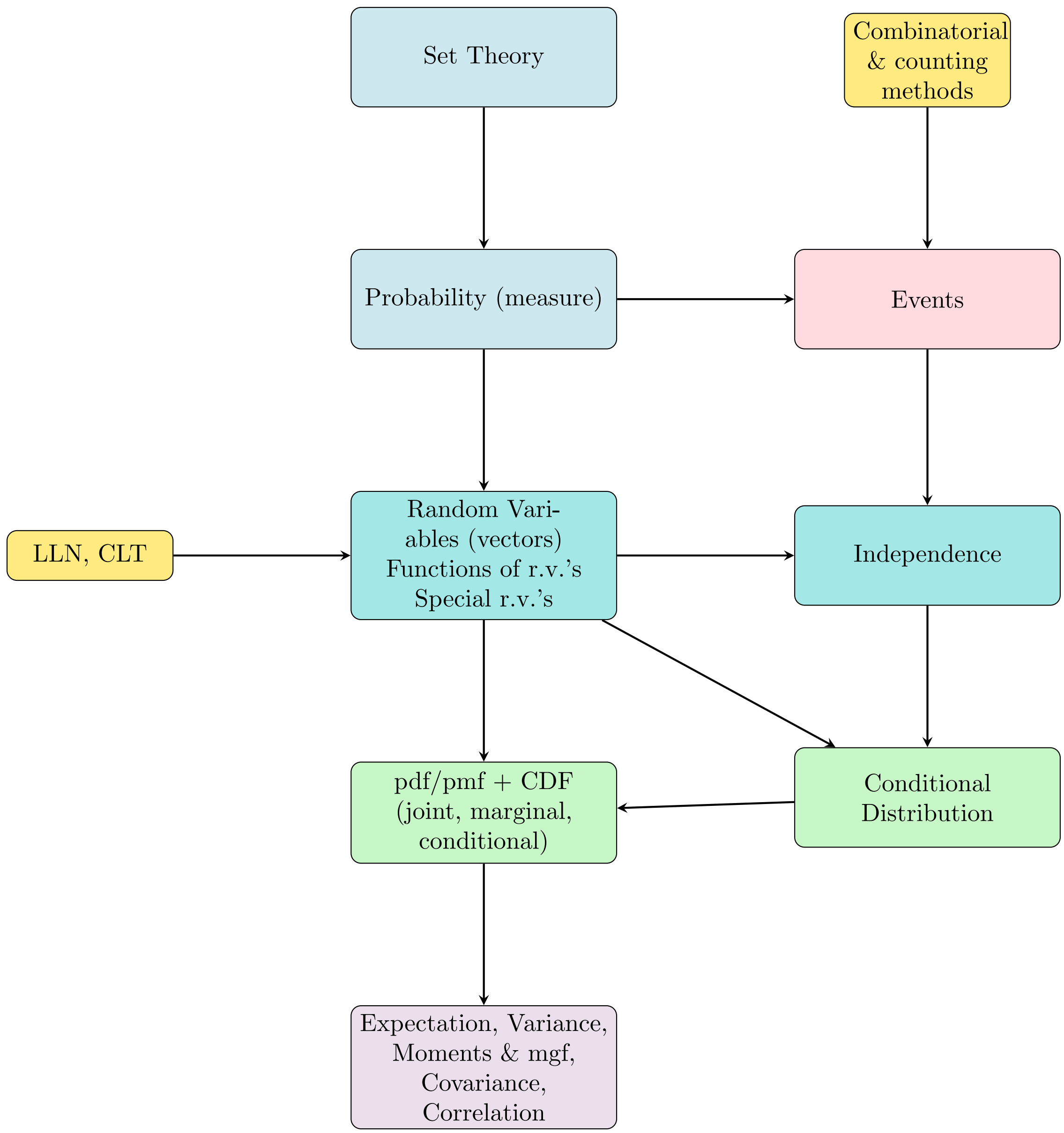
STAT 131 - Probability Theory
Course Map



Probability Theory: Properties, Definitions, and Theorems

Basic Properties of Probability

- $0 \leq P(A) \leq 1$ *(Probability bounds)*
- $P(A^c) = 1 - P(A)$ *(Complement rule)*
- $P(A \cap B) = 0$ *(Mutually exclusive events)*
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ *(Union of two events)*
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Conditional Probability and Multiplication Rule

- $P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad P(B) > 0$ *(Conditional probability)*
- $P(A \cap B) = P(A \mid B)P(B)$ *(Multiplication rule for two events)*
- $P(A \cap B \cap C) = P(A \mid B \cap C)P(B \mid C)P(C)$ *(Multiplication for three events)*

Independence

Two events A and B are independent if:

- $P(A \cap B) = P(A)P(B)$ *(Independence of two events)*
- $P(A \mid B) = P(A)$ *(Alternative definition of independence)*

Law of Total Probability

Let $A_1, A_2, \ldots, A_n$ be a partition of the sample space Ω such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n} A_i = \Omega$. Then, for any event B:

- $P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B \mid A_i)P(A_i)$

Bayes' Theorem

- $P(A_i \mid B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}$ *(Bayes' theorem)*

De Morgan's Laws

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

Inclusion-Exclusion Principle

- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

[redacted]

3. Suppose that a box contains five coins and that for each coin there is a different probability that a head will be obtained when the coin is tossed. Let p_i denote the probability of a head when the i^{th} coin is tossed ($i = 1, ..., 5$), and suppose that $p_1 = 0$, $p_2 = 1/4$, $p_3 = 1/2$, $p_4 = 3/4$, and $p_5 = 1$.

(a) In the case of tossing once of a randomly selected coin, if a head is obtained, what is the posterior probability that the ith coin was selected ($i = 1, ..., 5$)?

$$\Pr\left(\begin{smallmatrix}\text{The } i\text{-th coin}\\ \text{is selected}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{Head is}\\ \text{obtained}\end{smallmatrix}\right) \overset{BT}{=} \frac{\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } i\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)\Pr\left(\begin{smallmatrix}\text{The } i\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)}{\sum_{k=1}^{5}\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)\Pr\left(\begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)}$$

Prior

$$= \frac{\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } i\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)(1/5)}{\sum_{k=1}^{5}\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)(1/5)}$$

$$= \frac{\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } i\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)}{\sum_{k=1}^{5}\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)}$$

$$= \frac{2}{5}\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } i\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)$$

$$= \begin{cases} 0 & i=1 \\ 1/10 & i=2 \\ 1/5 & i=3 \\ 3/10 & i=4 \\ 2/5 & i=5 \end{cases}$$

k	$\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)$
1	0
2	1/4
3	1/2
4	3/4
5	1

(b) What would be the probability of getting another head if the same coin were tossed once more?

$$\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 2nd toss}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix}\right) = \frac{\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 2nd toss}\end{smallmatrix}, \begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix}\right)}{\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix}\right)}$$

$$\overset{LTP}{=} \frac{\sum_{k=1}^{5}\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 2nd toss}\end{smallmatrix}, \begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}, \begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix}\right)}{\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix}\right)}$$

$$= \frac{\sum_{k=1}^{5}\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 2nd toss}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}, \begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix}\right)\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)\Pr\left(\begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)}{\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix}\right)}$$

$$= \sum_{k=1}^{5}\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 2nd toss}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}, \begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix}\right)\frac{\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)\Pr\left(\begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}\right)}{\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix}\right)}$$

$$= \sum_{k=1}^{5}\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 2nd toss}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix}, \begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix}\right)\Pr\left(\begin{smallmatrix}\text{The } k\text{-th coin}\\ \text{is selected}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{Head is}\\ \text{obtained}\end{smallmatrix}\right)$$

$$\overset{\text{why?}}{=} \sum_{k=1}^{5} \Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 2nd toss}\end{smallmatrix} \;\middle|\; \begin{smallmatrix}\text{The k-th coin}\\ \text{is selected}\end{smallmatrix}\right) \Pr\left(\begin{smallmatrix}\text{The k-th coin}\\ \text{is selected}\end{smallmatrix} \;\middle|\; \begin{smallmatrix}\text{Head is}\\ \text{obtained}\end{smallmatrix}\right)$$

$$= 0 \cdot 0 + 1/10 \cdot 1/4 + 1/5 \cdot 1/2 + 3/10 \cdot 3/4 + 2/5 \cdot 1$$

(c) If the first toss of the chosen coin resulted in a tail, and the same coin were tossed again, what would be the probability of getting a head on the second toss?

I. $\Pr\left(\begin{smallmatrix}\text{The i-th coin}\\ \text{is selected}\end{smallmatrix} \;\middle|\; \begin{smallmatrix}\text{Tail is}\\ \text{obtained}\end{smallmatrix}\right) = \frac{2}{5} \Pr\left(\begin{smallmatrix}\text{Tail is}\\ \text{obtained}\end{smallmatrix} \;\middle|\; \begin{smallmatrix}\text{The i-th coin}\\ \text{is selected}\end{smallmatrix}\right)$

II. $\Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 2nd toss}\end{smallmatrix} \;\middle|\; \begin{smallmatrix}\text{Tail is}\\ \text{obtained}\\ \text{in the 1st toss}\end{smallmatrix}\right) = \sum_{k=1}^{5} \Pr\left(\begin{smallmatrix}\text{Head is}\\ \text{obtained}\\ \text{in the 2nd toss}\end{smallmatrix} \;\middle|\; \begin{smallmatrix}\text{The k-th coin}\\ \text{is selected}\end{smallmatrix}\right) \Pr\left(\begin{smallmatrix}\text{The i-th coin}\\ \text{is selected}\end{smallmatrix} \;\middle|\; \begin{smallmatrix}\text{Tail is}\\ \text{obtained}\end{smallmatrix}\right)$

d. Suppose that one coin is selected at random from the box and is tossed repeatedly until a head is obtained.

a) If the first head is obtained on the fourth toss, what is the posterior probability that the i^{th} coin was selected $(i = 1, ..., 5)$? Hint: You got 1 head, 3 tails

b) If we continue to toss the same coin until another head is obtained, what is the probability that exactly three additional tosses will be required? Hint: use LTP to condition on all possible coins

$\boxed{\text{For you}}$

1.5 Independence

$A \perp\!\!\!\perp B$ iff $P(A\cap B) = P(A)\cdot P(B)$

$P(A|B) = P(A)$

• Prop. $A \perp\!\!\!\perp B$, then $A^c \perp\!\!\!\perp B$, $B^c \perp\!\!\!\perp A$, $A^c \perp\!\!\!\perp B^c$

LTP.

$A_1, \dots, A_n$, Partition of Ω

$P(B) = \sum P(B\cap A_i) = \sum P(B|A_i)P(A_i)$

1. If three balanced dice are rolled, what is the probability that all three numbers will be the same?

$$\begin{aligned}
Pr(\text{all 3 are the same \#}) &= \sum_{i=1}^{6} Pr(\text{all 3 are the same \#, \#} = i) \\
&= \sum_{i=1}^{6} Pr(\text{die } 1 = i, \text{die } 2 = i, \text{die } 3 = i) \\
&\overset{\text{indep}}{=} \sum_{i=1}^{6} Pr(\text{die } 1 = i) Pr(\text{die } 2 = i) Pr(\text{die } 3 = i) \\
&= \sum_{i=1}^{6} \left(\tfrac{1}{6}\right)^3 \\
&= 6 \times \left(\tfrac{1}{6}\right)^3 \\
&= \frac{1}{6^2}
\end{aligned}$$

Another way:

$$\frac{6}{6^3}$$

(1,1,1) (2,2,2) ... (6,6,6)

All possibilities

2. Suppose that a person rolls two balanced dice three times in succession. Determine the probability that on each of the three rolls, the sum of the two numbers that appear will be 7.

$$\begin{aligned}
Pr(\text{in all 3 rolls both sum } 7) &= Pr(\text{in the first roll both sum } 7, \text{ in the second roll both sum } 7, \text{ in the third roll both sum } 7) \\
&\overset{\text{indep}}{=} Pr(\text{in the first roll both sum } 7) \\
&\quad \times Pr(\text{in the second roll both sum } 7) \\
&\quad \times Pr(\text{in the third roll both sum } 7) \\
&= Pr(\text{in the first roll both sum } 7)^3
\end{aligned}$$

$$\begin{aligned}
Pr(\text{in the first roll both sum } 7) &= \sum_{i=1}^{6} Pr(\text{in the first roll both sum } 7, \text{ die "one" is } i) \\
&= \sum_{i=1}^{6} Pr(\text{die "one" is } i, \text{ die "two" is } 7-i) \\
&\overset{\text{indep}}{=} \sum_{i=1}^{6} Pr(\text{die "one" is } i) Pr(\text{die "two" is } 7-i) \\
&= \sum_{i=1}^{6} \tfrac{1}{6}\cdot\tfrac{1}{6} \\
&= \frac{6}{36} \\
&= 1/6
\end{aligned}$$

(1,6)
(2,5)
(3,4)
(4,3)
(5,2)
(6,1)

3. Consider an experiment in which a fair coin is tossed until a head is obtained for the first time. If this experiment is performed three times, what is the probability that exactly the same number of tosses will be required for each of the three performances?

$\boxed{\text{For you}}$

4. The probability that any child in a certain family will have blue eyes is 1/4, and this feature is inherited independently by different children in the family. Suppose there are five children in the family.

I. Define events.

$A_i :=$ The i-th child (ordered by age) has blue eyes , $i \in \{1,2,3,4,5\}$

* Notice that if at least 1 blue eyes, then, the following event is given: $\boxed{A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5}$

"known to happened"

(a) What is the probability that at least three of the children have blue eyes?

$$\Pr(\text{At least } 3 \mid \text{At least } 1) = \frac{\Pr(\text{At least } 3 \text{ and } \text{At least } 1)}{\Pr(\text{At least } 1)}$$

$$\overset{\text{why?}}{=} \frac{\Pr(\text{At least } 3)}{\Pr(\text{At least } 1)}$$

But,

- $\Pr(\text{At least } 1) = \Pr(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5)$

$$= 1 - \Pr((A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5)^c)$$

$$= 1 - \Pr(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \cap A_5^c)$$

$$= 1 - (3/4)^5$$

• $Pr(\text{At least } 3) = 1 - Pr(\text{not "At least 3"})$

why?.

$$= 1 - Pr(\text{none or exactly 1 or exactly 2})$$

$$= 1 - [Pr(\text{none}) + Pr(\text{exactly 1}) + Pr(\text{exactly 2})]$$

* $Pr(\text{none}) = 1 - (3/4)^5$
* $Pr(\text{exactly 1}) = 5 \times (3/4)^4 (1/4)^1$
* $Pr(\text{exactly 2}) = \binom{5}{2} \times (3/4)^3 (1/4)^2$

$\boxed{\text{Can we derive a general expresion for } Pr(\text{exactly } j)\text{?}}$ $\rightarrow$ $Pr(\text{exactly } j) = \binom{\overset{\#\text{ of children}}{n}}{j} (3/4)^{n-j} (1/4)^j$

(b) If it is known that the youngest child in the family has blue eyes, what is the probability that at least three of the children have blue eyes?

$$Pr(\text{At least } 3 \mid A_1) \overset{\text{why?.}}{=} Pr(\text{At least 2 out of 4 children})$$

$$= Pr(\text{exactly } 2) + Pr(\text{exactly } 3) + Pr(\text{exactly } 4)$$

$$= \binom{4}{2} (3/4)^2 (1/4)^2 + \binom{4}{3} (3/4)(1/4)^3 + (1/4)^4$$

(c) Explain why the answer in part (a) is different from the answer in Problem 5.

- In a), we asked: $\Pr(\text{At least } 3 \mid \text{At least } 1)$
- In b), we asked: $\Pr(\text{At least } 3 \mid A_1)$

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5. Suppose that A, B, and C are three independent events such that Pr(A) = 1/4, Pr(B) = 1/3, and Pr(C) = 1/2.

   (a) Determine the probability that none of these three events will occur.

$$\Pr(\text{none}) = P(A^c \cap B^c \cap C^c)$$
$$= P(A^c)P(B^c)P(C^c)$$
$$= (3/4)(2/3)(1/2)$$

   (b) Determine the probability that exactly one of these three events will occur.

$$\Pr(\text{exactly one}) = \Pr(A \cap B^c \cap C^c) + \Pr(A^c \cap B \cap C^c) + \Pr(A^c \cap B^c \cap C)$$
$$= P(A)P(B^c)P(C^c) + P(A^c)P(B)P(C^c) + P(A^c)P(B^c)P(C)$$

c) Prob. at least 1 occur

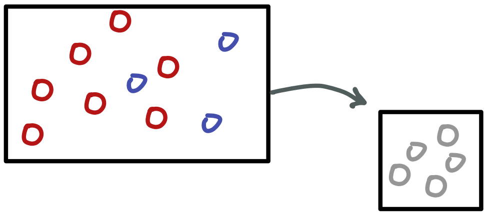
$$\Pr(\text{at least } 1) = 1 - \Pr(\text{none})$$

2 Random Variables and Distribution Functions

2.1 Discrete and continuous distributions

1. Suppose that a box contains seven red balls and three blue balls. If five balls are selected at random, without replacement, determine the p.f. of the number of red balls that will be obtained.

$$\frac{\binom{7}{\ }\binom{3}{5-}}{\binom{10}{5}}$$



Let $X :=$ # of red balls obtained

I. Det. the support of X

$$Sup(X) = \{2, 3, 4, 5\}$$

II. Prob of getting $x \in Sup(X)$ red balls.

$$Pr(X=2) = \frac{\binom{7}{2}\binom{3}{3}}{\binom{10}{5}} \leftarrow \text{\# of possible samples of size 5, from a total of 10}$$

$$Pr(X=3) = \frac{\binom{7}{3}\binom{3}{2}}{\binom{10}{5}}$$

$$\vdots$$

$$Pr(X=x) = \frac{\binom{7}{x}\binom{3}{5-x}}{\binom{10}{5}}$$

$X \sim$ HyperGeom

III. $f(x) = \dfrac{\binom{7}{x}\binom{3}{5-x}}{\binom{13}{5}}, \quad x \in Sup(X)$

2. Suppose that a random variable X has the binomial distribution with parameters $n = 15$ and $p = 0.5$. Find $Pr(X < 6)$.

$X \sim Bin(n, p)$

$\boxed{\text{For you}}$

$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n$

3. Suppose that a random variable X has a discrete distribution with the following p.f.:

$$f(x) = \begin{cases} \frac{c}{2^x}, & \text{for } x = 0, 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Find the value of the constant c.

$$1 = \sum_{x=0}^{\infty} f(x)$$
$$= \sum_{x=0}^{\infty} c \cdot \frac{1}{2^x}$$
$$= c \sum_{x=0}^{\infty} \frac{1}{2^x}$$
$$= c \sum_{x=0}^{\infty} \left(\frac{1}{2}\right)^x$$
$$= c \frac{1}{1 - 1/2}$$
$$= c \cdot 2$$

Then, $c = \frac{1}{2}$

Que tengan bonito día! ☺