Probability Theory: Properties, Definitions, and Theorems

Basic Properties of Probability

- $0 \leq P(A) \leq 1$ *(Probability bounds)*
- $P(A^c) = 1 - P(A)$ *(Complement rule)*
- $P(A \cap B) = 0$ *(Mutually exclusive events)*
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ *(Union of two events)*
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Conditional Probability and Multiplication Rule

- $P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad P(B) > 0$ *(Conditional probability)*
- $P(A \cap B) = P(A \mid B)P(B)$ *(Multiplication rule for two events)*
- $P(A \cap B \cap C) = P(A \mid B \cap C)P(B \mid C)P(C)$ *(Multiplication for three events)*

Independence

Two events A and B are independent if:

- $P(A \cap B) = P(A)P(B)$ *(Independence of two events)*
- $P(A \mid B) = P(A)$ *(Alternative definition of independence)*

Law of Total Probability

Let $A_1, A_2, \ldots, A_n$ be a partition of the sample space Ω such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^n A_i = \Omega$. Then, for any event B:

- $P(B) = \sum_{i=1}^n P(B \cap A_i) = \sum_{i=1}^n P(B \mid A_i)P(A_i)$

Bayes' Theorem

- $P(A_i \mid B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}$ *(Bayes' theorem)*

De Morgan's Laws

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

Inclusion-Exclusion Principle

- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$