STAT 131 Problems

Note:

- The problems are selected from the textbook.

1 Classical Probability Theory

1.1 Basic set theory

1. Suppose $A \subseteq B$, show $B^C \subseteq A^C$.

2. Suppose that one card is to be selected from a deck of 20 cards that contains 10 red cards numbered from 1 to 10 and 10 blue cards numbered from 1 to 10.

   Let A be the event that a card with an even number is selected, let B be the event that a blue card is selected, and let C be the event that a card with a number less than 5 is selected.

   Describe the sample space Ω and describe each of the following events both in words and as subsets of Ω:

   (a) $A \cap B \cap C$
   (b) $B \cap C^C$
   (c) $A \cup B \cup C$
   (d) $A \cap (B \cup C)$
   (e) $A^C \cap B^C \cap C^C$.

1.2 Definition of probability

1. Consider two events A and B such that $P(A) = 1/3$ and $P(B) = 1/2$. Determine the value of $P(B \cap A^C)$ for each of the following conditions:

   (a) A and B are disjoint
   (b) $A \subseteq B$
   (c) $P(A \cap B) = 1/8$.

2. If the probability that student A will fail a certain statistics exam is 0.5, the probability that student B will fail the exam is 0.2, and the probability that both student A and student B will fail the exam is 0.1,

   (a) what is the probability that at least one of these two students will fail the exam?
   (b) what is the probability that neither student A nor student B will fail the exam?
   (c) what is the probability that exactly one of these two students will fail the exam?

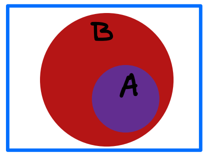
3. Consider two events A and B with $P(A) = 0.4$ and $P(B) = 0.7$. Determine the maximum and minimum possible values of $P(A \cap B)$ and the conditions under which each of these values is attained.

$\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{$even and smaller than $5\} = \{2, 4\}$, $A^c = \{1, 3, 5, 6, 7, 8, 9, 10\}$

$B = \{$smaller than $5\} = \{1, 2, 3, 4\}$ $B^c = \{5, 6, 7, 8, 9, 10\}$

1.1 Basic set theory



1. Suppose $A \subseteq B$, show $B^C \subseteq A^C$

$A \subseteq B \leftrightarrow \forall x\, (x \in A \rightarrow x \in B)$

$\leftrightarrow \forall x\, ((x \in A) \rightarrow (x \in B))$

$\boxed{p \rightarrow q \equiv \neg q \rightarrow \neg p}$

$\leftrightarrow \forall x\, (\neg(x \in B) \rightarrow \neg(x \in A))$

$\leftrightarrow \forall x\, (x \notin B \rightarrow x \notin A)$

$A^c \subseteq B^c$

$\leftrightarrow \forall x\, (x \in B^c \rightarrow x \in A^c)$

$\leftrightarrow B^c \subseteq A^c$

[redacted]

2. Suppose that one card is to be selected from a deck of 20 cards that contains 10 red cards numbered from 1 to 10 and 10 blue cards numbered from 1 to 10.

   Let A be the event that a card with an even number is selected, let B be the event that a blue card is selected, and let C be the event that a card with a number less than 5 is selected.

   Describe the sample space Ω and describe each of the following events both in words and as subsets of Ω:

   Ω

   1 2 3 4 5 6 7 8 9 10

   1 2 3 4 5 6 7 8 9 10

   (a) $A \cap B \cap C$
   (b) $B \cap C^C$
   (c) $A \cup B \cup C$
   I. (d) $A \cap (B \cup C)$
   (e) $A^C \cap B^C \cap C^C$.

$A = \{x \in \Omega \mid x$ # is even$\} = \{2\ 4\ 6\ 8\ 10\ 2\ 4\ 6\ 8\ 10\}$

$B = \{x \in \Omega \mid x$ color is blue$\} = \{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\}$

$C = \{x \in \Omega \mid x$ # is less than $5\} = \{1\ 2\ 3\ 4\ 1\ 2\ 3\ 4\}$

II.

(a) $A \cap B \cap C = \{x \in \Omega \mid$ x # is even and x color is blue and x # is less than 5 $\}$

$= \{x \in \Omega \mid$ x color is blue and x # is 2 or 4 $\}$

$= \{\boxed{2}\,\boxed{4}\}$

(b) $B \cap C^C = \{x \in \Omega \mid$ x color is blue and x # is **not** less than 5 $\}$

$= \{x \in \Omega \mid$ x color is blue and x # is larger or equal to 5 $\}$

$= \{\boxed{5}\,\boxed{6}\,\boxed{7}\,\boxed{8}\,\boxed{9}\,\boxed{10}\}$

Another way of thinking about it

$\boxed{B \setminus C := B \cap C^c}$

(d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$= \{x \in \Omega \mid$ x # is even and x color is blue $\} \cup \{x \in \Omega \mid$ x # is even and x # is less than 5 $\}$

$= \{\boxed{2}\,\boxed{4}\,\boxed{6}\,\boxed{8}\,\boxed{10}\} \cup \{\boxed{2}\,\boxed{4}\}$

$= \{\boxed{2}\,\boxed{4}\,\boxed{6}\,\boxed{8}\,\boxed{10}\}$

(c) $A \cup B \cup C = \{\boxed{2}\,\boxed{4}\,\boxed{6}\,\boxed{8}\,\boxed{10}\,\boxed{2}\,\boxed{4}\,\boxed{6}\,\boxed{8}\,\boxed{10}\}$

$\cup \{\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{5}\,\boxed{6}\,\boxed{7}\,\boxed{8}\,\boxed{9}\,\boxed{10}\}$

$\cup \{\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\}$

$= \{\boxed{2}\,\boxed{4}\,\boxed{6}\,\boxed{8}\,\boxed{10}\,\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{5}\,\boxed{6}\,\boxed{7}\,\boxed{8}\,\boxed{9}\,\boxed{10}\}$

$\cup \{\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\}$

$= \{\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{6}\,\boxed{8}\,\boxed{10}\,\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{5}\,\boxed{6}\,\boxed{7}\,\boxed{8}\,\boxed{9}\,\boxed{10}\}$

(e) $A^C \cap B^C \cap C^C = (A \cup B \cup C)^c$

$= \{\boxed{5}\,\boxed{7}\,\boxed{9}\}$

1.2 Definition of probability

2. If the probability that student A will fail a certain statistics exam is 0.5, the probability that student B will fail the exam is 0.2, and the probability that both student A and student B will fail the exam is 0.1,

I. $Pr(A_f) = 0.5 \longrightarrow Pr(A_f^c) = 0.5$

$Pr(B_f) = 0.2 \longrightarrow Pr(B_f^c) = 0.8$

$Pr(A_f \cap B_f) = 0.1 \longrightarrow Pr(A_f^c \cup B_f^c) = 0.9$

$P(A) = 1 - P(A^c)$

A_f = A fails
B_f = B fails

$Pr(A_f^c \cap B_f^c) = 1 - P(A_f \cup B)$

(a) what is the probability that at least one of these two students will fail the exam?

$Pr($at least 1 fails$) = 1 - Pr($not at least 1 fails$)$
$= 1 - Pr($none fails$)$
$= 1 - Pr($not A fails and not B fails$)$

$Pr($not A fails and not B fails$) = Pr($not A fails$) + Pr($not B fails$) - Pr($not A fails or not B fails$)$
$= 0.5 + 0.8 - 0.9$
$= 0.4$

$Pr(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A \cap B) = P(A) + P(B) - Pr(A \cup B)$

$Pr($not A fails and not B fails$) = Pr(A_f^c \cap B_f^c)$
$= Pr(A_f^c) + Pr(B_f^c) - Pr(A_f^c \cup B_f^c)$

Then $Pr($at least 1 fails$) = 1 - Pr($not A fails and not B fails$)$
$= 1 - 0.4$
$= 0.6$

(b) what is the probability that neither student A nor student B will fail the exam?

$Pr($not A fails and not B fails$) = 0.4$

(c) what is the probability that exactly one of these two students will fail the exam?

$$\Pr(\text{exactly 1 fails}) = \Pr\left(\begin{array}{c}\text{not A fails}\\ \text{and}\\ \text{B fails}\end{array} \;\text{or}\; \begin{array}{c}\text{A fails}\\ \text{and}\\ \text{not B fails}\end{array}\right)$$

$$= \Pr\left(\begin{array}{c}\text{not A fails}\\ \text{and}\\ \text{B fails}\end{array}\right) + \left(\begin{array}{c}\text{A fails}\\ \text{and}\\ \text{not B fails}\end{array}\right)$$

$$= ?$$

$$\Pr(\text{exactly 1 fails}) = 1 - \Pr(\text{not exactly 1 fails})$$

$$= 1 - \Pr\left(\begin{array}{c}\text{both fail}\\ \text{or}\\ \text{both not fail}\end{array}\right)$$

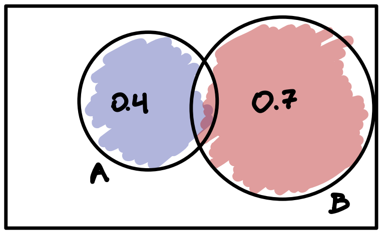
$$\Pr\left(\begin{array}{c}\text{both fail}\\ \text{or}\\ \text{both not fail}\end{array}\right) = \Pr(\text{both fail}) + \Pr(\text{both not fail})$$

$$= 0.1 + 0.4$$

$$= 0.5$$

3. Consider two events A and B with $P(A) = 0.4$ and $P(B) = 0.7$. Determine the maximum and minimum possible values of $P(A \cap B)$ and the conditions under which each of these values is attained.

I. $P(A) = 0.4$
$P(B) = 0.7$



II. $0 \leq P(A \cap B) \leq 1$, can we do better? YES

Using the same Theorem as before,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
$$= 1.1 - P(A \cup B)$$

The largest $P(A \cap B)$ is achieved at the smallest $P(A \cup B)$ and the smallest $P(A \cap B)$ is achieved at the largest $P(A \cup B)$

Case 1: Smallest $P(A \cup B)$

$A \subseteq B$
then, $P(A \cup B) = P(B) = 0.7$
then, $P(A \cap B) = 0.4$

Note: If $A \subseteq B \rightarrow$ $A \cup B = B$ and $A \cap B = A$

Case 2: largest $P(A \cup B)$

It is $P(A \cup B) = 1$

then, $P(A \cap B) = 0.1$

More exercises ?

1. Suppose that a number x is to be selected from the real line S, and let A, B, and C be the events represented by the following subsets of S, where the notation $\{x : ---\}$ denotes the set containing every point x for which the property presented following the colon is satisfied:

$$A = \{x : 1 \leq x \leq 5\}$$

$$B = \{x : 3 < x \leq 7\}$$

$$C = \{x : x \leq 0\}$$

Describe each of the following events as a set of real numbers:

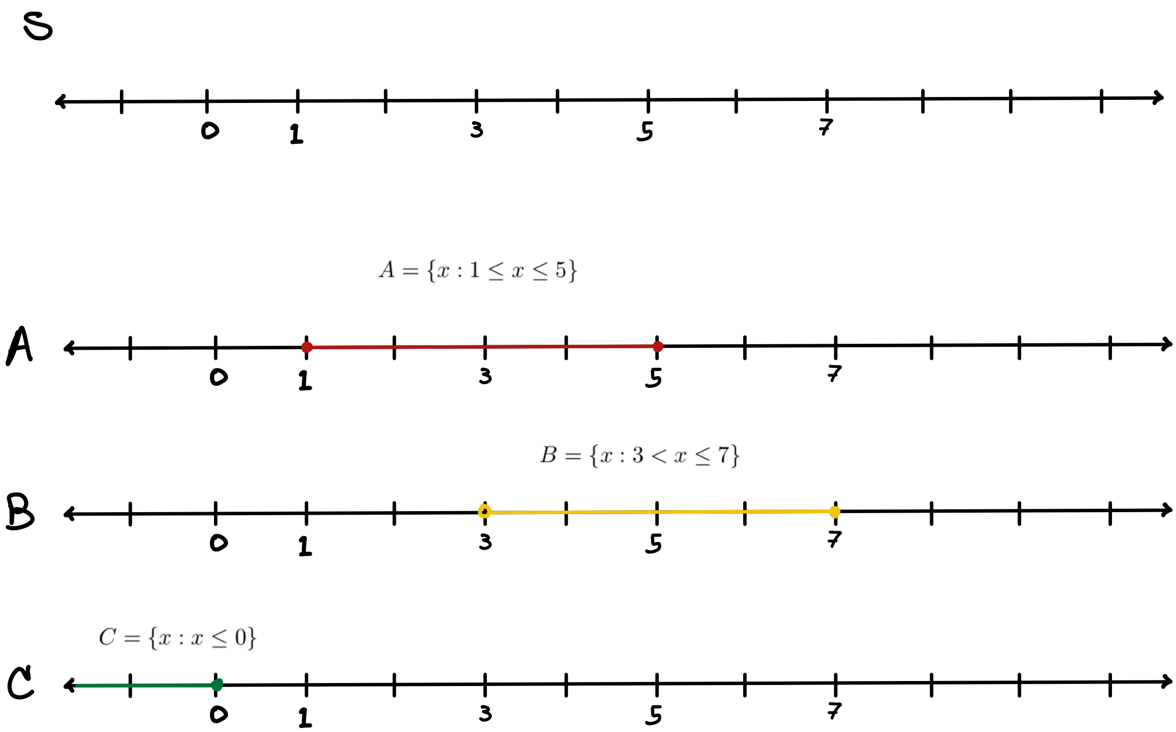
(a) A^c

(b) $A \cup B$

(c) $B \cap C^c$

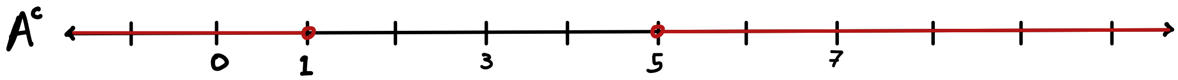
(d) $A^c \cap B^c \cap C^c$

(e) $(A \cup B) \cap C$



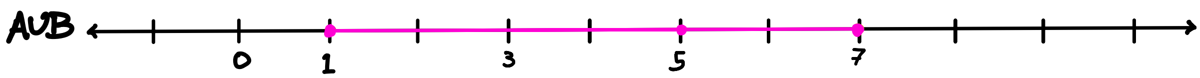
(a) A^c

$$A^c = \{x : 1 \le x \le 5\}^c$$
$$= \{x : 1 \le x \text{ and } x \le 5\}^c \quad (D\cap E)^c = D^c \cup E^c$$
$$= \{x : x<1 \text{ or } x>5\}$$



(b) $A \cup B$

$$A\cup B = \{x : 1\le x\le 5\} \cup \{x : 3<x\le 7\}$$
$$= \{x : 1\le x\le 5 \text{ or } x : 3<x\le 7\}$$
$$= \{x : 1\le x\le 7\}$$

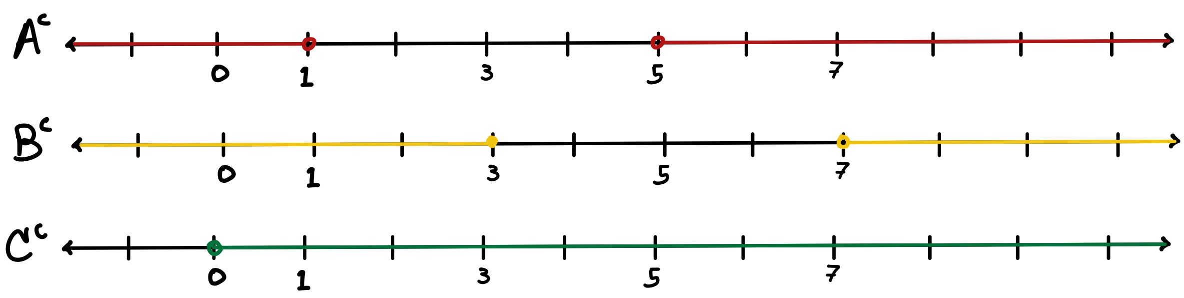


(c) $B \cap C^c$

$$B\cap C^c = \{x : 3<x\le 7\} \cap \{x : x\le 0\}^c$$
$$= \{x : 3<x\le 7\} \cap \{x : x>0\}$$
$$= \{x : 3<x\le 7\}$$
$$= B$$

(d) $A^c \cap B^c \cap C^c$

$$A^c\cap B^c\cap C^c = \{x : 1\le x\le 5\}^c \cap \{x : 3<x\le 7\}^c \cap \{x : x\le 0\}^c$$
$$= \{x : x<1 \text{ or } x>5\} \cap \{x : 3\ge x \text{ or } x\ge 7\} \cap \{x : x>0\}$$
$$= \{x : (x<1 \text{ or } x>5) \text{ and } (3\ge x \text{ or } x\ge 7) \text{ and } x>0\}$$
$$= ?.$$



$$= \{x : 0<x<1 \text{ or } x>7\}$$

(e) $(A \cup B) \cap C$

$$(A \cup B) \cap C = (A \cup B) \cap \{x : x \leq 0\}$$

$$= \{x : 1 \leq x \leq 7\} \cap \{x : x \leq 0\}$$

$$= \emptyset$$

2. If the probability that student A will fail a certain statistics examination is 0.5, the probability that student B will fail the examination is 0.2, and the probability that both student A and student B will fail the examination is 0.1, what is the probability that at least one of these two students will fail the examination?

I. A:= Student "A" fails
B:= Student "B" fails

II. $P(A)=0.5$
$P(B)=0.2$
$P(A\cap B)=0.1$

III. P(at least 1 student fails) $\overset{!}{=}$ P(either only A fails or either only B fails or both fail)

$= P(A\cup B)$

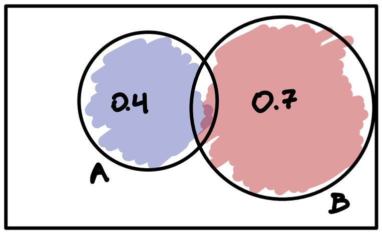
Theorem $= P(A)+P(B)-P(A\cap B)$

$= 0.5+0.2-0.1$

$= 0.6$

3. Consider two events A and B with $P(A) = 0.4$ and $P(B) = 0.7$. Determine the maximum and minimum possible values of $P(A \cap B)$ and the conditions under which each of these values is attained.

I. $P(A) = 0.4$
$P(B) = 0.7$



II. $0 \le P(A \cap B) \le 1$, can we do better? YES

Using the same Theorem as before,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
$$= 1.1 - P(A \cup B)$$

The largest $P(A \cap B)$ is achieved at the smallest $P(A \cup B)$ and the smallest $P(A \cap B)$ is achieved at the largest $P(A \cup B)$

Case 1: Smallest $P(A \cup B)$

$A \subseteq B$
then, $P(A \cup B) = P(B) = 0.7$
then, $P(A \cap B) = 0.4$

Note: If $A \subseteq B \rightarrow$ $A \cup B = B$ and $A \cap B = A$

Case 2: largest $P(A \cup B)$

It is $P(A \cup B) = 1$

then, $P(A \cap B) = 0.1$

4. If two balanced dice are rolled, what is the probability that the sum of the two numbers that appear will be odd?

I. How does the outcomes of this experiment look like?

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

36 possibilities

II.

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

16 out of 32 have an odd sum.

Therefore, $\frac{16}{32} = \frac{1}{2}$

Diff. Aproach.

I. P(Sum is odd)

= P(one die is odd and the other even)

$= P\left(\begin{matrix}\text{First odd}\\ \text{and}\\ \text{second even}\end{matrix} \text{ or } \begin{matrix}\text{second odd}\\ \text{and}\\ \text{First even}\end{matrix}\right)$

$= P\left(\begin{matrix}\text{First odd}\\ \text{and}\\ \text{second even}\end{matrix}\right) + P\left(\begin{matrix}\text{second odd}\\ \text{and}\\ \text{First even}\end{matrix}\right)$

= P(First odd) P(second even) + P(second odd) P(First even)

$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}$

$= 2\cdot\frac{1}{2}\frac{1}{2}$

$= \frac{1}{2}$