Expectations Review - Fall 2024

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Introduction

This document is a mini recap for understanding Expectations in probability theory. The goal is to clarify the concepts of expectation, variance, and covariance.

1. Expectation

The expectation is the "average" or "mean" of a random variable.

Continuous Case

- For a random variable X with density $f_X(x)$:

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

- For a function $g(X)$:

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Discrete Case

- For a random variable X with probabilities $P(X = x)$:

$$\mathbb{E}(X) = \sum_{x \in \text{support of } X} x P(X = x)$$

- For a function $g(X)$:

$$\mathbb{E}(g(X)) = \sum_{x \in \text{support of } X} g(x) P(X = x)$$

2. Variance

Variance measures the spread or variability of a random variable.

Formula

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

Alternative Form

$$\mathrm{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$$

Properties

$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

3. Standard Deviation

The standard deviation is the square root of the variance:

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}$$

4. Joint Expectation

For two random variables X and Y, the expectation of a function $g(X, Y)$ is:

Continuous Case

$$\mathbb{E}(g(X, Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dx\, dy$$

Discrete Case

$$\mathbb{E}(g(X, Y)) = \sum_{x}\sum_{y} g(x, y) P(X = x, Y = y)$$

5. Conditional Expectation

The conditional expectation refines the expectation given information about another variable.

Continuous Case

Using the conditional density $f_{X|Y=y}(x)$:

$$\mathbb{E}(X \mid Y = y) = \int_{-\infty}^{\infty} x f_{X|Y=y}(x)\, dx$$

Discrete Case

Using conditional probabilities $P(X = x \mid Y = y)$:

$$\mathbb{E}(X \mid Y = y) = \sum_x xP(X = x \mid Y = y)$$

6. Properties of Expectations

Linearity of Expectation

$$\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$$

Law of Total Expectation

$$\mathbb{E}(X) = \mathbb{E}[\mathbb{E}(X \mid Y)]$$

7. Covariance

Covariance measures the relationship between two random variables.

Definition

$$\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

Alternative Form

$$\mathrm{Cov}(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y)))$$

Variance of a Sum

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y)$$

EXERCISE 1: Continuous case

1. Summary of Densities

Given the joint probability density function:

$$f_{X,Y}(x,y) = \frac{1}{2}(3x^2 + 2y), \quad 0 \le x \le 1,\, 0 \le y \le 1.$$

The marginal and conditional densities are:

Marginal Densities

- Marginal density of X:

$$f_X(x) = \frac{1}{2}(3x^2 + 1), \quad 0 \le x \le 1.$$

- Marginal density of Y:

$$f_Y(y) = \frac{1}{2}(1 + 2y), \quad 0 \le y \le 1.$$

Conditional Densities

- Conditional density of $Y \mid X = x$:

$$f_{Y|X=x}(y) = \frac{3x^2 + 2y}{3x^2 + 1}, \quad 0 \le y \le 1.$$

- Conditional density of $X \mid Y = y$:

$$f_{X|Y=y}(x) = \frac{3x^2 + 2y}{1 + 2y}, \quad 0 \le x \le 1.$$

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2. Computations

2.1 Expectations

a) **Expectation of X:**

$$\mathbb{E}(X) = \int_0^1 x f_X(x)\,dx, \quad f_X(x) = \frac{1}{2}(3x^2 + 1).$$

b) **Expectation of Y:**

$$\mathbb{E}(Y) = \int_0^1 y f_Y(y)\,dy, \quad f_Y(y) = \frac{1}{2}(1 + 2y).$$

c) **Expectation of the Product XY:**

$$\mathbb{E}(XY) = \int_0^1 \int_0^1 xy f_{X,Y}(x,y)\,dx\,dy, \quad f_{X,Y}(x,y) = \frac{1}{2}(3x^2 + 2y).$$

2.2 Second Moments

a) **Second Moment of X:**

$$\mathbb{E}(X^2) = \int_0^1 x^2 f_X(x)\, dx.$$

b) **Second Moment of Y:**

$$\mathbb{E}(Y^2) = \int_0^1 y^2 f_Y(y)\, dy.$$

c) **Second Moment of the Product X^2Y^2:**

$$\mathbb{E}(X^2Y^2) = \int_0^1 \int_0^1 x^2 y^2 f_{X,Y}(x,y)\, dx\, dy.$$

2.3 Variances

a) **Variance of X:**

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

b) **Variance of Y:**

$$\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2.$$

2.4 Covariance

The covariance between X and Y is:

$$\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

2.5 Conditional Expectations

a) **Conditional Expectation of $Y \mid X = x$:**

$$\mathbb{E}(Y \mid X = x) = \int_0^1 y f_{Y|X=x}(y)\, dy, \quad f_{Y|X=x}(y) = \frac{3x^2 + 2y}{3x^2 + 1}.$$

b) **Conditional Expectation of $X \mid Y = y$:**

$$\mathbb{E}(X \mid Y = y) = \int_0^1 x f_{X|Y=y}(x)\, dx, \quad f_{X|Y=y}(x) = \frac{3x^2 + 2y}{1 + 2y}.$$

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3. Additional Quantities

3.1 Variance of a Conditional Distribution

The variance of $Y \mid X = x$ is:

$$\mathrm{Var}(Y \mid X = x) = \mathbb{E}(Y^2 \mid X = x) - (\mathbb{E}(Y \mid X = x))^2 .$$

Similarly, the variance of $X \mid Y = y$ is:

$$\mathrm{Var}(X \mid Y = y) = \mathbb{E}(X^2 \mid Y = y) - (\mathbb{E}(X \mid Y = y))^2 .$$

3.2 Law of Total Expectation and Variance

- **Total Expectation**:

$$\mathbb{E}(Y) = \mathbb{E}\big[\mathbb{E}(Y \mid X)\big], \quad \mathbb{E}(X) = \mathbb{E}\big[\mathbb{E}(X \mid Y)\big].$$

- **Total Variance**:

$$\mathrm{Var}(Y) = \mathbb{E}\big[\mathrm{Var}(Y \mid X)\big] + \mathrm{Var}\big(\mathbb{E}(Y \mid X)\big),$$

$$\mathrm{Var}(X) = \mathbb{E}\big[\mathrm{Var}(X \mid Y)\big] + \mathrm{Var}\big(\mathbb{E}(X \mid Y)\big).$$

EXERCISE 2: Discrete case

1. Joint Probability Table

The joint probability distribution of two discrete random variables X and Y is given by the table below:

$X\backslash Y$	1	2	3	Row Totals
1	0.32	0.03	0.01	0.36
2	0.06	0.24	0.02	0.32
3	0.02	0.03	0.27	0.32
Column Totals	0.40	0.30	0.30	

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2. Marginal Probabilities

The marginal probabilities for X and Y are:

Marginal of X:

$$P(X = 1) = 0.36, \quad P(X = 2) = 0.32, \quad P(X = 3) = 0.32.$$

Marginal of Y:

$$P(Y = 1) = 0.40, \quad P(Y = 2) = 0.30, \quad P(Y = 3) = 0.30.$$

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3. Conditional Probabilities

Conditional Probabilities of $X \mid Y$:

For each value of Y:

$P(X = x \mid Y = y) =$

$X\backslash Y$	1	2	3
1	0.8	0.10	0.0333
2	0.15	0.80	0.0667
3	0.05	0.10	0.90

Conditional Probabilities of $Y \mid X$:

For each value of X:

$P(Y = y \mid X = x) =$

$Y\backslash X$	1	2	3
1	0.8889	0.1875	0.0625
2	0.0833	0.75	0.0938
3	0.0278	0.0625	0.8438

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1. Joint Probability Table

The joint probability distribution of two discrete random variables X and Y is given by:

$X \backslash Y$	1	2	3	Row Totals
1	0.32	0.03	0.01	0.36
2	0.06	0.24	0.02	0.32
3	0.02	0.03	0.27	0.32
Column Totals	0.40	0.30	0.30	1.00

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2. Marginal Expectations

Expectation of X:

$$\mathbb{E}(X) = \sum_x x \cdot P(X = x) = 1(0.36) + 2(0.32) + 3(0.32)$$

Expectation of Y:

$$\mathbb{E}(Y) = \sum_y y \cdot P(Y = y) = 1(0.40) + 2(0.30) + 3(0.30)$$

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3. Second Moments

Second Moment of X:

$$\mathbb{E}(X^2) = \sum_x x^2 \cdot P(X = x) = 1^2(0.36) + 2^2(0.32) + 3^2(0.32)$$

Second Moment of Y:

$$\mathbb{E}(Y^2) = \sum_y y^2 \cdot P(Y = y) = 1^2(0.40) + 2^2(0.30) + 3^2(0.30)$$

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4. Variances

Variance of X:

$$\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

Variance of Y:

$$\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2$$

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5. Expectation of the Product

$$\mathbb{E}(XY) = \sum_x \sum_y x \cdot y \cdot P(X = x, Y = y)$$

Substituting the values:

$$\mathbb{E}(XY) = 1{\cdot}1{\cdot}0.32+1{\cdot}2{\cdot}0.03+1{\cdot}3{\cdot}0.01+2{\cdot}1{\cdot}0.06+2{\cdot}2{\cdot}0.24+2{\cdot}3{\cdot}0.02+3{\cdot}1{\cdot}0.02+3{\cdot}2{\cdot}0.03+3{\cdot}3{\cdot}0.27$$

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6. Conditional Expectations

Conditional Expectation of $Y \mid X = x$:

For each x:
- When $X = 1$:
$$\mathbb{E}(Y \mid X = 1) = 1 \cdot \frac{0.32}{0.36} + 2 \cdot \frac{0.03}{0.36} + 3 \cdot \frac{0.01}{0.36}$$
- When $X = 2$:
$$\mathbb{E}(Y \mid X = 2) = 1 \cdot \frac{0.06}{0.32} + 2 \cdot \frac{0.24}{0.32} + 3 \cdot \frac{0.02}{0.32}$$
- When $X = 3$:
$$\mathbb{E}(Y \mid X = 3) = 1 \cdot \frac{0.02}{0.32} + 2 \cdot \frac{0.03}{0.32} + 3 \cdot \frac{0.27}{0.32}$$

Conditional Expectation of $X \mid Y = y$:

For each y:
- When $Y = 1$:
$$\mathbb{E}(X \mid Y = 1) = 1 \cdot \frac{0.32}{0.40} + 2 \cdot \frac{0.06}{0.40} + 3 \cdot \frac{0.02}{0.40}$$
- When $Y = 2$:
$$\mathbb{E}(X \mid Y = 2) = 1 \cdot \frac{0.03}{0.30} + 2 \cdot \frac{0.24}{0.30} + 3 \cdot \frac{0.03}{0.30}$$
- When $Y = 3$:
$$\mathbb{E}(X \mid Y = 3) = 1 \cdot \frac{0.01}{0.30} + 2 \cdot \frac{0.02}{0.30} + 3 \cdot \frac{0.27}{0.30}$$

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7. Covariance

$$\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X) \cdot \mathbb{E}(Y)$$