2.3 Bivariate distributions

1. Suppose that X and Y have a discrete joint distribution for which the joint p.f. is defined as follows:

$$f(x,y) = \begin{cases} c|x+y|, & \text{for } x, y \in \{-2,-1,0,1,2\} \\ 0, & \text{otherwise.} \end{cases}$$

Determine

- Is $f(x,y)$ a prob. or a density?
- $\mathrm{Supp}(X,Y) = \{(x,y) \in \mathbb{R}^2 \mid x \in \{-2,-1,0,1,2\} \text{ and } y \in \{-2,-1,0,1,2\}\} = \{-2,-1,0,1,2\} \times \{-2,-1,0,1,2\} = \{-2,-1,0,1,2\}^2$

(a) the value of the constant c;

$$1 = \sum_x \sum_y c|x+y| = c \sum_x \sum_y |x+y| = c \sum_x (|x-2| + |x-1| + |x+0| + |x+1| + |x+2|) = c\,40 \longrightarrow c = \frac{1}{40}$$

$$\begin{array}{ccccc} 4 & 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{array}$$

(b) $Pr(X = 0 \text{ and } Y = -2)$;

$$P_r(X=0, Y=-2) = \frac{1}{40}|0-2| = 0.05$$

(c) $Pr(X = 1)$;

$$P_r(X=1) = \sum_y P_r(X=1, Y=y) = \frac{1}{40} \sum_y |1+y| = \frac{7}{40}$$

- $P_r(X=x) = \frac{1}{40} \sum_y |x+y|$, this is the marginal of X
- whats $P_r(Y=y)$?

(d) $Pr(|X - Y| \leq 1)$.

$$\begin{aligned} P_r(|X-Y| \leq 1) &= \sum_y P_r(|X-Y| \leq 1, Y=y) \\ &= \sum_y P_r(|X-Y| \leq 1 \mid Y=y) P_r(Y=y) \\ &= \sum_y P_r(|X-y| \leq 1) P_r(Y=y) \\ &= \sum_y P_r(-1+y \leq X \leq y+1) P_r(Y=y) \\ &= P_r(-1-2 \leq X \leq 2+1) P_r(Y=-2) \\ &\quad + P_r(-1-1 \leq X \leq 1+1) P_r(Y=-1) \\ &\quad + P_r(-1+0 \leq X \leq 0+1) P_r(Y=0) \\ &\quad + P_r(-1+1 \leq X \leq 1+1) P_r(Y=1) \\ &\quad + P_r(-1+2 \leq X \leq 2+1) P_r(Y=2) \\ &= P_r(-3 \leq X \leq -1) P_r(Y=-2) \\ &\quad + P_r(-2 \leq X \leq 0) P_r(Y=-1) \\ &\quad + P_r(-1 \leq X \leq 1) P_r(Y=0) \\ &\quad + P_r(0 \leq X \leq 2) P_r(Y=1) \\ &\quad + P_r(1 \leq X \leq 3) P_r(Y=2) \end{aligned}$$

2. Suppose that X and Y have a continuous joint distribution for which the joint p.d.f. is defined as follows:

$$f(x,y) = \begin{cases} cy^2, & \text{for } 0 \le x \le 2, \text{ and } 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Determine

- Is $f(x,y)$ a prob. or a density?
- $Supp(X,Y) = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2,\ 0 \le y \le 1\} = [0,2] \times [0,1]$

(a) the value of the constant c;

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^2 cy^2\,dx\,dy = \int_0^1\int_0^2 cy^2\,dx\,dy = c\int_0^1\int_0^2 y^2\,dx\,dy = c\int_0^1 y^2 \int_0^2 dx\,dy = c\int_0^1 y^2 \cdot 2\,dy = c\cdot 2\int_0^1 y^2\,dy$$

$$= c \cdot 2 \left.\tfrac{1}{3}y^3\right|_0^1 = c \cdot 2/3 \quad \longrightarrow c = \frac{3}{2}$$

(b) $Pr(X + Y > 2)$

$$P_r(X+Y>2) = \int_0^1 P_r(X+Y>2 \mid Y=y) f_Y(y)\,dy = \int_0^1 P_r(X+y>2)\, f_Y(y)\,dy = \int_0^1 \overbrace{P_r(X>2-y)}^{\int_{2-y}^{2} f_X(x)\,dx} f_Y(y)\,dy$$

$$= \int_0^1 [1 - P_r(X \le 2-y)]\, f_Y(y)\,dy = \int_0^1 [1 - F_X(2-y)] f_Y(y)\,dy = 1 - \int_0^1 F_X(2-y) f_Y(y)\,dy$$

Lets find $f_X(x)$, $F_X(x)$, and $f_Y(y)$

$\boxed{f_X(x)}$ $\quad f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 cy^2\,dy = c\int_0^1 y^2\,dy = \frac{3}{2}\int_0^1 y^2\,dy = \frac{1}{2} \rightarrow X \sim Unif(0,2)$

$\boxed{F_X(x)}$ $\quad F_X(x) = \frac{1}{2}x$

$\boxed{f_Y(y)}$ $\quad f_Y(y) = \int_0^2 c\cdot y^2\,dx = c\cdot y^2 \int_0^2 dx = \frac{3}{2}y^2 \int_0^2 dx = 3y^2 \quad \rightarrow Y \sim Beta(3,1)$

Therefore,

$$P_r(X+Y>2) = 1 - \int_0^1 F_X(2-y) f_Y(y)\,dy = 1 - \int_0^1 \tfrac{1}{2}(2-y)3y^2\,dy = 1 - \int_0^1 \tfrac{3}{2}(2-y)y^2\,dy = 1 - \int_0^1 \tfrac{3}{2}(2y^2 - y^3)\,dy$$

$$= 1 - \int_0^1 \tfrac{3}{2}(2y^2 - y^3)\,dy = 1 - \tfrac{3}{2}\left.\left(2\tfrac{1}{3}y^3 - \tfrac{1}{4}y^3\right)\right|_0^1 = 1 - \tfrac{3}{2}\left(2\tfrac{1}{3} - \tfrac{1}{4}\right)$$

$$= 1 - \tfrac{3}{2}\left(2\tfrac{1}{3} - \tfrac{1}{4}\right) = 1 - 1 + 3/8 = 3/8$$

(c) $Pr(Y < 1/2)$

$$P_r(Y < 1/2) = P_r(Y \le 1/2) = \int_{-\infty}^{1/2} f_Y(y)\,dy = \int_0^{1/2} 3y^2\,dy$$

(d) $Pr(X \le 1)$

$$P_r(X \le 1) = \int_0^1 f_X(x)\,dx = F_X(1)$$

(e) $Pr(X = 3Y)$

$$P_r(X = 3Y) = P_r(X - 3Y = 0) = \int_0^1 P_r(X - 3Y = 0 \mid Y=y) f_Y(y)\,dy = \int_0^1 P_r(X - 3Y = 0 \mid Y=y) f_Y(y)\,dy$$

$$= \int_0^1 P_r(X - 3y = 0) f_Y(y)\,dy = \int_0^1 \underline{P_r(X = 3y)}\, f_Y(y)\,dy = 0!$$

3. Suppose that the joint p.d.f. of two random variables X and Y is as follows:

$$f(x,y) = \begin{cases} c(x^2+y), & \text{for } 0 \le y \le 1-x^2 \\ 0, & \text{otherwise.} \end{cases}$$

Determine

(a) the value of the constant c;

(b) $Pr(0 \le X \le 1/2)$

(c) $Pr(Y \le X+1)$

(d) $Pr(Y = X^2)$.

$\boxed{\text{For you!}}$

2.4 Marginal distributions

1 Suppose that X and Y have a discrete joint distribution for which the joint p.f. is defined as follows:

$$f(x,y) = \begin{cases} \frac{1}{30}(x+y), & \text{for } x \in \{0,1,2\} y \in \{0,1,2,3\}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the marginal p.f.'s of X and Y.

$$f_X(x) = P_r(X=x) = \sum_y f(x,y) = \sum_y \frac{1}{30}(x+y) = \frac{1}{30}\sum_y (x+y)$$
$$= \frac{1}{30}[(x+0)+(x+1)+(x+2)+(x+3)]$$
$$= \frac{1}{30}(4x+6) = \underline{\frac{2}{15}x+\frac{1}{5}} = \frac{1}{15}(2x+3)$$

$$f_Y(y) = P_r(Y=y) = \sum_x f(x,y) = \frac{1}{30}(3+3y) = \underline{\frac{1}{10}+\frac{1}{10}y}$$

(b) Are X and Y independent?

I. $f(x,y) = f_X(x)f_Y(y)$?

II. Does $f_{Y|X=x}(y)$ depend on x ?
or Does $f_{X|Y=y}(x)$ depend on y ?

Lets check!

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{30}(x+y)}{\frac{1}{15}(2x+3)} = \frac{1}{2}\frac{x+y}{2x+3}$$, depends on x !

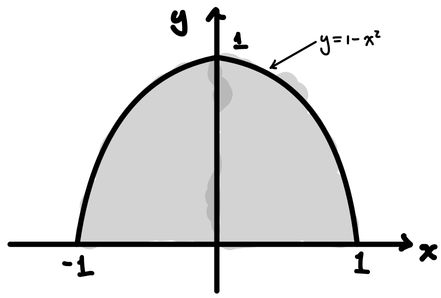
Therefore, They are not independent.

2 Suppose that the joint p.d.f. of X and Y is as follows:

$$f(x,y) = \begin{cases} \frac{15}{4}x^2, & \text{for } 0 \le y \le 1 - x^2, \\ 0, & \text{otherwise.} \end{cases}$$

I. Draw the support

$Supp(X,Y) = \{ (x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 - x^2 \}$



Is $(\sqrt{2}, 1)$ in the Support?

Is $(\frac{1}{\sqrt{2}}, 0.1)$ in the Support?

What values can y take?
What values can x take?

(a) Determine the marginal pdf's of X and Y.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{15}{4}x^2\,dx = \frac{5}{4}x^3 \Big|_{-\sqrt{1-y}}^{\sqrt{1-y}} = \frac{5}{2}(1-y)^{3/2}, \quad y \in (0,1)$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{1-x^2} \frac{15}{4}x^2\,dy = \frac{15}{4}x^2(1-x^2), \quad x \in (-1,1)$$

(b) Are X and Y independent?

Does $f_{Y|X=x}(y)$ depend on x?

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{15}{4}x^2}{\frac{15}{4}x^2(1-x^2)} = \frac{1}{(1-x^2)}, \quad y \in (0, 1-x^2)$$

Therefore, they are not independent.

① X and Y are continuous

joint probability density function

$f(x, y)$

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(r,s)\,ds\,dr$$

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x\, \partial y}$$

joint cumulative distribution function

$F(x, y) = \Pr(X \le x \text{ and } Y \le y)$

$$F_1(x) = \lim_{y\to\infty} F(x,y)$$

$$F_2(y) = \lim_{x\to\infty} F(x,y)$$

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

marginal probability density functions

$f_1(x)$

$f_2(y)$

$$F_1(x) = \int_{-\infty}^{x} f_1(r)\,dr$$

$$F_2(y) = \int_{-\infty}^{y} f_2(s)\,ds$$

marginal cumulative distribution functions

$F_1(x)$

$F_2(y)$

$$f_1(x) = \frac{dF_1(x)}{dx}$$

$$f_2(y) = \frac{dF_2(y)}{dy}$$

② X and Y are discrete

joint probability function

$f(x, y)$

$$F(x,y) = \sum_{r=-\infty}^{x}\sum_{s=-\infty}^{y} f(r,s)$$

joint cumulative distribution function

$F(x, y) = \Pr(X \le x \text{ and } Y \le y)$

$$f(x,y) = F(x,y) - F(x,y^-) - F(x^-,y) + F(x^-,y^-)$$

$$= \Pr(X \le x, Y \le y) - \Pr(X \le x, Y < y) - \Pr(X < x, Y \le y) + \Pr(X < x, Y < y)$$

jump size of $F(x,y)$ at $X = x$ & $Y = y$

$$F_1(x) = \lim_{y\to\infty} F(x,y)$$

$$F_2(y) = \lim_{x\to\infty} F(x,y)$$

$$f_1(x) = \sum_{y=-\infty}^{\infty} f(x,y)$$

$$f_2(y) = \sum_{x=-\infty}^{\infty} f(x,y)$$

marginal probability functions

$f_1(x)$

$f_2(y)$

$$F_1(x) = \sum_{r=-\infty}^{x} f_1(r)$$

$$F_2(y) = \sum_{s=-\infty}^{y} f_2(s)$$

marginal cumulative distribution functions

$F_1(x)$

$F_2(y)$

$$f_1(x) = F_1(x) - F_1(x^-)$$

$$= \Pr(X \le x) - \Pr(X < x) \Leftarrow \text{jump size of } F_1(x) \text{ at } X = x$$

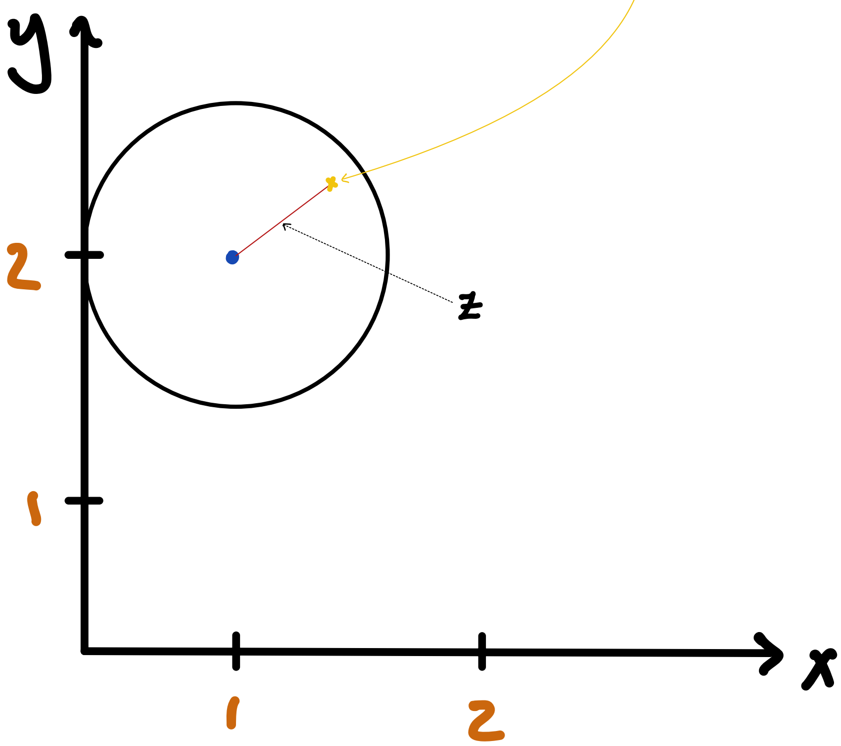
$$f_2(y) = F_2(y) - F_2(y^-)$$

$$= \Pr(Y \le y) - \Pr(Y < y) \Leftarrow \text{jump size of } F_2(y) \text{ at } Y = y$$

Que tengan bonito día! ☺

More ?

2. Consider a point chosen uniformly at random within a circle in the xy-plane, described by $(x-1)^2 + (y-2)^2 = 1$. Let Z denote a random variable representing the distance between the circle's center and this point. Find and sketch the c.d.f. of Z.



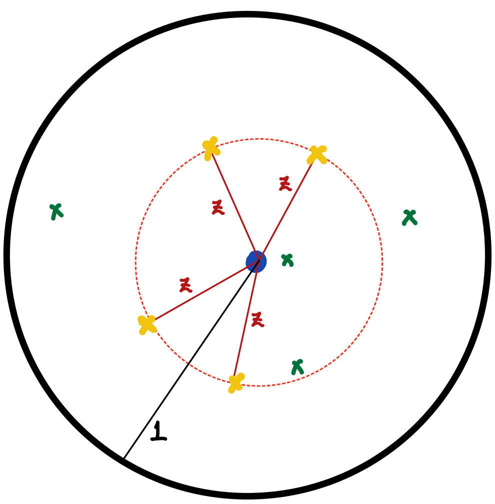
$$A := \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + (y-2)^2 \le 1\}$$

I. What's the smallest and largest Z?

$\text{Sup}(Z) = [0,1]$

II. CDF of Z:

$\Pr(Z \le z) = \Pr($"Distance between random point within A and its center is less than z"$)$



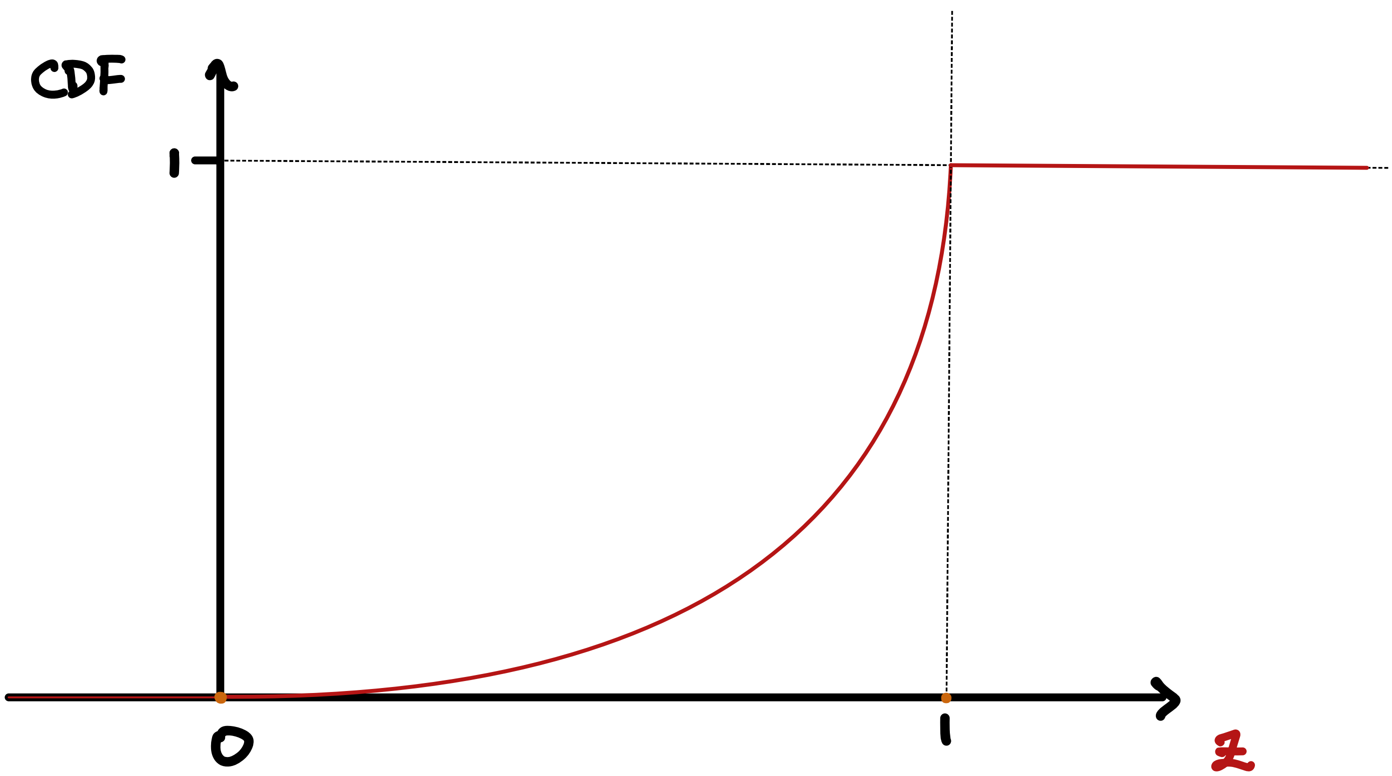
$$\Pr(Z \le z) = \Pr\left(\text{"Distance between random point within } A \text{ and its center is less than } z\text{"}\right)$$

$$= \Pr\left(\text{"All random points inside of the circle of radius } z\text{"}\right)$$

$$= \frac{\text{Area of circle of radius } z}{\text{Area of } A}$$

$$= \frac{\pi z^2}{\pi 1^2}$$

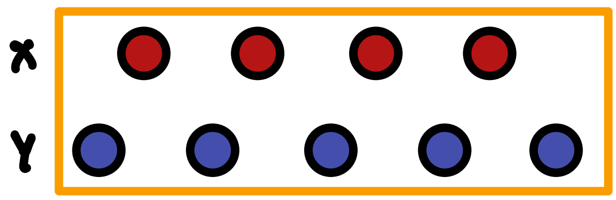
$$= z^2$$



4. Consider an illuminated signboard where the top row has four bulbs and the bottom row contains five. Use X to indicate the count of non-working bulbs in the top row at a specific time t, and Y for the bottom row at that same time. The joint probability function of X and Y can be found in the subsequent table.

X	Y 0	Y 1	Y 2	Y 3	Y 4
0	0.08	0.07	0.05	0.02	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.11	0.02	0.03
3	0.04	0.01	0.03	0.03	0.04

Determine the following probabilities:



(a) $P(X = 1)$

X	Y 0	Y 1	Y 2	Y 3	Y 4
0	0.08	0.07	0.05	0.02	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.11	0.02	0.03
3	0.04	0.01	0.03	0.03	0.04

$$P(X=1)=\sum_{y=0}^{4} P(X=1, Y=y)$$

(b) $P(Y \geq 1)$

X	Y 0	Y 1	Y 2	Y 3	Y 4
0	0.08	0.07	0.05	0.02	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.11	0.02	0.03
3	0.04	0.01	0.03	0.03	0.04

$$P(Y\geq 1)=\sum_{y=1}^{4}\sum_{x=0}^{3} P(X=x, Y=y)$$

$$=1-P(Y=0)$$

Topic: Joint distribution: marginalization

Goal: Get $f_X(x)$ and $f_Y(y)$ from $f_{X,Y}(x,y)$

$X \perp\!\!\!\perp Y \leftrightarrow$

$f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$= h_1(x) h_2(y)$

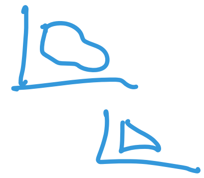
I? check if $f_{X,Y}$ factors

II? $f_{X|Y=y}(x)$ depends on y

1. Suppose random variables X and Y have a continuous joint distribution whose joint pdf is:

$$f(x,y) = \begin{cases} \frac{3}{2}y^2 & \text{for } 0 \le x \le 2 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} = \boxed{\frac{3}{2}y^2 \mathbb{1}^{(y)}_{(0,1)}}\boxed{\mathbb{1}^{(x)}_{(0,2)}}$$

Determine the marginal pdf of X and Y.

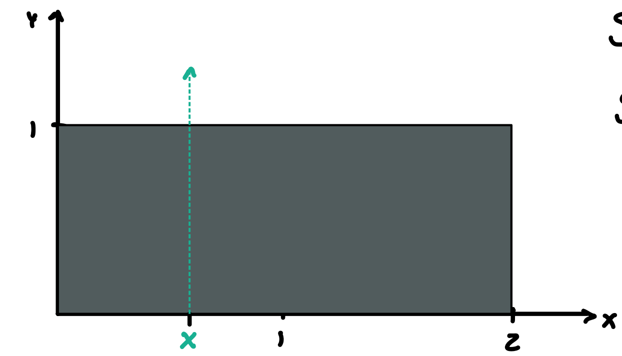


Recall

Marginal of X $\quad f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

Marginal of Y $\quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

I. Understand $\mathrm{Sup}(X,Y)$



$\mathrm{Sup}(X) = ?$

$\mathrm{Sup}(Y) = ?$

$Z \sim \mathrm{Unif}(a,b)$

$f_Z(z) = \frac{1}{b-a}\mathbb{1}_{(a,b)}$

II. Compute the marginals

For $x \in \mathrm{Sup}(X)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_{-\infty}^{\infty} \frac{3}{2}y^2 \mathbb{1}^{(x)}_{(0,2)} \mathbb{1}^{(y)}_{(0,1)}\,dy$$

$$= \int_{-\infty}^{\infty} \frac{3}{2}y^2 \mathbb{1}^{(y)}_{(0,1)}\,dy$$

$\frac{3}{2}\int y^2\,dy$

$= \frac{3}{2}\frac{1}{3}y^3$

$$= \int_0^1 \frac{3}{2}y^2\,dy$$

$$= \frac{3}{2}\frac{1}{3}y^3\Big|_0^1$$

$$= \frac{1}{2}$$

Then $f_X(x) = \frac{1}{2}\mathbb{1}^{(x)}_{(0,2)}$

$X \sim ?$

For $y \in \mathrm{Sup}(Y)$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{3}{2}y^2 \mathbb{1}^{(x)}_{(0,2)} \mathbb{1}^{(y)}_{(0,1)}\,dx$$

$$= \frac{3}{2}y^2 \int_{-\infty}^{\infty} \mathbb{1}^{(x)}_{(0,2)}\,dx$$

$$= \frac{3}{2}y^2 \int_0^2 dx$$

$$= \frac{3}{2}y^2\, x\Big|_0^2$$

$$= 3y^2$$

Then, $f_Y(y) = 3y^2 \mathbb{1}^{(y)}_{(0,1)}$

$Y \sim ?$

Topic: Joint distribution: marginalization

Goal:
- "Find" the joint
- Work with marginals
- Work with conditional dist.

2. Suppose either of two instruments can be used to make a measurement. Instrument 1 yields a measurement whose pdf is:

$$h_1(x) = \begin{cases} 2x & \text{for } 0 < x < 1, \\ 0 & \text{otherwise} \end{cases}$$

and instrument 2 yields a measurement whose pdf is:

$$h_2(x) = \begin{cases} 3x^2 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise} \end{cases}$$

Now, suppose that one of the two instruments is chosen at random and a measurement X is made with it.

(a) Determine the marginal pdf of X?

(b) If the value of the measurement is $X = 0.25$, what is the probability that instrument 1 was used?

I. Let $Y \sim Ber(\frac{1}{2})$ s.t. $Y = \begin{cases} 1 & \text{w.p. } \frac{1}{2}, \text{ using inst } 1 \\ 0 & \text{w.p. } \frac{1}{2}, \text{ using inst } 2 \end{cases}$

$$X \mid Y=y \sim \begin{cases} h_1, & \text{if } y=1 \\ h_2, & \text{if } y=0 \end{cases}$$

Then, $f_{X|Y=y}(x) = h_1(x)\mathbb{1}_{\{1\}}(y) + h_2(x)\mathbb{1}_{\{0\}}(y)$

Fun Facts
- Hierarchical models
- Mixture models
  ↳ show pic

II. Find the Joint!

Notice this is a joint of a cont. r.v. and a discrete r.v.

Recall

$$f_{X,Y}(x,y) = f_{X|Y=y}(x) \cdot f_Y(y)$$

Then, $f_{X,Y}(x,y) = \left[h_1(x)\mathbb{1}_{\{1\}}(y) + h_2(x)\mathbb{1}_{\{0\}}(y)\right](\frac{1}{2})^y(\frac{1}{2})^{1-y}$

Notice $f_{X,Y}(x,1) = \frac{1}{2}h_1(x)$
and $f_{X,Y}(x,0) = \frac{1}{2}h_2(x)$

Then $f_{X,Y}(x,y) = \frac{1}{2}\mathbb{1}_{\{1\}}(y)h_1(x) + \frac{1}{2}\mathbb{1}_{\{0\}}(y)h_2(x)$

$$= \left[h_1(x)\tfrac{1}{2}\right]^y\left[h_2(x)\tfrac{1}{2}\right]^{1-y}$$

This step is not "necessary" but simplifies computations

(a) Determine the marginal pdf of X?

$$f_X(x) = \cancel{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy} = \boxed{\sum_{y=0}^{1} f_{XY}(x,y) = f(x,0) + f(x,1)}$$

$$= \tfrac{1}{2} h_1(x) + \tfrac{1}{2} h_2(x)$$

(b) If the value of the measurement is $X = 0.25$, what is the probability that instrument 1 was used?

why?

$$\Pr(\text{using inst. } 1 \mid X = 0.25) = \Pr(Y = 1 \mid X = 0.25)$$

I.

$$\Pr(Y = y \mid X = 0.25) \overset{\text{B.T.}}{=} \frac{\overbrace{f_{X|Y=y}(0.25)\Pr(Y=y)}^{f_{X,Y}(0.25,\,y)}}{\underbrace{f_{X|y=1}(0.25)\Pr(Y=1) + f_{X|y=0}(0.25)\Pr(Y=0)}_{f_X(0.25)}}$$

$$= \frac{f_{X|Y=y}(0.25)\,0.5}{f_{X|Y=1}(0.25)\,0.5 + f_{X|Y=0}(0.25)\,0.5}$$

$$= \frac{f_{X|Y=y}(0.25)}{f_{X|Y=1}(0.25) + f_{X|Y=0}(0.25)}$$

$$= \frac{f_{X|Y=y}(0.25)}{h_1(0.25) + h_2(0.25)}$$

$$= \frac{16}{11} f_{X|Y=y}(0.25)$$

$$f_{X|Y=y}(x) = h_1(x)\mathbb{1}_{\{1\}}(y) + h_2(x)\mathbb{1}_{\{0\}}(y)$$

II. $\Pr(Y = 1 \mid X = 0.25) = \frac{16}{11}\,\frac{1}{2}$

$$= \frac{8}{11}$$

$$> 0.5$$

Topic: Joint distribution: marginalization

Goal:
- "Find" the joint
- Work with marginals
- Work with conditional dist.

3. Suppose that pdf of random variable X is:

$$f(x) = \begin{cases} x^n e^{-x}\frac{1}{n!} & \text{for } 0 < x < \cancel{1}\infty, \\ 0 & \text{otherwise} \end{cases} \quad \Big| \quad X \sim \text{Gamma}(n,1)$$

and for any given value $X = x$ $(x > 0)$, the n i.i.d. random variables $Y_1, \cdots, Y_n$ and their conditional pdf is:

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & \text{for } 0 < y < x, \\ 0 & \text{otherwise} \end{cases} \quad \Big| \quad Y|X=x \overset{iid}{\sim} \text{Unif}(0,x)$$

(a) Marginal joint pdf of $Y_1, \cdots, Y_n$.

(b) Conditional pdf of X given some values of $Y_1, \cdots, Y_n$.

(a) Marginal joint pdf of $Y_1, \cdots, Y_n$.

I. Find the joint.

$$\boxed{\text{What is } \text{Supp}(Y_1,\dots,Y_n)\text{?}}$$

$$\begin{aligned} f_{Y_1,\dots,Y_n,X}(x,y_1,\dots,y_n) &= f_{Y_1,\dots,Y_n|X=x}(x,y_1,\dots,y_n) f_X(x) \\ &\overset{\text{why?}}{=} \prod_{i=1}^n f_{Y_i|X=x}(y_i)\, f_X(x) \\ &= \prod_{i=1}^n (1/x)\, 1_{(0,x)}(y_i)\, f_X(x) \\ &= x^{-n} \prod_{i=1}^n 1_{(0,x)}(y_i)\, f_X(x) \\ &= x^{-n} \prod_{i=1}^n 1_{(0,x)}(y_i)\, x^n e^{-x} 1_{(0,\infty)}(x) \\ &= e^{-x} 1_{(0,\infty)}(x) \prod_{i=1}^n 1_{(0,x)}(y_i) \end{aligned}$$

II. Find the marginal joint of $(Y_1,\dots,Y_n)$

$$\begin{aligned} f_{Y_1,\dots,Y_n}(y_1,\dots,y_n) &= \int_{-\infty}^{\infty} e^{-x} 1_{(0,\infty)}(x) \prod_{i=1}^n 1_{(0,x)}(y_i)\, dx \\ &= \int_0^{\infty} e^{-x} \prod_{i=1}^n 1_{(0,x)}(y_i)\, dx \end{aligned}$$

Notice that

$$\begin{array}{|l|} \hline 0 < y_1 < x \\ 0 < y_2 < x \\ \vdots \\ 0 < y_n < x \\ \hline \end{array} \longleftrightarrow \max\{y_i\} < x$$

$$= \int_0^\infty e^{-x} \mathbb{1}_{(\max\{y_i\}, \infty)}(x)\, dx$$

$$= \int_{\max\{y_i\}}^\infty e^{-x}\, dx$$

$$= -e^{-x} \Big|_{\max\{y_i\}}^{\infty}$$

$$= e^{-x} \Big|_{\infty}^{\max\{y_i\}}$$

$$= e^{-\max\{y_i\}}, \quad \forall_i\ y_i > 0$$

(b) Conditional pdf of X given some values of $Y_1, \cdots, Y_n$.

$$f_{X|Y_1=y_1,\dots,Y_n=y_n}(x) \overset{\text{Def.}}{=} \frac{f_{Y_1,\dots,Y_n,X}(x, y_1,\dots,y_n)}{f_{Y_1,\dots,Y_n}(y_1,\dots,y_n)}$$

$$= \frac{e^{-x}\mathbb{1}_{(0,\infty)}(x) \prod_{i=1}^{n} \mathbb{1}_{(0,x)}(y_i)}{e^{-\max\{y_i\}} \prod_{i=1}^{n} \mathbb{1}_{(0,x)}(y_i)}$$

$$= e^{-(x-\max\{y_i\})} \mathbb{1}_{(\max\{y_i\},\infty)}(x)$$

What is Sop$(X \mid Y_1,\dots,Y_n)$?