Functions of a Random Variable

X r.v. with pdf $f_X(x)$

$$X \longrightarrow Y := g(X)$$

$f_Y(y)$?

$$\boxed{\begin{aligned} f(x) &= g(r(x)) \\ f'(x) &= g'(r(x)) \cdot r'(x) \end{aligned}}$$

$F_X(x) = P_r(X \leq x) = 1 - P_r(X > x)$

$F_X(x) = P_r(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

$\frac{\partial CDF(x)}{\partial x} = pdf(x)$

[redacted] Functions of a Random Variable

$\mathbb{1}_{(0,2)}(x) = \begin{cases} 1 & x \in (0,2) \\ 0 & \text{o.w.} \end{cases}$

$F(x) = \int_0^x \frac{1}{2} t\,dt$

1. Suppose that the p.d.f. of a random variable X is as follows:

step 0.

$$f(x) = \begin{cases} \frac{1}{2}x, & \text{for } 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

$\boxed{f(x) = \frac{1}{2} x \mathbb{1}_{(0,2)}(x)}$

$\boxed{F(x) = \frac{1}{4} x^2, \quad x \in (0,2)}$

(a) suppose that $Y = X(2 - X)$. Determine the c.d.f. and the p.d.f. of Y.

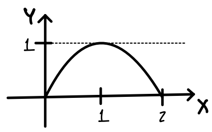
(b) suppose that $Z = 4 - X^3$. Determine the pdf of Z. $\longrightarrow$ Find CDF of Z $\longrightarrow$ obtain pdf of Z taking its derivative

(c) suppose that $W = 3X + 2$. Determine the pdf of W.

---

a) $Y = X(2-X) = -(X-1)^2 + 1$

I. Find Sup(Y)



$Sup(Y) = (0,1)$

II. Find $F_Y(y)$

Let $y \in Sup(Y)$

Then, $F_Y(y) = P_r(Y \le y) = P_r(X(2-X) \le y)$

$= P_r(X \le 1 - \sqrt{1-y}) + P_r(X \ge 1 + \sqrt{1-y})$

$= F_X(1 - \sqrt{1-y}) + [1 - F_X(1 + \sqrt{1-y})]$

III. Find $f_Y(y)$

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(F_X(1 - \sqrt{1-y}) + [1 - F_X(1 + \sqrt{1-y})]\right)$

$= \frac{d}{dy} F_X(1 - \sqrt{1-y}) - \frac{d}{dy} F_X(1 + \sqrt{1-y})$

$= f_X(1 - \sqrt{1-y})(\frac{1}{2})(1-y)^{-1/2} + f_X(1 + \sqrt{1-y})(\frac{1}{2})(1-y)^{-1/2}$

$= \frac{1}{4}((1-y)^{-1/2} - 1) + \frac{1}{4}((1-y)^{-1/2} + 1)$

$= \frac{1}{2}(1-y)^{-1/2}, \quad y \in (0,1)$

---

b) $Z = 4 - X^3$

I. Find Sup(Z)

$x \in Sup(X) \leftrightarrow 0 < x < 2$

$\leftrightarrow 0 < x^3 < 8$

III. Find $f_Z(z)$

Let $z \in Sup(Z)$

$f_Z(z) = \frac{d}{dz} F_Z(z)$

$\boxed{f_X(x) = \frac{1}{2} x \mathbb{1}_{(0,2)}(x)}$

$$\leftrightarrow -8 < -x^3 < 0$$
$$\leftrightarrow -4 < 4 - x^3 < 4$$
$$\leftrightarrow -4 < z < 4$$
$$\leftrightarrow z \in (-4,4)$$

Then $\operatorname{Sup}(Z) = (-4,4)$

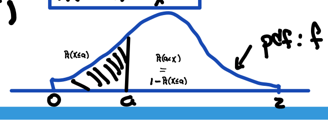
II. Find $F_Z(z)$

Let $z \in \operatorname{Sup}(Z)$

$$\begin{aligned} F_Z(z) &= \Pr(Z \le z) \\ &= \Pr(4 - X^3 \le z) \\ &= \Pr(4 - z \le X^3) \\ &= \Pr((4-z)^{1/3} \le X) \\ &= 1 - \Pr(X \le (4-z)^{1/3}) \\ &= 1 - F_X((4-z)^{1/3}) \end{aligned}$$

$\Pr(a \le Y) = 1 - \Pr(Y < a)$
$= 1 - \Pr(Y \le a)$

$\boxed{\Pr(X \le a) = F_X(a)}$



$$\begin{aligned} &= \frac{d}{dz}\left(1 - F_X((4-z)^{1/3})\right) \\ &= -\frac{d}{dz} F_X((4-z)^{1/3}) \\ &= -\left[F_X'((4-z)^{1/3}) \cdot (1/3)(4-z)^{1/3-1}(-1)\right] \\ &= -\left[f_X((4-z)^{1/3}) \cdot (1/3)(4-z)^{-2/3}(-1)\right] \\ &= f_X((4-z)^{1/3})(1/3)(4-z)^{-2/3} \\ &= \frac{1}{6}(4-z)^{-1/3}, \quad z \in (-4,4) \end{aligned}$$

c) $W = 3X + 2$

$$f(x) = \frac{1}{2} x \, \mathbb{1}_{(0,2)}(x)$$

I. Find $\operatorname{Sup}(W)$

$$\begin{aligned} x \in \operatorname{Sup}(X) &\leftrightarrow 0 < x < 2 \\ &\leftrightarrow 0 < 3x < 6 \\ &\leftrightarrow 2 < 3x + 2 < 8 \\ &\leftrightarrow 2 < w < 8 \\ &\leftrightarrow w \in (2,8) \end{aligned}$$

Then $\operatorname{Sup}(W) = (2,8)$

II. Find $F_W(w)$

Let $w \in \operatorname{Sup}(W)$

$$\begin{aligned} F_W(w) &= \Pr(W \le w) \\ &= \Pr(3X + 2 \le w) \\ &= \Pr\left(X \le (w-2)\tfrac{1}{3}\right) \\ &= F_X\left((w-2)\tfrac{1}{3}\right) \\ &= F_X\left((w-2)\tfrac{1}{3}\right) = \frac{1}{4}\frac{1}{3^2}(w-2)^2 \end{aligned}$$

III. Find $f_W(w)$

$$\begin{aligned} f_W(w) &= \frac{d}{dw} F_W(w) \\ &= \frac{d}{dw} F_X\left((w-2)\tfrac{1}{3}\right) \\ &= f_X\left((w-2)\tfrac{1}{3}\right)\tfrac{1}{3} \\ &= \frac{1}{18}(w-2), \quad w \in (2,8) \end{aligned}$$

2. Suppose that a random variable X has the uniform distribution on the interval $[0, 1]$. Determine the p.d.f. of

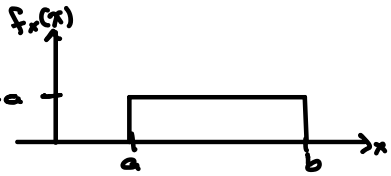
(a) X^2

(b) $-X^3$, and

(c) $\sqrt{X}$.

Let $X \sim Unif(0,1) \longrightarrow$ $f_X(x) = 1_{(0,1)}(x)$, $F_X(x) = x$

$X \sim Unif(a,b) \longrightarrow$ $f_X(x) = \frac{1}{b-a} 1_{(a,b)}(x)$, $F_X(x) = \frac{x-a}{b-a} 1_{(a,b)}(x)$



---

a) Denote $Y = X^2$, find $f_Y(y)$

I. Find Sup(Y)

$$0 \le X \le 1 \longleftrightarrow 0 \le X^2 \le 1 \longleftrightarrow 0 \le Y \le 1$$

$$Sup(Y) = (0,1)$$

II. Find $F_Y(y)$

Let $y \in Sup(Y)$,

then, $F(Y) = P_r(Y \le y)$
$= P_r(X^2 \le y)$
$= P_r(\sqrt{X^2} \le \sqrt{y})$
$= P_r(|X| \le \sqrt{y})$
$= P_r(X \le \sqrt{y})$
$= F_X(\sqrt{y})$
$= \sqrt{y}$

III. Find $f_Y(y)$

$$f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = \frac{\partial}{\partial y}[\sqrt{y}]$$

Then, $\boxed{f_Y(y) = \frac{1}{2\sqrt{y}} 1_{(0,1)}(y)}$

b) Denote $Y=-X^3$, Find $f_Y(y)$

I. Find Sup(Y)

$0 \le X \le 1$

$\leftrightarrow -1 \le -X \le 0$

$\leftrightarrow -1 \le -X^3 \le 0$

$\leftrightarrow -1 \le Y \le 0$

Sup(Y) = (-1, 0)

II. Find $F_Y(y)$

Let $y \in Sup(Y)$,

then, $F(Y) = P_r(Y \le y)$

$= P_r(-X^3 \le y)$

$= P_r(X^3 \ge -y)$

$= P_r(X \ge -y^{1/3})$

$= 1 - P_r(X \le -y^{1/3})$

$= 1 - F_X(-y^{1/3})$

$= 1 + y^{1/3}$

III. Find $f_Y(y)$

$$f_Y(y) = \frac{\partial F_Y(y)}{\partial y}$$

$$= \frac{\partial}{\partial y}[1 + y^{1/3}]$$

$$= 1/3 \, y^{-2/3}$$

Then, $\boxed{f_Y(y) = 1/3 \, y^{-2/3} \, \mathbb{1}_{(-1,0)}(y)}$

c) Denote $Y=\sqrt{X}$, Find $f_Y(y)$

I. Find Sup(Y)

$0 \le X \le 1$

$\leftrightarrow 0 \le \sqrt{X} \le 1$

$\leftrightarrow 0 \le Y \le 1$

Sup(Y) = (0, 1)

II. Find $F_Y(y)$

Let $y \in Sup(Y)$,

then, $F(Y) = P_r(Y \le y)$

$= P_r(\sqrt{X} \le y)$

$= P_r(X \le y^2)$

$= F_X(y^2)$

$= y^2$

III. Find $f_Y(y)$

$$f_Y(y) = \frac{\partial F_Y(y)}{\partial y}$$

$$= \frac{\partial}{\partial y}[y^2]$$

$$= 2y$$

Then, $\boxed{f_Y(y) = 2y \, \mathbb{1}_{(0,1)}(y)}$

3. Let Z be the rate at which customers are served in a queue. Assume that Z has the p.d.f.

$$f(z) = \begin{cases} 2e^{-2z}, & \text{for } z > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the p.d.f. of the average waiting time $T = 1/Z$.

For you!

4. Suppose that X has the uniform distribution on the interval $[-1, 1]$. The p.d.f. of $Y = X^2$ is much larger for values of y near 0 than for values of y near 1 despite the fact that the p.d.f. of X is flat. Give an intuitive reason why this occurs in this example.

For you!

6

5. Suppose that the p.d.f. of a random variable X is as follows:

$$f(x) = \begin{cases} 3x^2, & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases} = 3x^2 \mathbb{1}_{(0,1)}(x)$$

Also, suppose that $Y = 1 - X^2$. Determine the p.d.f. of Y.

I. Find Sup(Y)

$x \in \mathrm{Sup}(X) \leftrightarrow 0 < x < 1$

$\leftrightarrow 0 < x^2 < 1$

$\leftrightarrow 0 < 1 - x^2 < 1$

$\leftrightarrow 0 < y < 1$

$\leftrightarrow y \in (0,1)$

$\mathrm{Sup}(Y) = (0,1)$

II. Find $F_Y(y)$

$F_Y(y) = P_r(Y \leq y)$

$= P_r(1 - X^2 \leq y)$

$= P_r(1 - y \leq X^2)$

$= P_r(\sqrt{1-y} \leq X)$

$= 1 - P_r(X \leq \sqrt{1-y})$

$= 1 - F_X(\sqrt{1-y})$

III. Find $f_Y(y)$

$f_Y(y) = \dfrac{\partial F_Y(y)}{\partial y}$

$= 1 - F_X(\sqrt{1-y})$

$= -\dfrac{\partial F_X(\sqrt{1-y})}{\partial y}$

$= f_X(\sqrt{1-y})(\tfrac{1}{2})(1-y)^{-1/2}$

$= \tfrac{3}{2}\sqrt{1-y}, \quad y \in (0,1)$

2.6 Multivariate distributions

1. Suppose that three random variables X_1, X_2, and X_3 have a continuous joint distribution with the following joint p.d.f.:

$$f(x_1,x_2,x_3) = \begin{cases} c(x_1+2x_2+3x_3), & \text{for } 0 \le x_i \le 1, i=1,2,3, \\ 0, & \text{otherwise.} \end{cases}$$

$$\boxed{f(x_1,x_2,x_3) = c(x_1+2x_2+3x_3)\,\mathbb{1}^{(x_1)}_{(0,1)}\mathbb{1}^{(x_2)}_{(0,1)}\mathbb{1}^{(x_3)}_{(0,1)}}$$

(a) the value of the constant c;

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(x_1+2x_2+3x_3)\,\mathbb{1}^{(x_1)}_{(0,1)}\mathbb{1}^{(x_2)}_{(0,1)}\mathbb{1}^{(x_3)}_{(0,1)}\,dx_1\,dx_2\,dx_3$$

$$= c\int_0^1\int_0^1 \boxed{\int_0^1 (x_1+2x_2+3x_3)\,dx_1}\,dx_2\,dx_3$$

$$= c\int_0^1\int_0^1 \left(\tfrac{1}{2}x_1^2+2x_2x_1+3x_3x_1\right)\Big|_0^1\,dx_2\,dx_3$$

$$= c\int_0^1 \boxed{\int_0^1 \left(\tfrac{1}{2}+2x_2+3x_3\right)dx_2}\,dx_3$$

$$= c\int_0^1\int_0^1 \left(\tfrac{1}{2}x_2+2\tfrac{1}{2}x_2^2+3x_3x_2\right)\Big|_0^1\,dx_3$$

$$= \boxed{c\int_0^1 \left(\tfrac{1}{2}+2\tfrac{1}{2}+3x_3\right)dx_3}$$

$$= c\left(\tfrac{1}{2}+2\tfrac{1}{2}+3\tfrac{1}{2}\right)$$

$$= c\,\tfrac{6}{2}$$

$$= c\cdot 3 \rightarrow c = \tfrac{1}{3}$$

(b) the marginal joint p.d.f. of X_1 and X_3;

$$f_{X_1,X_3}(x_1,x_3) = \int_{-\infty}^{\infty} f(x_1,x_2,x_3)\,dx_2$$

$$= \int_{-\infty}^{\infty} c(x_1+2x_2+3x_3)\,\mathbb{1}^{(x_1)}_{(0,1)}\mathbb{1}^{(x_2)}_{(0,1)}\mathbb{1}^{(x_3)}_{(0,1)}\,dx_2$$

$$= c\int_0^1 (x_1+2x_2+3x_3)\,\mathbb{1}^{(x_1)}_{(0,1)}\mathbb{1}^{(x_3)}_{(0,1)}\,dx_2$$

$$= c\int_0^1 (x_1+2x_2+3x_3)\,dx_2\,\mathbb{1}^{(x_1)}_{(0,1)}\mathbb{1}^{(x_3)}_{(0,1)}$$

$$= c\,(x_1+1+3x_3)\,\mathbb{1}^{(x_1)}_{(0,1)}\mathbb{1}^{(x_3)}_{(0,1)}$$

$$= \frac{x_1+1+3x_3}{3}\,\mathbb{1}^{(x_1)}_{(0,1)}\mathbb{1}^{(x_3)}_{(0,1)}$$

(c) $P(X_3 < \frac{1}{2}|X_1 = 1/4, X_2 = 3/4)$.

$$P(X_3 < \tfrac{1}{2}\,|\,X_1=\tfrac{1}{4}, X_2=\tfrac{3}{4}) = \int_{-\infty}^{1/2} f^{(x_3)}_{X_3|X_1=\frac{1}{4},X_2=\frac{3}{4}}\,dx_3$$

$$f^{(x_3)}_{X_3|X_1=\frac{1}{4},X_2=\frac{3}{4}} = \frac{f(\frac{1}{4},\frac{3}{4},x_3)}{f_{X_1,X_2}(\frac{1}{4},\frac{3}{4})}$$

$$= \frac{c(\frac{1}{4}+2\frac{3}{4}+3x_3)\,\mathbb{1}^{(x_3)}_{(0,1)}}{f_{X_1,X_2}(\frac{1}{4},\frac{3}{4})}$$

$$= \frac{c(\frac{1}{4}+2\frac{3}{4}+3x_3)\,\mathbb{1}^{(x_3)}_{(0,1)}}{c(\frac{1}{4}+2\frac{3}{4}+\frac{3}{2})\,\mathbb{1}^{(x_1)}_{(0,1)}\mathbb{1}^{(x_2)}_{(0,1)}}$$

$$= \frac{(\frac{1}{4}+2\frac{3}{4}+3x_3)}{(\frac{1}{4}+2\cdot\frac{3}{4}+\frac{3}{2})}\,\mathbb{1}^{(x_3)}_{(0,1)}$$

$$= \frac{\frac{1}{4}+2\frac{3}{4}+3x_3}{\frac{1}{4}+2\cdot\frac{3}{4}+\frac{3}{2}}\,\mathbb{1}^{(x_3)}_{(0,1)}$$

$$= \frac{7/4+3x_3}{13/4}\,\mathbb{1}^{(x_3)}_{(0,1)}$$

$$= \left(\tfrac{7}{13}+\tfrac{12}{13}x_3\right)\mathbb{1}^{(x_3)}_{(0,1)}$$

Therefore

$$P(X_3 < \tfrac{1}{2}\,|\,X_1=\tfrac{1}{4}, X_2=\tfrac{3}{4}) = \int_{-\infty}^{1/2} \left(\tfrac{7}{13}+\tfrac{12}{13}x_3\right)\mathbb{1}^{(x_3)}_{(0,1)}\,dx_3$$

$$= \int_0^{1/2} \left(\tfrac{7}{13}+\tfrac{12}{13}x_3\right)dx_3$$

$$= \tfrac{7}{13}x_3+\tfrac{6}{13}x_3^2\Big|_0^{1/2}$$

$$= \tfrac{7}{13}\tfrac{1}{2}+\tfrac{6}{13}\tfrac{1}{4}$$

$$\approx 0.384$$

X_1, X_2, X_3 with joint $f_{X_1,X_2,X_3}(x_1, x_2, x_3)$

$$f_{X_2 | X_1 = x_1, X_3 = x_3}(x_2) = \frac{f_{X_1,X_2,X_3}(x_1, x_2, x_3)}{f_{X_1,X_3}(x_1, x_3)}$$

$$f_{X_1,X_3}(x_1, x_3) = \int_{x_2 \in \mathrm{Supp}(X_2)} f_{X_1,X_2,X_3}(x_1, x_2, x_3)\, dx_2$$

Que tengan bonito día! ☺

Extra!

- Suppose X is a random variable whose pdf is f, and let $Y = X^2$. Show the pdf of Y is:

$$g(y) = \frac{1}{2\sqrt{y}}\left[f(\sqrt{y}) + f(-\sqrt{y})\right].$$

Lets generalize what we did in 2. a)

I. Find $F_Y(y)$

Let $y \in Sup(Y)$,

then, $F(Y) = P_r(Y \leq y) = P_r(X^2 \leq y) = P_r(\sqrt{X^2} \leq \sqrt{y})$

$= P_r(|X| \leq \sqrt{y}) = P_r(-\sqrt{y} \leq X \leq \sqrt{y})$

$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$

II. Find $f_Y(y)$

$f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = \frac{\partial}{\partial y}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right] = \frac{\partial}{\partial y}F_X(\sqrt{y}) - \frac{\partial}{\partial y}F_X(-\sqrt{y})$

$= f_X(\sqrt{y})\frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\frac{1}{2\sqrt{y}}(-1) = f_X(\sqrt{y})\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\frac{1}{2\sqrt{y}}$

$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$

$\square$

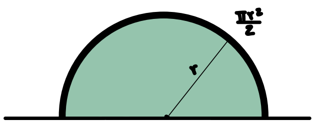
2. Suppose that a point (X_1, X_2, X_3) is chosen at random, that is, in accordance with the uniform p.d.f., from the following set S:

$$S = \{(x_1, x_2, x_3) : 0 \le x_i \le 1 \text{ for } i = 1, 2, 3\}.$$

$$f(x_1, x_2, x_3) = \mathbb{1}_{(0,1)}^{(x_1)} \mathbb{1}_{(0,1)}^{(x_2)} \mathbb{1}_{(0,1)}^{(x_3)}$$

(a) $P\left((X_1 - 1/2)^2 + (X_2 - 1/2)^2 + (X_3 - 1/2)^2 \le 1/4\right)$

$$\begin{aligned}
P_r\left((X_1-\tfrac{1}{2})^2+(X_2-\tfrac{1}{2})^2+(X_3-\tfrac{1}{2})^2 \le \tfrac{1}{4}\right) &= \int_0^1\int_0^1 P_r\left((X_1-\tfrac{1}{2})^2+(X_2-\tfrac{1}{2})^2+(X_3-\tfrac{1}{2})^2 \le \tfrac{1}{4},\ X_2 = x_2,\ X_3 = x_3\right) dx_2\, dx_3 \\
&= \int_0^1\int_0^1 P_r\left((X_1-\tfrac{1}{2})^2+(X_2-\tfrac{1}{2})^2+(X_3-\tfrac{1}{2})^2 \le \tfrac{1}{4} \mid X_2 = x_2,\ X_3 = x_3\right) f_{X_2,X_3}^{(x_2,x_3)}\, dx_2\, dx_3 \\
&= \int_0^1\int_0^1 P_r\left((X_1-\tfrac{1}{2})^2+(X_2-\tfrac{1}{2})^2+(X_3-\tfrac{1}{2})^2 \le \tfrac{1}{4} \mid X_2 = x_2,\ X_3 = x_3\right) dx_2\, dx_3 \\
&= \int_0^1\int_0^1 P_r\left((X_1-\tfrac{1}{2})^2+(x_2-\tfrac{1}{2})^2+(x_3-\tfrac{1}{2})^2 \le \tfrac{1}{4}\right) dx_2\, dx_3 \\
&= \int_0^1\int_0^1 P_r\left((X_1-\tfrac{1}{2})^2+(x_2-\tfrac{1}{2})^2+(x_3-\tfrac{1}{2})^2 \le \tfrac{1}{4}\right) dx_2\, dx_3 \\
&= \int_0^1\int_0^1 P_r\left(\tfrac{1}{2} - \sqrt{\tfrac{1}{4}-(x_3-\tfrac{1}{2})^2-(x_2-\tfrac{1}{2})^2} \le X_1 \le \tfrac{1}{2} + \sqrt{\tfrac{1}{4}-(x_3-\tfrac{1}{2})^2-(x_2-\tfrac{1}{2})^2}\right) dx_2\, dx_3 \\
&= \int_0^1\int_0^1 2\sqrt{\tfrac{1}{4}-(x_3-\tfrac{1}{2})^2-(x_2-\tfrac{1}{2})^2}\, dx_2\, dx_3 \\
&= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} 8\sqrt{\tfrac{1}{4}-x_3^2-x_2^2}\, dx_2\, dx_3 \\
&= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} 8\frac{\pi}{2}\left(\tfrac{1}{4}-x_3^2\right)^2 dx_2\, dx_3 \\
&= \int_0^{\frac{1}{2}} 4\pi\left(\tfrac{1}{4}-x_3^2\right)^2 dx_3 \\
&= 4\pi\int_0^{\frac{1}{2}}\left(\tfrac{1}{16}-\tfrac{1}{2}x_3^2+x_3^4\right) dx_3 \\
&= \frac{4\pi}{60} \\
&= \tfrac{1}{15}\pi
\end{aligned}$$



(b) $P(X_1^2 + X_2^2 + X_3^2 \le 1)$

For you!

[redacted]

2.8 Functions of several random variables

2. Suppose that X_1 and X_2 have a continuous joint distribution for which the joint p.d.f. is as follows:

$$f(x,y) = \begin{cases} x+y, & \text{for } 0 \le x \le 1, 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\boxed{f(x,y) = (x+y)\, \mathbb{1}_{(0,1)}^{(x)} \mathbb{1}_{(0,1)}^{(y)}}$$

(a) Find the p.d.f. of $Y = X_1X_2$

I. Define $Y := \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = g\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_1X_2 \end{pmatrix}$

II. Find $\text{Supp}(Y) = \text{Supp}\left(\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}\right)$

Notice that $Y_1 = X_1$, then $\text{Supp}(Y_1) = (0,1)$

and $Y_2 = X_1X_2$, but $X_1X_2 \in (0,1)$
and $X_1X_2 \le X_1$

that means, $Y_2 \le Y_1$

Therefore $\boxed{\text{Supp}(Y) = \{(y_1,y_2) \mid 0 \le y_2 \le y_1 \le 1\}}$

III. Get g^{-1}

$\begin{cases} y_1 = x_1 \\ y_2 = x_1x_2 \end{cases} \leftrightarrow \begin{array}{l} x_1 = y_1 \\ x_2 = y_2/x_1 = y_2/y_1 \end{array}$ then $g^{-1}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2/Y_1 \end{pmatrix}$

IV. Get $J(g^{-1}) = \begin{vmatrix} 1 & 0 \\ -\frac{y_2}{y_1^2} & \frac{1}{y_1} \end{vmatrix} = \frac{1}{y_1}$

V. Get the joint of $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}(g^{-1}(y_1,y_2)) \cdot J \cdot \mathbb{1}_{\text{Supp}(Y)}(y_1,y_2) \\ &= f_{X_1,X_2}(y_1, y_2/y_1) \cdot \frac{1}{y_1} \cdot \mathbb{1}_{\text{Supp}(Y)}(y_1,y_2) \\ &= \left(y_1 + \frac{y_2}{y_1}\right) \cdot \frac{1}{y_1} \cdot \mathbb{1}_{\text{Supp}(Y)}(y_1,y_2) \\ &= \left(1 + \frac{y_2}{y_1^2}\right) \cdot \mathbb{1}_{\text{Supp}(Y)}(y_1,y_2) \end{aligned}$$

IV. Get the marginal of Y_2 (this is the pdf of X_1X_2)

$$\begin{aligned} f_{Y_2}(y_2) &= \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\, dy_1 \\ &= \int^{\infty} \left(1 + \frac{y_2}{y_1^2}\right) \cdot \mathbb{1}_{\text{Supp}(Y)}(y_1,y_2)\, dy_1 \end{aligned}$$

$$= \int_{y_2}^{1} \left(1 + \frac{y_2}{y_1^2}\right) dy_1$$

$$= \left(y_1 - \frac{y_2}{y_1}\right)\Big|_{y_2}^{1}$$

$$= (1 - y_2) - (y_2 - 1)$$

$$= 2(1 - y_2), \quad y_2 \in (0,1)$$

(b) Find the p.d.f. of $Z = X_1/X_2$

For you!

2.9 Important discrete r.v.'s

3. Suppose that X_1 and X_2 are independent random variables and that X_i has the Poisson distribution with mean λ_i $(i = 1, 2)$. For each fixed value of k $(k = 1, 2, ...)$, determine the conditional distribution of X_1 given that $X_1 + X_2 = k$.

For you!

2.10 Important continuous r.v.'s

3. Suppose that X_1,..., X_n form a random sample of size n from the exponential distribution with parameter β. Determine the distribution of the sample mean $\bar{X}_n$.

For you!

5. Suppose that X_1 and X_2 form a random sample of two observed values from the exponential distribution with parameter β. Determine the distribution of $X_1/(X_1 + X_2)$.

For you!