MAA107 Project: Traffic Modelling

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Abstract

In this project, we will be analysing the LWR traffic model. LWR is a macroscopic model created by Ligthill and Whitham in 1955 and then extended by Richards one year later. The model relies on the assumption that the traffic is comparable to fluid steam, thus the three fundamental variables are flow q, density ρ, and mean speed u. We will also track the number of cars N on a section of a road. We will first introduce the mathematical methods we need to analyse the model. After that, we will derive the fundamental equations in the LWR, and provide the underlining intuition. We will study the fundamental diagram and discuss the insights it provides. This will be followed with an introduction and the interpretation of shockwaves in traffic flow theory. Then, we will turn to analytically solving the model. We will include our cellular automata simulations of traffic, and discuss how it relates to numerically obtained results of LWR solution. Finally, we will conclude by by commenting on the nature of the obtained results. One can access to the source code here.

1 Introduction

1.1 Traffic Modeling

In mathematics, traffic flow is the study of interactions between travellers and infrastructure, with the aim of understanding and developing an optimal transport networks with efficient movement of traffic and minimal traffic congestion problems. When we talk about traffic models, we can talk about two types: Macroscopic models and Microscopic models. The later represents the study of individual vehicles and their parameters such as position or velocity, whereas the former focuses on the traffic as a whole, and tries to identify relationships between traffic flow characteristics like density, flow, mean speed of a traffic stream, etc...In general, the macroscopic models rely on the assumption of similarity with the fluid streams, which enables us to use several different tools developed by physicists. One of those is the use of waves (and shock-waves) to interpret changes in traffic behavior. This is explained in [2] "Slight changes in flow are propagated back through the stream of vehicles along kinematic waves, whose velocity relative to the road is the slope of the graph of flow against concentration". As is the case with most branches of mathematical modeling, in traffic flow theory, the empirical data (such as the fundamental diagram) is often used to provide insights or explain certain results.

2 Mathematical Tools: Multivariable Functions and Partial Differential Equations

Since most of our work will be done with multivariable functions, we will first introduce tools from multivariable calculus for solving partial differential equations (PDEs).

2.1 Partial Derivatives

***Definition* 1.** Let $f : \mathbf{R}^n \to \mathbf{R}$ be a smooth function of n variables $(x_1, ..., x_n)$. Then, its partial derivative with respect to x_i is defined as

$$\frac{\partial f}{\partial x_i}(x_1, ..., x_n) = \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ..., x_n) - f(x_1, ..., x_i, ..., x_n)}{h} \tag{1}$$

We will denote the partial derivative $\frac{\partial f}{\partial x_i}$ as f_{x_i} from now on and assume that we are working with smooth functions for which all the partial derivatives exists.

Using this definition, the second and higher order partial derivatives can be defined analogously. It is simply done by applying the same limits multiple times which exist under the condition that the function is smooth. For example, the second partial derivative with respect to x_i is given as follows:

$$f_{x_i x_i} = \frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_i} \tag{2}$$

Similarly, we can have mixed partial derivatives

$$f_{x_i x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i} \tag{3}$$

We will now state the theorem that provides us with the relation between the two mixed partial derivatives $f_{x_i x_j}$ and $f_{x_j x_i}$.

Theorem 1 (Clairaut's Theorem)**.** Let $f\colon\ \Omega \to \mathbf{R}$ be a smooth function defined on a set $\Omega \subset \mathbf{R}^n$. Then, $\forall i, j \in \{1, 2...n\}$

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} \tag{4}$$

Proof. For simplicity we will omit denoting all the other variables other than the variables of interest. In addition, we will denote $x := x_i$, $y := x_j$, $h := \Delta y$, $l := \Delta x$:

$$\begin{aligned}
f_{xy}(x,y) &= lim_{h\to 0} \frac{f_x(x,y+h) - f_x(x,y)}{h} \\
&= \lim_{h\to 0} \frac{\lim_{l\to 0} \frac{f(x+l,y+h)-f(x,y+h)}{l} - \lim_{l\to 0} \frac{f(x+l,y)-f(x,y)}{l}}{h} \\
&= \lim_{h\to 0} \frac{\lim_{l\to 0} \frac{f(x+l,y+h)-f(x,y+h)-f(x+l,y)+f(x,y)}{l}}{h} \\
&= \lim_{h\to 0}\lim_{l\to 0} \frac{f(x+l,y+h) - f(x,y+h) - f(x+l,y) + f(x,y)}{hl}
\end{aligned}$$

On the other hand, we also get

$$\begin{aligned}
f_{yx}(x,y) &= \lim_{l\to 0} \frac{f_y(x+l,y) - f_y(x,y)}{l} \\
&= \lim_{l\to 0} \frac{\lim_{h\to 0} \frac{f(x+l,y+h)-f(x+l,y)}{h} - \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{l}}{h} \\
&= \lim_{l\to 0} \frac{\lim_{h\to 0} \frac{f(x+l,y+h)-f(x+l,y)-f(x,y+h)+f(x,y)}{h}}{l} \\
&= \lim_{l\to 0}\lim_{h\to 0} \frac{f(x+l,y+h) - f(x+l,y) - f(x,y+h) + f(x,y)}{hl}
\end{aligned}$$

So we can conclude that $f_{xy} = f_{yx}$. □

2.2 Partial Differential Equations (PDEs)

[1] Partial differential equation (PDE) is an equation which imposes relations between the various partial derivatives of a multivariable function. A general PDE has the form:

$$F(\mathbf{x}, f(\mathbf{x}), \frac{\partial f}{\partial x_j}\mathbf{x}, ..., \frac{\partial^m f}{\partial x_{j_1}...\partial x_{j_m}}\mathbf{x}) = 0 \tag{5}$$

where m is the order of the equation, and $\mathbf{x} \in \mathbf{R}^n$ is the vector of variables of f. If F can be written as a linear combination of its variables where coefficients are functions of $\mathbf{x}$ in addition to a constant term, then we say the PDE is *linear* PDE. If the constant term is zero, then we say the PDE is *homogeneous*. There is a wide variety of PDEs and there is no general method that allows us to solve all of them. Hence, we will first explain a method called the *Method of Characteristics* to solve linear PDEs and then apply it to a specific case for our modelling purposes.

2.3 Method of Characteristics

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a smooth function of two variables. Let us consider the following linear PDE for such a function:

$$a(x, y)f_x + b(x, y)f_y = c(x, y) \tag{6}$$

Theorem 2. Suppose that we can find a solution $f(x, y)$. Then, the vector $M := (a(x, y), b(x, y), c(x, y))$ lies in the tangent of the surface $S := \{(x, y, f(x, y))\}$.

Proof. If $f(x, y)$ is a solution to this PDE, then we have:

$$MV^T = 0$$

where $V = (f_x(x, y), f_y(x, y), -1)$. But also recall from geometry of surfaces that normal to the surface created by S is given by:

$$N(x, y) = (f_x(x, y), f_y(x, y), -1) = V$$

Hence, the vector M is perpendicular to V which is normal to the surface S. We conclude that M lies in the tangent of the surface. □

To construct such a surface, we start by a curve $C = \{(x(s), y(s), z(s))\}$ parametrized by s such that the vector M is tangent to the curve. Indeed, this means the curve C should satisfy the following relations

$$\frac{dx}{ds} = a(x(s), y(s)) \quad \frac{dy}{ds} = b(x(s), y(s)) \quad \frac{dz}{ds} = c(x(s), y(s)) \tag{7}$$

which ensures that M is tangent to the surface of solutions. This curve is known as the *integral curve* for the vector field M and in the case of PDEs

they are known as the *characteristic curves* of the PDE. The set of equations above are known as the *characteristic equations* of the PDE.
All in all, this method allows us to reduce a linear PDE to a set of ordinary differential equations which can be solved using the tools in the ODE. After finding such curves by solving the equations above, we can construct a solution by taking union of these characteristic curves to create a surface, which will provides us with *a solution* to the PDE.

3 LWR Traffic Flow Model

Focus of this section will be introducing and deriving the Lighthill-Whitham-Richards model (LWR Model) for the traffic flow. This will be done using the mathematical tools introduced in Section 2.

3.1 Conservation equation

We consider a lane with vehicles moving in one direction. It is assumed that all the vehicles are the same length which does not vary over time and that it has a continuous distribution, i.e. we neglect the discrete properties of vehicle flow. More precisely, let $N(x,t)$ be the number of all vehicles with position larger than x at time t. We will assume that $N \in \mathcal{C}^2$, which means it is twice differentiable with continuous derivatives up to the second order. Then, the number of cars between x and $x + \Delta x$ can be given as follows:

$$\Delta N_x(x,t) = N(x,t) - N(x + \Delta x, t) \tag{8}$$

Hence, as we look to a smaller region with letting $\Delta x \to 0$, we can define the linear density of vehicles at position x at time t as follows:

$$\rho(x,t) = \lim_{\Delta x \to 0} \frac{\Delta N(x,t)}{\Delta x} = -\frac{\partial N(x,t)}{\partial x}. \tag{9}$$

Similarly, the amount of vehicles that passed through x between times t and $t + \Delta t$ can be given as follows given that there are no cars travelling backwards:

$$\Delta N_t(x,t) = N(x, t + \Delta t) - N(x,t) \tag{10}$$

Similarly, as we look to a smaller time interval by letting $\Delta t \to 0$, we can define the flow rate $q(x,t)$ as the number of vehicles that pass through x per unit time at time t:

$$q(x,t) = \lim_{\Delta t \to 0} \frac{\Delta N_t(x,t)}{\Delta t} = \frac{\partial N(x,t)}{\partial t}. \tag{11}$$

Under the previous assumption that $N \in \mathcal{C}^2$, we have the following relation by the Clairaut's Theorem:

$$\frac{\partial^2 N(x,t)}{\partial x \partial t} = \frac{\partial^2 N(x,t)}{\partial t \partial x}. \tag{12}$$

Combining with the previous equations, we obtain the *conservation equation*:

$$\frac{\partial q}{\partial x} + \frac{\partial \rho}{\partial t} = 0. \tag{13}$$

We define the local speed of the cars at time t and position x as follows:

$$u(x,t) = \frac{q(x,t)}{\rho(x,t)}. \tag{14}$$

Let us consider this relation in edge cases. In the case where $q = 0$ which means that there are no flow, we need either speed or density to be zero. If $\rho = 0$, it means that there are no cars and if we have $u = 0$ then it means that a *jam density* $\rho = \rho_j$ is reached. Based on this observation, we conclude that $q = 0$ if $\rho = 0$ or $\rho = \rho_j$ and we make the assumption that speed of the vehicles is a known function of the local density, which gives us the following relation:

$$q(\rho) = u(\rho)\rho = Q(\rho). \tag{15}$$

Hence, we conclude that the density can vary between 0 and ρ_j and there is a critical density ρ_c between them for which the flow takes a maximum value. The final goal of our model is to find all the variables mentioned above at any position x at any time t given a inital condition $\rho(x,0) = \phi(x)$. This can be done through the tools provided in Section 2, but before doing so, let us discuss the implications of our final assumption.

3.2 Fundamental Diagrams

As previously stated, the LWR model is based on the assumption that traffic flow, density and speed are related as stated above. At the end of the last section, we assumed that the speed is a function of only density, making flow also a function of density. In this section we will discuss this and similar assumptions about macroscopic elements of traffic models that will help us solve the LWR. In traffic flow theory, the relations between macroscopic characteristics such as (15) are represented by *fundamental diagrams*.

There are several types of fundamental diagrams depending on which variables are we considering. Namely, we have: Speed-Density, Flow-Density and Speed-Flow diagrams. It is important to note that all three diagrams represent the same information - it is reasonable to assume that under the same conditions, drivers behave the same way -. We will focus on the Flow-Density diagram as it provides us with important insights about the model. The diagram will depend on plenty of factors, such as external conditions, the different properties of the road, the different types of drivers and vehicles as well as traffic regulations, but the main properties that interest us are mostly invariant to these changes. There are two main approaches to construction of the fundamental diagrams. The first one relies on driving behaviour theories (like the ones we use in the cellular automata section) of microscopic traffic models, whereas the second approach uses empirical data and curve fitting techniques. We will now look at an example of a fundamental diagram, and define the important details.
Let us consider the flow-density fundamental diagram below which plots $Q(\rho)$, the flow with respect to density. First, we will analyse the point at which the flow is zero. These are the points where either $\rho = 0$ which means that there are no cars, or $\rho = \rho_j$ which represents the jam density (the speed is zero). Now we will look at the point which maximizes the flow. We denote it as the critical density point $\rho = \rho_c$, the traffic flow will reach its maximum q_{max}. The region in which $\rho < \rho_c$, is what we refer to as 'free-flow region' because the nature of the traffic is such that the drivers do not influence each other, namely the traffic is free. On the other hand the region where the density is greater than the critical density is called congestion region, because here the traffic is dense enough for congestions to take place, making the drivers impact each other's behaviour.

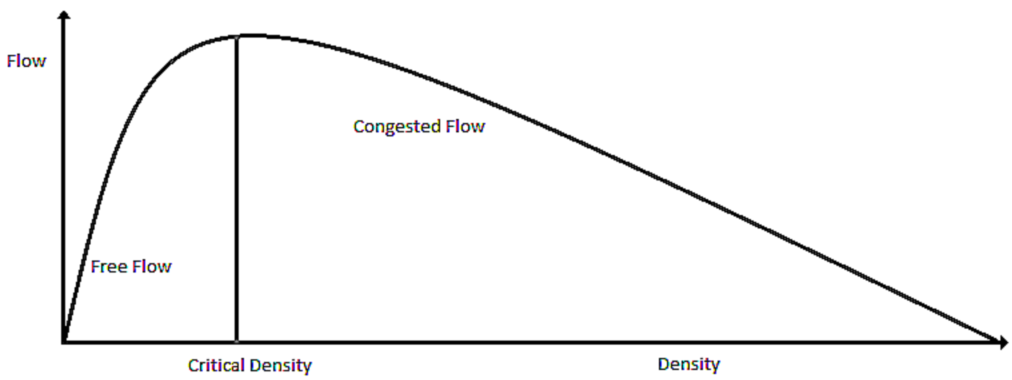

Figure 1: Flow - Density diagram

Now, we will try to incorporate (14) into the Flow-Density diagram and obtain additional information about the relations of the macro variables. The geometrical meaning of (14) is presented on the diagram below. The speed at density ρ can be interpreted as the slope of the line connecting the origin and the point $(\rho, Q(\rho))$ (satisfies (14)). At the uncongested part of the diagram, we notice that the speed does not change a lot, which can be explained by the assumption that when the road is almost empty every driver drives at the maximum speed allowed. We call this speed u_{max}, and graphicaly we can interpret it as the derivative of the $Q(\rho)$ at density 0. Contrary to that, on the congested part of the diagram even the slightest change in density changes the speed. This supports the intuition that in a near jam situation every increase in density heavily decreases the speed.

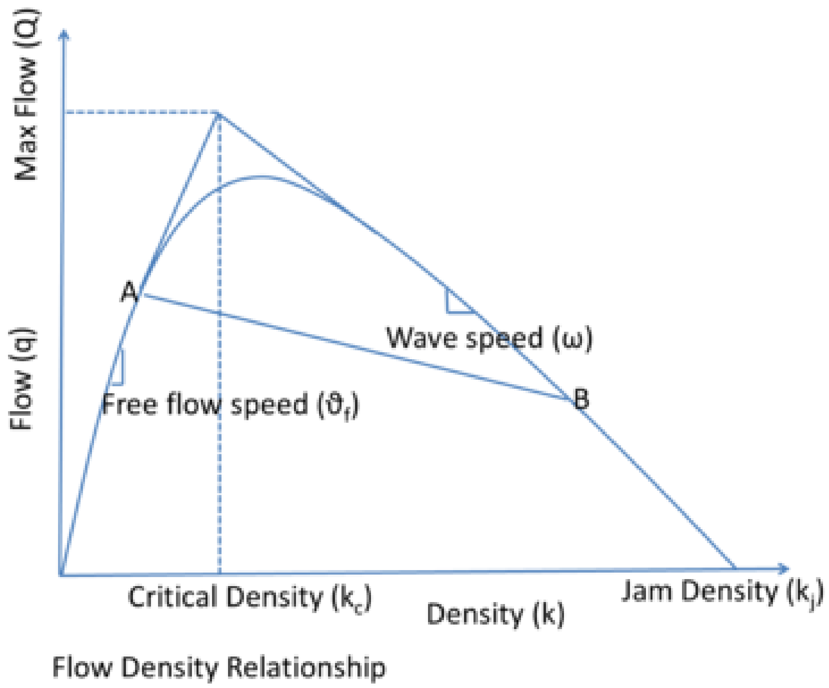

Figure 2: Speed and Wave speed

In the next section we explain the derivation of the fact that the "speed of information travelling" or wave speed can be seen as the derivative of the flow rate $Q'(\rho)$. In the Figures 1 and 2 we can see that the fundamental diagram can have different shapes. However, every fundamental diagram has certain characteristics that allow the results that we previously derived to always

hold. For example, the part of the diagram for low values of ρ is assumed to be straight because the speed is constant due to lack of traffic. Also, it is assumed that for the density less than the the critical value, that part of fundamental diagram is concave. Now, we will state one very important implications of the fundamental diagram that will primarily serve as a "Sanity check" for our model. From the assumption on concavity that we made, we see that the speed of the traffic $u(x,t)$ is always larger than the derivative of the flow rate $Q'(\rho)$. In the first part of the diagram this is the consequence of the concavity assumption, and on the second part the derivative is negative and slope is positive, so the inequality is trivial. This can be interpreted as the fact that traffic information never travels faster than the traffic that carries it, meaning that in general, nothing that happens behind a vehicle can affect the behaviour of that vehicle.
The analysis and the discussion above served as proof that if we make an assumption that $q = Q(\rho)$, with some restriction for Q, it will remain consistent with our intuition behind the model, and will not break simple logical rules of traffic. From this point onward, for the purpose of simulation, we will assume fundamental diagram in the triangular shape. This means that derivative of $Q(\rho)$ takes a constant values for each given region.

3.3 Analytical Solution for Non-Congested Case

In this section, we are going to theoretically solve the model for the uncongested part of the triangular fundamental diagram, which means that we will have a constant for $Q'(\rho)$ in the given region. The general case is hard to tackle theoretically so we provide a qualitative explanation based on the fundamental diagrams for the congestion.
Based on our assumptions in the LWR model, we have the three following equations for the analytical solution:

$$q(x,t) - \rho(x,t)u(x,t) = 0 \tag{16}$$

$$q(x,t) - Q(\rho(x,t)) = 0 \tag{17}$$

$$\frac{\partial q}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \tag{18}$$

By the chain rule, we have:

$$\frac{\partial q}{\partial x} = \frac{\partial q}{\partial \rho}\frac{\partial \rho}{\partial x} = \frac{dQ(\rho)}{d\rho}\frac{\partial \rho}{\partial x} = Q'(\rho)\frac{\partial \rho}{\partial x} \tag{19}$$

Hence, we can write our PDE as:

$$Q'(\rho)\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \tag{20}$$

This is exactly the same type of PDE we studied in the Section 2.3, so we utilize the Method of Characteristics again to reduce this PDE into a set of ODEs. We have

$$a(x,t) = Q'(\rho) \quad b(x,t) = 1 \quad c(x,t) = 0 \tag{21}$$

Hence, the set of ODEs to solve this PDE is given by:

$$\frac{dx}{ds} = Q'(\rho) \quad \frac{dt}{ds} = 1 \quad \frac{dz}{ds} = 0 \tag{22}$$

From this point we assume for theoretical ease that $Q'(\rho)$ takes a constant positive value $Q'(\rho) = a$. Then, these equations can be solved without much difficulty:

$$x(s) = as + c_1 \quad t(s) = s + c_2 \quad z(s) = c_3 \tag{23}$$

By eliminating s using this equations, we observe that the curves are defined by $x_0 = x - at$ and $z = k$ for some constants x_0, k. To find the integral surface of all of these curves, we first observe that z is constant along $x - at$, hence $z(x,t) = f(x - at)$. Then, by letting $\rho(x,t) = z(x,t) = f(x - at)$, for any smooth function f, we find a solution to PDE which can be verified as follows:

$$\rho_t + a\rho_x = -af'(x - at) + af'(x - at) = 0 \tag{24}$$

Given an inital condition $\rho(x,0) = \phi(x)$, we need to satisfy $\phi(x) = \rho(x,0) = f(x - a0) = f(x)$. Hence, given an inital condition, we have the following solution:

$$\rho(x,t) = \phi(x - at) \tag{25}$$

This means that under the given assumptions, the distribution of the cars will stay the same and move rightwards with velocity a! Due to this phenomenon, sometimes this speed a is known as the wave speed. For example, if $\phi(x) = e^{-x^2}$, we have the same normal distribution centered at $x = at$ at time t:

$$\rho(x,t) = e^{-(x-at)^2} \tag{26}$$

However, we have to be careful with our assumption that $Q'(\rho)$ is a positive constant, which is not the case in the fundamental diagrams. There are many shortcomings of the model due to the fluid-like assumption, which will be discussed more in detail in Discussion and Conclusion.

3.4 Numerical Solution for Non-Congested Case

In this section, we are going to introduce a numerical scheme to solve the following PDE that we found above as a result of our model:

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0 \tag{27}$$

where $Q'(\rho) = a$ is a constant as assumed in the model. Assume that we know the distribution at time t as $\rho(x, t)$. For sufficiently small Δt, we can write the above equation as follows:

$$\frac{\rho(x, t + \Delta t) - \rho(x, t)}{\Delta t} = -a\frac{\partial \rho}{\partial x} \tag{28}$$

Also, for sufficiently small Δx, we can approximate the partial derivative on the RHS as follows:

$$\frac{\partial \rho}{\partial x} = \frac{\rho(x + \Delta x, t) - \rho(x, t)}{\Delta x} \tag{29}$$

Hence, given the state $\rho(x, t)$, we can calculate the next step $\rho(x, t + \Delta t)$ numerically with the following formula:

$$\rho(x, t + \Delta t) = \rho(x, t) - a\frac{\rho(x + \Delta x, t) - \rho(x, t)}{\Delta x}\Delta t \tag{30}$$

Using such iterations and an inital condition for distribution given as $\rho(x, 0) = \phi(x)$, we can calculate the state of the system at any x and $t = N\Delta t$ for some $N \in \mathbf{N}$. Indeed, we obtain the following plot when we simulate a system with $\Delta x = 0.1$, $\Delta t = 0.001$, $a = 10$, and $\phi(x) = e^{-0.01x^2}$:

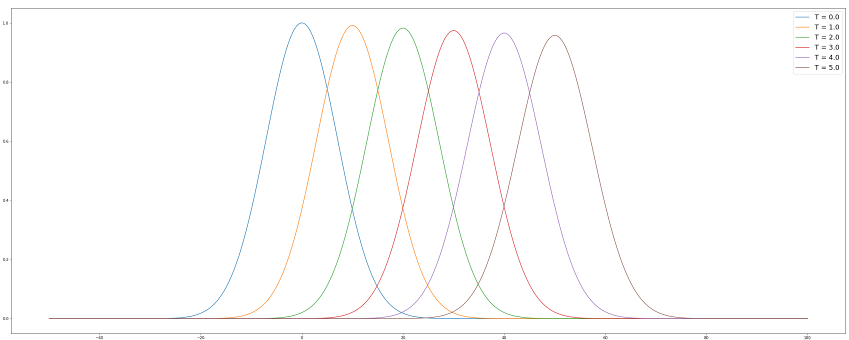

Figure 3: Numerical Solution

This results confirms our theoretical result that in the case of congestion the cars move forward with a given constant velocity.

3.5 How Does a Congestion Occur: Shock Waves

We have already established that flow, speed, and density are the main variables that uniquely define a state in our macroscopic model. In order to describe the transition from the flow-speed-density state to the time and space, we will use the analogy with the theory of Kinematic Waves. More precisely, in the boundary between the two different flow states, we observe what will be called a shock wave. Let us now consider what would happen in the traffic if one vehicle were to suddenly stop moving. We can describe this behaviour as a sudden change of Flow-Speed-Density states using the shock wave theory. As one vehicle stops, every vehicle behind it will have to stop and a traffic blockade occurs. On the following figure each curve represents one vehicle and its path in the (x, t) plane. We observe the influence that one car that stopped had on the traffic as a whole.

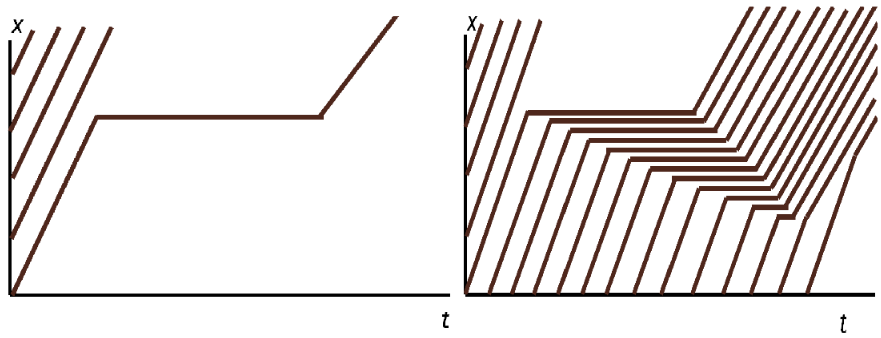

Figure 4: Shockwave Formation

The regions where density and flow are continuous can be identified on the following figure, with the boundaries between those regions representing shock waves. We differentiate between four areas and three shock waves. In the region 1 (Figure below) we see that each curve (vehicle) is moving freely. Then at time t_0, at position x_0, one vehicle stops and a blockade is formed. Area 2 is composed the vehicles that arrive to the tail of the blockade after t_0 and stop moving. Emptiness of area 3 represents the part of the road starting at x_0 after t_0 where the lack of traffic caused by the blockade. The line between areas 2 and 3 represents the "head of queue" shock wave, showing us the stationary position of the vehicle that created the blockade from t_1 to t_2. The line between areas 1 and 2 is called the "stop wave", and gives us the position of the tail of the blockade, meaning the last vehicle that is directly influenced by the blockade. Area 4 explains the behavior of the vehicles after t_1, the point where the front of blockade started moving. We notice that the distance between vehicles in area 4 is considerably smaller than in area 1. This follows from the assumption that the speed does not change in the first part of the fundamental diagram (free-flow state and full capacity state have roughly the same speed), so the vehicles will maintain the minimum distance allowing them to travel at maximum speed allowed. The line between areas 3 and 4 represents the "start wave", showing us the time and space where each care "left" the blockade by beginning to move. Time t_2, position x_2 represent a point where the "start wave" caught on to the "stop wave". This happened because the distance between vehicles in area 1 is bigger than the distance in area 3, thereby, we have more vehicles per second leaving the blockade, than joining it. We can interpret this as one shock wave having a bigger "speed" than the other. Speed of the shock wave is the speed at which the state is changing. To make it more clear, we will give an example of a shock wave. Consider a situation where a vehicle is on the road, and a crash happens at some distance in front. This causes a shok wave similar to the one we showed on the graph above. Naturally the driver will now see a wave of red break lights appearing in front. The speed at which the brake lights appeared to "travel" backwards is the speed of the shock wave.

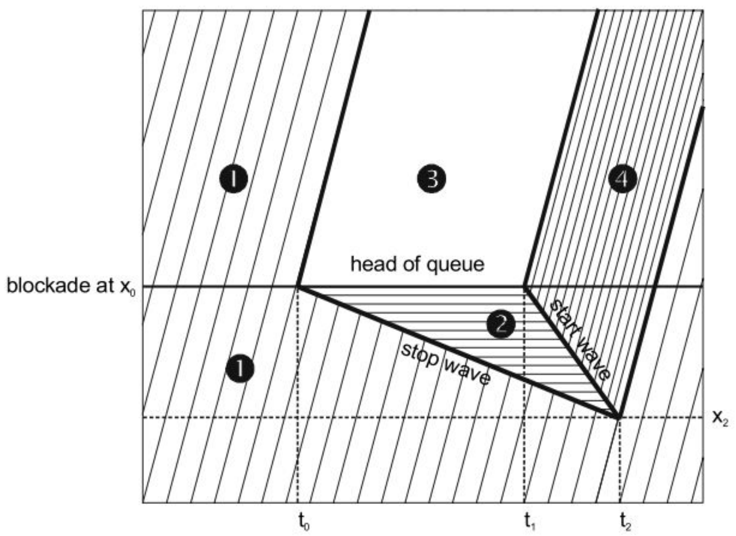

Figure 5

We will now consider an arbitrary shock wave and determine its speed. We remind ourselves that a shock wave separates two states which can be described with macro variables flow and desnity. Let the speed of the wave be ω and (ρ_1, u_1) be the state of the point at which the vehicle is entering the shock wave. Then the flow-rate equation representing the influence of the wave at that point is:

$$q_1 = \rho_1(u_1 - \omega) \tag{31}$$

Similarly, for a vehicle exiting the shockwave at state (ρ_2, u_2) the equation is:

$$q_2 = \rho_2(u_2 - \omega) \tag{32}$$

Since no vehicles disappear, flow into the shock must equal the flow out of the shock i.e $q_1 = q_2$. Thus we can conclude the speed of the shockwave equals to

$$\omega = \frac{q_1 - q_2}{\rho_1 - \rho_2}. \tag{33}$$

4 Case Study: Simulation for Fundamental Diagram

In this section, we are going to build a computational simulation to replicate the traffic flow using a cellular automata and observe what kind of fundamental diagram best reflects such a system's behaviour.

4.1 Cellular Automata

Cellular automata are discrete computational dynamical systems that typically consists of an infinite regular grid in which every cell is described by a state from a common finite set Ω. The states of the cells evolve deterministically and synchronously based on some common local update rules. Such structures emerged in the late 40s from the work of Ulam and Von Neumann and gained increasing popularity in modelling dynamical systems with the increasing computational power over the years. Most notable examples include the Conway's Game of Life as an example of Turing Complete system, urban evolution automaton to explain segregation in the neighbourhoods, and simulation of the Ising Model to explain interaction of a grid of magnetic dipoles.

The power of such cellular automata comes from the fact that setting very simple local rules for evolution can lead to complicated macro behaviour. For example, even though the Conway's Game of Life consists of very simple rules, it can be proven that it is Turing-complete which means that anything that can be calculated algorithmically on a Turing Machine can be computed in the Game of Life.

4.2 Cellular Automaton for Traffic Flow

Similarly, we will build a cellular automaton to model the traffic flow. Let us consider a grid of h rows and l columns which represents a piece of road with h lanes that can contain $h \times l$ cars in total. Let us place a coordinate system on the top left of this grid and locate each cell with coordinates $i \in X = \{0, ..., l-1\}$ and $j \in Y = \{0, ..., h-1\}$. Then, each cell at (i, j) takes a state $\omega_{ij} \in \Omega = \{0, 1\}$ where 1 denotes the car and 0 denotes empty place.

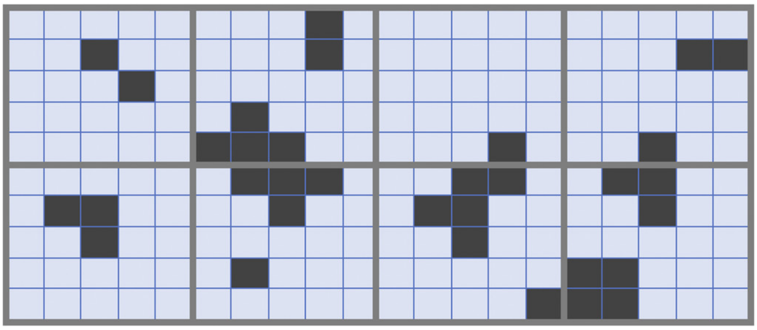

Figure 6: A Road with parameters $h = 10$, $l = 20$

Now, we need to find a transition rule about how these cars will move. To do so, we utilize a straightforward micro-behaviour: a car moves by one cell if the spot in front of it is empty and stays still otherwise. New cars keeps coming to the empty lanes at the beginning of the road with a random Bernoulli process with a given parameter p and cars freely exit at the end of the road. Although the idea is quite simple, we will formalize it as follows.
Let us denote the state of the cellular automata at time t by the function $s_t : (i, j) \to \Omega$ and assume that we know its inital value at $t = 0$ for all $i \in X$, $j \in Y$. Then, the transition rule to calculate s_{t+1} based on s_t is given as follows:

1. First, let us consider the cells (i, j) such that $i \in X/\{0, l-1\}$, $j \in Y$. We have $s_{t+1}(i, j) = 1$ if $s_t(i-1, j) = 1$ and $s_t(i, j) = 0$ or $s_t(i, j) = 1$ and $s_t(i+1, j) = 1$. Otherwise, $s_{t+1}(i, j) = 0$.

2. Secondly, let us consider the cells $(0, j)$ where $j \in Y$. If $s_t(0, j) = 1$ and $s_t(1, j) = 0$, then $s_{t+1}(0, j) = 0$. If $s_t(0, j) = 1$ and $s_t(1, j) = 1$, then $s_{t+1}(0, j) = 1$. Otherwise, in the case $s_t(0, j) = 0$, a car joins the traffic with a given probability p, i.e. we have $s_{t+1}(i, j) = \text{Binom}(p)$.

3. Finally, let us consider the cells $(l-1, j)$ where $j \in Y$. If $s_t(l-2, j) = 1$ and $s_t(l-1, j) = 0$, we have $s_{t+1}(l-1, j) = 1$. Otherwise $s_{t+1}(l-1, j) = 0$.

Note that all the phrases with both 'and' and 'or' operators are separated into two logical expressions by the phrase 'or', however, there are no parentheses in the writing for aesthetic purposes.
Since, we have everything ready, we can run the simulation to see what happens. In the simulation, we define the density to be the ratio of cells in the state 1 and the flow rate to be the number of cars that moved in the given density. We obtain the following plot for the Fundamental Diagram when we run the simulation with parameters $l = 100$, $h = 20$, and $p = 0.4$ for $N = 350$ rounds. We run 350 lines since we observe that the density starts to become constant around that point since we reach full saturation.

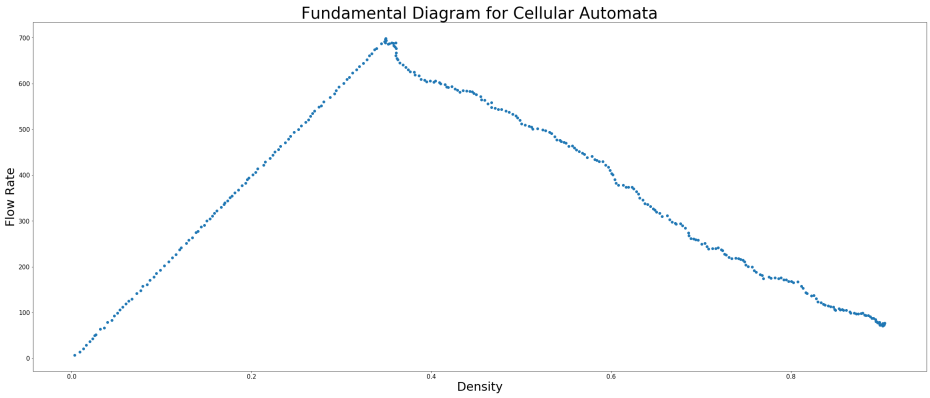

Figure 7: Fundamental Diagram of Cellular Automata

We observe that the cellular automata system results in a triangular fundamental diagram. This means that we can apply the results from the previous theoretical analysis!

5 Discussion and Conclusion

Although we were able to explain the traffic flow and how congestions occur based on some models that included PDEs and diagrams, it is worthwhile to discuss the shortcomings of the model and phenomenon it cannot explain well.
Our model implicitly assumes that vehicle flow is similar to fluid flow with some additional assumptions. When there is a hypothetical congestion in the fluid, it can possibly cause the fluid to flow backwards, which is explained

by the case in which a is a negative constant in the congested part of the triangular diagram. However, this is obviously not the case for vehicles and it is a shortcoming of our model. Indeed, when one has two regions, one that is above and one that is below the critical density, being next to each other, numerical solutions deteriorate and do not give meaningful results.
Even though we obtained a Fundamental Diagram for Cellular Automata, it doesn't mean that the theoretical results will hold with the simulation data. This is due to the fact that LWR is different from the cellular automata by its continuous nature. Hence, when we plot $\rho(x,t)$, we still see similar results but with small distortions due to the discrete reality of the simulation. Note that this can be largely avoided by using a very large grid and a huge spatial interval length for calculation of the density to minimize the effect of discretization.
To conclude, although our model doesn't explain all the phenomenon accurately, it reveals the most important fundamental results and provides a clear framework on how one can proceed to model traffic for further research in its optimization.

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