Analiză Curs 13.

Funcțiile gama și beta ale lui Euler.

Teorema 1. Pentru orice $p\in(0,\infty)$ integrala improprie $\int_{0+0}^{\infty} x^{p-1}e^{-x}dx$ este convergentă

Teorema 2. Pentru orice $p,q\in(0,\infty)$ integrala improprie $\int_{0+0}^{1-0} x^{p-1}(1-x)^{q-1}dx$ este convergentă.

Definiția 1: a) Funcția $\Gamma:(0,\infty)\to\mathbb{R}$ definită prin $\Gamma(p)=\int_{0+0}^{+\infty} x^{p-1}e^{-x}dx$ se numește funcția gama a lui Euler.

b) Funcția $B:(0,\infty)\times(0,\infty)\to\mathbb{R}$ definită prin $B(p)=\int_{0+0}^{1+0} x^{p-1}(1-x)^{q-1}dx$ se numește funcția beta a lui Euler.

Proprietățile funcției Γ.

1) $\Gamma(p)>0\ \forall p\in(0,\infty)$

2) $\Gamma(p+1)=p\Gamma(p)\ \forall p\in(0,\infty)$

3) $\Gamma(1)=\int_0^{\infty} e^{-x}dx=(-e^{-x})\Big|_0^{\infty}=\lim_{x\to\infty}-e^{-x}+1=1.$

$\Gamma(2)=\Gamma(1+1)=1\cdot\Gamma(1)=1$

$\Gamma(3)=\Gamma(2+1)=2\cdot\Gamma(2)=2$

$\Gamma(4)=\Gamma(3+1)=3\cdot\Gamma(3)=3\cdot2=3!$

$\Gamma(m+1)=m!\ \forall m\in\mathbb{N}.$

4) $\Gamma(p)\cdot\Gamma(1-p)=\frac{\pi}{\sin(p\cdot\pi)}\quad\forall p\in(0,1)$

Exercițiul 1 $\Gamma(\frac{1}{2})=?$

$p=\frac{1}{2}\overset{(4)}{\Longrightarrow}\Gamma(\frac{1}{2})\cdot\Gamma(\frac{1}{2})=\frac{\pi}{\sin\frac{\pi}{2}}\Rightarrow\Gamma^2(\frac{1}{2})=\pi\Rightarrow\Gamma(\frac{1}{2})=\sqrt{\pi}$ sau $\Gamma(\frac{1}{2})=-\sqrt{\pi}\overset{(1)}{\Longrightarrow}\boxed{\Gamma(\frac{1}{2})=\sqrt{\pi}}$

Exercițiul 2: $\Gamma(\frac{m+1}{2})=?\ \forall m\in\mathbb{N}$

Cazul 1: $m=2k+1,\ k\in\mathbb{N}$. $\Gamma(\frac{m+1}{2})=\Gamma(k+1)=k!$

Cazul 2: $m=2k\quad k\in\mathbb{N}$. $\Gamma(\frac{m+1}{2})=\Gamma(\frac{2k+1}{2})=\Gamma(k+\frac{1}{2})=\Gamma(1+\frac{2k-1}{2})=$

$=\frac{2k-1}{2}\cdot\Gamma(\frac{2k-1}{2})=\frac{2k-1}{2}\Gamma(1+\frac{2k-3}{2})=\frac{2k-1}{2}\cdot\frac{2k-3}{2}\cdot\Gamma(\frac{2k-3}{2})=\dots=$

$=\frac{(2k-1)(2k-3)(2k-5)\dots3\cdot1}{2k}\cdot\Gamma(\frac{1}{2})=\frac{1\cdot3\cdot5\cdot\dots\cdot(2k-1)}{2k}\cdot\sqrt{\pi}$

Proprietățile funcției B

1) $B(p,q)>0\ \forall p,q\in(0,\infty)$

2) $B(p,q)=B(q,p)\ \forall p\in(0,\infty)$

3) $B(p+1,q)=\frac{p}{p+q}B(p,q)$

$B(p,q+1)=\frac{q}{p+q}B(p,q)\ \forall p,q\in(0,\infty)$

4) $B(p,q)=2\int_{0+0}^{\frac{\pi}{2}-0}\sin^{2p-1}x\cos^{2q-1}x\,dx.$

5) $B(p,q)=\int_{0+0}^{+\infty}\frac{x^{p-1}}{(1+x)^{p+q}}dx$

Formula de legătură între Γ și B

$$B(p,q)=\frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)} \quad \forall p,q\in(0,\infty)$$

Exercițiu. $\int_0^\infty e^{-x^2}dx = I$

Schimbare de variabilă $x^2=t \Rightarrow 2x\,dx=dt$. sau $x=\sqrt{t}\Rightarrow dx=(\sqrt{t})'dt$.
$dx=\frac{1}{2\sqrt{t}}dt$. $\quad x=0\Rightarrow t=0 \quad ; \quad x\to\infty\Rightarrow t\to\infty$.

$$I=\int_{0+0}^\infty \frac{e^{-t}}{2\sqrt{t}}dt=\frac{1}{2}\int_{0+0}^\infty t^{-\frac{1}{2}}\cdot e^{-t}dt.$$

$p-1=-\frac{1}{2}\Rightarrow p=\frac{1}{2}\Rightarrow I=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$

Exercițiu $\quad I=\int_0^{\pi/2}\sin^7x\cos^8x\,dx,\quad J=\int_0^{\frac{\pi}{2}-0}\sqrt{tgx}.$

$\begin{cases}2p-1=7\\2q-1=8\end{cases}\Rightarrow \begin{matrix}p=4\\q=\frac{9}{2}\end{matrix} \Rightarrow I=\frac{1}{2}B\left(4,\frac{9}{2}\right)=\frac{1}{2}\cdot\frac{\Gamma(4)\cdot\Gamma(\frac{9}{2})}{\Gamma(4+\frac{9}{2})}=\frac{1}{2}\cdot\frac{3!\cdot\Gamma(\frac{9}{2})}{\Gamma(\frac{17}{2})}=$

$$\Gamma\left(\frac{17}{2}\right)=\Gamma\left(1+\frac{15}{2}\right)=\frac{15}{2}\cdot\Gamma\left(\frac{15}{2}\right)=\frac{15}{2}\cdot\Gamma\left(1+\frac{13}{2}\right)=\frac{15}{2}\cdot\frac{13}{2}\Gamma\left(\frac{13}{2}\right)=\frac{15}{2}\cdot\frac{13}{2}\cdot\frac{11}{2}\cdot\frac{9}{2}\cdot\Gamma\left(\frac{9}{2}\right)$$

$$I=\frac{\frac{1}{\cancel{2}}\cdot\overset{3}{\cancel{6}}\cdot\cancel{\Gamma(\frac{9}{2})}}{\frac{9\cdot11\cdot13\cdot15\cdot\cancel{\Gamma(\frac{9}{2})}}{2^4}}=\frac{\cancel{3}\cdot2^4}{\underset{3}{\cancel{9}}\cdot11\cdot13\cdot15}=\frac{16}{33\cdot195}$$

$15\cdot13=195$

$$J=\int_0^{\frac{\pi}{2}-0}\sqrt{\frac{\sin x}{\cos x}}dx=\int_0^{\frac{\pi}{2}-0}\frac{\sin^{\frac{1}{2}}x}{\cos^{\frac{1}{2}}x}dx=\int_0^{\frac{\pi}{2}-0}\sin^{\frac{1}{2}}x\cdot\cos^{-\frac{1}{2}}x\,dx$$

$2p-1=\frac{1}{2}\Rightarrow p=\frac{3}{4}$
$2q-1=-\frac{1}{2}\Rightarrow q=\frac{1}{4}$
$\Rightarrow J=\frac{1}{2}\cdot B\left(\frac{3}{4},\frac{1}{4}\right)=\frac{1}{2}\frac{\Gamma(\frac{3}{4})\cdot\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4}+\frac{1}{4})}=\frac{1}{2}\cdot\frac{\Gamma(\frac{3}{4})\cdot\Gamma(\frac{1}{4})}{\Gamma(1)}=\frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$

$$J=\frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(1-\frac{3}{4}\right)=\frac{1}{2}\cdot\frac{\pi}{\sin\frac{3\pi}{4}}=\frac{1}{2}\cdot\pi\cdot\frac{2}{\sqrt{2}}=\frac{\pi}{\sqrt{2}}$$

Exercițiu: $\quad I=\int_0^{2-0}\frac{x^2}{\sqrt{2-x}}dx \qquad [0,2)\longrightarrow[0,1)$

Schimbare de variabilă: $x=2t\Rightarrow dx=2dt$.
$x=0\Rightarrow t=0 \quad x\nearrow2\Rightarrow t\nearrow1 \quad t\nearrow1\Rightarrow$

$$\Rightarrow I=\int_0^{1-0}\frac{4t^2}{\cancel{\sqrt{2-2t}}\ \sqrt{1-t}}\cdot\overset{\sqrt{2}}{\cancel{2}}dt=4\sqrt{2}\cdot\int_0^{1-0}\frac{t^2}{\sqrt{1-t}}dt=4\sqrt{2}\int_0 t^2(1-t)^{-\frac{1}{2}}dt.$$

$2p-1=2\Rightarrow p=\frac{3}{2}$
$2q-1=-\frac{1}{2}\Rightarrow q=\frac{1}{2}$
$\Rightarrow I=4\sqrt{2}\cdot B\left(3,\frac{1}{2}\right)=4\cdot\sqrt{2}\frac{\Gamma(3)\cdot\Gamma(\frac{1}{2})}{\Gamma(3+\frac{1}{2})}=4\sqrt{2}\cdot\frac{2!\cdot\sqrt{\pi}}{\Gamma(\frac{7}{2})}$

$$\Gamma\left(\frac{7}{2}\right)=\Gamma\left(\frac{5}{2}+1\right)=\frac{5}{2}\cdot\Gamma\left(\frac{5}{2}\right)=\frac{5}{2}\cdot\left(1+\frac{3}{2}\right)=\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{15}{8}\cdot\sqrt{\pi}$$

$$I=\frac{4\sqrt{2}\cdot2\cdot\cancel{\sqrt{\pi}}}{\frac{15}{8}\cdot\cancel{\sqrt{\pi}}}=\frac{64\sqrt{2}}{15}.$$

$$\int\frac{(2-t)^2}{\sqrt{t}}(-dt)=-\int\frac{4-4t+t^2}{\sqrt{t}}dt=$$