Geometrie
S5

Ex1:

$q(x) = 2x_1x_2 + x_2^2 - x_1x_3 + \frac{3}{4}x_3^2$

coef lui x_1^2 ; coef lui $x_1x_2/2$

$$A = \begin{array}{c} \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{4} \end{pmatrix} \begin{array}{l} \leftarrow x_1 \\ \leftarrow x_2 \\ \leftarrow x_3 \end{array} \\ \uparrow \quad \uparrow \quad \uparrow \\ x_1 \quad x_2 \quad x_3 \end{array}$$

$(i, j) \to$ coef lui $\frac{x_ix_j}{2}$, $i \neq j$

x_i^2 $i = j$

Metoda Jacobi

$\Delta_1 = 0$

$\Delta_2 = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$

$\Delta_3 = \begin{vmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{4} \end{vmatrix} = -\frac{1}{4} - \frac{3}{4} = -1 = \det A$

Dacă tot Δ ~~sunt nenuli (sau~~ sunt > 0)

$q(x') = \frac{1}{\Delta_1}(x_1')^2 + \frac{\Delta_1}{\Delta_2}(x_2')^2 + \frac{\Delta_2}{\Delta_3}(x_3')^2$

- $q(x) = x_1^2 + x_3^2 + 4x_1x_2 - 4x_1x_3$

$A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

$\Delta_1 = 1$

$\Delta_2 = -4$

$\Delta_3 = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 0 & 0 \\ -2 & 0 & 1 \end{vmatrix} = -4$

$q(x) = x_1^2 - x_3^2 - 4x_1x_2 - 4x_1x_3$

$A = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix}$

$\Delta_1 = 1$

$\Delta_2 = 4$

$\Delta_3 = 4$

$$q(x') = \frac{1}{\Delta_1}(x_1')^2 + \frac{\Delta_1}{\Delta_2}(x_2')^2 + \frac{\Delta_2}{\Delta_3}(x_3')^2$$

$$= (x_1')^2 + \frac{1}{4}(x_2')^2 + \cancel{}\ 4\,(x_3')^2 \quad \leftarrow \text{forma canonică}$$

$\exists \{f_1, \dots, f_n\}$ bază a.î. $x = \sum_{i=1}^{n} x_i' f_i$

$f_i = c_{1i} e_1 + \cancel{} + c_{2i} e_2 + c_{3i} e_3 + \dots c_{i,i} e_i$

unde
$$\begin{pmatrix} a_{11} & \dots & a_{1i} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{ii} \end{pmatrix}\begin{pmatrix} c_{1i} \\ \vdots \\ c_{ii} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$\exists \{f_1, f_2, f_3\}$ bază în $\mathbb{R}^3$

$f_1 = c_{11} \cdot e_1$

$$\left.\begin{array}{l} a_{11} \cdot c_{11} = 1 \\ a_{11} = 1 \end{array}\right| \Rightarrow c_{11} = 1$$

$f_1 = e_1 = (1, 0, 0)$

$f_2 = c_{21} \cdot e_1 + c_{22} e_2 = -\frac{1}{2}(1,0,0) - \frac{1}{4}(0,1,0) = (-\frac{1}{2}, -\frac{1}{4}, 0)$

$$\begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$c_{21} - 2c_{22} = 0$

$-2c_{21} = 1 \qquad \Rightarrow c_{21} = -\frac{1}{2}$

$c_{22} = -\frac{1}{4}$

$f_3 = c_{31} e_1 + c_{32} e_2 + c_{33} e_3$

$$\begin{pmatrix} 1 & -2 & -2 \\ -2 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_{31} \\ c_{32} \\ c_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$c_{31} - 2c_{32} - 2c_{33} = 0$

$-2c_{31} = 0 \Rightarrow c_{31} = 0$

$-2c_{31} - c_{33} = 1 \Rightarrow c_{33} = -1$

$-2c_{32} = -2 \Rightarrow c_{32} = 1 \qquad f_3 = (0, 1, -1)$

$(x_1, x_2, x_3) = x_1' \cdot f_1 + x_2' \cdot f_2 + x_3' \cdot f_3 = x_1'(1,0,0) + x_2'(-\frac{1}{2}, -\frac{1}{4}, 0) + x_3'(0,1,-1) =$

$= (x_1' - \frac{1}{2}x_2', -\frac{1}{4}x_2' + x_3', -x_3')$

$x_1 = x_1' - \frac{1}{2}x_2'$

$x_2 = -\frac{1}{4}x_2' + x_3' \qquad \Rightarrow x_2' = -4x_2 - 4x_3$

$x_3 = -x_3' \qquad \Rightarrow x_3' = -x_3$

$\Rightarrow x_1' = x_1 + \frac{1}{2}(-4x_2 - 4x_3) = x_1 - 2x_2 - 2x_3$

- V sp vect$/\mathbb{R}$

  $\langle , \rangle : V \times V \to \mathbb{R}$ biliniară

  $\langle x, y \rangle = \langle y, x \rangle$ (simetrică)

  $\langle x, x \rangle \geq 0$

  $\langle x, x \rangle = 0 \Rightarrow x = 0_V$

- $\mathbb{R}^n$, $x = (x_1, \dots x_n)$
  $y = (y_1, \dots y_n)$

  $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

- $\mathcal{C}([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f\text{-cont}\}$ ← fc. continue pe intervalul $[a,b]$

  $\langle f, g \rangle = \int_a^b f(x) g(x) dx$

  $\langle f + f', g \rangle = \langle f, g \rangle + \langle f', g \rangle$

  $\langle f + f', g \rangle = \int_a^b [f(x) + f'(x)] \cdot g(x) dx = \int_a^b [f(x) g(x) + f'(x) g(x)] dx =$

  $= \int_a^b f(x) g(x) dx + \int_a^b f'(x) g(x) dx = \langle f, g \rangle + \langle f', g \rangle$

  $\langle f, g + g' \rangle = \langle f, g \rangle + \langle f, g' \rangle$

  $\langle \alpha \cdot f, g \rangle = \alpha \cdot \langle f, g \rangle$

  $\alpha \in \mathbb{R}$

$$\langle f,f\rangle = \int_a^b f(x) f(x)\, dx = \int_a^b f^2(x)\, dx \quad \Big| \Rightarrow \int_a^b f^2(x)\, dx \geq 0$$
$$f^2(x) \geq 0$$

$$\langle f,f\rangle = 0 \Leftrightarrow \int_a^b f^2(x)\, dx = 0$$

$$\int_a^b f^2(x) = 0$$

$f : [a,b] \to [0,\infty)$ cont

$\exists x_0 \in [a,b],\ f(x_0) > 0$

$\Rightarrow \int_a^b f(x)\, dx > 0$

$f^2(x) \geq 0 \quad \forall x \in [a,b]$

Pp $f \not\equiv 0 \Rightarrow \exists x_0 \in [a,b]$ a.î. $f(x_0) \neq 0 \Rightarrow f^2(x_0) > 0$

$\Rightarrow \underbrace{\int_a^b f^2(x)}_{=0} > 0$ $\cancel{\alpha}$ $\Rightarrow f \equiv 0$

- Norma

$\|\ \| : V \to \mathbb{R}$

$\|x\| \geq 0\ \forall x \in V$

$\|x\| = 0 \Leftrightarrow x = 0_V$

$\|\alpha x\| = \alpha \cdot \|x\|,\ \forall \alpha \in \mathbb{R},\ \forall x \in V$

$\|x+y\| \leq \|x\| + \|y\| \quad \forall x,y \in V$

$\|x\|^2 = \langle x,x\rangle$

$\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$

$x \perp y \Leftrightarrow \langle x,y\rangle = 0$
(ortogonali)

$\|x\| = 1 \quad x$ s.n. vector normat

$x \in \mathbb{R}^2,\quad x = (2,1)$

$\|x\| = \sqrt{2^2+1^2} = \sqrt{5}$

$x' = \frac{1}{\|x\|} \cdot x = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \qquad \|x'\| = 1$

Teorema:

$\{e_1, \dots e_n\}$ bază a lui $V \Rightarrow \exists \cdot \{f_1', \dots f_n'\}$ baza ortonormată

(i.e. : $\langle f_i', f_j' \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ vreau să ajung la baza asta

Alg Gram - Schmidt

Pas 1 : Găsesc o bază ortogonală

$f_1 = e_1$

$f_2 = e_2 + \alpha_{21} \cdot f_1$

$\langle f_2, f_1 \rangle = 0 \Rightarrow \langle e_2 + \alpha_{21} \cdot f_1, f_1 \rangle = 0$

$\langle e_2, f_1 \rangle + \alpha_{21} \langle f_1, f_1 \rangle = 0$

$\alpha_{21} = \dfrac{-\langle e_2, f_1 \rangle}{\langle f_1, f_1 \rangle}$

$f_2 = e_2 - \dfrac{\langle e_2, f_1 \rangle}{\langle f_1, f_1 \rangle} \cdot f_1$

$f_i = e_i - \dfrac{\langle e_i, f_1 \rangle}{\langle f_1, f_1 \rangle} \cdot f_1 - \dots - \dfrac{\langle e_i, f_{i-1} \rangle}{\langle f_{i-1}, f_{i-1} \rangle} \cdot f_{i-1}$

$f_i \perp f_j \quad \forall i \neq j$

$f_i' = \dfrac{1}{\|f_i\|} \cdot f_i \Rightarrow \| f_i' \| = 1$

Ex

$B = \{ e_1 = (1,1,1), e_2 = (0,1,1), e_3 = (0,0,1) \}$ bază în $\mathbb{R}^3$

Găsiți o baza ortogonală a lui $\mathbb{R}^3$

$f_1 = e_1 = (1,1,1)$

$f_2 = e_2 - \dfrac{\langle e_2, f_1 \rangle}{\langle f_1, f_1 \rangle} \cdot f_1 = (0,1,1) - \dfrac{\overset{0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1}{\langle (0,1,1), (1,1,1) \rangle}}{\langle (1,1,1), (1,1,1) \rangle} \cdot (1,1,1) =$

$= (0,1,1) - \dfrac{\cancel{\langle (0,1,1) \rangle}}{\cancel{\langle (1,1,1) \rangle}} \dfrac{2}{3} \cdot (1,1,1) = (0,1,1) - \left(\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right) = \left(-\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$

E

$$f_3 = e_3 - \frac{\langle e_3, f_1\rangle}{\langle f_1, f_1\rangle}\cdot f_1 - \frac{\langle e_3, f_2\rangle}{\langle f_2, f_2\rangle}\cdot f_2 =$$

$$= (0,0,1) - \frac{\langle (0,0,1),(1,1,1)\rangle}{\langle (1,1,1),(1,1,1)\rangle}\cdot(1,1,1) - \frac{\langle (0,0,1),(-\frac{2}{3},\frac{1}{3},\frac{1}{3})\rangle}{\langle (-\frac{2}{3},\frac{1}{3},\frac{1}{3}),(-\frac{2}{3},\frac{1}{3},\frac{1}{3})\rangle}\cdot\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right) =$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{\frac{1}{3}}{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}\cdot\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right) =$$

$$= \left(-\frac{1}{3},-\frac{1}{3},\frac{2}{3}\right) - \frac{1}{\cancel{3}}\cdot\frac{\cancel{9}\,3}{\cancel{6}\,2}\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right) =$$

$$= \left(-\frac{1}{3},-\frac{1}{3},\frac{2}{3}\right) - \left(-\frac{1}{3},\frac{1}{6},\frac{1}{6}\right) = \left(0,-\frac{3}{6},\frac{3}{6}\right) = \left(0,-\frac{1}{2},\frac{1}{2}\right)$$

$$\|f_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\|f_2\| = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}}$$

$$\|f_3\| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}}$$

$$f_1' = \frac{1}{\sqrt{3}}\cdot(1,1,1) = \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

$$f_2' = \frac{1}{\sqrt{\frac{2}{3}}}\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right) = \frac{\sqrt{3}}{\sqrt{2}}\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right) = \left(-\frac{2\sqrt{3}}{3\sqrt{2}},\frac{\sqrt{3}}{3\sqrt{2}},\frac{\sqrt{3}}{3\sqrt{2}}\right) = \left(-\frac{\sqrt{2}}{\sqrt{3}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$$

$$f_3' = \sqrt{2}\left(0,-\frac{1}{2},\frac{1}{2}\right) = \left(0,-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$$

- $x = (x_1, x_2, x_3)$

$y = (y_1, y_2, y_3)$

$y_1\cdot\vec{i} + y_2\cdot\vec{j} + y_3\cdot\vec{k}$

$$\underset{\text{prod vect}}{x \times y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (\ldots)\vec{i} + (\ldots)\vec{j} + (\ldots)\vec{k}$$