Curs 10
Algebră

Algebră liniară K - corp comutativ, $n \in \mathbb{N}; n \geq 2$

$A \in M_n(K)$

Polinomul $f(x) = \det(xI_n - A) \in K[x]$ s.n. polin. caract. al mat. A

↳ polinom de grad n cu coef. în K

Teoremă: (Hamilton - Cayley)

$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ $\quad \boxed{f(A) = O_n}$.

$f(A) \underset{def}{=} A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 \cdot I_n$

Ex teoremă: $A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$; $f(x) = \det(xI_2 - A) = \det\begin{pmatrix} x-1 & -1 \\ -1 & x-3 \end{pmatrix} = (x-1)(x-3) - 1$

$f(x) = x^2 - 4x + 2$ -> polin. caract. al mat. A

$A^2 - 4A + 2I_2 = O_2$ $\quad$ **t.H.C** $\quad$ Obs

$A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 10 \end{pmatrix}$ $\quad$ 1) $\boxed{a_{n-1} = -\operatorname{tr} A}$ -> urma matricii A

$\begin{pmatrix} 2 & 4 \\ 4 & 10 \end{pmatrix} - \begin{pmatrix} 4 & 4 \\ 4 & 12 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = O_2$ $\quad$ $\operatorname{tr} A = a_{11} + a_{22} + \dots + a_{nn}$

$\operatorname{tr} A = 1 + 3 = 4$ $\quad$ 2) $\boxed{a_0 = (-1)^n \det A}$

$\det A = 1 \cdot 3 - 1 \cdot 1 = 2$

$$f(x) = \det\begin{pmatrix} x - a_{11}, & -a_{12}, & \dots, & -a_{1n} \\ -a_{21}, & x - a_{22}, & \dots, & a_{2n} \\ \vdots & & & \\ -a_{n1}, & \dots & \dots, & x - a_{nn} \end{pmatrix} = +(x - a_{11})(x - a_{22}) \dots (x - a_{nn}) - \dots$$

Polinomul minimal al unei matrici

Def: S.n. —"— : g - pol. min. al mat. A; $g \in K[x]$, g monic;

$A \in M_n(K)$ $\quad g(A) = O_n$

$\nexists\, h \in K[x], h \neq 0$, grad $h <$ grad g a.î. $h(A) = O_n$ (Existǎ g)

T: $h \in K[x], h(A) = O_n \Rightarrow g \mid h$ (în inelul $K[x]$ $\quad g = h \cdot h_1$) $h_1 \in K[x]$

Dem: T.Î.R. $h : g$; $h = g \cdot q + r$; $q, r \in K[x]$; $\boxed{\text{grad } r < \text{grad } g}$ $O_n = h(A) = g(A) \cdot q(A) + r(A)$

$O_n = r(A) \Rightarrow r = 0$ (def lui g)

Consecințe: $g \mid f$; f ↳ pol caract al matricii ($xI_n - A$)
g ↳ polin minimal al lui A

Teoremă: (FROBENIUS)

g și f au aceeași factori ireductibili

Memento: $h \in K[x]$, h - polinom ireduct
grad $h \geq 1$
$h = h_1 \cdot h_2$; $h_1, h_2 \in K[x]$ $\Rightarrow$ grad $h_1 = 0$ sau grad $h_2 = 0$

Valori proprii ale matricii A

+ Cas $\Rightarrow$ $\exists$. L corp comut, $K \subseteq L$ a.î. f are n răd. în L ;

$x_1, \dots, x_n$ = valori proprii ale mat A

$f(x) = \det(xI_n - A) = (x - x_1)(x - x_2)\dots(x - x_n)$

$x_j \in L \ \forall j = \overline{1,n}$

$(-1)^n x_1 x_2 \dots x_n = (-1)^n \det A$

$-(x_1 + \dots + x_n) = -\operatorname{Tr} A$

$\boxed{x_1 x_2 \dots x_n = \det A}$

$\boxed{x_1 + x_2 + \dots + x_n = \operatorname{Tr} A}$ $= a_{11} + a_{22} + \dots + a_{nn}$

Problemă: K corp finit (comutativ) ; $\mathbb{N} \ni n \geq 2$

$|\underbrace{U(M_n(K))}_{\text{elem inversabile din inelul } M_n(K)}| = ?$

□ Alg. liniară

T: $\forall$ corp finit este comutativ

$|U(M_2(\mathbb{Z}_{26}))| = 32 \cdot 168 \cdot 156$
↳ nu e corp

$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \in U(M_n(K))$

K corp comut

$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| x_j \in K \ \forall j = \overline{1,n} \right\}$

V - sp. vect. de dimensiune n peste K ;

$(V, +)$ ⊕ $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$

⊙ $\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$

$e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \to j$

$e_1, \dots, e_n$ bază pt V/K

T: $c_1, c_2, \dots, c_n$ formează o bază a lui $V \Leftrightarrow \det A \neq 0$

$\det A = 0$
$\Leftrightarrow \exists \alpha_1, \alpha_2, \dots, \alpha_n \in K$ nu toți 0
a.î. $\alpha_1 c_1 + \alpha_2 c_2 + \dots + \alpha_n c_n = 0$

$0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Algebră (10.2

Continuare pb:

În câte moduri pot alege prima coloană? $(2^n - 1)$

$\boxed{\lambda c_1}$ După ce am fixat prima coloană, în câte feluri pot alege coloana a II-a?

$(2^n - 2)$

$C_3 \neq \lambda_1 C_1 + \lambda_2 C_2$ După ce am fixat primele 2 coloane, — // — a III-a?

$(2^n - 2^n)$

$C_n \neq \lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3 + \dots + \lambda_{n-1} C_{n-1}$

$|U(M_n(K))| = (2^n-1)(2^n-2)(2^n-2^2)\dots(2^n-2^{n-1})$

Exemplu: $K = \mathbb{Z}_2$ $n=2$

$|M_2(\mathbb{Z}_2)| = 16$

$q=2;\ n=2$

$(2^2-1)(2^2-2)=6$

$\to$ $\begin{pmatrix}0&0\\0&1\end{pmatrix}$ $\begin{pmatrix}0&0\\1&1\end{pmatrix}$ ... $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ ... $\begin{pmatrix}0&1\\1&1\end{pmatrix}$

$\begin{pmatrix}1&1\\1&1\end{pmatrix}$

$|U(M_2(\mathbb{Z}_{26}))| = ?$

$|\mathbb{Z}_{26}| = |\mathbb{Z}_2 \times \mathbb{Z}_{13}| = 26$ $\to$ au același cardinal $\Rightarrow$ pot să arăt doar inj // surj;

$\mathbb{Z}_{26} \simeq \mathbb{Z}_2 \times \mathbb{Z}_{13}$

R, S inele

$R \times S = \{(r, s) \mid r \in R, s \in S\}$

$(0_R, 0_S)$ | $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$

$(1_R, 1_S)$ | $(r_1, s_1) \cdot (r_2, s_2) = (r_1 r_2, s_1 s_2)$

$f: \mathbb{Z}_{26} \to \mathbb{Z}_2 \times \mathbb{Z}_{13}$

morfism
$$\begin{cases} f(\bar{x}) = (\hat{x}, \tilde{x}) \\ f(\bar{x_1} + \bar{x_2}) = f(\bar{x_1}) + f(\bar{x_2}) \\ f(\bar{x_1} \cdot \bar{x_2}) = f(\bar{x_1}) \cdot f(\bar{x_2}) \end{cases}$$

$\underline{f \text{ inj}}$

$(\hat{x}, \tilde{x}) = f(\bar{x}) = f(\bar{y}) = (\hat{y}, \tilde{y})$

f bij. $\Leftrightarrow$ f inj $\Leftrightarrow$ f surj.

$f: R \to K_1 \times K_2$; f izomorf. de inele

$R \simeq K_1 \times K_2$ $\quad$ K_1, K_2 corpuri

(Ex) $M_n(R) \simeq M_n(K_1) \times M_n(K_2)$

$A = \begin{pmatrix} r_{11} & \dots & r_{1n} \\ \vdots & & \\ r_{n1} & \dots & r_{nn} \end{pmatrix}$ $\quad$ $g(A) = (B, C)$ $\quad$ g izom. de inele

$f(r_{ij}) = (\underset{\in K_1}{k_{ij}}, \underset{\in K_2}{t_{ij}})$

$B = (k_{ij})_{1 \le i,j \le n}$

$C = (t_{ij})_{1 \le i,j \le n}$

$\Rightarrow$

$$\Rightarrow U(M_n(R)) \simeq (M_n(k_1)\times M_n(k_2)) \cong U(M_n(k_1))\times U(M_n(k_2))$$

$$|U(M_2(\mathbb{Z}_{26}))| = \underbrace{|U(M_2(\mathbb{Z}_2))|}_{6} \cdot \underbrace{|U(M_2(\mathbb{Z}_{13}))|}_{(13^2-1)(13^2-13)} = 6\cdot 168\cdot 156$$

1) k corp finit $\Leftrightarrow$ k comutativ

2) k corp finit $\Rightarrow |k|$

$y+y=0$

$x(1+x)=x+x^2=$

$=x+(1+x)=1$

$x^2=0$
1
x
$1+x = x^2$

$|k|=4$ k corp $\boxed{0,1,x,1+x}$

(Fals)

p prim $|\mathbb{Z}_p|=p$

$|k|=4$; k corp

$\boxed{0,1,x,1+x}$

k corp cu 4 elem

$1+x\neq 1$

$\neq x$

$1+x=0$

$x+x=0$ $\Big| \Rightarrow x=1$

$\mathbb{Z}_2\times\mathbb{Z}_2$ nu e corp

$(0,1)\cdot(a,b)=(0,b)\neq(1,1)$

nu are invers

$1+1=0$

$1+1+1+1=0$

$(1+1)(1+1)=0 \Rightarrow (1+1)=0$

$\nexists$ corp cu 10 elem

P_p că $\exists k$ corp cu 10 elem

$(1+1)(1+1+1+1+1)=\underbrace{1+\ldots+1}_{10}=0$

$1+1=0$ sau $1+1+1+1+1=0$

nu pot fi adev. simultan