S1 Geometrie vectorială

- Sp. vect
- Lin ind; Lin dep
- Sist de gen; Baze

Sp vect (def) $V \neq \emptyset$, K - corp comutativ

$+ : V \times V \to V$ adunarea vect

$(v_1, v_2) \to v_1 + v_2$ (op. int)

$\cdot : K \times V \to V$ înmulțirea vect cu scalari

$(\alpha, v) \to \alpha v$ (op. ext)

I $(V, +)$ grup comutativ

II 1. $(\alpha_1 + \alpha_2) \cdot v = \alpha_1 v + \alpha_2 v$

2. $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$ $\forall v, v_1, v_2 \in V$

3. $(\alpha_1 \alpha_2) v = \alpha_1(\alpha_2 v)$ $\forall \alpha, \alpha_1, \alpha_2 \in K$

4. $1 \cdot v = v$

$(V/K, +, \cdot)$ sp vect peste K

$K = \mathbb{R}$ sp vect real

$K = \mathbb{C}$ sp vect complex

Ap 1 $(\mathbb{R}^3/\mathbb{R}, +, \cdot)$ st de sp vect real

$+ : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

$(x_1, x_2, x_3) + (y_1, y_2, y_3) \overset{def}{=} (x_1 + y_1, x_2 + y_2, x_3 + y_3)$, $\forall x_1, x_2, x_3, y_1, y_2, y_3$ elem din $\mathbb{R}^3$

$\cdot : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$

$\alpha(x_1, x_2, x_3) \overset{def}{=} (\alpha x_1, \alpha x_2, \alpha x_3)$, $(\forall) \alpha \in \mathbb{R}$, $\forall (x_1, x_2, x_3) \in \mathbb{R}^3$

I $(\mathbb{R}^3, +)$ grup comutativ
- asoc
- com
- $\exists$ elem neutru
- $(\forall)$ elem admite un opus

I asoc $(x+y)+z = x+(y+z)$, $(\forall) x, y, z \in \mathbb{R}^3$

$((x_1, x_2, x_3) + (y_1, y_2, y_3)) + (z_1, z_2, z_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3) + (z_1, z_2, z_3) = ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, (x_3 + y_3) + z_3) =$

$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), x_3 + (y_3 + z_3)) =$

$= (x_1, x_2, x_3) + (y_1 + z_1, y_2 + z_2, y_3 + z_3) = (x_1, x_2, x_3) + (y_1, y_2, y_3) + (z_1, z_2, z_3)$

2. com. $x+y=y+x, \forall x,y\in\mathbb{R}^3$

$(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3)=(y_1+x_1,y_2+x_2,y_3+x_3)=(y_1,y_2,y_3)+(x_1,x_2,x_3)$

3. $(\exists)$ elem neutru

$(\exists)\ e\in\mathbb{R}^3$ a.î. $\forall x\in\mathbb{R}^3: x+e=e+x=x$

$e(e_1,e_2,e_3)$

$(x_1+e_1,x_2+e_2,x_3+e_3)=(x_1,x_2,x_3)$

$$\Rightarrow \begin{cases} x_1+e_1=x_1 \Rightarrow e_1=0 \\ x_2+e_2=x_2 \Rightarrow e_2=0 \\ x_3+e_3=x_3 \Rightarrow e_3=0 \end{cases} \qquad e=(0,0,0)=0_{\mathbb{R}^3}$$

4. $(\forall)\ x\in\mathbb{R}^3, (\exists)\ x'\in\mathbb{R}^3$ a.î. $x+x'=x'+x=e$

$(x_1,x_2,x_3)+(x_1',x_2',x_3')=(x_1+x_1',x_2+x_2',x_3+x_3')=(0,0,0)$

$$\Rightarrow \begin{cases} x_1+x_1'=0 \Rightarrow x_1'=-x_1 \\ x_2+x_2'=0 \Rightarrow x_2'=-x_2 \\ x_3+x_3'=0 \Rightarrow x_3'=-x_3 \end{cases}$$

II. 1. $(\alpha_1+\alpha_2)x=\alpha_1x+\alpha_2x, (\forall)x\in\mathbb{R}^3; \alpha_1,\alpha_2\in\mathbb{R}$

$(\alpha_1+\alpha_2)(x_1,x_2,x_3)=((\alpha_1+\alpha_2)x_1,(\alpha_1+\alpha_2)x_2,(\alpha_1+\alpha_2)x_3)=$

$=(\alpha_1x_1+\alpha_2x_1,\alpha_1x_2+\alpha_2x_2,\alpha_1x_3+\alpha_2x_3)$

$=(\alpha_1x_1,\alpha_1x_2,\alpha_1x_3)+(\alpha_2x_1,\alpha_2x_2,\alpha_2x_3)=$

$=\alpha_1(x_1,x_2,x_3)+\alpha_2(x_1,x_2,x_3)$

2. $\alpha(x+y)=\alpha x+\alpha y, \forall x,y\in\mathbb{R}^3$

$\alpha((x_1,x_2,x_3)+(y_1,y_2,y_3))=\alpha(x_1+y_1,x_2+y_2,x_3+y_3)=$

$=(\alpha(x_1+y_1),\alpha(x_2+y_2),\alpha(x_3+y_3))=(\alpha x_1+\alpha y_1,\alpha x_2+\alpha y_2,\alpha x_3+\alpha y_3)=(\alpha x_1,\alpha x_2,\alpha x_3)+(\alpha y_1,\alpha y_2,\alpha y_3)=$

$=\alpha(x_1,x_2,x_3)+\alpha(y_1,y_2,y_3)$

3. $(\alpha_1\alpha_2)x=\alpha_1(\alpha_2x), \forall x\in\mathbb{R}$

$(\alpha_1\alpha_2)(x_1,x_2,x_3)=((\alpha_1\alpha_2)x_1,(\alpha_1\alpha_2)x_2,(\alpha_1\alpha_2)x_3)=$

$=(\alpha_1(\alpha_2x_1),\alpha_1(\alpha_2x_2),\alpha_1(\alpha_2x_3))=\alpha_1(\alpha_2x_1,\alpha_2x_2,\alpha_2x_3)=$

$=\alpha_1(\alpha_2(x_1,x_2,x_3))$

4. $1x=x, \forall x\in\mathbb{R}$ sp vect real

Deci $(\mathbb{R}^3/\mathbb{R},+,\cdot)$ sp vect real

Temă $(M_3(\mathbb{R})/\mathbb{R},+,\cdot)$

$+ : \mathcal{M}_3(\mathbb{R}) \times \mathcal{M}_3(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$

$A + B = (a_{ij} + b_{ij})_{i,j=\overline{1,3}}$ $(\forall) A = (a_{ij})_{i,j=\overline{1,3}}$

$B = (b_{ij})_{i,j=\overline{1,3}}$

$\cdot : \mathbb{R} \times \mathcal{M}_3(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$

$\alpha A = (\alpha a_{ij})_{i,j=\overline{1,3}}, \forall \alpha \in \mathbb{R}, (\forall) A = (a_{ij})_{i,j=\overline{1,3}}$

Lin ind. Lin. dep

~~Def~~ Fie V/K sp vect

$S = \{v_1, \dots, v_n\} \subset V$

S s.n. s.v. liniar ind dacă,

$(\forall) \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0_V \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

$\alpha_i \in K, (\forall) i = \overline{1,n}$

{$\forall$ comb liniară nulă se se ~~obține~~ realizează nul cu scalari nuli}

b) S' s.n. s.v lin. dep. dacă nu este lin ind

{$(\exists)$ comb liniară nulă care se realizează cu scalari, nu toți nuli}

Ap 1. Stabiliți dacă urm sist vectoriale sunt liniar independente sau liniar dep.

S.V. $\begin{cases} L.I \\ L.D \end{cases}$

$\boxed{\mathbb{R}^3}$ În cazul l.d, precizați rel de dep liniară

a) $S_1 = \{v_1 = (1,-2,2), v_2 = (3,2,1)\}$

b) $S_2 = \{v_1 = (1,1,-1), v_2 = (1,-1,1), v_3 = (-1,1,1)\}$

c) $S_3 = \{v_1 = (2,3,-1), v_2 = (1,1,2), v_3 = (1,2,-3)\}$

d) $S_4 = \{v_1 = (3,-1,0), v_2 = (3,2,1), v_3 = (1,2,1), v_4 = (7,3,2)\}$

a) Fie $\alpha_1 v_1 + \alpha_2 v_2 = 0_{\mathbb{R}^3}$

$\alpha_1, \alpha_2 \in \mathbb{R}$

$\alpha_1(1,-2,2) + \alpha_2(3,2,1) = (0,0,0)$

$\Rightarrow \begin{cases} \alpha_1 + 3\alpha_2 = 0 \\ -2\alpha_1 + 2\alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0.$

$\begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} = 2+6 = 8 \neq 0 \Rightarrow \alpha_1 = \alpha_2 = 0$ sol. unică

$\Rightarrow S_1$ s.v. lin. indep

b) Fie $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{\mathbb{R}^3}$

$\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

$\alpha_1(1,1,-1) + \alpha_2(1,-1,1) + \alpha_3(-1,1,1) = (0,0,0)$

$$\begin{cases} \alpha_1 + \alpha_2 - \alpha_3 = 0 \\ \alpha_1 - \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases}$$

$\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -1 - 1 + 1 - 1 - 1 - 1 = -4$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$ sol unică $\Rightarrow S_2$ e lin ind

$\boxed{R_1}$ $\mathbb{R}^n$

$S = \{v_1, v_2, \dots, v_m\} \subset \mathbb{R}^n$ $\quad A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & & v_m \end{pmatrix} \in \mathcal{M}_{(n,m)}(\mathbb{R})$

a) S - s.v. lin ind dacă $rg A = m \leq n$ ← dim. spațiului

b) S - s.v. lin. dep dacă $rg A \neq m$

c) $A = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ -1 & 2 & -3 \end{vmatrix} \in \mathcal{M}_3(\mathbb{R})$

$\det A = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ -1 & 2 & -3 \end{vmatrix} = -6 + 6 - 2 + 1 - 8 + 9 = 0$

$\Rightarrow rg A \leq 2 \neq 3 \Rightarrow S_3$ s.v. lin. dep.

V1 $\quad v_1 - v_2 - v_3 = 0_{\mathbb{R}^3}$

V2 $\quad \alpha_1(2,3,-1) + \alpha_2(1,1,2) + \alpha_3(1,2,-3) = (0,0,0) \Rightarrow$

$\Rightarrow \begin{cases} 2\alpha_1 + \alpha_2 + \alpha_3 = 0 \\ 3\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \\ \alpha_1 + 2\alpha_2 - 3\alpha_3 = 0 \end{cases}$ $\quad \det A = 0 \Rightarrow rg A \leq 2$.

$\Delta_p = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1 \neq 0 \Rightarrow rg A = 2$

$\alpha_3 = \lambda, \lambda \in \mathbb{R}$ nec sc.

$$\begin{cases} 2\alpha_1+\alpha_2=-\lambda \\ 3\alpha_1+\alpha_2=-2\lambda \end{cases} \Leftrightarrow \alpha_1=-\lambda,\ \alpha_2=\lambda,\ \cancel{\alpha_3=\lambda},\ \lambda\in\mathbb{R}$$

$-\lambda v_1+\lambda v_2+\lambda v_3=0$

$\lambda=-1 \Rightarrow v_1-v_2-v_3=0_{\mathbb{R}^3}$

d) $A=\begin{pmatrix} 3 & 3 & 1 & 7 \\ -1 & 2 & 2 & 7 \\ 0 & 1 & 1 & 2 \end{pmatrix} \in \mathcal{M}_{(3,4)}(\mathbb{R})$

$\operatorname{rg} A \le 3 \ne 4 \xRightarrow{\boxed{R_1}}$ S.v. lin dep

$v_1+v_2+v_3=v_4$

$v_1+v_2+v_3-v_4=0_{\mathbb{R}^3}$

$\boxed{\text{Ap 1}}$ $m=?\ (\in\mathbb{R})$

$S=\{v_1=(1,1,m),\ v_2=(1,2,3),\ v_3=(-1,2,1)\}$

a) lin. dep

b) lin indep

S.s.v. lin dep $\Leftrightarrow \operatorname{rg} A\ne 3 \Leftrightarrow \det A=0$

$A=\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 2 \\ m & 3 & 1 \end{pmatrix}$ ~~ord~~

$\det A=\begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 2 \\ m & 3 & 1 \end{vmatrix} = 2+2m-3+$
$+2m-6-1=$
$=4m-8 \Rightarrow m=2$

b) S.s.v. lin indep $\Leftrightarrow \operatorname{rg} A=3 \Leftrightarrow \det A\ne 0$

$\Leftrightarrow m\ne 2 \Leftrightarrow m\in\mathbb{R}\setminus\{2\} \Leftrightarrow \begin{cases} \alpha_1=? \\ \alpha_2=? \\ \alpha_3=? \end{cases}$

__Sist de gen__

__Def__ Fie V/K sp. vect finit generat.

$S=\{v_1,v_2,\dots,v_n\}\subset V$

S - sist de gen pt V dacă $(\forall)\ v\in V$,

$(\exists)\ \alpha_1,\alpha_2,\dots,\alpha_n\in K$ a.î. $v=\alpha_1v_1+\alpha_2v_2+\dots+\alpha_nv_n$

(R_2) $\mathbb{R}^n$ $\quad S = \{v_1, \dots, v_n\} \subset \mathbb{R}^n$

S - sist de gen pt $\mathbb{R}^n$ dacă $\operatorname{rg} A = n \leq m$

S nu este sist de gen pt $\mathbb{R}^n$ dacă $\operatorname{rg} A \neq m$

Ap 1) S.G.$_{\mathbb{R}^3}$ - sunt sau nu sist de gen pt sp vect din care fac parte

a) $S_1 = \{v_1 = (1,0,1), v_2 = (2,3,4)\}$

b) $S_2 = \{v_1 = (1,1,1), v_2 = (1,1,0), v_3 = (1,0,0)\}$

c) $S_3 = \{v_1 = (2,-1,3), v_2 = (0,2,5), v_3 = (2,1,8)\}$

d) $S_4 = \{v_1 = (-1,2,1), v_2 = (0,1,-1), v_3 = (3,2,1), v_4 = (1,-3,2)\}$

Rez. a) $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 4 \end{pmatrix} \in \mathcal{M}_{(3,2)}(\mathbb{R})$

$\operatorname{rg} A \leq 2 \neq 3 \overset{R_2}{\Rightarrow} S_1$ nu este sist de gen pt sp vect $\mathbb{R}^3$

b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$\det A = -1 \Rightarrow \operatorname{rg} A = 3 = \dim_{\mathbb{R}} \mathbb{R}^3 \overset{R_2}{\Rightarrow} S_2$ sist de gen pt sp vect $\mathbb{R}^3$