Seminar 6

$(E, < >)$ sp euclidian, $x \perp y \Leftrightarrow <x, y> = 0$

$W \leq E$ subspațiu

$W^{\perp} = \{ x \in E \mid <x, w> = 0 \ \forall \ w \in W \} \cancel{= \{0\}}$

$W \cap W^{\perp} = \{ w \mid w \in W$ și $w \in W^{\perp} \} = \{ 0_E \}$

$\underset{\underset{W}{\in}}{<\overset{\shortparallel}{w}}, \underset{\underset{W}{\in}}{x}> = 0 ; \forall x \in W$

$<w, w> = 0 \Leftrightarrow \boxed{w = 0.}$

$\left.\begin{array}{l} \underset{\psi}{E} = W + W^{\perp} \\ x = w + v \end{array}\right\} \Rightarrow \star$

$W, W^{\perp}$ subspații $\leq E$

$\dim (W + W^{\perp}) = \dim (W) + \dim (W^{\perp}) - \dim (W \cap W^{\perp})$

$\dim (W + W^{\perp}) = \dim (W) + \dim (W^{\perp})$ $\quad \overset{\shortparallel}{0}$

$(E, <,>)$ sp. euclidian

$f: E \to E$ apl ortogonală s.n. transformare ortogonală

- Valorile proprii ale unei transf. ortogonale pot 1 sau -1.

- Matricea asoc. unei transf. ort. are det. $= \pm 1$

$\dim E = 1$

$\{e\}$ baza in E

$f: E \to E$ transf ortog

~~$f = a \cdot e$; $a \neq 0$~~

$f(e) = a \cdot e$; $a \neq 0$

e - vect. propriu $\Rightarrow$ a val. proprie pt $f \to a = \pm 1$

$a = 1 \Rightarrow f(e) = e \Rightarrow f = id_E$

$a = -1 \Rightarrow f(e) = -1 \Rightarrow f = -id_E$

$\dim E = 2$

$f: E \to E$ transf ort.

$\Downarrow$

$A \to \det A = \pm 1.$

$\det A = 1.$

$\exists$ un un reper $\{e_1, e_2\}$

f are matricea

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} ; \theta \in [0, 2\pi]$$

$\det A = -1$ (simetrie)

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$\dim E = 3$

$\det A = 1$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$\det A = -1.$

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Spații afine

$\mathcal{A}$ = mult. (nevidă) de puncte

V - sp. vect. peste un corp K

$\varphi : \mathcal{A} \times \mathcal{A} \longrightarrow V$

- $\varphi(A,B) + \varphi(B,C) = \varphi(A,C) \quad \forall A,B,C \in \mathcal{A}$

$\exists\, O \in \mathcal{A}$ a.î. $\varphi_O : \mathcal{A} \to V$ bijectivă

$\varphi_O(A) = \varphi(O, 1)$

$\varphi(B,A) = -\varphi(A,B) \quad \forall A,B \in \mathcal{A}$

$\varphi(A,A) = 0$

$\varphi(A,B) = \varphi(A,A) + \varphi(A,B) \Longrightarrow \varphi(A,A) = 0$

$\varphi(A,B) \cancel{+} = -\varphi(B,A)$

Ex: K^n

$\varphi(x,y) = y - x$

Prop: $\forall A \in \mathcal{A} \Rightarrow \varphi_A : \mathcal{A} \to V$

$\varphi_A(B) = \varphi(A,B)$ bijectie

$\varphi_A(A) = 0 \qquad \varphi(A,A) = 0$

$\vdots$

$j = s \Rightarrow k_s \langle e_s, e_s \rangle = 0 \Rightarrow k_s = 0$

$\Rightarrow v = \sum_{i=s+1}^{n} b_i e_i \Rightarrow \{e_{s+1}, \dots, e_n\}$ sist de gen. pt. $W^{\perp}$

Deci $\left.\begin{array}{l} E = W + W^{\perp} \\ W \cap W^{\perp} = \{0_E\} \end{array}\right| \Rightarrow E = W \oplus W^{\perp}$ (sumă directă)

$\dim E = \dim W + \dim W^{\perp}$

complementul ort al lui W

Ex: $\mathbb{R}^3$ sp vect $\mathbb{R}$

$S = \langle e_1 = (1, 0, 1), e_2 = (2, 1, -3) \rangle$

Det. complementul ortogonal al lui S ($S^{\perp}$)

$S^{\perp} = \{x \in \mathbb{R}^3 \mid \langle x, s \rangle = 0 \; \forall s \in S\} = \star$

$\langle x, e_1 \rangle = 0 \Leftrightarrow x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

$\langle x, e_2 \rangle = 0 \Leftrightarrow 2x_1 + x_2 - 3x_3 = 0 \Rightarrow -5x_3 + x_2 = 0 \Rightarrow x_2 = 5x_3$

$\star = \{x = (-x_3, 5x_3, x_3) \mid x_3 \in \mathbb{R}\}$

Baza: $(-1, 5, 1)$ baza in $S^{\perp} \Rightarrow \dim S^{\perp} = 1$ (elem din baza)

$\dim S = 2$

Aplicații

$(E_1 \langle , \rangle_1), (E_2, \langle , \rangle_2)$ sp euclidian

$f: E_1 \to E_2$ apl liniară

$\langle f(x), f(y) \rangle_2 = \langle x, y \rangle_1$

S.n. apl. ortogonală

norma

$\| \cdot \|_1 = \sqrt{\langle , \rangle_1}$

$\| \cdot \|_2 = \sqrt{\langle , \rangle_2}$

$\| f(x) \|_2 \overset{?}{=} \| x \|_1 \to$ DA

„$\Rightarrow$"

Fie $\{e_1, \dots, e_s\}$ bază ortogonală în W ($W \leq E$)

Completez $\{e_1, e_2, \dots, e_s, \dots, e_n\}$ la o bază în E

Presupunem $\{e_1, \dots, e_n\}$ bază ortogonală

Afirm $\{e_{s+1}, \dots, e_n\}$ baza in $W^{\perp}$

$e_{s+1} \leq i \leq n$

$e_i \in W^{\perp}$

$\langle e_i, w \rangle = 0$

$$w \in W \Rightarrow w = \sum_{j=1}^{s} a_j e_j$$

$$\langle e_i, w \rangle = \langle e_i, \sum_{j=1}^{s} a_j e_j \rangle =$$

$$= \sum_{j=1}^{s} a_j \langle e_i, e_j \rangle = 0.$$

$\{e_{s+1}, \dots, e_n\}$ sist de gen $W^{\perp}$

Vreau: $v \in W^{\perp}$, $v = \sum_{i=s+1}^{n} b_i e_i$

in E.

$$v = \sum_{i=1}^{n} b_i e_i$$

$$v \in W^{\perp} \Rightarrow \langle v, e_j \rangle = 0 \ \forall j = \overline{1,s}$$

$$\langle \sum_{i=1}^{n} b_i e_i, e_j \rangle = 0, \ \forall j = \overline{1,s}$$

$$\sum_{i=1}^{n} b_i \langle e_i, e_j \rangle = 0, \ \forall j \in \overline{1,s}$$

$$j = 1 \Rightarrow b_1 \underbrace{\langle e_1, e_1 \rangle}_{\neq 0} = 0 \Rightarrow b_1 = 0$$

$$j = 2 \Rightarrow b_2 \underbrace{\langle e_2, e_2 \rangle}_{\neq 0} = 0 \Rightarrow b_2 = 0$$

Deci, pot privi $\mathcal{A}$ ca fiind un sp. vect. cu elem. null A.

Spațiii afine

$\Rightarrow$ Pe $\mathcal{A}$ $\exists$ atâtea structuri de sp. vect câte p

sunt în $\mathcal{A}$.

Toate sunt izomorfe cu V.

Combinații afine

$P_1, \dots, P_n \in \mathcal{A}$

$$\sum_{i=1}^{n} a_i P_i$$

Fixăm $O \in \mathcal{A}$

$\varphi_O : \mathcal{A} \to V$

$\varphi_O(P_i) = \overrightarrow{OP_i}$

$$\overrightarrow{OP} = \sum_{i=1}^{n} a_i \overrightarrow{OP_i}$$

Alegem $O' \in \mathcal{A}$

$$\sum_{i=1}^{n} a_i \overrightarrow{O'P_i} = \overrightarrow{O'P'}$$

$$P = P' \iff \sum_{i=1}^{n} a_i \overrightarrow{OP_i} = \sum_{i=1}^{n} a_i \overrightarrow{O'P_i}$$

$$\overrightarrow{O'P_i} = \overrightarrow{O'O} + \overrightarrow{OP_i}$$

$$\sum_{i=1}^{n} a_i \overrightarrow{OP_i} = \sum_{i=1}^{n} a_i \left(\overrightarrow{O'O} + \overrightarrow{OP_i}\right) = \sum_{i=1}^{n} a_i \overrightarrow{O'O} +$$

$P = P' \Longleftrightarrow \overrightarrow{OP} = \overrightarrow{OP'}$

$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P'} \Big| \Rightarrow \sum_{i=1}^{n} a_i \overrightarrow{OP_i} = \overrightarrow{OO'} + \sum_{i=1}^{n} a_i \cdot \overrightarrow{O'P_i}$

$\overrightarrow{O'P_i} = \overrightarrow{OP_i} - \overrightarrow{OO'}$

$\cancel{\sum_a a_i \overrightarrow{OP_i}} = \overrightarrow{OO'} - \sum a_i \overrightarrow{OO'} + \cancel{\sum a_i \overrightarrow{OP_i}}$

$\Rightarrow \left(1 - \sum_{i=1}^{n} a_i\right) \overrightarrow{OO'} = 0$

$O \neq O' \Rightarrow OO' \neq 0$

$$\boxed{1 = \sum_{i=1}^{m} a_i}$$

$M \subseteq A$

$Af(M) = \left\{ \sum_{i=1}^{m} a_i P_i \;\middle|\; P_i \in M, \sum_{i=1}^{n} a_i = 1 \;\middle|\; n \in \mathbb{N} \right\}$

acoperirea afină a lui M

Ex: $M = \{(1,0,1), (2,1,-3)\} \subseteq \mathbb{R}^3$

$Af(M) = \{ a_1 \cdot P_1 + a_2 \cdot P_2 \mid a_1 + a_2 = 1 \} = \{ (a_1 + 2a_2, a_2, a_1 - 3a_2) \mid a_1 + a_2 = 1 \}$

a) $M_1 \subseteq M_2 \Rightarrow Af(M_1) \subseteq Af(M_2)$

b) $M \subseteq Af(M)$

c) $Af(Af(M)) = Af(M)$

a) $Af(M_1) = \{ \sum a_i P_i \mid P_i \in M_1 ; \sum a_i = 1, n \in \mathbb{N} \}$

$\subseteq \{ \sum a_i P_i \mid P_i \in M_2 , \sum a_i = 1 ; n \in \mathbb{N} \}$

b) $\forall P \in M \Rightarrow 1 \cdot P \in Af(M) \Rightarrow M \subseteq Af(M)$

c) $_n \supseteq _n^n$ - evidentà

$$\sum_{j=1}^{nc} b_j \sum_{i=1}^{m} a_i \cdot P_i = \sum_{k=1}^{s} c_k \cdot P_k \in Af(M)$$

$$\sum_{i=1}^{m} a_i = 1.$$

$$\sum_{j=1}^{t} \left(b_j \sum_{i=1}^{n} a_i \right) = \sum_{j=1}^{t} b_j = 1$$