

51

Inele și corpuri

Def Un inel este un triplet de forme $(A, +, \cdot)$, unde A este o mulțime nevoidă și $+$, \cdot sunt opere de pe A cu proprietăți:

1) $(A, +)$ grup abelian

2) (A, \cdot) monoid

3) Înmulțirea este distribuțională de adunare

$$\begin{aligned} a(b+c) &= ab+ac \\ (b+c)a &= ba+ca \end{aligned} \quad \forall a, b, c \in A$$

Def Un inel numit în casnice element numit este inversabil și numește apăr.

$$(\mathbb{Z}, +, \cdot)$$

$$(\mathbb{Q}, +, \cdot)$$

$$(\mathbb{R}, +, \cdot)$$

$$(\mathbb{C}, +, \cdot)$$

$$(\mathbb{Z}_n, +, \cdot)$$

inele comutative

$$V(\mathbb{Z}) = \{ \pm 1 \}$$

$(\mathbb{Q}, +, \cdot)$ corp $V(\mathbb{Q}) = \mathbb{Q}$

$(\mathbb{R}, +, \cdot), (\mathbb{C}, +, \cdot)$ corpuri

$\forall n \geq 2$, $(\mathbb{Z}_n, +, \cdot)$ este corp $\Leftrightarrow n$ este nr prim

$$\mathbb{Z}_n \text{ este corp } \Leftrightarrow V(\mathbb{Z}_n) = \mathbb{Z}_n \setminus \{0\} \Leftrightarrow$$

\Leftrightarrow nu există ca n să dividă cu $m < n$ prim

$$(\mathbb{Z}_3, +, \cdot), (\mathbb{Z}_5, +, \cdot), \dots \text{ corpuri}$$

$$\mathbb{Z}[i] = \{a+bi \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$$

$(\mathbb{Z}[i], +, \cdot)$ inel

$$V(\mathbb{Z}[i]) = ?$$

$\forall z_1 = a+bi \in \mathbb{Z}[i] : a, b \in \mathbb{Z}$

$$z_1 \in V(\mathbb{Z}[i]) \Leftrightarrow \exists z_2 \in \mathbb{Z}[i], z_2 = c+di, z_2 \neq 0$$

$$\text{ai } (a+bi)(c+di) = 1$$

$$\begin{aligned} |(a+bi)(c+di)|^2 &= 1 \quad \text{cu } c \in \mathbb{N} \\ \underbrace{(a^2+b^2)}_{c \in \mathbb{N}} \underbrace{(c^2+d^2)}_{c \in \mathbb{N}} &= 1 \Rightarrow a^2+b^2=c^2+d^2 \quad \begin{array}{l} a^2=0, b^2=1 \Rightarrow a=0, b=\pm 1 \\ a^2=1, b=0 \Rightarrow a=\pm 1, b=0 \end{array} \\ &\text{cifra } \cup(\mathbb{Z}[i]) = \{\pm 1, \pm i\} \end{aligned}$$

Def: Spunem că inelul A are divizori ai lui zero, dacă $\exists x, y \in A$, $x, y \neq 0$ astfel încât $x \cdot y = 0$.

$$\text{ex } \mathbb{Z}_6 : 2 \cdot 3 = 6 = 0 \rightarrow (\mathbb{Z}_6, +, \cdot) \text{ are divizori ai lui } 0.$$

$$(\mathbb{Z} \times \mathbb{Z}, +, \cdot) \quad (0, 1) \cdot (1, 0) = (0, 0)$$

Rinse) $\forall n > 2$, $(M_n(R), +, \cdot)$ este inel multiplicativ de ordin n cu elem din R

$\forall n > 12$, $(M_n(R), +, \cdot)$ este un inel neicomutativ, cu divizori ai lui zero

$$m=2 \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow M_2(R) \text{ este nocom}$$

$$\neq 0_2 \neq 0_2 \quad \Rightarrow M_2(R) \text{ are divizori ai lui zero}$$

$\cup(M_n(R)) = GL_n(R)$ - grupul general linear de ordin n cu elem din R
nch. Inversibile

$$GL_n(R) = \{A \in M_n(R) / \det A \neq 0\}$$

$$GL_n(\mathbb{Z}_6) = \{A \in M_n(\mathbb{Z}_6) \mid \text{condc}(\det A, 6) = 1\}$$

① Fie $a, b \in R^*$, $c \in R$

Def pe R legătură de comp

$$x * y = ax + by \quad \forall x, y \in R$$

Dacă $a, b, c \in (\mathbb{R}, +, \cdot)$ și fie inel. Ce obținem?

$$\begin{aligned}
 \text{Vorfae. lui } * : & (\cancel{x} * y) * z = \cancel{x} * (y * z) \\
 & (\cancel{ax + by + c}) * z = x * (\cancel{ay + bz + c}) \\
 & \cancel{(ax + by + c) * z} + b = \\
 & (cx + by + c) + bz + c = \\
 & = ax + b(ay + bz + c) + c \\
 & ax + by + ac + bz = ax + by + bz + bc \\
 & \cancel{ac} = bc
 \end{aligned}$$

$$\begin{aligned}
 & ax(a-1) + c(a-b) + 2b(1-b) = 0 \\
 a^2 & = a \Rightarrow a = 1 \\
 ac & = bc \\
 b^2 & = b \Rightarrow b = 1 \\
 0, b & \neq 0
 \end{aligned}$$

$x * y = x + y + c$
 Com este evidentă
 Elém neutră, $x * e = x$.

$$\begin{aligned}
 x + x + c &= x \\
 e + c &= 0 \Rightarrow e = -c \\
 \text{then in, } & \forall x \in \mathbb{R}^*, \exists x^{-1} \in \mathbb{R} \text{ s.t. } x * x^{-1} = -c \\
 & x + x^{-1} + c = -c \\
 & x^{-1} = -x - 2c \in \mathbb{R}.
 \end{aligned}$$

$(\mathbb{R}, *)$ grup com
 Veroare asoc pt \mathbb{R} , elem neutră
 este $(x-y) * z = x * (y-z)$

Distributivitate:

$$\begin{aligned}
 (\cancel{x} * y) * z &= x * \cancel{y} * z \\
 (x + y + c) * z &= x * z + y * z + c \\
 x * z + y * z + c &= x * z + y * z + c \Rightarrow c = c + 0 \in \mathbb{R} \\
 &\Rightarrow c = 0
 \end{aligned}$$

$$\text{Deci } x * y = x + y$$

În concluzie, \exists un singur $\text{inel } (\mathbb{R}, *, +)$ cu
 proprietatea $x * y = ax + by + c$, $a, b, c \in \mathbb{R}^*$,
 $x, y \in \mathbb{R}$ și anume $(\mathbb{R}, +, *)$

$\tilde{a}^2 = a$ dem. idempotente

② Se se rezolvă sist de ec. în cadrul \mathbb{Z}_{12}

$$a) \begin{cases} \hat{3}x + \hat{2}y = \hat{1} \\ \hat{5}x + \hat{3}y = \hat{2} \end{cases} \quad b) \begin{cases} \hat{2}x + \hat{3}y = \hat{2} \\ \hat{5}x + \hat{6}y = \hat{3} \end{cases}$$

$$a) \begin{cases} (-)\hat{x} + \hat{y} = \hat{1} \\ \hat{5}x + \hat{2}y = \hat{1} \end{cases} \quad \left| \begin{array}{l} 1 \cdot 2 - (\hat{2}x + \hat{2}y = \hat{2}) \\ \hat{3}x + \hat{2}y = \hat{1} \end{array} \right. \quad \left| \begin{array}{l} \hat{2}x = \hat{1} \Rightarrow x = \hat{1} \\ \hat{3}x = \hat{1} \Rightarrow x = \hat{1} \end{array} \right. \Rightarrow x = \hat{1}, y = \hat{2}$$

$$b) \cancel{\text{c)} } \quad 12 \div 7 = 1 \cdot 1 + 5$$

$$12 = 1 \cdot 7 + 5$$

$$\frac{5}{7} = 1 \cdot 5 + 2$$

$$5 = 2 \cdot 2 + 1.$$

$$2 = 1 \cdot 2 + 0$$

$$\cancel{\text{rezolvare}} \quad 1 + \frac{1}{1 + \frac{1}{2}} = 1 + \frac{1}{\frac{3}{2}} = 1 + \frac{2}{3} = \frac{5}{3}$$

$j=0$	s_j	g_j	t_j
$j=0$	12	-	0
$j=1$	2	1	1
$j=2$	5	1	-1
$j=3$	2	2	2
$j=4$	1	2	-5

$$t_j = t_{j-1} - \frac{s_j}{g_j}, \quad j > 1.$$

$$t_0 = 0, \quad t_1 = 1.$$

$$(t_j)_{j \geq 0} \quad t_2 = 1 - 1 \cdot 1 = -1$$

$$t_3 = 1 + 1 \cdot 1 = 2.$$

$$t_4 = -1 - 2 \cdot 2 = -5$$

$$7^{-1} \pmod{12} = -5 = 7.$$

$$7x + 3y \equiv 2 \pmod{12} \quad \left(\begin{matrix} 7 \\ 1 \end{matrix} \right)^{-1} = 2$$

$$x + 2y \equiv 1 \pmod{3}$$

$$x + 9y \equiv 2 \pmod{3}$$

$$4 \left(\begin{matrix} 1 \\ 2 \end{matrix} - \begin{matrix} 2 \\ 9 \end{matrix} y \right) + 2y \equiv 3 \pmod{3}$$

$$8 - 36y \equiv 6y \equiv 3 \pmod{3}$$

$$8 + 6y \equiv 3 \pmod{3}$$

$$8 \equiv -5 \pmod{3} \quad \Rightarrow \quad 6y \equiv 2 \pmod{3} \Rightarrow \text{am o soluție} \\ \text{6y} \not\equiv 0 \pmod{3} \Rightarrow \text{sist e incompletabil}$$

Def A un inel

O submult $\emptyset \neq B \subseteq A$ se numeste subinel dacă

- 1) $B \subseteq (A, +)$, adică $x - y \in B, \forall x, y \in B$
- 2) B este parte stabilită în (A, \cdot) dacă

" 3) $1 \in B$

$$\forall [1] \subseteq \mathbb{C} \quad (a+bi) - (c+di) = (a-c) + (b-d)i \quad \forall a, b, c, d \in \mathbb{C}$$

$$\{a+bi | a, b \in \mathbb{Z}\}$$

$$\forall a, b \in \mathbb{Z}$$

$$\mathbb{Z}[i]$$

$$a+bi(c+di) = (ac-bd) + (ad+bc)i \quad \forall a, b, c, d \in \mathbb{C}$$

$$1 = 1 + 0 \cdot i \in \mathbb{Z}[i]$$

$\Rightarrow \mathbb{Z}[i]$ subinel al lui \mathbb{C} .

Alt exemplu:

$$\left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\} \subseteq \mathbb{Q}$$

subinel

Def Def A un inel. Se numește ideal stăruitor o mulțime

$$\emptyset \neq J \subseteq A$$

- i) $x - y \in J, \forall x, y \in J$
- ii) $a \in J, \forall a \in A, x \in J$

$A =$ comun $\Rightarrow ax = xa$ - nu de ideal stăruitor \Rightarrow id. drept (idealuri)

③ Să se studieze, dacă term. submultimi sunt ideale ale unui inel respectiv:

a) $J_1 = \{ f \in \mathbb{Z}[x] \mid \text{grad}(f) \geq 6 \}$, $R = \mathbb{Z}[x]$.

fie $f = x^6 + x \in J_1$,

$$g = x^6 \in J_1$$

$$f - g = x^6 + x - x^6 = x \notin J_1 \Rightarrow J_1$$
 nu e ideal în $\mathbb{Z}[x]$

$0 \notin J_1 \Rightarrow J_1$ nu e subinel

b) $J_2 = \{ f \in \mathbb{Z}[x] \mid \text{grad } f = 5 \} \cup \{0\}$.

$$f = x^5 + x^2 \in J_2$$

$$g = x^5 \in J_2$$

$$f - g = x^5 + x^2 - x^5 = x^2 \in J_2 \Rightarrow J_2$$
 nu e ideal $\Rightarrow \mathbb{Z}[x]$

$$a) \mathcal{I}_3 = \{ f \in \mathbb{Q}[x] \mid \deg f \leq 3 \}$$

$$\begin{cases} i = x^2 \in \mathcal{I}_3 \\ c = x^3 + x \in \mathbb{Q}[x] \end{cases} \quad \left\{ \begin{array}{l} \deg c = 3 \\ \deg x^3 = 3 \end{array} \right.$$

$$d) \mathcal{I}_4 = \{ f \in \mathbb{Q}[x] \mid 2 \text{ es el n\'umero n\'ulo de } f \}$$

$$\begin{aligned} & \text{Sea } f, g \in \mathcal{I}_4 \Rightarrow f(2) = 0 \\ & \quad g(2) = 0 \end{aligned}$$

$$(f \cdot g)(2) = f(2) \cdot g(2) = 0 \cdot 0 = 0 \Rightarrow f \cdot g \in \mathcal{I}_4$$

$$\text{Sea } f \in \mathcal{I}_4, P \in \mathbb{Q}[x]$$

$$(fP)(2) = f(2)P(2) = 0 \Rightarrow fP \in \mathcal{I}_4$$

$$e) \mathcal{I}_5 = \{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \mid a, b \in \mathbb{N} \}$$

$$\text{Sea } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathcal{I}_5$$

$$AB = \begin{pmatrix} ae & af \\ ce & cf \end{pmatrix} \in \mathcal{I}_5$$

-ideal simple

$$\text{Sea } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{I}_5, \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in M_2(\mathbb{R})$$

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ma & mb \\ pa & pb \end{pmatrix} \in \mathcal{I}_5 \rightarrow \text{ideal simple}$$

-ideal direct

$$\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ am+bp & an+bq \end{pmatrix} \in \mathcal{I}_5 \rightarrow \mathcal{I}_5 \text{ e ideal direct}$$

Algebra Seminar 2,

$$\mathbb{U}(\mathbb{Z}[i]) = \{\pm n, \pm i\}$$

$$\mathbb{U}(\mathbb{Z}) = \{\pm 1\}$$

$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$$

$$\mathbb{U}(\mathbb{Z}[\sqrt{2}]) = \{\pm (1 + \sqrt{2})^m \mid m \in \mathbb{Z}\}$$

Notez că $\mu = 1 + \sqrt{2}$ este un element inversabil $|1 + \sqrt{2}| = 1$.

$$R = \mathbb{Z}[\sqrt{2}]$$

$$\text{Pun } W = \{a + b\sqrt{2} \in \mathbb{U}(R) \mid a, b \in \mathbb{N}\}$$

că să uită că $x \cdot \mu^{-m}$, $x \in W$ ($\mu^{-m} \in R \Rightarrow \mu^{-m} \in R$)

Notează $x \cdot \mu^{-m} = c + d\sqrt{2} > 0$ (o algebră)

$$1. cd > 0 \Rightarrow c, d > 0 \Rightarrow x \cdot \mu^{-m} \in W$$

$$x \in W \Rightarrow \exists m \in \mathbb{N} \text{ a.s.t. } x \in [\mu^m, \mu^{m+1})$$

$$x \cdot \mu^{-m} \in W \Rightarrow x \cdot \mu^{-m} \geq 1 + \sqrt{2} = \mu \Rightarrow x \geq \mu^{m+1} \quad \times$$

$$2. cd < 0 \Rightarrow \frac{x \cdot \mu^{-m}}{\in \mathbb{U}(R)} = \frac{c^2 - 2cd^2}{c + d\sqrt{2}} = \frac{c^2 - 2d^2}{c - d\sqrt{2}}$$

$$\text{Observe că } x \cdot \mu^{-m} \text{ este ul. inversabil} \Rightarrow |x \cdot \mu^{-m}| = 1$$

$$\forall y \in R \text{ a.s.t. } (x \cdot \mu^{-m}) \cdot y = 1 \Rightarrow |(x \cdot \mu^{-m}) \cdot y| = |\underbrace{x \cdot \mu^{-m}}_{\in \mathbb{U}(R)} \cdot \underbrace{y}_{\in \mathbb{U}(R)}| = 1 \Rightarrow |\mu^{-m} \cdot x| = 1$$

$$|\mu^{-m}| = 1 \Rightarrow |c + d\sqrt{2}| = 1 \Rightarrow |(c + d\sqrt{2})(c - d\sqrt{2})| = 1 \Rightarrow |c^2 - d^2 \cdot 2| = 1 \Rightarrow c^2 - d^2 \cdot 2 = \pm 1$$

$$x \cdot \mu^{-m} = \frac{c^2 - 2d^2}{c - d\sqrt{2}} \stackrel{(*)}{=} \frac{1}{|c| + |d|\sqrt{2}} \leq \frac{1}{|c|} \Rightarrow x \cdot \mu^{-m} \leq \frac{1}{|c|} \Rightarrow x \leq \mu^{m-1} \quad \times$$

$$3. c \cdot d = 0. \quad c \neq 0 \text{ și } d = 0 \Rightarrow$$

$\Rightarrow x \cdot \mu^{-m} = c \Rightarrow x \cdot \mu^{-m} = 1 \Rightarrow x = \mu^m \Rightarrow$ oricare element inversabil este de

putere de $1 + \sqrt{2}$

$$\mathbb{U}(\mathbb{Z}[\sqrt{2}]) = \{\pm (1 + \sqrt{2})^m \mid m \in \mathbb{Z}\}$$

$$\mathbb{U}(\mathbb{Z}[\sqrt{3}]) = ?$$

$$\sqrt{3} = \overline{1 + (\sqrt{3} - 1)}$$

$$\frac{1}{\sqrt{3} - 1} = \frac{\sqrt{3} + 1}{2} = 1 + \left(\frac{\sqrt{3} + 1}{2} - 1\right) = \boxed{1} + \frac{\sqrt{3} - 1}{2}$$

$$\frac{2}{\sqrt{3} - 1} = 2 \cdot \frac{\sqrt{3} + 1}{2} = \sqrt{3} + 1 = 2 + (\sqrt{3} + 1 - 2) = \boxed{2} + \boxed{\sqrt{3} - 1} \text{ STOP}$$

$\sqrt{3} = (1; \overline{1; 2})$ - primul întreg se pun normal, iar a următoare divizie se repeta.

$1 + \frac{1}{2} = \boxed{2} \rightsquigarrow 2 + \sqrt{3}$

ultimul din perioada se face

$2 + \sqrt{3}$ unitate fundamentală.

$$\mathbb{U}(\mathbb{Z}[\sqrt{3}]) = \{\pm (2 + \sqrt{3})^m \mid m \in \mathbb{Z}\}$$

$$(1; 1, 2, 3, \cancel{4})$$

↓
 +
 ↓
 1 +
 ↓
 2 +
 ↓
 3

→ Metoda fractiilor continue.

Ex2) $U(\mathbb{Z}[\sqrt{29}]) = ?$

$$\sqrt{29} = 5 + (\sqrt{29} - 5)$$

$$\frac{1}{\sqrt{29}-5} = \frac{\sqrt{29}+5}{4} = 2 + \left(\frac{\sqrt{29}+5}{4} - 2 \right) = 2 + \left(\frac{\sqrt{29}-3}{4} \right)$$

$$\frac{4}{\sqrt{29}-3} = \frac{4(\sqrt{29}+3)}{20} = \frac{\sqrt{29}+3}{5} = 1 + \left(\frac{\sqrt{29}+3}{5} - 1 \right) = 1 + \left(\frac{\sqrt{29}-2}{5} \right)$$

$$\frac{5}{\sqrt{29}-2} = \frac{5(\sqrt{29}+2)}{25} = \frac{\sqrt{29}+2}{5} = 1 + \left(\frac{\sqrt{29}+2}{5} - 1 \right) = 1 + \left(\frac{\sqrt{29}-3}{5} \right)$$

$$\frac{5}{\sqrt{29}-3} = \frac{5(\sqrt{29}+3)}{20} = \frac{\sqrt{29}+3}{4} = 2 + \left(\frac{\sqrt{29}+3}{4} - 2 \right) = 2 + \left(\frac{\sqrt{29}-5}{4} \right)$$

$$\frac{4}{\sqrt{29}-5} = \frac{4(\sqrt{29}+5)}{4} = \sqrt{29}+5 = 10 + (\sqrt{29}-5) \rightarrow \text{STOP.}$$

$$\sqrt{29} = (5; 2, \underbrace{1, 1, 2, \cancel{1}}_{\text{perioada}})$$

$$5 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2}}}}} = 5 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2}}} = 5 + \frac{1}{2 + \frac{3}{5}} = 5 + \frac{1}{\frac{13}{5}} = \frac{13}{5} + \frac{5}{13} =$$

$$= \frac{70}{13} \rightsquigarrow 70 + 13\sqrt{29}$$

$$U(\mathbb{Z}[\sqrt{29}]) = \left\{ \pm (70 + 13\sqrt{29})^m \mid m \in \mathbb{Z} \right\}$$

$$70^2 + 28 \cdot 13^2 = \pm 1$$

169.
 29
 49
 169
 338
 1301

A determina elementele inversabile din $\mathbb{Z}[\sqrt{29}]$ e tot suma cu o singura solutie a ecuatiei $x^2 - 29y^2 = \pm 1$.

$$\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$$

A determina elem. inversabile din $\mathbb{Z}[\sqrt{d}] \Leftrightarrow$ sa gasesti solutiile ec. $x^2 - dy^2 = \pm 1$.

C. Ideale lui \mathbb{C} ?

$R = \text{un corp comutativ}$

Fie I un ideal. $\{0\} \neq I \subseteq R$.

Fie $0 \neq x \in I \Rightarrow x \in R \Rightarrow x^{-1} \in R \Rightarrow x^{-1} \cdot x \in I \Rightarrow 1 \in I \Rightarrow \forall n \in R \quad n = n \cdot 1 \in I$.
 $\Rightarrow R \subseteq I \Rightarrow R = I$.

Morală: Singurul ideal într-un corp comutativ sunt $\{0\}, R$.
 $\boxed{I \in \mathcal{I} \Rightarrow I = R}$

Ideale din $(\mathbb{Z}, +, \cdot)$

Oraș subgrup al lui $(\mathbb{Z}, +, \cdot)$ și de forma $m\mathbb{Z}$, $m \in \mathbb{N}$ și acestea sunt chiar ideale lui $(\mathbb{Z}, +, \cdot)$

Fie $a, b \in m\mathbb{Z} \Rightarrow \begin{cases} a = m a_1 \\ b = m b_1 \end{cases} \quad a_1, b_1 \in \mathbb{Z} \Rightarrow \begin{cases} a - b = m(a_1 - b_1) \in m\mathbb{Z} \\ m \in \mathbb{N} \end{cases}$

$\Rightarrow a - b \in m\mathbb{Z}$

Fie $a \in m\mathbb{Z}$
 $\begin{cases} a = m a_1 \\ r \in \mathbb{Z} \end{cases} \Rightarrow a = m a_1$

$$ra = a \cdot r = (m a_1) \cdot r = m(a_1 \cdot r) \in m\mathbb{Z}$$

$(\mathbb{Z}, +, \cdot)$ este comutativ

$\Rightarrow m\mathbb{Z}$ sunt ideale ale lui $(\mathbb{Z}, +, \cdot)$

Ideale lui $(\mathbb{Z}_m, +, \cdot)$

Subgrupurile lui $(\mathbb{Z}_m, +, \cdot)$ sunt de forma $d\mathbb{Z}_m$ cu d/m și acestea sunt chiar ideale lui $(\mathbb{Z}_m, +, \cdot)$

Fie $\bar{a}, \bar{b} \in d\mathbb{Z}_m \quad \bar{a} = d \cdot \bar{a}_1 \quad \bar{b} = d \cdot \bar{b}_1$

$$\bar{a} \cdot \bar{b} = d \cdot \bar{a}_1 \cdot d \cdot \bar{b}_1 = d(\bar{a}_1 \cdot \bar{b}_1) = d(\bar{a}_1 \cdot \bar{b}_1) \in d\mathbb{Z}_m \quad \textcircled{1}$$

$$\text{Fie } \bar{r} \in \mathbb{Z}_m \Rightarrow \bar{a} \cdot \bar{r} = r \cdot \bar{a} \in d\mathbb{Z}_m \quad \bar{a} = d \cdot \bar{a}_1 \quad (\mathbb{Z}_m, +, \cdot) \text{ comutativ} \quad \textcircled{2}$$

$$\bar{r} \cdot (d \cdot \bar{a}_1) = d \cdot \bar{r} \cdot \bar{a}_1 = d(\bar{r} \cdot \bar{a}_1) \in d\mathbb{Z}_m \quad \textcircled{3}$$

Din $\textcircled{1}$ și $\textcircled{2}$ și $\textcircled{3}$ $d\mathbb{Z}_m$ este ideal al lui $(\mathbb{Z}_m, +, \cdot)$

Problema:

1) Fie $I = 20\mathbb{Z}$, $J = 36\mathbb{Z}$. Calcula $I \cap J$, $I + J$, $I \cdot J$.

Propozitie: I, J ideale în R , R inel.

1) $I \cap J$ este ideal în R

2) $I + J = \{i+j \mid i \in I, j \in J\}$ ideal în R

3) $I \cdot J = \left\{ \sum_{i=1}^m x_i y_i \mid x_i \in I, y_i \in J \text{ m} \in \mathbb{N} \right\} = (xy \mid x \in I, y \in J)$

$$I = 20\mathbb{Z}, J = 36\mathbb{Z}$$

$$a\mathbb{Z} \cap b\mathbb{Z} = [a, b]\mathbb{Z} \quad a\mathbb{Z} \cdot b\mathbb{Z} = ab\mathbb{Z}$$

$$a\mathbb{Z} + b\mathbb{Z} = (a, b)\mathbb{Z}$$

$$I \cap J = [20, 36]\mathbb{Z} = 180\mathbb{Z}$$

$$I + J = (20, 36)\mathbb{Z} = 4\mathbb{Z}$$

$$I \cdot J = 20 \cdot 36\mathbb{Z} = 720\mathbb{Z}$$

Elemente speciale în inel

1) inversabile ✓ $\mathbb{U}(R)$

2) divizorii lui 0 ✓ $D(R)$

3) (x se numește divizor al lui 0 dacă $\exists y \neq 0, y \in R$ a.t. $x \cdot y = 0$)

3) idempotenți $\text{Idem}(R)$

$x \in R$ s.m. idempotent dacă $x^2 = x$

4) nulpotenți $\mathcal{N}(R)$

$x \in R$ s.m. nulpotent dacă $\exists m \in \mathbb{N}^*$ a.t. $x^m = 0$

a) Dacă $R = (\mathbb{Z}, +, \cdot)$

$$\mathbb{U}(\mathbb{Z}) = \{\pm 1\} \quad \text{Idem}(\mathbb{Z}) = \{0, 1\} \quad x^k = x \text{ în } \mathbb{Z} = 1 \quad x(x-1) = 0 = \bigcup_{x=0}^{x=0} \mathbb{Z} = 1$$

$$D(\mathbb{Z}) = \{0\} \quad \mathcal{N}(\mathbb{Z}) = \{0\}$$

b) $R = (\mathbb{Z}_6, +, \cdot)$

$$\mathbb{U}(\mathbb{Z}_6) = \{\bar{1}, \bar{5}\} \quad \text{Idem}(\mathbb{Z}_6) = \{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$$

$$D(\mathbb{Z}_6) = \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\} \quad \mathcal{N}(\mathbb{Z}_6) = \{\bar{0}\}$$

$$D(\mathbb{Z}_m) = \mathbb{Z}_m - \mathbb{U}(\mathbb{Z}_m)$$

$$\mathcal{N}(\mathbb{Z}_m) = ? \quad m = p_1^{d_1} \cdot p_2^{d_2} \cdots p_t^{d_t} \quad p_1^{d_1} p_2^{d_2} \cdots p_t^{d_t} \mid \bar{x}^k \Rightarrow p_i^{d_i} \mid \bar{x}^k, i \in \overline{1, t}$$

$$\bar{x} \in \mathcal{N}(\mathbb{Z}_m) \Leftrightarrow \exists k \in \mathbb{N}^* \text{ a.s. } \bar{x}^k = \bar{0} \Leftrightarrow m \mid \bar{x}^k$$

$$\Rightarrow p_1 p_2 \cdots p_t \mid \bar{x} \Rightarrow x \in \underbrace{p_1 \cdots p_t}_{\mathbb{Z}_m} \mathbb{Z}_m$$

Arg Se
page

$$\text{defe } d\mathbb{Z}_m \Rightarrow x = p_1 \dots p_t \cdot m, m \in \mathbb{Z}_m$$

$$\Rightarrow ? \exists N \text{ s.t. } (\overline{x})^N = \overline{0} \Rightarrow m / (\overline{x})^N \Rightarrow m / (p_1 p_2 \dots p_t)^N =$$

$$m = p_1^{d_1} p_2^{d_2} \dots p_t^{d_t} / (p_1 p_2 \dots p_t)^N$$

$$p_1^{d_1} p_2^{d_2} \dots p_t^{d_t} / p_1^N p_2^N \dots p_t^N$$

$$N \geq \max(d_1, d_2, \dots, d_t)$$

$$N(\mathbb{Z}_m) = d\mathbb{Z}_m, d = p_1 \dots p_t \quad m = p_1^{d_1} \dots p_t^{d_t}$$

3) A) (aceea ceva) monomialelor
pt a fi tot ceva A și B sămănuți.

$$\begin{array}{ll} x_1=1 & s_1=2 \\ x_2=1 & s_2=1 \\ x_3=0 & s_3=0 \end{array} \quad J(1,1,0) = 2 = u + A = 1A = -2$$

$$\begin{array}{ll} x_1=1 & s_1=3 \\ x_2=1 & s_2=3 \\ x_3=1 & s_3=1 \end{array} \quad J(1,1,1) = G = 27 + (-21 \cdot 9 + 3B) = 9 + 3B$$

Seminar 3

Rezultat: R_1, R_2 reale comutative

$$U(R_1 \times R_2) = U(R_1) \times U(R_2)$$

$$\text{Idem}(R_1 \times R_2) = \text{Idem}(R_1) \times \text{Idem}(R_2)$$

$$W(R_1 \times R_2) = W(R_1) \times W(R_2)$$

$$\mathcal{Q}(R_1 \times R_2) = ?$$

Prop:

Pie R un inel comut. și $f = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ un polinom. $\text{grad}(f) = n$, $a_n \neq 0$. Atunci:

$$i) f \in U(R[x]) \iff a_0, a_1, \dots, a_n \in U(R)$$

$$ii) f \in V(R[x]) \iff a_0, \dots, a_1 \in V(R), a_0 \in U(R)$$

$$iii) f \in \mathcal{Q}(R[x]) \iff \exists d \in R \text{ a.s. } d f = 0 \quad d \neq 0.$$

Ex: Câte polinoame inv sunt în $\mathbb{Z}_5[x]$?

$$|U(\mathbb{Z}_5[x])| = ?$$

$$\text{Pie } f = a_n x^n + \dots + a_1 x + a_0; a_0, a_1, \dots, a_n \in \mathbb{Z}_5$$

$$f \in U(\mathbb{Z}_5[x]) \iff a_0, \dots, a_1 \in U(\mathbb{Z}_5) \text{ și } a_0 \in U(\mathbb{Z}_5) \quad ①$$

$$U(\mathbb{Z}_5) = \{1, 2, 3, 4\} \quad ②$$

$$U(\mathbb{Z}_5) = \{0\} \quad ③$$

Din ① ② ③ = 4 polinoame inversabile

$$U(\mathbb{Z}_{54}[x]) = ?$$

$\text{Re } J > a_n x^n + \dots + a_1 x + a_0 \text{ cu } a_0, a_1, \dots, a_n \in U(\mathbb{Z}_{54})$

$$U(\mathbb{Z}_{54}) = 2 \cdot 3 \mathbb{Z}_{54} = 6 \mathbb{Z}_{54} = \{0, 6, 12, \dots, 48\}$$

$$54 = 2 \cdot 3^3$$

$$d = 2 \cdot 3$$

Polinoame nilpotente in $\mathbb{Z}_{54}[x]$. Exemplu:

$$J = 12x^4 + 6x^3 + 48$$

② Factorizare num. polinoame in $\mathbb{Z}[x]$, $\mathbb{R}[x]$, $\mathbb{C}[x]$

a) $J_1(x) = x^4 + x^3 - x - 1$

b) $J_2(x) = x^3 + 2x^2 - 5x - 6$

c) $J_3(x) = x^3 + 2x^2 - 4x + 1$

d) $J_4(x) = 5x^4 + 12x^3 + x^2 + 12x + 4$

✓ Sg. polinoame ireducibile din $\mathbb{C}[x]$ sunt cele de gradul I. Sg. polinoame ireducibile din $\mathbb{R}[x]$ sunt cele de gradul I + cele de gradul II cu $\Delta < 0$.

a) $J_1(x) = x^4 + x^3 - x - 1$
 $J_1(x) = x^3(x+1) - (x+1) = (x+1)(x^3 - 1) = (x+1)(x-1)(x^2 + x + 1)$
 $x^2 + x + 1 \Rightarrow \Delta = -3 \Rightarrow x_1 = \frac{-1 - i\sqrt{3}}{2}, x_2 = \frac{-1 + i\sqrt{3}}{2}$
 $\Delta < 0 \Rightarrow x^2 + x + 1 \text{ ireducibil in } \mathbb{R}[x]$

$J_1(x) = (x+1)(x-1)\left(x - \frac{-1 - i\sqrt{3}}{2}\right)\left(x - \frac{-1 + i\sqrt{3}}{2}\right)$
 in $\mathbb{C}[x]$

Continuare Seminar 3

b) $f_2(x) = x^3 + 2x^2 - 5x - 6$ care sunt soluțiile?

$$f_2(x) = 0 \Leftrightarrow x = 2$$

$$\begin{array}{r} x^3 + 2x^2 - 5x - 6 \\ -x^3 + 2x^2 \\ \hline 4x^2 - 5x - 6 \\ -4x^2 + 8x \\ \hline 3x - 6 \\ -3x + 6 \\ \hline \end{array}$$

$$f_2(x) = (x-2)(x^2 + 4x + 3)$$

$$\Delta = 16 - 12 = 4$$

$$x_{1,2} = \frac{-4 \pm 2}{2} \quad \begin{cases} x_1 = -3 \\ x_2 = -1 \end{cases}$$

$$f_2(x) = (x-2)(x+3)(x+1)$$

c) $f_3(x) = x^3 + 2x^2 - 4x + 1$

$$f_3(1) = 0$$

$$\begin{array}{r} x^3 + 2x^2 - 4x + 1 \\ -x^3 - x^2 \\ \hline 3x^2 - 4x + 1 \\ -3x^2 + 3x \\ \hline -x + 1 \\ \hline \end{array}$$

$$f_3(x) = (x-1)(x^2 + 3x - 1) \text{ - lusc peste } \mathbb{Z}[x]$$

$$\Delta = 9 + 4 - 13$$

$$x_{1,2} = \frac{-3 \pm \sqrt{13}}{2} \quad \begin{cases} x_1 = \frac{-3 + \sqrt{13}}{2} \\ x_2 = \frac{-3 - \sqrt{13}}{2} \end{cases}$$

$$f_3(x) = (x-1) \left(x - \frac{(-3 + \sqrt{13})}{2} \right) \left(x - \frac{(-3 - \sqrt{13})}{2} \right)$$

d) $f_4(x) = 4x^4 - 12x^3 + x^2 + 12x + 4$

$$f_4(2) = 0$$

$$\begin{array}{r} 4x^4 - 12x^3 + x^2 + 12x + 4 \\ -4x^3 + x^2 + 12x + 4 \\ \hline 4x^3 - 8x^2 \\ \hline -7x^2 + 12x + 4 \\ -7x^2 - 14x \\ \hline -2x + 4 \\ -2x + 4 \\ \hline \end{array}$$

111

$$f_4(x) = (x-2)(4x^3 - 4x^2 - 7x - 2)$$

$\overbrace{\quad\quad\quad}^{2\text{ sol}}$

$$\begin{array}{r} 4x^3 - 4x^2 - 7x - 2 \\ - 4x^3 + 8x^2 \\ \hline - 9x^2 - 7x - 2 \\ - 4x^2 + 8x \\ \hline x - 2 \\ - x + 2 \end{array}$$

$$f_4(x) = (x-2)(x-2)(4x^2 + 4x + 1)$$

$$f_4(x) = (x-2)^2(2x+1)^2$$

desc peste toate mult.

SEMINAR 4

Forme fundamentale de morfism pt mule

Fie R un mule, fie I un ideal al mulului

$\Rightarrow I$ este un subgrup (normal)

$\Rightarrow \left(\frac{R}{I}, +\right)$ grup factor $\forall r = r + i \in R$ un element

Definire

$$\left(\frac{R}{I}, +, \cdot\right) = \overline{x, y} = \overline{x, y}$$

mul factor

Fie $(R_1, +, \circ), (R_2, *, \odot)$ mule

$f: R_1 \rightarrow R_2$ dacă morfism de mule

$$\begin{cases} f(x+y) = f(x) * f(y) \\ f(xy) = f(x) \circ f(y) \\ f(1_{R_1}) = 1_{R_2} \end{cases} \quad \forall x, y \in R_1$$

f^{-1}

Fie R_1, R_2 mule, $f: R_1 \rightarrow R_2$ morfism de mule

Asternu $\frac{R_1}{Ker f} \cong Im f$

(fisură): $Im f = R_2$

$$\frac{R_1}{Ker f} \cong R_2$$

$$\textcircled{2} \quad \frac{\mathbb{Z}[x]}{n\mathbb{Z}[x]} \simeq \mathbb{Z}_n[x] \quad \text{dim. di mille}$$

Cant f: $\mathbb{Z}[x] \rightarrow \mathbb{Z}_n[x]$ morfism surjectiv de mille cu $\ker f = (n)$
 $(n) = (n)\mathbb{Z}[x] = \{ nh(x) \mid h(x) \in \mathbb{Z}[x] \}$

$$f(P) = \hat{P}$$

$$P = a_m x^m + \dots + a_0 \in \mathbb{Z}[x]$$

schemb wif den intagi im restori ale deci n

$$\hat{P} = \hat{a}_m x^m + \dots + \hat{a}_0 \quad \hat{a}_i \in \mathbb{Z}_n$$

f morphism de mille

$$f(P+Q) = \hat{P+Q} = \hat{P} + \hat{Q} = f(P) + f(Q)$$

$$f(PQ) = \hat{P}\hat{Q} = \hat{P} \cdot \hat{Q} = f(\hat{P}) + f(\hat{Q})$$

$$f(1) = \hat{1}$$

$$f \text{ wif } \forall P \in \mathbb{Z}_n[x], \exists \text{ eva } \in \mathbb{Z}[x] \text{ ast. } f(\text{eva}) = \hat{P}$$

wif $\text{eva} \in \mathbb{Z}[x]$ ast. $f(\text{eva}) = \hat{P}$ doar $f(P) = \hat{P} \Rightarrow f \text{ wif}$

$$\ker f = (n)$$

$$\ker f = \{ P \in \mathbb{Z}[x] \mid f(P) = 0 \} = \{ P \in \mathbb{Z}[x] \mid \hat{P} = 0 \} = \{ P \in \mathbb{Z}[x] \mid \text{wif } P = 0 \}$$

$$\ker f \subseteq (n)\mathbb{Z}[x] \quad \textcircled{1}$$

$$P = n(-)$$

$$\text{fix } P \in (n)\mathbb{Z}[x] \Rightarrow P = n h(x)$$

$$h(x) \in \mathbb{Z}[x]$$

$$\rightarrow f(P) = \hat{P} = n \cdot \hat{h}(x) = \hat{n} \cdot \hat{h}(x) = 0$$

$$\text{f}(P) = 0 \Rightarrow P \in \ker f \Rightarrow (n) \subseteq \ker f \quad \textcircled{2}$$

$$\ker f = (n)$$

$$\frac{\mathbb{Z}[x]}{n\mathbb{Z}[x]} \simeq \mathbb{Z}_n[x]$$

"durf"

Polinoamele Simetrie Fundamentale

$K[x_1 - x_n]$ nucleul de pe în determinante permutări

$$\Delta_1 = x_1 + x_2 + \dots + x_n$$

$$\Delta_2 = x_1x_2 + x_1x_3 + \dots = \sum_{i < j} x_i x_j$$

$$\Delta_K = \sum_{1 < i_1 < i_2 < \dots < i_K} x_{i_1} x_{i_2} \dots x_{i_K}$$

$$\Delta_n = x_1 x_2 \dots x_n$$

Teorema Fundamentală a Polinoamelor Simetrice

✓ $f \in K[x_1 - x_n]$ polinom simetric \Leftrightarrow scrie f ca o combinație de polinoame de pe simetrice fundamentale

✓ $f \in K[x_1 - x_n]$ și g astfel că $f(x_1, \dots, x_n) = g(x_1 - \Delta_1)$

$$1) f(x, y) = x^2 + y^2 = (x+y)^2 - 2xy = \Delta_1^2 - 2\Delta_2$$

$$\Delta_1 = x+y$$

$$\Delta_2 = xy$$

$$2) f(x, y) = x^3 + 2xy^2 + 2xy + x + y + y^3 = f(y, x)$$

$$f(x, y) = x^3 + y^2 + 2xy(y+x) + x + y =$$

$$= (x+y)(x^2 - xy + y^2) + 2\Delta_1\Delta_2 + \Delta_1$$

$$f(x, y) = \Delta_1((x+y)^2 - 3xy) + 2\Delta_1\Delta_2 + \Delta_2$$

$$= \Delta_1(\Delta_1^2 - 3\Delta_2) + 2\Delta_1\Delta_2 + \Delta_1$$

$$= \Delta_1^3 - \Delta_1\Delta_2 + \Delta_1$$

$$3) f(x, y, z) = x^3 + y^3 + z^3$$

raport componente omogene la fel de good

$$T(f) = c x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \rightsquigarrow c \Delta_1^{a_1-a_2} \Delta_2^{a_2-a_3} \dots \Delta_n^{a_n}$$

$$T(f) = x^3 \rightsquigarrow 1, \Delta_1^{3-0} \Delta_2^{0-0} \Delta_3^0 = \Delta_1^3$$

$$\Rightarrow f(x, y, z) = \Delta_1^3 - \Delta_1^3 + x^3 + y^3 + z^3$$

$$\textcircled{1} \quad \frac{\mathbb{R}[x]}{(x-a)} \underset{a \in \mathbb{R}}{\simeq} \mathbb{R}$$

$$(x-a) = \{(x-a) h(x) \mid h(x) \in \mathbb{R}[x]\}$$

$$\| (x-a) \mathbb{R}[x]$$

Cant $f: \mathbb{R}[x] \rightarrow \mathbb{R}$ m鈔fom surjectiv de mule u Kerf = $(x-a)$

$$f(P) = P(a) \in \mathbb{R}$$

$$P \in \mathbb{R}[x]$$

cf m鈔fom de mule

$$f(P+Q) = (P+Q)(a) = P(a) + Q(a) = f(P) + f(Q)$$

$$f(PQ) = (PQ)(a) = (P(a))Q(a) = f(P) \cdot f(Q)$$

$$f(1) = 1(a) = 1$$

f nrj

$$\forall b \in \mathbb{R} \exists c \in \mathbb{R}[x] \text{ ac. } f(c) = b$$

$$f(b) = b(a) = b$$

b polinom constant

$$\text{Kerf} = (x-a)$$

$$\text{Kerf} = \{P \in \mathbb{R}[x] \mid f(P) = 0\} = \{P \in \mathbb{R}[x] \mid P(a) = 0\}$$

$$\text{Kerf} = \{P \in \mathbb{R}[x] \mid x-a \mid P\} \stackrel{\text{sen la +}}{\subseteq} (x-a) \quad \textcircled{1}$$

$$\text{for } P \in (x-a) \Rightarrow P = (x-a) h(x)$$

$$h(x) \in \mathbb{R}[x]$$

$$P(a) = (\underbrace{x-a}_{(x-a)}) h(a) = 0 \Rightarrow f(P) = 0 \Rightarrow P \in \text{Kerf}$$

$$\textcircled{1} + \textcircled{2} \Rightarrow \text{Kerf} = (x-a)$$

$$\frac{\mathbb{R}[x]}{\text{Kerf}} \simeq$$

$$\frac{\mathbb{R}[x]}{(x-a)} \simeq \mathbb{R}$$

$(x-a) =$ ideal generat de polinomul $x-a$

$$\textcircled{3} \quad \frac{\mathbb{Z}[x]}{(x^2+1)} \simeq \mathbb{Z}[i] \quad \text{deo de multe}$$

Cant $f: \mathbb{Z}[x] \rightarrow \mathbb{Z}[i]$ multiform dorj de multe cu $\text{Kerf} = (x^2+1)$

$$f(P) = \in \mathbb{Z}[i]$$

$$P = a_m x^m + \dots + a_0 \underset{a_i \in \mathbb{Z}}{\sim} a_m i^m + \dots + a_1 i + a_0 \in \mathbb{Z}[i]$$

f multiform

f dorj

$\forall a+bi \in \mathbb{Z}[i]$, f cu $a+bi$ in $\mathbb{Z}[x]$ al $A(\text{cav}) = a+bi$.

$$f(a+bx) = a+bi \in \mathbb{Z}[i]$$

$$\text{Kerf} = (x^2+1)$$

$$\text{Kerf} = \{ P \in \mathbb{Z}[x] \mid f(P) = 0 \} = \{ P \in \mathbb{Z}[x] \mid P(i) = 0 \} = \{ P \in \mathbb{Z}[x] \mid P(-i) = 0 \} = \{ P \in \mathbb{Z}[x] \mid x^2+1 \mid P \} \subseteq (x^2+1) \quad \textcircled{1}$$

$$\text{deo } P \in (x^2+1) \Rightarrow P = (x^2+1)h(x)$$

$$f(P) = P(i) = (i^2+1) h(i) = 0$$

$$\Rightarrow P \in \text{Kerf} \quad \text{dorj}$$

$$\textcircled{1} + \textcircled{2} \Rightarrow \text{Kerf} = (x^2+1) \quad \textcircled{2}$$

$$\frac{\mathbb{Z}[x]}{(x^2+1)} \simeq \text{dorj} = \mathbb{Z}[i]$$

Poliuame simetrice

$$f(x) = x^3 + 2x^2 + 5 \in \mathbb{Z}[x] \quad \text{pol simetric} \quad (\text{pol cu } x=0 \text{ si} \\ \text{root este simetric})$$

$$f(x,y) = x^2 - y^2 + x + y \in \mathbb{Z}[x,y]$$

$$f(y,x) = y^2 - x^2 + y + x \quad \text{Nu pol simetric}$$

$$f(x,y) = x + y^2 + x + y = f(y,x) \quad \text{DA}$$

$$f(x,y,z) = x^3 + 2x^2 + 5 \in \mathbb{Z}[x,y,z] \quad \text{pol simetric m}$$

Seminar 5

$$f(x, y, z) = (x-y)^2 (x-z)^2 (y-z)^2$$

scrieți f în funcție de pol. simetrică fundamentală $\alpha_1, \alpha_2, \alpha_3$.

Atunci am găsit

$$f = a \alpha_1^2 \alpha_2^2 + b \alpha_1^2 \alpha_3^2 + c \alpha_2^2 \alpha_3^2 + d \alpha_1 \alpha_2 \alpha_3 + e \alpha_3^2$$

Pas I T(f) = $x^2 \cdot x^2 y^2 = x^4 y^2$ (în ordine lexicografică funcția f)

$$T(f) = x^4 y^2 z^0 \rightsquigarrow$$

$$x^{\alpha_1} y^{\alpha_2} z^{\alpha_3} \rightsquigarrow \begin{matrix} \alpha_1 - \alpha_2 & \alpha_2 - \alpha_3 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{matrix}$$

$$T(f) = \alpha_1^{4-2} \alpha_2^{2-0} \alpha_3^0 = \alpha_1^2 \alpha_2^2$$

Pas II Găsiți monomiale de același grad ca termenul dominant dar mai mici lexicografic.

$$(4, 2, 0) \rightarrow \alpha_1^2 \cdot \alpha_2^2$$

$$(4, 1, 1) \rightarrow x^4 y^2 z^1 \rightsquigarrow \alpha_1^{4-2} \alpha_2^{1-1} \alpha_3^1 = \alpha_1^2 \alpha_3^1$$

$$(4, 0, 2) \rightarrow \alpha_1^4 y^{-2} \alpha_3^2 \rightarrow \underline{\underline{NLL}}$$

$$(3, 3, 0) \rightarrow x^3 y^3 \rightarrow \alpha_1^{3-3} \alpha_2^0 \alpha_3^0 = \alpha_2^3$$

$$(3, 2, 1) \rightarrow x^3 y^2 z^1 \rightarrow \alpha_1^{3-2} \alpha_2^{2-1} \alpha_3^1 = \alpha_1^1 \alpha_2 \alpha_3^1$$

$$(2, 2, 2) \rightarrow x^2 y^2 z^2 \rightarrow \alpha_1^{2-2} \alpha_2^{2-2} \alpha_3^2 = \alpha_3^2$$

$$f(x, y, z) = a \cdot \alpha_1^2 \alpha_2^2 + b \cdot \alpha_1^2 \alpha_3^2 + c \alpha_2^2 \alpha_3^2 + d \alpha_1 \alpha_2 \alpha_3 + e \alpha_3^2$$

$$a, b, c, d = ?$$

$$a = \text{coef. termenului dominant} = \text{coef } T(f) = 1$$

$$\begin{array}{cccccc} x & y & z & \alpha_1 & \alpha_2 & \alpha_3 \end{array} \quad \begin{array}{c} f(x, y, z) \\ f(1, 1, 0) \end{array}$$

$$\begin{array}{cccccc} 1 & 1 & 0 & 2 & 1 & 0 \end{array} \quad f(1, 1, 0) = (-1)^2 \cdot (1-0) \cdot (0-1)^2 = 0$$

aleg convenireală

$$2^2 \cdot 1 + c \cdot 1 = 0 \Rightarrow 4 + c = 0 \Rightarrow c = -4$$

$$\begin{aligned} \alpha_2 &= xy^2 yz + xz \\ \alpha_3 &= x \cdot y \cdot z \\ \alpha_1 &= x + y + z \end{aligned}$$

$$\begin{array}{cccccc} x & y & z & \alpha_1 & \alpha_2 & \alpha_3 \\ -1 & 1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 2 & 0 & -3 & 2 \end{array}$$

$$\begin{array}{c} f(x, y, z) \\ f(-1, 1, 0) \end{array} \quad \begin{array}{c} x=2 \\ z=-1 \\ y=0 \end{array}$$

$$(-4)(-3)^2 + 2 \cdot 2^2 \Rightarrow 2 = -27$$

$$\begin{array}{ccccccc} x & y & z & \alpha_1 & \alpha_2 & \alpha_3 & f(x, y, z) \\ 1 & 1 & 1 & 3 & 3 & 1 & 0 \\ 1 & 1 & -1 & 1 & -1 & -1 & 0 \\ 8 & 1 & b & 27 & 4 \cdot 27 + d \cdot 9 & -27 = 0 & \cancel{\text{f}(1, 1, 1)} \\ \hline 27b + 9d = 54 \end{array}$$

$$\begin{aligned} 1 + b(c-1) + (-4) \cdot (-1) + d \cdot -27 &= 0 \\ 1 - b + 4 + d - 27 &= 0 \\ b + d &= 22 \end{aligned}$$

$$\begin{cases} -b + d = 22 \\ 27b + 9d = 54 \end{cases}; \quad b = -4 \quad \rightarrow \quad f(x, y, z) = \alpha_1^2 \alpha_2^2 - 4 \alpha_1^3 \alpha_3 - 4 \alpha_2^3 + 18 \alpha_1 \alpha_2 \alpha_3 - 27 \alpha_3^2$$

Pt. n=2 : $x_1 \neq x_2$

$$ax^2 + bx + c = 0$$

$$a = 0, a, b, c \in \mathbb{F}$$

are real distincte $\Leftrightarrow \Delta \neq 0$.

$$\Delta = b^2 - 4ac$$

$$\left. \begin{array}{l} x_1 = x_1 + x_2 = -\frac{b}{a} \\ x_2 = x_1 \cdot x_2 = \frac{c}{a} \end{array} \right\} \text{Vieta}$$

$$\Delta = b^2 - 4ac = a^2 \left(\frac{b^2}{a^2} - \frac{4c}{a} \right) = a^2 (\alpha_1^2 - 4\alpha_2)$$

$$\alpha_1^2 - 4\alpha_2 = (x_1 + x_2)^2 - 4(x_1 x_2) = (x_1 - x_2)^2$$

$$\Delta = a^2 (x_1 - x_2)^2 \neq 0 \Leftrightarrow x_1 \neq x_2$$

Pb:

Nat un polinom: $a_n x^n + \dots + a_1 x + a_0$, $a_n \neq 0$.

Daca sunt toate real distincte

$$\Delta = \prod_{i < j} (x_i - x_j)^2, \quad x_i \neq x_j, i \neq j = g(\alpha_1, \dots, \alpha_n)$$

$$\alpha_1 = \frac{-a_n - 1}{a_n}$$

$$\alpha_2 = \frac{-a_n - 2}{a_n}$$

$$\alpha_3 = \frac{-a_n - 3}{a_n}$$

$\Delta \neq 0 \Leftrightarrow$ toate real sunt distincte

② Fix $f(t) = 2t^3 - \frac{7}{3}t^2 + 7t + 1 \in \mathbb{Z}_{11}[t]$
Det. dacă f are rădăcini distincte sau căte două.

$$S_1 = -(-5)(2)^{-1} = \frac{7}{3} \cdot 6 = 30 = 8$$

$$S_2 = 4 \cdot (2)^{-1} = 4 \cdot 6 = 24 = 2$$

$$S_3 = (-1) \cdot (2)^{-1} = 10 \cdot 6 = 60 = 8$$

$$\Delta = (x-y)^2(x-z)^2(y-z)^2 = 01^2 \Delta_2^2 - 4 \Delta_3^3 \Delta_2^3 + 18 \Delta_1 \Delta_2 \Delta_3 - 275$$
$$\Delta = 236 - \dots$$

$\Delta \neq 0 \Rightarrow f(t)$ are trei rădăcini dist.

③ Fix x_1, x_2, x_3 rădăcini pol. $x^3 + 5x + 6$.
Constr. un pol. cu rădăcini y_1, y_2, y_3 unde:

$$a) y_i = x_i + 2$$

$$b) y_i = \frac{1}{x_i + 1}$$

$$a) S_1 y = y_1 + y_2 + y_3 = S_1 x + 6$$

$$S_1 x = 0$$

$$S_2 x = \frac{1}{2}$$

$$S_3 = -3$$

$$S_2 y = y_1 y_2 + y_1 y_3 + y_2 y_3 = (x_1 + 2)(x_2 + 2) + (x_1 + 2) + (x_2 + 2)(x_3 + 2)$$
$$= x_1 x_2 + 2x_2 + 4 + x_1 x_3 + 2x_1 + 4 + x_2 x_3 + 2x_3 + 2x_2 + 4$$
$$= S_2 x + 12 + 4x_1 + 4x_2 + 4x_3 = 12 + S_2 x + 4S_1 = 12 + \frac{1}{2} + 0 = \frac{29}{2}$$

$$\begin{aligned}
 S_3y &= y_1 \cdot y_2 \cdot y_3 = (x_1+1)(x_2+1)(x_3+1) \\
 &= (x_1x_2 + 2x_1 + 2x_2 + 4)(x_3+1) \quad \leftarrow x_1x_2x_3 + 2x_1x_3 + 2x_2x_3 + 8 \cdot S_3x \\
 &\approx x_1x_2x_3 + 2x_1x_3 + 2x_2x_3 + 4x_3 + 2x_1x_2 + 4x_1 + 4x_2 + 8 = \\
 &= S_3x + 2S_2x + 4S_1x + 8 \\
 &= -3 + \cancel{x \cdot \frac{5}{2}} + 4 \cdot 0 + 8 = 10
 \end{aligned}$$

$$P(y) = y^3 - S_1y^2 + S_2y - S_3y = y^3 - 6y^2 + \frac{29}{2}y - 10.$$

$$\text{b)} S_1y = y_1 + y_2 + y_3 = \frac{1}{x_1+1} + \frac{1}{x_2+1} + \frac{1}{x_3+1} =$$

$$= \frac{(x_2+1)(x_3+1) + (x_1+1)(x_3+1) + (x_1+1)(x_2+1)}{(x_1+1)(x_2+1)(x_3+1)}$$

$$= \frac{x_2x_3 + x_1x_3 + x_1x_2 + x_1 + x_2 + x_3 + 1}{(x_1+1)(x_2+1)(x_3+1)}$$

$$= \frac{S_2x + 2S_1x + 3}{(x_1+1)(x_2+1)(x_3+1)}$$

④ Calculate $x^5y^5z^5$, where x, y, z are roots of the equation $t^3 + t + 1 = 0$.

$$x, y, z \text{ are roots of } \begin{cases} x^3 + x + 1 = 0 & |x^2 \\ y^3 + y + 1 = 0 & |y^2 \\ z^3 + z + 1 = 0 & |z^2 \end{cases}$$

$$\overbrace{x^3 + y^3 + z^3}^{+} = -(x + y + z) - 3 \Rightarrow x^3 + y^3 + z^3 = -3$$

$$\Leftrightarrow \begin{cases} x^5 + x^3 + x^2 = 0 \\ y^5 + y^3 + y^2 = 0 \\ z^5 + z^3 + z^2 = 0 \end{cases}$$

$$\overbrace{x^5 + y^5 + z^5}^{-3} = -(\underbrace{x^3 + y^3 + z^3}_{-1}) - (\underbrace{x^2 + y^2 + z^2}_{S_1^2 - 2S_2}) = +3 + 2 = 5.$$

⑤ Calc. $x_1^{2018} + x_2^{2018} + x_3^{2018} \pmod{11}$, where x_1, x_2, x_3 are roots of the equation $x^3 + 2x + 3 = 0$.

$$\begin{cases} x_1^3 + 2x_1 + 3 = 0 \\ x_2^3 + 2x_2 + 3 = 0 \\ x_3^3 + 2x_3 + 3 = 0 \end{cases}$$

$$\overbrace{x_1^3 + x_2^3 + x_3^3}^{+} = -2(x_1 + x_2 + x_3) - 9$$

DA

$$\begin{cases} x_1^3 + 2x_1 + 3 = 0 \text{ } | \cdot x_1 \\ x_2^3 + 2x_2 + 3 = 0 \text{ } | \cdot x_2 \\ x_3^3 + 2x_3 + 3 = 0 \text{ } | \cdot x_3 \end{cases} \Rightarrow x_1^4 + x_2^4 + x_3^4 = -2 \underbrace{(x_1^2 + x_2^2 + x_3^2)}_{0x^2 - 2\Delta_2} - 3 \underbrace{s_1}_{= 8} = 0$$

la potenzia 5:

$$x_1^5 + x_2^5 + x_3^5 - 2 \underbrace{(x_1^3 + x_2^3 + x_3^3)}_{-9} - 3 \underbrace{(0x^2 - 2\Delta_2)}_{-4} = 18 + 12 = 30$$

la potenzia 6:

$$x_1^6 + x_2^6 + x_3^6 = -2(x_1^4 + x_2^4 + x_3^4) - 3 \underbrace{(x_1^3 + x_2^3 + x_3^3)}_{-9} = -16 + 27 = 11 \equiv 0 \pmod{11}$$

2018: 6 = 33 rest 2

$$\begin{aligned} x_1^{2018} + x_2^{2018} + x_3^{2018} &= -2(x_1^{2016} + x_2^{2016} + x_3^{2016}) - 3(x_1^{2015} + x_2^{2015} + x_3^{2015}) \\ x_1^7 + x_2^7 + x_3^7 &= -2(x_1^5 + x_2^5 + x_3^5) - 3(x_1^4 + x_2^4 + x_3^4) \\ &= -60 - 24 = -84 \equiv -4 \pmod{11} \end{aligned}$$