(S1) Geometrie vectorială

{ Sp. vect
{ Lin ind ; Lin dep
{ Sist de gen ; Baze

Sp vect (def) $V \neq \emptyset$, K - corp comutativ

$+ : V \times V \to V$ adunarea vect

$(v_1, v_2) \to v_1 + v_2$ (op. int)

$\cdot : K \times V \to V$ înmulțirea vect cu scalari

$(\alpha, v) \to \alpha v$ (op. ext)

I $(V, +)$ grup comutativ

II 1. $(\alpha_1 + \alpha_2) \cdot v = \alpha_1 v + \alpha_2 v$

2. $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$ $\quad \forall\, v, v_1, v_2 \in V$

3. $(\alpha_1 \alpha_2)\, v = \alpha_1(\alpha_2 v)$ $\quad \forall\, \alpha, \alpha_1, \alpha_2 \in K$

4. $1 \cdot v = v$

$(V/K, +, \cdot)$ sp vect peste K

$K = \mathbb{R}$ sp vect real

$K = \mathbb{C}$ sp vect complex

$\boxed{Ap\ 1}$ $(\mathbb{R}^3/\mathbb{R}, +, \cdot)$ st de sp vect real

$+ : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

$(x_1, x_2, x_3) + (y_1, y_2, y_3) \overset{def}{=} (x_1 + y_1, x_2 + y_2, x_3 + y_3)$, $\forall\, x_1, x_2, x_3, y_1, y_2, y_3$ elem din $\mathbb{R}^3$

$\cdot : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$

$\alpha(x_1, x_2, x_3) \overset{def}{=} (\alpha x_1, \alpha x_2, \alpha x_3)$, $(\forall)\, \alpha \in \mathbb{R}$, $\forall\, (x_1, x_2, x_3) \in \mathbb{R}^3$

I $(\mathbb{R}^3, +)$ grup comutativ
- asoc
- com
- $\exists$ elem neutru
- $(\forall)$ elem admite un opus

I asoc $\quad (x+y)+z = x+(y+z)$, $(\forall)\, x, y, z \in \mathbb{R}^3$

$((x_1, x_2, x_3) + (y_1, y_2, y_3)) + (z_1, z_2, z_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3) + (z_1, z_2, z_3) = ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, (x_3 + y_3) + z_3) =$

$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), x_3 + (y_3 + z_3)) =$

$= (x_1, x_2, x_3) + (y_1 + z_1, y_2 + z_2, y_3 + z_3) = (x_1, x_2, x_3) + (y_1, y_2, y_3) + (z_1, z_2, z_3)$

2. com. $x+y=y+x, \forall x,y\in\mathbb{R}^3$

$(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3)=(y_1+x_1,y_2+x_2,y_3+x_3)=(y_1,y_2,y_3)+(x_1,x_2,x_3)$

3. $(\exists)$ elem neutru

$(\exists)\, e\in\mathbb{R}^3$ aî $\forall x\in\mathbb{R}^3: x+e=e+x=x$

$e(e_1,e_2,e_3)$

$(x_1+e_1,x_2+e_2,x_3+e_3)=(x_1,x_2,x_3)$

$\Rightarrow \begin{cases} x_1+e_1=x_1 \Rightarrow e_1=0 \\ x_2+e_2=x_2 \Rightarrow e_2=0 \\ x_3+e_3=x_3 \Rightarrow e_3=0 \end{cases}$ $e=(0,0,0)=0_{\mathbb{R}^3}$

4. $(\forall)\, x\in\mathbb{R}^3, (\exists)\, x'\in\mathbb{R}^3$ aî $x+x'=x'+x=e$

$(x_1,x_2,x_3)+(x_1',x_2',x_3')=(x_1+x_1',x_2+x_2',x_3+x_3')=(0,0,0)$

$\Rightarrow \begin{cases} x_1+x_1'=0 \Rightarrow x_1'=-x_1 \\ x_2+x_2'=0 \Rightarrow x_2'=-x_2 \\ x_3+x_3'=0 \Rightarrow x_3'=-x_3 \end{cases}$

II. 1. $(\alpha_1+\alpha_2)x=\alpha_1x+\alpha_2x, (\forall)x\in\mathbb{R}^3; \alpha_1,\alpha_2\in\mathbb{R}$

$(\alpha_1+\alpha_2)(x_1,x_2,x_3)=((\alpha_1+\alpha_2)x_1,(\alpha_1+\alpha_2)x_2,(\alpha_1+\alpha_2)x_3)=$

$=(\alpha_1x_1+\alpha_2x_1,\alpha_1x_2+\alpha_2x_2,\alpha_1x_3+\alpha_2x_3)$

$=(\alpha_1x_1,\alpha_1x_2,\alpha_1x_3)+(\alpha_2x_1,\alpha_2x_2,\alpha_2x_3)=$

$=\alpha_1(x_1,x_2,x_3)+\alpha_2(x_1,x_2,x_3)$

2. $\alpha(x+y)=\alpha x+\alpha y, \forall x,y\in\mathbb{R}^3$

$\alpha((x_1,x_2,x_3)+(y_1,y_2,y_3))=\alpha(x_1+y_1,x_2+y_2,x_3+y_3)=$

$=(\alpha(x_1+y_1),\alpha(x_2+y_2),\alpha(x_3+y_3))=(\alpha x_1+\alpha y_1,\alpha x_2+\alpha y_2,\alpha x_3+\alpha y_3)=(\alpha x_1,\alpha x_2,\alpha x_3)+(\alpha y_1,\alpha y_2,\alpha y_3)=$

$=\alpha(x_1,x_2,x_3)+\alpha(y_1,y_2,y_3)$

3. $(\alpha_1\alpha_2)x=\alpha_1(\alpha_2x), \forall x\in\mathbb{R}$

$(\alpha_1\alpha_2)(x_1,x_2,x_3)=((\alpha_1\alpha_2)x_1,(\alpha_1\alpha_2)x_2,(\alpha_1\alpha_2)x_3)=$

$=(\alpha_1(\alpha_2x_1),\alpha_1(\alpha_2x_2),\alpha_1(\alpha_2x_3))=\alpha_1(\alpha_2x_1,\alpha_2x_2,\alpha_2x_3)=$

$=\alpha_1(\alpha_2(x_1,x_2,x_3))$

4. $1\cdot x=x, \forall x\in\mathbb{R}$

Deci $(\mathbb{R}^3/\mathbb{R},+,\cdot)$ sp vect real

Temă $(M_3(\mathbb{R})/\mathbb{R},+,\cdot)$ sp vect real

$+ : \mathcal{M}_3(\mathbb{R}) \times \mathcal{M}_3(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$

$A + B = (a_{ij} + b_{ij})_{i,j=\overline{1,3}}$ $(\forall) A = (a_{ij})_{i,j=\overline{1,3}}$

$B = (b_{ij})_{i,j=\overline{1,3}}$

$\cdot : \mathbb{R} \times \mathcal{M}_3(\mathbb{R}) \to \mathcal{M}_3(\mathbb{R})$

$\alpha A = (\alpha a_{ij})_{i,j=\overline{1,3}}, \forall \alpha \in \mathbb{R}, (\forall) A = (a_{ij})_{i,j=\overline{1,3}}$

Lin ind. Lin. dep

Def Fie V/K sp vect

$S = \{v_1, \dots, v_n\} \subset V$

S s.n. s.v. liniar ind dacă,

$(\forall)\ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0_V \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

$\alpha_i \in K, (\forall) i = \overline{1,n}$

{$\forall$ comb liniară nulă se se ~~[illegible]~~ realizează nulă cu scalari nuli}

b) S' s.n. s.v lin. dep. dacă nu este lin ind

{$(\exists)$ comb liniară nulă care se realizează cu scalari, nu toți nuli}

Ap 1) Stabiliți dacă urm sist vectoriale sunt liniar independente sau liniar dep.

S.V. $\begin{cases} L.I \\ L.D \end{cases}$

$\boxed{\mathbb{R}^3}$ În cazul l.d, precizați rel de dep liniară

a) $S_1 = \{v_1 = (1,-2,2), v_2 = (3,2,1)\}$

b) $S_2 = \{v_1 = (1,1,-1), v_2 = (1,-1,1), v_3 = (-1,1,1)\}$

c) $S_3 = \{v_1 = (2,3,-1), v_2 = (1,1,2), v_3 = (1,2,-3)\}$

d) $S_4 = \{v_1 = (3,-1,0), v_2 = (3,2,1), v_3 = (1,2,1), v_4 = (7,3,2)\}$

a) Fie $\alpha_1 v_1 + \alpha_2 v_2 = 0_{\mathbb{R}^3}$

$\alpha_1, \alpha_2 \in \mathbb{R}$

$\alpha_1(1,-2,2) + \alpha_2(3,2,1) = (0,0,0)$

$\Rightarrow \begin{cases} \alpha_1 + 3\alpha_2 = 0 \\ -2\alpha_1 + 2\alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0.$

$\begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} = 2+6 = 8 \neq 0 \Rightarrow \alpha_1 = \alpha_2 = 0$ sol. unică

$\Rightarrow S_1$ s.v. lin. indep

b) Fie $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{\mathbb{R}^3}$

$\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}^3$

$\alpha_1(1,1,-1) + \alpha_2(1,-1,1) + \alpha_3(-1,1,1) = (0,0,0)$

$\begin{cases} \alpha_1 + \alpha_2 - \alpha_3 = 0 \\ \alpha_1 - \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases}$

$\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -1 \cancel{+1} + 1 \cancel{+1} + 1 - 1 - 1 = -4$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$ sol unică $\Rightarrow S_2$ e lin ind

$\boxed{R_1}$ $\mathbb{R}^n$

$S = \{v_1, v_2, \dots, v_m\} \subset \mathbb{R}^n$

$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & & v_m \end{pmatrix} \in \mathcal{M}_{(n,m)}(\mathbb{R})$

a) S - s.v. lin ind dacă $rg A = m \leq n$ (dim. spațiului)

b) S - s.v. lin. dep dacă $rg A \neq m$

c) $A = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ -1 & 2 & -3 \end{vmatrix} \in \mathcal{M}_3(\mathbb{R})$

$\det A = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ -1 & 2 & -3 \end{vmatrix} = -6 + 6 \cancel{m} - 2 + 1 - 8 + 9 = 0$

$\Rightarrow rg A \leq 2 \neq 3 \Rightarrow S_3$ s.v. lin. dep.

(V1) $v_1 - v_2 - v_3 = 0_{\mathbb{R}^3}$

(V2) $\alpha_1(2,3,-1) + \alpha_2(1,1,2) + \alpha_3(1,2,-3) = (0,0,0) \Rightarrow$

$\Rightarrow \begin{cases} 2\alpha_1 + \alpha_2 + \alpha_3 = 0 \\ 3\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \\ \alpha_1 + 2\alpha_2 - 3\alpha_3 = 0 \end{cases}$ $\det A = 0 \Rightarrow rg A \leq 2$.

$\Delta_p = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1 \neq 0 \Rightarrow rg A = 2$

$\alpha_3 = \lambda$, $\lambda \in \mathbb{R}$ nec. sec.

$\begin{cases} 2\alpha_1 + \alpha_2 = -\lambda \\ 3\alpha_1 + \alpha_2 = -2\lambda \end{cases} \Leftrightarrow \alpha_1 = -\lambda, \ \alpha_2 = \lambda, \ \cancel{\alpha_3 = \lambda}, \ \lambda \in \mathbb{R}$

$-\lambda v_1 + \lambda v_2 + \lambda v_3 = 0$

$\lambda = -1 \Rightarrow v_1 - v_2 - v_3 = 0_{\mathbb{R}^3}$

d) $A = \begin{pmatrix} 3 & 3 & 1 & 7 \\ -1 & 2 & 2 & 7 \\ 0 & 1 & 1 & 2 \end{pmatrix} \in \mathcal{M}_{(3,4)}(\mathbb{R})$

$\operatorname{rg} A \leq 3 \neq 4 \overset{(R_1)}{\Rightarrow}$ S.v. lin dep

$v_1 + v_2 + v_3 = v_4$

$v_1 + v_2 + v_3 - v_4 = 0_{\mathbb{R}^3}$

$\boxed{\text{Ap 1}}$ $m = ? (\in \mathbb{R})$

$S = \{v_1 = (1, 1, m), v_2 = (1, 2, 3), v_3 = (-1, 2, 1)\}$

a) lin. dep

b) lin indep

S.s.v. lin dep $\Leftrightarrow \operatorname{rg} A \neq 3 \Leftrightarrow \det A = 0$

$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 2 \\ m & 3 & 1 \end{pmatrix}$ ~~ordo~~

$\det A = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 2 \\ m & 3 & 1 \end{vmatrix} = 2 + 2m - 3 + 2m - 6 - 1 = 4m - 8 \Rightarrow m = 2.$

b) S. s.v. lin indep $\Leftrightarrow \operatorname{rg} A = 3 \Leftrightarrow \det A \neq 0$

$\Leftrightarrow m \neq 2 \Leftrightarrow m \in \mathbb{R} \setminus \{2\} \Leftrightarrow \begin{cases} \alpha_1 = -\lambda \\ \alpha_2 = \lambda \\ \alpha_3 = \lambda \end{cases}$

<u>Sist de gen</u>

<u>Def</u> Fie V/K sp. vect finit generat.

$S = \{v_1, v_2, \dots, v_n\} \subset V$

S - sist de gen pt V dacă $(\forall) v \in V$,

$(\exists) \alpha_1, \alpha_2, \dots, \alpha_n \in K$ a.î. $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

$\boxed{R_2}$ $\mathbb{R}^n$ $S = \{v_1, ..., v_n\} \subset \mathbb{R}^n$
S - sist de gen pt $\mathbb{R}^n$ dacă $rg A = n \leq m$

S nu este sist de gen pt $\mathbb{R}^n$ dacă $rg A \neq m$

$\boxed{Ap 1}$ $S.G._{\mathbb{R}^3}$ - sunt sau nu sist de gen pt sp vect din care fac parte

a) $S_1 = \{v_1 = (1,0,1), v_2 = (2,3,4)\}$
b) $S_2 = \{v_1 = (1,1,1), v_2 = (1,1,0), v_3 = (1,0,0)\}$
c) $S_3 = \{v_1 = (2,-1,3), v_2 = (0,2,5), v_3 = (2,1,8)\}$
d) $S_4 = \{v_1 = (-1,2,1), v_2 = (0,1,-1), v_3 = (3,2,1), v_4 = (1,-3,2)\}$

Rez. a) $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 4 \end{pmatrix} \in \mathcal{M}_{(3,2)}(\mathbb{R})$

$rg A \leq 2 \neq 3 \overset{R_2}{\Rightarrow} S_1$ nu este sist de gen pt sp vect $\mathbb{R}^3$

b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$\det A = -1 \Rightarrow rg A = 3 = \dim_{\mathbb{R}} \mathbb{R}^3 \overset{\boxed{R_2}}{\Rightarrow} S_2$ sist de gen pt sp vect $\mathbb{R}^3$

GEOMETRIE

SEMINAR 2

1) $v_1, \dots, v_n \in V$ s.n liniar independenti

$\alpha v_1 + \dots + \alpha v_n = 0$

$\Rightarrow \alpha_1 = \dots = \alpha_n = 0$

2) $v_1, \dots, v_n$ sist de gen

$\Leftrightarrow \forall v \in V \; \exists a_1, \dots, a_n \in K$

a.î. $v = a_1 v_1 + \dots + a_n v_n$

① $\mathbb{R}^3$, $v_1 = (1,0,1)$, $v_2 = (0,1,0)$, $v_3 (2,0,0)$

Arătați că v_1, v_2, v_3 l. ind

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -2 \Rightarrow \text{sist lin ind}$$

Dimensiunea $(\dim_{\mathbb{R}} \mathbb{R}^3) = 3 \Rightarrow v_1, v_2, v_3$ bază pt $\mathbb{R}^3$

② $w_1 = (1,0,-1)$ $w_2 = (-1,1,0)$ $w_3 = (0,1,1)$

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 2 \neq 0 \Rightarrow \text{sist lin ind.}$$

$\dim_{\mathbb{R}} \mathbb{R}^3 = 3 \Rightarrow$ bază pt $\mathbb{R}^3$

$$\left.\begin{array}{l} B_1 = \{v_1, v_2, v_3\} \\ B_2 = \{w_1, w_2, w_3\} \end{array}\right\} \text{baze în } \mathbb{R}^3$$

$w_1 \in \mathbb{R}^3$

$w_1 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$w_2 = \beta_1 v_1 + \beta_2 v_3 + \beta_3 v_3$

$w_3 = \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3$

$\alpha_1 \cdot (1,0,1) + \alpha_2 (-1,1,0) + \alpha_3 (2,0,0) = (1,0,-1)$

$$\left.\begin{array}{l} \alpha_1 + 2\alpha_3 = 1 \\ \alpha_2 = 0 \\ \alpha_1 = -1 \end{array}\right\} \Rightarrow \alpha_3 = 1$$

$\beta_1(1,0,1)+\beta_2(0,1,0)+\beta_3(2,0,0)=(-1,1,0)$

$$\left.\begin{array}{l}\beta_1+2\beta_3=-1\\ \beta_2=1\\ \beta_1=0\end{array}\right\} \Rightarrow \beta_3=\frac{-1}{2}$$

$\gamma_1(1,0,1)+\gamma_2(0,1,0)+\gamma_3(2,0,0)=(0,1,1)$

$$\left.\begin{array}{l}\gamma_1+2\gamma_3=0\\ \gamma_2=1\\ \gamma_1=1\end{array}\right\} \Rightarrow \gamma_3=\frac{-1}{2}$$

$$A=\begin{pmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{pmatrix}=\begin{pmatrix}-1 & 0 & 1\\ 0 & 1 & 1\\ 1 & -\frac{1}{2} & -\frac{1}{2}\end{pmatrix}$$ schimbarea bazei

$[w_1]_{B_1}=(\alpha_1,\alpha_2,\alpha_3)$

$$(v_1\ \ v_2\ \ v_3)\cdot\begin{pmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{pmatrix}=(w_1\ \ w_2\ \ w_3)$$

③ $B=\{(2,2,-1),(2,-1,2),(-1,2,2)\}$

a) Arătați că B este bază pt $\mathbb{R}^3$

b) Scrieți coord. vectorului $v=(1,1,-1)$ în raport cu baza B

Fie $\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}$ a.î. $\alpha_1(2,2,-1)+\alpha_2(2,-1,2)+\alpha_3(-1,2,2)=(0,0,0)$

$$\begin{cases}2\alpha_1+2\alpha_2-\alpha_3=0\\ 2\alpha_1-\alpha_2+2\alpha_3=0\\ -\alpha_1+2\alpha_2+2\alpha_3=0\end{cases} \Rightarrow \begin{vmatrix}2 & 2 & -1\\ 2 & -1 & 2\\ -1 & 2 & 2\end{vmatrix}=-4-4-4+1-8-8$$

$=-27\neq 0\Rightarrow$ este sist liniar ind

b) $+(1,1,1)$

$|$sist lin ind$|\leq|B|=|$sist de gen$|$

$$\begin{cases}2\alpha_1+2\alpha_2-\alpha_3=1\\ 2\alpha_1-\alpha_2+2\alpha_3=0\\ -\alpha_1+2\alpha_2+2\alpha_3=1\end{cases}\quad \Delta\alpha_1=\begin{vmatrix}1 & 2 & -1\\ 1 & -1 & 2\\ 1 & -2 & 2\end{vmatrix}=-2-2+4-1-4-4=-9$$

$\Delta=-27$

$$\Delta x_2 = \begin{vmatrix} 2 & 1 & -1 \\ 2 & 1 & 2 \\ -1 & 1 & 2 \end{vmatrix} = 4-2-2-1-4-4=-9$$

$$\Delta x_3 = \begin{vmatrix} 2 & 2 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -2+4-2-1-4-4=-9$$

$$x_1 = x_2 = x_3 = \frac{1}{3}$$

$$v = \frac{1}{3}(1,1,1)$$

$$[v]_B = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

V spatiu vect.

$S \subseteq V$ subspatiu

$(S,+) \Rightarrow$ grup comutativ

$(S, K, \cdot)$ sp vectorial

a) $\forall x, y \in S \Rightarrow x+y \in S$

b) $\forall x \in S, \forall a \in K \Rightarrow a \cdot x \in S$

Exemple: 1) $S = \{(x_1, x_2, x_3, 0) / x_1, x_2, x_3 \in \mathbb{R}\} \subseteq \mathbb{R}^4$

a) $\forall (x_1, x_2, x_3, 0) + (y_1, y_2, y_3, 0) = (x_1+y_1, x_2+y_2, x_3+y_3, 0) \in S$

b) $\forall (t_1, t_2, t_3, 0) = (at_1, at_2, at_3, 0) \in S$

Din a) si b) $\Rightarrow S \subseteq \mathbb{R}^4$

ex: $\mathbb{R}_n[X] = \{f \in \mathbb{R}[X] / \text{grad} f \leq n\}$

multimea pol.

$S = \{f \in \mathbb{R}[X] / \text{grad} f = n\}$

Este S subspatiu vect on $\mathbb{R}_n[X]$?

Pp S e subspatiu vectorial

Fie 2 polinoame din S

$f = x^n + 1$ si $g = -x^n + 1$

$\Rightarrow f+g = 2 \notin S \Rightarrow$ presupunere falsă $\Rightarrow S$ nu e subspatiu vect

Dacă $0 \notin S \Rightarrow S$ nu e subspatiu

$S=\{(x_1,x_2,x_3) / x_1,x_2,x_3 \in \mathbb{R}$ și $x_1+x_2-x_3+1=0\}$

Este o subspațiu vect. în $\mathbb{R}^3$?

$$\underbrace{x_1+y_1}_{=z_1\in\mathbb{R}}+\underbrace{x_2+y_2}_{=z_2\in\mathbb{R}}-\underbrace{(x_3+y_3)}_{=z_3\in\mathbb{R}}+2=0$$

$$\underbrace{z_1+z_2-z_3+1=0}_{0}$$

$$\Rightarrow \underbrace{z_1+z_2-z_3+1}_{0}+1=0 \Rightarrow 1=0 \text{ X}$$

ex: $S_1, S_2 \subset V$

$S_1+S_2=\{u_1+u_2 / u_1\in S_1, u_2\in S_2\}$

$S_1+S_2 \subset V$ subsp vect

$v=u_1+u_2$, $u_1\in S_1, u_2\in S_2$

$w=u_1'+u_2'$, $u_1'\in S_1, u_2'\in S_2$

$$v+w = \underbrace{u_1+u_1'}_{\in S_1}+\underbrace{u_2+u_2'}_{\in S_2}$$

$$\forall a\in K \Rightarrow a\cdot v=a(u_1+u_2)=\underbrace{a\cdot u_1}_{\in S_1}+\underbrace{a\cdot u_2}_{\in S_2}$$

$S_1\cap S_2=\{x / x\in S_1$ și $x\in S_2\}$

$S_1\cap S_2$ e subsp.

$u, v\in S_1\cap S_2$

I $\underbrace{u}_{\in S_1}+\underbrace{v}_{\in S_1}\in S_1$ II $\underbrace{u}_{\in S_2}+\underbrace{v}_{\in S_2}\in S_2$

Dm I și II $u+v\in S_1\cap S_2$

$\forall a\in K, u\in S_1\cap S_2 \Rightarrow u\in S_1$ și $u\in S_2$

$\underbrace{a\cdot u}_{\in S_1}\in S_1$ și $\underbrace{a\cdot u}_{\in S_2}\in S_2 \Rightarrow a\cdot u\in S_1\cap S_2$

Deci $S_1\cap S_2$ e subsp

Teorema Grassmann

$S_1, S_2 \subseteq V$ subsp.

$\Rightarrow \dim(S_1+S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$

Ex: $\mathbb{R}^4$ $e_1=(1,0,0,0)$ $e_2=(0,1,0,0)$ $e_3=(0,0,1,0)$ $e_4=(0,0,0,1)$

$S_1 = \{(x_1, x_2, x_3, 0) / x_1, x_2, x_3 \in \mathbb{R}\}$

$S_2 = \{(y_1, 0, 0, y_4) / y_1, y_4 \in \mathbb{R}\}$

a) $S_1 \cap S_2$

b) $S_1 + S_2$

a) Fie $v \in S_1 \cap S_2 \Rightarrow v \in S_1$ și $v \in S_2$

$\Rightarrow v = (x_1, x_2, x_3, 0)$ și $v = (y_1, 0, 0, y_4) \Rightarrow v = (t_1, 0, 0, 0)$, $t_1 \in \mathbb{R}$

$t_1 = x_1 = y_1$

$S_1 \cap S_2 = \{(t_1, 0, 0, 0) / t_1 \in \mathbb{R}\}$

b) $\dim(S_1+S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$

$= 3 + 2 - 1 = 4 \Rightarrow S_1 + S_2 = \mathbb{R}^4$

23.03.2018

Seminar 3

Aplicații liniare

Fie V, W - spații vectoriale peste K.
$f: V \to W$ se numește aplicație liniară dacă $f(x+y) = f(x) + f(y)$, $\forall x, y \in V$.

$f(ax) = a \cdot f(x)$, $\forall a \in K$, $\forall x \in V$.

$Ker(f) = \{x \in V \mid f(x) = 0\}$

$Im(f) = \{y \in W \mid \exists x \in V \text{ a.î. } f(x) = y\}$

Teoremă $\dim_K V = \dim_K(Ker(f)) + \dim_K(Im(f))$

E1 $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1, 0)$.

a) Arătați că f este aplicație liniară

b) Calculați $\dim_{\mathbb{R}}(Ker(f))$, $\dim_{\mathbb{R}}(Im(f))$.

a) 1) $f((x_1, x_2) + (y_1, y_2)) = f(x_1 + y_1, x_2 + y_2)$
$= f(x_1 + y_1, 0)$ (1)

$f(x_1, x_2) + f(y_1, y_2) = f(x_1, 0) + f(y_1, 0)$
$= f(x_1 + y_1, 0)$ (2)

Din (1) și (2) $\Rightarrow f((x_1, x_2) + (y_1, y_2)) = f(x_1, x_2) + f(y_1, y_2)$

2) $f(a(x_1,x_2)) = f(ax_1, ax_2) = (ax_1, 0) =$
$= a(x_1, 0) = a \cdot f(x_1, x_2)$.

Din 1), 2) $\Rightarrow$ f este o aplicație liniară

b) $\operatorname{Ker} f = ?$

$$f(x_1,x_2)=0 \Rightarrow (x_1,0)=0 \Rightarrow \begin{cases} x_1=0 \\ x_2 \in \mathbb{R} \end{cases}$$

$$\operatorname{Ker} f = \{ (x_1,x_2) \mid x_1=0 \text{ și } x_2 \in \mathbb{R} \}$$
$$= \{ (0,x_2) \mid x_2 \in \mathbb{R} \}$$

$B = \{(0,1)\}$ bază în Ker f $\Rightarrow \dim \operatorname{Ker} f = 1$

$\operatorname{Im} f = \{ y \in W \mid \exists\, x \in V \text{ a.î. } f(x) = y \}$

$\operatorname{Im} f = \{ (x_1, 0) \mid x_1 \in \mathbb{R} \}$

$B = \{(1,0)\} \Rightarrow \dim_K (\operatorname{Im} f) = 1$

E2 $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x_1,x_2,x_3) =$
$= (x_1 + 2x_2 - x_3, x_1 + x_3)$

a) Arătați că f aplicație liniară

b) Calculați dim(Ker f), dim(Im f) = ?

1) $f((x_1,x_2,x_3)+(y_1,y_2,y_3)) = f(x_1+y_1, x_2+y_2, x_3+y_3)$
$= (x_1+y_1+2(x_2+y_2)-(x_3+y_3), x_1+y_1+x_3+y_3)$
$= (x_1+2x_2+y_1+2y_2-x_3-y_3, x_1+x_3+y_1+y_3)$
$=$

$= f(x_1, x_2, x_3) + f(y_1, y_2, y_3)$

$f(x_1, x_2, x_3) = (0, 0) \Rightarrow \begin{cases} x_1 + 2x_2 - x_3 = 0 \\ x_1 + x_3 = 0 \end{cases}$

2) $f(a(x_1, x_2, x_3)) = f(ax_1, ax_2, ax_3)$

$\operatorname{Ker} f = \{(x_1, -x_1, -x_1) \mid x_1 \in \mathbb{R}\}$

$B = \{(1, -1, -1)\} \Rightarrow \dim(\operatorname{Ker}(f)) = 1$

$\dim(\operatorname{Im}(f)) = \dim(\mathbb{R}^3) - \dim(\operatorname{Ker} f)$

$= 3 - 1 = 2.$

- $f: V \to V$ aplicație liniară s.n. endomorfism.
- valoare proprie λ.

$\lambda \in K$ cu prop. că $\exists v \in V, v \neq 0$ a.î. $f(v) = \lambda \cdot v$. v se numește vector propriu asociat valorii proprii λ.

E3. $f: \mathbb{R}^3 \to \mathbb{R}^3$.

$f(x_1, x_2, x_3) = (x_1 - 3x_2 + 3x_3, 3x_1 - 5x_2 + 3x_3, 6x_1 - 6x_2 + 4x_3)$

Pot să îi asociez lui f matricea

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

$$f(x_1, x_2, x_3) = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{cases} x_1 - 3x_2 + 3x_3 = 4x_1 \\ 3x_1 - 5x_2 + 3x_3 = 4x_2 \\ 6x_1 - 6x_2 + 4x_3 = -2x_3 \end{cases} \Leftrightarrow$$

$$\begin{cases} -3x_1 - 3x_2 + 3x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \\ 6x_1 - 6x_2 = 0 \end{cases}$$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

$-6x_1 + 3x_3 = 0 \Rightarrow -2x$

$V_4 = \{(x_1, x_1, 2x_1) \mid x_1 \in \mathbb{R}\}$

$\Rightarrow \dim(V_4) = 1 = m_4$

$\Rightarrow A / f$ e diagonalizabilă

$\downarrow$ sau

$A = C \cdot A' \cdot C^{-1}$

A - diagonalizabilă $\Rightarrow A' = C^{-1} \cdot A \cdot C$

$$A' = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$B = \{(1,1,0), (-1,0,1)\}$ bază pt V_{-2}

$B = \{(1,1,2)\}$ bază pt V_4.

$$P = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$= (1-\lambda)(-2-\lambda)(1-\lambda) + 3(-2-\lambda)(-3)$

$= (-2-\lambda)\left[(1-\lambda)^2 - 9\right] = -(2+\lambda)\left[(1-\lambda)\cancel{\ \ }-9\right]$

$= -(2+\lambda)(1-\lambda-3)(1-\lambda+3)$

$= -(2+\lambda)(-2-\lambda)(4-\lambda)$

$= (2+\lambda)^2(4-\lambda)$

Valori proprii : $-2, 4$.

$(P(\lambda) = 0 \iff (\lambda+2)^2(4-\lambda) = 0)$

$m_{-2} = 2$

$m_4 = 1$.

• Găsim vectorii proprii asociați valorilor proprii.

$\lambda = -2$

$x = (x_1, x_2, x_3)$ vector pr. asoc. lui λ.

dacă $f(x) = \lambda \cdot x$.

$$\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{cases} x_1 - 3x_2 + 3x_2 = -2x_1 \\ 3x_1 - 5x_2 + 3x_3 = -2x_2 \\ 6x_1 - 6x_2 + 4x_3 = -2x_3 \end{cases} \iff$$

$$\begin{cases} 3x_1 - 3x_2 + 3x_3 = 0 \\ 3x_1 - 3x_2 + 3x_3 = 0 \\ 6x_1 - 6x_2 + 6x_3 = 0 \end{cases} \iff x_1 - x_2 + x_3 = 0$$

$x_1 = x_2 - x_3 \Rightarrow V_\lambda = V_2 = \{(x_2 - x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\} \Rightarrow \dim(V_{-2}) = 2 = m_{-2}$

• λ valoare proprie ptr. A deci

$P(\lambda) := \det(A - \lambda \cdot I_n) = 0.$

$V_\lambda = \{ v \in V \mid f(v) = \lambda \cdot v \}$

$\dim_K V_\lambda$ s.n. multiplicitatea geometrică a lui λ.

• A este diagonalizabilă $\Leftrightarrow$

$\begin{cases} \text{• toate valorile proprii sunt din } K \\ \text{• } \forall\ \lambda \text{ val. proprie } \dim V_\lambda = m_\lambda. \end{cases}$

$$f(x_1, x_2, x_3) = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

a) Calculați val proprii și vectorii proprii asociați

b) Este A diagonalizabilă?

a) $P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix}$

$$\xrightarrow{C_2 = C_2 + C_3} \begin{vmatrix} 1-\lambda & 0 & 3 \\ 3 & -2-\lambda & 3 \\ 6 & -2-\lambda & 4-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -2-\lambda & 3 \\ -2-\lambda & 4-\lambda \end{vmatrix}$$

$$+ 3 (-1)^{1+3} \begin{vmatrix} 3 & -2-\lambda \\ 6 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) \begin{vmatrix} 1 & 3 \\ 1 & 4-\lambda \end{vmatrix}$$

$+ 3(-2-\lambda) \begin{vmatrix} 3 & 1 \\ \ldots \end{vmatrix}$

Geometrie
S5

Ex1:

$q(x) = 2x_1x_2 + x_2^2 - x_1x_3 + \frac{3}{4}x_3^2$

coef lui x_1^2 coef lui $x_1x_2/2$

$$A = \begin{array}{c} \\ \left(\begin{array}{ccc} 0 & 1 & -\frac{1}{2} \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{4} \end{array}\right) \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array} \begin{array}{l} \leftarrow x_1 \\ \leftarrow x_2 \\ \leftarrow x_3 \end{array}$$

$(i,j) \to$ coef lui $\frac{x_ix_j}{2}$, $i \neq j$

x_i^2 $i = j$

Metoda Jacobi

$\Delta_1 = 0$

$\Delta_2 = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$

$\Delta_3 = \begin{vmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{4} \end{vmatrix} = -\frac{1}{4} - \frac{3}{4} = -1 = \det A$

Dacă tot Δ ~~sunt nenuli (sau~~ sunt > 0)

$q(x') = \frac{1}{\Delta_1}(x_1')^2 + \frac{\Delta_1}{\Delta_2}(x_2')^2 + \frac{\Delta_2}{\Delta_3}(x_3')^2$

- $q(x) = x_1^2 + x_3^2 + 4x_1x_2 - 4x_1x_3$

$A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

$\Delta_1 = 1$

$\Delta_2 = -4$

$\Delta_3 = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 0 & 0 \\ -2 & 0 & 1 \end{vmatrix} = -4$

$q(x) = x_1^2 - x_3^2 - 4x_1x_2 - 4x_1x_3$

$A = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix}$

$\Delta_1 = 1$

$\Delta_2 = 4$

$\Delta_3 = 4$

$$q(x') = \frac{1}{\Delta_1}(x_1')^2 + \frac{\Delta_1}{\Delta_2}(x_2')^2 + \frac{\Delta_2}{\Delta_3}(x_3')^2$$

$$= (x_1')^2 + \frac{1}{4}(x_2')^2 + \cancel{\ } 1 (x_3')^2$$ ← forma canonică

$\exists \{f_1, \dots, f_n\}$ bază a.î. $x = \sum_{i=1}^{n} x_i' f_i$

$f_i = c_{1i} e_1 + \cancel{c_{2i} e_1} + c_{2i} e_2 + c_{3i} e_3 + \dots c_{i,i} e_i$

unde
$$\begin{pmatrix} a_{11} & \dots & a_{1i} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{ii} \end{pmatrix} \begin{pmatrix} c_{1i} \\ \vdots \\ c_{ii} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$\exists \{f_1, f_2, f_3\}$ bază în $\mathbb{R}^3$

$f_1 = c_{11} \cdot e_1$

$$\left.\begin{aligned} a_{11} \cdot c_{11} &= 1 \\ a_{11} &= 1 \end{aligned}\right| \Rightarrow c_{11} = 1$$

$f_1 = e_1 = (1, 0, 0)$

$f_2 = c_{21} \cdot e_1 + c_{22} e_2 = -\frac{1}{2}(1,0,0) - \frac{1}{4}(0,1,0) = (-\frac{1}{2}, -\frac{1}{4}, 0)$

$$\begin{pmatrix} 1, & -2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} c_{21} \\ c_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$c_{21} - 2c_{22} = 0$

$-2c_{21} = 1 \quad \Rightarrow c_{21} = -\frac{1}{2}$

$c_{22} = -\frac{1}{4}$

$f_3 = c_{31} e_1 + c_{32} e_2 + c_{33} e_3$

$$\begin{pmatrix} 1 & -2 & -2 \\ -2 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{31} \\ c_{32} \\ c_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$c_{31} - 2c_{32} - 2c_{33} = 0$

$-2c_{31} = 0 \Rightarrow c_{31} = 0$

$-2c_{31} - c_{33} = 1 \Rightarrow c_{33} = -1$

$-2c_{32} = -2 \Rightarrow c_{32} = 1$ $\qquad f_3 = (0, 1, -1)$

$(x_1, x_2, x_3) = x_1' \cdot f_1 + x_2' \cdot f_2 + x_3' \cdot f_3 = x_1'(1,0,0) + x_2'(-\frac{1}{2}, -\frac{1}{4}, 0) + x_3'(0,1,-1) =$

$= (x_1' - \frac{1}{2}x_2', -\frac{1}{4}x_2' + x_3', -x_3')$

$x_1 = x_1' - \frac{1}{2}x_2'$

$x_2 = -\frac{1}{4}x_2' + x_3' \quad \Rightarrow x_2' = -4x_2 - 4x_3$

$x_3 = -x_3' \quad \Rightarrow x_3' = -x_3$

$\Rightarrow x_1' = x_1 + \frac{1}{2}(-4x_2 - 4x_3) = x_1 - 2x_2 - 2x_3$

- V sp vect$/\mathbb{R}$

$\langle , \rangle : V \times V \to \mathbb{R}$ biliniară

$\langle x, y \rangle = \langle y, x \rangle$ (simetrică)

$\langle x, x \rangle \geq 0$

$\langle x, x \rangle = 0 \Rightarrow x = 0_V$

- $\mathbb{R}^n$, $x = (x_1, \dots x_n)$

$y = (y_1, \dots y_n)$

$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

- $\mathcal{C}([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f\text{-cont}\}$ ← fc. continue pe intervalul $[a,b]$

$\langle f, g \rangle = \int_a^b f(x) g(x) dx$

$\langle f + f', g \rangle = \langle f, g \rangle + \langle f', g \rangle$

$\langle f + f', g \rangle = \int_a^b [f(x) + f'(x)] \cdot g(x) dx = \int_a^b [f(x) g(x) + f'(x) g(x)] dx =$

$= \int_a^b f(x) g(x) dx + \int_a^b f'(x) g(x) dx = \langle f, g \rangle + \langle f', g \rangle$

$\langle f, g + g' \rangle = \langle f, g \rangle + \langle f, g' \rangle$

$\langle \alpha \cdot f, g \rangle = \alpha \cdot \langle f, g \rangle$

$\alpha \in \mathbb{R}$

$<f,f> = \int_a^b f(x) f(x)\, dx = \int_a^b f^2(x)\, dx$ $\Big|$ $\Rightarrow \int_a^b f^2(x)\, dx \geq 0$

$f^2(x) \geq 0$

$<f,f> = 0 \Leftrightarrow \int_a^b f^2(x)\, dx = 0$

$\int_a^b f^2(x) = 0$

$f : [a,b] \to [0,\infty)$ cont

$\exists x_0 \in [a,b],\ f(x_0) > 0$

$\Rightarrow \int_a^b f(x)\, dx > 0$

$f^2(x) \geq 0 \quad \forall x \in [a,b]$

Pp $f \not\equiv 0 \Rightarrow \exists x_0 \in [a,b]$ a.î. $f(x_0) \neq 0 \Rightarrow f^2(x_0) > 0$

$\Rightarrow \underbrace{\int_a^b f^2(x)}_{0} > 0$ $\quad$ **contradicție** $\quad \Rightarrow f \equiv 0$

- Norma

$\| \ \| : V \to \mathbb{R}$

$\|x\| \geq 0 \ \forall x \in V$

$\|x\| = 0 \Leftrightarrow x = 0_V$

$\|\alpha x\| = \alpha \cdot \|x\| \ , \forall \alpha \in \mathbb{R} , \forall x \in V$

$\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$

$\|x\|^2 = <x, x>$

$\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$

$x \perp y \Leftrightarrow <x, y> = 0$

(ortogonali)

$\|x\| = 1 \quad x$ s.n. vector normat

$x \in \mathbb{R}^2, \quad x = (2, 1)$

$\|x\| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$x' = \frac{1}{\|x\|} \cdot x = \left(\frac{2}{\sqrt{5}} , \frac{1}{\sqrt{5}} \right) \qquad \|x'\| = 1$

Teorema:

$\{e_1, \dots e_n\}$ bază a lui V $\Rightarrow \exists \cdot \{f_1', \dots f_n'\}$ baza ortonormată

(i.e. : $<f_i', f_j'> = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ vreau să ajung la baza asta

Alg Gram - Schmidt

Pas 1 : Găsesc o bază ortogonală

$f_1 = e_1$

$f_2 = e_2 + \alpha_{21} \cdot f_1$

$<f_2, f_1> = 0 \Rightarrow <e_2 + \alpha_{21} \cdot f_1, f_1> = 0$

$<e_2, f_1> + \alpha_{21} <f_1, f_1> = 0$

$\alpha_{21} = \frac{-<e_2, f_1>}{<f_1, f_1>}$

$f_2 = e_2 - \frac{<e_2, f_1>}{<f_1, f_1>} \cdot f_1$

$f_i = e_i - \frac{<e_i, f_1>}{<f_1, f_1>} \cdot f_1 - \dots - \frac{<e_i, f_{i-1}>}{<f_{i-1}, f_{i-1}>} \cdot f_{i-1}$

$f_i \perp f_j \ \forall i \neq j$

$f_i' = \frac{1}{\|f_i\|} \cdot f_i \Rightarrow \| f_i' \| = 1$

Ex

$B = \{ e_1 = (1,1,1), e_2 = (0,1,1), e_3 = (0,0,1)\}$ bază în $\mathbb{R}^3$

Găsiți o baza ortogonală a lui $\mathbb{R}^3$

$f_1 = e_1 = (1,1,1)$

$f_2 = e_2 - \frac{<e_2, f_1>}{<f_1, f_1>} \cdot f_1 = (0,1,1) - \frac{\overset{0\cdot 1 + 1\cdot 1 + 1 \cdot 1}{\overset{\|}{<(0,1,1),(1,1,1)>}}}{<(1,1,1),(1,1,1)>} \cdot (1,1,1) =$

$= (0,1,1) - \cancel{\frac{<(0,1,1)>}{<(1,1,1)>}} \frac{2}{3} \cdot (1,1,1) = (0,1,1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

E

$$f_3 = e_3 - \frac{\langle e_3, f_1\rangle}{\langle f_1, f_1\rangle}\cdot f_1 - \frac{\langle e_3, f_2\rangle}{\langle f_2, f_2\rangle}\cdot f_2 =$$

$$= (0,0,1) - \frac{\langle (0,0,1),(1,1,1)\rangle}{\langle (1,1,1),(1,1,1)\rangle}\cdot(1,1,1) - \frac{\langle (0,0,1),(-\frac{2}{3},\frac{1}{3},\frac{1}{3})\rangle}{\langle (-\frac{2}{3},\frac{1}{3},\frac{1}{3}),(-\frac{2}{3},\frac{1}{3},\frac{1}{3})\rangle}\cdot\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right) =$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{\frac{1}{3}}{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}\cdot\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right) =$$

$$= \left(-\frac{1}{3},-\frac{1}{3},\frac{2}{3}\right) - \frac{1}{\cancel{3}}\cdot\frac{\cancel{9}3}{\cancel{6}2}\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right) =$$

$$= \left(-\frac{1}{3},-\frac{1}{3},\frac{2}{3}\right) - \left(-\frac{1}{3},\frac{1}{6},\frac{1}{6}\right) = \left(0,-\frac{3}{6},\frac{3}{6}\right) = \left(0,-\frac{1}{2},\frac{1}{2}\right)$$

$$\|f_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\|f_2\| = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}}$$

$$\|f_3\| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}}$$

$$f_1' = \frac{1}{\sqrt{3}}\cdot(1,1,1) = \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

$$f_2' = \frac{1}{\sqrt{\frac{2}{3}}}\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right) = \frac{\sqrt{3}}{\sqrt{2}}\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right) = \left(-\frac{2\sqrt{3}}{3\sqrt{2}},\frac{\sqrt{3}}{3\sqrt{2}},\frac{\sqrt{3}}{3\sqrt{2}}\right) = \left(-\frac{\sqrt{2}}{\sqrt{3}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$$

$$f_3' = \sqrt{2}\left(0,-\frac{1}{2},\frac{1}{2}\right) = \left(0,-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$$

- $x = (x_1, x_2, x_3)$

$y = (y_1, y_2, y_3)$

$y_1\cdot\vec{i} + y_2\cdot\vec{j} + y_3\cdot\vec{k}$

$$\underset{\times}{x \cdot y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (\dots)\,\vec{i} + (\dots)\,\vec{j} + (\dots)\,\vec{k}$$

prod vect

Seminar 6

$(E, <>)$ sp euclidian , $x \perp y \iff <x, y> = 0$

$W \leq E$ subspațiu

$W^{\perp} = \{ x \in E \mid <x, w> = 0 \ \forall \ w \in W \} \cancel{= \{0\}}$

$W \cap W^{\perp} = \{ w \mid w \in W$ și $w \in W^{\perp} \} = \{ 0_E \}$

$\underset{\underset{W}{\in}}{<\overset{\shortparallel}{w}}, \underset{\underset{W}{\in}}{x}> = 0 \ ; \forall \ x \in W$

$<w, w> = 0 \iff \boxed{w = 0.}$

$$\left. \begin{array}{l} \underset{\psi}{E} = W + W^{\perp} \\ x = w + v \end{array} \right\} \Rightarrow (\star)$$

$W, W^{\perp}$ subspații $\leq E$

$\dim (W + W^{\perp}) = \dim (W) + \dim (W^{\perp}) - \dim (\underset{\overset{\shortparallel}{0}}{W \cap W^{\perp}})$

$\dim (W + W^{\perp}) = \dim (W) + \dim (W^{\perp})$

$(E, <,>)$ sp. euclidian

$f: E \to E$ apl ortogonală s.n. transformare ortogonală

- Valorile proprii ale unei transf. ortogonale pot 1 sau -1.

- Matricea asoc. unei transf. ort. are det. $= \pm 1$

$\dim E = 1$

$\{e\}$ baza in E

$f: E \to E$ transf ortog

~~$f = a \cdot e$; $a \neq 0$~~

$f(e) = a \cdot e$; $a \neq 0$

e - vect. propriu $\Rightarrow$ o val. proprie pt $f \to a = \pm 1$

$a = 1 \Rightarrow f(e) = e \Rightarrow f = id_E$

$a = -1 \Rightarrow f(e) = -1 \Rightarrow f = -id_E$

$\dim E = 2$

$f: E \to E$ transf ort.

$\Downarrow$

$A \to \det A = \pm 1.$

$\det A = 1.$

$\exists$ un un reper $\{e_1, e_2\}$

f are matricea

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \theta \in [0, 2\pi]$$

$\det A = -1$ (simetrie)

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$\dim E = 3$

$\det A = 1$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$\det A = -1.$

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Spații afine

$\mathcal{A}$ = mult. (nevidă) de puncte

V - sp. vect. peste un corp K

$\varphi : \mathcal{A} \times \mathcal{A} \longrightarrow V$

- $\varphi(A, B) + \varphi(B, C) = \varphi(A, C) \quad \forall A, B, C \in \mathcal{A}$

$\exists\, O \in \mathcal{A}$ a.î. $\varphi_O : \mathcal{A} \to V$ bijectivă

$\varphi_O(A) = \varphi(O, A)$

$\varphi(B, A) = -\varphi(A, B) \quad \forall A, B \in \mathcal{A}$

$\varphi(A, A) = 0$

$\varphi(A, B) = \varphi(A, A) + \varphi(A, B) \Longrightarrow \varphi(A, A) = 0$

$\varphi(A, B) = -\varphi(B, A)$

Ex: K^n

$\varphi(x, y) = y - x$

Prop: $\forall A \in \mathcal{A} \Rightarrow \varphi_A : \mathcal{A} \to V$

$\varphi_A(B) = \varphi(A, B)$ bijectie

$\varphi_A(A) = 0 \quad \varphi(A, A) = 0$

$\vdots$

$j=s \Rightarrow k_s < e_s, e_s> = 0 \Rightarrow k_s = 0$

$\Rightarrow v = \sum_{i=s+1}^{n} b_i e_i \Rightarrow \{e_{s+1}, \dots, e_n\}$ sist de gen. pt. $W^{\perp}$

Deci $\left.\begin{array}{l} E = W + W^{\perp} \\ W \cap W^{\perp} = \{0_E\} \end{array}\right| \Rightarrow E = W \oplus W^{\perp}$
(sumă directă)

$\dim E = \dim W + \dim W^{\perp}$
complementul ort al lui W

Ex: $\mathbb{R}^3$ sp vect $\mathbb{R}$

$S = < e_1 = (1, 0, 1), e_2 = (2, 1, -3) >$

Det. complementul ortogonal al lui S ($S^{\perp}$)

$S^{\perp} = \{x \in \mathbb{R}^3 \mid <x, s> = 0 \ \forall s \in S\} = \star$

$<x, e_1> = 0 \Leftrightarrow x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

$<x, e_2> = 0 \Leftrightarrow 2x_1 + x_2 - 3x_3 = 0 \Rightarrow -5x_3 + x_2 = 0 \Rightarrow x_2 = 5x_3$

$\star = \{x = (-x_3, 5x_3, x_3) \mid x_3 \in \mathbb{R}\}$

Baza: $(-1, 5, 1)$ baza în $S^{\perp} \Rightarrow \dim S^{\perp} = 1$ (elem din bază)
$\dim S = 2$

Aplicații

$(E_1 <, >_1), (E_2, <, >_2)$ sp euclidian

$f: E_1 \to E_2$ apl liniară

$< f(x), f(y) >_2 = <x, y>_1$

S.n. apl. ortogonală

norma

$\| \cdot \|_1 = \sqrt{<, >_1}$

$\| \cdot \|_2 = \sqrt{<, >_2}$

$\| f(x) \|_2 \overset{?}{=} \| x \|_1 \to$ DA

„ $\Rightarrow$"

Fie $\{e_1, \dots, e_s\}$ bază ortogonală în W ($W \leq E$)
Completez $\{e_1, e_2, \dots, e_s, \dots, e_n\}$ la o bază în E
Presupunem $\{e_1, \dots, e_n\}$ bază ortogonală
Afirm $\{e_{s+1}, \dots, e_n\}$ baza in $W^\perp$

e
$s+1 \leq i \leq n$
$e_i \in W^\perp$
$\langle e_i, w \rangle = 0$

$$w \in W \Rightarrow w = \sum_{j=1}^{s} a_j e_j$$

$$\langle e_i, w \rangle = \langle e_i, \sum_{j=1}^{s} a_j e_j \rangle =$$

$$= \sum_{j=1}^{s} a_j \langle e_i, e_j \rangle = 0.$$

$\{e_{s+1}, \dots, e_n\}$ sist de gen $W^\perp$

Vrean: $v \in W^\perp$, $v = \sum_{i=s+1}^{n} b_i e_i$

in E.

$$v = \sum_{i=1}^{n} b_i e_i$$

$$v \in W^\perp \Rightarrow \langle v, e_j \rangle = 0 \ \forall j = \overline{1, s}$$

$$\langle \sum_{i=1}^{n} b_i e_i, e_j \rangle = 0, \ \forall j = \overline{1, s}$$

$$\sum_{i=1}^{n} b_i \langle e_i, e_j \rangle = 0, \ \forall j \in \overline{1, s}$$

$$j = 1 \Rightarrow b_1 \underbrace{\langle e_1, e_1 \rangle}_{\neq 0} = 0 \Rightarrow b_1 = 0$$

$$j = 2 \Rightarrow b_2 \langle e_2, e_2 \rangle = 0 \Rightarrow b_2 = 0$$

Deci, pot privi A ca fiind un sp. vect. cu elem. null A.

Spațiii afine

$\Rightarrow$ Pe A $\exists$ atâtea structuri de sp. vect câte p[...] sunt în A.

Toate sunt izomorfe cu V.

Combinații afine

$P_1, \dots, P_n \in A$

$$\sum_{i=1}^{n} a_i P_i$$

Fixăm $O \in A$

$\varphi_O : A \to V$

$\varphi_O(P_i) = \overrightarrow{OP_i}$

$$\overrightarrow{OP} = \sum_{i=1}^{n} a_i \overrightarrow{OP_i}$$

Alegem $O' \in A$

$$\sum_{i=1}^{n} a_i \overrightarrow{O'P_i} = \overrightarrow{O'P'}$$

$$P = P' \iff \sum_{i=1}^{n} a_i \overrightarrow{OP_i} = \sum_{i=1}^{n} a_i \overrightarrow{O'P_i}$$

$$\overrightarrow{O'P_i} = \overrightarrow{O'O} + \overrightarrow{OP_i}$$

$$\sum_{i=1}^{n} a_i \overrightarrow{OP_i} = \sum_{i=1}^{n} a_i \left(\overrightarrow{O'O} + \overrightarrow{OP_i}\right) = \sum_{i=1}^{n} a_i \overrightarrow{O'O} +$$

$P = P' \Longleftrightarrow \overrightarrow{OP} = \overrightarrow{OP'}$

$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P'} \Big| \Rightarrow \sum_{i=1}^{n} a_i \overrightarrow{OP_i} = \overrightarrow{OO'} + \sum_{i=1}^{n} a_i \cdot \overrightarrow{O'P_i}$

$\overrightarrow{O'P_i} = \overrightarrow{OP_i} - \overrightarrow{OO'}$

$\cancel{\sum a_i \overrightarrow{OP_i}} = \overrightarrow{OO'} - \sum a_i \overrightarrow{OO'} + \cancel{\sum a_i \overrightarrow{OP_i}}$

$\Rightarrow \left(1 - \sum_{i=1}^{n} a_i\right) \overrightarrow{OO'} = 0$

$O \neq O' \Rightarrow \overrightarrow{OO'} \neq 0$

$$\boxed{1 = \sum_{i=1}^{m} a_i}$$

$M \subseteq A$

$Af(M) = \left\{ \sum_{i=1}^{m} a_i P_i \;\middle|\; P_i \in M, \sum_{i=1}^{n} a_i = 1 \;\middle|\; n \in \mathbb{N} \right\}$

acoperirea afină a lui M

Ex: $M = \{(1, 0, 1), (2, 1, -3)\} \subseteq \mathbb{R}^3$

$Af(M) = \{ a_1 P_1 + a_2 \cdot P_2 \mid a_1 + a_2 = 1 \} = \{ (a_1 + 2a_2), a_2, a_1 - 3a_2 \mid a_1 + a_2 = 1 \}$

a) $M_1 \subseteq M_2 \Rightarrow Af(M_1) \subseteq Af(M_2)$

b) $M \subseteq Af(M)$

c) $Af(Af(M)) = Af(M)$

a) $Af(M_1) = \{ \sum a_i P_i \mid P_i \in M_1; \sum a_i = 1, n \in \mathbb{N} \}$
$\subseteq \{ \sum a_i P_i \mid P_i \in M_2, \sum a_i = 1; n \in \mathbb{N} \}$

b) $\forall P \in M \Rightarrow 1 \cdot P \in Af(M) \Rightarrow M \subseteq Af(M)$

c) $_n \supseteq {}^m_n$ - evidentă

$$\sum_{j=1}^{nC} b_j \sum_{i=1}^{m} a_i \cdot P_i = \sum_{k=1}^{D} c_{k_1} \cdot P_K \in Af(M)$$

$$\sum_{i=1}^{m} a_i = 1.$$

$$\sum_{j=1}^{t} \left(b_j \sum_{i=1}^{n} a_i\right) = \sum_{j=1}^{t} b_j = 1$$

25.05.2018

SEMINAR 7

$P(x_1^0, \ldots\ldots, x_n^0)$

$v = (v_1, \ldots, v_n)$

$$d: \frac{X_1 - x_1^0}{v_1} = \frac{X_2 - x_2^0}{v_2} = \ldots = \frac{X_n - x_n^0}{v_n} = t$$

Ec. param.
$$\begin{cases} X_1 = t \cdot v_1 + x_1^0 \\ \vdots \\ X_n = t \cdot v_n + x_n^0 \end{cases}$$

$P_1(x_1^1, \ldots, x_n^1)$

$P_2(x_1^2, \ldots, x_n^2)$

$$P_1P_2: \frac{X_1 - x_1^1}{x_1^2 - x_1^1} = \ldots = \frac{X_n - x_n^1}{x_n^2 - x_n^1} = t$$

Ex: Ec. param. pt. dreapta ce trece prin punctul $P = (2, 0, 4)$ și are vectorul director $v = (1, 0, -1)$

$$d: \frac{X_1 - 2}{1} = \frac{X_2 - 0}{0} = \frac{X_3 - 4}{-1} = t$$

$$d: \begin{cases} X_1 = t + 2 \\ X_2 = 0 \\ X_3 = -t + 4 \end{cases}$$

$A=(1,-1,2)$, $B=(3,1,-4)$

$$AB: \frac{x_1-1}{3-1} = \frac{x_2+1}{1+1} = \frac{x_3-2}{-4-2}$$

$$AB: \frac{x_1-1}{2} = \frac{x_2+1}{2} = \frac{x_3-2}{-6}$$

Ex 1) Ec. planului ce trece prin $P=(2,0,4)$ și are ca subspațiu director subspațiul generat de vectorii $(1,0,-1)$, $(1,1,-1)$

$$\begin{vmatrix} x_1-2 & 1 & 1 \\ x_2-0 & 0 & 1 \\ x_3-4 & -1 & -1 \end{vmatrix} = 0$$

$\cancel{(x_1-2)\cdot 1+(x_2-0)\cdot}$

$(x_1-2)\cdot(1)+(-1)\cdot(x_2-0)\cdot 0+(x_3-4)\cdot 1=0$

$x_1-2+x_3-4=0$

$$\boxed{\pi: x_1+x_3-6=0}$$

~~Ex 2~~ $\mathcal{A}_1=(A_1, V_1, \varphi_1)$

$\mathcal{A}_1 \parallel \mathcal{A}_2 \overset{def}{\Longleftrightarrow} V_1 \subseteq V_2$ sau $V_2 \subseteq V_1$

$d \parallel \pi$ (în $\mathbb{R}^3$)

$$d: \frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p} = t$$

$\pi: ax+by+cz+d=0$

$$\begin{cases} x = mt + x_0 \\ y = nt + y_0 \\ z = pt + z_0 \\ ax + by + cz + d = 0 \end{cases}$$

$$a(mt + x_0) + b(nt + y_0) + c(pt + z_0) + d = 0$$

$$t(am + bn + cp) + \underbrace{ax_0 + by_0 + cz_0 + d}_{\neq 0} = 0 \quad (\text{pt. că } P \notin \pi)$$

$$P = (x_0, y_0, z_0) \in d$$

$$d \parallel \pi \Leftrightarrow am + bn + cp = 0$$

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j + 2 \sum_{i=1}^{n} b_i x_i + c = 0$$

$$A = (a_{ij})_{i,j=\overline{1,n}} \qquad \cancel{b = (b_1, \dots, b_n)}$$

$$\bar{A} = \left(\begin{array}{c|c} A & B \\ \hline {}^tB & c \end{array}\right) \qquad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\delta = \det A$$

$$\Delta = \det \bar{A}$$

$$d(A, B) = \|\overrightarrow{AB}\|$$

$$\overrightarrow{AB}$$

Ex: $\Gamma: 5x^2+8xy+5y^2-18x-18y+9=0$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ elipsă

$\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$ hiperbolă

$x^2=2py \quad , p\geq 0$

Consider forma pătratică $\cancel{g_h(x,y)=5x^2+8xy}$

$g_h(x,y)=5x^2+8xy+5y^2$ (Folosesc un algoritm ~~...~~ la alegere la forma canonică ... de ex Gauss)

$$A=\begin{pmatrix} 5 & 4 \\ 4 & \boxed{5} \end{pmatrix}$$

· Aflu valorile proprii

$$\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix}=0 \Leftrightarrow (5-\lambda)^2-16=0$$

$$(5-4-\lambda)(5-\lambda+4)=0$$

$$(1-\lambda)(9-\lambda)=0$$

$$\lambda_1=1 \quad , \lambda_2=9$$

$$V_{\lambda}=\left\{(x,y)\in\mathbb{R}^2 \mid A\cdot\begin{pmatrix}x\\y\end{pmatrix}=\lambda\begin{pmatrix}x\\y\end{pmatrix}\right\}$$

$$V_{\lambda_1=1}=\left\{(x,y)\in\mathbb{R}^2 \mid \begin{cases}5x+4y=x\\4x+5y=y\end{cases}\right\}$$

$$\Leftrightarrow 4x+4y=0$$

$$x=-y$$

$V_{\lambda_1} = \{ (x, -x) \mid x \in \mathbb{R} \}$

Alegem o bază ~~ortogonală~~ ortonormată în V_{λ_1}

$x \cdot (1, -1)$

$\|(1,-1)\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\Rightarrow$ baza $B_{\lambda_1} = \{ (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \} = e_1$

$V_{\lambda_2} = \{ (x,y) \in \mathbb{R}^2 \mid A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 9 \begin{pmatrix} x \\ y \end{pmatrix} \}$

$= \{ (-x, x) \mid x \in \mathbb{R} \}$

aleg o bază ortonormată în V_{λ_2}: $\{ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \} = e_2$

$$R = \begin{pmatrix} e_1 & e_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\det R = \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow R$ e o rotație.

Efectuăm rotația

$$\begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \Leftrightarrow \begin{cases} x = \frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{2}} y' \\ y = -\frac{1}{2} x' + \frac{1}{\sqrt{2}} y' \end{cases}$$

$$x \cdot y = \left(\frac{1}{\sqrt{2}} y'\right)^2 - \left(\frac{1}{\sqrt{2}} x'\right)^2$$

Γ va avea ~~ecuația~~ urm. formă: $\Gamma : 9(x')^2 + (y')^2 -$
$- 18\sqrt{2} x' + 9 = 0$

$$(3x')^2 - 2\cdot 3x'\cdot 3\sqrt{2} + (3\sqrt{2})^2 - (3\sqrt{2})^2$$

$$= (3x' - 3\sqrt{2})^2 - 18$$

$$\Gamma: (3x' - 3\sqrt{2})^2 - 18 + (y')^2 + 9 = 0$$

$$\Gamma: 9(x' - \sqrt{2})^2 + (y')^2 - 9 = 0$$

Facem translația:

$$\begin{cases} x'' = x' - \sqrt{2} \\ y'' = y' \end{cases} \Rightarrow \Gamma: (9x'')^2 + (y')^2 - 9 = 0$$

$$\Gamma \Big| \cdot \frac{1}{9}$$

$$\Rightarrow \Gamma: \frac{(x'')^2}{1} + \frac{(y'')^2}{9} - 1 = 0$$

$x = x''$

$y = y''$

$\Gamma: \frac{x^2}{1} + \frac{y^2}{9} - 1 = 0$ (elipsă)

$P(0, 3) \in \Gamma$

$T_P\Gamma$ (dreapta tg în P la Γ)

$T_P\Gamma$

$\cancel{\mathcal{E}: \frac{x^2}{a^2} \; x^2}$

$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

$P(x_0, y_0) \in \mathcal{E}$

$T_{P,\mathcal{E}}: \frac{x \cdot x_0}{a^2} + \frac{y \cdot y_0}{b^2} - 1 = 0$

$T_{P,\Gamma}: \frac{x \cdot 0}{1} + \frac{y \cdot 3}{9} - 1 = 0$

$\frac{y}{3} - 1 = 0$

$\boxed{y = 3}$

$\Gamma: \underbrace{x^2 + y^2 + 5z^2 - 6xy + 2xz - 2yz} - 4x + 8y - 12z + 14 = 0$

Consider forma patratică $q(x,y,z) = x^2 + y^2 + 5z^2 - 6xy + 2xz - 2yz$

și o aduc la forma canonică.

$$A = \begin{pmatrix} 1 & -3 & 1 \\ -3 & 1 & -1 \\ 1 & -1 & 5 \end{pmatrix}$$

· Val. proprii $\begin{vmatrix} 1-\lambda & -3 & 1 \\ -3 & 1-\lambda & -1 \\ 1 & -1 & 5-\lambda \end{vmatrix} = 0$

$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0$

$\lambda_1 = -2$

$\lambda_2 = 3$

$\lambda_3 = 6$

$V_{\lambda_1} = \{(x,y,z) \in \mathbb{R}^3 \mid A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}\}$

$= \{(x, x, 0) \mid x \in \mathbb{R}\}$

O bază ortonormată în V_{λ_1}: $\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)\}$
$ \shortparallel e_1$

$V_{\lambda_2}: \{(x,y,z) \in \mathbb{R}^3 \mid A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}\}$

$V_{\lambda_2}: \begin{cases} x - 3y + z = 3x \\ -3x + y - z = 3y \\ x - y + 5z = 3z \end{cases}$

$\begin{cases} -2x - 3y + z = 0 \\ -3x - 2y - z = 0 \\ x - y + 2z = 0 \end{cases}$ (+)

$-5x - 5y = 0 \Rightarrow x = -y$

$2x + 2z = 0$

$2x = -2z$

$x = -z$

$V_{\lambda_2} = \{(x, -x, -x) \mid x \in \mathbb{R}\}$

bază ortonormată: $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$
$ \shortparallel e_2$

$V_{\lambda_3} = \{(x, -x, 2x) \mid x \in \mathbb{R}\}$

bază ortonormată: $(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$
$ \shortparallel e_3$

$$R = (e_1 \; e_2 \; e_3) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix}$$

$\det R = 1 \Rightarrow R$ e rotație

* OBSV: dacă îmi dă $R = -1$ schimb

2 coloane între a.î. să dea 1 !

Efectuăm rotația $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

$$X = \frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{3}} y' + \frac{1}{\sqrt{6}} z'$$

$$Y = \frac{1}{\sqrt{2}} x' - \frac{1}{\sqrt{3}} y - \frac{1}{\sqrt{6}} z'$$

$$Z = -\frac{1}{\sqrt{3}} y' + \frac{2}{\sqrt{6}} z'$$

$$\Gamma: -2(x')^2 + 3(y')^2 + 6(z')^2$$
$$-4\left(\frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{3}} y' + \frac{1}{\sqrt{6}} z'\right) + 8\left(\frac{1}{\sqrt{2}} x' - \frac{1}{\sqrt{3}} y' - \frac{1}{\sqrt{6}} z'\right)$$
$$+(-12)\left(-\frac{1}{\sqrt{3}} y' + \frac{2}{\sqrt{6}} z'\right) + 14 = 0$$

$$\Gamma: -2(x')^2 + 3(y')^2 + 6(z')^2 + \frac{4}{\sqrt{2}} x' - \frac{24}{\sqrt{6}} z' + 14 = 0$$

$$\frac{\cancel{4}2\sqrt{2}}{\cancel{2}} x' \qquad \frac{4\cancel{24}\sqrt{6}}{\cancel{6}} z'$$

$$-2\left((x')^2 + \sqrt{2}x' + \left(\frac{\sqrt{2}}{2}\right)^2\right) + 1$$

$$6(z')^2 - 4\sqrt{6} z'$$

$$(\sqrt{6} z')^2 - 2\sqrt{6} z' \cdot 2 + 4 - 4$$

$$\Gamma: -2\cdot\left(x' + \frac{\sqrt{2}}{2}\right)^2 + 3(y')^2 + (\sqrt{6} z' - 2)^2 + 1 - 4 + 14 = 0$$

$$x'' = x' + \frac{\sqrt{2}}{2} \qquad 6\left(z' - \frac{2}{\sqrt{6}}\right)^2$$

$$y'' = y'$$

$$z'' = z' - \frac{2}{\sqrt{6}}$$

$$\Gamma: -2(x'')^2 + 3(y')^2 + 6(z'')^2 + 11 = 0 \quad \Big| \cdot \left(-\frac{1}{11}\right)$$

$$\Gamma : \frac{(x'')^2}{\frac{11}{2}} - \frac{(y'')^2}{\frac{11}{3}} - \frac{(z'')^2}{\frac{11}{6}} - 1 = 0$$

hiperboloid cu două pânze.