# GEOMETRIE
## SEMINAR 2

1) $v_1, \dots, v_m \in V$ s.n liniar independenti

$\alpha v_1 + \dots + \alpha v_m = 0$

$\Rightarrow \alpha_1 = \dots = \alpha_m = 0$

2) $v_1, \dots, v_m$ sist de gen

$\Leftrightarrow \forall v \in V \; \exists a_1, \dots, a_m \in K$

a.î. $v = a_1 v_1 + \dots + a_m v_m$

① $\mathbb{R}^3$, $v_1 = (1,0,1)$, $v_2 = (0,1,0)$, $v_3 (2,0,0)$

Arătați că $v_1, v_2, v_3$ l. ind

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -2 \Rightarrow \text{sist lin ind}$$

Dimensiunea $(\dim_{\mathbb{R}} \mathbb{R}^3) = 3 \Rightarrow v_1, v_2, v_3$ bază pt $\mathbb{R}^3$

② $w_1 = (1,0,-1)$ $w_2 = (-1,1,0)$ $w_3 = (0,1,1)$

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 2 \neq 0 \Rightarrow \text{sist lin ind.}$$

$\dim_{\mathbb{R}} \mathbb{R}^3 = 3 \Rightarrow$ bază pt $\mathbb{R}^3$

$$\left.\begin{array}{l} B_1 = \{v_1, v_2, v_3\} \\ B_2 = \{w_1, w_2, w_3\} \end{array}\right\} \text{baze în } \mathbb{R}^3$$

$w_1 \in \mathbb{R}^3$

$w_1 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$w_2 = \beta_1 v_1 + \beta_2 v_3 + \beta_3 v_3$

$w_3 = \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3$

$\alpha_1 \cdot (1,0,1) + \alpha_2 (-1,1,0) + \alpha_3 (2,0,0) = (1,0,-1)$

$$\left.\begin{array}{l} \alpha_1 + 2\alpha_3 = 1 \\ \alpha_2 = 0 \\ \alpha_1 = -1 \end{array}\right\} \Rightarrow \alpha_3 = 1$$

$\beta_1(1,0,1)+\beta_2(0,1,0)+\beta_3(2,0,0)=(-1,1,0)$

$$\left.\begin{array}{l}\beta_1+2\beta_3=-1\\ \beta_2=1\\ \beta_1=0\end{array}\right\} \Rightarrow \beta_3=\frac{-1}{2}$$

$\gamma_1(1,0,1)+\gamma_2(0,1,0)+\gamma(2,0,0)=(0,1,1)$

$$\left.\begin{array}{l}\gamma_1+2\gamma_3=0\\ \gamma_2=1\\ \gamma_1=1\end{array}\right\} \Rightarrow \gamma_3=\frac{-1}{2}$$

$$A=\begin{pmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{pmatrix}=\begin{pmatrix}-1 & 0 & 1\\ 0 & 1 & 1\\ 1 & -\frac{1}{2} & -\frac{1}{2}\end{pmatrix}$$ schimbarea bazei

$[w_i]_{B_1}=(\alpha_1,\alpha_2,\alpha_3)$

$$(v_1\ v_2\ v_3)\cdot\begin{pmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{pmatrix}=(w_1\ w_2\ w_3)$$

③ $B=\{(2,2,-1),(2,-1,2),(-1,2,2)\}$

a) Arătați că B este bază pt $\mathbb{R}^3$

b) Scrieți coord. vectorului $v=(1,1,-1)$ în raport cu baza B

Fie $\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}$ a.î. $\alpha_1(2,2,-1)+\alpha_2(2,-1,2)+\alpha_3(-1,2,2)=(0,0,0)$

$$\begin{cases}2\alpha_1+2\alpha_2-\alpha_3=0\\ 2\alpha_1-\alpha_2+2\alpha_3=0\\ -\alpha_1+2\alpha_2+2\alpha_3=0\end{cases} \Rightarrow \begin{vmatrix}2 & 2 & -1\\ 2 & -1 & 2\\ -1 & 2 & 2\end{vmatrix}=-4-4-4+1-8-8$$

$=-27\neq0\Rightarrow$ este sist liniar ind

b) $\dagger(1,1,1)$

$|$sist lin ind$|\leq|B|\leq|$sist de gen

$$\begin{cases}2\alpha_1+2\alpha_2-\alpha_3=1\\ 2\alpha_1-\alpha_2+2\alpha_3=\varphi\\ -\alpha_1+2\alpha_2+2\alpha_3=1\end{cases}$$

$$\Delta\alpha_1=\begin{vmatrix}1 & 2 & -1\\ 1 & -1 & 2\\ 1 & -2 & 2\end{vmatrix}=-2-2+4-1-4-4=-9$$

$\Delta=-27$

$\Delta_2=$

$D\Delta_3=$

$\Delta x_2 = \begin{vmatrix} 2 & 1 & -1 \\ 2 & 1 & 2 \\ -1 & 1 & 2 \end{vmatrix} = 4-2-2-1-4-4 = -9$

$\Delta x_3 = \begin{vmatrix} 2 & 2 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -2+4-2-1-4-4 = -9$

$x_1 = x_2 = x_3 = \frac{1}{3}$

$v = \frac{1}{3}(1,1,1)$

$[v]_B = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

---

V spatiu vect.

$S \subseteq V$ subspatiu

$(S,+)$ > grup comutativ

$(S, K, \cdot)$ sp vectorial

a) $\forall x, y \in S \Rightarrow x+y \in S$

b) $\forall x \in S, \forall a \in K \Rightarrow a \cdot x \in S$

Exemple: 1) $S = \{(x_1, x_2, x_3, 0) / x_1, x_2, x_3 \in \mathbb{R}\} \subseteq \mathbb{R}^4$

a) $\forall (x_1, x_2, x_3, 0) + (y_1, y_2, y_3, 0) = (x_1+y_1, x_2+y_2, x_3+y_3, 0) \in S$

b) $\forall (t_1, t_2, t_3, 0) = (at_1, at_2, at_3, 0) \in S$

Din a) si b) $\Rightarrow S \subseteq \mathbb{R}^4$

ex: $\mathbb{R}_n[X] = \{f \in \mathbb{R}[X] / \text{grad} f \leq n\}$

multimea pol.

$S = \{f \in \mathbb{R}[X] / \text{grad} f = n\}$

Este S subspatiu vect in $\mathbb{R}_n[X]$?

Pp S e subspatiu vectorial

Fie 2 polinoame din S

$f = x^n + 1$ si $g = -x^n + 1$

$\Rightarrow f + g = 2 \notin S \Rightarrow$ presupunere falsa $\Rightarrow$ S nu e subspatiu vect

Daca $0 \notin S \Rightarrow$ S nu e subspatiu

$S = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}$ şi $x_1 + x_2 - x_3 + 1 = 0\}$
Este o subspaţiu vect. în $\mathbb{R}^3$?

$$\underbrace{x_1 + y_1}_{=t_1 \in \mathbb{R}} + \underbrace{x_2 + y_2}_{=t_2 \in \mathbb{R}} - \underbrace{(x_3 + y_3)}_{=t_3 \in \mathbb{R}} + 2 = 0$$

$$\underbrace{t_1 + t_2 - t_3 + 1 = 0}_{\infty}$$

$$\Rightarrow \underbrace{t_1 + t_2 - t_3 + 1}_{0} + 1 = 0 \Rightarrow 1 = 0 \;\%$$

ex: $S_1, S_2 \subset V$

$S_1 + S_2 = \{u_1 + u_2 \mid u_1 \in S_1, u_2 \in S_2\}$

$S_1 + S_2 \subset V$ subsp vect

$v = u_1 + u_2$ , $u_1 \in S_1, u_2 \in S_2$

$w = u'_1 + u'_2$ , $u'_1 \in S_1, u'_2 \in S_2$

$$v + w/2 = \underbrace{u_1 + u'_1}_{\in S_1} + \underbrace{u_2 + u'_2}_{\in S_2}$$

$$\forall a \in K \Rightarrow a \cdot v = a(u_1 + u_2) = \underbrace{a \cdot u_1}_{\in S_1} + \underbrace{a \cdot u_2}_{\in S_2}$$

$S_1 \cap S_2 = \{x \mid x \in S_1$ şi $x \in S_2\}$

$S_1 \cap S_2$ e subsp.

$u, v \in S_1 \cap S_2$

I $\underset{\in S_1}{u} + \underset{\in S_1}{v} \in S_1$ II $\underset{\in S_2}{u} + \underset{\in S_2}{v} \in S_2$

Dm I şi II $u + v \in S_1 \cap S_2$

$\forall a \in K, u \in S_1 \cap S_2 \Rightarrow u \in S_1$ şi $u \in S_2$

$\underset{\in S_1}{a \cdot u} \in S_1$ şi $\underset{\in S_2}{a \cdot u} \in S_2 \Rightarrow a \cdot u \in S_1 \cap S_2$

Deci $S_1 \cap S_2$ e subsp

Teorema Grassmann

$S_1, S_2 \subseteq V$ subsp.

$\Rightarrow \dim(S_1+S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$

Ex: $\mathbb{R}^4$ $e_1=(1,0,0,0)$ $e_2=(0,1,0,0)$ $e_3=(0,0,1,0)$ $e_4=(0,0,0,1)$

$S_1 = \{(x_1, x_2, x_3, 0) / x_1, x_2, x_3 \in \mathbb{R}\}$

$S_2 = \{(y_1, 0, 0, y_4) / y_1, y_4 \in \mathbb{R}\}$

a) $S_1 \cap S_2$

b) $S_1 + S_2$

a) Fie $v \in S_1 \cap S_2 \Rightarrow v \in S_1$ și $v \in S_2$

$\Rightarrow v = (x_1, x_2, x_3, 0)$ și $v = (y_1, 0, 0, y_4) \Rightarrow v = (t_1, 0, 0, 0)$, $t_1 \in R$

$t_1 = x_1 = y_1$

$S_1 \cap S_2 = \{(t_1, 0, 0, 0) / t_1 \in \mathbb{R}\}$

b) $\dim(S_1+S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$

$= 3 + 2 - 1 = 4 \Rightarrow S_1 + S_2 = \mathbb{R}^4$