SEMINAR 4

Forma fundamentală de izomorfism pt inele

Fie R un inel, fie J un ideal al inelului
$\Rightarrow J$ este un subgrup (normal)
$\Rightarrow \left(\frac{R}{J}, +\right)$ grup factor $\quad \hat{r} = r + J$ un element $r \in R$

Definesc $\widehat{x \cdot y} = \hat{x} \cdot \hat{y}$

$\left(\frac{R}{J}, +, \cdot\right)$ inel factor

Fie $(R_1, +, \cdot)$, $(R_2, *, \circledcirc)$ inele

$f : R_1 \to R_2$ s.n. morfism de inele

dacă
$$\begin{cases} f(x+y) = f(x) * f(y) \\ f(xy) = f(x) \circ f(y) \\ f(1_{R_1}) = 1_{R_2} \end{cases} \quad \forall x, y \in R_1$$

TFI

Fie R_1, R_2 inele, $f : R_1 \to R_2$ morfism de inele

Atunci $\frac{R_1}{\mathrm{Ker} f} \simeq \mathrm{Im} f$

(f surj : $\mathrm{Im} f = R_2$)
$\hookrightarrow \frac{R_1}{\mathrm{Ker} f} \simeq R_2$

② $\dfrac{\mathbb{Z}[x]}{n\mathbb{Z}[x]} \simeq \mathbb{Z}_n[x]$ izom. de inele

Caut $f: \mathbb{Z}[x] \to \mathbb{Z}_n[x]$ morfism surjectiv de inele cu $\operatorname{Ker} f = (n)$

$(n) = (n)\mathbb{Z}[x] = \{ n\,h(x) \mid h(x) \in \mathbb{Z}[x] \}$

$\boxed{\underset{\in \mathbb{Z}[x]}{f(P)} = \hat{P}}$ schimb coef din întregi în resturi ale lui n

$P = a_m x^m + \dots + a_0 \in \mathbb{Z}[x] \to \hat{P} = \hat{a}_m x^m + \dots + \hat{a}_0$, $\hat{a}_i \in \mathbb{Z}_n$

$a_i \in \mathbb{Z}$

f morfism de inele

$f(P+Q) = \widehat{P+Q} = \hat{P} + \hat{Q} = f(P) + f(Q)$

$f(PQ) = \widehat{PQ} = \hat{P} \cdot \hat{Q} = f(\hat{P}) + f(\hat{Q})$

$f(1) = \hat{1}$

f surj $\forall \hat{P} \in \mathbb{Z}_n[x]$, $\exists$ ceva $\in \mathbb{Z}[x]$ a.î. $f(\text{ceva}) = \hat{P}$

dar $f(P) = \hat{P} \Rightarrow$ f surj

$\operatorname{Ker} f = (n)$

$\operatorname{Ker} f = \{ P \in \mathbb{Z}[x] \mid f(P) = 0 \} = \{ P \in \mathbb{Z}[x] \mid \hat{P} = 0 \} = \{ P \in \mathbb{Z}[x] \mid$ coef lui $P / n \}$

$\operatorname{Ker} f \subseteq (n)\mathbb{Z}[x]$ ① $\qquad P = n(\dots)$

Fie $P \in (n)\mathbb{Z}[x] \Rightarrow P = n\,h(x)$

$h(x) \in \mathbb{Z}[x]$

$\to f(P) = \hat{P} = \widehat{n \cdot h(x)} = \hat{n} \cdot \widehat{h(x)} = 0$

$f(P) = 0 \Rightarrow P \in \operatorname{Ker} f \Rightarrow (n) \subseteq \operatorname{Ker} f$ ②

① + ② $\Rightarrow$

$\operatorname{Ker} f = (n)$

TFI $\Rightarrow$ $\dfrac{\mathbb{Z}[x]}{n\mathbb{Z}[x]} \simeq \underset{\operatorname{Im} f}{\mathbb{Z}_n[x]}$

Polinoamele Simetrice Fundamentale

$K[x_1, \dots, x_n]$ inelul de pol în ~~n~~ nedeterminate peste corpul K

$s_1 = x_1 + x_2 + \dots + x_n$

$s_2 = x_1x_2 + x_1x_3 + \dots + \; = \sum_{i<j} x_ix_j$

$\vdots$

$s_k = \sum_{i_1<\dots<i_k} x_{i_1} x_{i_2} \dots x_{i_k}$

$s_n = x_1x_2 \dots x_n$

Teorema Fundamentală a ~~polinoamelor simetrice~~ simetrice

$\forall f \in K[x_1 \dots x_n]$ polin. simetric se scrie în mod unic ca o expresie polinomială de pol simetrice fundamentale adică $\forall f \in K[x_1 \dots x_n]$ $\exists! g$ a.î. $f(x_1, \dots x_n) = g(s_1 \dots s_n)$

1) $f(x,y) = x^2+y^2 = (x+y)^2 - 2xy = s_1^2 - 2s_2$

$s_1 = x+y$

$s_2 = xy$

2) $f(x,y) = x^3 + 2xy^2 + 2x^2y + x + y + y^3 = f(y,x)$

$f(x,y) = x^3 + y^3 + 2xy(y+x) + x + y =$

$= (x+y)(x^2 - xy + y^2) + 2s_1s_2 + s_1$

$f(x,y) = s_1((x+y)^2 - 3xy) + 2s_1s_2 + s_2$

$= s_1(s_1^2 - 3s_2) + 2s_1s_2 + s_1$

$= s_1^3 - s_1s_2 + s_1$

3) $f(x,y,z) = x^3+y^3+z^3$

împart componentele omogene în fct de grad

$T(f) = cx_1^{a_1}x_2^{a_2} \dots x_n^{a_n} \rightsquigarrow cs_1^{a_1-a_2} s_2^{a_2-a_3} \dots s_n^{a_n}$

$T(f) = x^3 \rightsquigarrow 1 \cdot s_1^{3-0} s_2^{0-0} s_3^{0} = s_1^3$

$x^3y^0z^0$

$\Rightarrow f(x,y,z) = s_1^3 - s_1^3 + x^3 + y^3 + z^3$

④ $\frac{\mathbb{R}[x]}{(x-a)} \simeq \mathbb{R}$
$a \in \mathbb{R}$

$(x-a)$ - idealul generat de polinomul x-...

$(x-a) = \{(x-a)h(x) \mid h(x) \in \mathbb{R}[x]\}$
$\parallel$
$(x-a)\mathbb{R}[x]$

Caut $f: \mathbb{R}[x] \to \mathbb{R}$ $\boxed{\text{morfism surjectiv}}$ de inele cu $\boxed{\ker f = (x-a)}$

$\boxed{f(P) = P(a) \in \mathbb{R}, \quad P \in \mathbb{R}[x]}$

DEM
⇒ f morfism de inele
$f(P+Q) = (P+Q)(a) = P(a) + Q(a) = f(P) + f(Q)$
$f(PQ) = (PQ)(a) = P(a)Q(a) = f(P) \cdot f(Q)$
$f(1) = 1(a) = 1$

⇒ f surj
$\forall\, b \in \mathbb{R}\ \exists$ ceva $\in \mathbb{R}[x]$ a.î. $f(\text{ceva}) = b$
$f(b) = b(a) = b$
b polinom constant

$\ker f = (x-a)$
$\ker f = \{P \in \mathbb{R}[x] \mid f(P) = 0\} = \{P \in \mathbb{R}[x] \mid P(a) = 0\}$
Bezout
$\ker f = \{P \in \mathbb{R}[x] \mid x-a \mid P\} \subseteq (x-a)$ ①
Fie $P \in (x-a) \Rightarrow P = (x-a)h(x)$
$h(x) \in \mathbb{R}[x]$
$P(a) = \underbrace{(a-a)}_{0} h(a) = 0 \Rightarrow f(P) = 0 \Rightarrow P \in \ker f$
$\Rightarrow (x-a) \in \ker f$ ②

① + ② $\Rightarrow \ker f = (x-a)$

TFI ⇒ $\frac{\mathbb{R}[x]}{\ker f} \simeq$

$\frac{\mathbb{R}[x]}{(x-a)} \simeq \mathbb{R}$

③ $\frac{\mathbb{Z}[x]}{(x^2+1)} \simeq \mathbb{Z}[i]$ izo de inele

* Caut $f: \mathbb{Z}[x] \to \mathbb{Z}[i]$ morfism surj de inele cu $\operatorname{Ker} f = (x^2+1)$

$f(P) =$
$\in \mathbb{Z}[x]$

$P = a_m x^m + \dots + a_0 \rightsquigarrow a_m i^m + \dots + a_1 i + a_0 \in \mathbb{Z}[i]$
$a_i \in \mathbb{Z}$

f morfism

f surj

$\forall a+bi \in \mathbb{Z}[i]$, $\exists$ ceva în $\mathbb{Z}[x]$ a.î. $f(\text{ceva}) = a+bi$

$f(\underbrace{a+bx}_{\in \mathbb{Z}[x]}) = a+bi$

$\operatorname{Ker} f = (x^2+1)$

$\operatorname{Ker} f = \{P \in \mathbb{Z}[x] \mid f(P) = 0\} = \{P \in \mathbb{Z}[x] \mid P(i) = 0\} = \cancel{\{P \in \mathbb{Z}[x]\}}$ $P(-i)=0$

$= \{P \in \mathbb{Z}[x] \mid x^2+1 \mid P\} \subseteq (x^2+1)$ ①

Fie $P \in (x^2+1) \Rightarrow P = (x^2+1)h(x)$

$f(P) = P(i) = \underbrace{(i^2+1)}_{=0} h(i) = 0$

$\Rightarrow P \in \operatorname{Ker} f \Rightarrow (x^2+1) \subseteq \operatorname{Ker} f$ ②

① + ② $\Rightarrow \operatorname{Ker} f = (x^2+1)$

$\xRightarrow{TFI} \frac{\mathbb{Z}[x]}{(x^2+1)} \simeq \operatorname{Im} f = \mathbb{Z}[i]$
f surj

Polinoame Simetrice

$f(x) = x^3 + 2x^2 + 5 \in \mathbb{Z}[x]$ pol simetric ($\forall$ pol într-o sg nedet e simetric)

$f(x,y) = x^2 - y^2 + x + y \in \mathbb{Z}[x,y]$

$f(y,x) \neq y^2 - x^2 + y + x$ Nu ~~pol~~ e simetric

$f(x,y) = x + y^2 + x + y = f(y,x)$ DA

$f(x,y,z) = x^3 + 2x^2 + 5 \in \mathbb{Z}[x,y,z]$ ~~nu~~ e simetric în