Analiză S3

Serii de puteri

Exercițiul 1:

Să se calculeze $\overset{\text{raza de convergență}}{R}$, A și f în cazul următoarelor serii de puteri

a) $\sum\limits_{n=0}^{+\infty} \frac{x^{n+1}}{n+1}$

b) $\sum\limits_{n=0}^{+\infty} \frac{2^n \cdot x^{n+1}}{n+1}$

c) $\sum\limits_{n=0}^{n} (-1)^n (n+1) x^n$

d) $\sum\limits_{n=0}^{\infty} (-1)^n n \cdot x^n$

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a) $x_0 = 0$ $\quad a_{n+1} = \frac{1}{n+1} \Rightarrow a_n = \frac{1}{n} \ \forall n \in \mathbb{N}^*$

$a_0 = 0$

$l = \overline{\lim} \sqrt[n]{|a_n|} = \overline{\lim} \sqrt[n]{\frac{1}{n}}$

$\lim \sqrt[n]{\frac{1}{n}} = ?$ (cum o calculăm?)

Crit radicalului

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \underline{1} \Rightarrow$$

$$\Rightarrow \exists \lim_{n\to\infty} \sqrt[n]{\frac{1}{n+1}} = 1 \Rightarrow \overline{\lim} \sqrt[n]{\frac{1}{n+1}} = 1 = l$$

$$l \in (0; +\infty) \Rightarrow R = \frac{1}{l} = \frac{1}{1} = 1$$

$$\begin{cases} (x_0 - R, x_0 + R) \subseteq A \subseteq [x_0 - R; x_0 + R] \\ A \subseteq \mathbb{R} \end{cases} \Rightarrow$$

Studiem ce interval e, sunt 4 cazuri

$$\Rightarrow \begin{cases} (-1, 1) \subseteq A \subseteq [-1, 1] \\ A \subseteq \mathbb{R} \end{cases}$$

$$x = -1 \Rightarrow \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{n+1}$$ Dacă e convergentă $\Rightarrow -1 \in A$

Dacă e divergentă $\Rightarrow -1 \notin A$

$$= \sum_{n=0}^{+\infty} \frac{1}{n+1} \cdot (-1)^{n+1}$$

$$\lim_{n\to\infty} \frac{1}{n+1} = 0 \quad , \quad \frac{1}{n+1} - \frac{1}{n} = \frac{n - n - 1}{(n+1) \cdot n} = -\frac{1}{n^2 + n} < 0$$

verificăm dacă e descrescător

$\overset{\text{crit Leibniz}}{\Longrightarrow}$ seria e convergentă $\Rightarrow -1 \in A$

$$x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \overset{n+1=m}{=} \sum_{m}^{\infty} \frac{1}{m} \Rightarrow$$

seria putere $\sum_{n=0}^{\infty} \frac{1}{n^a}$

$\Rightarrow \alpha = 1 \Rightarrow$ seria este divergentă $\Rightarrow$
$1 \notin A$

$A = [-1, 1)$

$f : [-1, 1) \longrightarrow \mathbb{R} \quad f(x) = \sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1}$

- $f(x_0) = a_0 \Leftrightarrow f(0) = 0$ (condiția inițială)
- f este indefinit derivabilă pe $(-1, 1)$
- f continuă în -1

$\left(\frac{x^{n+1}}{n+1}\right)' = x^n$ (seamănă cu o serie putere cunoscută, prima)

$$f'(x) = \sum_{n=0}^{+\infty} \left(\frac{x^{n+1}}{n+1}\right)' = \sum_{n=0}^{+\infty} x^n = \frac{1}{1-x} \; \forall x \in (-1,1)$$

$f'(x) = \frac{1}{1-x} \; \forall x \in (-1, 1)$

$\Rightarrow f(x) = \int \frac{1}{1-x} dx + \mathcal{C} = -\ln(1-x) + c$

$f(x) = -\ln(1-x) + c$
$f(0) = 0 \Leftrightarrow -\ln(1-0) + c = 0 \Rightarrow \underline{c = 0}$

$f(x) = -\ln(1-x)\ \forall x \in (-1,1)$

f continuă în $-1 \Rightarrow f(-1) = \lim_{x \searrow -1} f(x) = \lim_{x \searrow -1} -\ln(1-x)$

$= -\ln(2)$

$$f(x) = \begin{cases} -\ln(1-x) & x \in (-1,1) \\ -\ln(2) & x = -1 \end{cases}$$

b) $\sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n+1} = g(x)$

$x_0 = 0,\ a_0 = 0$

$a_{n+1} = \frac{2^n}{n+1} \Rightarrow a_n = \frac{2^{n-1}}{n}\ \forall n \in \mathbb{N}^*$

$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = f(x)\ \forall x \in [-1,1)$

$f(2x) = \sum_{n=0}^{\infty} \frac{2^{n+1} x^{n+1}}{n+1}\ \forall 2x \in [-1;1) \mid :2 \Rightarrow$

$\Rightarrow \frac{f(2x)}{2} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n+1}\ \forall x \in [-\frac{1}{2}; \frac{1}{2}) \Rightarrow g(x) = \frac{f(2x)}{2}$

$$g(x) = \begin{cases} -\frac{\ln(1-2x)}{2}, & x \in (-\frac{1}{2}, \frac{1}{2}) \\ -\frac{\ln 2}{2}, & x = -\frac{1}{2} \end{cases}$$

$A=[-\frac{1}{2};\frac{1}{2})$

$R=\frac{1}{2}$

c) $\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

$x_0=0$, $a_0=1$

$a_n=(-1)^n(n+1)$

$R=\overline{\lim}\sqrt[n]{|a_n|}=\overline{\lim}\sqrt[n]{n+1}$

$\lim\limits_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim\limits_{n\to\infty}\frac{n+2}{n+1}=1\Rightarrow\exists\lim\limits_{n\to\infty}\sqrt[n]{n+1}=1\Rightarrow$

$\Rightarrow\overline{\lim}\sqrt[n]{n+1}=1\ \forall n\in\mathbb{N}$

$l=1\Rightarrow 1\in(0,\infty)\Rightarrow R=\frac{1}{l}=1$

$\begin{cases}(x_0-R,x_0+R)\\ A\subseteq\mathbb{R}\end{cases}\Leftrightarrow\begin{cases}(-1,1)\subseteq A\subseteq[-1,1]\\ A\subseteq\mathbb{R}\end{cases}$

Luăm $\boxed{x=-1}\Rightarrow\sum_{n=0}^{\infty}(-1)^n\cdot(n+1)(-1)^n=\sum_{n}^{\infty}(n+1)$

$\lim\limits_{n\to\infty}n+1=\infty\Rightarrow\sum_{0}^{\infty}(n+1)$ e divergentă $\Rightarrow -1\notin A$ I

Luăm $\boxed{x=1}\Rightarrow\sum_{n=0}^{\infty}(-1)^n(n+1)(1)^n=\sum_{n=0}^{\infty}(-1)^n(n+1)$

Nu există $\lim\limits_{n\to\infty}(-1)^n(n+1)\Rightarrow\sum(-1)^n(n+1)$ - e divergentă $\Rightarrow 1\notin A$ II

$\forall x\in[-\frac{1}{2},\frac{1}{2})$

Din I, II $\Rightarrow A=(-1,1)$

$f:(-1,1)\to\mathbb{R}$

$$f(x)=\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$$

$-f(x_0)=a_0 \iff f(0)=1$

- f e definit deriv pe $(-1,1)$

$$\int f(x)\,dx=\int(-1)^n(n+1)x^n\,dx=(-1)^n\int(n+1)x^n\,dx$$
$$=(-1)^n x^{n+1}+C$$

$g'(x)=f(x)$

$$g(x)=\sum_{n=0}^{+\infty}(-1)^n x^{n+1}+c=\sum_{n=0}^{+\infty}(-1)^n x^n\cdot x+c=$$

$$=x\cdot\sum_{n=0}^{\infty}(-1)^n x^n+c=\frac{x}{1+x}+c\quad\forall x\in(-1,1)$$

$$f(x)=\left(\frac{x}{1+x}+c\right)'=\frac{x+1-x}{(1+x)^2}=\frac{1}{(1+x)^2}\ \forall x\in(-1,1)$$

d) $$\sum_{n=0}^{\infty}(-1)^n n x^n=\sum_{n=0}^{+\infty}(-1)^n(n+1-1)x^n=$$
$$=\sum_{n=0}^{\infty}\left[(-1)^n(n+1)x^n-(-1)^n x^n\right]=$$
$$=\sum(-1)^n(n+1)x^n-\sum(-1)^n x^n=\frac{1}{(1+x)^2}-\frac{1}{1+x}$$
$$=-\frac{x}{(1+x)^2}\ \forall x\in(-1,1)$$

$A=(-1,1)$

$R=1$

Temă: $\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1}$; $R, A, f = ?$

$x_0 = 0$

$a_0 = 0$

$\frac{1}{2n+1} = a_{2n+1}$ -> impari

$a_{2n} = 0$ $\forall n \in \mathbb{N}^*$

-> pari