Phillips Curve derivation

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$$\max_{P_t(j)} E_t \sum_{s=0}^{\infty} Q_{t,t+s}\alpha_k^s \left[P_t(j)Y_{t+s}(j) - W_{K,t+s}Y_{t+s}(j)\right]$$

The firm also knows that:

$$Y_{t+s}(j) = C_{t+s}(j) + G_{t+s}(j)$$

$$C_{t+s}(j) = C_{1,t}\left(\frac{p_t(j)}{p_{1,t}}\right)^{-\theta} * \frac{1}{n}$$

$$C_{t+s}(j) = C_{2,t}\left(\frac{p_t(j)}{p_{2,t}}\right)^{-\theta} * \frac{1}{1-n}$$

the FOC w.r.t. $p_t(j)$ will be:

$$E_t \sum_{s=0}^{\infty} Q_{t,t+s}\alpha_k^s \left[P_{k,t}^* \frac{\partial Y_{t+s}(j)}{\partial P_{k,t}^*} + Y_{t+s}(j) - W_{k,t+s}\frac{\partial Y_{t+s}(j)}{\partial P_{k,t}^*}\right] = 0$$

substituting in $\dfrac{\partial Y_{t+s}(j)}{\partial P_{k,t}^*} = C_{t+s}(j)\dfrac{-\theta}{P_{k,t}^*}$

$$E_t \sum_{s=0}^{\infty} Q_{t,t+s}\alpha_k^s \left[(1-\theta)C_{t+s}(j)\right] = E_t \sum_{s=0}^{\infty} Q_{t,t+s}\alpha_k^s \left[-\theta \frac{W_{K,t+s}}{P_{k,t}^*} C_{t+s}(j) - G_{t+s}(j)\right]$$

Where the desired markup $M = \dfrac{\theta}{\theta - 1}$

$$E_t \sum_{s=0}^{\infty} Q_{t,t+s}\alpha_k^s \left[C_{t+s}(j)\frac{P_{k,t}^*}{P_{K,t}}\right] = E_t \sum_{s=0}^{\infty} Q_{t,t+s}\alpha_k^s \left[M\frac{W_{K,t+s}}{P_{K,t+s}}\frac{P_{K,t+s}}{P_{K,t}} C_{t+s}(j) + \frac{1}{\theta-1} G_{t+s}(j)\frac{P_{k,t}^*}{P_{K,t}}\right]$$

Log linearizing around the symmetric steady state:

$$E_t \sum_{s=0}^{\infty} (\beta\alpha_k)^s \left[\bar{C}\frac{\bar{P}_{k,t}^*}{\bar{P}_{K,t}}\Big(1 + \hat{c}_{t+s}(j) + \hat{p}_{k,t}^* - \hat{p}_{K,t}\Big)\right] =$$

$$E_t \sum_{s=0}^{\infty} (\beta\alpha_k)^s \left[M\frac{\bar{W}}{\bar{P}_k}\bar{C}\Big(1 + \hat{\psi}_{t+s} + \hat{p}_{k,t+s} - \hat{p}_{k,t} + \hat{c}_{t+s}(j)\Big) + \frac{1}{\theta-1}\bar{G}_k\frac{\bar{P}_{k,t}^*}{\bar{P}_{K,t}}\Big(1 + \hat{g}_{t+s}(j) + \hat{p}_{k,t}^* - \hat{p}_{K,t}\Big)\right]$$

Knowing that, at the steady state, $\dfrac{\bar{P}^*_{k,t}}{\bar{P}_{K,t}} = 1$, $\dfrac{\bar{W}}{\bar{P}_k} = \dfrac{1}{M}$, and dividing everything by $\bar{C}$:

$$E_t \sum_{s=0}^{\infty} (\beta\alpha_k)^s \Big[\hat{p}^*_{k,t} - \hat{p}_{K,t}\Big] = E_t \sum_{s=0}^{\infty} (\beta\alpha_k)^s \Big[\hat{\psi}_{t+s} + \hat{p}_{k,t+s} - \hat{p}_{k,t} + (1-A)\big(\hat{g}_{t+s}(j) + \hat{p}^*_{k,t} - \hat{p}_{K,t}\big)\Big]$$

Defining $A = 1 - \dfrac{1}{\theta-1}\dfrac{\bar{G}_k}{\bar{C}} = 1 - \dfrac{1}{\theta-1} s^g_k \dfrac{1-s^p}{s^p}$

$$\frac{A}{1-\alpha_k\beta}\Big(\hat{p}^*_{k,t} - \hat{p}_{K,t}\Big) = E_t \sum_{s=0}^{\infty} (\beta\alpha_k)^s \Big[\hat{\psi}_{t+s} + (1-A)\hat{g}_{t+s}(j)\Big] + E_t \sum_{s=0}^{\infty} \sum_{\tau=0}^{s-1} \hat{\pi}_{k,t+\tau+1}$$

Since $\displaystyle\sum_{\tau=0}^{\infty} (\beta\alpha_k)^{\tau} \sum_{l=0}^{\tau-1} \pi_{k,t+1+l} = \frac{\alpha_k\beta}{1-\alpha_k\beta} \sum_{\tau=0}^{\infty} (\beta\alpha_k)^{\tau} \pi_{k,t+1+\tau}$

$$\Big(\hat{p}^*_{k,t} - \hat{p}_{K,t}\Big) = \frac{1-\alpha_k\beta}{A} E_t \sum_{s=0}^{\infty} (\beta\alpha_k)^s \Big[\hat{\psi}_{t+s} + (1-A)\hat{g}_{t+s}(j)\Big] + \frac{\alpha_k\beta}{A} E_t \sum_{s=0}^{\infty} \hat{\pi}_{k,t+s+1}$$

In difference form:

$$\Big(\hat{p}^*_{k,t} - \hat{p}_{K,t}\Big) = \alpha_k\beta E_t\Big(\hat{p}^*_{k,t+1} - \hat{p}_{K,t+1}\Big) + \frac{1-\alpha_k\beta}{A}\Big(\hat{\psi}_t + (1-A)\hat{g}_t(j)\Big) + \frac{\alpha_k\beta}{A} E_t \hat{\pi}_{k,t+1}$$

Given that $p^*_{kt} - p_{kt} = \dfrac{\alpha_k}{1-\alpha_k}\pi_{kt}$

$$\hat{\pi}_{k,t} = \beta\left(\alpha_k + \frac{1-\alpha_k}{A}\right) E_t \hat{\pi}_{k,t+1} + \frac{(1-\alpha_k)(1-\alpha_k\beta)}{\alpha_k A}\Big(\hat{\psi}_t + (1-A)\hat{g}_t(j)\Big)$$

Substituting back in for A, the final equation becomes:

$$\hat{\pi}_{k,t} = \beta\left(\alpha_k + \frac{1-\alpha_k}{1 - \dfrac{s^g_k}{\theta-1}\dfrac{1-s^p}{s^p}}\right) E_t \hat{\pi}_{k,t+1} + \frac{(1-\alpha_k)(1-\alpha_k\beta)}{\alpha_k\left(1 - \dfrac{s^g_k}{\theta-1}\dfrac{1-s^p}{s^p}\right)}\Big(\hat{\psi}_t + \frac{s^g_k}{\theta-1}\frac{1-s^p}{s^p}\hat{g}_t(j)\Big)$$

Derivation of Optimal Government Spending Policy

The Welfare function $\mathcal{W}$ is:

$$\mathcal{W} = \frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \sum_k n_k \left[\frac{\theta * (1-\chi_k)}{\lambda_k} * \pi_{k,t}^2 + (1+\phi)\hat{y}_{k,t}^2 + \frac{\chi_k}{(1-\chi_k)(1-\chi)} (\hat{y}_{k,t} - \hat{g}_{k,t})^2 - \frac{2\chi}{1-\chi} \hat{y}_{1,t} + \frac{2 s_k^g}{s^p} \hat{g}_{1,t} \right] \right.$$

Constraints

$$\hat{\pi}_{k,t} - \beta \left(\alpha_k + \frac{1-\alpha_k}{D_k} \right) E_t \hat{\pi}_{k,t+1} - \frac{(1-\alpha_k)(1-\alpha_k \beta)}{\alpha_k D_k} \left(\hat{\psi}_{k,t} + \frac{s_k^g}{\theta - 1} \frac{1-s^p}{s^p} \hat{g}_{k,t} \right) = 0$$
$$p_{k,t} - \pi_{k,t} + p_{k,t-1} = 0$$
$$s_1^p p_{1,t} + (1-s_1^p) p_{2,t} - p_t^c = 0$$
$$s_1^p \pi_{1,t} + (1-s_1^p) \pi_{2,t} - \pi_t^c = 0$$
$$\hat{y}_t - n\hat{y}_{1,t} - (1-n) = 0$$
$$\hat{\tau}_t - \hat{p}_{1,t} + \hat{p}_{2,t} = 0$$
$$\sum_{k=1}^{2} \nu_{k,t} \left(\hat{y}_{k,t} - s^p \hat{c}_t - s_k^g \hat{g}_{k,t} \right) = 0$$
$$\hat{\pi}_t - n\hat{\pi}_{1,t} - (1-n)\hat{\pi}_{2,t} = 0$$
$$\hat{c}_t - E_t \hat{c}_{t+1} + i_t - E_t \hat{\pi}_{t+1} = 0$$
$$\hat{\psi}_{1,t} - \hat{c}_t - \phi \hat{y}_{1,t} + (1 - s_{p1})\hat{\tau}_t = 0$$
$$\hat{\psi}_{2,t} - \hat{c}_t - \phi \hat{y}_{2,t} - s_{p1} \hat{\tau}_t = 0$$