ASSIGNMENT 4

A-10-2.

Consider a completely state controllable system

$$\dot{x} = Ax + Bu$$

Define the controllability matrix as M:

$$M = [B \vdots AB \vdots \cdots \vdots A^{n-1}B]$$

Show that

$$M^{-1}AM = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & & 0 & -a_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}$$

where $a_1, a_2, \ldots, a_n$ are the coefficients of the characteristic polynomial

$$|sI - A| = s^n + a_1 s^{n-1} + \cdots + a^{n-1}s + a_n$$

Solution. **Let us consider the case where n=3. We shall show that**

$$AM = M\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \qquad \text{(10-140)}$$

The left-hand side of Equation (10-140) is

$$AM = A[B \vdots AB \vdots A^2B] = [AB \vdots A^2B \vdots A^3B]$$

The right-hand side of Equation (10-140) is

$$[B \vdots AB \vdots A^2B]\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} = [AB \vdots A^2B \vdots -a_3B - a_2AB - a_1A^2B] \qquad \text{(10-141)}$$

The Cayley-Hamilton theorem states that matrix A satisfies its own characteristic equation or, in the case of n=3.

$$A^3 + a_1A^2 + a_2A + a_3I = 0 \qquad \text{(10-142)}$$

Using Equation (10-142), the third column of the right-hand side of Equation (10-141) becomes

$$-a_3B - a_2AB - a_1A^2B = (-a_3I - a_2A - a_1A^2)B = A^3B$$

Thus Equation (10-141) becomes

$$[\boldsymbol{B} \vdots \boldsymbol{AB} \vdots \boldsymbol{A}^2\boldsymbol{B}]\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} = [\boldsymbol{AB} \vdots \boldsymbol{A}^2\boldsymbol{B} \vdots \boldsymbol{A}^3\boldsymbol{B}]$$

Hence, the left-hand side and the right-hand side of Equation (10-140) are the same.

We have thus shown that Equation (10-140_) is true. Consequently,

$$\boldsymbol{M}^{-1}\boldsymbol{AM} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$$

The preceding derivation can be easily extended to the general case of any positive integer n.

A-10-4.

Consider the state equation

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

where

$$A = \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The rank of the controllability matrix M,

$$M = [\boldsymbol{B} \vdots \boldsymbol{AB}] = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix}$$

is 2. Thus, the system is completely state controllable. Transform the given state equation into the controllable canonical form.

Solution. **Since**

$$|\boldsymbol{sI} - \boldsymbol{A}| = \begin{vmatrix} s-1 & -1 \\ 4 & s+3 \end{vmatrix} = (\mathrm{s}-1)(\mathrm{s}+3) + 4 = \mathrm{s}^2 + 2\mathrm{s} + 1 = \mathrm{s}^2 + a_1 s + a_2$$

we have

$$a_1 = 3, \qquad a_2 = 1$$

Define

$$T = MW$$

where

$$M = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix}, \quad W = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Then

$$T = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix}$$

and

$$T^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

Define

$$x = T\hat{x}$$

Then the state equation becomes

$$\dot{\hat{x}} = T^{-1}AT\hat{x} + T^{-1}Bu$$

Since

$$T^{-1}AT = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

we have

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

which is in the controllable canonical form.

A-10-6.

A regulator system has a plant

$$\frac{Y(s)}{U(s)} = \frac{10}{(s+1)(s+2)(s+3)}$$

Define state variable as

$$x_1 = y$$

$$x_2 = \dot{x}_1$$

$$x_3 = \dot{x}_2$$

By use of the state-feedback control $u = -Kx$, it is described to place the closed-loop poles at

$$s = -2 + j2\sqrt{3}, \quad s = -2 - j2\sqrt{3}, \quad s = -10$$

Obtain the necessary state-feedback gain matrix K with MATLAB.

Solution. **The state-space equations for the system become**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0u]$$

Hence,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0], \qquad D = [0]$$

(Note that, for the pole placement, matrices C and D do not affect the state-feedback gain matrix K.)

Two MATLAB programs for obtaining state-feedback gain matrix K are given in MATLAB Programs 10-24 and 10-25

MATLAB Program 10–24
A = [0 1 0;0 0 1;-6 -11 -6]; B = [0;0;10]; J = [-2+j*2*sqrt(3) -2-j*2*sqrt(3) -10]; K = acker(A,B,J) K = 15.4000 4.5000 0.8000

MATLAB Program 10–25
A = [0 1 0;0 0 1; -6 -11 -6]; B = [0;0;10]; J = [-2+j*2*sqrt(3) -2-J*2*Sqrt(3) -10]; K = place(A,B,J) place: ndigits= 15 K = 15.4000 4.5000 0.8000

A-10-9.

Consider the system defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

$$\boldsymbol{y} = \boldsymbol{Cx}$$

where

$$A = \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad C = [1 \quad 1]$$

The rank of the observability matrix N,

$$N = [\boldsymbol{C}^* \vdots \boldsymbol{A}^*\boldsymbol{C}^*] = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}$$

is 2. Hence, the system is completely observable. Transform the system equations into the observable canonical form.

Solution.

Since

$$|\boldsymbol{sI} - \boldsymbol{A}| = \boldsymbol{s}^2 + 2\boldsymbol{s} + 1 = \boldsymbol{s}^2 + \boldsymbol{a}_1\boldsymbol{s} + \boldsymbol{a}_2$$

we have

$$\boldsymbol{a}_1 = 2, \quad \boldsymbol{a}_2 = 1$$

Define

$$\boldsymbol{Q} = (\boldsymbol{WN}^*)^{-1}$$

where

$$\boldsymbol{N} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}, \quad W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$Q^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

Define

$$x = Q\hat{x}$$

Then the state equation becomes

$$\dot{\hat{x}} = Q^{-1}AQ\hat{x} + Q^{-1}Bu$$

or

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -4 & -3\end{bmatrix}\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\ 2\end{bmatrix}u$$

$$= \begin{bmatrix}0 & -1\\ 1 & -2\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}0\\ 2\end{bmatrix}u \qquad (10-157)$$

The output equation becomes

$$y = CQ\hat{x}$$

or

$$\boldsymbol{y} = [\mathbf{1} \quad \mathbf{1}]\begin{bmatrix}-\mathbf{1} & \mathbf{0}\\ \mathbf{1} & \mathbf{1}\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{x}}_1\\ \hat{\boldsymbol{x}}_2\end{bmatrix} = [\mathbf{0} \quad \mathbf{1}]\begin{bmatrix}\hat{\boldsymbol{x}}_1\\ \hat{\boldsymbol{x}}_2\end{bmatrix} \qquad (\mathbf{10-158})$$

A-10-10.

For the system defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

$$\boldsymbol{y} = \boldsymbol{Cx}$$

consider the problem of designing a state observer such that the desired eigenvalues for the observer gain matrix are $\mu_1, \mu_2 \dots, \mu_n$.

Show that the observer gain matrix given by Equation (10–61), rewritten as

$$\boldsymbol{K_e} = (\boldsymbol{WN^*})^{-1}\begin{bmatrix}\boldsymbol{\alpha_n - a_n}\\ \boldsymbol{\alpha_{n-1} - a_{n-1}}\\ \vdots\\ \boldsymbol{\alpha_1 - a_1}\end{bmatrix} \qquad \textbf{(10-159)}$$

can be obtained from Equation (10-13) by considering the dual problem. That is, the matrix K_e can be determined by considering the pole-placement problem for the dual system, obtaining the state-feedback matrix K, and taking its conjugate transpose, or $K_E = K^*$.

Solution.

The dual of the given system is

$$\dot{\boldsymbol{z}} = \boldsymbol{A * z + C * v} \qquad (10\text{-}160)$$

$$n = B * z$$

Using the state-feedback control

$$v = -Kz$$

Equation (10-160) becomes

$$\dot{\boldsymbol{z}} = (A^* - C * K)z$$

Equation (10-13), which is rewritten here, is

$$K = [\boldsymbol{\alpha_n - a_n} \vdots \boldsymbol{\alpha_{n-1} - a_{n-1}} \vdots \cdots \vdots \boldsymbol{\alpha_2 - a_2} \vdots \boldsymbol{\alpha_1 - a_1}]T^{-1} \qquad (10\text{-}161)$$

where

$$T = MW = [\boldsymbol{C^*} \vdots \boldsymbol{A^*C^*} \vdots \cdots \vdots (\boldsymbol{A^*})^{\boldsymbol{n-1}}\boldsymbol{C^*}] = N$$

Hence, matrix T can also be written as

$$T = NW$$

Since $W = W^*$, we have

$$T^* = W^*N^* = WN^*$$

and

$$(T^*)^{-1} = (WN^*)^{-1}$$

Taking the conjugate transpose of both sides Equation (10-146), we have

$$K^* = (T^{-1}) * \begin{bmatrix} \boldsymbol{\alpha_n - a_n} \\ \boldsymbol{\alpha_{n-1} - a_{n-1}} \\ \vdots \\ \boldsymbol{\alpha_1 - a_1} \end{bmatrix} = (\boldsymbol{T^*})^{-1} \begin{bmatrix} \boldsymbol{\alpha_n - a_n} \\ \boldsymbol{\alpha_{n-1} - a_{n-1}} \\ \vdots \\ \boldsymbol{\alpha_1 - a_1} \end{bmatrix} = (\boldsymbol{WN^*})^{-1} \begin{bmatrix} \boldsymbol{\alpha_n - a_n} \\ \boldsymbol{\alpha_{n-1} - a_{n-1}} \\ \vdots \\ \boldsymbol{\alpha_1 - a_1} \end{bmatrix}$$

Since $K_e = K^*$, this last equation is the same as Equation (10-159). Thus, we obtained Equation (10-159) by considering the dual program.