ASSIGNMENT 5

A-10-3.

Consider a completely state controllable system

$$\dot{x} = Ax + Bu$$

Define

$$M = [B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{n-1}B]$$

and

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where a_i's are the coefficients of the characteristic polynomial

$$|sI - A| = s^n + a_1 s^{n-1} + \cdots + a^{n-1}s + a_n$$

Define also

$$T = MW$$

Show that

$$T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Solution. **Let us consider the case where n=3. We shall show that**

$$T^{-1}AT = (MW)^{-1}A(MW) = W^{-1}(M^{-1}AM)W = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \qquad (10\text{-}143)$$

Referring to Problem A-10-2, we have

$$M^{-1}AM = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$$

Hence Equation (10-143) can be rewritten as

$$W^{-1}\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} W = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Therefore, we need to show that

$$\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} W = W \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \qquad (10\text{-}144)$$

The left-hand side of Equation (10-144) is

$$\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The right-hand side of Equation (10-144) is

$$\begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} = \begin{bmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Clearly, Equation (10-144) holds true. Thus, we have shown that,

$$T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Next, we shall show that

$$T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that Equation (10-145) can be written as

$$B = T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = MW \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Nothing that

$$T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [B \vdots AB \vdots A^2B] \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [B \vdots AB \vdots A^2B] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = B$$

we have

$$T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The derivation shown here can be easily extended to the general case of any positive integer n.

A-10-12.

Consider a completely state controllable system defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu} \qquad \textbf{(10-167)}$$

$$\boldsymbol{y} = \boldsymbol{Cx}$$

where $x = state\ vector(n - vector)$

$u = control\ signal(scalar)$

$y = output\ signal(scalar)$

$A = n \times n\ constanct\ matrix$

$B = n \times 1\ constant\ matrix$

$C = 1 \times n\ constant\ matrix$

Suppose that the rank of the following $(\mathbf{n+1}) \times (\mathbf{n+1})$ matrix

$$\begin{bmatrix} A & B \\ -C & 0 \end{bmatrix}$$

is $\mathbf{n+1}$. Show that the system defined by

$$\dot{e} = \hat{A}e + \hat{B}u_e \qquad (10\text{-}168)$$

where

$$\hat{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad u_e = u(t) - u(\infty)$$

is completely state controllable.

Solution. **Define**

$$\boldsymbol{M} = [\boldsymbol{B} \vdots \boldsymbol{AB} \vdots \cdots \vdots \boldsymbol{A}^{n-1}\boldsymbol{B}]$$

Because the system given by Equation (10-167) is completely state controllable, the rank of matrix M is n. Then the rank of

$$\begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}$$

is n+1. Consider the following equation:

$$\begin{bmatrix} A & B \\ -C & 0 \end{bmatrix}\begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} AM & B \\ -CM & 0 \end{bmatrix}$$

Since matrix

$$\begin{bmatrix} A & B \\ -C & 0 \end{bmatrix}$$

Is of rank n+1, the left-hand side of Equation (10-169) is of rank n+1. Therefore, the right-hand side of Equation (10-169) is also of rank n+1. Since

$$\begin{bmatrix} AM & B \\ -CM & 0 \end{bmatrix} = \begin{bmatrix} A[\boldsymbol{B} \vdots \boldsymbol{AB} \vdots \cdots \vdots \boldsymbol{A}^{n-1}\boldsymbol{B}] & B \\ -C[\boldsymbol{B} \vdots \boldsymbol{AB} \vdots \cdots \vdots \boldsymbol{A}^{n-1}\boldsymbol{B}] & 0 \end{bmatrix}$$

$$= \begin{bmatrix} AB \vdots & \boldsymbol{A}^2\boldsymbol{B} & \vdots \cdots \vdots & \boldsymbol{A}^n\boldsymbol{B} & \vdots \boldsymbol{B} \\ -CB \vdots & -\boldsymbol{CAB} & \vdots \cdots \vdots & -\boldsymbol{CA}^{n-1}\boldsymbol{B} & \vdots \boldsymbol{0} \end{bmatrix}$$

$$= [\hat{A}\hat{B} \vdots \ \hat{A}^2\hat{B} \vdots \cdots \vdots \hat{A}^n\hat{B} \vdots \hat{B}]$$

We find that the rank of

$$[\hat{A}\hat{B} \vdots \ \hat{A}^2\hat{B} \vdots \cdots \vdots \hat{A}^n\hat{B} \vdots \hat{B}]$$

is n+1. Thus, the system defined by Equation (10-168) is completely state controllable.

B–10–5.

Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Show that this system cannot be stabilized by state-feedback control $u = -Kx$, whatever matrix K is chosen.

Solution. **Substituting**

$$u = -Kx = -[k_1 \quad k_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

into the state equation, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \quad k_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -k_1 & 1-k_2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The characteristic equation becomes

$$|\boldsymbol{sI} - \boldsymbol{A}| = \begin{vmatrix} s+k_1 & -1+k_2 \\ 0 & s-2 \end{vmatrix} = (s+k_1)(s-2) = 0$$

Because of the presence of one eigenvalue (s=2) in the right-half s plane, the system is unstable whatever values k_1 and k_2 may assume.

B–10–17.

Consider the system defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

Where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -a \end{bmatrix}$$

$$a = adjustable\ parameter > 0$$

Determine the value of the parameter a so as to minimize the following performance index:

$$J = \int_0^\infty x^T x\, dt$$

Assume that the initial state $\boldsymbol{x}(\boldsymbol{0})$ is given by

$$x(0) = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

Solution.

To determine the parameter a in matrix A, we first determine matrix

$$\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{I}$$

or

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -a \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -a \end{bmatrix} = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The result is

$$P = \begin{bmatrix} \frac{a^2+5a-1}{2(2a-1)} & \frac{2a^2+3}{2(2a-1)} & \frac{1}{2} \\ \frac{2a^2+3}{2(2a-1)} & \frac{a^3+a^2+a+7}{2(2a-1)} & \frac{a^2+a+1}{2(2a-1)} \\ \frac{1}{2} & \frac{a^2+a+1}{2(2a-1)} & \frac{a+3}{2(2a-1)} \end{bmatrix}$$

Then we can obtain the optimal value of the parameter a that minimizes the performance index J for any given initial condition x(0). Since x(0) is given by

$$\boldsymbol{x}(\boldsymbol{0}) = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

the performance index J can be simplified to

$$J = x^T(0)Px(0) = p_{11}c_1^2$$

Therefore, we obtain

$$J = \frac{a^2+5a-1}{4a-2}c_1^2$$

To minimize J, we determine a from dJ/da=0, or

$$\frac{4a^2-4a-6}{(4a-2)^2} = 0$$

from which we get

$$a = 1.823, a = -0.823$$

Since a is specified to be positive, we discard the negative value of a.

Thus we choose $a = 1.823$. Nothing that $a = 1.823$ satisfies the condition for the minimum (d^2J/da^2>0), the optimal value of a is 1.823.

B-10-19.

Determine the optimal control signal u for the system defined by

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

such that the following performance index is minimized:

$$J = \int_0^\infty (x^T x + u^2) dt$$

Solution.

The optimal control signal u will have the form $u = -Kx$. Therefore, the performance index J becomes

$$\boldsymbol{J} = \int_0^\infty (x^T x + u^2) dt = \int_0^\infty x^T (I + K^T K) x dt$$

Since R=I in this problem, Equation (10-15) becomes

$$(A - BK)^T P + P(A - BK) = -(I + K^T K)$$

and Equation (10-117) becomes

$$K = P^{-1} B^T P = B^T P$$

where P is determined from the reduced matrix Riccati equation:

$$A^T P + PA - PBB^T P + I = 0$$

Solving for P, requiring that it be positive definite, we obtain

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

The optimal feedback gain matrix K becomes

$$K = B^T P = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Thus, the optimal control signal u is given by

$$u = -Kx = -x_1 - x_2$$