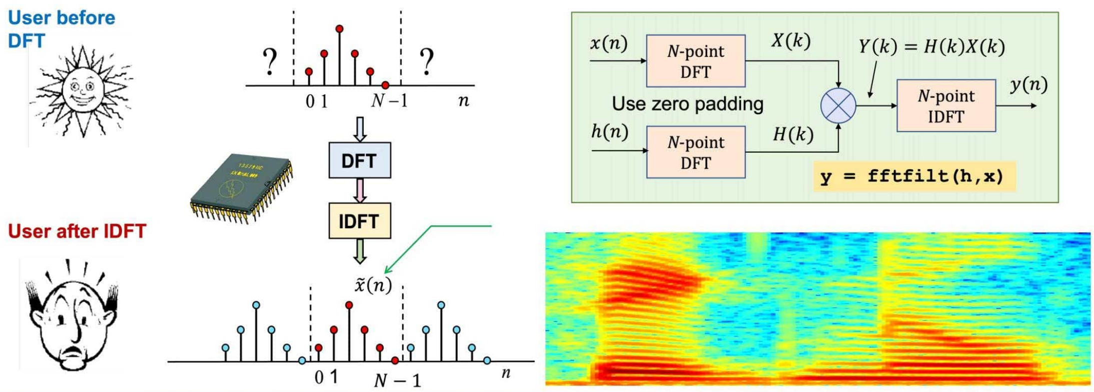
The Discrete Fourier Transform



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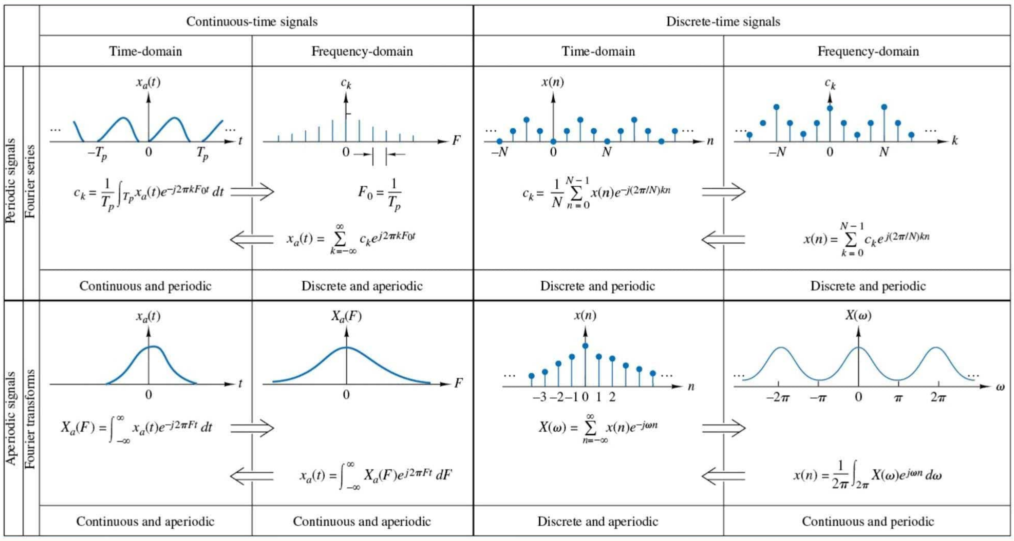
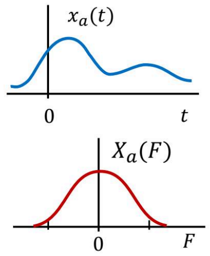
Fourier Representation of Signals

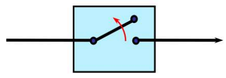


1

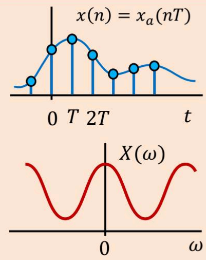
Time-Domain Sampling



Ideal Sampler



$F_s = \frac{1}{T}$



This is what we have for DSP!

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi Ft} dF \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi Ft} dt$$

$$x(n) = \frac{1}{2\pi} \int_{-1/2}^{1/2} X(\omega) e^{j\omega n} d\omega \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

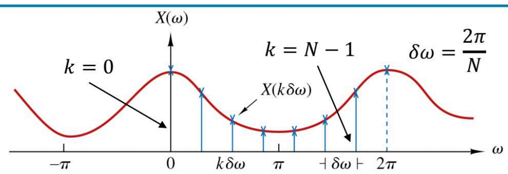
What happens if we sample $X(\omega)$?

$$x(n) = x_a(nT) \Rightarrow t = nT \Rightarrow \omega = 2\pi \frac{F}{F_s} \qquad X(F) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X_a(F - mF_s)$$

2

Frequency-Domain Sampling

$$X(k\delta\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi}{N}kn}, \qquad k = 0,1,\dots,N-1$$



$$X(k\delta\omega) = \cdots + \sum_{n=-N}^{-1} x(n) e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} + \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi}{N}kn} + \cdots$$

$n \leftarrow n - lN$

$$X(k\delta\omega) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi}{N}kn} \Rightarrow X(k\delta\omega) = \sum_{n=0}^{N-1} \underbrace{\left[\sum_{l=-\infty}^{\infty} x(n - lN) \right]}_{x_p(n)} e^{-j\frac{2\pi}{N}kn}$$

$x_p(n)$ Periodic with period N

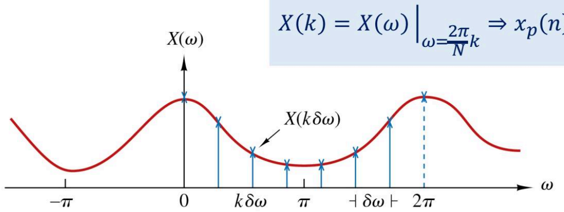
$$x_p(n) = \sum_{n=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn} \quad \overset{DTFS}{\longleftrightarrow} \quad c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}kn}$$

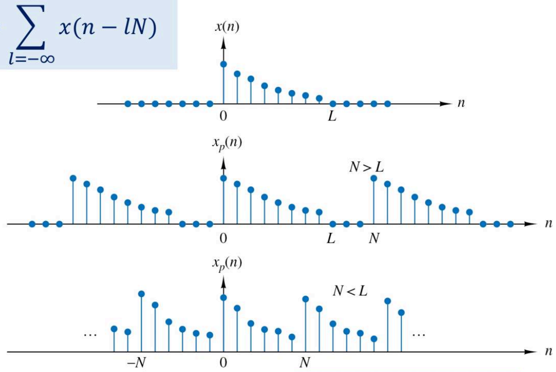
$$c_k = \frac{1}{N} X\left(\frac{2\pi}{N} k\right)$$

3

Time-Domain Aliasing

$$X(k) = X(\omega)\Big|_{\omega=\frac{2\pi}{N}k} \Rightarrow x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$





$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn}, n = 0,1,\ldots,N-1$$

If $x(n) = 0$ for $n \notin [0, L-1]$ and $N > L \Rightarrow$

$$x(n) = \begin{cases} x_p(n), 0 \le n \le N-1 \\ 0, \quad \text{elsewhere} \end{cases}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn}$$

No time-domain aliasing $\Rightarrow$ Perfect reconstruction of $x(n)$ from $X(k\delta\omega)$

4

Reconstruction of $X(\omega)$ from $X(2\pi k/N)$

$$X(\omega) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn}\right] e^{-j\omega n}$$

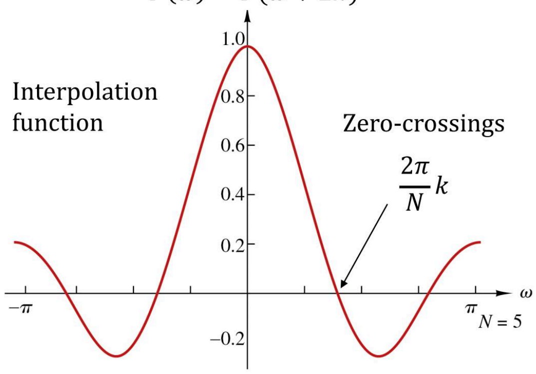
$$= \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{-j(\omega-\frac{2\pi}{N}k)n}\right]$$

$$P(\omega) = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

$$= \frac{\sin(\omega N/2)}{N\sin(\omega/2)} e^{-j\omega(N-1)/2}$$ Dirichlet's function

Periodic counterpart of $\frac{\sin x}{x}$!

$$P(\omega) = P(\omega + 2\pi)$$



$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) P\left(\omega - \frac{2\pi}{N}k\right)$$

5

Example of Sampling the DTFT $X(\omega)$

$x(n) = a^n u(n) \Rightarrow X(\omega) = \frac{1}{1 - ae^{-j\omega}}, 0 < a < 1$

Sampling with $\delta\omega = \frac{2\pi}{N} \Rightarrow$

$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) = \frac{a^n}{1 - a^N}, 0 \le n \le N - 1$

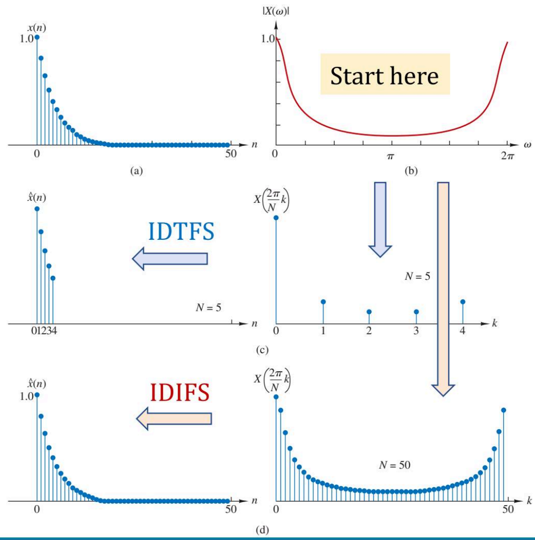
Aliasing!

Time-domain aliasing decreases as N increases!

$\hat{X}(\omega) = \sum_{n=0}^{N-1} x_p(n) e^{-j\omega n} = \frac{1}{1 - a^N} \frac{1 - a^N e^{-j\omega N}}{1 - ae^{-j\omega}}$

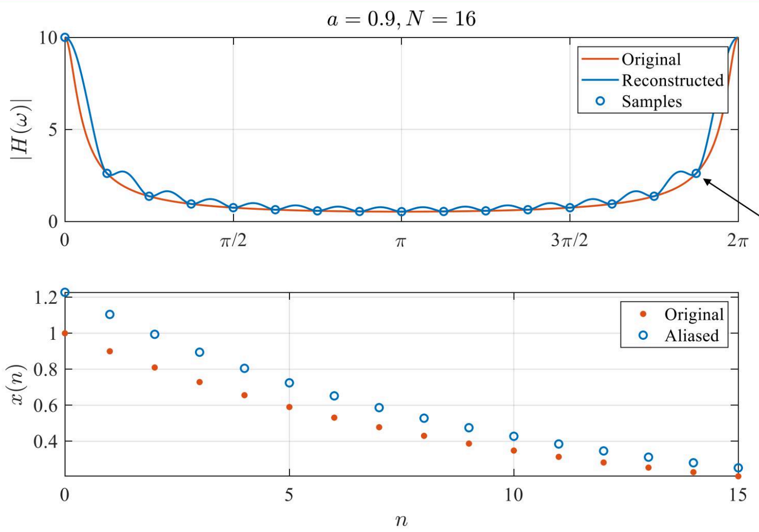
Reconstructed $X(\omega)$

$\hat{X}(\omega) \to X(\omega)$ as $N \to \infty$



6

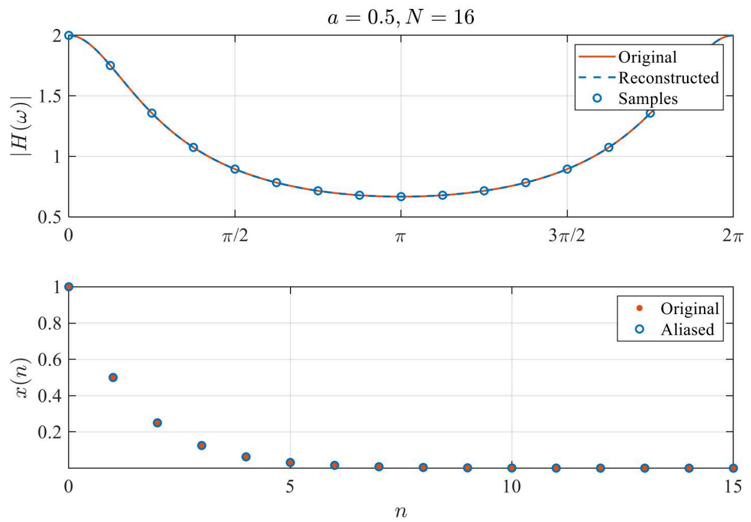
Significant Time-Domain Aliasing



Agreement at the sampling points!

7

Insignificant Time-Domain Aliasing

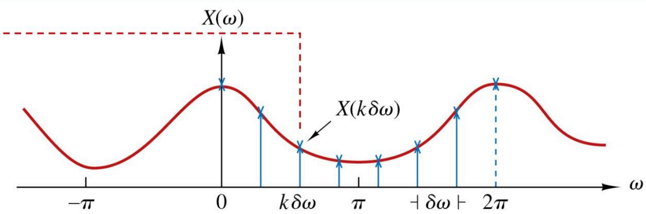


8

The Discrete Fourier Transform: Motivation

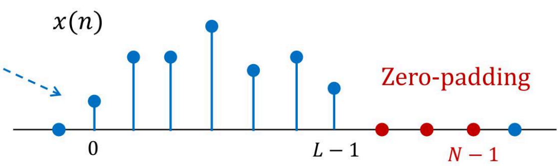
Let $x(n) = 0,\ n \notin [0, L-1] \Rightarrow$

$$X(\omega) = \sum_{n=0}^{L-1} x(n)e^{-j\omega n}$$



$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)e^{j\frac{2\pi}{N}kn} = \sum_{l=-\infty}^{\infty} x(n - lN)$$

If $N \geq L \Rightarrow \quad x_p(n) = \{x(0) \cdots x(L-1)\ \underbrace{0 \cdots 0}_{\text{Zero-padding} \rightarrow N-L}\}$



$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{L-1} x(n)e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}$$

Red formulas provide a useful computational Fourier analysis tool for finite length sequences!

9

The Discrete Fourier Transform (DFT)

Given N samples $\{x(n)\}_0^{N-1}$ of a sequence $x(n)$ (periodic or aperiodic) , the N-point Discrete Fourier Transform (DFT) $\{X(k)\}_0^{N-1}$ is defined by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}, \qquad k = 0,1,\dots,N-1$$

Given the N-DFT coefficients $\{X(k)\}_0^{N-1}$, the N-samples $\{x(n)\}_0^{N-1}$ of a sequence $x(n)$ (periodic or aperiodic), can be recovered using the N-point Inverse DFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}, \qquad n = 0,1,\dots,N-1$$

The DFT uniquely specifies N samples $\{x(n)\}_0^{N-1}$ of any sequence by N-DFT coefficients $\{X(k)\}_0^{N-1}$

10

Matrix Formulation of DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn}, \qquad W_N = e^{-j\frac{2\pi}{N}}$$

$$\underbrace{\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}}_{\boldsymbol{X}_N} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N & \cdots & W_N^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}}_{\substack{\boldsymbol{W}_N \\ \text{Symmetric}}} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\boldsymbol{x}_N}$$

$$\boldsymbol{W}_N^H \boldsymbol{W}_N = (\boldsymbol{W}_N^*)^T \boldsymbol{W}_N = \boldsymbol{I} \qquad \boldsymbol{X}_N = \boldsymbol{W}_N \boldsymbol{x}_N \Rightarrow \boldsymbol{x}_N = \frac{1}{N}\boldsymbol{W}_N^* \boldsymbol{X}_N$$

Unitary Matrix

11

How to Think about the DFT

The DFT takes N consecutive samples $\{x(0), x(1), \dots, x(N-1)\}$ of any sequence $x(n)$, treats them as a finite length sequence, and computes N equi-spaced samples of its DTFT $X(k) = X_N(\omega_k)$ at $\omega_k = \frac{2\pi}{N}k,\ k = 0,1,\dots,N-1$

The inverse DFT takes the samples $X(k) = X_N\left(\frac{2\pi k}{N}\right), k = 0,1,\dots,N-1$ and recovers the original values $\{x(0), x(1), \dots, x(N-1)\}$

The physical meaning of the DFT coefficients $\{X(0), X(1), \dots, X(N-1)\}$ depends on how we select the samples $\{x(0), x(1), \dots, x(N-1)\}$ of $x(n)$, and whether the sequence $x(n)$ is aperiodic or periodic

Interpretation of DFT Coefficients

Aperiodic Signals

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$X_N(\omega) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn} = X_N\left(\frac{2\pi}{N}k\right)$$

$X(\omega) = X_N(\omega)$ if $x(n) = 0, n \notin [0, L-1], L \le N$

$X(\omega) \neq X_N(\omega)$ otherwise

Periodic Signals

$$c_k = \frac{1}{N}\sum_{n=0}^{N_p-1} x(n)e^{-j\frac{2\pi}{N_p}kn}$$

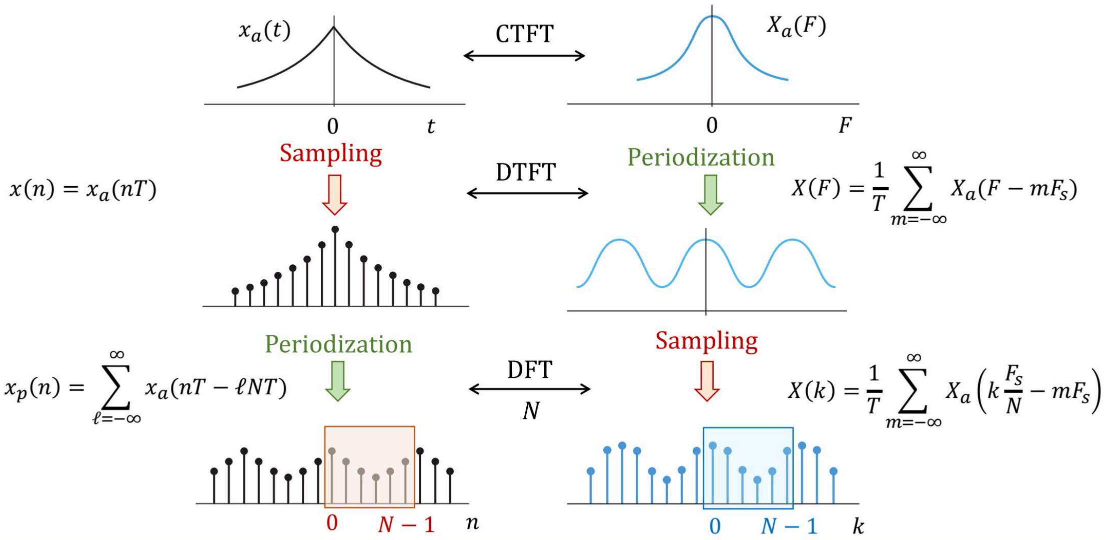
$$X_N(\omega) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn} = X_N\left(\frac{2\pi}{N}k\right)$$

$$N = N_p \Rightarrow c_k = \frac{1}{N}X(k)$$

$$N \neq N_p \Rightarrow c_k \neq \frac{1}{N}X(k)$$

Relationship Between DFT and CTFT



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14

The Importance of Zero-Padding

$$x(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases} \Rightarrow X(\omega) = \sum_{n=0}^{L-1} x(n)e^{-j\omega n} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\frac{\omega(L-1)}{2}}$$

$$\text{If } N \ge L \Rightarrow \quad X_N(\omega) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n} = \sum_{n=0}^{L-1} x(n)e^{-j\omega n} = X(\omega)$$

$$\{x(0), x(1), \dots, x(L-1), \underbrace{0, \dots, 0}_{N-L}\} \xleftrightarrow[N]{\text{DFT}} X(k) = \sum_{n=0}^{N-1} x(n)e^{j\frac{2\pi}{N}kn}, \qquad k = 0,1,\dots,N-1$$

Zero-Padding

$$X(k) = X_N\left(\frac{2\pi k}{N}\right) = X\left(\frac{2\pi k}{N}\right), k = 0,1,\dots,N-1$$

Use zero-padding to control the sampling period $2\pi/N$ of $X_N(\omega)$

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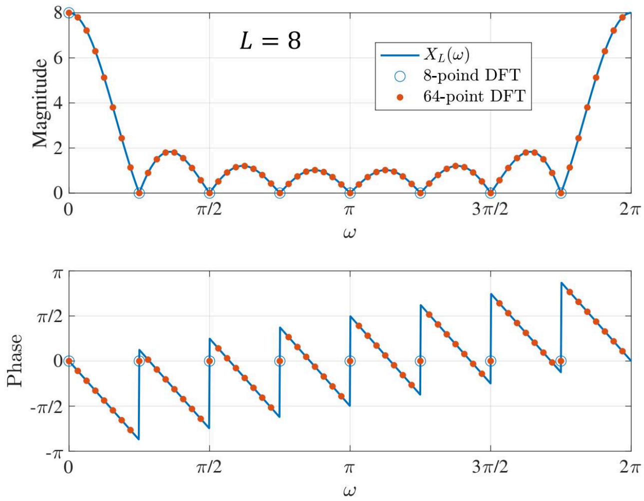
15

Zero-Padding ⇒ Denser DTFT Sampling

Zero-padding ⇒ Ideal "Time-Limited" spectral interpolation



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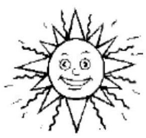
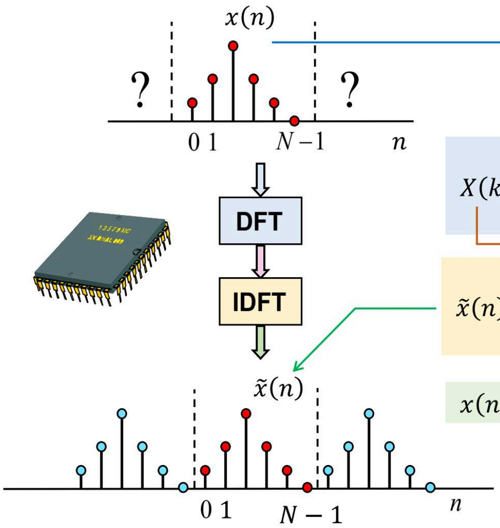
16

User View Versus DFT View of Data

User before DFT



User after IDFT



$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} = X(k+N)$$

$$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} = \tilde{x}(n+N)$$

$$x(n) = \tilde{x}(n), \qquad 0 \le n \le N-1$$

- Compute in $0 \le k, n \le N-1$
- Think assuming periodicity!

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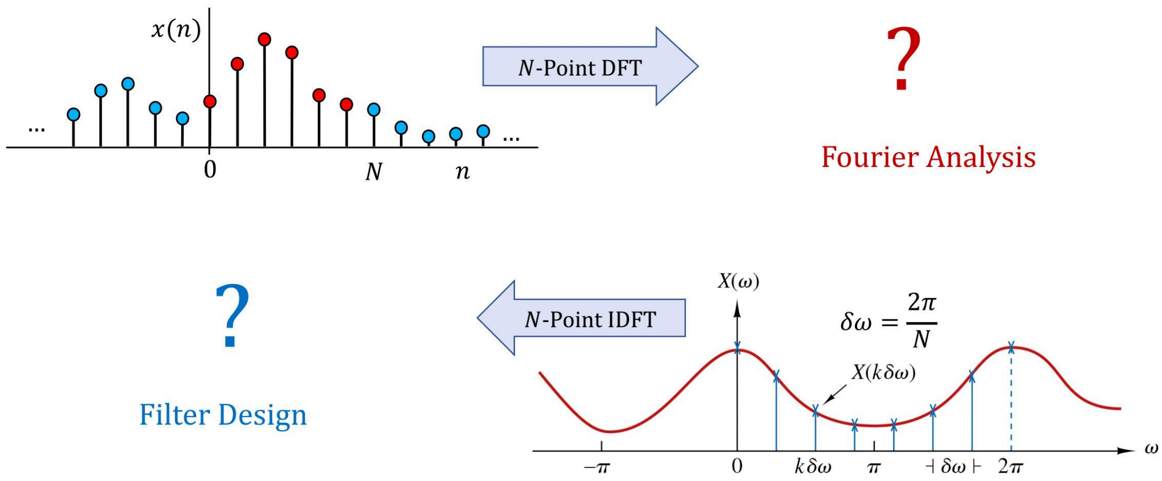
17

Applications of DFT



18

Summary

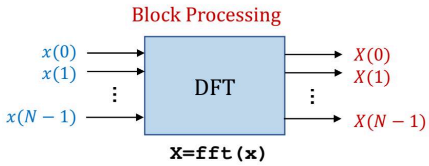
- The DFT is a finite orthogonal transform which provides a unique representation of N consecutive samples from any sequence (periodic or not) through a set of N DFT coefficients
- The physical meaning of the DFT coefficients depends upon what we know or what we assume about all the remaining samples of the sequence
- The DFT cannot provide any information about the unavailable (that is, unused in the computation) samples of the sequence
- The value of the DFT stems from:
  - Its relation to CTFT, CTFS, DTFT, DTFS and z-Transform
  - Its relation to Convolution and Correlation operations
  - The existence of very efficient algorithms for its computation. These algorithms are collectively known as Fast Fourier Transform (FFT) algorithms.
- Note: The FFT is not another new transform; it is simply an efficient computational algorithm for computing the DFT

19

The Discrete Fourier Transform (DFT)

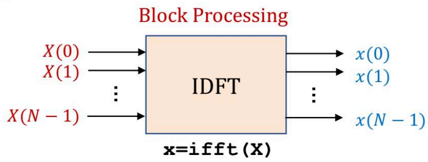
Given N samples $\{x(n)\}_0^{N-1}$ of a sequence $x(n)$ (periodic or aperiodic) , the N-point Discrete Fourier Transform (DFT) $\{X(k)\}_0^{N-1}$ is defined by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}, \qquad k = 0,1,\ldots,N-1$$



Given the N-DFT coefficients $\{X(k)\}_0^{N-1}$, the N-samples $\{x(n)\}_0^{N-1}$ of a sequence $x(n)$ (periodic or aperiodic), can be recovered using the N-point Inverse DFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}, \qquad n = 0,1,\ldots,N-1$$



The DFT uniquely specifies N samples $\{x(n)\}_0^{N-1}$ of any sequence by N-DFT coefficients $\{X(k)\}_0^{N-1}$

The Inherent DFT Periodicity

N-Point DFT Pair

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad \xleftrightarrow[N]{\text{DFT}} \quad x(n) = \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$k = 0,1,\ldots,N-1 \qquad\qquad n = 0,1,\ldots,N-1$$

Periodic in k

Twiddle factor $W_N = e^{j\frac{2\pi}{N}} \Rightarrow \quad W_N^{kn} = W_N^{(k+N)n} = W_N^{k(n+N)}$

Periodic in n

Implications

Properties Computation

Access values of $x(n)$ and $X(k)$ outside the range $[0, N-1]$ using periodic extension

Compute finitely but think periodically!

Real Sequences and the Meaning of Symmetry

$x(n) = x_R(n) + jx_I(n)$

$X(k) = X_R(k) + jX_I(k)$

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi}{N} kn + x_I(n) \sin \frac{2\pi}{N} kn \right]$$

$$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n) \sin \frac{2\pi}{N} kn - x_I(n) \cos \frac{2\pi}{N} kn \right]$$

$$x_R(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \cos \frac{2\pi}{N} kn - X_I(k) \sin \frac{2\pi}{N} kn \right]$$

$$x_I(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \sin \frac{2\pi}{N} kn + X_I(k) \cos \frac{2\pi}{N} kn \right]$$

Real signals: $x_I(n) = 0 \Rightarrow$

$$X_R(k) = \sum_{n=0}^{N-1} x(n) \cos \frac{2\pi}{N} kn$$

$$X_I(k) = -\sum_{n=0}^{N-1} x(n) \sin \frac{2\pi}{N} kn$$

$X_R(-k) = X_R(k)$

$X_I(-k) = -X_I(k)$

Symmetry is defined with respect to 0!

What is the meaning of symmetry in $[0, N-1]$?

22

Symmetry of DFT of Real Sequences

$X_R(-k) = X_R(k)$ and Periodic extension $\Rightarrow$

$X_R(N-k) = X_R(N+k) = X_R(k)$

$k = 0, 1, \ldots, N-1$

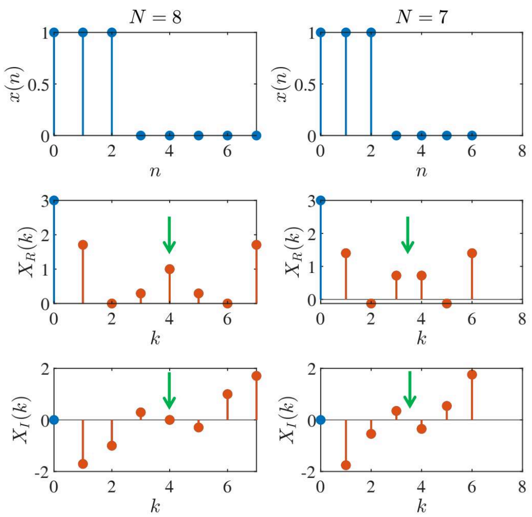
$X_I(-k) = -X_I(k)$ and Periodic extension $\Rightarrow$

$X_I(-0) = -X_I(0) \Rightarrow X_I(0) = 0$ (always!)

$X_I(N-k) = -X_I(N+k) = -X_I(k)$

$k = 0, 1, \ldots, N-1$

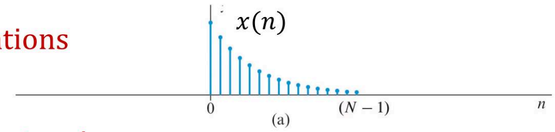
Check symmetry with respect to the center of the interval $[1, N-1]$



23

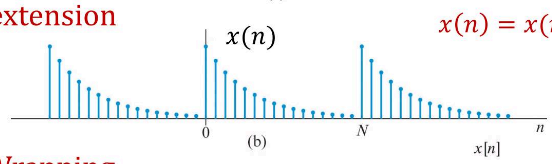
Periodic Extension and Circular Wrapping

Finite segment used for computations



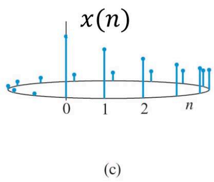
Periodic extension

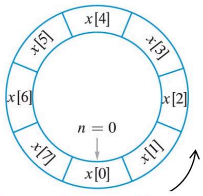
$x(n) = x(n + rN)$



Visualizations used for interpretations and manipulations

Circular Wrapping





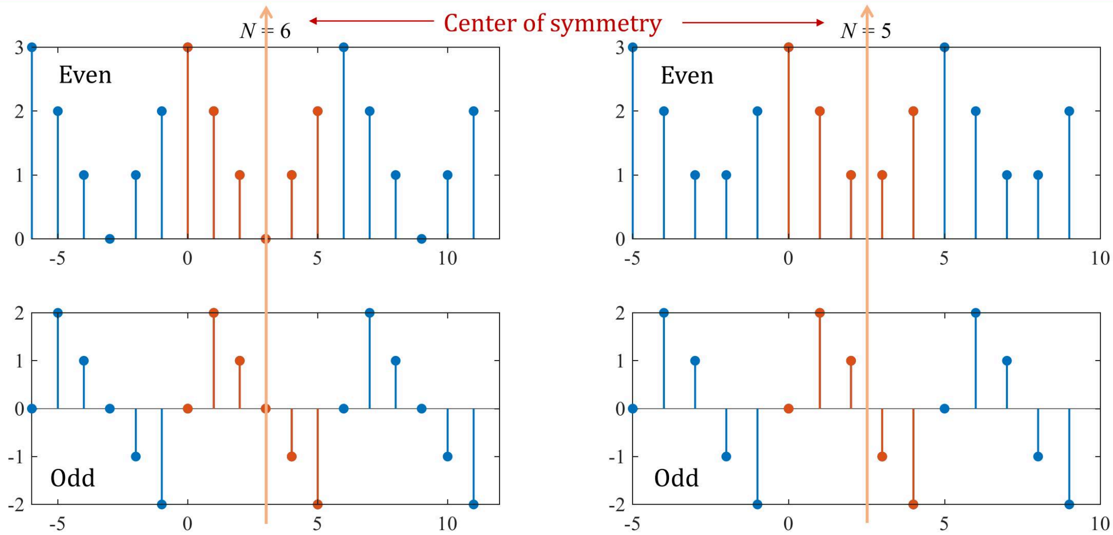
Modulo-N addressing

Even and odd symmetries defined as for the DFT coefficients

$x(n) = x(n + rN) = x(\langle n \rangle_N)$

24

Symmetry of Finite Length Sequences



25

Symmetry Properties of DFT

N-Point Sequence $x(n)$, $0 \le n \le N-1$	N-Point DFT
$x(n)$	$X(k)$
$x^*(n)$	$X^*(N-k)$
$x^*(N-n)$	$X^*(k)$
$x_R(n)$	$X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$
$jX_I(n)$	$X_{co}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$
$x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x^*(N-n)]$	$jX_I(k)$
Real Signals	
Any real signal	$X(k) = X^*(N-k)$
$x(n)$	$X_R(k) = X_R(N-k)$
	$X_I(k) = -X_I(N-k)$
	$\lvert X(k)\rvert = \lvert X(N-k)\rvert$
	$\angle X(k) = -\angle X(N-k)$
$x_{ce}(n) = \frac{1}{2}[x(n) + x(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x(N-n)]$	$jX_I(k)$

26

Multiplication of Two N-Point DFTs

$$X(k) = \sum_{m=0}^{N-1} x(m) W_N^{km} \qquad H(k) = \sum_{\ell=0}^{N-1} h(\ell) W_N^{k\ell} \qquad Y(k) = H(k)X(k) \xrightarrow[N]{\text{IDFT}} y(n) = ?$$

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)X(k)W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{\ell=0}^{N-1} h(\ell)W_N^{k\ell} \sum_{m=0}^{N-1} x(m) W_N^{km}\right) W_N^{-kn}$$

$$y(n) = \sum_{\ell=0}^{N-1} h(\ell) \sum_{m=0}^{N-1} x(m) \underbrace{\left(\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-\ell-m)}\right)}_{\substack{1 \text{ if } n-\ell-m = rN \\ 0 \text{ if } n-\ell-m \neq rN}}$$

$$y(n) = \sum_{\ell=0}^{N-1} h(l)x(n-\ell-rN)$$

Periodic convolution

$$y(n) = y(n+N)$$

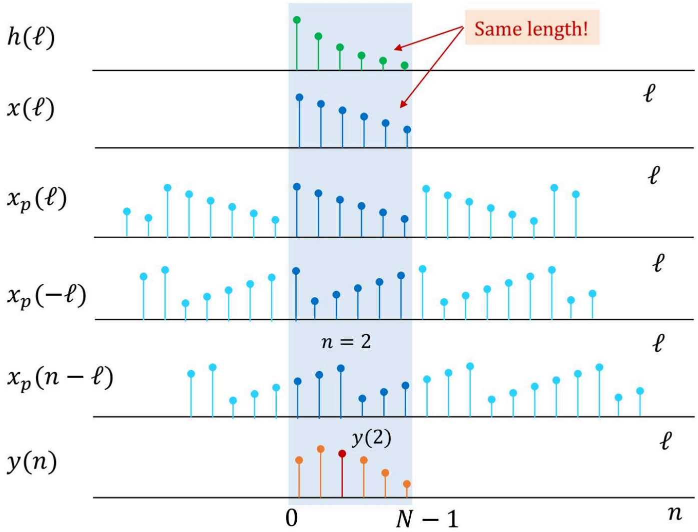
$$m = n - \ell - rN = \langle n-\ell \rangle_N$$

$$y(n) = \sum_{\ell=0}^{N-1} h(l)x(\langle n-\ell \rangle_N)$$

Circular convolution

27

Illustration of Circular Convolution



$$y(n) = \sum_{\ell=0}^{N-1} h(\ell)x(\langle n-\ell \rangle_N)$$

$$x_p(n) = \sum_{r=-\infty}^{\infty} x(n-rN)$$

$$y(n) = \sum_{\ell=0}^{N-1} h(\ell)x_p(n-l)$$

We never use these formulas in practice!

Properties of DFT

Property	Time Domain	Frequency Domain
Notation	$x(n), y(n)$	$X(k), Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular convolution	$x_1(n) \circledN x_2(n)$	$X_1(k)X_2(k)$
Circular correlation	$x(n) \circledN y^*(-n)$	$X(k)Y^*(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \circledN X_2(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

FIR Filtering Using the DFT

Linear convolution

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

$x(n) = 0, \quad n \notin [0, L-1] \Rightarrow$

$y(n) = 0, \quad n \notin [0, L+M-2]$

$$X(\omega) = \sum_{n=0}^{L-1} x(n)e^{-j\omega n}$$

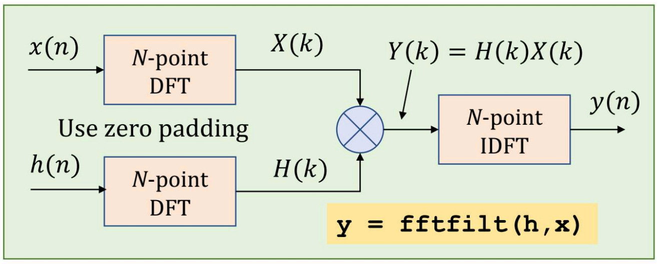
$$H(\omega) = \sum_{n=0}^{M-1} h(n)e^{-j\omega n}$$

$$Y(\omega) = H(\omega)X(\omega)$$

Let $N \geq L + M - 1$

$$Y\left(\frac{2\pi}{N}k\right) = H\left(\frac{2\pi}{N}k\right)X\left(\frac{2\pi}{N}k\right), k = 0,1,\dots,N-1$$

IDFT $\{y(0), y(1), \dots, y(L+M-2), 0, \dots, 0\}$



$$N < L + M - 1 \Rightarrow \text{IDFT}\{Y(k)\} = \sum_{m=-\infty}^{\infty} y(n - mN) \qquad \text{Aliasing!}$$

30

Example

$h(n) = \{1 \ 0 \ 1 \ 1\}, \qquad M = 4 \qquad\qquad H(z) = 1 + z^{-2} + z^{-3}$

$x(n) = \{0 \ 1 \ 2 \ 3\}, \qquad L = 4 \qquad\qquad X(z) = z^{-1} + 2z^{-2} + 3z^{-3}$

$$Y(z) = H(z)X(z) = z^{-1} + 2z^{-2} + 4z^{-3} + 3z^{-4} + 5z^{-5} + 3z^{-6}$$

$y(n) = \{0 \ 1 \ 2 \ 4 \ 3 \ 5 \ 3\}$

$N = L + M - 1 = 7$

```
X=fft(h,N);
H=fft(h,N);
Y=H.*X;
y=real(ifft(Y))
```

$N = 4$

```
y =
     2     2     3     2
```

$N = 8$

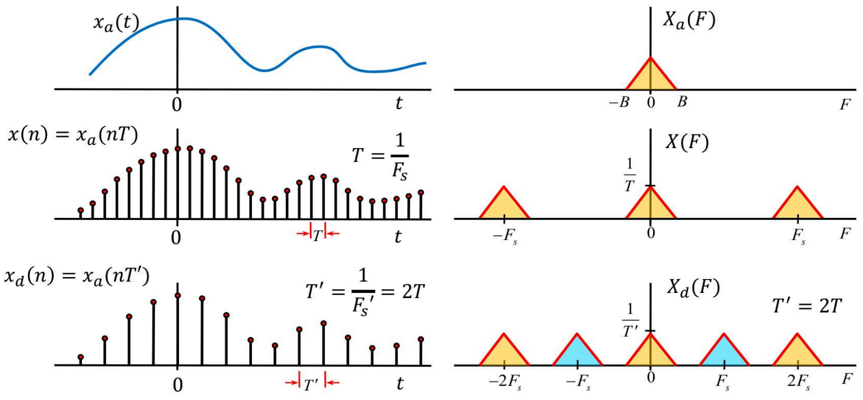
```
y =
  Columns 1 through 5

         0    1.0000    2.0000    4.0000    3.0000
  Columns 6 through 8

    5.0000    3.0000         0
```


31

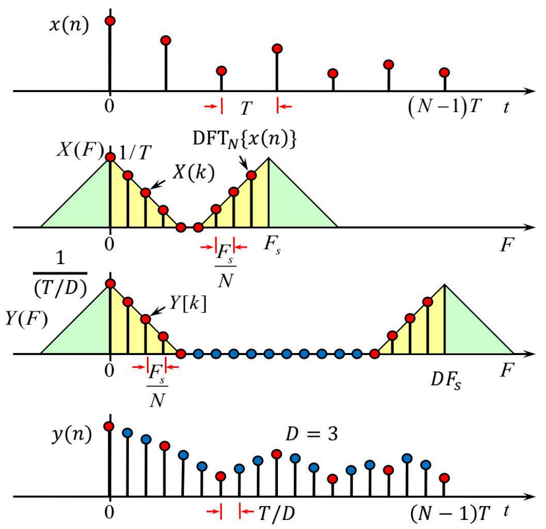
Decimation and Interpolation



$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\,\frac{\sin\left[\frac{\pi}{T}(t-nT)\right]}{\frac{\pi}{T}(t-nT)} \Rightarrow \hat{x}_a(mT')\Big|_{T'=\frac{T}{D}} = \sum_{n=0}^{N-1} x_a(nT)\,\frac{\sin\left[\frac{\pi}{D}(m-nD)\right]}{\frac{\pi}{D}(m-nD)}$$

32

DFT–Based BL Interpolation



Algorithm

Compute: $X(k) = \text{DFT}_N\{x(n)\}$

Set $M = DN$ and compute:

$$Y(k) = X(k), \quad 0 \le k \le \left\lfloor \frac{N}{2} \right\rfloor - 1$$

$$Y(k) = \frac{X(k)}{2}, k = \frac{N}{2}, (N \text{ even})$$

$$Y(k) = X(k), \quad k = \left\lfloor \frac{N}{2} \right\rfloor, (\text{N odd})$$

$$Y(k) = 0, \quad \left\lfloor \frac{N}{2} \right\rfloor \le k \le \left\lfloor \frac{M}{2} \right\rfloor$$

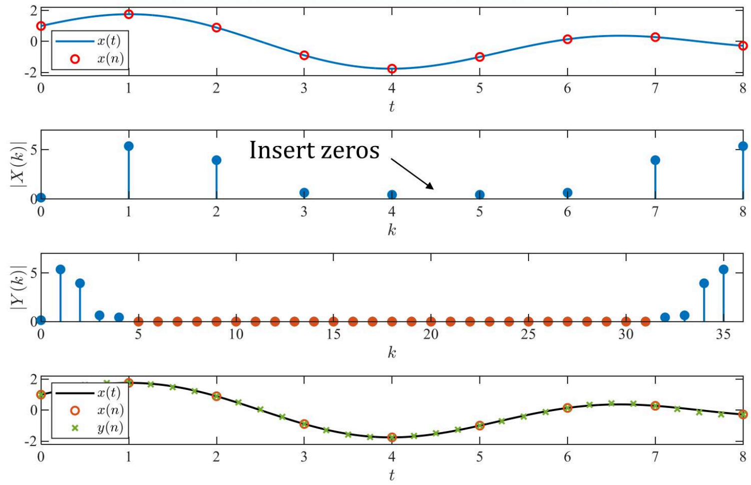
$$Y(M-k) = Y^*(k), \quad 1 \le k \le \left\lfloor \frac{M}{2} \right\rfloor$$

Compute: $y(n) = \text{IDFT}_N\{Y(k)\}$

33

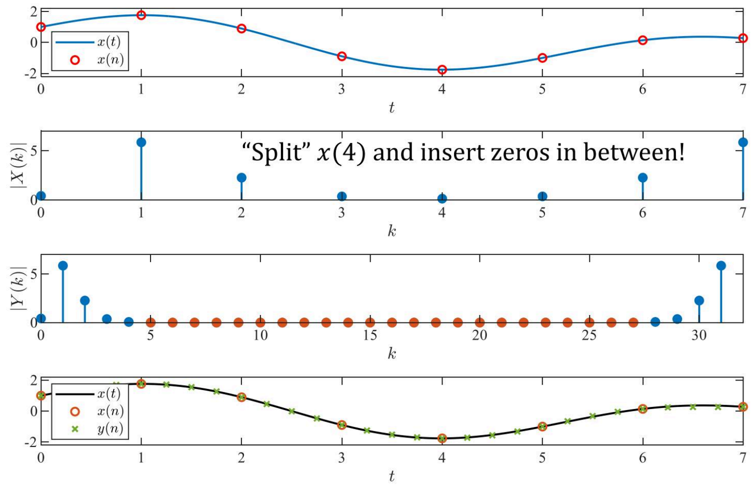
Example of DFT-Based Interpolation ($N = 9$)

$x_a(t) = \cos(2\pi 0.1t) + \cos(2\pi 0.2t - \pi/2)$



Example of DFT-Based Interpolation ($N = 8$)

$x_a(t) = \cos(2\pi 0.1t) + \cos(2\pi 0.2t - \pi/2)$



Summary

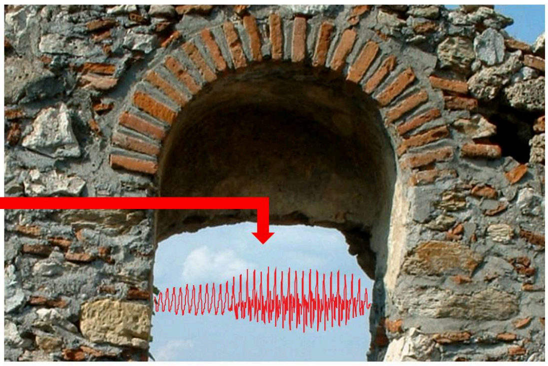
- The N-point DFT treats the N-input samples as one period of a periodic sequence
- Symmetry in DFT/IDFT inputs and outputs is defined about $n = k = 0$ for the "periodic" viewpoint and about $n = k = N/2$ for the "circular" viewpoint
- Shift and fold operations should be considered in a periodic or circular context
- DFT of real sequences
  - Real part has even symmetry
  - Imaginary part has odd symmetry
  - Real and even sequences have a real and even DFT
- The DFT can be used for the efficient computation of circular convolution and the implementation of "block" FIR filters

Fourier or Spectral Analysis of Signals

The objective of Fourier or spectral analysis is to determines the spectrum (Fourier transform or Fourier series) of a signal from a set of observed signal values (data)

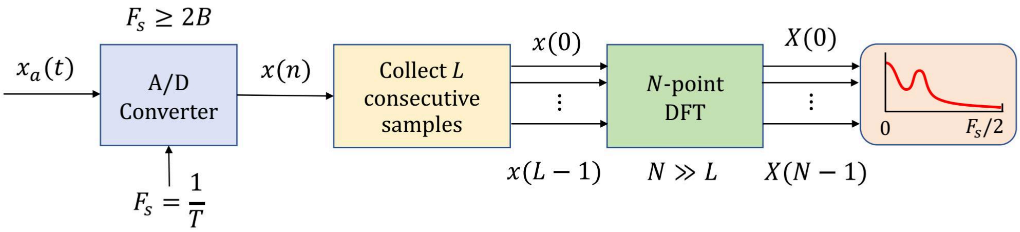
Theory: The entire signal is available
⇒ Compute exact ("true") spectrum

Practice: Only a finite segment of the signal is available ⟺ Windowing
⇒ Compute approximate spectrum



Every practical DSP application involves some kind of windowing!

Fourier Analysis Using the DFT

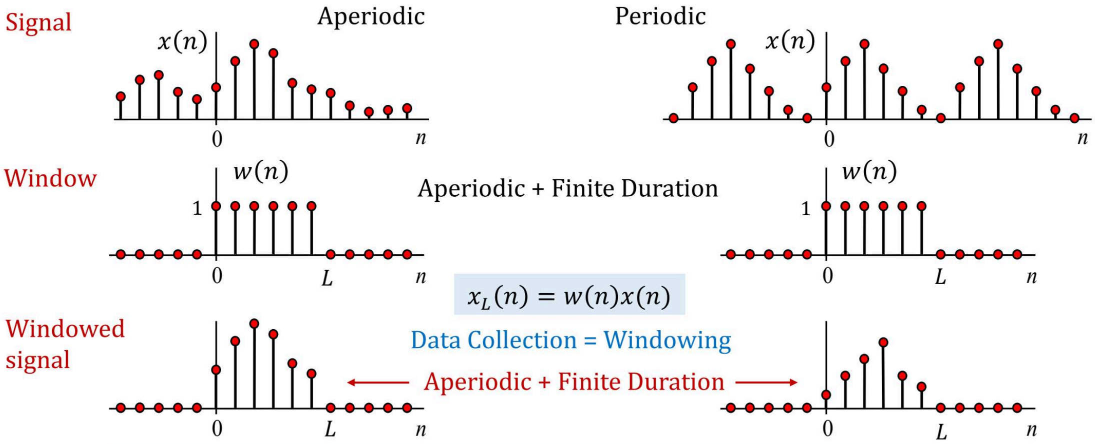


- The sampling frequency F_s determines the highest frequency component than can be distinguished ("seen") in the spectrum
- The duration LT of the segment determines the lowest frequency component (except DC) $F_0 = \frac{1}{LT}$ that can be "seen" in the spectrum
- The most challenging aspect of spectral analysis is to understand how the DFT coefficients are related to the "true" spectrum of the signal
- We typically assume that sampling is done properly and we focus on the sequence $x(n)$

EECE-5666: Digital Signal Processing The Discrete Fourier Transform 38/61

38

Practical Spectral Analysis



Evaluate and use the continuous function $X_L(\omega)$ to approximate the DTFS (discrete spectrum) or DTFT (continuous spectrum) of $x(n)$

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39

The Signal Windowing Theorem

Windowing: $s(n) = x(n)w(n) \Rightarrow$ $\quad S(\omega) = \sum_{n=-\infty}^{\infty} s(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(n)w(n)e^{-j\omega n}$

Periodic $x(n)$

$$S(\omega) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}\right] w(n)e^{-j\omega n}$$

$$S(\omega) = \sum_{k=0}^{N-1} c_k \left[\sum_{n=-\infty}^{\infty} w(n)e^{j\left(\omega-\frac{2\pi}{N}k\right)n}\right]$$

$$S(\omega) = \sum_{k=0}^{N-1} c_k W\left(\omega - \frac{2\pi}{N}k\right)$$

Superposition of window's spectra

Aperiodic $x(n)$

$$S(\omega) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{2\pi} X(\theta)e^{j\theta n}d\theta\right] w(n)e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{2\pi} X(\theta)\left[\sum_{n=-\infty}^{\infty} w(n)e^{-j(\omega-\theta)n}\right] d\theta$$

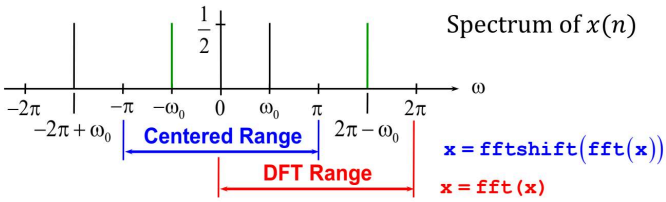
$$S(\omega) = \frac{1}{2\pi}\int_{2\pi} X(\theta)W(\omega-\theta)d\theta$$

Periodic convolution

40

Truncating a Sinusoidal Signal

$$x(n) = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \qquad -\infty < n < \infty$$



Spectrum of $x(n)$

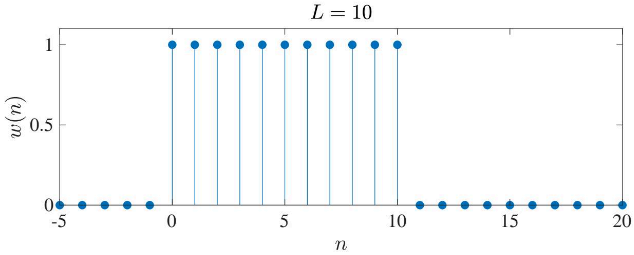
$$x_L(n) = w(n)\cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n}\,w(n) + \frac{1}{2}e^{-j\omega_0 n}\,w(n)$$

$$X_L(\omega) = \frac{1}{2}W(\omega+\omega_0) + \frac{1}{2}W(\omega-\omega_0)$$

41

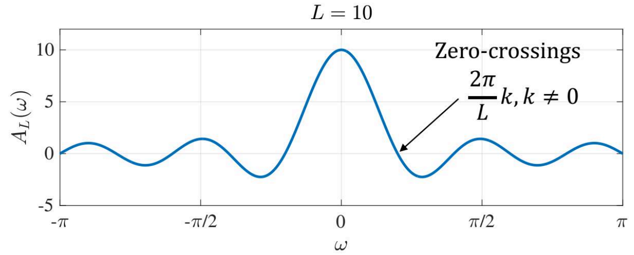
Rectangular Window

$$w(n) = \begin{cases} 1, 0 \le n \le L-1 \\ 0, \quad \text{otherwise} \end{cases}$$



Dirichlet's function

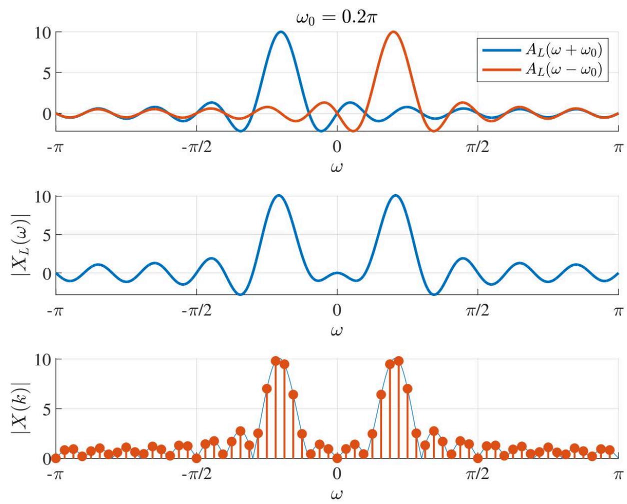
$$W(\omega) = \underbrace{\frac{\sin(\omega L/2)}{\sin(\omega/2)}}_{A_L(\omega)} e^{-j\omega(L-1)/2}$$



42

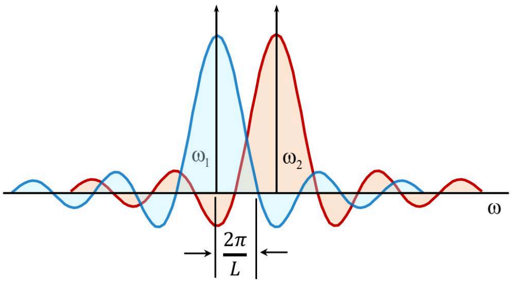
Spectrum of Truncated Sinusoidal Signal

From now on we will use zero-padding to compute and display $X_L(\omega)$



43

Spectral Resolution

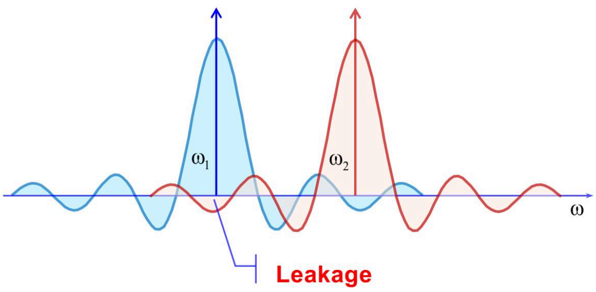


- To resolve two peaks using a rectangular window requires:

$$|\omega_1 - \omega_2| \geq \frac{2\pi}{L} \Rightarrow |f_1 - f_2| \geq \frac{1}{L} \Rightarrow L \geq \frac{1}{|f_1 - f_2|} = \frac{1}{\Delta f}$$

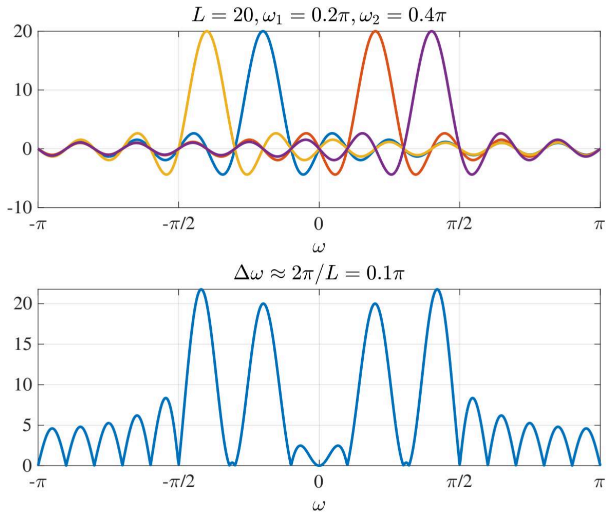
- The ability to distinguish peaks (resolvability) depends on the width of the "main" lobe which is determined by the window length L

Spectral Leakage



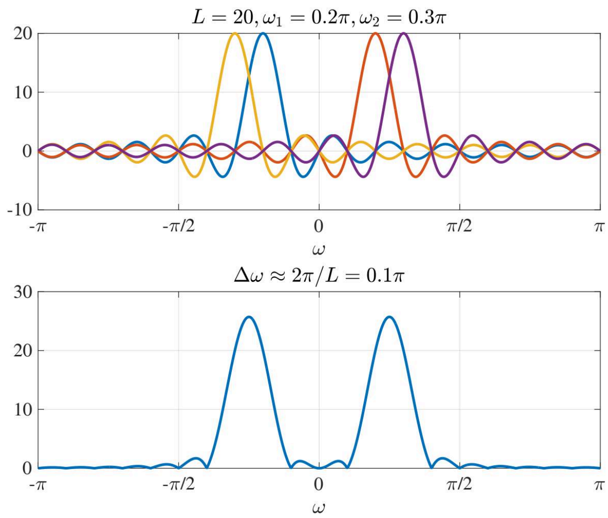
- Power from one frequency component leaks into the vicinity of another frequency component through the window side lobes
- The degree of leakage depends on the relative amplitude of the main lobe and the side lobes of the window transform
- Leakage creates "false" peaks, that is, peaks at wrong frequencies, non-existing peaks, or changes the amplitude of existing peaks

Spectral Resolution: Sufficient



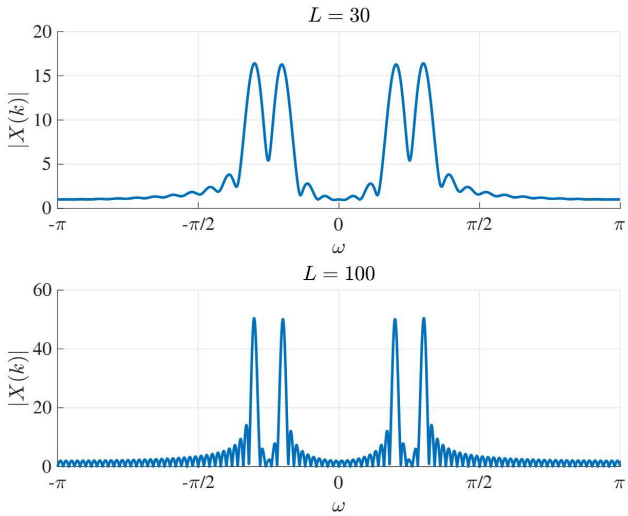
46

Spectral Resolution: Insufficient



47

Spectral Resolution: Observation Length



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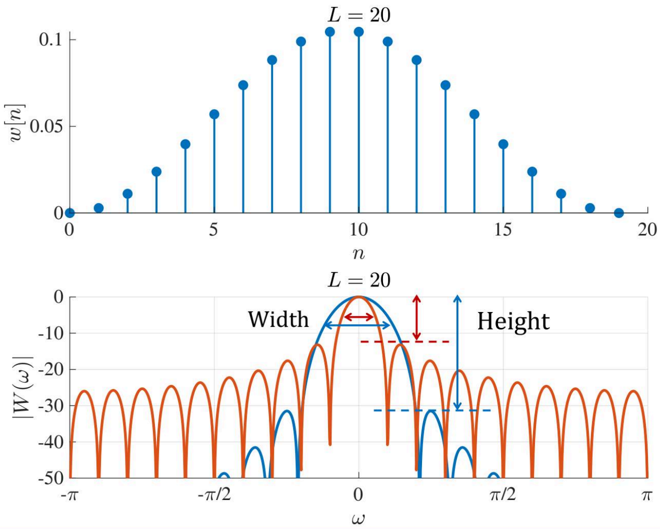
48

Non-Rectangular Windowing: Hann Window



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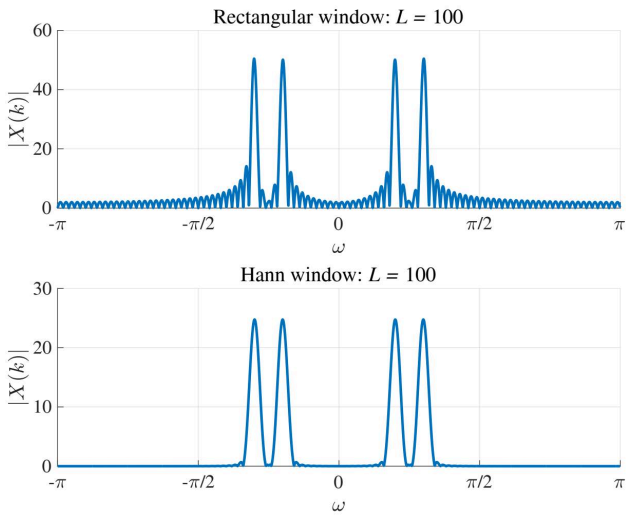
49

Spectral Resolution: Effect of Window Shape



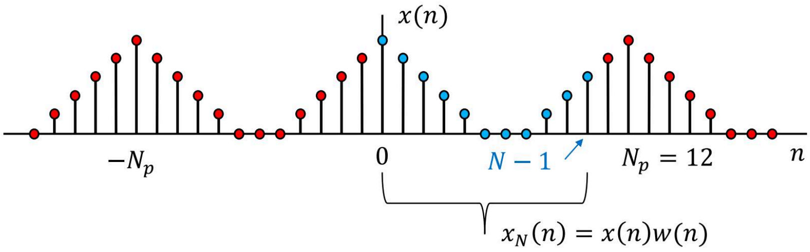
50

Spectral Resolution versus DFT "Resolution"

- Spectral resolution is related to the ability to distinguish ("resolve") two closely spaced-in-frequency spectral lines (sinusoids or complex exponentials)
- DFT resolution is the sampling interval $2\pi/N$ used by DFT to sample the DTFT of a windowed sequence in the fundamental frequency interval $0 \leq \omega < 2\pi$
- Zero padding improves the "visual appearance" by computing more closely spaced samples of the DTFT curve
- Zero padding does not improve the spectral resolution because the shape of the DTFT curve depends upon the "actual" size of the window, that is, the non zero samples of the signal

51

Using the DFT for Periodic Sequences



$$c_k = \frac{1}{N_p}\sum_{n=0}^{N_p-1} x(n)e^{-j\frac{2\pi}{N_p}kn}$$

$$X_N(\omega) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n} = \sum_{k=0}^{N_p-1} c_k W\left(\omega - \frac{2\pi}{N_p}k\right)$$

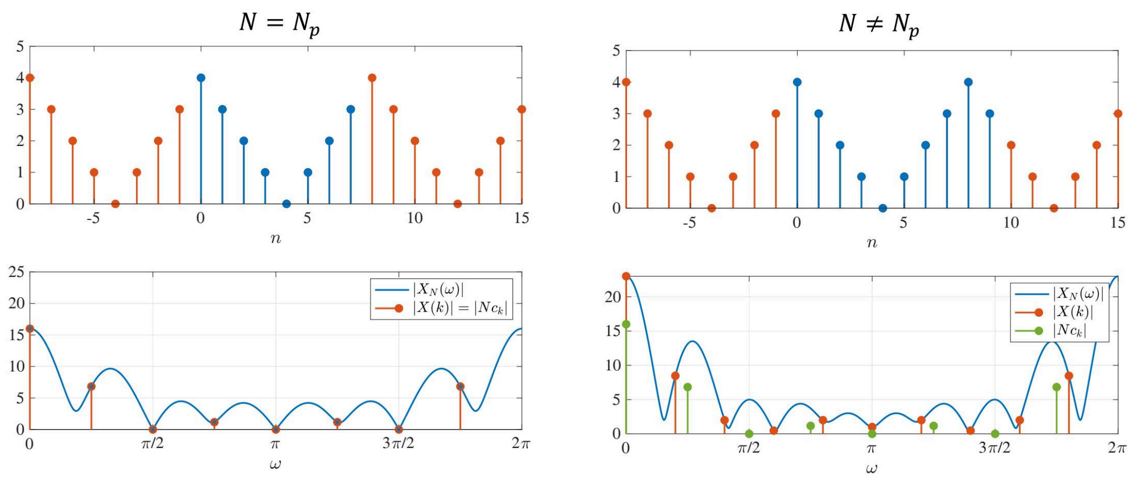
$$N = N_p \Rightarrow c_k = \frac{1}{N}X(k)$$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn} = X_N\left(\frac{2\pi}{N}k\right)$$

$$N \neq N_p \Rightarrow c_k \neq \frac{1}{N}X(k)$$

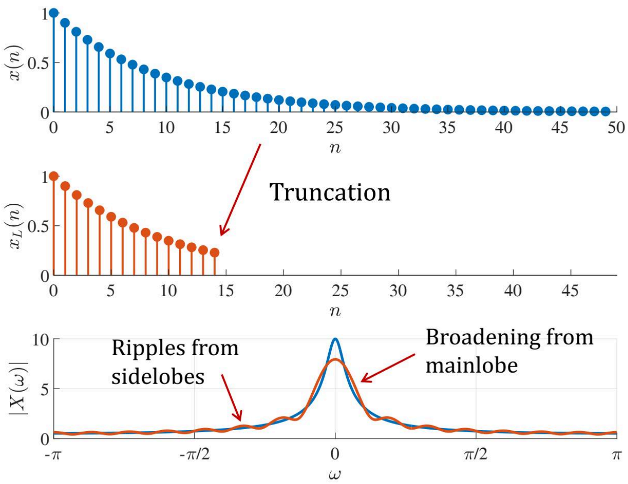
52

Example



53

Example: Windowing an Aperiodic Sequence



54

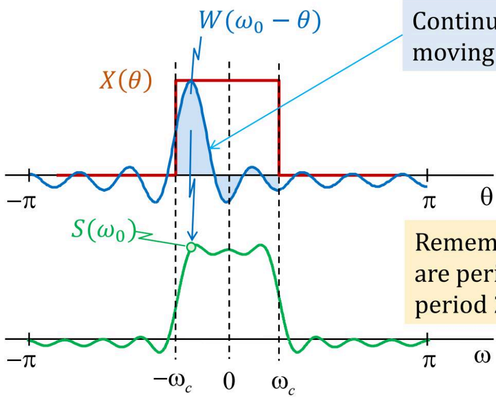
Illustration of Periodic Frequency-Domain Convolution

$$S(\omega) = \frac{1}{2\pi}\int_{2\pi} X(\theta)W(\omega-\theta)d\theta = X(\omega) \otimes W(\omega)$$

Linear combination of window spectrum copies

$$S(\omega) \approx \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) W\left(\omega - \frac{2\pi}{N}k\right)$$

Windowing is an "unintended and unavoidable" operation in Fourier analysis and filter design!

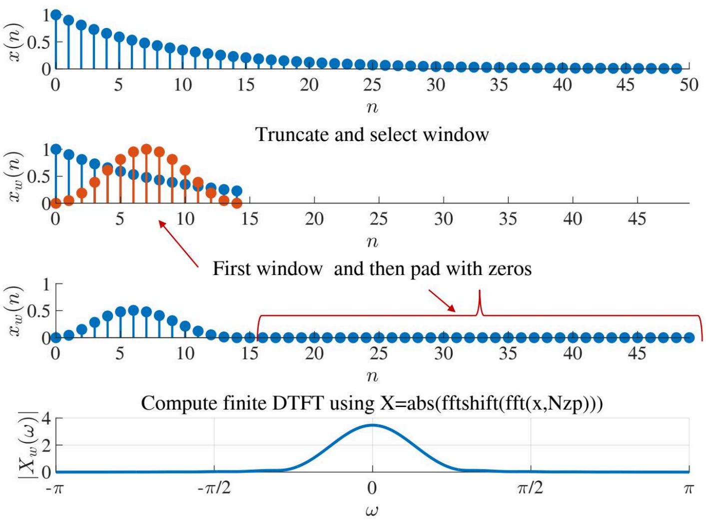


Continuous weighted moving average

Remember: Spectra are periodic with period 2π

55

Practical Fourier Analysis



56

Signals with Time-Varying Spectra

- Assumption for DFT spectral analysis: the amplitudes, frequencies, and phases of the sinusoidal components of the analyzed signal do not chance with time within the analysis window
- Spectral resolution × Window length ≈ Constant ⇒ We cannot use long windows for signal with time-varying spectra
- A practical solution is to break a long signal into small segments and analyze each one using the DFT ⇒ time-dependent DFT or short-time DFT

$$X(k,n) = \sum_{m=0}^{L-1} w(m)\underbrace{x(n+m)}_{\text{Shift signal}} e^{-j\frac{2\pi}{N}km}, \qquad k = 0,1,\ldots,K-1$$

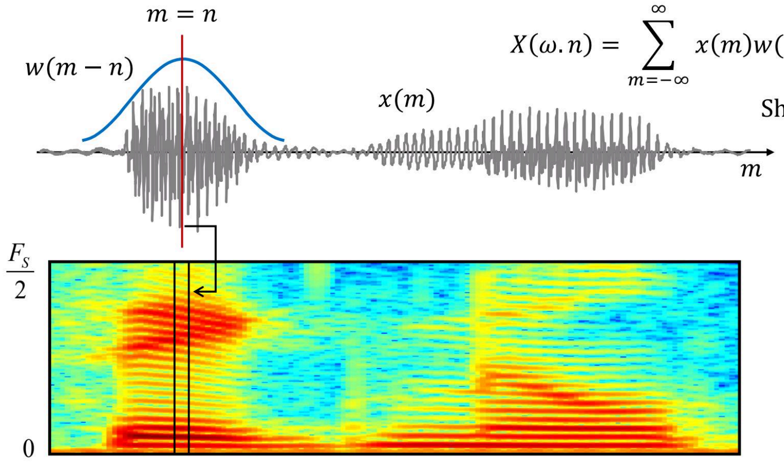
Finite window: $w(m)$, $0 \le m \le L-1$

57

Spectrogram of a Speech Signal

$$X(\omega.n) = \sum_{m=-\infty}^{\infty} x(m)w(m-n)\,e^{-j\omega m}$$

Shift window

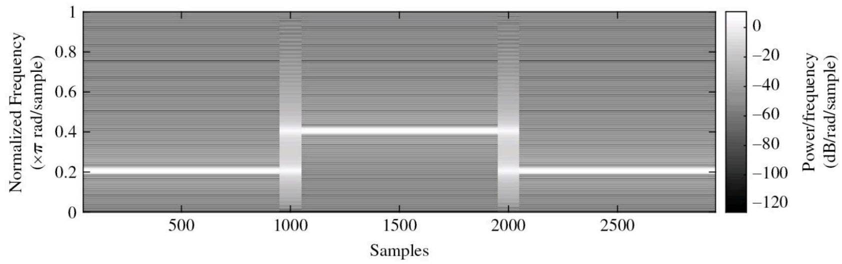


`specgram(x,NFFT,Fs,hanning(L),Noverlap)`

58

Spectrogram of Sinusoid with Step-Changes in Frequency

$L = 200$

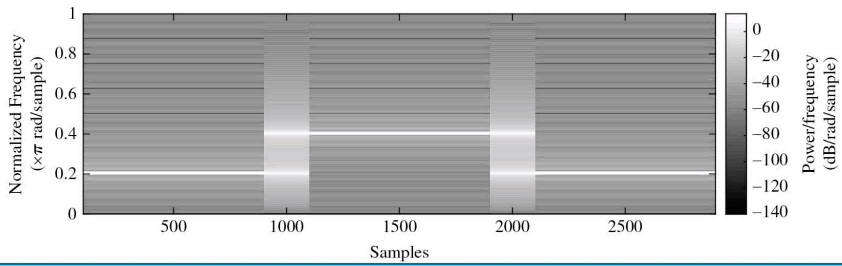


Hamming window

Overlap $= L/2$

$N_{FFT} = 1024$

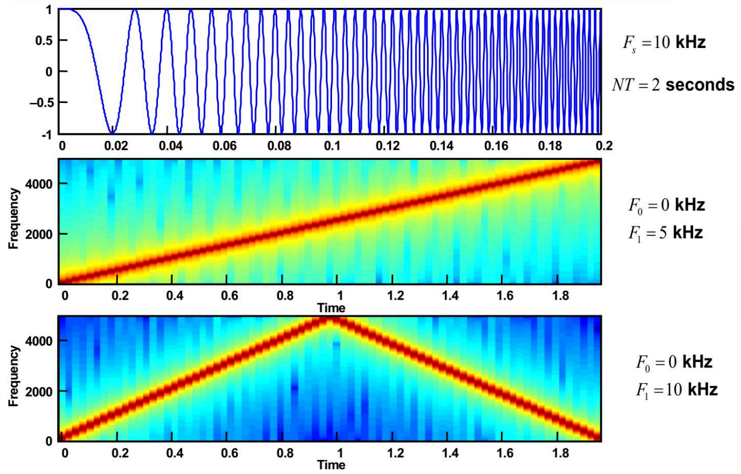
$L = 400$



59

Linear Chirp Spectrogram

$$x(n) = \cos\left[2\pi\left(F_0 + \frac{F_1 - F_0}{NT} nT\right) nT\right], 0 \le n \le N-1, T = 1/F_s$$



Visual "proof" of sampling theorem!

60

Summary

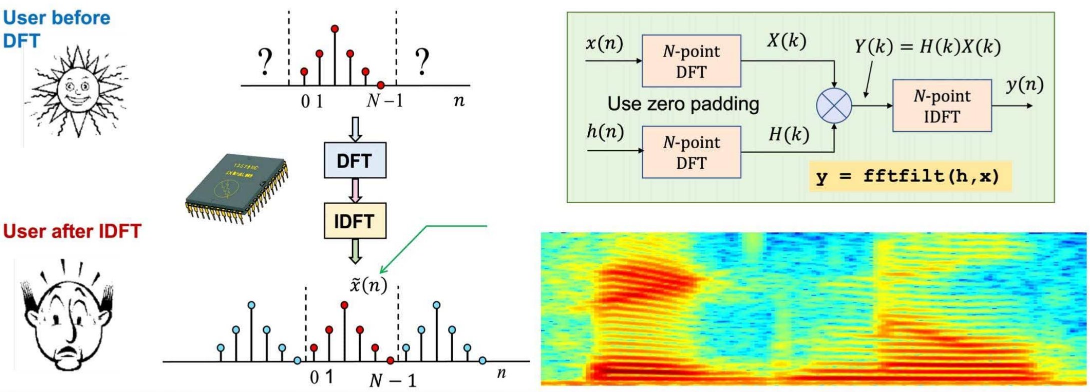
Theory		Practice	
$x(n), \quad -\infty < n < \infty$	Entire signal	$x_L(n) = x(n)w_L(n)$	Windowed segment
			Blurring + Leakage
$c_k = \dfrac{1}{N_p}\displaystyle\sum_{n=0}^{N_p-1} x(n)e^{j\frac{2\pi}{N_p}kn}$	Fourier series	$X_L(\omega) = \displaystyle\sum_{n=0}^{L-1} x_L(n)e^{-j\omega n}$	"Finite" Fourier transform
$X(\omega) = \displaystyle\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$	Fourier transform	$\mathrm{DFT}_N\{x_L(n)\} = X(k) = X_L\left(\dfrac{2\pi}{N}k\right)$, $k = 0,1,\dots,N-1$	

The challenge is interpretation not computing!

Use $N = 2^M \gg L$ with zero-padding to compute $X(k)$ and then plot it as a "continuous" line

61

The Discrete Fourier Transform



62