EECE-5666 : Midterm-2 Exam Solutions: 2023-SPRG

[60-Points]

Default Plot Parameters:

```
set(0,'defaultfigurepaperunits','points','defaultfigureunits','points');
set(0,'defaultaxesfontsize',10);
set(0,'defaultaxestitlefontsize',1.4,'defaultaxeslabelfontsize',1.2);
```

Problem-1 (20-points) The Discrete Fourier Transform

The following three parts (a), (b), and (c) are not related.

Use only direct calculations and/or DFT properties to solve each part. Use of Symbolic Math toolbox or Mathematica or Wolfram Alpha, etc. will not be graded or credited.

(a) [6-Points] Determine the N-point DFT $X(k)$ of the N-point sequence

$$x(n) = 4 + \cos^2\left(\frac{2\pi n}{N}\right) \qquad n = 0, 1, \cdots, N-1$$

Solution: Expressing the cosine function in terms of complex-exponentials. we have

$$\begin{aligned} x(n) &= 4 + \left[\frac{e^{j2\pi n/N} + e^{-j2\pi n/N}}{2}\right]^2 = 4 + \frac{1}{4}\left[e^{j4\pi n/N} + 2e^{j2\pi n/N}e^{-j2\pi n/N} + e^{-j4\pi n/N}\right] \\ &= \frac{9}{4} + \frac{1}{4}\exp\left(j\frac{2\pi}{N}(2)n\right) + \frac{1}{4}\exp\left(j\frac{2\pi}{N}(-2)n\right) \end{aligned}$$

Efficient approach: Compare the right-hand side above with the IDFT $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N}$ to identify $X(k)$.

$$\begin{aligned} x(n) &= \frac{9}{4} + \frac{1}{4}\exp\left(j\frac{2\pi}{N}(2)n\right) + \frac{1}{4}\exp\left(j\frac{2\pi}{N}(-2)n\right) \\ &= \frac{9}{4} + \frac{1}{4}\exp\left(j\frac{2\pi}{N}(2)n\right) + \frac{1}{4}\exp\left(j\frac{2\pi}{N}(N-2)n\right) \qquad (\text{since } \langle -2\rangle_N = N-2) \\ \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N} &= \underbrace{\frac{9}{2}\exp\left(j\frac{2\pi}{N}(0)n\right)}_{k=0 \text{ term}} + \underbrace{\frac{1}{4}\exp\left(j\frac{2\pi}{N}(2)n\right)}_{k=2 \text{ term}} + \underbrace{\frac{1}{4}\exp\left(j\frac{2\pi}{N}(N-2)n\right)}_{k=N-2 \text{ term}} \end{aligned}$$

Comparing the $X(k)$ terms on both sides above, we obtain

$$X(k) = \begin{cases} \frac{9N}{2}, & k = 0 \\ \frac{N}{4}, & k = 2 \text{ and } k = N-2 \\ 0, & \text{otherwise} \end{cases}$$

<u>Direct computation approach</u>: DFT of a complex exponential with harmonic frequency k_0 is $N\delta(k - k_0)$, that is,

$$\sum_{n=0}^{N-1} \left(e^{\jmath 2\pi (k_0) n/N}\right) e^{-\jmath 2\pi kn/N} = \sum_{n=0}^{N-1} e^{-\jmath 2\pi (k-k_0) n/N} = \begin{cases} N, & k = k_0 \\ 0, & k \neq k_0 \end{cases} = N\delta(k - k_0)$$

Hence,

$$\sum_{n=0}^{N-1} \left(\frac{9}{2} e^{\jmath 2\pi (0) n/N} + \frac{1}{4} e^{\jmath 2\pi (2) n/N} + \frac{1}{4} e^{\jmath 2\pi (N-2) n/N}\right) e^{-\jmath 2\pi kn/N} = \frac{9N}{2}\delta(k) + \frac{N}{4}\delta(k-2) + \frac{N}{4}\delta(K - N + 2)$$

(b) [9-Points] Let $x_1(n) \underset{N\text{-point}}{\overset{\text{DFT}}{\longleftrightarrow}} X_1(k)$ and $x_2(n) \underset{N\text{-point}}{\overset{\text{DFT}}{\longleftrightarrow}} X_2(k)$ be two N-point DFT pairs.

i. [4-Points] Show that the sum (there was a typographic error in the forllowing expression in the exam)

$$S = \sum_{n=0}^{N-1} x_1(n) x_2^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2^*(k).$$

You must provide all steps to get full credit.

Solution: Consider

$$\begin{aligned} S = \sum_{n=0}^{N-1} x_1(n) x_2^*(n) &= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X_1(k) e^{\jmath 2\pi nk/N}\right] x_2^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \left[\sum_{n=0}^{N-1} x_2^*(n) e^{\jmath 2\pi nk/N}\right] \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \left[\sum_{n=0}^{N-1} x_2(n) e^{-\jmath 2\pi nk/N}\right]^* = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2^*(k) \end{aligned}$$

which is the right-hand side.

---

ii. [5-Points] Evaluate the sum S above if $x_1(n) = \cos\left(\frac{2\pi n k_1}{N}\right)$ and $x_2(n) = \cos\left(\frac{2\pi n k_2}{N}\right)$, where k_1 and k_2 are integers between $0 \leq k_1, k_2 \leq N-1$.

Solution: To obtain the sum, we use the DFT values. From part (a), we have

$$\begin{aligned} X_1(k) = \underset{N\text{-point}}{\text{DFT}} \left[\cos\left(\frac{2\pi n k_1}{N}\right)\right] &= \frac{N}{2}\delta(k - k_1) + \frac{N}{2}\delta(k - N + k_1) \\ X_2(k) = \underset{N\text{-point}}{\text{DFT}} \left[\cos\left(\frac{2\pi n k_2}{N}\right)\right] &= \frac{N}{2}\delta(k - k_2) + \frac{N}{2}\delta(k - N + k_2) \end{aligned}$$

If $k_1 = k_2$ (or, equivalently, $k_1 = N - k_2$), the sum S will be

$$S = \frac{1}{N}\left[\frac{N^2}{4} + \frac{N^2}{4}\right] = \frac{N}{2}$$

Otherwise, $S = 0$. Hence

$$S = \begin{cases} \frac{N}{2}, & k_1 = k_2 \text{ or } k_1 = N - k_2 \\ 0, & \text{otherwise} \end{cases}$$

(c) [5-Points] Let $x_1(n)$ and $x_2(n)$ be two causal and finite-length sequences. Their values or their lengths are not known except that their lengths are ≤ 10. However, their linear convolution is known and is given by

$$x_1(n) * x_2(n) = \{\underset{\uparrow}{6}, 19, 40, 70, 110, 125, 140, 155, 170, 185, 184, 166, 130, 75\}.$$

Let $y(n)$ be their 8-point circular convolution. Determine $y(n)$. You must provide clear reasoning and/or steps, not just an answer, to get full credit.

Solution: If sequence $x_1(n)$ is N_1-point and if sequence $x_2(n)$ is N_2-point, then their linear convolution is of length $N = N_1 + N_2 - 1$, Since the length of the above linear convolution is $N = 14$, we have $N_1 + N_2 = 15$. Since the circular convolution $y(n)$ is 8-point, which should be $\geq \max(N_1, N_2)$. Thus, $N_1 = 8$ and $N_2 = 7$ or $N_1 = 7$ and $N_2 = 8$. Then the circular convolution is a time-domian aliased version (aliased by 8 samples) of the linear convolution, or

$$\begin{aligned} y(n) &= \{\underset{\uparrow}{6 + 170}, 19 + 185, 40 + 184, 70 + 166, 110 + 130, 125 + 75, 140, 155\} \\ &= \{\underset{\uparrow}{176}, 204, 224, 236, 240, 200, 140, 155\} \end{aligned}$$

Problem-2 (20-Points) The Fast Fourier Transform

The following 3 parts (a), (b), and (c) are not related.

(a) [5-Points] A causal signal has length $N = 2431 = 11 \times 13 \times 17$. Determine the total number of complex multiplications needed in computing its DFT using the divide-and-conquer approach. Assume that the twiddle factor $W_L^0 = 1 + \jmath 0$ is a complex-valued number. You must provide clear reasoning or proof to get full credit.

Hint: Twice use the divide-and-conquer computational complexity formula $(8.1.19)$ given on page-525 of the textbook *Digital Signal Processing* (by Proakis and Manolakis).

Solution: Let $N = N_1 M$ where $M = N_2 N_3$.Consider the divide and conquer equation $(8.1.15)$. To compute N-point DFT via $(8.1.15)$, we need M-point DFT computed N_1times, N_1-point DFT computed Mtimes, and N twiddle factor multiplications. Thus, the total number of complex multiplications are

$$\underset{(N\text{-point DFT})}{\text{ComplexMults}} = N_1(M^2) + M(N_1^2) + N. \qquad (2\text{-}a.1)$$

Using $(8.1.15)$ again, we can compute each $M = N_2 N_3$-point DFT via divide and conquer approach. Each computation requires

$$\begin{aligned}\underset{(M\text{-point DFT})}{\text{ComplexMults}} &= N_2(N_3^2) + N_3(N_2^2) + M \\ &= N_2 N_3(N_2 + N_3 + 1). \qquad (2\text{-}a.2)\end{aligned}$$

Replacing M^2 in $(2\text{-}a.1)$ by $(2\text{-}a.2)$ and M by $N_2 N_3$, we obtain

$$\begin{aligned}\underset{(N\text{-point DFT})}{\text{ComplexMults}} &= N_1(N_2 N_3(N_2 + N_3 + 1)) + N_2 N_3(N_1^2) + N_1 N_2 N_3 \\ &= N_1 N_2 N_3(N_1 + N_2 + N_3 + 2). \qquad (2\text{-}a.3)\end{aligned}$$

Using $N_1 = 11$, $N_2 = 13$, and $N_3 = 17$, we obtain

$$\underset{(N\text{-point DFT})}{\text{ComplexMults}} = 2431(11 + 13 + 17 + 2) = 104533$$

while the direct would require $2431^2 = 5909761$ complex multiplications,which is more than 50 times the divide and conquer approach.

(b) [11-Points] Consider a 16-point radix-2 decimation-in-time FFT (DIT-FFT) algorithm similar in structure to the one given in Figure 8.1.11 (Page-536) of the textbook *Digital Signal Processing* (by Proakis and Manolakis), but drawn for a 16-point FFT.

i. [2-Point] How many ***complex multiplications and additions*** would be required for direct evaluations of a 16-point DFT?

Answer: The formula for the number of complex multiplications in the direct DFT is N^2. For $N = 16$, the total number of complex multiplications are 256. The formula for the number of complex additions in the direct DFT is $N(N-1)$. For $N = 16$, the total number of complex additions are 240.

---

ii. [2-Point] How many ***complex multiplications and additions*** would be required in the 16-point Radix-2 DIT FFT algorithm?

Answer: The formula for the number of complex multiplications in radix-2 FFT is $(N/2)\log_2(N)$. For $N = 16$, the total number of complex multiplications are 32. The formula for the number of complex additions in radix-2 FFT is $N\log_2(N)$. For $N = 16$, the total number of complex additions are 64.

iii. [3-Point] What ***time index*** samples will be put in the third, seventh, and twelfth position indices for implementing the 16-point, Radix-2 DIT FFT algorithm? Fos example, the time sample in the first position is $x(0)$. You must provide justification for each answer and not just the final answer.

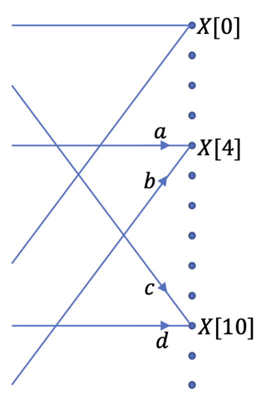
Answer: In DIT-FFT algorithm, the time indexes are bit reversed. The time index at the third position is 2, which in binary is 0010. The bit reversed binary is 0100, So the time index is 4. Similarly, other time indexes can be computed.

Third position: 4

Seventh position: 6

Twelfth position: 13

iv. [4-Points] Part of the signal flow graph for the last stage is shown below. What are the twiddle factors a, b, c, and d? Explain your answers to get full credit.



Answer: We want $X[4]$ and $X[10]$ for $N = 16$.

$$\begin{aligned} X[4] &= G[4] + W_{16}^{4} H[4] = G[4] + (-\jmath)H[4] \\ &\Rightarrow \; a = 1 \text{ and } b = -\jmath \end{aligned}$$

and

$$\begin{aligned} X[10] = G\left[10 - \frac{16}{2}\right] + W_{16}^{10} H\left[10 - \frac{16}{2}\right] &= G[2] + \left(-\frac{\sqrt{2}}{2} + \jmath\frac{\sqrt{2}}{2}\right) H[2] \\ &\Rightarrow \; c = 1 \text{ and } d = -\frac{\sqrt{2}}{2} + \jmath\frac{\sqrt{2}}{2} \end{aligned}$$

$a = 1$

$b = -\jmath$

$c = 1$

$d = -\frac{\sqrt{2}}{2} + \jmath\frac{\sqrt{2}}{2}$

(c) [4-Points] A continuous-time signal $x_a(t)$ is uniformly sampled over 1 second to generate 4096 samples.

i. (2-points) Suppose we are only interested in the DFT samples that correspond to frequencies in the range $200 \le F \le 300$ Hz. How many complex multiplies are required to evaluate these values by computing the DFT directly, and how many complex multipliers are required if a DIT-FFT is used? Justify your answers.

Solution: The sampling frequency is $F_s = 4096$ sam/sec. With 4096-point DFT, each frequency sample represents 1Hz. Over the frequency range from 200 to 300 Hz, we have 101 DFT samples. The number of multiplications necessary to evaluate only these samples using direct DFT is $101 \times 4096 = 413,696$. On the otherhand the number of multiplications required if FFT is used is $(4096/2)\log_2(4096) = 24,576$.

Complex multiplies for direct DFT: 413696

Complex multiplies for DIT-FFT: $24,576$

ii. (2-points) How many frequency samples $M < N$ would be needed in order for the N-point FFT algorithm to be more efficient than evaluating the N-point DFT directly?

Solution: To evaluate M frequency samples from N smples, we need MN multiplivations and using FFT we need $(N/2)\log_2 N$ multiplications. Thus we want

$$MN \ge (N/2)\log_{10} N \quad \Rightarrow \quad M \ge \frac{1}{2}\log_2 N$$

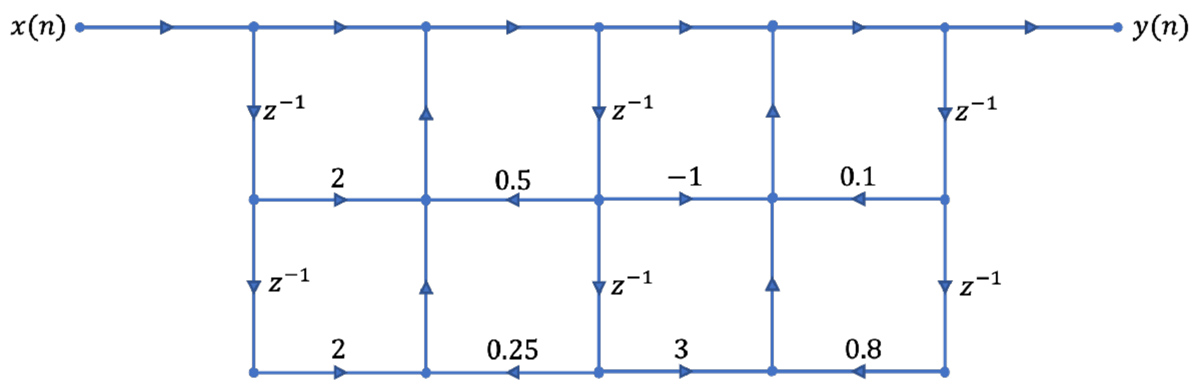
for FFT to be more efficient. For $N = 4096$, we need $M \ge 6$ for FFT to be efficient.

Problem-3 (20-Points) Digital Filter Structures

In this problem, use of MATLAB is strongly recommended for coefficient calculations.

The parts (a) and (b) are not related.

(a) (10-points) Consider the following IIR filter structure:



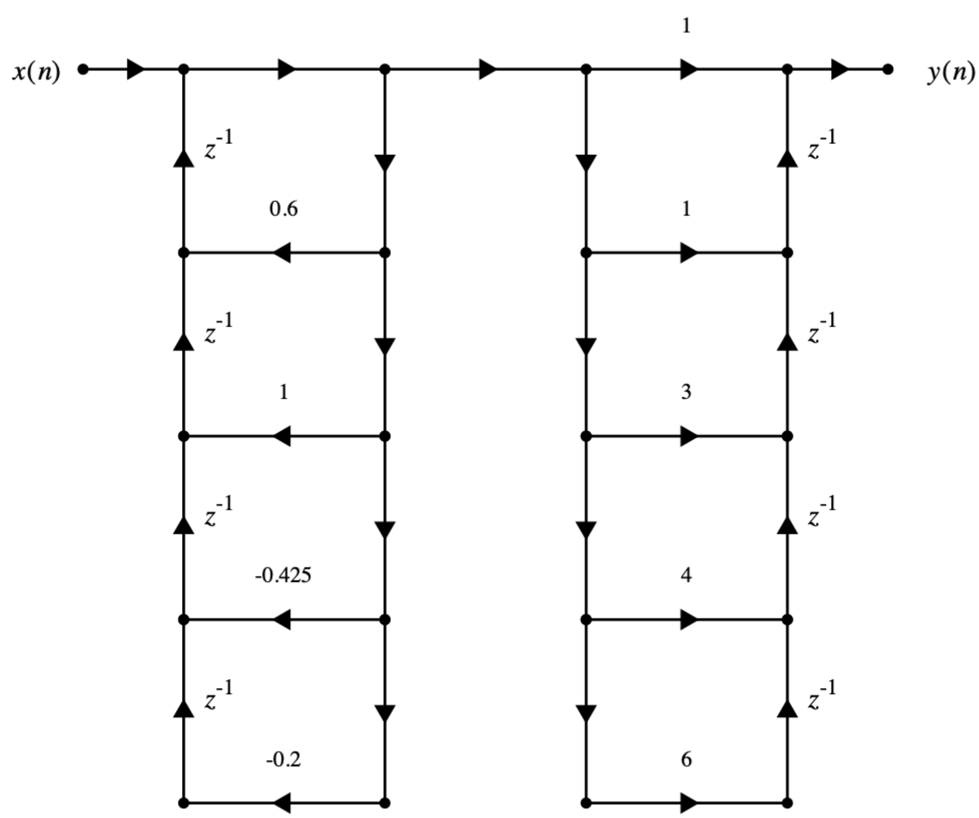
i. [5-Points] Determine and draw a transposed direct form-I signal flow graph (SFG) with coefficients accurate up to 2 decimal places.

MATLAB script: A careful inspection of the above SFG indicates that it is a cascade of two second-order sections. Hence, we will use the sos2tf to obtain the rational function to draw the directform-II SFG.

```
clc; close all; clear;
sos = [1,-1,3,1,-0.5,-0.25;1,2,2,1,-0.1,-0.8];
[b,a] = sos2tf(sos)
```

```
b = 1×5
      1      1      3      4      6
a = 1×5
    1.0000    -0.6000    -1.0000     0.4250     0.2000
```

Signal Flow Graph: Neatly draw your structure on a paper, scan or photograph it, and submit as an image. The image must fit within the page margin of the exported PDF file.



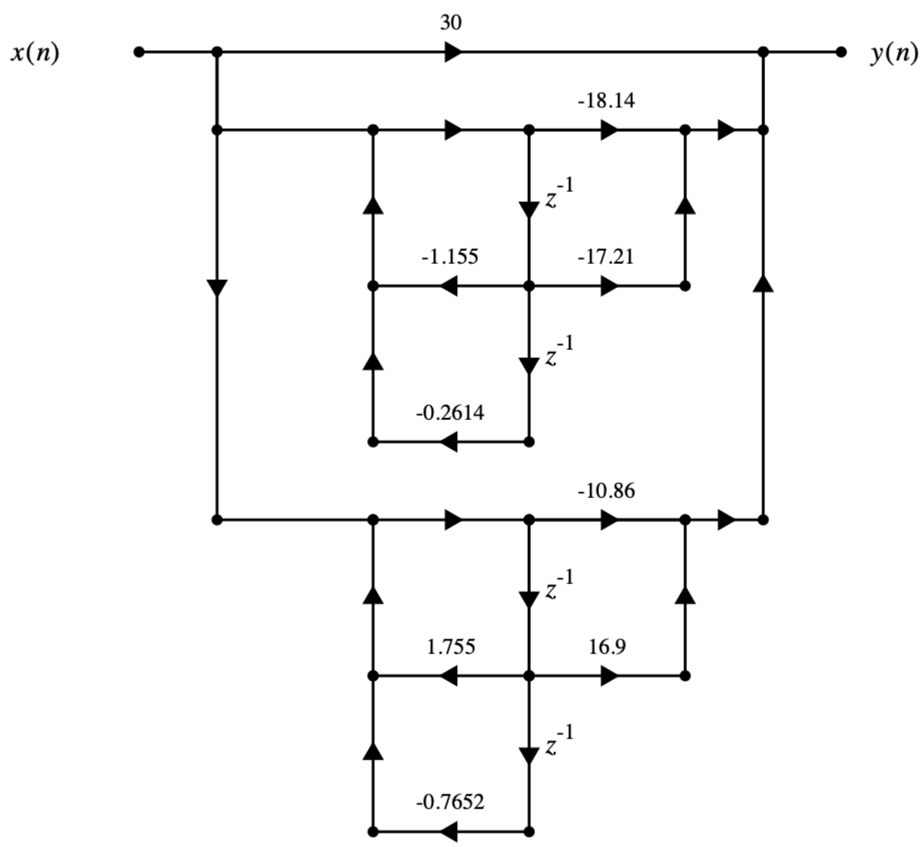
ii. (5-points) Determine and draw a parallel structure SFG containing second order sections in normal direct-II form with coefficients accurate up to 2 decimal places.

MATLAB script:

```
[C,B,A] = dir2par(b,a)
```

```
C = 30
B = 2×2
  -18.1447  -17.2093
  -10.8553   16.9043
A = 2×3
    1.0000    1.1548    0.2614
    1.0000   -1.7548    0.7652
```

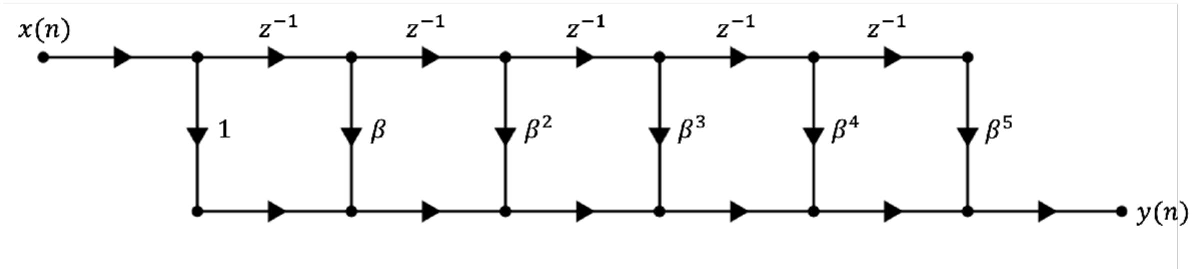
Signal Flow Graph: Neatly draw your structure on a paper, scan or photograph it, and submit as an image. The image must fit within the page margin of the exported PDF file.



(b) (10-points) The unit impulse response of an FIR filter is $h(n) = \beta^n[u(n) - u(n-6)]$, where β is a constant.

i. [3-Points] Draw the normal direct form structure.

Signal Flow Graph: Neatly draw your structure on a paper, scan or photograph it, and submit as an image. The image must fit within the page margin of the exported PDF file.

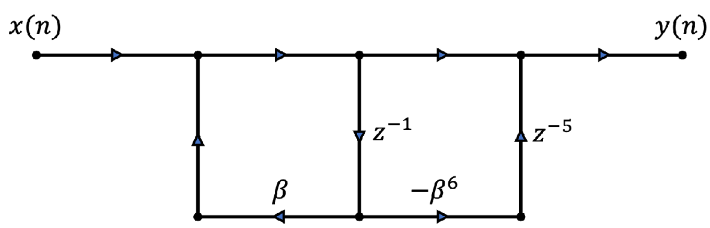


ii. [3-Points] Show that this FIR filter can also be implemented as a cascade of an FIR system and an IIR system, that is, show that the system function $H(z)$ can be expressed as a product of an FIR filter and an IIR filter. Draw the signal flow graph (normal form) of this structure,

Solution: The system function is given by

$$H(z) = \sum_{n=0}^{5} h(n)z^{-1} = \sum_{n=0}^{5} \beta^n z^{-1} = \frac{1-\beta^6 z^{-6}}{1-\beta z^{-1}} \quad \Rightarrow \quad H_{\text{FIR}}(z) = 1-\beta^6 z^{-6}, \quad H_{\text{IIR}}(z) = \frac{1}{1-\beta z^{-1}}$$

Signal Flow Graph: Neatly draw your structure on a paper, scan or photograph it, and submit as an image. The image must fit within the page margin of the exported PDF file.



iii. [4-Points] For each of the above implementations in sub-parts **i** and **ii**, determine the total number of multiplications and additions required to compute each output value and the number of unit-delay elements that are required.

Solution: The direct form SFG requires 5 delays, 5 multiplications, and 5 additions. The cascade on the other hand, requires 1 additional delay but only two multiplications and 2 additions.