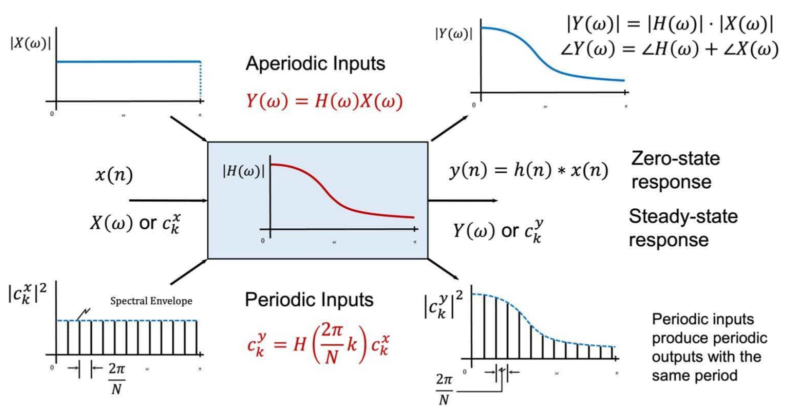
DTFT System Analysis



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Frequency Response Function

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = H_R(\omega) + jH_I(\omega) = |H(\omega)|e^{j\angle H(\omega)}$$

If $h(n) = \text{Real} \Rightarrow H_R(-\omega) = H_R(\omega)$ (Even) and $H_I(-\omega) = -H_I(\omega)$ (Odd)

If $h(n)$ is real, then for all $n, -\infty < n < \infty \Rightarrow$

$$x(n) = A\cos(\omega_0 n + \theta) \rightarrow y(n) = A|H(\omega_0)|\cos[\omega_0 n + \theta + \angle H(\omega_o))$$

True for LTI systems only!

Physical meaning: Magnitude response Phase response

Linearity implies that this property holds for any linear combination of sinusoidal or complex exponential sequences

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Periodicity and Symmetry of $H(\omega)$

- $H(\omega)$ is always a periodic function with period 2π

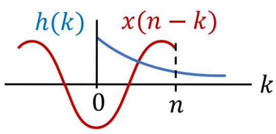
$$H(\omega + 2\pi) = \sum_k h(k)\mathrm{e}^{\mathrm{j}(\omega+2\pi)k} = \sum_k h(k)\mathrm{e}^{\mathrm{j}\omega k}\,\mathrm{e}^{\mathrm{j}2\pi k} = H(\omega)$$

- It is sufficient to specify $H(\omega)$ over any interval of one period
- If $h(n)$ is real, that is, $h^*(n) = h(n)$, we have

$$H(-\omega) = H^*(\omega) \qquad \text{(conjugate symmetry)}$$
$$|H(-\omega)| = |H(\omega)| \qquad \text{(even symmetry)}$$
$$\angle H(-\omega) = -\angle H(\omega) \qquad \text{(odd symmetry)}$$
$$H_{\mathrm{R}}(-\omega) = H_{\mathrm{R}}(\omega) \qquad \text{(even symmetry)}$$
$$H_{\mathrm{I}}(-\omega) = -H_{\mathrm{I}}(\omega) \qquad \text{(odd symmetry)}$$

- It is sufficient to specify $H(\omega)$ in the interval $0 \leq \omega \leq \pi$

Causal Complex Exponential Inputs



$x(n) = e^{j\omega n}u(n) \longrightarrow$ LTI $h(n)$ $\longrightarrow y(n)$

$h(n) = 0, n < 0$ (Causal)

$\sum_n |h(n)| < \infty$ (Stable)

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k) = \sum_{k=0}^{n} h(k)e^{-j\omega(n-k)} = \left(\sum_{k=0}^{\infty} h(k)e^{-j\omega k}\right)e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h(k)e^{-j\omega k}\right)e^{j\omega n}$$

$$y(n) = \underbrace{H(\omega)e^{j\omega n}}_{\substack{\text{Steady-state}\\ y_{ss}(n)}} - \underbrace{\left(\sum_{k=n+1}^{\infty} h(n)e^{-j\omega k}\right)e^{j\omega n}}_{\substack{\text{Transient}\\ y_{tr}(n)}}$$

Eigenfunction property holds for the steady-state response of stable systems!

$$|y_{tr}(n)| \leq \sum_{k=n+1}^{\infty} |h(k)| \leq \sum_{k=0}^{\infty} |h(k)| < \infty \quad \text{(Stability)}$$

- IIR systems $\Rightarrow y_{tr}(n) \to 0$ as $n \to \infty$
- FIR systems $\Rightarrow y_{tr}(n) = 0$ for $n \geq M$ (if $h(n) = 0$ for $n < 0$ and $n \geq M$)

Sinusoidal Response of Real LTI Systems

- Consider the sinusoidal sequence

$$x(n) = A_x \cos(\omega n + \phi_x) = \frac{A_x}{2} \mathrm{e}^{\mathrm{j}\phi_x} \mathrm{e}^{\mathrm{j}\omega n} + \frac{A_x}{2} \mathrm{e}^{-\mathrm{j}\phi_x} \mathrm{e}^{-\mathrm{j}\omega n}.$$

- Eigenfunction property of LTI systems

$$x_1(n) = \frac{A_x}{2} \mathrm{e}^{\mathrm{j}\phi_x} \mathrm{e}^{\mathrm{j}\omega n} \xmapsto{\mathcal{T}} y_1(n) = |H(\omega)| \frac{A_x}{2} \mathrm{e}^{\mathrm{j}\phi_x} \mathrm{e}^{\mathrm{j}[\omega n + \angle H(\omega)]}$$

$$x_2(n) = \frac{A_x}{2} \mathrm{e}^{-\mathrm{j}\phi_x} \mathrm{e}^{-\mathrm{j}\omega n} \xmapsto{\mathcal{T}} y_2(n) = |H(-\omega)| \frac{A_x}{2} \mathrm{e}^{-\mathrm{j}\phi_x} \mathrm{e}^{\mathrm{j}[-\omega n + \angle H]}$$

- Linearity: $x(n) = x_1(n) + x_2(n) \xmapsto{\mathcal{T}} y(n) = y_1(n) + y_2(n)$ and real $h(n)$:

$$|H(-\omega)| = |H(\omega)| \text{ and } \angle H(-\omega) = -\angle H(\omega) \Rightarrow$$

$$x(n) = A_x \cos(\omega n + \phi_x) \xmapsto{\mathcal{T}} y(n) = A_y \cos(\omega n + \phi_y)$$

$$A_y = |H(\omega)| A_x, \quad \phi_y = \angle H(\omega) + \phi_x$$

Moving Average Filter

$$y(n) = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x(n-k)$$

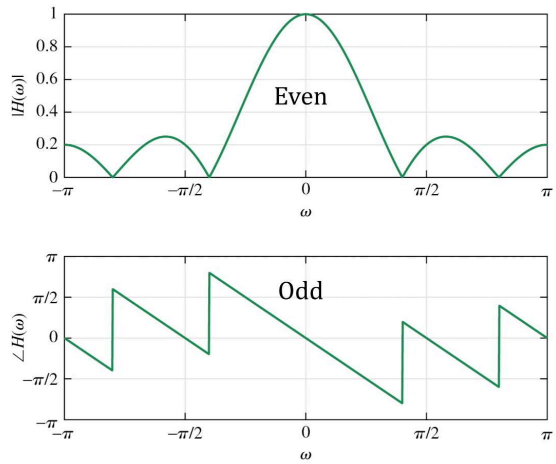
$$H(\omega) = \frac{1}{M_1 + M_2 + 1} e^{-\frac{j\omega(M_2 - M_1)}{2}}$$

$$\times \frac{\sin[\omega(M_1 + M_2 + 1)/2]}{\sin\left(\frac{\omega}{2}\right)}$$

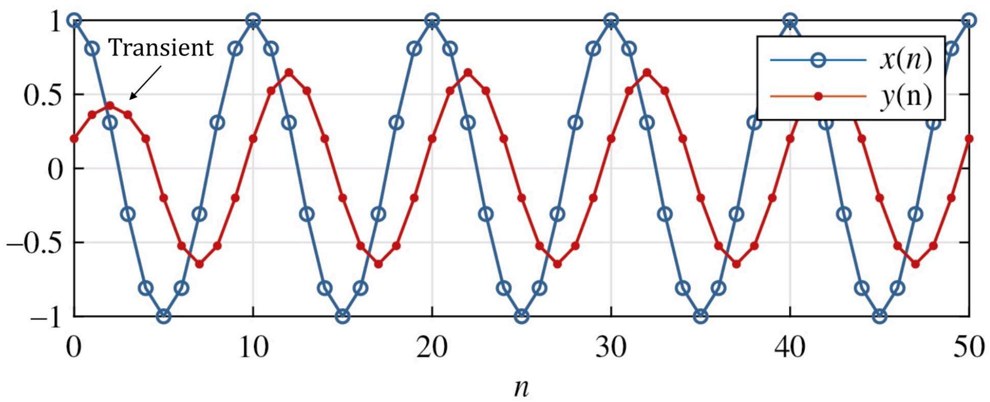
$M_1 = 0, M_2 = M \Rightarrow h(\mathrm{n})$ causal, $H(\omega)$ complex

$M_1 = M_2 = M \Rightarrow h(\mathrm{n})$ non−causal, $H(\omega)$ real

$M_1 = 0, M_2 = 4$



Sinusoidal Response



Response of a MA filter with $M = 4$ to the sinusoid $x(n) = \cos(0.2\,\pi\, n)\, u(n)$

First-Order Recursive System

$y(n) = ay(n-1) + bx(n), \quad -1 < a < 1, \quad b > 0$

$Y(\omega) = ae^{-j\omega}Y(\omega) + bX(\omega)$ $\qquad H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{b}{1 - ae^{-j\omega}}$

$|H(\omega)| = \dfrac{b}{\sqrt{1 - 2a\cos\omega + a^2}}$ $\qquad \angle H(\omega) = \Theta(\omega) = \angle b - \tan^{-1}\dfrac{a\sin\omega}{1 - a\cos\omega}$

$0 < a < 1 \Rightarrow |H(\omega)|_{\max} = |H(0)| = \dfrac{b}{1-a}$ (Low−pass)

$-1 < a < 0 \Rightarrow |H(\omega)|_{\max} = |H(\pi)| = \dfrac{b}{1+a}$ (High−pass)

To normalize the gain of the filter we set $b = 1 - |a|$

Graphical Representation of the FRF

- A plot of $|H(e^{j\omega})|$ (magnitude) and $\angle H(e^{j\omega})$ (phase) versus ω shows at a glance what the system does to complex exponential signals and sinusoids of different frequencies
- In MATLAB this is done using the following code:

```
>> omega=linspace(-pi,pi,1000);
>> b=[1];
>> a=[1 -0.8];
>> H=freqz(b,a,omega);
>> subplot(2,1,1), plot(omega/pi,abs(H))
>> subplot(2,1,2), plot(omega/pi,angle(H)/pi)
```

- We divide by `pi` to plot frequency and phase in multiples of π for easier interpretation of results
- MATLAB function `angle` provides the principal value of the angle in the range $-\pi < \angle H(e^{j\omega}) \leq +\pi$ using `atan2`

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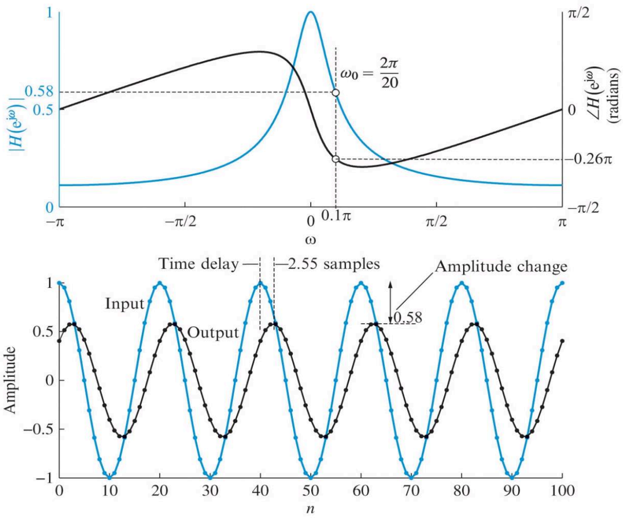
Plots of Frequency Response Function



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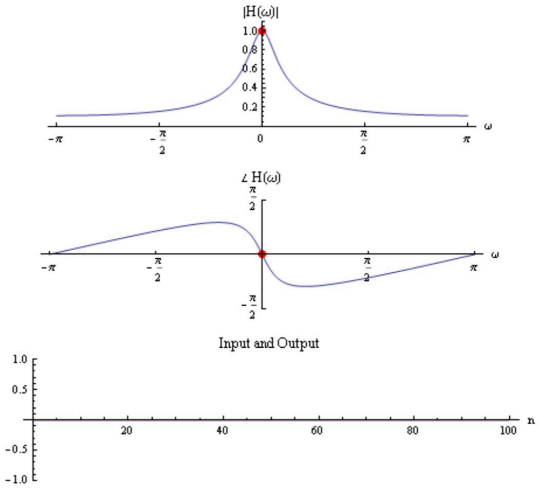
Interpretation of Frequency Response: Live



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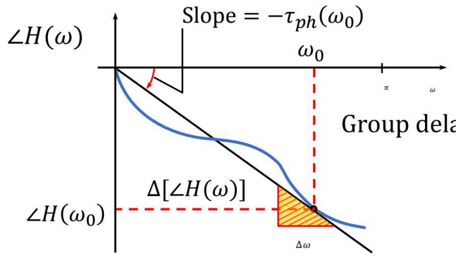
Magnitude and Phase Distortions

Distortion-less Response $y(n) = cx(n - n_d) \Rightarrow H(\omega) = ce^{-j\omega n_d} \Rightarrow \begin{cases} |H(\omega)| = c & \text{(constant)} \\ \angle H(\omega) = -\omega n_d & \text{(linear)} \end{cases}$

Phase delay $\tau_{ph}(\omega) = -\dfrac{\angle H(\omega)}{\omega}$ (carrier delay)

Group delay $\tau_g(\omega) = -\dfrac{d}{d\omega}[\angle H(\omega)]$ (envelope delay)



Group delay $= \lim\limits_{\Delta\omega \to 0} \dfrac{\Delta[\angle H(\omega)]}{\Delta\omega}$

Envelop delay in sinusoidal modulation (see CP 5.17)

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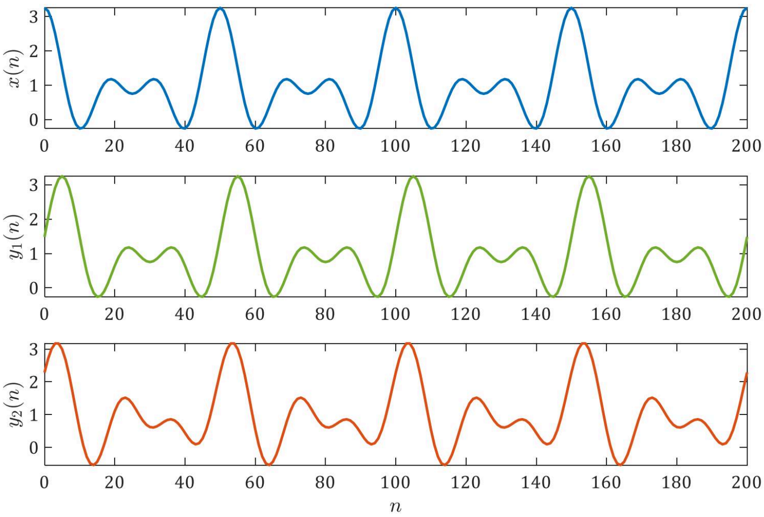
Illustration of Phase Response Effects

Input signal

Linear phase
MA filter output
$M = 5$

Nonlinear phase
PZ(1,1) filter output
$a = 0.9$



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MATLAB Computation of $H(\omega)$

Function	Description
`H=freqz(b,a,N)`	Computes $H(\omega)$ at N equispaced frequencies at $0 \le \omega \le \pi$
`H=freqz(b,a,om)`	Computes $H(\omega)$ at the frequencies specified by vector om $= [\omega_1 \omega_2 \ldots \omega_N]$
`p=angle(H)`	Computes $\angle H(\omega)$ modulo 2π (principal value) which results in a discontinuous graph
`[ph om]=phasez(b,a,om)`	Compute the unwrapped phase response, phase delay, and group delay at the frequencies specified by vector om $= [\omega_1 \omega_2 \ldots \omega_N]$ using the function p = unwrap(p)
`[pd om]=phasedelay(b,a,om)`	
`[gd om]=grpdelay(b,a,om)`	

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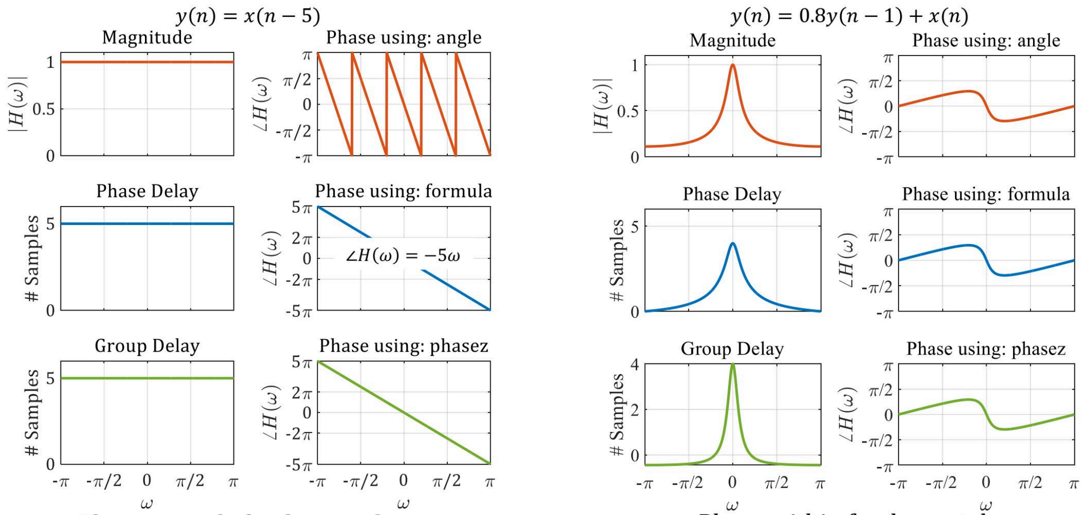
Examples of Phase Response Calculations



Phase exceeds fundamental range

Phase within fundamental range

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Response LTI Systems in the Frequency Domain

$x(n)$, c_k^x or $X(\omega)$ → Stable LTI System, $h(n), H(\omega)$ → $y(n) = h(n) * x(n)$, c_k^y or $Y(\omega)$

$$y(n) = \sum_k h(k)x(n-k)$$

Aperiodic inputs

Replace k by m

Periodic inputs $\quad x(n) = \sum_{k=0}^{N-1} c_k^x e^{j\frac{2\pi}{N}kn}$

$$Y(\omega) = \sum_k h(k)\mathcal{F}\{x(n-k)\}$$

$$= \sum_k h(k)e^{-j\omega k}\, X(\omega)$$

$$y(n) = \sum_m h(m) \sum_k c_k^x\, e^{j\frac{2\pi}{N}k(n-m)}$$

$$= \sum_k c_k^x\, e^{j\frac{2\pi}{N}kn} \underbrace{\sum_m h(m)\, e^{-j\frac{2\pi}{N}km}}_{H\left(\frac{2\pi}{N}k\right)}$$

$Y(\omega) = H(\omega)X(\omega)$ | Convolution theorem | $c_k^y = H\left(\frac{2\pi}{N}k\right) c_k^x$

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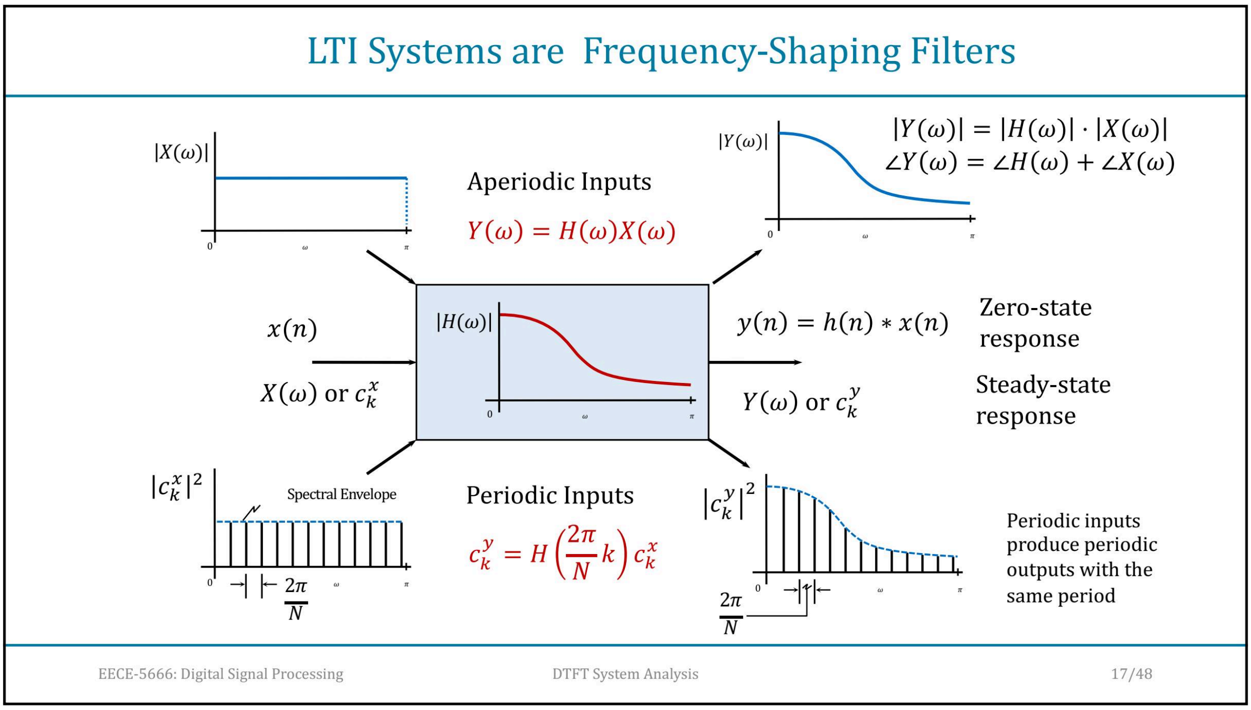
LTI Systems are Frequency-Shaping Filters
$|X(\omega)|$
Aperiodic Inputs
$Y(\omega) = H(\omega)X(\omega)$
$|Y(\omega)|$
$|Y(\omega)| = |H(\omega)| \cdot |X(\omega)|$
$\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)$
$x(n)$
$X(\omega)$ or c_k^x
$|H(\omega)|$
$y(n) = h(n) * x(n)$
$Y(\omega)$ or c_k^y
Zero-state response
Steady-state response
$|c_k^x|^2$
Spectral Envelope
$\frac{2\pi}{N}$
Periodic Inputs
$c_k^y = H\left(\frac{2\pi}{N}k\right) c_k^x$
$|c_k^y|^2$
$\frac{2\pi}{N}$
Periodic inputs produce periodic outputs with the same period
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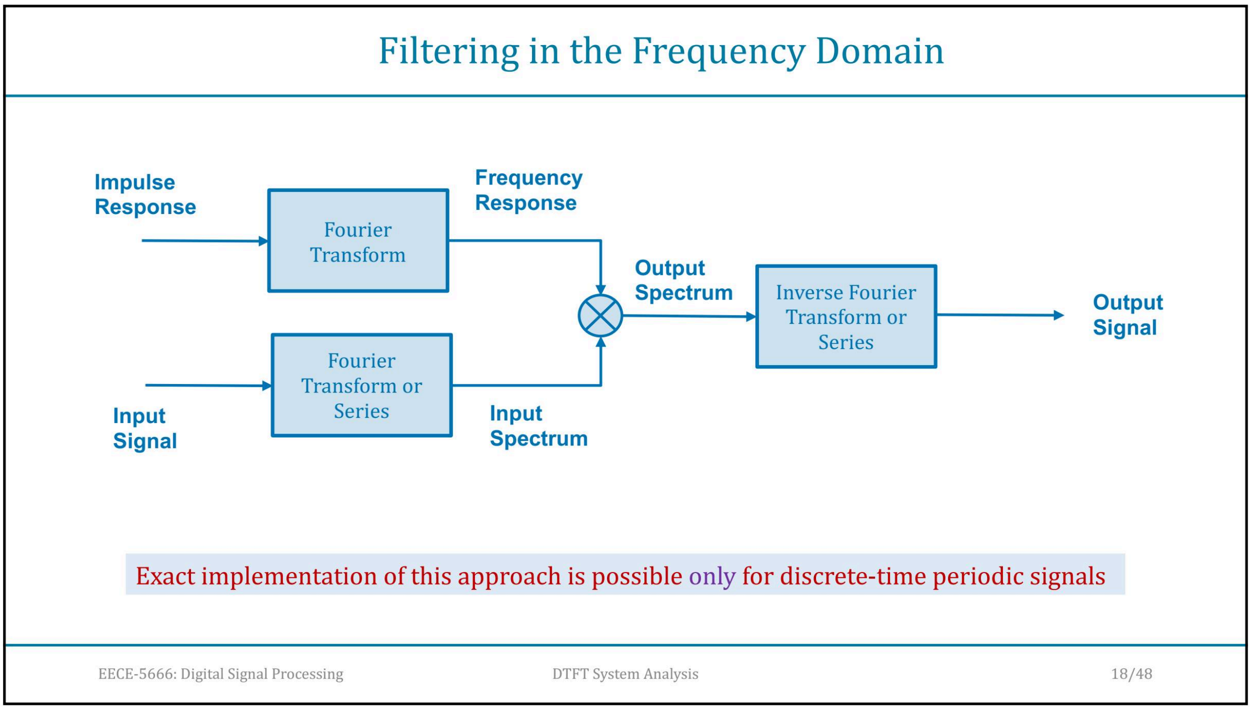
Filtering in the Frequency Domain
Impulse Response
Fourier Transform
Frequency Response
Output Spectrum
Inverse Fourier Transform or Series
Output Signal
Input Signal
Fourier Transform or Series
Input Spectrum
Exact implementation of this approach is possible only for discrete-time periodic signals
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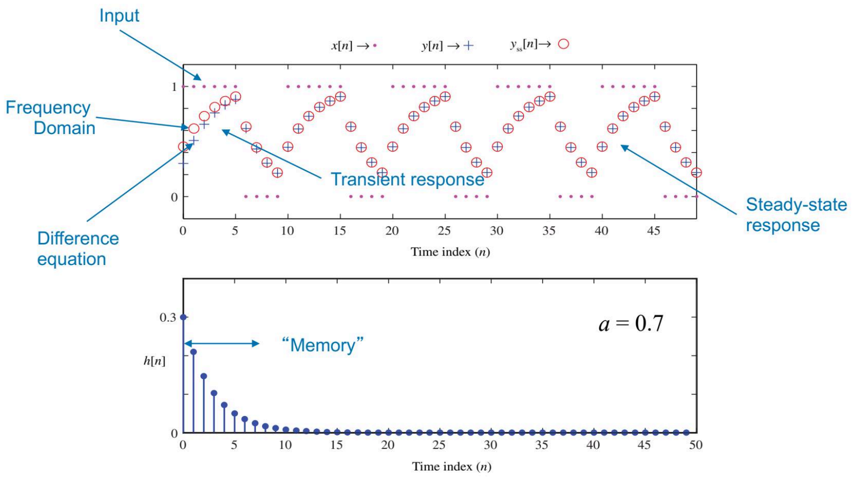
Filter With "Short-Memory"



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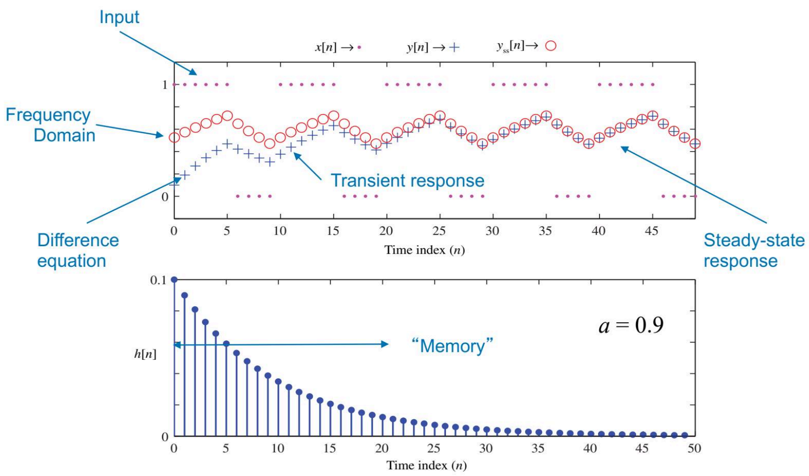
Filter With "Long-Memory"



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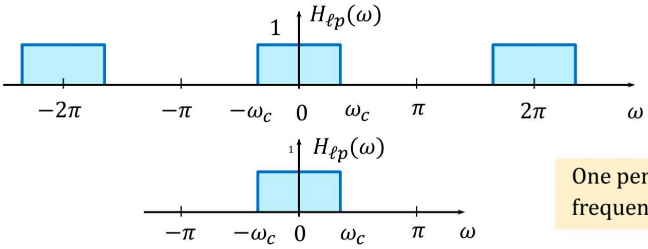
The Ideal Low-Pass Filter



One period of the periodic frequency response

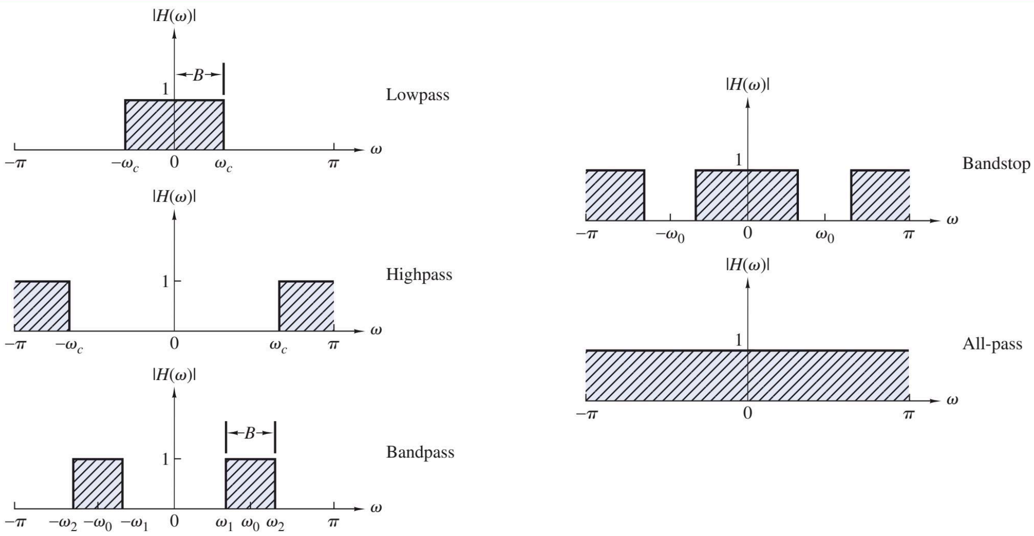
$$H_{\ell p}(\omega)=\begin{cases}1, & |\omega|<\omega_c\\ 0, & \omega_c<|\omega|<\pi\end{cases}\Rightarrow$$

$$h_{\ell p}(n)=\begin{cases}\dfrac{\omega_c}{\pi}, & n=0\\ \dfrac{\sin\omega_c n}{\pi n}, & n\neq 0\end{cases}\Rightarrow \text{IIR}$$

- $h_{\ell p}(n)\neq 0, n<0 \Rightarrow$ Non−causal
- $\sum_{n=-\infty}^{\infty}\left|h_{\ell p}(n)\right|=\infty \Rightarrow$ Unstable
- Non−rational system function ⇒ Unrealizable

How we design “good” practical filters?

Ideal Frequency-Selective Filter Responses



Response of LTI Systems: The Three Domains

Input signal

Any type $\quad x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) \quad$ Zero-state response

$n-\text{Domain}$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Aperiodic $\quad X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)X(z) \quad$ Zero-state response

$z-\text{Domain}$

Aperiodic $\quad X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega) = H(\omega)X(\omega) \quad$ Steady-state response

Periodic $\quad c_k^x \qquad c_k^y = H\left(\frac{2\pi}{N}k\right)c_k^x$

$\omega-\text{Domain}$

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Evaluation of System Function and Frequency Response

Convolution Summation

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$Y(z) = H(z)X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=-\infty}^{\infty} h(k)z^{-k}$$

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \frac{Y(\omega)}{X(\omega)} = \sum_{k=-\infty}^{\infty} h(k)e^{-jk\omega}$$

Difference Equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{1 + \sum_{k=1}^{N} a_k e^{-jk\omega}}$$

$H(\omega)$ exists for stable systems (Unit circle within ROC)

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Rational System Functions: Poles and Zeros

$$H(z)=\frac{Y(z)}{X(z)}=\frac{B(z)}{A(z)}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}=\frac{z^{-M}\sum_{k=0}^{M} b_k z^{M-k}}{z^{-N}\left(z^{N}+\sum_{k=1}^{N} a_k z^{N-k}\right)}=b_0 z^{(N-M)}\frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

Zeros

$$H(z)=\frac{B(z)}{A(z)}=b_0 z^{(N-M)}\frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}=b_0\frac{\prod_{k=1}^{M}(1-z_k z^{-1})}{\prod_{k=1}^{N}(1-p_k z^{-1})}$$

Poles

Stability: All poles must be inside the unit circle in the z −plane

How does the location of poles and zeros affect the shape of magnitude and phase responses?

Invertibility and Minimum-Phase Systems

- An LTI system $H(z)$ with input $x(n)$ and output $y(n)$ is said to be invertible if we can uniquely determine $x(n)$ from $y(n)$
- The cascade of a system and its inverse is equal to the identity system

$$h(n) * h_{inv}(n) = \delta(n)$$

$$H(z)H_{inv}(z) = 1$$

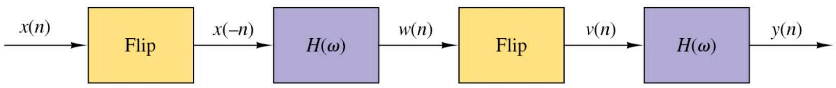
- The inverse of a system with rational system function is

$$H(z)=\frac{B(z)}{A(z)} \Rightarrow H_{inv}(z)=\frac{1}{H(z)}=\frac{A(z)}{B(z)}$$

Poles become zeros and vice-versa!

- A causal and stable system with a causal and stable inverse system is called a minimum-phase system
- A rational minimum-phase system must have all poles and zeros inside the unit circle

Zero-Phase Filtering with a Causal Filter



Let $H(\omega) = \mathcal{F}\{h(n)\} = \sum_{n=0}^{\infty} h(n)e^{-j\omega n}$

If $G(\omega) = |H(\omega)|^2 \Rightarrow \angle G(\omega) = 0$ (zero-phase)

If $h(n)$ is real-valued, we have

$$H^*(\omega) = \sum_n h(n)e^{j\omega n} = H(-\omega) = H(1/z)\Big|_{z=e^{j\omega}}$$

$$G(\omega) = H(\omega)H(-\omega) = H(z)H(1/z)\Big|_{z=e^{j\omega}}$$

Spectral Factorization

$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-n}$$

$$\mathcal{Z}\{h(-n)\} = \sum_n h(-n)z^{-n} = \underbrace{\sum_m h(m)(z^{-1})^{-m}}_{H(1/z)}$$

$$g(n) = h(n) * h(-n)$$

$$y(n) = g(n) * x(n) = h(n) * [h(-n) * x(n)]$$

$$\left.\begin{aligned} v(n) &= h(-n) * x(n) \\ w(n) &= h(n) * x(-n) \end{aligned}\right\} \; v(n) = w(-n)$$

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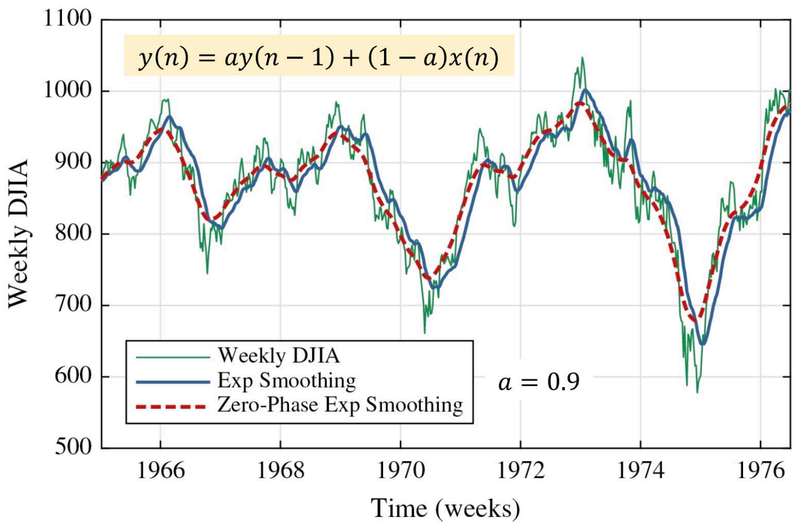
Zero-Phase Exponential Smoothing



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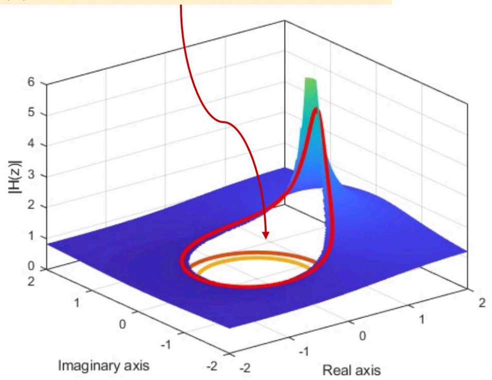
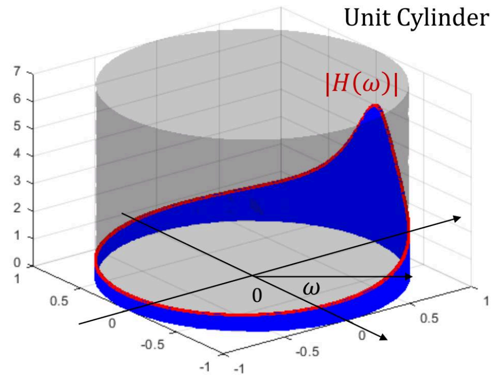
From System Function to Frequency Response

$H(z)$ is not defined outside the ROC!





$$H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$$

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \sum_{k=-\infty}^{\infty} h(k) e^{-jk\omega}$$

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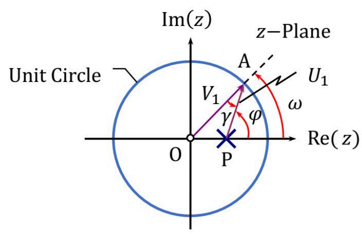
Example: The Simplest Pole-Zero System

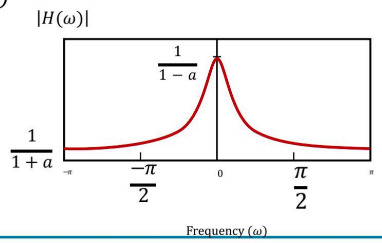
$y(n) = ay(n-1) + x(n)$

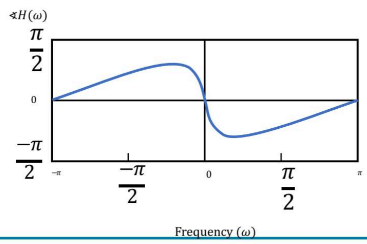
$$Y(z) = az^{-1}Y(z) + X(z) \Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$H(\omega) = \frac{e^{j\omega}}{e^{j\omega}-a} = \frac{\overrightarrow{OA}}{\overrightarrow{OA}-\overrightarrow{OP}} = \frac{\overrightarrow{OA}}{\overrightarrow{PA}} = \frac{\overrightarrow{V_1}(\omega)}{\overrightarrow{U_1}(\omega)}$$



$$|H(\omega)| = \frac{V_1(\omega)}{U_1(\omega)} \qquad \angle H(\omega) = \angle V_1(\omega) - \angle U_1(\omega) = \omega - \varphi = -\gamma$$





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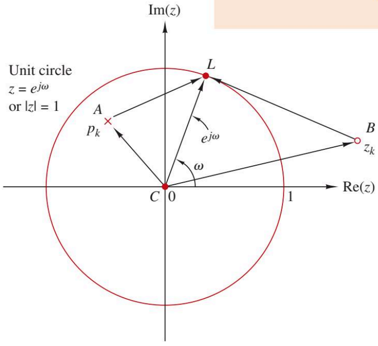
Geometrical Computation of Frequency Response

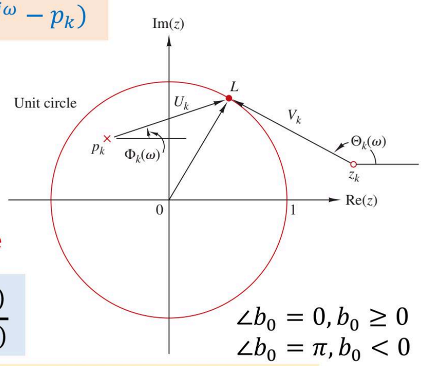
$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$



Unit circle must be inside ROC

Poles must be inside the unit circle

Zeros can be everywhere



$$|H(\omega)| = |b_0| \frac{\prod_{k=1}^{M} V_k(\omega)}{\prod_{k=1}^{N} U_k(\omega)}$$

$\angle b_0 = 0, b_0 \geq 0$
$\angle b_0 = \pi, b_0 < 0$

$$\overrightarrow{BL} = e^{j\omega} - z_k = V_k(\omega) e^{j\Theta_k(\omega)}$$

$$\overrightarrow{AL} = e^{j\omega} - p_k = U_k(\omega) e^{j\Phi_k(\omega)}$$

$$\angle H(\omega) = \angle b_0 + \omega(N-M) + \sum_{k=1}^{M} \Theta_k(\omega) - \sum_{k=1}^{M} \Phi_k(\omega)$$

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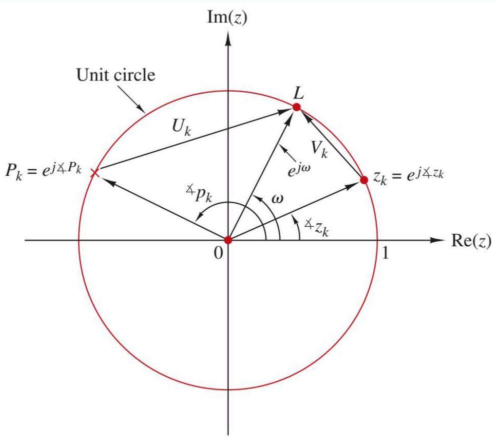
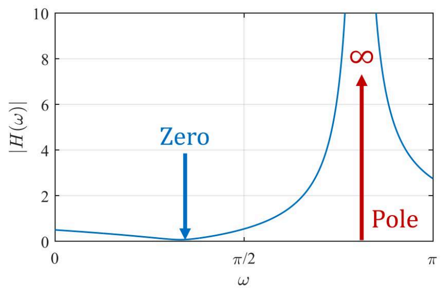
Significance of Poles and Zeros



There is an intuitive relationship between the location of poles and zeros and the shape of magnitude frequency response!



- Poles close to (but inside) the unit circle amplify frequency components
- Zeros on the unit circle eliminate frequency components
- Zeros close to (but inside) the unit circle attenuate frequency components

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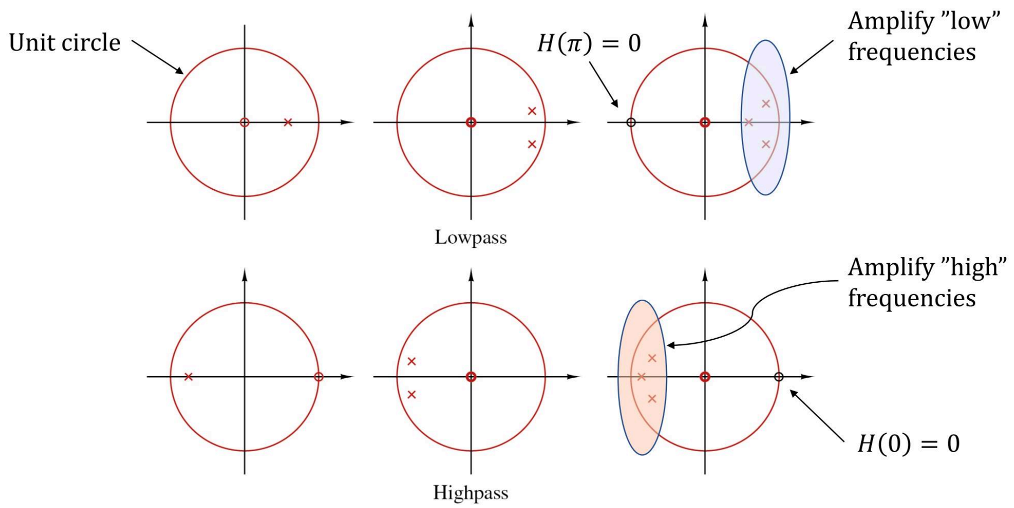
Examples of Pole-Zero Patterns



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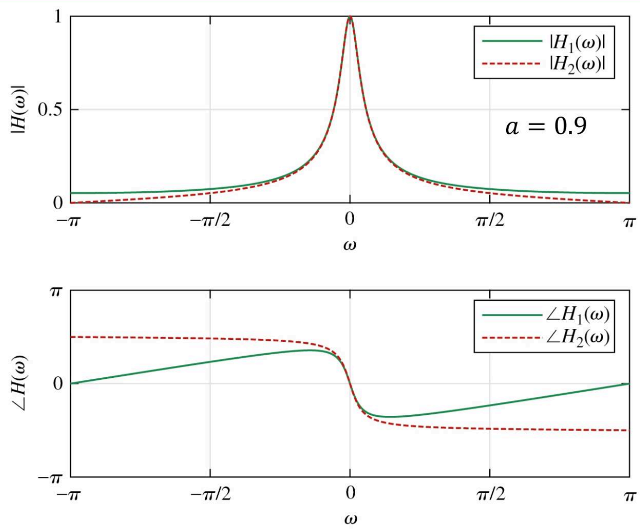
Single Real Pole and Zero

$$H_1(z) = \frac{1-a}{1-az^{-1}}$$

$$H_2(z) = \frac{1-a}{2}\frac{1+z^{-1}}{1-az^{-1}}$$



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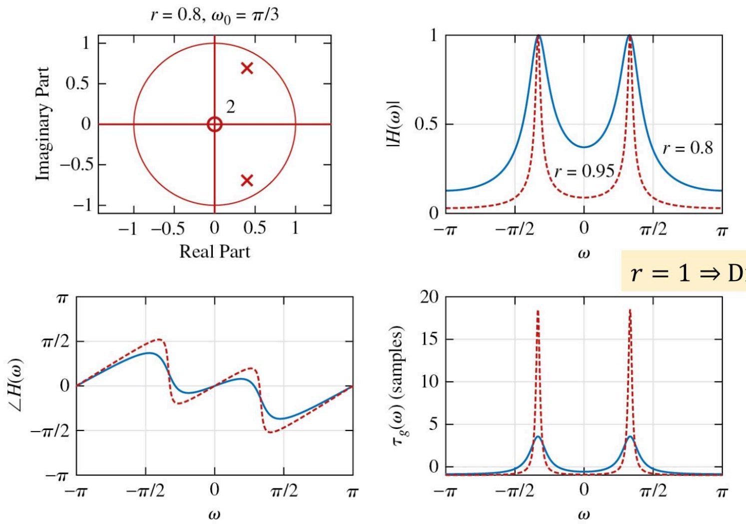
Digital Resonators: Zeros at Zero



$r = 1 \Rightarrow$ Digital Oscillator

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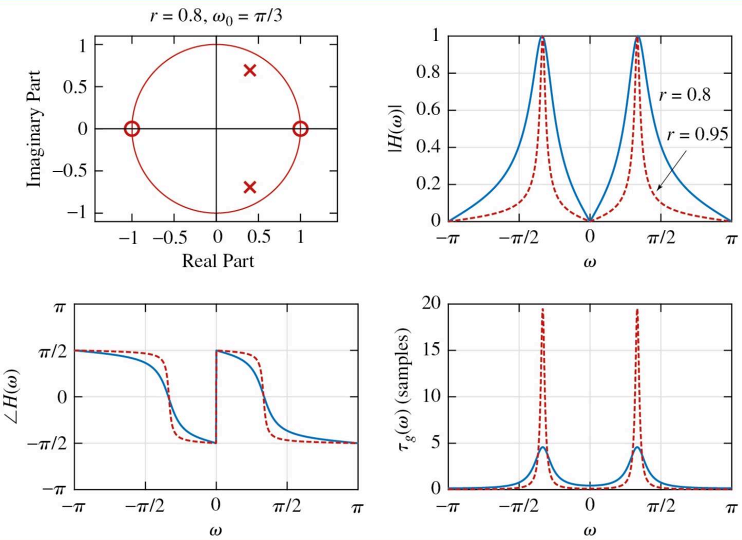
Digital Resonators: Zeros at ± 1



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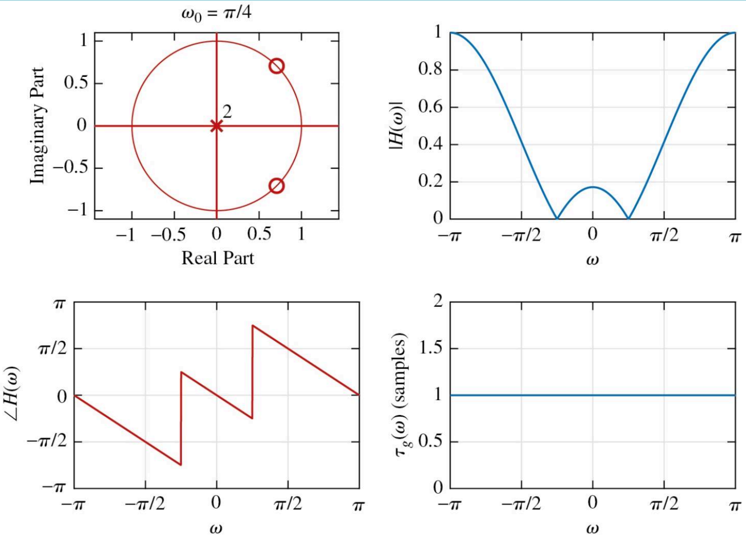
All-Zero Notch Filters



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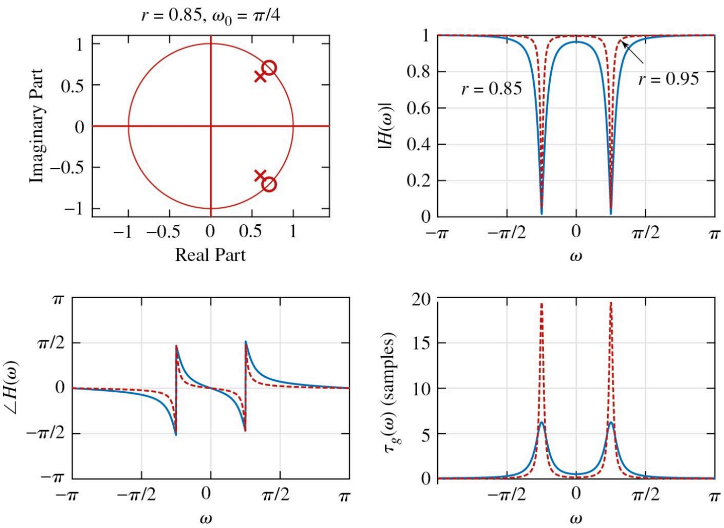
Pole-Zero Notch Filters



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Removing Seasonal or Cyclical Trends

Data exhibit yearly periodicity

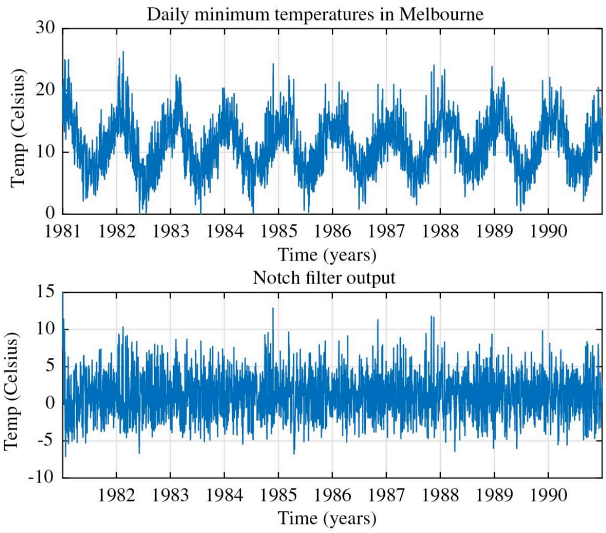
$T_P = 1 \text{ year} = 365 \text{ days}$

Sampling period

$T = 1 \text{ day}$

Apply pole-zero notch filter with

$$\omega_0 = 2\pi \frac{T}{T_p}, \qquad r = 0.95$$



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Comb Filters

Prototype filter

$$H(z) = \sum_n h(n) z^{-n}$$

Comb filter

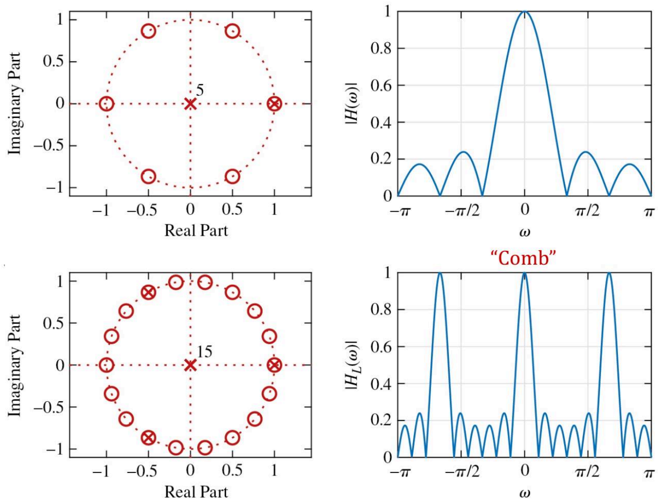
$$G(z) = H(z^L) = \sum_n h(n) z^{-Ln}$$

$$G(z) = \cdots + h(-1)z^L + h(0) + h(1)z^{-L} +$$

$$g(n) = \begin{cases} h(n/L), & n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$G(\omega) = H(L\omega)$$

Periodic in ω with period $2\pi/L$



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Separation of Solar and Lunar Spectral Components

Solar components

$F_S = k\frac{1}{T_S}, \quad T_S = 24$ hours

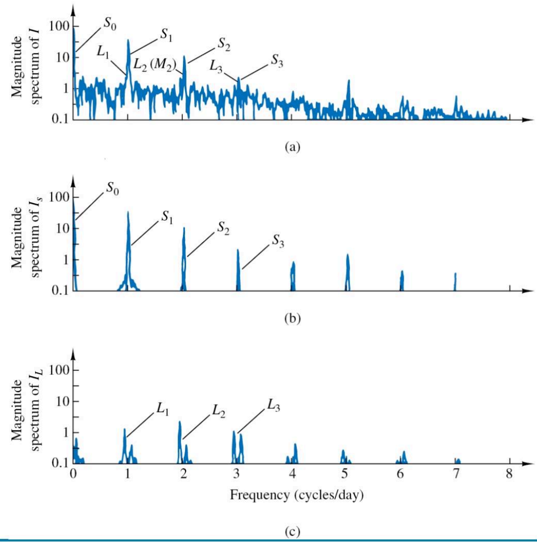
$F_0 = \frac{1}{T_S} = 1$ cycle per day

Lunar components

$F_L = k\frac{1}{T_L}, \quad T_L = 24.84$ hours

$F_0 = \frac{1}{T_L} = 0.96618$ cycles per day

Note that the weak lunar spectral components are almost hidden by the strong solar spectral components



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Reverberation Filters

First artificial reverberator (Schroeder, 1962)

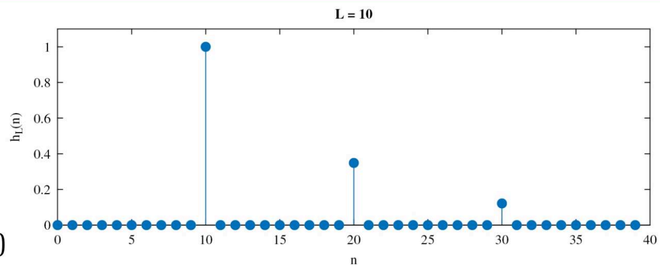
$$y(n) = \sum_{k=1}^{\infty} a^k x(n-kL), \qquad 0 < a < 1$$

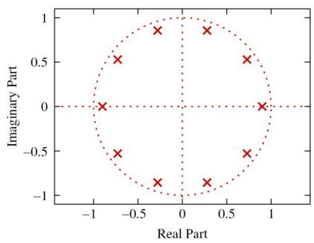
$$H_L(z) = \frac{Y(z)}{X(z)} = \frac{az^{-L}}{1-az^{-L}} \quad \text{(Comb filter)}$$

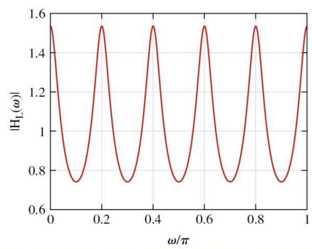
$$H(z) = \frac{az^{-1}}{1-az^{-1}}, \; H_L(z) = H(z^L)$$

$$y(n) = ay(n-L) + ax(n-L)$$

$$p_k = re^{j2\pi k/L}, \qquad r = a^{1/L}$$







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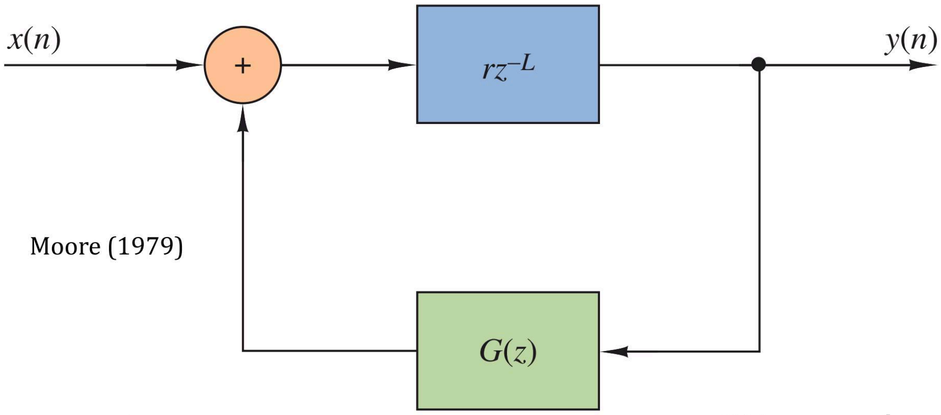
Low-Pass Reverberator



Moore (1979)

$$G(z) = \frac{1-a}{1-az^{-1}}, \qquad 0 < a < 1$$

$$C(z) = \frac{Y(z)}{X(z)} = \frac{rz^{-L} - raz^{-(L+1)}}{1 - az^{-1} - r(1-a)z^{-L}}$$

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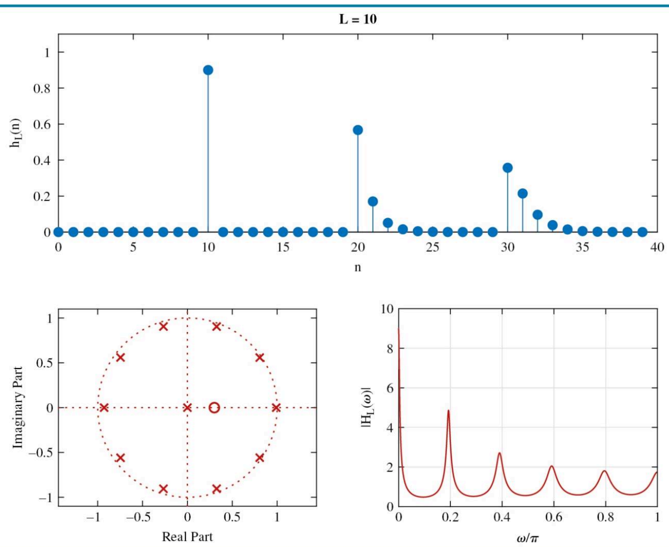
Low-Pass Reverberator



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All-Pass Filters

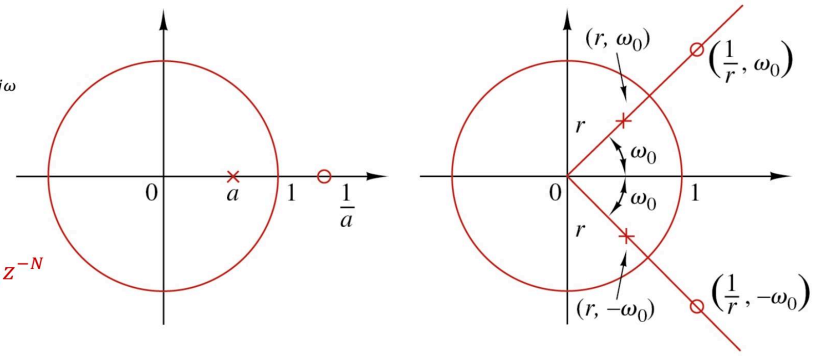
$|H(\omega)| = 1$ (all-pass)

$|H(\omega)|^2 = H(z)H(z^{-1})\Big|_{z=e^{j\omega}}$

$H_{ap}(z) = z^{-N}\dfrac{A(z^{-1})}{A(z)}$

$A(z) = 1 + a_1 z^{-1} + \cdots + a_N z^{-N}$

$\Rightarrow |H_{ap}(\omega)| = 1$



(a) (b)

All-pass filters are used as phase equalizers

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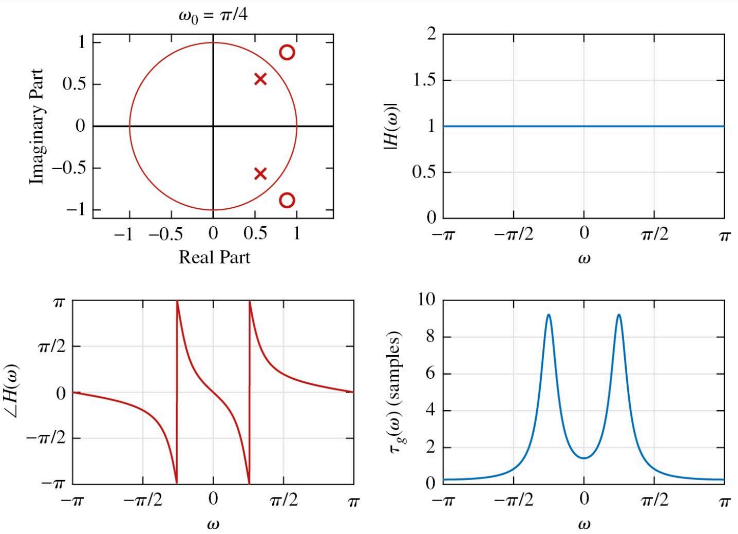
Example of a Second-order All-Pass Filter



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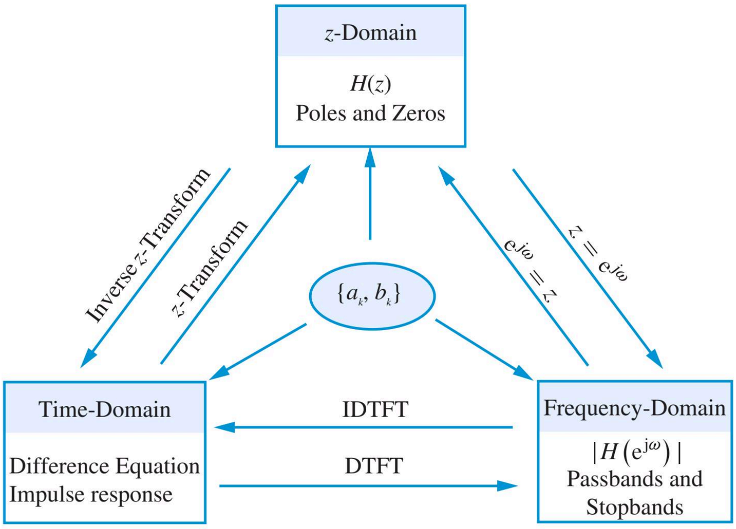
The Three Domains



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Summary

- The frequency response function can be used to understand the effects of an LTI system upon the input signals
- Both the amplitude and phase responses are important and should be specified for the design of practical filters (frequency-selective filters, amplitude equalizers, phase equalizers)
- There is no clear connection between the shape of impulse response or filter coefficients and the shape of frequency response
- The location of poles and zeros determine the shape of frequency response
- By appropriate pole-zero placements, we can design many types of simple filters.

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