EEECE5666 (DSP) : Homework-4 Solutions

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Default Plot Parameters

```
set(0,'defaultfigurepaperunits','points','defaultfigureunits','points');
set(0,'defaultaxesfontsize',10); set(0,'defaultaxeslinewidth',1.5);
set(0,'defaultaxestitlefontsize',1.4,'defaultaxeslabelfontsize',1.2);
```

Problem 4.1

Text Problem 7.4 (Page 509)

From the sequences $x_1(n) = \cos\left(\frac{2\pi}{N}n\right), \quad x_2(n) = \sin\left(\frac{2\pi}{N}n\right), \quad 0 \le n \le N-1$, analytically determine the N-point

(a) Circular convolution $x_1(n) \, (N) \, x_2(n)$.

Solution: With $x_1(n) = \cos\left(\frac{2\pi}{N}n\right) = \frac{1}{2}\left(e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}\right)$, we have $X_1(k) = \frac{N}{2}[\delta(k-1) + \delta(k+1)]$.

Similarly, $X_2(k) = \frac{N}{2j}[\delta(k-1) - \delta(k+1)]$.

The product is defined as $X_3(k) = X_1(k)X_2(k) = \frac{N^2}{4j}[\delta(k-1) - \delta(k+1)]$, leading to

$$x_3(n) = \frac{N}{2}\sin\left(\frac{2\pi}{N}n\right).$$

MATLAB Verification:

```
N = 4; n = 0:N-1;
x1 =  cos(2*pi/N.*n); x2 = sin(2*pi/N.*n);
x3 = cconv(x1, x2, N)
```

```
x3 = 1×4
   -0.0000    2.0000    0.0000   -2.0000
```

```
x3 = N/2.*sin(2*pi/N.*n)
```

```
x3 = 1×4
         0    2.0000    0.0000   -2.0000
```

(b) Circular correlation of $x_1(n)$ and $x_2(n)$.

Solution: Following similar procedure in (a), we have

$$X_4(k) = X_1(k)X_2^*(k) = -\frac{N^2}{4j}[\delta(k-1) - \delta(k+1)]$$

$$x_4(n) = -\frac{N}{2}\sin\left(\frac{2\pi}{N}n\right)$$

MATLAB Verification:

```
N = 4; n = 0:N-1;
x1 =  cos(2*pi/N.*n); x2 = sin(2*pi/N.*n);
X1 = fft(x1,N); X2 = fft(x2,N); X3 = X1.*conj(X2);
x4 = (ifft(X3,4))
```

```
x4 = 1×4
    0.0000   -2.0000   -0.0000    2.0000
```

```
x4 = -N/2.*sin(2*pi/N.*n)
```

```
x4 = 1×4
         0   -2.0000   -0.0000    2.0000
```

(c) Circular autocorrelation of $x_1(n)$.

Solution: Following similar procedure in (a), we have

$$X_5(k) = X_1(k)X_1^*(k) = \frac{N^2}{4}[\delta(k-1) + \delta(k+1)]$$

$$x_5(n) = \frac{N}{2}\cos\left(\frac{2\pi}{N}n\right)$$

MATLAB Verification:

```
x1 =  cos(2*pi/N.*n); x2 = sin(2*pi/N.*n);
```

```
X1 = fft(x1,N); X2 = fft(x2,N); X3 = X1.*conj(X1);
x5 = (ifft(X3,4))
```

```
x5 = 1×4
     2     0    -2     0
```

```
x5 = N/2.*cos(2*pi/N.*n)
```

```
x5 = 1×4
    2.0000    0.0000   -2.0000   -0.0000
```

(d) Circular autocorrelation of $x_2(n)$.

Solution:

Following similar procedure in (a), we have

$$X_6(k) = X_2(k)X_2^*(k) = \frac{N^2}{4}[\delta(k-1)+\delta(k+1)]$$

$$x_6(n) = \frac{N}{2}\cos\left(\frac{2\pi}{N}n\right)$$

MATLAB Verification:

```
N = 4; n = 0:N-1;
x1 =  cos(2*pi/N.*n); x2 = sin(2*pi/N.*n);
X1 = fft(x1,N); X2 = fft(x2,N); X3 = X2.*conj(X2);
x6 = (ifft(X3,N))
```

```
x6 = 1×4
     2     0    -2     0
```

```
x6 = N/2.*cos(2*pi/N.*n)
```

```
x6 = 1×4
    2.0000    0.0000   -2.0000   -0.0000
```

Problem 4.2

Text Problem 7.6 (Page 510)

Consider the Blackman window given by

$$w(n) = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right),\ 0 \le n \le N-1 \qquad (4.2.1)$$

```
clc; close all; clear;
```

(a) Determine the N-point DFT $W(k)$ of $w(n)$.

Solution: To obtain the required N-point DFT into a compact expression, we will need the following four results:

- N-point DFT of a constant A: Letting $\omega_k = (2\pi/N)k$, we have

$$\begin{aligned}\underset{N\text{-point}}{\mathrm{DFT}}[A] &= \sum_{n=0}^{N-1} A\mathrm{e}^{-\jmath\omega_k n} = A\frac{1-\mathrm{e}^{-\jmath\omega_k N}}{1-\mathrm{e}^{-\jmath\omega_k}} = A\frac{\mathrm{e}^{-\jmath\omega_k(N/2)}}{\mathrm{e}^{-\jmath\omega_k(1/2)}}\frac{\sin(\omega_k N/2)}{\sin(\omega_k/2)} \\ &= (AN)\mathrm{e}^{-\jmath\omega_k(N-1)/2}\left[\frac{\sin(\omega_k N/2)}{N\sin(\omega_k/2)}\right] \triangleq (AN)\mathrm{diric}[\omega_k, N]\mathrm{e}^{-\jmath\omega_k(N-1)/2} \quad \text{(P4.2a)}\end{aligned}$$

where $\mathrm{diric}[\omega_k, N] \triangleq \left[\frac{\sin(\omega_k N/2)}{N\sin(\omega_k/2)}\right]$ is the Dirichlet or periodic sinc function (MATLAB definition).

- N-point DFT of a complex exponential $\mathrm{e}^{\jmath(\omega_0 n)}$, $-\pi < \omega_0 \le \pi$: Using the frequency shifting property and $(P4.2.a)$

$$\begin{aligned}\underset{N\text{-point}}{\mathrm{DFT}}\left[A\mathrm{e}^{\jmath\omega_0 n}\right] &= \underset{N\text{-point}}{\mathrm{DFT}}[A]\Big|_{\omega_k \to (\omega_k - \omega_0)} \\ &= (AN)\mathrm{diric}[\omega_k - \omega_0, N]\mathrm{e}^{-\jmath(\omega_k-\omega_0)(N-1)/2} \\ &= \left[(AN)\mathrm{diric}[\omega_k - \omega_0, N]\mathrm{e}^{\jmath\omega_0(N-1)/2}\right]\mathrm{e}^{-\jmath\omega_k(N-1)/2} \quad \text{(P4.2b)}\end{aligned}$$

- Substituting $\omega_0 = \dfrac{2\pi}{N-1}$ in $(P4.2.b)$, we obtain

$$\begin{aligned}\underset{N\text{-point}}{\mathrm{DFT}}\left[A\mathrm{e}^{\jmath\frac{2\pi}{N-1}n}\right] &= \left[(AN)\mathrm{diric}[\omega_k - 2\pi/(N-1), N]\mathrm{e}^{\jmath\left(\frac{2\pi}{N-1}\right)\left(\frac{N-1}{2}\right)}\right]\mathrm{e}^{-\jmath\omega_k(N-1)/2} \\ &= -\left[(AN)\mathrm{diric}[\omega_k - 2\pi/(N-1), N]\right]\mathrm{e}^{-\jmath\omega_k(N-1)/2} \quad \text{(P4.2c)}\end{aligned}$$

since $\mathrm{e}^{\jmath\left(\frac{2\pi}{N-1}\right)\left(\frac{N-1}{2}\right)} = \mathrm{e}^{\jmath\pi} = -1$.

- Similarly, substituting $\omega_0 = \dfrac{4\pi}{N-1}$ in $(P4.2.b)$, we obtain

$$\begin{aligned}\underset{N\text{-point}}{\mathrm{DFT}}\left[A\mathrm{e}^{\jmath\frac{4\pi}{N-1}n}\right] &= \left[(AN)\mathrm{diric}[\omega_k - 4\pi/(N-1), N]\mathrm{e}^{\jmath\frac{4\pi}{N-1}\frac{N-1}{2}}\right]\mathrm{e}^{-\jmath\omega_k(N-1)/2} \\ &= \left[(AN)\mathrm{diric}[\omega_k - 4\pi/(N-1), N]\right]\mathrm{e}^{-\jmath\omega_k(N-1)/2} \quad \text{(P4.2d)}\end{aligned}$$

since $\mathrm{e}^{\jmath\left(\frac{4\pi}{N-1}\right)\left(\frac{N-1}{2}\right)} = \mathrm{e}^{\jmath 2\pi} = +1$.

Now, we can express the N-point DFT $W(k)$ of $w(n)$ by appropriately combining above four results into the final compact expression

$$W(k)=\underbrace{\begin{bmatrix}(0.42N)\text{diric}\left[\frac{2\pi}{N}k,N\right]\\ +(0.25N)\left(\text{diric}\left[\frac{2\pi}{N}k-\frac{2\pi}{N-1},N\right]+\text{diric}\left[\frac{2\pi}{N}k+\frac{2\pi}{N-1},N\right]\right)\\ +(0.04N)\left(\text{diric}\left[\frac{2\pi}{N}k-\frac{4\pi}{N-1},N\right]+\text{diric}\left[\frac{2\pi}{N}k+\frac{4\pi}{N-1},N\right]\right)\end{bmatrix}}_{\text{Zero-phase (real-valued) component}} e^{j\frac{2\pi}{N}\left(\frac{N-1}{2}\right)k} \qquad \text{(P4.2f)}$$

Note that the expression in the square backet on the right-hand side above is a zero-phase term, i.e., a real function of k while the term outside of it is a linear phase term.

(b) Verify your answer in part (a) using MATLAB for $N=21$. To do this, first synthesize $w(n)$ given in $(4.2.1)$ and compute its 21-point DFT (use the **`fft`** function). Then synthesize $W(k)$ from your answer in (a) for $N=21$. Finally, compute the maximum of the absolute difference between the two DFT arrays. This difference should be less than 10^{-10}. Printout this difference.

MATLAB script: For comparison, we will compare the zero-phase component in the square brackets above in (P4.2f) using the **`zerophase`** function.

```
N = 21; n = 0:N-1; k = 0:N-1;
omk = (2*pi/N)*k; % sampled frequency array
beta = 2*pi/(N-1); % digital frequency of the first cosine term
wB = 0.42 - 0.5*cos(2*pi*n/(N-1)) + 0.08*cos(4*pi*n/(N-1)); % Blackman window
wBnumZP = zerophase(wB,1,omk); % Numerical zero-phase component of the N-point
                               % DFT of w(n) for comparison
WB0 = 0.42*N*diric(omk,N); % DFT of the first term in w(n)
WB1 = 0.25*N*(diric(omk-beta,N) + diric(omk+beta,N)); % DFT of the second term in w(n)
WB2 = 0.04*N*(diric(omk-2*beta,N) + diric(omk+2*beta,N)); % DFT of the third term in w(
WBanaZP = WB0+WB1+WB2; % Analytic zero-phase component of the N-point DFT
                       % of w(n) for comparison
Check = max(abs(WBanaZP-wBnumZP))
```

```
Check = 1.8474e-13
```

Clearly, the DFT $W(k)$ expression in (P4.2f) is verified.

Problem 4.3

Text Problems 7.8 and 7.9 (Page 510)

```
clc; close all; clear;
```

(a) Text Problem 7.8: Determine the circular convolution of the sequences $x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$ and $x_2(n) = \{\underset{\uparrow}{4}, 3, 2, 2\}$ using the time-domain formula in (7.2.39). This must a hand calculation.

Solution:

To compute the 7-point circular convolution, let us consider the two sequences as 7-point sequences using zero-padding. Let $x_1[n] = \{\underset{\uparrow}{1}, 2, 3, 1\}$ and $x_2[n] = \{\underset{\uparrow}{4}, 3, 2, 2\}$ Let $x_3[n]$ be the 4-point circular convolution between $x_1[n]$ and $x_2[n]$. It is given by

$$x_3[n] = \sum_{k=0}^{3} x_1[n]x_2\big[\langle n - k\rangle_4\big]$$

Then

$$x_3[0] = \sum_{k=0}^{3} x_1[k]x_2\big[\big\langle -k\rangle_4\big] = \sum[\underset{\uparrow}{1}, 2, 3, 1\} \times \{\underset{\uparrow}{4}, 2, 2, 3\}]$$
$$x_3[0] = \sum[4, 4, 6, 3] = 17$$

Next,

$$x_3[1] = \sum_{k=0}^{3} x_1[k]x_2\big[\big\langle 1 - k\rangle_3\big] = \sum[\{\underset{\uparrow}{1}, 2, 3, 1\} \times \{\underset{\uparrow}{3}, 4, 2, 2\}]$$
$$x_3[1] = \sum[3, 8, 6, 2] = 19$$

Similarly,

$$x_3[2] = \sum[\{\underset{\uparrow}{1}, 2, 3, 1\} \times \{\underset{\uparrow}{2}, 3, 4, 2\}] = \sum[2, 6, 12, 2] = 22,$$
$$x_3[3] = \sum[\{\underset{\uparrow}{1}, 2, 3, 1\} \times \{\underset{\uparrow}{2}, 2, 3, 4\}] = \sum[2., 4, 9, 4] = 19,$$

Thus $x_3[n] = \{\underset{\uparrow}{17}, 19, 22, 19\}$.

(b) Text Problem 7.9: Use the four-point DFT and IDFT to determine the sequence $x_3(n) = x_1(n) \, (N) \, x_2(n)$ where $x_1(n)$ and $x_2(n)$ are the sequence given in Problem 7.8. This part can be done using MATLAB.

MATLAB script: Computation is done using the following script.

```
x1 = [1, 2, 3, 1]'; x2 = [4, 3, 2, 2]';
X1 = fft(x1,4); X2 = fft(x2,4); X3 = X1.*X2;
x3 = real(ifft(X3,4)); x3 = round(x3(:)')
```

```
x3 = 1×4
```

```
    17    19    22    19
```

which agrees with hand calculations in (a).

(c) Verify your calculations in (a) using the **cconv** function from Signal Processing toolbox.

MATLAB script: Computation is done using the following script.

```
x1 = [1, 2, 3, 1]'; x2 = [4, 3, 2, 2]';
x3 = cconv(x1,x2, 4); x3 = x3'; display(x3);
```

```
x3 = 1×4
    17    19    22    19
```

which also agrees with hand calculations in (a).

Problem 4.4

Text Problem 7.13 (Page 511)

Let $x_p(n)$ be a periodic sequence with fundamental period N. Consider the following DFTs:

$$x_p(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X_1(k)$$

$$x_p(n) \underset{3N}{\overset{\text{DFT}}{\longleftrightarrow}} X_3(k)$$

```
clc; close all; clear;
```

(a) Determine the relationship between $X_1(k)$ and $X_3(k)$?

Solution: Define $W_N^{kn} = e^{-j\frac{2\pi}{N}nk}$, the two DFTs would be

$$X_1(k) = \sum_{n=0}^{N-1} x_p(n)W_N^{kn}, \text{ and } X_3(k) = \sum_{n=0}^{3N-1} x_p(n)W_{3N}^{kn}.$$

After expanding, we have

$$\begin{aligned}
X_3(k) &= \sum_{n=0}^{3N-1} x_p(n)W_{3N}^{kn} \\
&= \sum_{n=0}^{N-1} x_p(n)W_{3N}^{kn} + \sum_{n=N}^{2N-1} x_p(n)W_{3N}^{kn} + \sum_{n=2N}^{3N-1} x_p(n)W_{3N}^{kn}. \\
&= \sum_{n=0}^{N-1} x_p(n)W_N^{\frac{k}{3}n} + \sum_{n=N}^{2N-1} x_p(n)W_N^{\frac{k}{3}n} + \sum_{n=2N}^{3N-1} x_p(n)W_N^{\frac{k}{3}n}
\end{aligned}$$

Given the periodicity property of W_N^{kn}, we have

$$W_N^{\frac{k}{3}(n+N)} = W_N^{\frac{k}{3}(n)} W_N^{\frac{k}{3}(N)} = W_N^{\frac{k}{3}(n)} W_{3N}^{k(N)} = W_N^{\frac{k}{3}(n)} W_3^k.$$

Therefore, the equation above can be expressed as:

$$\begin{aligned} X_3(k) &= \sum_{n=0}^{N-1} x_p(n) W_N^{\frac{k}{3}n} + \sum_{n=N}^{2N-1} x_p(n) W_N^{\frac{k}{3}n} + \sum_{n=2N}^{3N-1} x_p(n) W_N^{\frac{k}{3}n} \\ &= \sum_{n=0}^{N-1} x_p(n) W_N^{\frac{k}{3}n} + \sum_{n=0}^{N-1} x_p(n) W_N^{\frac{k}{3}n} W_3^k + \sum_{n=0}^{N-1} x_p(n) W_N^{\frac{k}{3}n} W_3^{2k} \\ &= X_1\left(\frac{k}{3}\right) + X_1\left(\frac{k}{3}\right) W_3^k + X_1\left(\frac{k}{3}\right) W_3^{2k} \\ &= X_1\left(\frac{k}{3}\right)\left(1 + W_3^k + W_3^{2k}\right) = \begin{cases} 3X_1(k/3), & k = 0, 3 \\ 0, & k = 1, 2, 4, 5 \end{cases} \end{aligned}$$

(b) Verify the result in part (a) using the sequence $x_p(n) = \{\ldots, 1, 2, 1, \underset{\uparrow}{2}, 1, 2, 1, 2, \ldots\}$. You do not need MATLAB to do this part.

Solution: Consider the 2-point DFT of $x_p(n)$:

$$\begin{aligned} X_1(k) &= \sum_{n=0}^{1} x_p(n) e^{-j\frac{2\pi}{2}nk} = 2 + e^{-j\pi k}, \quad k = 0, 1. \\ &= \{\underset{\uparrow}{3}, 1\} \end{aligned}$$

and the 6-point DFT of $x_p(n)$:

$$\begin{aligned} X_3(k) &= \sum_{n=0}^{5} x_p(n) e^{-j\frac{2\pi}{6}nk} = 2 + e^{-j\frac{\pi}{3}k} + 2e^{-j\frac{2\pi}{3}k} + e^{-j\pi k} + 2e^{-j\frac{4\pi}{3}k} + e^{-j\frac{5\pi}{3}k}, \quad k = 0, 1, 2, 3, 4, 5 \\ &= \{\underset{\uparrow}{9}, 0, 0, 3, 0, 0\} \end{aligned}$$

Thus, $X_3(k) = \begin{cases} 3X_1(k/3), & k = 0, 3 \\ 0, & k = 1, 2, 4, 5 \end{cases}$, which verifies the result in part (a).

Problem 4.5

MATLAB Problem CP7.3 (Page 515)

Consider the discrete-time signal: $x(n) = (0.8)^n \left[u(n) - u(n-20)\right]$.

```
% clc; close all; clear;
```

(a) Determine the z-transform $X(z)$ of $x(n)$.

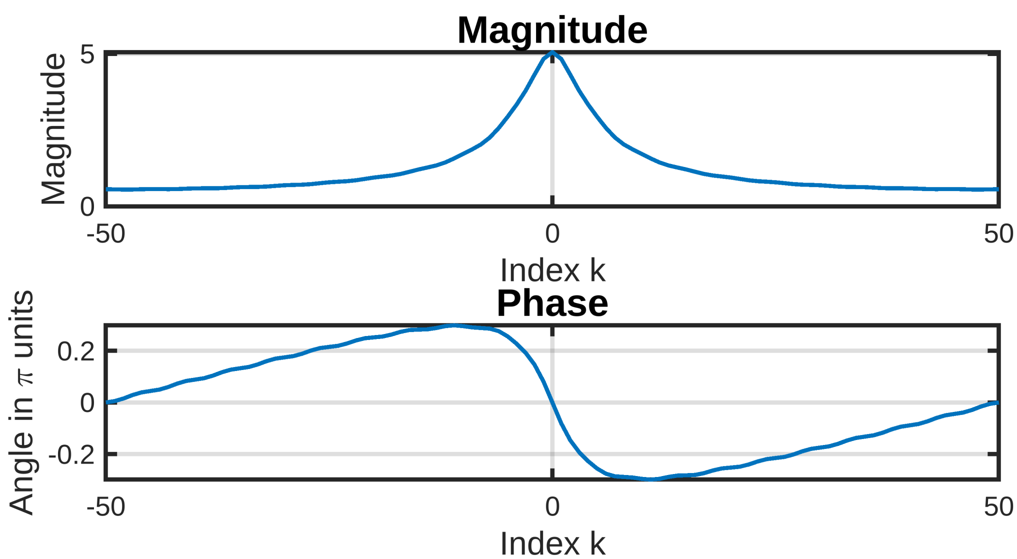
Solution: According to the table and the shift property, we have

$$\begin{aligned} X(z) &= \frac{1}{1-0.8z^{-1}} - \frac{0.8^{20}z^{-20}}{1-0.8z^{-1}} \qquad \text{ROC:}|z| > 0.8 \\ &= \frac{1-0.8^{20}z^{-20}}{1-0.8z^{-1}} \qquad \text{ROC:}|z| > 0.8 \end{aligned}$$

(b) Determine the spectrum $X(\omega)$ of $x(n)$ from the z-transform $X(z)$ and plot $|X(\omega)|$ and $\angle X(\omega)$ by sampling $X(\omega)$ at the frequencies $\omega_k = (2\pi/100)k$, $-50 \le k \le 50$. Do not use **`stem`** plot. Provide a figure with (2×1) subplots.

MATLAB script:

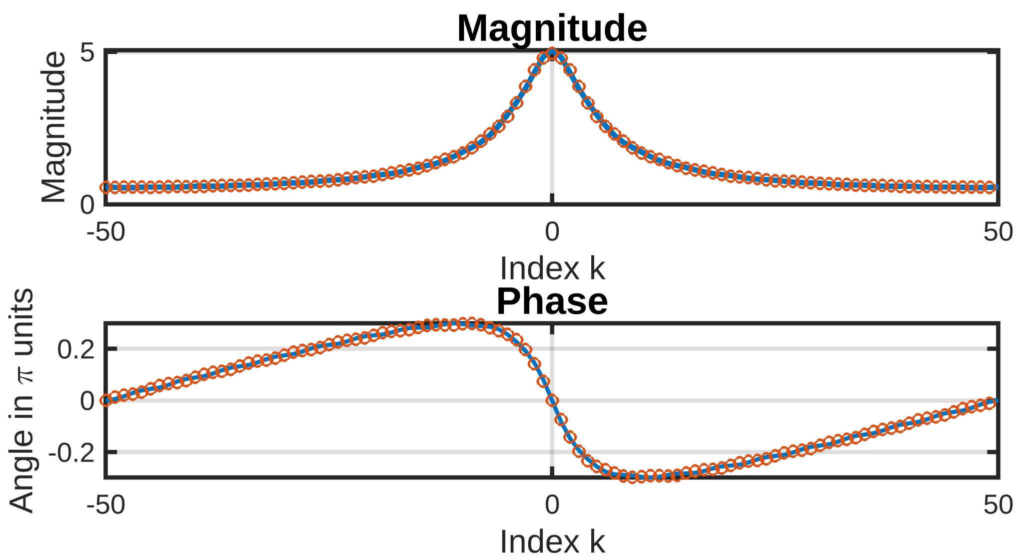
```
k = -50:50;
X1 = 1./(1-0.8*exp(-1j.*2*pi/100.*k)) + ...
    (0.8^20*exp(-1j.*40*pi/100.*k))./(1-0.8*exp(-1j.*2*pi/100.*k));

figure('position',[0,0,8,4]*72);
subplot(2,1,1); plot(k,abs(X1),'linewidth',1.5);
ylabel('Magnitude'); title('Magnitude');
xlabel('Index k'); grid;
subplot(2,1,2); plot(k,angle(X1)/pi,'linewidth',1.5);
ylabel('Angle in \pi units'); title('Phase');
xlabel('Index k');  grid;
```



(c) Use zero-padding to extend the length of $x(n)$ to 100 points, and compute the $N = 100$ DFT of the resulting sequence. Plot the magnitude and phase of the DFT $X(k)$ for $-50 \le k \le 50$, as circles (use **'o'** in your plot command**)** over the magnitude and phase spectra of $X(\omega)$ using the '**hold on**' command. You will have to plot the continuous spectra again in this part in a figure with (2×1) subplots.

MATLAB script:

```
N = 100; n = 0:N-1; u = n>=0;
u_20 = n-20 >=0; x = 0.8.^n .*(u-u_20);
X2 = fftshift(fft(x, N));
figure('position',[0,0,8,4]*72);
subplot(2,1,1); plot(k,abs(X1),'linewidth',2); hold on;
plot(k(1:end-1),abs(X2),'o','linewidth',1.,'Markersize', 4);
ylabel('Magnitude'); title('Magnitude');
xlabel('Index k'); grid;
subplot(2,1,2); plot(k,angle(X1)/pi,'linewidth',1.5); hold on;
plot(k(1:end-1),angle(X2)/pi,'o','linewidth',1.,'Markersize', 4);
ylabel('Angle in \pi units'); title('Phase');
xlabel('Index k');  grid;
```



(d) Compare and comment on your results in part **(b)** and **(c)**.

The results at (b) and (c) are the same. This is because that, the DFT consideres the limited-length aperiodic signal as the periodic signal.

Problem 4.6

Analysis of error between linear and circular convolutions

Let $x_1(n)$, $0 \le n < N_1$, be an N_1-point sequence. Let $x_2(n)$, $0 \le n < N_2$, be an N_2-point sequence. Let $x_3(n) \triangleq x_1(n) * x_2(n)$ be the **linear** convolution. Let $x_4(n) \triangleq x_1(n)(N)x_2(n)$ be an N-point **circular** convolution where $N \ge \max(N_1, N_2)$.

(a) Show that, in general, the circular convolution is an aliased version of the linear convolution, that is,

$$x_4(n) = \left(\sum_{r=-\infty}^{\infty} x_3\big(n - rN\big) \right) p_N(n) \qquad (4.6.1)$$

where $p_N \triangleq u(n) - u(n-N)$ is the rectangular window.

Proof: Consider the circular convolution

$$\begin{aligned} x_4[n] &= x_1[n](N)x_2[n] = \left[\sum_{m=0}^{N-1} x_1[m] x_2\big[\langle n-m \rangle_N\big] \right] p_N[n] \\ &= \left[\sum_{m=0}^{N-1} x_1[m] \sum_{r=-\infty}^{\infty} x_2\big[n - m - rN\big] \right] p_N[n] \\ &= \Bigg[\sum_{r=-\infty}^{\infty} \underbrace{\left(\sum_{m=0}^{N-1} x_1[m] x_2\big[(n - rN) - m\big] \right)}_{\text{Linear convolution } x_3[n-rN]} \Bigg] p_N[n] \\ &= \left(\sum_{r=-\infty}^{\infty} x_3\big[n - rN\big] \right) p_N[n]. \end{aligned}$$

This proves $(4.6.1)$ which means that, in general, the circular convolution is an aliased version of the linear convolution.

(b) Let $N \ge L \triangleq (N_1 + N_2 - 1)$ which is the length of the linear convolution. From $(4.6.1)$ show that

$$x_4(n) = x_3(n) \quad 0 \le n \le (N-1), \qquad (4.6.2)$$

which means that the circular convolution is same as the linear convolution over the given interval.

Solution: Since $x_3[n]$ is an $N = (N_1 + N_2 - 1)$-point sequence, there is no overlap between shifted replicas of $x_3[n]$ in the aliasing sum in $(4.6.1)$. Hence $(4.6.2)$ follows.

(c) Let $\max(N_1, N_2) \le N < L$ and let $e_N(n) \triangleq x_4(n) - x_3(n)$, $0 \le n \le N-1$ be an error between the circular and the linear convolutions. From $(4.6.1)$ show that

$$e_N(n) = x_3(n + N), \quad 0 \le n \le (N-1). \qquad (4.6.3)$$

Thus, under the proper conditions, the error at each n is given by the linear convolution N samples away.

Solution: Using the definition of the error sequence $e[n]$ and $(4.6.1)$ we have

$$\begin{aligned} e_N[n] &= x_4[n] - x_3[n] = \left(\sum_{r=-\infty}^{\infty} x_3\big[n - rN\big] \right) - \underbrace{x_3[n]}_{r=0 \text{ term}}, \quad 0 \le n \le N-1 \\ &= \sum_{r \ne 0} x_3[n - rN], \quad 0 \le n \le N-1. \qquad (4.6.4) \end{aligned}$$

Since $N \ge \max(N_1, N_2)$, only the first left $(r = -1)$ and the first right $(r = 1)$ aliases (or images) remain in $(4.6.4)$ over the $0 \le n \le (N-1)$ interval. Hence

$$e[n] = x_3[n+N] + x_3[n-N], \quad 0 \le n \le N-1. \qquad (4.6.5)$$

Finally, since $x_1[n]$ and $x_2[n]$ are causal sequences, $x_3[n]$ is also causal. Then the second term in $(4.6.5)$ is zero, proving $(4.6.3)$.

(d) Let $N = \max(N_1, N_2)$ and $M = \min(N_1, N_2)$. Then, using $(4.6.3)$, show that

$$e_N(n) = 0 \ \text{ for } \ n \ge (M-1), \qquad (4.6.6)$$

that is, that the first $M-1$ samples in $x_3(n)$ are in error but the remaining samples are the correct linear convolution samples $x_4(n)$, $(M-1) \le n \le (N-1)$.

Solution: Without loss of generality, let $N_1 = \max(N_1, N_2) = N$ which means that $M = N_2$. Now the linear convolution is $L = (N_1 + N_2 - 1)$-point long. Then from $(4.6.3)$, the error sequence is zero after $n + N \ge L$ or $n + N_1 \ge N_1 + N_2 - 1$ or $n \ge N_2 - 1 = M - 1$ proving $(4.6.6)$.

Problem 4.7

This problem numerically verifies equations (4.6.3) and (4.6.6) from Problem 4.6 above.

For the following sequences compute (i) the linear convolution $x_3(n) = x_1(n) * x_2(n)$, (ii) the N-point circular convolution $x_3(n) = x_1(n)(N)x_2(n)$, and (iii) the error sequence $e_N(n)$ in $(4.6.3)$.

(a) $x_1(n) = (0.8)^n p_{10}(n), \quad x_2(n) = (-0.8)^n p_{10}(n); \quad N = 15$. Do this part using MATLAB.

MATLAB script: The convolutions and the corresponding error sequences are computed using the following \ml script.

```
clc; close all; clear;
N1 = 10; n1 = 0:N1-1; x1 = 0.8.^n1;                 % Sequence x1[n]
N2 = 10; n2 = 0:N2-1; x2 = (-0.8).^n2;              % Sequence x2[n]
```

```
L = N1+N2-1; n3 = 0:L-1; x3 = conv(x1,x2);        % Linear conv x3[n]
N = 15; x4 = real(ifft(fft(x1,N).*fft(x2,N),N)); % Circular conv x4[n]
en = x4 - x3(1:N); en(1:7)                         % Error sequence e[n]
```

```
ans = 1×7
   -0.0000   -0.0281    0.0000   -0.0180         0    0.0000   -0.0000
```

```
x3N = x3(N+1:19), % Sequence x3[n+N]
```

```
x3N = 1×4
    0.0000   -0.0281    0.0000   -0.0180
```

(b) $x_1(n) = \{1, -1, 1\}, \quad x_2(n) = \{1, 0, -1, 0, 1\}; \quad N = 5$. Do this part using hand calculations.

Solution: We will use matrix computational approach to do these hand calculations. For linear convolution, we will convert $x_2[n]$ into a banded Toeplitz matrix and $x_1[n]$ into a column vector, Then $x_3[n]$ is obtained as a column vector using

$$\begin{bmatrix} x_3[0] \\ x_3[1] \\ x_3[2] \\ x_3[3] \\ x_3[4] \\ x_3[5] \\ x_3[6] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

or $x_3[n] = \{1, -1, 0, 1, 0, -1, 1\}$.

For circular convolution, we will pad $x_1[n]$ with two zeros to make it a 5-point sequence and convert $x_2[n]$ into a 5×5 circulant matrix to obtain $x_4[n]$ as a column vector

$$\begin{bmatrix} x_4[0] \\ x_4[1] \\ x_4[2] \\ x_4[3] \\ x_4[4] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

or $x_4[n] = \{0, 0, 0, 1, 0\}$. Hence

$$e[n] = \big(x_4[n] - x_3[n]\big)\,p_5[n] = \{-1, 1, 0, 0, 0\} = x_3[n+5]$$

which satisfies $(4.6.3)$. Also note that the first $M - 1 = 3 - 1 = 2$ samples in the circular convolution are in error which verifies $(4.6.6)$.

Problem 4.8

Let $x_c(t) = 10te^{-20t}\cos(20\pi t)u(t)$.

```
clc; close all; clear;
```

(a) Determine the CTFT $X_c(F)$ of $x_c(t)$.

Solution: From the CTFT table in the "LS_Brief_Review" document, the CTFT of $y_c(t) = e^{-20t}\cos(20\pi t)u(t)$ is

$$Y_c(j\Omega) = \mathscr{F}\left[e^{-20t}\cos(20\pi t)u(t)\right] = \frac{20 + j\Omega}{(20 + j\Omega)^2 + (20\pi)^2}.$$

Since $x_c(t) = 10ty_c(t)$, then using the CTFT property $\mathscr{F}\{ty_c(t)\} = j\frac{dY_c(j\Omega)}{d\Omega}$, the CTFT of $x_c(t)$ is $10j\frac{dY_c(j\Omega)}{d\Omega}$ or

$$x_c(j\Omega) = 10j\frac{d}{d\Omega}\left(\frac{20 + j\Omega}{(20 + j\Omega)^2 + (20\pi)^2}\right) = \frac{10\left[(20 + j\Omega)^2 - (20\pi)^2\right]}{\left[(20 + j\Omega)^2 + (20\pi)^2\right]^2}$$

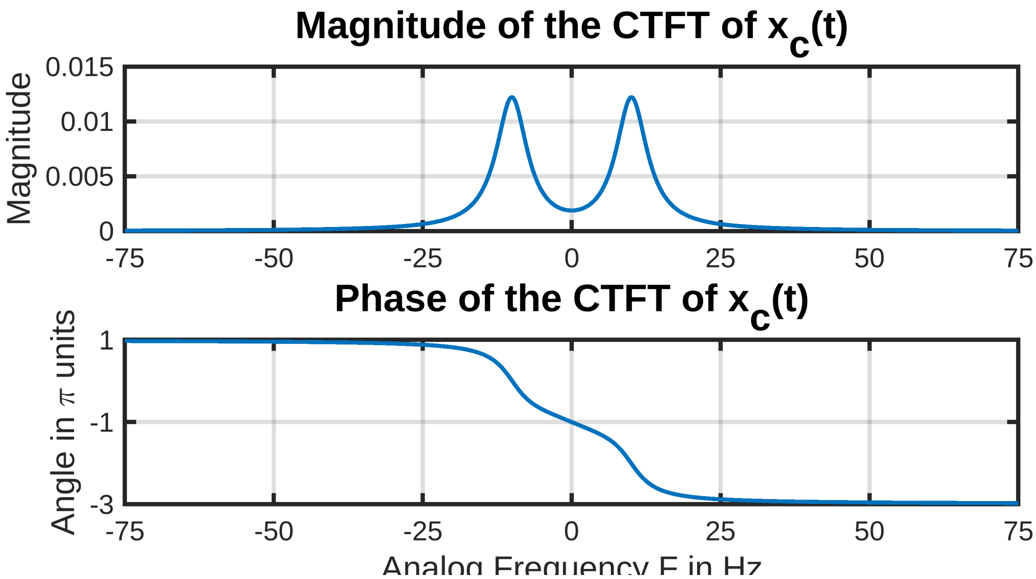
or

$$x_c(j2\pi F) = \frac{10\left[(20 + j2\pi F)^2 - (20\pi)^2\right]}{\left[(20 + j2\pi F)^2 + (20\pi)^2\right]^2}.$$

(b) Plot magnitude and phase of $X_c(F)$ over $-75 \le F \le 75$ Hz. Use (2×1) subplots.

MATLAB script: Computation and plotting of the CTFT is done using the following script.

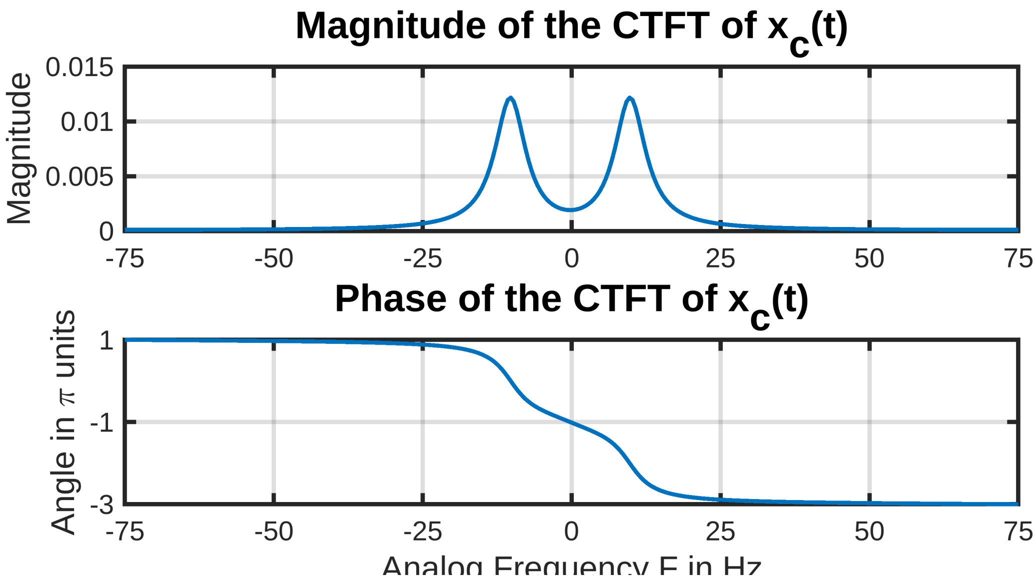
```
F = linspace(-75,75,1001);
Xc = -10*((20*pi)^2-(2j*pi*F+20).^2)./((20*pi)^2+(2j*pi*F+20).^2).^2;
figure('position',[0,0,8,4]*72);
subplot(2,1,1); plot(F,abs(Xc),'linewidth',1.5); axis([-75,75,0,0.015]);
ylabel('Magnitude'); title('Magnitude of the CTFT of x_c(t)');
set(gca,'xtick',(-75:25:75),'ytick',(0:0.005:0.015)); grid;
subplot(2,1,2); plot(F,unwrap(angle(Xc))/pi,'linewidth',1.5);
ylabel('Angle in \pi units'); title('Phase of the CTFT of x_c(t)');
xlabel('Analog Frequency F in Hz'); axis([-75,75,-3,1]);
set(gca,'xtick',(-75:25:75),'ytick',(-3:2:1)); grid;
```



(c) Use the **`fft`** function to approximate CTFT computations. Choose sampling rate to minimize aliasing and the number of samples to capture most of the signal waveform. Plot magnitude and phase of your approximation and compare it with the plot in (b) above.

Solution: From the above plot, a reasonable value of the bandwidth of $x_c(t)$ is 75 Hz. Thus, we will choose $F_s = 150$ Hz to minimize the frequency-domain aliasing. Also, since e^{-20t} is zero in approximately $5/20 = 0.25$ sec, we will choose time-width of about 2 seconds to capture most of the non-zero signal waveform which will minimize the time-domain aliasing. This will give us about 300 samples. Now we can use FFT to numerically compute the CTFT of $x_c(t)$. These calculations are given in the script below.

```
Fs = 150; T = 1/Fs; t1 =0; t2 = 2; nT = t1:T:t2; N = length(nT);
xnT = 10*nT.*exp(-20*nT).*cos(20*pi*nT);
X = fftshift(fft(xnT)); Xc_approx = T*[X,X(1)];
om = linspace(-1,1,N+1)*pi; F = om*Fs/(2*pi);
figure('position',[0,0,8,4]*72);
subplot(2,1,1); plot(F,abs(Xc_approx),'linewidth',1.5); axis([-75,75,0,0.015]);
ylabel('Magnitude'); title('Magnitude of the CTFT of x_c(t)');
set(gca,'xtick',(-75:25:75),'ytick',(0:0.005:0.015)); grid;
subplot(2,1,2); plot(F,unwrap(angle(Xc_approx))/pi,'linewidth',1.5);
ylabel('Angle in \pi units'); title('Phase of the CTFT of x_c(t)');
xlabel('Analog Frequency F in Hz'); axis([-75,75,-3,1]);
set(gca,'xtick',(-75:25:75),'ytick',(-3:2:1)); grid;
```



Comparison: The plots shown above agree with those in part (b).