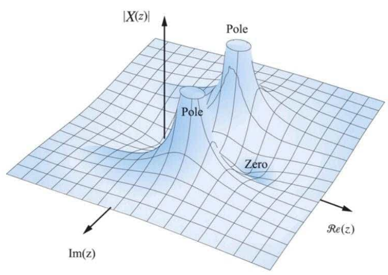
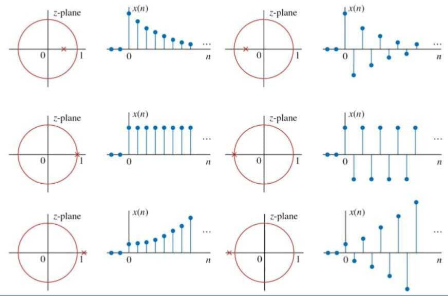
The z-Transform





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Motivation

Problem

Given the system $y(n) = y(n-1) + y(n-2) + x(n)$

- Find the value of $h(1000)$ (by hand!)
- Determine whether the system is linear
- Determine whether the system is stable
- Compute the output $y(n)$ if $x(n) = a^n u(n)$

Solution

Use the $z-$transform to convert the difference equation

$$y(n) = y(n-1) + y(n-2) + x(n)$$

to the algebraic equation

$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + X(z)$$

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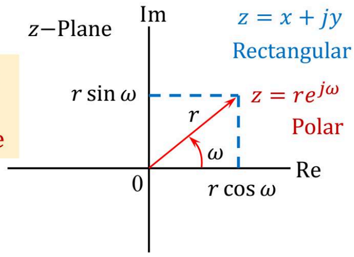
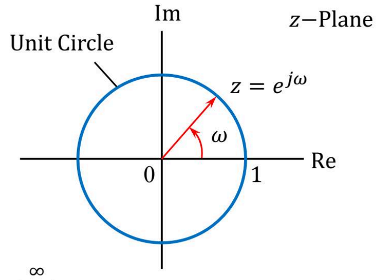
Eigen-Properties of LTI Systems

The complex z-Plane and the unit circle





$$x(n) = z^n, -\infty < n < \infty \text{ and } y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \Rightarrow$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h(k)\, z^{-k} = H(z)\, z^n, \qquad -\infty < n < \infty$$

Eigenvalues Eigenfunctions

$H(z_0) = 0 \Rightarrow y(n) = 0$ when $x(n) = z_0^n$!

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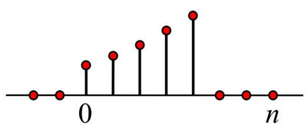
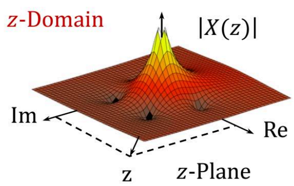
Definition of z-Transform

n-Domain



z-Domain



$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \quad \overset{z}{\longleftrightarrow} \quad X(z) = \sum_{k=-\infty}^{\infty} x(k)z^{-k}$$

- $X(z)$ is a Laurent Series; converges at $z = z_0$ iff $\sum_{k=-\infty}^{\infty} |x(n)\, {z_0}^{-n}| < \infty$
- ROC (Region of Convergence) $=$ All z such that $X(z)$ is finite
- $X(z_k) = 0 \Rightarrow z_k =$ zero; $X(p_k) = \infty \Rightarrow p_k =$ pole
- Notation: $X(z) = \mathcal{Z}\{x(n)\}$ or $x(n) \overset{z}{\longleftrightarrow} X(z)$

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Further Topics

Bilateral or two−sided z−transform:	$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$
Unilateral or one−sided z−transform:	$X^{+}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$
For causal sequences:	$x[n] = 0$ for $n < 0 \Rightarrow X(z) = X^{+}(z)$
The inverse z-transform, which is formally defined by a contour integral, it is seldom needed in practical applications:	$x[n] \triangleq \mathcal{Z}^{-1}\{X(z)\} \triangleq \frac{1}{2\pi j}\oint_{c} X(z)\ z^{n-1}dz$

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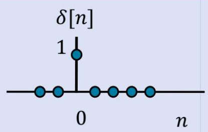
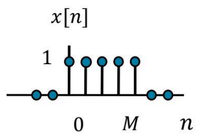
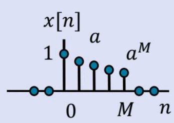
Finite Duration Sequences

Unit Impulse		$\Rightarrow$	$X(z) = \sum_{n=-\infty}^{\infty} \delta[n]\, z^{-n} = z^{0} = 1$, ROC: All z
Square-Pulse		$\Rightarrow$	$X(z) = \sum_{n=0}^{M} 1\, z^{-n} = \frac{1 - z^{-(M+1)}}{1 - z^{-1}}$, ROC: $\lvert z \rvert > 0$
Exponential-Pulse		$\Rightarrow$	$X(z) = \sum_{n=0}^{M} a^{n}z^{-n} = \sum_{n=0}^{M} (az^{-1})^{n} = \frac{1 - a^{M+1}z^{-(M+1)}}{1 - az^{-1}}$ ROC: $\lvert z \rvert > 0$

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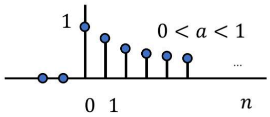
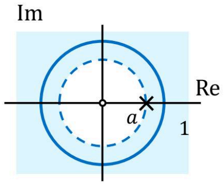
Causal Exponential Sequence

$x[n] = a^n u(n)$



$$X(z) = \frac{1}{1 - az^{-1}}$$



$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

If $|az^{-1}| < 1$or $|z| > |a| \Rightarrow$

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

zero: $z = 0$
pole: $p = a$

The ROC is the exterior of a circle in the z-Plane

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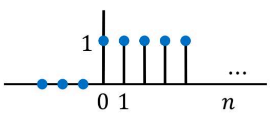
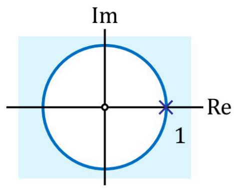
Causal Exponential Sequence (Cont.)

$a = 1 \Rightarrow$ Unit Step

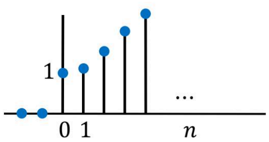
$x(n) = u(n)$



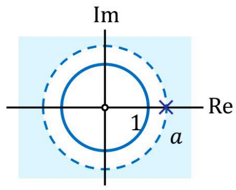
$$X(z) = \frac{1}{1 - z^{-1}}$$



$a > 1$



$$X(z) = \frac{1}{1 - az^{-1}}$$



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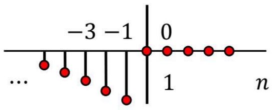
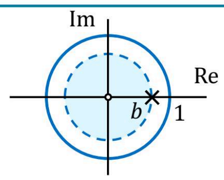
Anticausal Exponential Sequence

$$y(n) = \begin{cases} -b^n, n < 0 \\ 0, n \geq 0 \end{cases}$$



$$Y(z) = \frac{1}{1 - bz^{-1}}$$



$$Y(z) = -\sum_{n=-\infty}^{-1} b^n z^{-n} = -(b^{-1}z + b^{-2}z^2 + b^{-3}z^3 + \ldots)$$

$$= -b^{-1}z(1 + b^{-1}z + b^{-2}z^2 + \ldots) \text{ if } |b^{-1}z| < 1 \text{ or } |z| < |b| \Rightarrow$$

$$Y(z) = \frac{-bz^{-1}}{1 - b^{-1}z} = \frac{1}{1 - bz^{-1}}, \text{ ROC: } |z| < |b|$$

For $b = a \Rightarrow \mathcal{Z}\{a^n u(n)\} = \frac{1}{1 - az^{-1}} = \mathcal{Z}\{-a^n u[-n-1]\}$

$x(n) \neq y(n) \Rightarrow X(z) = Y(z)!$

Unique recovery of $x(\mathrm{n})$ from $X(z)$ requires both $X(z)$ and ROC

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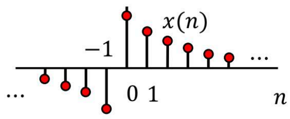
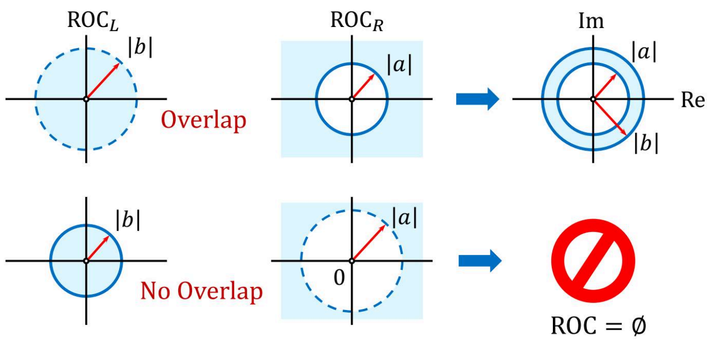
Two-Sided Exponential Sequence

$$x(n) = \begin{cases} a^n, & n \geq 0 \\ -b^n, & n < 0 \end{cases}$$



$$X(z) = \sum_{n=-\infty}^{-1} -b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \frac{1}{1 - bz^{-1}} + \frac{1}{1 - az^{-1}}$$



When $|a| < |b| \Rightarrow$
$\text{ROC} = \text{ROC}_\text{L} \cap \text{ROC}_\text{R}$

When $|a| > |b| \Rightarrow$
$X(z)$ does not exist!

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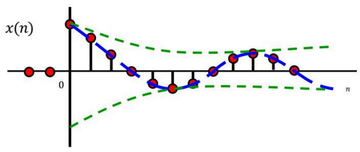
Exponentially Oscillating Sequence

$$x(n) = \begin{cases} r^n \cos(\omega_0 n), & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$r > 0, 0 \leq \omega_0 < 2\pi$$



- The constant r determines the behavior of the envelop
- The constant ω_0 determines the number of samples per period of oscillation.

Periodicity $\Leftrightarrow \cos\omega_0 n = \cos[\omega_0(n+N)]$

$\Rightarrow$ A period of 2π rad contains N samples, where $N = \dfrac{2\pi}{\omega_0}$, e.g., $\omega_0 = 45° = \dfrac{\pi}{4}$ rad $\Rightarrow N = 8$ samples

From the identity: $\cos\theta = \dfrac{1}{2}e^{j\theta} + \dfrac{1}{2}e^{-j\theta} \Rightarrow$

$$X(z) = \sum_{n=0}^{\infty} r^n (\cos \ \omega_0 n) z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty}(r\, e^{j\omega_0}z^{-1})^n + \frac{1}{2}\sum_{n=0}^{\infty}(re^{-j\omega_0}z^{-1})^n$$

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Exponentially Oscillating Sequence

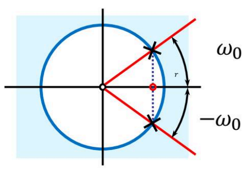
$$X(z) = \frac{1}{2}\sum_{n=0}^{\infty}(r\, e^{j\omega_0}z^{-1})^n + \frac{1}{2}\sum_{n=0}^{\infty}(re^{-j\omega_0}z^{-1})^n$$

Since $\left|e^{\pm j\theta}\right| = 1 \Rightarrow$ both series converge iff $|z| > r > 0$

$$X(z) = \frac{1}{2}\frac{1}{1 - re^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1 - re^{-j\omega_0}z^{-1}}, \text{ ROC: } |z| > r > 0$$

$$X(z) = \frac{1 - rz^{-1} \ \cos \ \omega_0}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})} = \frac{1 - rz^{-1} \ \cos \ \omega_0}{1 - 2rz^{-1} \ \cos \ \omega_0 + r^2 z^{-2}}$$

$$X(z) = \frac{z(z - r\cos \ \omega_0)}{z^2 - (2r\cos\omega_0)\, z + r^2}, |z| > r$$



$z_1 = 0, z_2 = r \ \cos \ \omega_0$
$p_1 = re^{j\omega_0}, \ p_2 = re^{-j\omega_0}$

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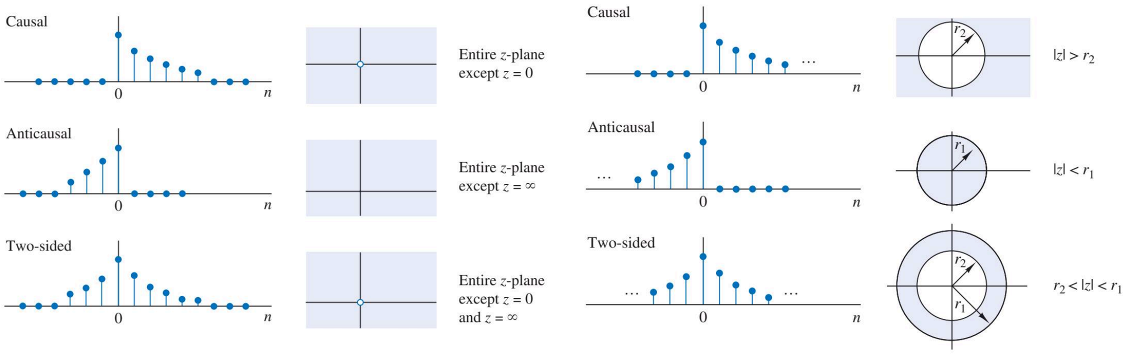
Typical Signals with Their Corresponding ROCs

Finite Duration Signals

Infinite Duration Signals



For finite duration sequences the ROC is the entire z-plane, with the possible exception of $z = 0$ or $z = \infty$

- The ROC cannot contain any poles
- The ROC is a connected (single contiguous) region

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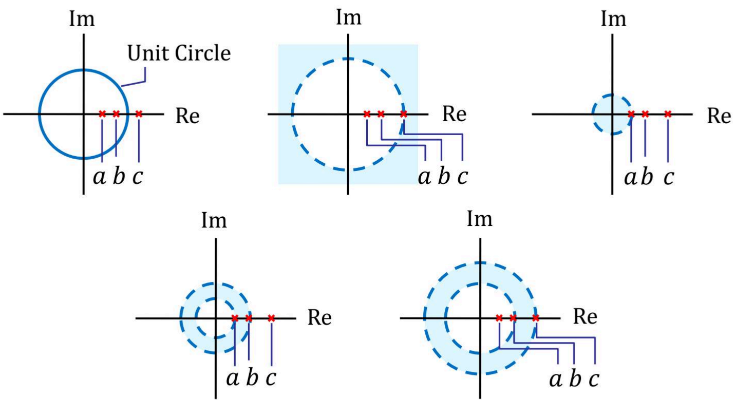
Important Observations



A z−transform is an algebraic formula and a ROC

$X(z)$ is legitimate only for z within the ROC

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Exponential Sequences and Rational z-Transforms

$$x[n] = \sum_{k=1}^{N} c_k p_k^n \overset{z}{\longleftrightarrow} X(z) = \sum_{k=1}^{N} \frac{c_k}{1 - p_k z^{-1}}, p_k \text{ distinct}$$

$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} = \frac{A_1(1 - p_2 z^{-1}) + A_2(1 - p_1 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

$$X(z) = \frac{\sum_{k=1}^{N} c_k \prod_{\substack{m=1 \\ m \neq k}}^{N} (1 - p_m z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})} = \frac{b_0 + b_1 z^{-1} + \ldots + b_{N-1} z^{-(N-1)}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

- Fundamental theorem of algebra:

$$A(z) = a_0 + a_1 z^{-1} + \cdots + a_N z^{-N} = a_0 \prod_{k=1}^{N} (1 - p_k z^{-1})$$

- Real coefficients $\Rightarrow$ real or complex–conjugate roots
  - Real poles $\Rightarrow$ p_k^n
  - Complex–conjugate pairs $\Rightarrow |p_k|^n \cos(\omega_k n + \theta_k)$, $p_k = |p_k| e^{j\omega_k}$

Inverse z-Transform

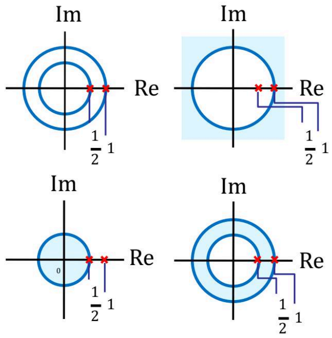
- Partial fraction expansion and table look-up
  - Sufficient for most practical problems
  - MATLAB implementation: **`residuez`**
- Expansion into a series of terms in the variables z and z^{-1}
  - For rational functions use long division
  - MATLAB implementation: **`deconv`**
- Complex integration using the method of residues

Example: Real Distinct Poles

$$X(z)=\frac{1+z^{-1}}{(1-z^{-1})(1-0.5z^{-1})}=\frac{A_1}{1-z^{-1}}+\frac{A_2}{1-0.5z^{-1}},\text{poles: } p_1=1, p_2=0.5$$

$$1+z^{-1}=A_1(1-0.5z^{-1})+A_2(1-z^{-1}) \qquad \Rightarrow z+1=A_1(z-0.5)+A_2(z-1)$$

$$z=1\Rightarrow 2=A_1(1-0.5)\Rightarrow A_1=0.4 \qquad z=0.5\Rightarrow 1.5=A_2(-0.5)\Rightarrow A_2=-3$$



$$x_{\mathrm{rs}}\,[n]=4u[n]-3\left(\frac{1}{2}\right)^n u[n]$$

$$x_{\mathrm{ls}}\,[n]=-4u[-n-1]+3\left(\frac{1}{2}\right)^n u[-n-1]$$

$$x_{\mathrm{ts}}\,[n]=-4u[-n-1]-3\left(\frac{1}{2}\right)^n u[n]$$

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Example: Complex-Conjugate Poles

$$X(z)=\frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}=\frac{A_1}{1-p_1z^{-1}}+\frac{A_2}{1-p_2z^{-1}}$$

$$z^2-z+0.5=0\Rightarrow p_1=\frac{1}{2}+j\frac{1}{2}, p_2=\frac{1}{2}-j\frac{1}{2}=p_1^*$$

$$1+z^{-1}=A_1(1-p_2z^{-1})+A_2(1-p_1z^{-1}) \qquad z+1=A_1(z-p_2)+A_2(z-p_1)$$

$$z=p_1\Rightarrow A_1=\frac{1}{2}-j\frac{3}{2}$$
$$z=p_2\Rightarrow A_2=\frac{1}{2}+j\frac{3}{2}=A_1^*$$

$$p_{1,2}=re^{\pm j\omega_0}, A_{1,2}=|A|e^{\pm j\theta}$$
$$r=1/\sqrt{2}, \omega_0=45^0, |A|=1.58$$
$$\theta=-71.56^0$$

$$x[n]=A_1p_1^nu[n]+A_1^*(p_1^*)^nu[n]=|A|e^{j\theta}r^ne^{j\omega_0n}u[n]+|A|e^{-j\theta}r^ne^{-j\omega_0n}u[n]$$

$$x[n]=|A|r^n\left(e^{j\omega_0n}e^{j\theta}+e^{-j\omega_0n}e^{-j\theta}\right)u[n]=2|A|r^n\cos(\omega_0n+\theta)u[n]$$

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Partial Fraction Expansion Using MATLAB

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{B(z)}{A(z)} = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

(distinct poles)

```
[A,p,C]=residuez(b,a)
[b,a]=residuez(A,p,C)
```

The functions can handle repeated poles

$$X(z) = \frac{6 - 10z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = 1 + \frac{2}{1 - z^{-1}} + \frac{3}{1 - 2z^{-1}}$$

b=[6 -10 2]; a=[1 -3 2] ⇒ A=[3 2]; p=[2 1]; C=1

ROC: $|z| > 2 \Rightarrow x(n) = \delta(n) + (2 + 3 \cdot 2^n)u(n)$
ROC: $|z| < 1 \Rightarrow x(n) = \delta(n) - (2 + 3 \cdot 2^n)u(-n-1)$
ROC: $1 < |z| < 2 \Rightarrow x(n) = \delta(n) - 2u(n) + 3 \cdot 2^n u(-n-1)$

z-Transform Properties

- There is a unique correspondence between a sequence and its z-Transform
- We can change a sequence by manipulating the formula of its z-Transform and vice-versa
- The z-Transform has several properties. The most useful properties to be discussed are:
  - Linearity
  - Time-shifting property
- Additional properties will be introduced as needed

Linearity and Time-Shifting Property

Linearity

$$\mathcal{Z}\{a_1\, x_1[n] + a_2\, x_2[n]\} = \sum_n \{a_1\, x_1[n] + a_2\, x_2[n]\}\, z^{-n}$$

$$= a_1 \sum_n x_1[n] z^{-n} + a_2 \sum_n x_2[n] z^{-n}$$

$$\mathcal{Z}\{a_1\, x_1[n] + a_2\, x_2[n]\} = a_1 X_1(z) + a_2 X_2(z)$$

Time-Shifting

$$\mathcal{Z}\{x[n-n_0]\} = \sum_n x[n-n_0]\, z^{-n} = \Bigg(\underbrace{\sum_m x[m] z^{-m}}_{m = n - n_0}\Bigg) z^{-n_0}$$

$$\mathcal{Z}\{x[n-n_0]\} = z^{-n_0} X(z)$$

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Example

Determine the z–Transform of the sequence: $x[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

- Using the definition:

$$X(z) = \sum_{n=0}^{N-1} 1 z^{-n} = 1 + z^{-1} + \cdots + z^{-N+1} = \begin{cases} N, & z = 1 \\ \dfrac{1 - z^{-N}}{1 - z^{-1}}, & z \neq 1 \end{cases}$$

ROC: All z–plane, except $z = 0$

$$X(z) = 1 + z^{-1} + \cdots + z^{-(N-1)} \Rightarrow (N-1) \text{ zeros}$$

- Using linearity and time–shifting properties:

$$X(z) = \frac{1 - z^{-N}}{1 - z^{-1}}, \qquad N \text{ zeros and one pole}$$

$x(n) = u(n) - u(n-N) \Rightarrow$
$X(z) = U(z) - z^{-N} U(z) = (1 - z^{-N}) U(z)$
$= \frac{1-z^{-N}}{1-z^{-1}}$

Pole-zero cancelations!

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Common z-Transform Pairs

	Signal $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

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Common z-Transform Properties: Part 1

Property	Time Domain	z-Domain	ROC
Notation	$x(n), x_1(n), x_2(n)$	$X(z), X_1(z), X_2(z)$	ROC: $r_2 < \lvert z\rvert < r_1$
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(z)+a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time Shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z=0$ if $k>0$ and $z=\infty$ if $k<0$
Scaling in the z-domain	$a^n u(n)$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z\rvert < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\mathrm{Re}\{x(n)\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Includes ROC
Imaginary part	$\mathrm{Im}\{x(n)\}$	$\frac{1}{2}j[X(z)-X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \lvert z\rvert < r_1$

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Common z-Transform Properties: Part 2

Property	Time Domain	z-Domain	ROC
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)\, X_2^*\left(\frac{z}{v^*}\right) v^{-1}dv$	At least, $r_{1l}r_{2l} < \lvert z\rvert < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1}dv$		

Summary

- Any sequence $x(n)$, defined for all n, can be uniquely described by its z-transform (Formula *and* ROC)
- Given a sequence $x(n)$ we discussed how to determine its z-transform $X(z)$
- Given a rational z-transform and its ROC, we explained how to determine the sequence $x(n)$ using partial fraction expansion
- In the next lecture we will discuss how to
  - Use the z-transform to compute and solve convolution equations
  - Analyze LTI systems described by linear constant-coefficient difference equations

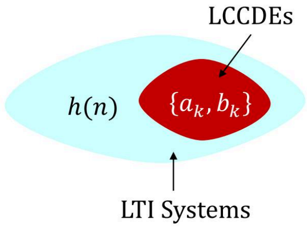
LTI System Review

- Every LTI system can be described by a convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = h(n) * x(n)$$

- A subclass of LTI systems can be described by a Linear Constant–Coefficient Difference Equation (LCCDE)

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$



- Conversion between representations

$$\{a_k, b_k\} \underset{\text{Difficult or impossible}}{\overset{\text{Easy}}{\rightleftarrows}} \{h(n)\}$$

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Some Interesting Problems

Given	Wanted	Problem
$x(n), h(n)$	$y(n)$	Convolution
$x(n), \{a_k, b_k\}$	$y(n)$	Solution of LCCDE
$x(n), y(n)$	$h(n)$ or $\{a_k, b_k\}$	System Identification
$y(n), h(n)$ or $\{a_k, b_k\}$	$x(n)$	Deconvolution or Inverse Filtering

Stability criteria

Given $h(n) \longrightarrow \sum_{n=-\infty}^{\infty} |h(n)| < \infty$

Given $\{a_k, b_k\} \longrightarrow$?

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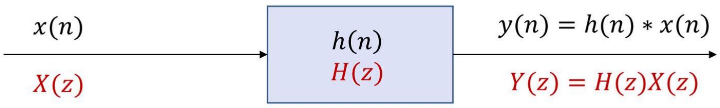
Convolution Theorem



$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \qquad \text{(Convolution)}$$

Taking z−transform with respect to n

$$Y(z) = \sum_{k=-\infty}^{\infty} h(k)\mathcal{Z}\{x(n-k)\} = \left(\sum_{k=-\infty}^{\infty} h(k)z^{-k}\right)X(z)$$

$$H(z) \triangleq \sum_{k=-\infty}^{\infty} h(k)z^{-k}$$ System Function

$Y(z) = H(z)X(z)$ Convolution ⇔ Polynomial multiplication

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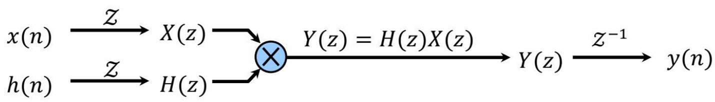
Example



$x(n) = \{\underset{\uparrow}{1}, -2, 1\} \quad h(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1, 1\}$

$X(z) = 1 + (-2)z^{-1} + z^{-2}$

$H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

$Y(z) = H(z)X(z) = 1 - z^{-1} - z^{-6} + z^{-7} \Rightarrow$

$y(n) = \{\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1\}$

Also: $H(z) = \dfrac{1 - z^{-6}}{1 - z^{-1}}, X(z) = (1 - z^{-1})^2$

$\Rightarrow Y(z) = (1 - z^{-6})(1 - z^{-1}) = 1 - z^{-1} - z^{-6} + z^{-7}$

Can we use convolution to multiply numbers?

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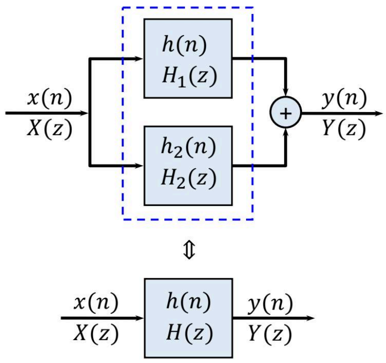
Parallel and Cascade Interconnection of LTI Systems

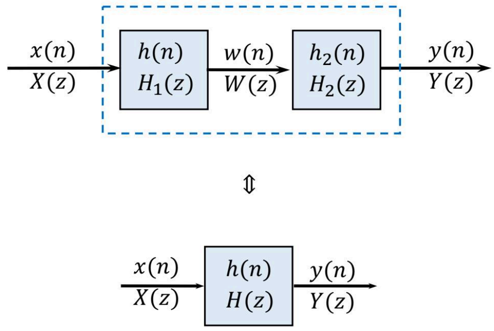
Parallel



$$h(n) = h_1(n) + h_2(n)$$

$$H(z) = H_1(z) + H_2(z)$$

Cascade



$$h(n) = h_1(n) * h_2(n) = h_2(n) * h_1(n)$$

$$H(z) = H_1(z)H_2(z) = H_2(z)H_1(z)$$

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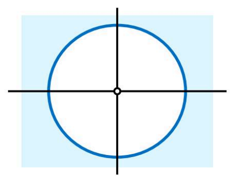
Causality and Stability Using the Impulse Response $h(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \qquad h(n) \overset{z}{\longleftrightarrow} H(z) = \sum_{k=-\infty}^{\infty} h(n)z^{-n}$$

Causality

$h(n) = 0, n < 0 \Leftrightarrow$ ROC of $H(z)$ is the exterior of a circle



Stability

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \Leftrightarrow |H(z)| \le \sum_{n=-\infty}^{\infty} |h(n)|\,|z|^{-n}\Big|_{|z|=1} < \infty \quad \Leftrightarrow$$

ROC of $H(z)$ *must* include the unit circle $|z| = 1$

Note: Causality and stability are independent properties

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LTI Systems Described by Difference Equations

$$y(n) = -\sum_{k=1}^{N} \underbrace{a_k}_{\text{Feedback Coefficients}} y(n-k) + \sum_{k=0}^{n} \underbrace{b_k}_{\text{Feed-Forward Coefficients}} x(n-k)$$

$x(n) = y(n) = 0, n < 0$ Initially at rest

$y(0), y(1), y(2), \ldots$ Computational order (causal)

Since $x(n), y(n)$ are defined for all $n \Rightarrow$

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z) \text{ or } Y(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} X(z)$$

If $x(n) = \delta(n), X(z) = 1 \Rightarrow y(n) = h(n), Y(z) = H(z)$

Rational System Functions - Poles and Zeros

Therefore $H(z) = \dfrac{Y(z)}{X(z)} \Rightarrow$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{z^{-M} \prod_{k=1}^{M} (z - z_k)}{z^{-N} \prod_{k=1}^{N} (z - p_k)} = \underbrace{b_0}_{\text{Gain}} \frac{\prod_{k=1}^{M} (1 - \overbrace{z_k}^{\text{Zeros}} z^{-1})}{\prod_{k=1}^{N} (1 - \underbrace{p_k}_{\text{Poles}} z^{-1})}$$

Rational System Function

- $\{a_k = 0\}_1^N \Rightarrow$ (all−zero) $\Rightarrow$ FIR system
- $\{b_k = 0\}_1^M \Rightarrow$ (all−pole) $\Rightarrow$ IIR system
- Poles or zeros at $z = 0$ are not counted
- `zplane(b,a)` or `zplane(z,p)` creates a zero−pole plot

Some Clarifications

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} \qquad M = \text{Order (\# zeros)}$$
$$M + 1 = \text{Length (\# coefficients)}$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} \qquad N = \text{Order (\# poles)}$$
$$N + 1 = \text{\# feedback coefficients}$$

		Impulse Response Length	
		Finite (FIR)	Infinite (IIR)
Implementation	Non-Recursive	$B(z)$	$h(n) = \frac{\sin\omega_0 n}{\omega_0 n}$
	Recursive	$\sum_{k=0}^{M} z^{-k} = \frac{1 - z^{-(M+1)}}{1 - z^{-1}}$	$\frac{B(z)}{A(z)}$

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LTI Systems with Rational System Functions

Consider a system with a rational system function

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-n}} \Leftrightarrow y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

The following computational process, with zero initial conditions, corresponds to a causal system

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x\,(n-k), \qquad n = 0,1,2,\ldots$$

In this case $h(n) = 0$ for $n < 0$ and the ROC of $H(z)$ is the exterior of a circle. Note that this system is causal by definition.

What about stability?

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Stability Using Poles and Zeros

A causal LTI is stable iff $\sum_{n=0}^{\infty} |h(n)| < \infty$

$$H(z) = \frac{B(z)}{A(z)} = \sum_{k=0}^{N-M} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

We assume distinct poles for simplicity

$$h(n) = \sum_{k=0}^{N-M} C_k \delta(n-k) + \sum_{k=1}^{N} A_k p_k^n u(n) \quad \text{(Causality)}$$

$$\sum_{n=0}^{\infty} |h(n)| \leq \sum_{k=0}^{N-M} |c_k| + \sum_{k=1}^{N} |A_k| \sum_{n=0}^{\infty} |p_k|^n < \infty \text{ iff } |p_k| < 1 \text{ for all } k$$

A casual pole−zero system is BIBO stable iff all poles are inside the unit circle. Zeros can be anywhere.

First- and Second-Order Systems

$$H(z) = \frac{B(z)}{A(z)} = \sum_{k=0}^{N-M} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \left(\begin{array}{c}\text{Assuming}\\ \text{distinct poles}\end{array}\right)$$

$$N = K_1 + 2K_2$$

$$H(z) = \frac{B(z)}{A(z)} = \sum_{k=0}^{N-M} C_k z^{-k} + \sum_{k=1}^{K_1} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{K_2} \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}$$

Real poles

Complex-conjugate poles

The behavior of a system with a rational system function can be understood in terms of the behavior of first-order systems with real poles and second-order systems with complex conjugate poles

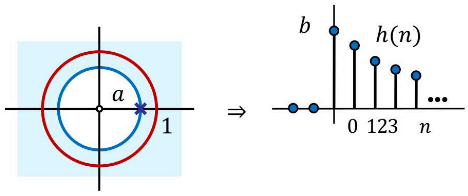
First-Order Pole-Zero Systems

Difference equation $y(n) = ay(n-1) + bx(n), \qquad -1 < a < 1$

System function $Y(z) = az^{-1}Y(z) + bX(z) \Rightarrow H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{b}{1-az^{-1}} = \dfrac{bz}{z-a}$

Since H is causal $\Rightarrow h(n) = ba^n u(n)$

The system is stable because $p = a$ and $|a| < 1$



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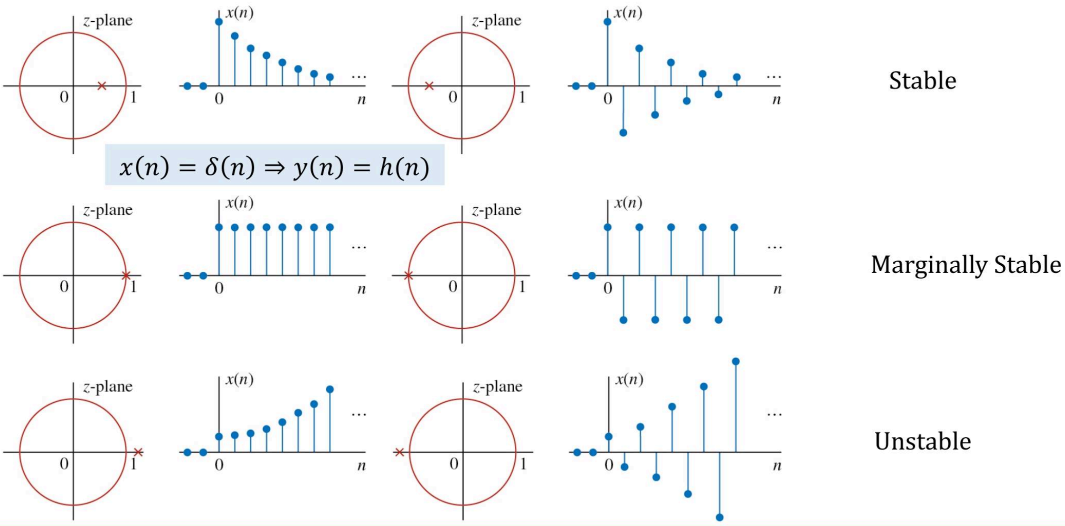
Pole Locations and Impulse Responses



$x(n) = \delta(n) \Rightarrow y(n) = h(n)$

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Second-Order Pole-Zero Systems

We shall analyze the system:

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n) + b_1 x(n-1)$$

System function $\Rightarrow$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Note: To simplify analysis we set $b_2 = 0$.
The case $b_2 \neq 0$ corresponds to the system
$H'(z) = C + H(z)$,
$C =$ constant, which can be easily analyzed

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{z(b_0 z + b_1)}{z^2 + a_1 z + a_2}$$

Zeros: $z_1 = 0, z_2 = -b_1/b_0$

Poles: $p_{1,2} = -\frac{a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_2}}{2}$

There are three cases:

Real and distinct poles: $a_1^2 > 4a_2$

Real and equal poles: $a_1^2 = 4a_2$

Complex conjugate poles: $a_1^2 < 4a_2$

Complex-Conjugate Poles ($a_1 < 4a_2$)

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + z^{-2}} = \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}}$$

$$= \frac{(A + A^*) - (Ap^* + A^* p)z^{-1}}{1 - (2r\cos\omega_o\ )z^{-1} + r^2 z^{-2}}$$

where $A = \frac{b_0 p + b_1}{p - p^*} = \frac{b_0 r e^{j\omega_0} + b_1}{2jr\sin\omega_0}$,

$p = re^{j\omega_0}, \qquad r > 0$

$\Rightarrow\ b_0 = 2\,\mathrm{Re}\{A\}\ \ b_1 = -2\,\mathrm{Re}\{Ap^*\}$

$a_1 = -2r\cos\omega_0\ \ a_2 = r^2 > 0$

$r = \sqrt{a_2}\ \ \cos\omega_0 = -\frac{a_1}{2\sqrt{a_2}}$

$$h(n) = Ap^n u(n) + A^*(p^*)^n u(n), A = |A|e^{j\theta}$$

$$= |A|e^{j\theta} r^n e^{j\omega_0 n} u(n) + |A|e^{-j\theta} r^n e^{-j\omega_0 n} u(n)$$

$$= |A| r^n \left[e^{j(\omega_0 n + \theta)} + e^{-j(\omega_0 n + \theta)}\right] u(n)$$

$$h(n) = 2|A| r^n \cos(\omega_0 n + \theta)\, u(n)$$

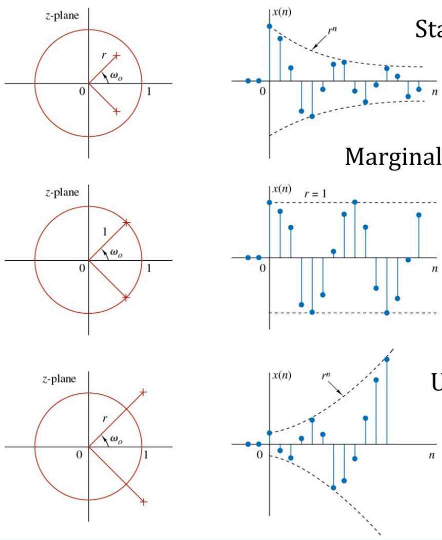
$r \Rightarrow$ Determines the envelope decay

$\omega_0 \Rightarrow$ Determines the frequency of oscillation

b_0, b_1, i.e., the location of zeros, affect only the phase shift and amplitude scaling factor

Note: Consider the cases $b_0 = 0$ or $b_1 = 0$.
Look at z–transform tables!

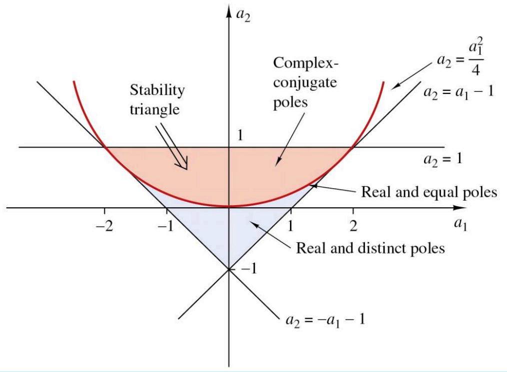
Pole Locations: Impulse Response Shape and Stability



Stability: $|p_1| < 1, |p_2| < 1$ (Poles inside Unit Circle)

$\Rightarrow\ |a_2| < 1$ and $|a_1| < 1 + a_2 \Rightarrow$

Stability Triangle (Coefficient Plane)



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Pole-Zero System in Practice

A practical pole-zero system is defined by

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k), \qquad n \geq 0$$

Zero initial conditions:

$x(-1) = \cdots = x(-M) = 0$

$y(-1) = \cdots = y(-N) = 0$

Input is "turned on" at $n = 0$

Causality is implied by the definition of the system and the computation flow

Difference Equation $\Rightarrow H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$

Stability: check if poles are inside unit circle

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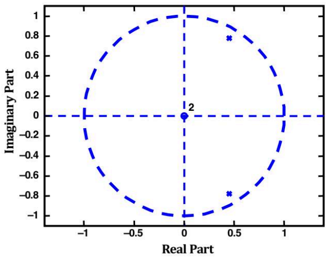
MATLAB Explorations

Compute the unit step response of the system

$y(n) = 0.9y(n-1) - 0.81y(n-2) + x(n),$ with $y(-1) = y(-2) = 0$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.9z^{-1} + 0.81z^{-2}} \qquad X(z) = \frac{1}{1 - z^{-1}}$$



b = 1;
a = (1 − 0.9 0.81);

$p_{1,2} = 0.9 \exp\left(\pm j \frac{\pi}{3}\right)$

System is stable

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Step Response

```
[A,p,C] = residuez(1,conv([1 -0.9 0.81],[1 -1]))
```

$$Y(z) = H(z)X(z) = \frac{\mathrm{A}(1)}{1 - \mathrm{p}(1)z^{-1}} + \frac{\mathrm{A}(2)}{1 - \mathrm{p}(2)z^{-1}} + \frac{\mathrm{A}(3)}{1 - \mathrm{p}(3)z^{-1}}$$

```
A =
  1.0989
 −0.0495 − 0.5425i
 −0.0495 + 0.5425i
```

```
p =
  1.0000
  0.4500 + 0.7794i
  0.4500 − 0.7794i
```

$$y(n) = \left\{1.0989 + 1.0894(0.9^n)\cos\left(\frac{\pi}{3}n - 1.6617\text{rads}\right)\right\}u(n)$$

```
u = ones(1,40); y = filter(b,a,u);
```


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Summary

- The z−Transform is a powerful tool for the analysis of discrete−time LTI systems
- Convolution of two sequences is equivalent to the multiplication of their z−Transforms
- Convolution provides the zero−state response of an LTI system
- For pole−zero systems the location of poles determines:
  - The nature of the impulse response
  - The stability of the system
- The z−Transform (Two−Sided) can be used to determine the zero−state response of an LTI system

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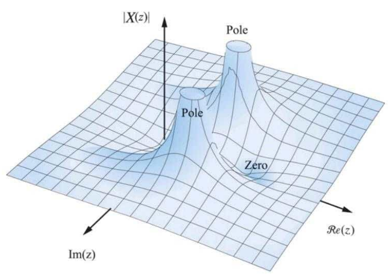
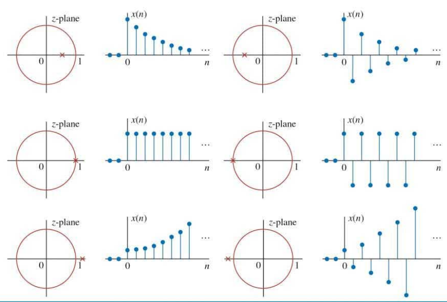
The z-Transform





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