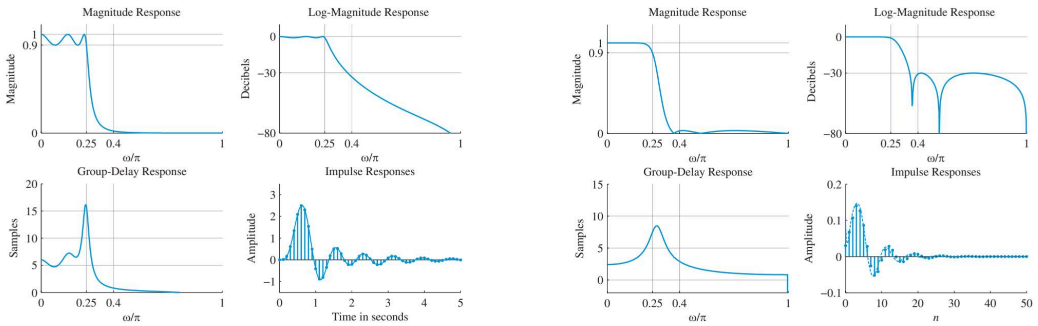
IIR Filter Design

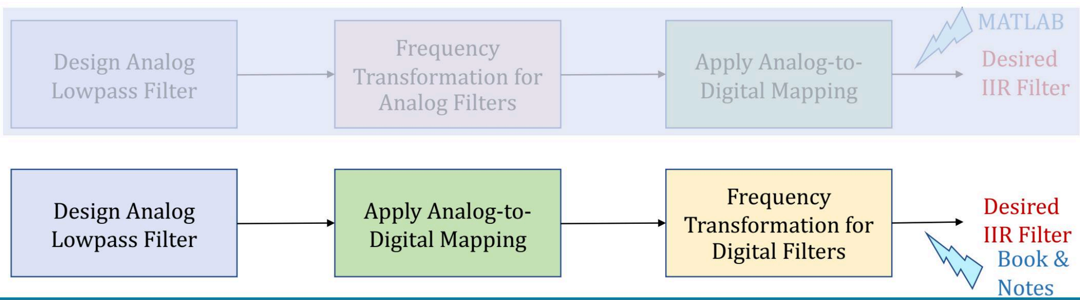
Analog to Digital Filter Transformations



0

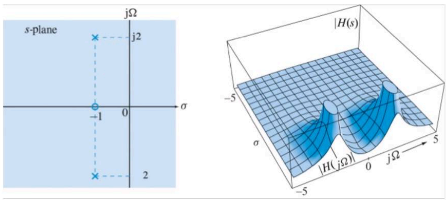
Design of Digital IIR Filters from Analog Filters

- Design techniques for analog filters are highly advanced
- Many practical analog filter design techniques involve closed form design formulas
- The approximation techniques used for analog filters do not lead to simple formulas when are applied directly to the design of digital IIR filters
- Two approaches:



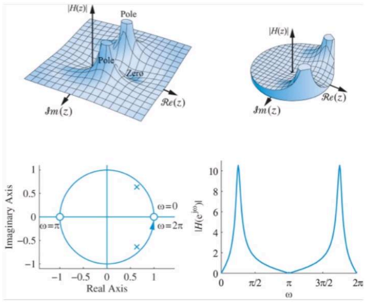
1

Challenges in Analog-to-Digital Filter Conversions



The frequency response of continuous-time systems is nonperiodic because it is evaluated on the infinite frequency axis (jΩ) in the complex plane

The frequency response of discrete-time systems is periodic because it is evaluated on the finite frequency circle ($e^{j\omega}$) in the complex plane



Requirements for CT to DT Mappings

Each transformation is equivalent to a mapping function $s = f(z)$ from the s-plane to the z-plane. Any useful mapping should satisfy three desirable conditions:

- A rational $H_c(s)$ should be mapped to a rational $H(z)$ (realizability):

$$\text{Rational } H_c(s) \longrightarrow H(z)$$

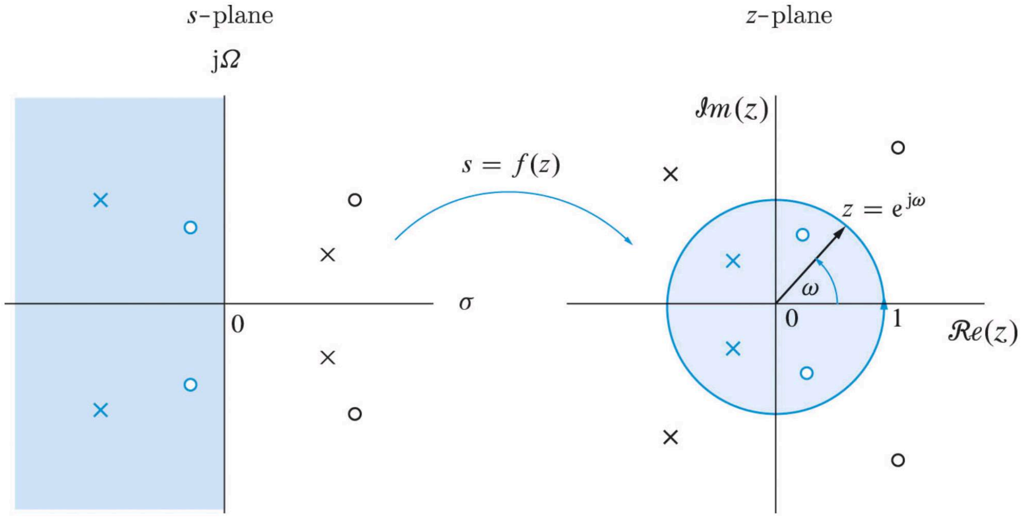
- The imaginary axis of the s-plane is mapped on the unit circle of the z-plane:

$$\{s = j\Omega\,, -\infty < \Omega < \infty\} \longrightarrow \{z = e^{j\omega}, -\pi < \omega \le \pi\}$$

- The left-half s-plane is mapped into the interior of the unit circle of the z-plane:

$$\{s \mid \text{Re}(s) < 0\} \longrightarrow \{z \mid |z| < 1\}$$

that is, stable analog filters are mapped into stable digital filters



4

Impulse Invariance Transformation

- The objective is to preserve the shape of the impulse response ⇒

$$h[n] \triangleq T_{\mathrm{d}} h_{\mathrm{c}}(nT_{\mathrm{d}})$$

  where T_{d} is called the design sampling period
- The frequency response of the resulting discrete-time filter is related to the frequency response of the continuous-time filter by

$$H(\omega) = \sum_{k=-\infty}^{\infty} H_{\mathrm{c}}\left(j\frac{\omega}{T_{\mathrm{d}}} + j\frac{2\pi}{T_{\mathrm{d}}}k\right)$$

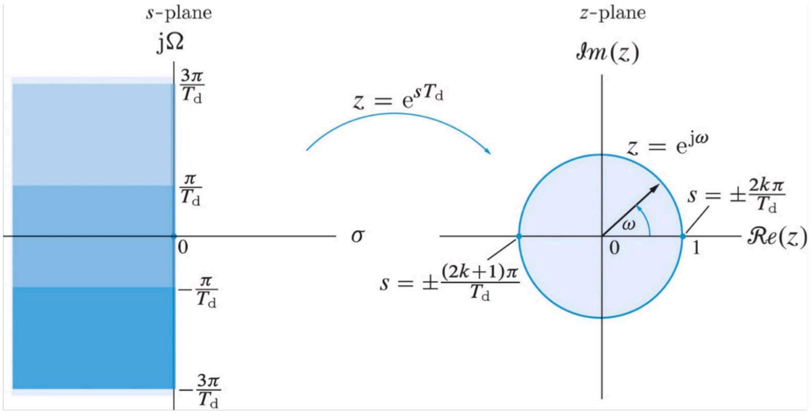
- The system functions are related by

$$H(z)\Big|_{z=e^{sT_{\mathrm{d}}}} = H_{\mathrm{c}}\left(s + j\frac{2\pi}{T_{\mathrm{d}}}k\right)$$

  since $\omega = \Omega T_{\mathrm{d}}$ and when z is on the UC s is on the imaginary axis
- The impulse invariance mapping is unique for bandlimited filters

5

s-Plane to z-Plane Mapping Illustration



Impulse Invariance can only be used for lowpass and bandpass filters

6

Impulse Invariance Transformation

Example: Let the impulse response of the analog system be $h_c(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$.
Then the system function is

$$H_c(s) = \frac{1/\tau}{s-(-1/\tau)}, \quad \mathrm{Re}(s) > -\frac{1}{\tau}.$$

To convert this analog system into a digital one, we sample at $t = nT_d$ to obtain

$$h[n] = T_d h_c(nT_d) = \frac{T_d}{\tau}e^{-nT_d/\tau}u[n] = \frac{T_d}{\tau}\left(e^{-T_d/\tau}\right)^n u[n]$$

$$\Rightarrow H(z) = \frac{T_d/\tau}{1-e^{-T_d/\tau}z^{-1}}, \quad |z| < e^{-T_d/\tau}.$$

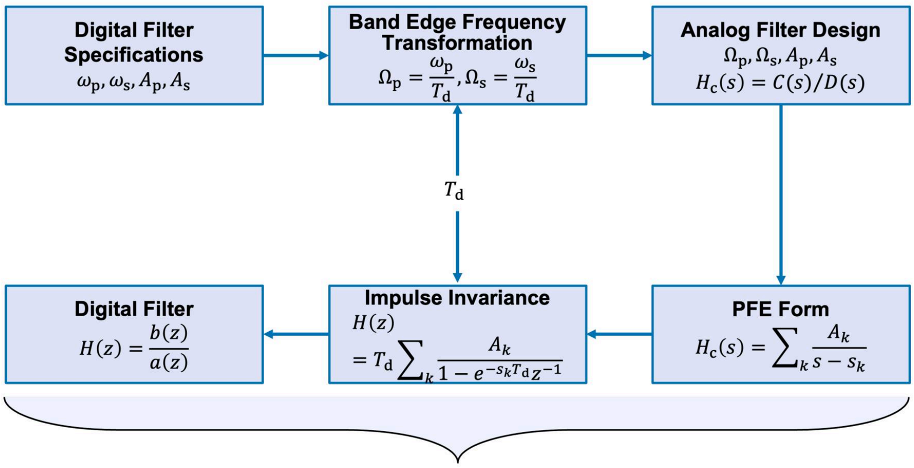
Pole Mapping: From the above example, we have $(s - p_1) \to (1 - e^{-p_1 T_d}z^{-1})$.

General Mapping: $H_c = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \quad \Rightarrow \quad H(z) = T_d \sum_{k=1}^{N} \frac{A_k}{1 - e^{-s_k T_d}z^{-1}}.$

MATLAB Function: `[C,D] = impinvar(b,a,1/Td)`

7

Impulse Invariance Transformation: Design Steps




```
[b,a] = impinvar(C,D,1/Td);
```


Digital Chebyshev-I Lowpass Filter Design Using Impulse Invariance

We want to design a lowpass digital Chebyshev-I filter to satisfy specifications:

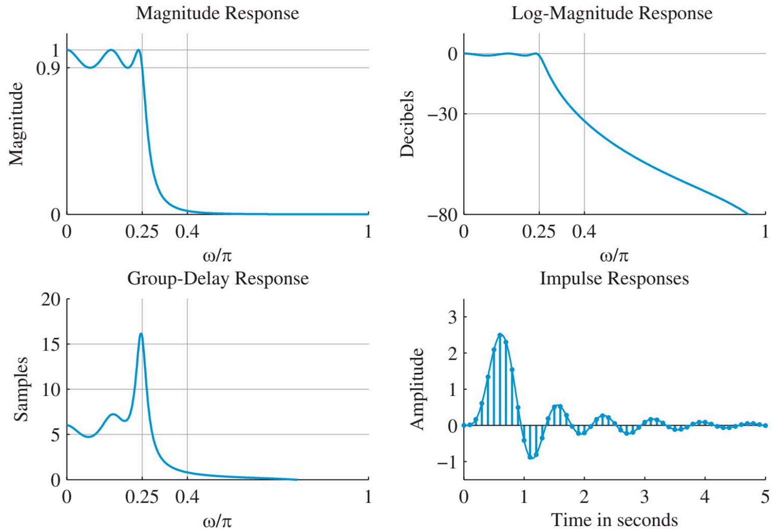
PB edge: $\omega_p = 0.25\pi$ rad, PB ripple: $A_p = 1$ dB,

SB edge: $\omega_s = 0.4\pi$ rad, SB attenuation: $A_s = 30$ dB.

MATLAB script:

```
>> omegap = 0.25*pi; omegas = 0.4*pi; Ap = 1; As = 30;
>> Td = 0.1; Omegap = omegap/Td; Omegas = omegas/Td;
>> [N,Omegac] = cheb1ord(Omegap,Omegas,Ap,As,'s'); N
N =
     5
>> [C,D] = cheby1(N,Ap,Omegac,'s');
>> [B,A] = impinvar(C,D,1/Td);
```

Digital Chebyshev-I Lowpass Filter Design Using Impulse Invariance



EECE-5666: Digital Signal Processing IIR Filter Design: A2D Filter Transformations 10/23

10

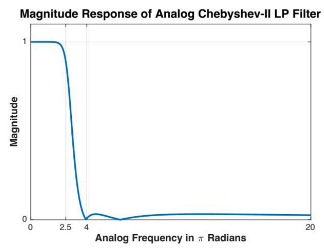
Digital Chebyshev-II Lowpass Filter Design Using Impulse Invariance

Again consider specifications given in the previous example. We want to design the digital filter using the Chebyshev II approximation.

$$\text{PB edge: } \omega_p = 0.25\pi \text{ rad}, \quad \text{PB ripple: } A_p = 1 \text{ dB},$$
$$\text{SB edge: } \omega_s = 0.4\pi \text{ rad}, \quad \text{SB attenuation: } A_s = 30 \text{ dB}.$$

MATLAB script:

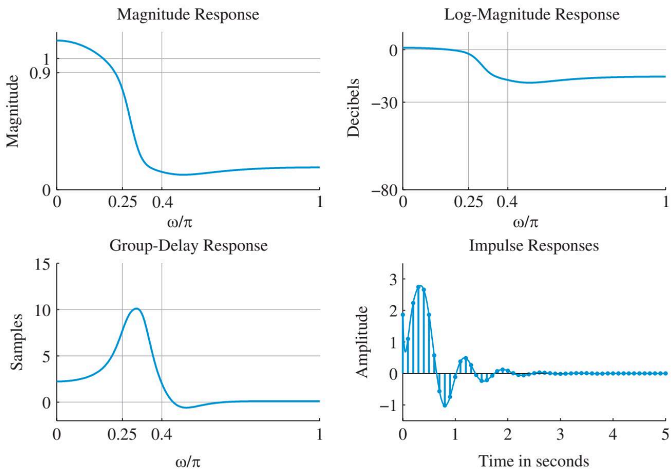
```
>> omegap = 0.25*pi; omegas = 0.4*pi; Ap = 1; As = 30;
>> Td = 0.1; Omegap = omegap/Td; Omegas = omegas/Td;
>> [N,Omegac] = cheb2ord(Omegap,Omegas,Ap,As,'s'); N
N =
     5
>> [C,D] = cheby2(N,As,Omegac,'s');
>> Om = linspace(0,20*pi,1001);
>> [Hc] = freqs(C,D,Om);
>> plot(Om/pi,abs(Hc),'line width',2);
>> [B,A] = impinvar(C,D,1/Td);
```



EECE-5666: Digital Signal Processing IIR Filter Design: A2D Filter Transformations 11/23

11

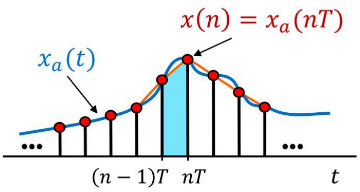
Digital Chebyshev-II Lowpass Filter Design Using Impulse Invariance



Clearly, The impulse Invariance method did not result in a satisfactory design

12

Analog-to-Digital Mapping by Bilinear Transformation



Analog Integrator

$$y_a(t) = \int_{-\infty}^{t} x_a(\tau)d\tau \Rightarrow H_a(s) = \frac{Y_a(s)}{X_a(s)} = \frac{1}{s}$$

Numerical approximation using trapezoidal rule

$$y(n) = y(n-1) + \frac{T}{2}[x(n) + x(n-1)]$$

$$H(z) = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$$

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$s = f(z) = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

Bilinear transformation

$$z = f^{-1}(s) = \frac{(2/T)+s}{(2/T)-s}$$

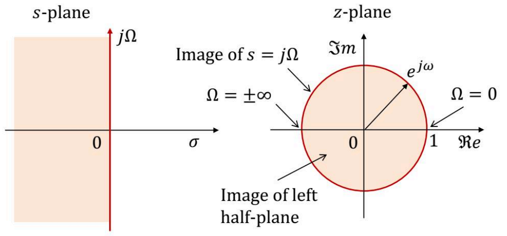
The parameter T, which does not have any effect on the design process, may be given any value that simplifies the derivations

13

Properties of Bilinear Transformation

$$z = re^{j\omega} = \frac{\frac{2}{T} + s}{\frac{2}{T} - s} = \frac{\frac{2}{T} + \sigma + j\Omega}{\frac{2}{T} - \sigma - j\Omega}$$

$$r = |z| = \left[\frac{\left(\frac{2}{T} + \sigma\right)^2 + \Omega^2}{\left(\frac{2}{T} - \sigma\right)^2 + \Omega^2}\right]^{\frac{1}{2}} \Rightarrow \begin{cases} \text{If } \sigma < 0 \text{ then } r < 1 \\ \text{If } \sigma = 0 \text{ then } r = 1 \\ \text{If } \sigma > 0 \text{ then } r > 1 \end{cases}$$



$s = j\Omega \Rightarrow z = e^{j\omega}$

$$\omega = \tan^{-1}\left(\frac{\Omega}{2/T + \sigma}\right) + \tan^{-1}\left(\frac{\Omega}{2/T - \sigma}\right)$$

$\sigma = 0 \Rightarrow$

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right) \Rightarrow \begin{cases} \Omega = 0 & \Rightarrow & \omega = 0 \\ \Omega \to \infty & \Rightarrow & \omega \to \pi \\ \Omega \to -\infty & \Rightarrow & \omega \to -\pi \end{cases}$$

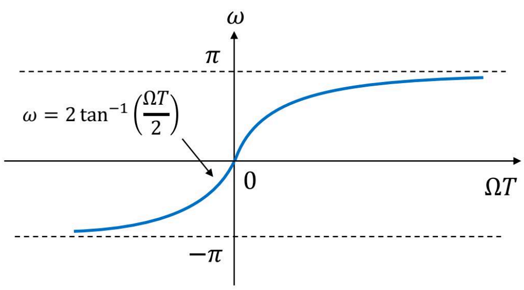
Rational $H_a(s) \Rightarrow$ Rational $H(z)$ (Realizability)

$\sigma < 0 \Rightarrow r < 1$ (Stability)

One-to-one mapping $\Rightarrow$ No aliasing!

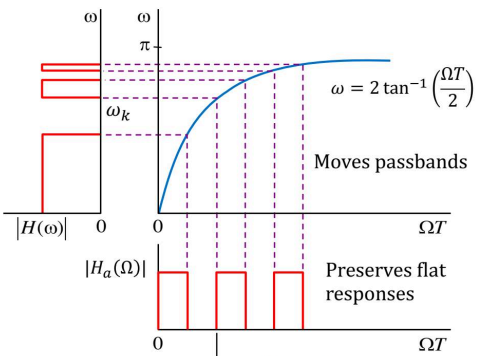
14

Frequency Axis Mapping of Bilinear Transformation



Monotonic mapping preserves type filter, i.e., LP→LP, HP→HP, etc.

Frequency Axis Warping



$\Omega_k = \frac{2}{T}\tan\left(\frac{\omega_k}{2}\right)$ Pre-warping!

The BL is used for the conversion of LP, HP, BP, and HP analog filters to similar digital filters

15

Example

Design LP digital filter with 3-dB bandwidth at $\omega_c = 0.2\pi$ using the analog filter

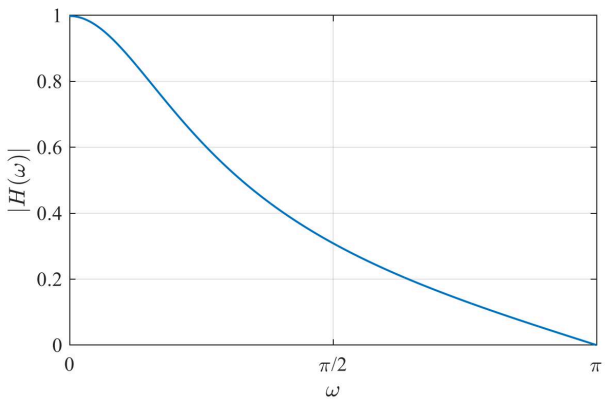
$$H_a(s) = \frac{\Omega_c}{s + \Omega_c}$$

$$\Omega_c = \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right) = \frac{0.65}{T}$$

$$H_a(s) = \frac{0.65}{sT + 0.65} \qquad s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

$$H(z) = 0.245\frac{1 + z^{-1}}{1 - 0.509z^{-1}}$$

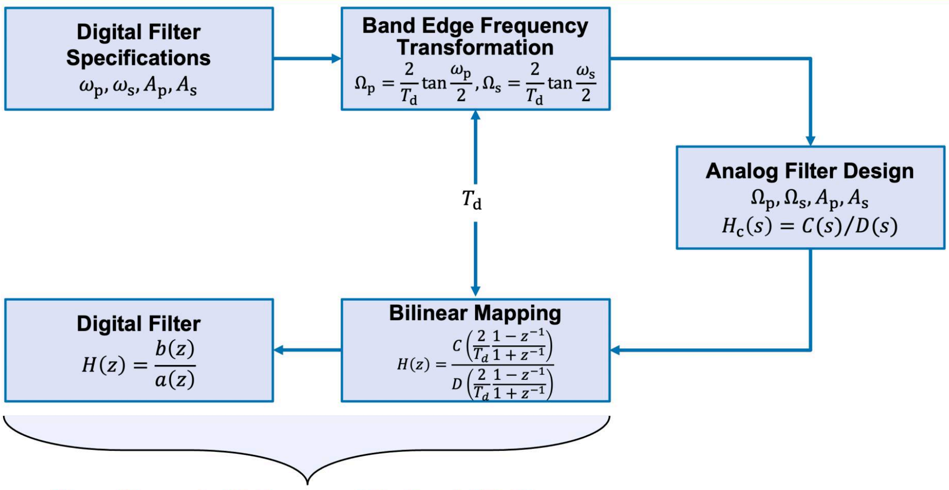
$|H(\omega)|_{\omega=0} = 1$ and $|H(\omega)|_{\omega=0.2\pi} = \frac{1}{\sqrt{2}}$ as desired



EECE-5666: Digital Signal Processing IIR Filter Design: A2D Filter Transformations 16/23

16

Bilinear Transformation: Design Steps




```
[b,a] = bilinear(C,D,1/Td);
```

EECE-5666: Digital Signal Processing IIR Filter Design: A2D Filter Transformations 17/23

17

Digital Chebyshev-II Lowpass Filter Design Using BL Mapping

Again consider specifications given in the previous example. We want to design the digital filter using the Chebyshev II approximation.

PB edge: $\omega_p = 0.25\pi$ rad, PB ripple: $A_p = 1$ dB,

SB edge: $\omega_s = 0.4\pi$ rad, SB attenuation: $A_s = 30$ dB.

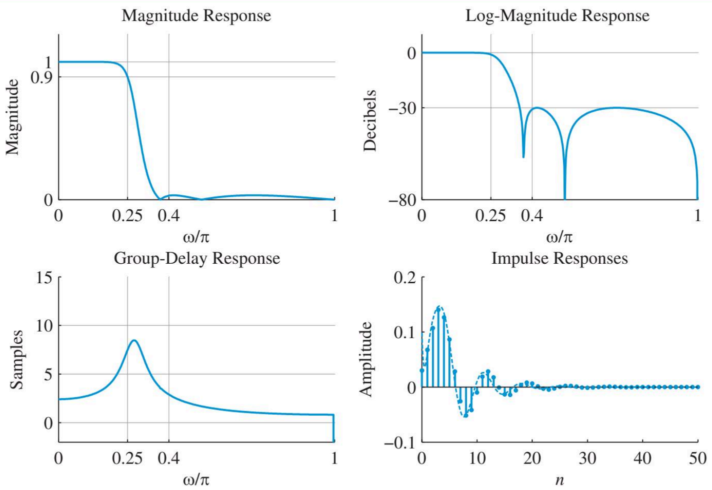
MATLAB script:

```
>> omegap = 0.25*pi; omegas = 0.4*pi; Ap = 1; As = 30;
>> Td = 2; Omegap = (2/Td)*tan(omegap/2);
>> Omegas = (2/Td)*tan(omegas/2);
>> [N,Omegac] = cheb2ord(Omegap,Omegas,Ap,As,'s'); N
N =
    5
>> [C,D] = cheby2(N,As,Omegac,'s');
>> [B,A] = bilinear(C,D,1/Td);
```

EECE-5666: Digital Signal Processing IIR Filter Design: A2D Filter Transformations 18/23

18

Digital Chebyshev-II Lowpass Filter Design Using BL Mapping



Now the bilinear transformation approach resulted in a satisfactory design.

EECE-5666: Digital Signal Processing IIR Filter Design: A2D Filter Transformations 19/23

19

Lowpass Digital Filter Desigan Functions in MATLAB SP Toolbox

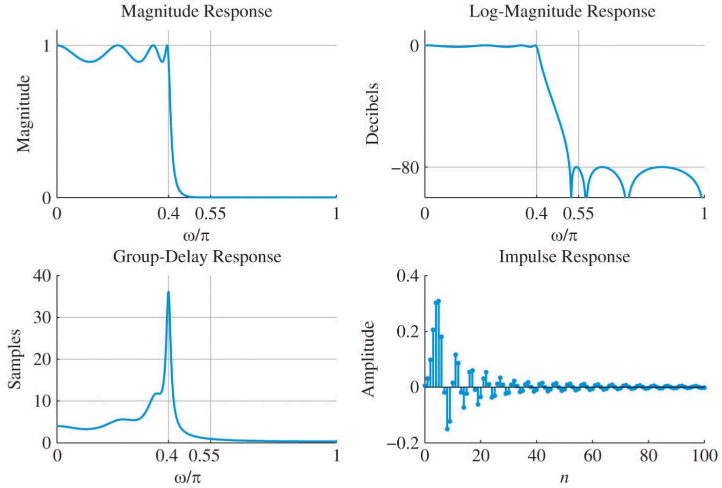
- Butterworth Lowpass Filter:
  - `[N,Omegac] = buttord(Omegap,Omegas,Ap,As)`
  - `[b,a] = butter(N,Omegac)`
- Chebyshev-1 Lowpass filter:
  - `[N,Omegap] = cheb1ord(Omegap,Omegas,Ap,As)`
  - `[b,a] = cheby1(N,Rp,Omegap)`
- Chebyshev-2 Lowpass filter:
  - `[N,Omegas] = cheb2ord(Omegap,Omegas,Ap,As)`
  - `[b,a] = cheby2(N,As,Omegas)`
- Elliptic Lowpass Filter:
  - `[N,Omegap] = ellip1ord(Omegap,Omegas,Ap,As)`
  - `[b,a] = ellip(N,Rp,As,Omegap)`

Important Note: In the above functions, all frequencies are normalized with respect to π radians, that is, use the given frequencies divided by π.

20

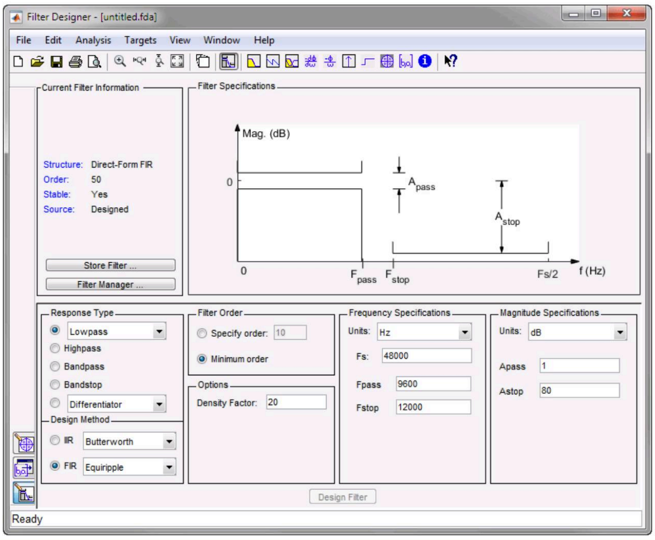
Digital Lowpass Elliptic Filter Design

`[N,Omegap] = ellip1ord(0.4,0.55,1,80); [b,a] = ellip(N,Rp,As,Omegap);`



21

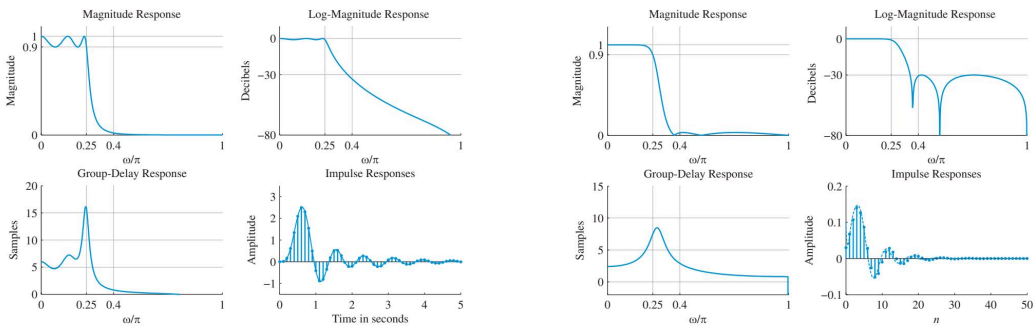
MATLAB Filter Design Tool: `filterDesigner`



22

IIR Filter Design

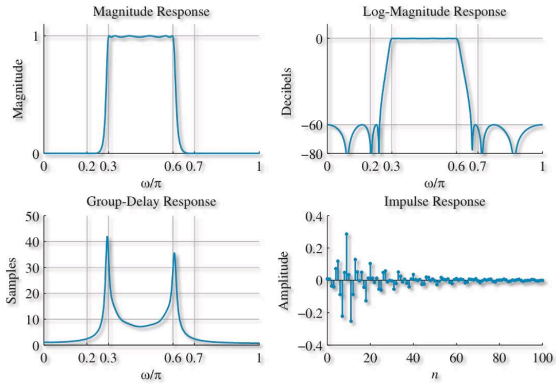
Analog to Digital Filter Transformations



23

IIR Filter Design

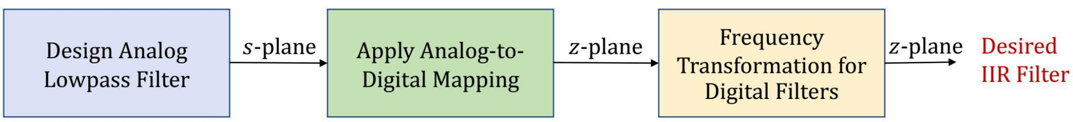
Frequency Band Transformations



0

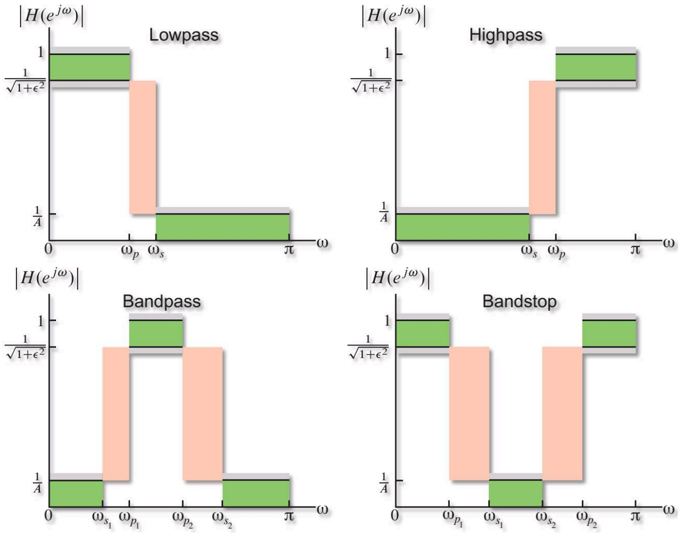
Design of Digital IIR Filters via Frequency-Band Transformations

- Using designed digital lowpass filter, we now consider designing other frequency selective digital filters via frequency band transformation
- The other frequency selective filters are:
  - Lowpass with a different passband cutoff frequency
  - Highpass
  - Bandpass
  - Bandstop or band-reject
- This requires a frequency-domain transformation that maps one z-plane into another z-plane:



1

Digital Filter Types



Frequency Transformations for Analog Filters

- As a motivation, let us first consider how the process works out in the s-domain
- Such transformations are very well-known

Type of transformation	Transformation	Band edge frequencies of new filter
Lowpass	$s \longrightarrow \frac{\Omega_p}{\Omega'_p} s$	Ω'_p
Highpass	$s \longrightarrow \frac{\Omega_p \Omega'_p}{s}$	Ω'_p
Bandpass	$s \longrightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$	Ω_l, Ω_u
Bandstop	$s \longrightarrow \Omega_p \frac{s(\Omega_u - \Omega_c)}{s^2 + \Omega_u \Omega_l}$	Ω_l, Ω_u

Example in s-Domain

Lowpass filter

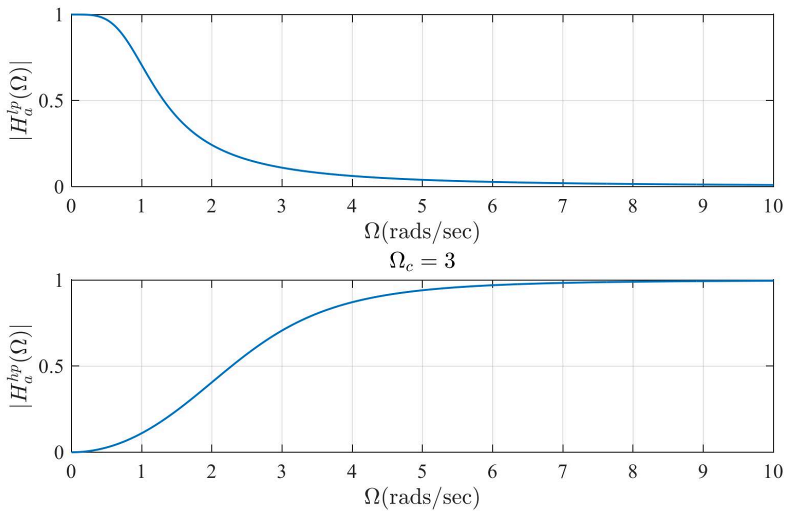
$$H_a^{lp}(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Frequency transformation

$$s \longmapsto \frac{\Omega_c}{s} \quad \text{or} \quad \Omega \longmapsto \frac{\Omega_c}{\Omega}$$

Highpass filter

$$H_a^{hp}(s) = \frac{s^2}{s^2 + \sqrt{2}\Omega_c s + \Omega_c^2}$$



EECE-5666: Digital Signal Processing IIR Filter Design: Filter Transformations 4/21

4

Requirements on the Frequency band Transformations

- Let $H_{\mathrm{lp}}(w)$ be the given prototype lowpass digital filter, and let $H(z)$ be the target (actual) frequency-selective digital filter
- The corresponding frequency variables are

$$w = q e^{j\omega} \text{ (Prototype domain)}$$
$$z = r e^{j\omega'} \text{ (Target domain)}$$

- Define a mapping of the form

$$w^{-1} = G(z^{-1}) \text{ such that } H(z) = H_{\mathrm{lp}}(w)\big|_{w^{-1}=G(z^{-1})}$$

- Given that $H_{\mathrm{lp}}(w)$ is a stable and causal filter, we also want $H(z)$ to be a stable and causal filter. This imposes the following requirements on $G(\cdot)$
  1. The function $G(\cdot)$ must be a rational function in z^{-1} so that $H(z)$ is realizable.
  2. The unit circle of the w-plane must map onto the unit circle of the z-plane.
  3. For stability, the inside of the UC of the w-plane must also map onto the inside of the z-plane

EECE-5666: Digital Signal Processing IIR Filter Design: Filter Transformations 5/21

5

- Consider the frequency variables ω and ω' of w and z, respectively, that is
$$w = \mathrm{e}^{\mathrm{j}\omega} \quad \text{and} \quad z = \mathrm{e}^{\mathrm{j}\omega'}$$
on their respective unit circles
- Requirement 2 (mapping of unit circle to unit circle) and $w^{-1} = G(z^{-1})$ implies that
$$w^{-1} = G(z^{-1}) \Rightarrow \mathrm{e}^{-\mathrm{j}\omega} = \left|G\left(\mathrm{e}^{-\mathrm{j}\omega'}\right)\right| \angle G\left(\mathrm{e}^{-\mathrm{j}\omega'}\right)$$
which means that $\left|G\left(\mathrm{e}^{-\mathrm{j}\omega'}\right)\right| = 1$ (an all-pass filter) and that $-\omega = \angle G\left(\mathrm{e}^{-\mathrm{j}\omega'}\right)$
- The most general all-pass filter expression that satisfies these requirements is
$$w^{-1} = G(z^{-1}) = \pm \prod_{k=1}^{N} \frac{z^{-1} - a_k^*}{1 - a_k z^{-1}}, \qquad |a_k| < 1$$
- Thus, there is a set of N parameters $\{a_k\}_1^N$, which must be determined using the given filter specifications.

6

Frequency Transformations for Digital Filters

Type of transformation	Transformation	Parameters
Lowpass	$z^{-1} \longrightarrow \dfrac{z^{-1} - a}{1 - az^{-1}}$	$\omega'_p =$ band edge frequency new filter $a = \dfrac{\sin[(\omega_p - \omega'_p)/2]}{\sin[(\omega_p + \omega'_p)/2]}$
Highpass	$z^{-1} \longrightarrow -\dfrac{z^{-1} + a}{1 + az^{-1}}$	$\omega'_p =$ band edge frequency new filter $a = -\dfrac{\cos[(\omega_p + \omega'_p)/2]}{\cos[(\omega_p - \omega'_p)/2]}$
Bandpass	$z^{-1} \longrightarrow -\dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	$\omega_l =$ lower band edge frequency $\omega_u =$ upper band edge frequency $a_1 = 2\alpha K/(K+1)$ $a_2 = (K-1)/(K+1)$ $\alpha = \dfrac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$ $K = \cot \dfrac{\omega_u - \omega_l}{2} \tan \dfrac{\omega_p}{2}$
Bandstop	$z^{-1} \longrightarrow \dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-1} - a_1 z^{-1} + 1}$	$\omega_l =$ lower band edge frequency $\omega_u =$ upper band edge frequency $a_1 = 2\alpha/(K+1)$ $a_2 = (1-K)/(1+K)$ $\alpha = \dfrac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$ $K = \tan \dfrac{\omega_u - \omega_l}{2} \tan \dfrac{\omega_p}{2}$

ω: prototype cutoff frequency
ω': target cutoff frequency

7

Example: Lowpass to Highpass Mapping

- Consider a lowpass digital filter design using Chebyshev-I prototype

$$\omega_p = 0.2\pi, \qquad A_p = 1\text{dB}, \qquad \omega_s = 0.3\pi, \qquad A_s = 15\text{dB}$$

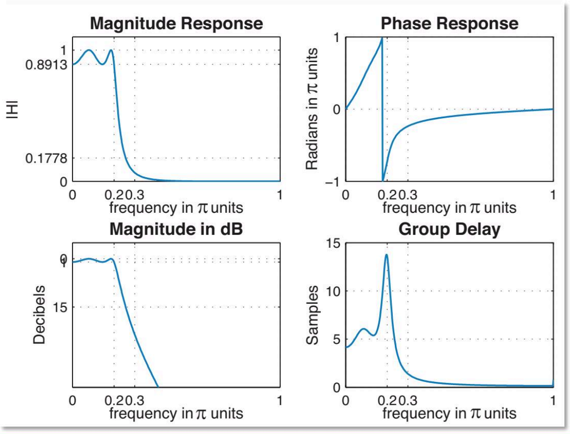
```
>> fp = 0.2; Ap = 1; fs = 0.3; As = 15;
>> [N,fc] = cheb1ord(fp,fs,Ap,As); N
N =
     4
>> [b,a] = cheby1(N,Ap,fc);
>> [sos,G] = tf2sos(b,a)
sos =
    1.0000    2.0000    1.0000    1.0000   -1.5548    0.6493
    1.0000    2.0000    1.0000    1.0000   -1.4996    0.8482
G =
    0.0018
```

$$H_{\text{lp}}(w) = \frac{0.0018\left(1 + w^{-1}\right)^4}{\left(1 - 0.1.4996w^{-1} + 0.8482w^{-2}\right)\left(1 - 0.1.5548w^{-1} + 0.6493w^{-2}\right)}$$

8

Example: Lowpass to Highpass Mapping

$$H_{\text{lp}}(w) = \frac{0.0018\left(1 + w^{-1}\right)^4}{\left(1 - 0.1.4996w^{-1} + 0.8482w^{-2}\right)\left(1 - 0.1.5548w^{-1} + 0.6493w^{-2}\right)}$$



9

Example: Lowpass to Highpass Mapping

- Using the frequency-band transformation Table (Slide-6), we want to transform this filter into a highpass filter with passband beginning at $\omega'_{\text{p}} = 0.6\pi$
- Thus, we want to map the passband cutoff frequency $\omega_p = 0.2\pi$ of the lowpass prototype filter into passband cutoff frequency $\omega'_{\text{p}} = 0.6\pi$ of the highpass filter
- We use the second row from the Table in which we need the parameter a, given by

$$a = -\frac{\cos[(\omega_p + \omega'_p)/2]}{\cos[(\omega_p - \omega'_p)/2]} = -\frac{\cos[(0.2\pi + 0.6\pi)/2]}{\cos[(0.2\pi - 0.6\pi)/2]} = -0.38197$$

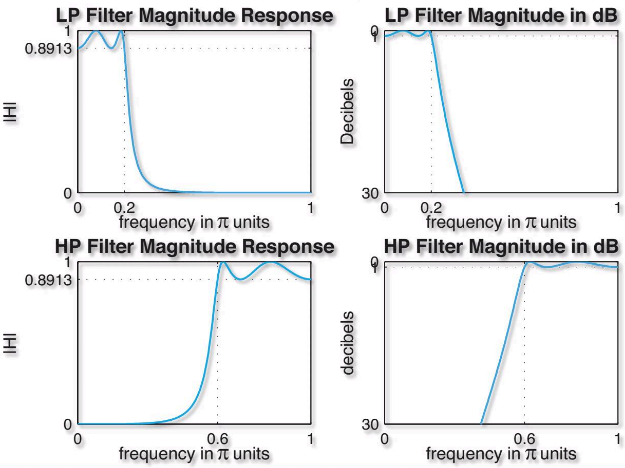
- Hence, the mapping is $w^{-1} = \frac{z^{-1} - 0.38197}{1 - 0.38197z^{-1}}$. Thus, the desired highpass filter is

$$H_{\text{hp}}(z) = H_{\text{lp}}(w)\Big|_{w^{-1} = \frac{z^{-1} - 0.38197}{1 - 0.38197z^{-1}}}$$

$$= \frac{0.02426(1 - z^{-1})^4}{(1 + 0.5561z^{-1} + 0.7657z^{-2})\big((1 + 1.0416z^{-1} + 0.4019z^{-2})\big)}$$

Example: Lowpass to Highpass Mapping

$$H_{\text{hp}}(z) = \frac{0.02426(1 - z^{-1})^4}{(1 + 0.5661z^{-1} + 0.7657z^{-2})\big((1 + 1.0416z^{-1} + 0.4019z^{-2})\big)}$$



Example: Lowpass to Highpass Mapping

- The resulting highpass filter has a stopband edge ω_s' that can also be computed from ω_s using the computed parameter a of the frequency-band transformation:

$$w^{-1} = -\frac{z^{-1}+a}{1+az^{-1}} \Rightarrow e^{j\omega} = -\frac{e^{-j\omega'}+a}{1+e^{-j\omega'}}$$

- which yields

$$\omega' = \tan^{-1}\left[-\frac{(1-a^2)\sin(\omega)}{2a+(1+a^2)\cos(\omega)}\right] \qquad (11.1)$$

$$\omega = \tan^{-1}\left[-\frac{(1-a^2)\sin(\omega')}{2a+(1+a^2)\cos(\omega')}\right] \qquad (11.2)$$

- Hence,

$$\omega_s' = \tan^{-1}\left[-\frac{(1-0.38197^2)\sin(0.3\pi)}{2(0.38197)+(1+0.38197^2)\cos(0.3\pi)}\right] = 0.4586\pi$$

12

Design of Lowpass to Highpass Mapping Function

- In the previous example, a lowpass digital filter was available that we transformed into a highpass filter so that the passband of lowpass was mapped to the passband of highpass through the LP → HP mapping.
- In practice, we have to first design such a lowpass filter from the specifications of the highpass filter. This is where equation (11.2) is useful.
- **Design procedure**: Given highpass filter specifications ω_s', A_s, ω_p', and A_p
  1. Choose ω_p of the lowpass prototype filter, say $\omega_p = 0.2\pi$
  2. Using ω_p' and ω_p, determine the design parameter a from the Table
  3. From a and ω_s', determine the stopband edge ω_s of the lowpass prototype using (11.2)
  4. Now design the lowpass prototype $H_{lp}(z)$ using specifications ω_p, A_p, ω_s, and A_s using any of the analog prototypes.
  5. Finally, using the transformation $w = -\frac{z^{-1}+a}{1+az^{-1}}$, map $H_{lp}(z)$ into $H_{hp}(z)$
- **MATLAB functions**: The previously discussed order-calculation and design functions can also implement frequency band transformations.

13

Example: Highpass Filter Design using FB Mapping

- Design a Chebyshev-I highpass digital filter with specifications:

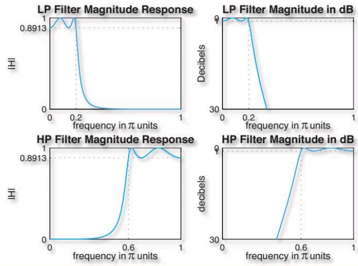
$$\omega_s = 0.4685\pi, \qquad A_s = 15\text{dB}, \qquad \omega_p = 0.6\pi, \qquad A_p = 1\text{dB}$$

- We will follow the procedure given on the previous slide:

```
oms = 0.45864*pi; As = 15; omp = 0.6*pi; Ap = 1;
% Determine the digital lowpass cutoff frequencies:
omplp = 0.2*pi; % This is arbitrary
alpha = -(cos((omplp+omp)/2))/(cos((omplp-omp)/2));
omslp = atan(-((1-alpha^2)*sin(oms))/...
    (2*alpha+(1+alpha^2)*cos(oms)));
% Compute Analog lowpass Prototype Specifications:
T = 1; Fs = 1/T;
OmegaP = (2/T)*tan(omplp/2);
OmegaS = (2/T)*tan(omslp/2);
% Design Analog Chebyshev-I Prototype Lowpass Filter:
[N,Wp] = cheb1ord(OmegaP,OmegaS, Ap, As,'s');
[C,D] = cheby1(N,Ap,Wp,'s');
% Perform BLT to obtain digital lowpass
[blp,alp] = bilinear(C,D,Fs);
% Transform digital lowpass into highpass filter
Nz = -[alpha,1]; Dz = [1,alpha];
[b,a] = zmapping(blp,alp,Nz,Dz);
% Cascade structure
[sos,G] = tf2sos(b,a)
```

```
sos =
    1.0000   -2.0000    1.0000    1.0000    1.0416    0.4019
    1.0000   -2.0000    1.0000    1.0000    0.5561    0.7647
G =
    0.0243
```

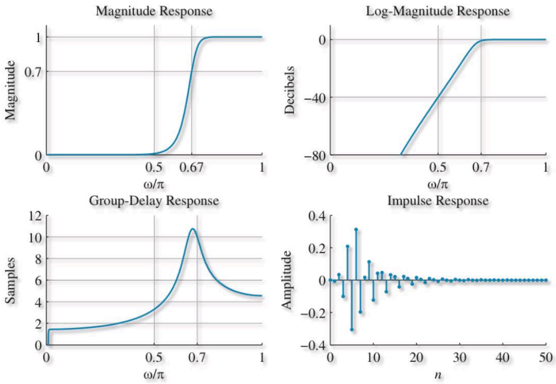
$$H(z) = \frac{0.0243(1-z^{-1})^4}{(1+0.5561z^{-1}+0.7647z^{-2})\big((1+1.0416z^{-1}+0.4019z^{-2})\big)}$$



14

Butterworth Highpass Filter Design

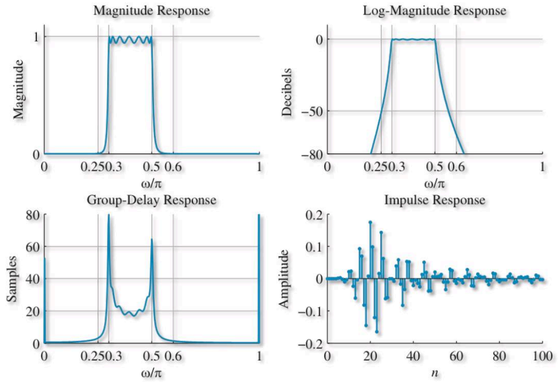
```
>> omegap = 0.7; Ap = 1; omegas = 0.5; As = 40;
>> [N,omegac] = buttord(omegap,omegas,Ap,As)
N =
     8
omegac =
    0.6739
>> [B,A] = butter(N,omegac,'high');
```



15

Chebyshev-I Bandpass Filter Design

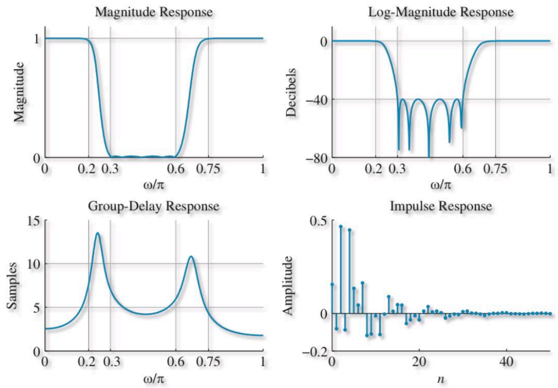
```
>> omegap = [0.3,0.5]; omegas = [0.25,0.6]; Ap = 0.5; As = 50;
>> [N,omegac] = cheb1ord(omegap,omegas,Ap,As); N
N =
     7
>> [B,A] = cheby1(N,Ap,omegac);
```



16

Chebyshev-II Bandstop Filter Design

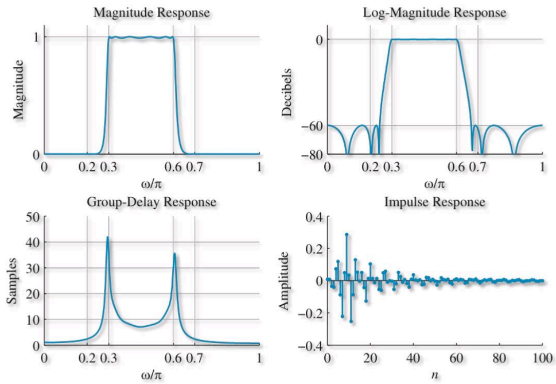
```
>> omegap = [0.2,0.75]; omegas = [0.3,0.6]; Ap = 0.5; As = 40;
>> [N,omegac] = cheb2ord(omegap,omegas,Ap,As); N
N =
     5
>> [B,A] = cheby2(N,As,omegac,'stop');
```



17

Elliptic Bandpass Filter Design

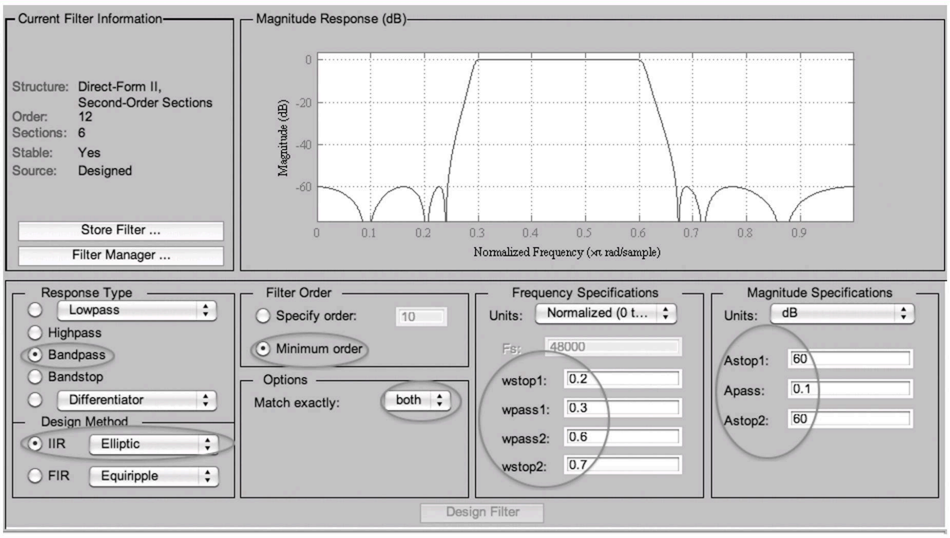
```
>> omegap = [0.3,0.6]; omegas = [0.2,0.7]; Ap = 0.1; As = 60;
>> [N,omegac] = ellipord(omegap,omegas,Ap,As); N
N =
     6
>> [B,A] = ellip(N,Ap,As,omegac);
```



EECE-5666: Digital Signal Processing IIR Filter Design: Filter Transformations 18/21

18

MATLAB Filter Design Tool: **filterDesigner**



EECE-5666: Digital Signal Processing IIR Filter Design: Filter Transformations 19/21

19

Comparison of FIR and IIR Filters

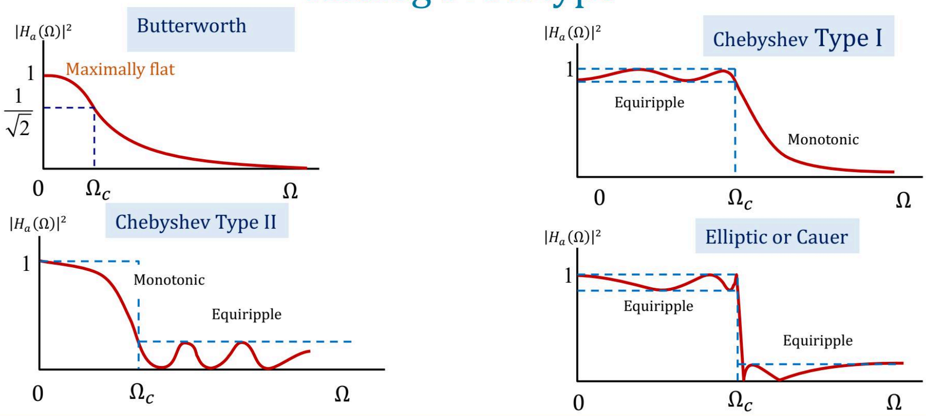
- In the case of FIR filters optimal filters are the equiripple filters designed via the parks-McClellan algorithm, while in the case of IIR filters optimal filters are the elliptic filters.
- For FIR filter the standard realization is the linear-phase direct form, while for elliptic filters cascade forms are widely used.
- Multiplications per output sample
  - Linear-phase FIR of order M: Number of mults/output sample $= M/2$
  - IIR filter of order N: Number of mults/output sample $= 4(N/2)$
- For most applications IIR elliptic filter are desirable from the computational point of view.
- If we take into account the phase equalizers, then FIR filter designs look good because of their exact linear-phase characteristics

IIR Filter Design Summary

- Causal and stable filters with rational system functions have a nonlinear phase response, which complicates filter design using optimization techniques.
- The IIR filter design problem consists of obtaining a causal and stable filter with a rational system function, whose frequency response best approximates the desired ideal magnitude responses within specified tolerances while the phase response is left unspecified.
- The most popular techniques for design of IIR filters start with the design of a continuous-time prototype lowpass filter (Butterworth, Chebyshev, or elliptic), which is subsequently converted to a discrete-time filter (lowpass, highpass, bandpass, or bandstop) using an appropriate set of transformations.
- The impulse invariance and bilinear mappings are two most popular transformations that convert analog into digital filters. The better and more versatile of the two is bilinear mapping.
- Frequency transformations of the allpass type are used to obtain a general frequency-selective filter from a prototype lowpass filter in the analog as well as the digital domain.

IIR Filter Design

Analog Prototype



0

Introduction

- IIR filter have infinite-duration impulse responses, hence they can be matched to analog filters, all of which generally have infinitely long impulse responses.
- The basic technique of IIR filter design transforms well-known analog filters into digital filters using complex-valued mappings.
- The advantage of this technique lies in the fact that
  - Design formulas have closed form expressions
  - Analog filter design (AFD) formulas are available extensively in literature and
  - the filter mappings functions are also extensively available in the literature.
- The basic technique requires Analog-to-Digital filter transformations.
- However, the AFD tables are available for lowpass filters only.
- For other frequency selective filters, we need to apply frequency-band mappings, which are also complex-valued and are available in literature.

1

Introduction

IIR Filter Design Steps

- Design analog lowpass filter.
- Study and apply filter transformations to obtain digital lowpass filter.
- Study and apply frequency-band transformations to obtain other digital filters from digital lowpass filter.

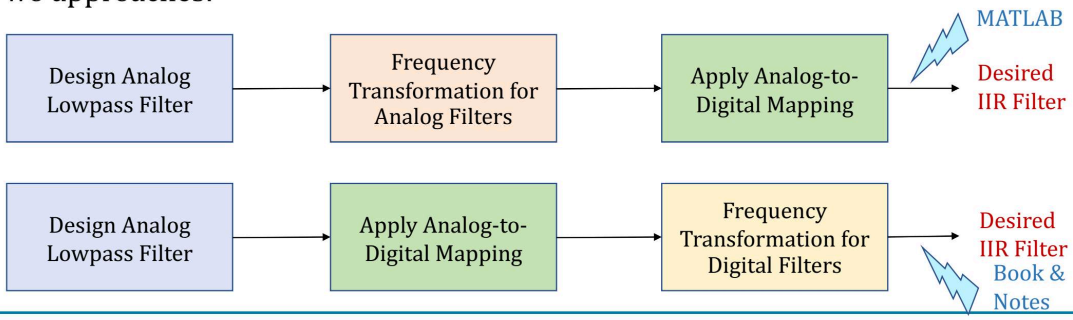
The Main Problem

- We have no control over the phase characteristics of the IIR filter.
- Hence IIR filter designs will be treated as magnitude-only designs.
- Accept whatever phase response we obtain from the design.

2

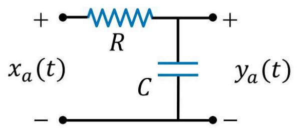
Design of Digital IIR Filters from Analog Filters

- Design techniques for analog filters are highly advanced
- Many practical analog filter design techniques involve closed form design formulas
- The approximation techniques used for analog filters do not lead to simple formulas when are applied directly to the design of digital IIR filters
- Two approaches:



3

Motivation: RC Circuit (Analog Filter)



Differential Equation

$$RC\frac{dy_a(t)}{dt} + y_a(t) = x_a(t)$$

Laplace Transform

$$X_a(s) = \int_{-\infty}^{\infty} x_a(t)\mathrm{e}^{-st}dt, \qquad s = \sigma + j\Omega$$

$$\frac{dx_a(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX_a(s)$$

System Function

$$RCsY_a(s) + Y_a(s) = X_a(s) \Rightarrow$$

$$H_a(s) = \frac{Y_a(s)}{X_a(s)} = \frac{\Omega_c}{s+\Omega_c}, \qquad \Omega_c = \frac{1}{RC}$$

Impulse Response

$$h_a(t) = \Omega_c \mathrm{e}^{-\Omega_c t}u_a(t) \qquad \text{Stability: } \Omega_c > 0$$

Convolution

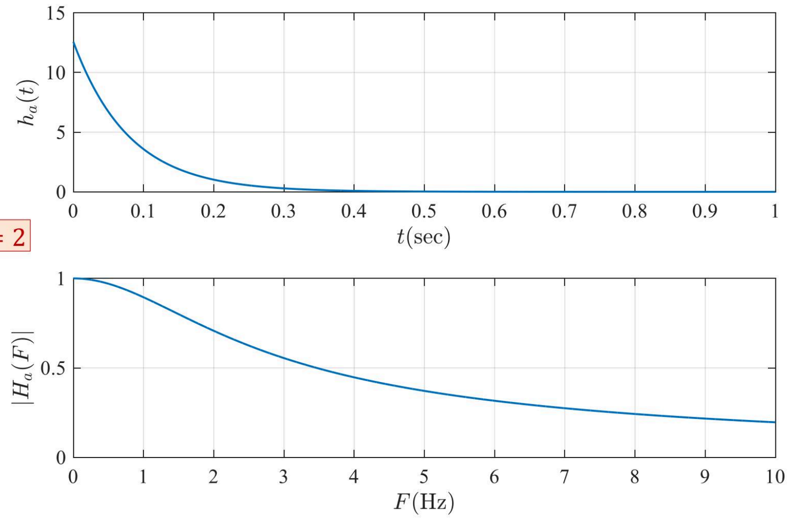
$$y_a(t) = \int_{-\infty}^{\infty} h_a(\tau)x_a(t-\tau)d\tau = h_a(t) * x_a(t)$$

Frequency Response Function

$$H_a(\Omega) = \int_{-\infty}^{\infty} h_a(t)\mathrm{e}^{-\mathrm{j}\Omega t}dt = H_a(s)\Big|_{s=\mathrm{j}\Omega}$$

Motivation: RC Circuit Responses

$RC = 0.5 \Rightarrow \Omega_c = 2$



Motivation: Digital Filter by Sampling Analog Impulse Response

$$h_a(t) = \Omega_c \mathrm{e}^{-\Omega_c t} u_a(t)$$

$$h(n) = h_a(nT) = \Omega_c \left(\mathrm{e}^{-\Omega_c T}\right)^n u(n)$$

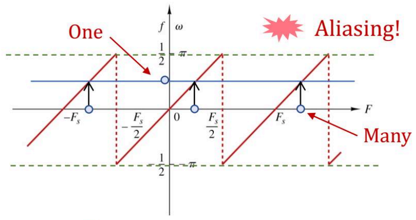
$$H(z) = \frac{\Omega_c}{1 - \mathrm{e}^{-\Omega_c T} z^{-1}} = \frac{b}{1 - az^{-1}}$$

$$y(n) = ay(n-1) + bx(n)$$

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \Rightarrow H(z) = \frac{b}{1 - \mathrm{e}^{-aT} z^{-1}}$$

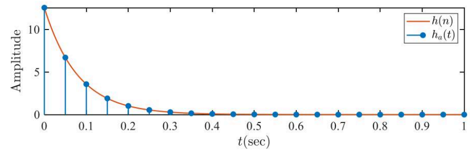
$$s = -\Omega_c \Rightarrow p = \mathrm{e}^{-\Omega_c T}$$

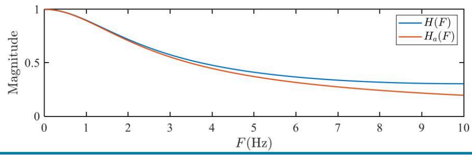
$$H(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a(F - kF_s), \qquad F_s = \frac{1}{T}$$



Many-to-one mapping

$z = \mathrm{e}^{sT}$

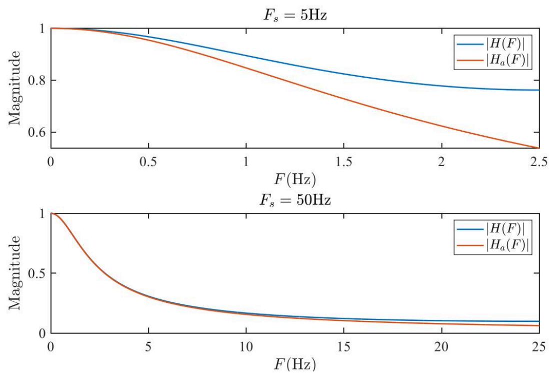




6

Motivation: Digital Filter by Sampling Analog Impulse Response

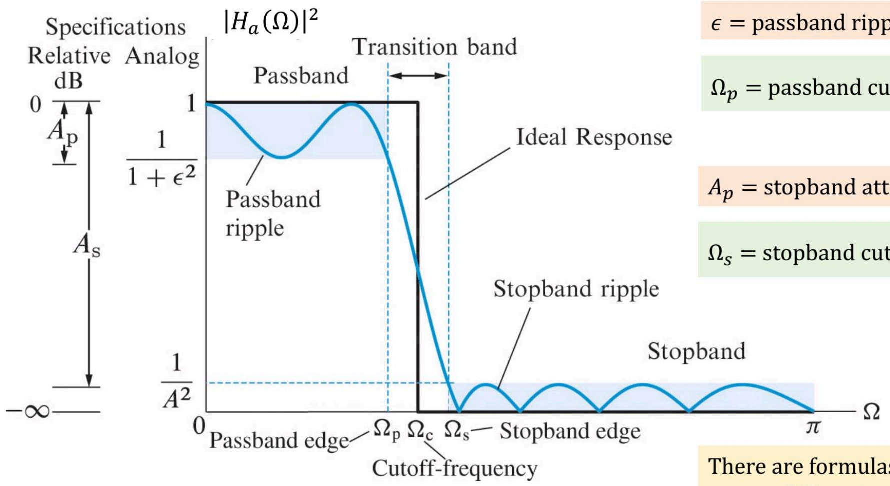
- Effect of Sampling Frequency



- **Conclusion**:
  It is possible to convert a known analog filter impulse response (or system function) into an implementable IIR digital filter
- **Problems**:
  - A simple RC circuit is a bad prototype lowpass filter
  - Sampling analog impulse response creates aliasing of the resulting digital frequency response.
- **Solution**:
  - We need better prototype analog filter
  - We need a better approach than simply sampling analog impulse response.

7

Practical Analog Filter Specifications



ϵ = passband ripple parameter

Ω_p = passband cutoff frequency in $\frac{\text{rads}}{\text{sec}}$

A_p = stopband attenuation parameter

Ω_s = stopband cutoff frequency in $\frac{\text{rads}}{\text{sec}}$

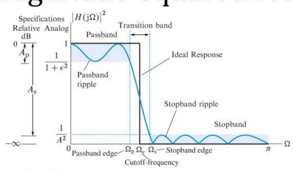
There are formulas to convert among different specs

8

Prototype Analog LPF Approximations

- The ideal (desired) LP filter prototype is specified by the magnitude-squared resp

$$|H_d(\Omega)|^2 = \begin{cases} 1, & 0 \le |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$



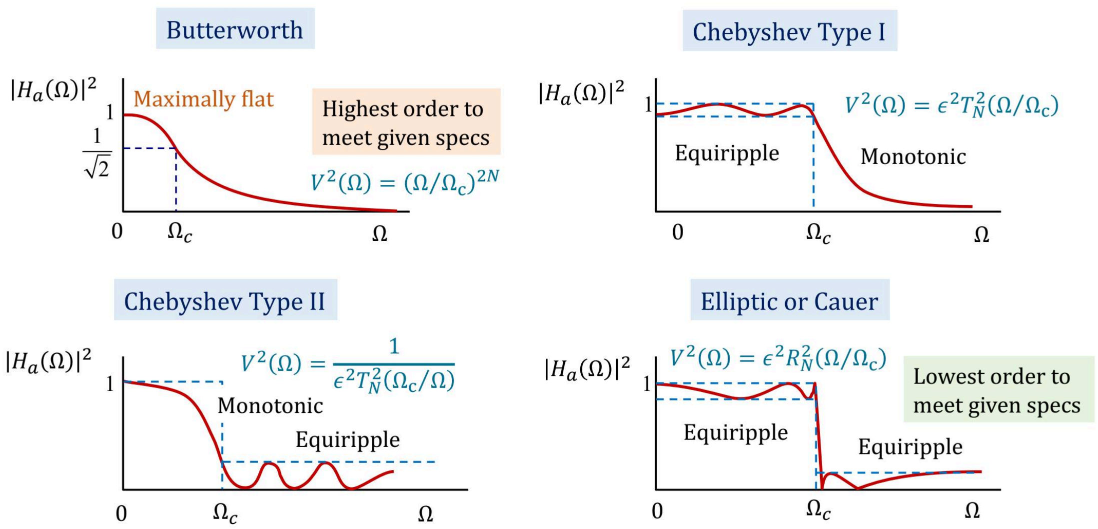
- The function used for the approximating filter is

$$|H_a(\Omega)|^2 = \frac{1}{1 + V^2(\Omega)}, \qquad \text{where} \quad V^2(\Omega) = \begin{cases} \ll 1, |\Omega| \le \Omega_c \\ \gg 1, |\Omega| > \Omega_c \end{cases}$$

- The three most widely used choices of $V(\Omega)$ lead to
  – Butterworth lowpass approximation
  – Chebyshev (I and II) lowpass approximation
  – Elliptic lowpass approximation
- The roots of Butterworth, Chebyshev, and Elliptic polynomials are specified by closed form formulas dependent directly or indirectly on ϵ and Ω_c

9

Analog Lowpass Filter Approximations



10

Spectral Factorization: $|H_a(\Omega)|^2 \Rightarrow H_a(s)$

Frequency Response Function

$$H_a(\Omega) = \int_{-\infty}^{\infty} h_a(t)e^{-j\Omega t}dt = H_a(s)\Big|_{s=j\Omega}$$

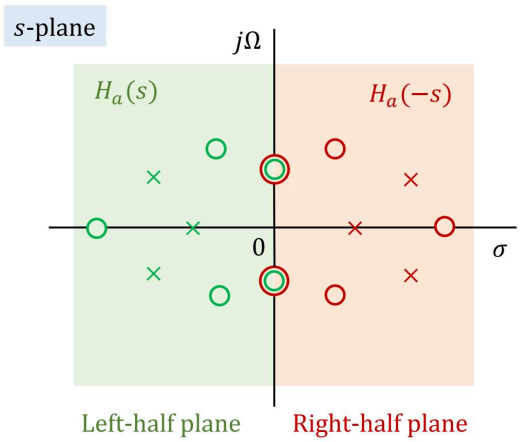
Magnitude Squared Response for real $h_a(t)$

$$|H_a(\Omega)|^2 = H_a(\Omega)H_a^*(\Omega) = H_a(\Omega)\,H_a(-\Omega)$$

$$|H_a(\Omega)|^2 = H_a(s)H_a(-s)\Big|_{s=j\Omega}$$

Spectral factorization

$|H_a(\Omega)|^2 \Rightarrow H_a(s)$ Causal and stable filter

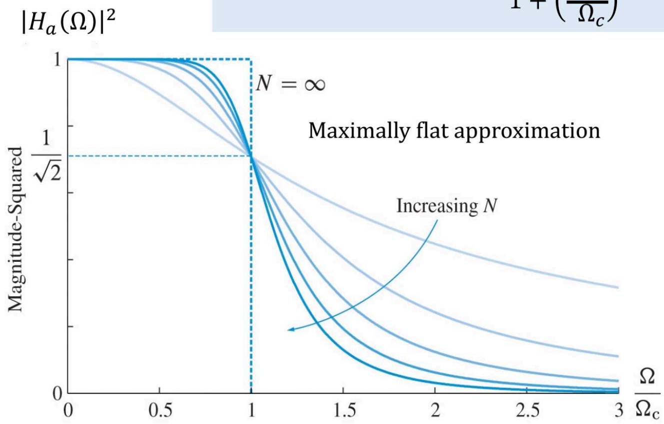


- Compute all poles and zeros of $H_a(s)H_a(-s)$
- Choose the green ones to form $H_a(s)$

11

The Butterworth Analog LP Filter Approximation

$$|H_a(\Omega)|^2 = \frac{1}{1+V^2(\Omega)} = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}, \qquad N = 1,2,\dots$$



1885-1958

The Butterworth filter was first described in 1930 by the British engineer and physicist Stephen Butterworth in his paper entitled "On the Theory of Filter Amplifiers."

12

The Butterworth Analog LP Filter Prototype

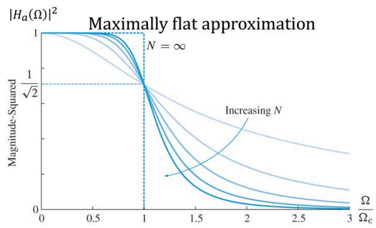
- This filter is characterized by the property that its magnitude response is maximally flat in both passband and stopband.
- Butterworth suggested that $V^2(\Omega) = (\Omega/\Omega_c)^{2N}$. Hence

$$H_B(s)H_B(-s)\Big|_{s=j\Omega} = |H_B(\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}}, \qquad N = 1,2,\cdots$$

or

$$H_B(s)H_B(-s) = \frac{1}{1+(s/j\Omega_c)^{2N}}, \qquad N = 1,2,\cdots$$

- Characteristics of Magnitude-Squared Response
  - $|H_B(0)|^2 = 1$ for all N
  - $|H_B(\Omega_c)|^2 = 0.5$ for all N (3dB attenuation at Ω_c)
  - $|H_B(\Omega)|^2$ monotonically decreases as $\Omega \to \infty$
  - Approaches an ideal LPF as $N \to \infty$



13

Poles of Butterworth Analog LP Filter Prototype

- Poles of $H_B(s)H_B(-s) = \frac{1}{1+(s/j\Omega_c)^{2N}}$ are given by

$$1 + (s/j\Omega_c)^{2N} = 0 \Rightarrow (s/j\Omega_c)^{2N} = -1 = 1e^{j(2k-1)\pi}, \qquad k = 1,2,\cdots,2N$$

$$\Rightarrow s^{2N} = (j\Omega_c)^{2N} e^{j(2k-1)\pi}$$

$$\Rightarrow s = (j\Omega_c)e^{\frac{j(2k-1)\pi}{2N}} = \Omega_c e^{\frac{j(2k-1)\pi}{2N}} e^{\frac{j\pi}{2}} = \Omega_c e^{\frac{j(2k+N-1)\pi}{2N}}$$

or

$$s_k = \Omega_c e^{\frac{j(2k+N-1)\pi}{2N}}, \qquad k = 1,2,\cdots,2N$$

or

$$|s_k| = \Omega_c$$

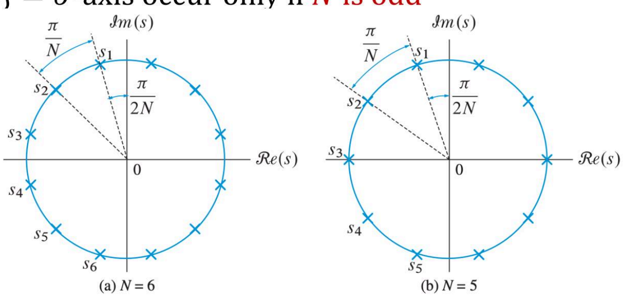
$$\angle s_k = \frac{\pi}{2N}(2k + N - 1), \qquad k = 1,2,\cdots,2N$$

- Thus, $2N$ poles of the product function $H_B(s)H_B(-s)$ are uniformly distributed around a circle of radius Ω_c with a spacing of $\frac{\pi}{N}$ starting at $\frac{\pi}{2N}$ radians.

Poles of Butterworth Analog LP Filter Prototype

$$s_k = \Omega_c e^{\frac{j(2k+N-1)\pi}{2N}}, \qquad k = 1,2,\cdots,2N$$

- There are $2N$ poles equally spaced in angle $\frac{\pi}{N}$ radians around a circle of radius Ω_c in the s-plane
- There are no poles on the $j\Omega$ axis
- Poles on the $\text{Re}\{s\} = \sigma$-axis occur only if N is odd



Stable and Causal Analog LP Butterworth Prototype

- A stale and causal analog filter $H_{\mathrm{B}}(s)$ can now be obtained by selecting poles in the left-half s-plane

$$H_{\mathrm{B}}(s) = \frac{\Omega_c^N}{\prod_1^N (s - s_k)}$$

- **Example**: Given that $|H_{\mathrm{B}}(s)|^2 = \frac{1}{1+64\Omega^6}$, determine $\mathrm{H_B}(s)$.

Solution: Consider

$$|H_{\mathrm{B}}(s)|^2 = \frac{1}{1+64\Omega^6} = \frac{1}{1+(\Omega/0.5)^{2(3)}} \Rightarrow N = 3, \Omega_c = 0.5$$

Hence

$$H_B(s) = \frac{\Omega_c^3}{(s-s_1)(s-s_2)(s-s_3)} = \frac{1/8}{(s+0.25-j0.433)(s+0.5)(s+0.25+j0.433)}$$
$$= \frac{0.125}{s^3+s^2+0.5s+0.125}$$

MATLAB Function: `[C,D] = butter(N,Omegac,'s');`

Design equations for Analog Lowpass Butterworth Prototype

- The analog lowpass filter is specified by four parameters:

$$\Omega_{\mathrm{p}}, \quad A_{\mathrm{p}}, \quad \Omega_{\mathrm{s}}, \quad A_{\mathrm{s}}$$

- Therefore, the essence of the design in the case of Butterworth prototype is to obtain the order N and the cutoff frequency Ω_c from the above parameters

$$|H_{\mathrm{B}}(\Omega)|^2 = \frac{1}{1+(\mathrm{j}\Omega/\mathrm{j}\Omega_c)^{2N}} : \begin{array}{l} \text{at } \Omega = \Omega_{\mathrm{p}}, \text{we want } -10\log_{10}\left|H_{\mathrm{B}}(\Omega_{\mathrm{p}})\right|^2 = A_{\mathrm{p}} \\ \text{at } \Omega = \Omega_{\mathrm{s}}, \text{we want } -10\log_{10}|H_{\mathrm{B}}(\Omega_{\mathrm{s}})|^2 = A_{\mathrm{s}} \end{array}$$

- Thus, we have two equations and two unknowns. The solution is

$$N = \left\lceil \frac{\log_{10}\left[(10^{A_{\mathrm{p}}/10}-1)/(10^{A_{\mathrm{s}}/10}-1)\right]}{2\log_{10}(\Omega_{\mathrm{p}}/\Omega_{\mathrm{s}})} \right\rceil \quad \text{and} \quad \frac{\Omega_p}{\sqrt[2N]{(10^{A_{\mathrm{p}}/10}-1)}} \le \Omega_c \le \frac{\Omega_s}{\sqrt[2N]{(10^{A_{\mathrm{s}}/10}-1)}}$$

- MATLAB Function:
`[N,Omegac] = buttord(Omegap,Omegas,Ap,As,'s')`

Design of an Analog Lowpass Butterworth Prototype

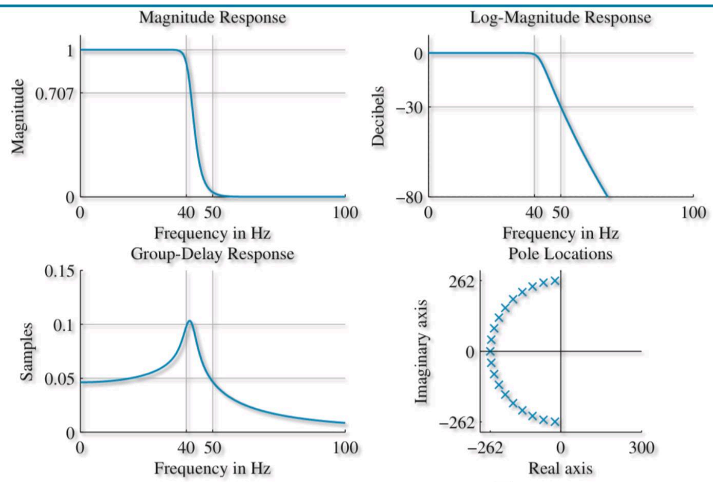
- Design an analog lowpass filter using the Butterworth prototype that satisfies:
  - Passband edge: $F_\mathrm{p} = 40$ Hz Passband ripple: $A_\mathrm{p} = 1$ dB
  - Stopband edge: $F_\mathrm{s} = 50$ Hz Passband ripple: $A_\mathrm{s} = 30$ dB

  Using the following MATLAB script

```
[N,Omegac] = buttord(2*pi*40,2*pi*50,1,30,'s'); N
N =
    19
Fc = Omegac/(2*pi)
Fc =
    41.6902
[C,D] = butter(N,Omegac,'s');
```

  The Butterworth prototype order is 19
- The frequency response plots are shown in the next slide.

Design Plots of an Analog Lowpass Butterworth Prototype



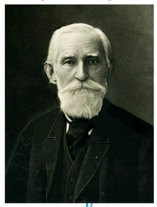
- At $F_c = 41.69$ Hz, the magnitude response is $1/\sqrt{2} = 0.707$.
- At $F_s = 50$ Hz, the log-magnitude response is $= 30$ dB.

The Chebyshev-I Analog LP Filter Approximation

The magnitude squared function of a LP Chebyshev approximation is

$$|H_C(\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)}, \qquad T_N(x) = \begin{cases} \cos(N\cos^{-1} x), & 0 \le |x| \le 1 \\ \cosh(N\cosh^{-1} x), & 1 < |x| < \infty \end{cases}$$

Pafnuty Chebyshev



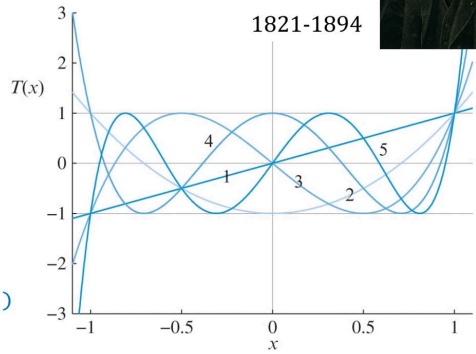
1821-1894

For $0 \le x \le 1$, $T_N(x)$ oscillates between -1 and $+1 \Rightarrow$

$$E_C^2(\Omega) = 1 - \frac{1}{1+\epsilon^2 T_N^2\left(\frac{\Omega}{\Omega_c}\right)} \approx \epsilon^2 T_N^2\left(\frac{\Omega}{\Omega_c}\right), |\Omega| \le \Omega_c$$

$\Rightarrow$ Equiripple optimality

For $1 < |x| < \infty$, $T_N(x)$ increases monotonically to infinity



MATLAB functions:

```
[N,Omegac] = cheb1ord (Omegap,Omegas,Ap,As,'s')
[C,D] = cheby1(N,Ap,Omegac,'s');
```


20

Design of an Analog Lowpass Chebyshev-I Prototype

- Design an analog lowpass filter using the Butterworth prototype that satisfies:
  - Passband edge: $F_p = 40$ Hz Passband ripple: $A_p = 1$ dB
  - Stopband edge: $F_s = 50$ Hz Passband ripple: $A_s = 30$ dB

  Using the following MATLAB script

```
[N,Omegac] = cheb1ord(2*pi*40,2*pi*50,1,30,'s'); N
N =
    7
Fc = Omegac/(2*pi)
Fc =
    40
[C,D] = cheby1(N,Ap,Omegac,'s');
```

  The Butterworth prototype order is $N = 19$ while for the Butterworth prototype the order is $N = 19$

- The frequency response plots are shown in the next slide.

21

Design of an Analog Lowpass Chebyshev-I Prototype

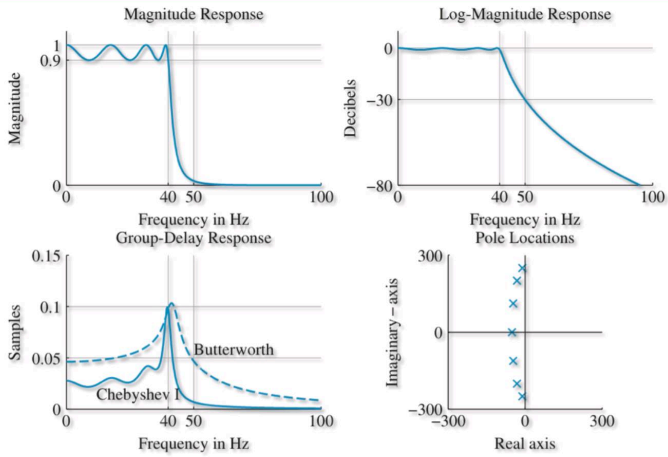
- Design an analog lowpass filter using the Chebyshev-I prototype that satisfies:
  – Passband edge: $F_p = 40$ Hz Passband ripple: $A_p = 1$ dB
  – Stopband edge: $F_s = 50$ Hz Passband ripple: $A_s = 30$ dB
  Using the following MATLAB script

```
[N,Omegac] = cheb1ord(2*pi*40,2*pi*50,1,30,'s'); N
N =
     7
Fc = Omegac/(2*pi)
Fc =
    40
[C,D] = cheby1(N,Ap,Omegac,'s');
```

  The Chebyshev-I prototype order is $N = 7$ while for the Butterworth prototype the order is $N = 19$

- The frequency response plots are shown in the next slide.

Design Plots of an Analog Lowpass Chebyshev-I Prototype



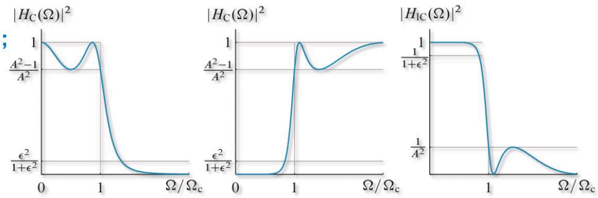
- At $F_c = F_p = 40$ Hz, the magnitude response is $1/\sqrt{1+\epsilon^2} = 0.89$.
- At $F_s = 50$ Hz, the log-magnitude response is > 30 dB.

The Chebyshev-II Analog LP Filter Approximation

- Related to the Chebyshev-I through a simple transformation
- It has a monotone passband and an equiripple stopband, which implies that the filter has both poles and zeros in the s-plane.
- Therefore, the group delay characteristics are better (and the phase response more linear) in the passband than the Chebyshev-I prototype.
- The magnitude response is defined as $|H_{\mathrm{IC}}(\mathrm{j}\Omega)|^2 = 1 - \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)} = \frac{\epsilon^2 T_N^2(\Omega/\Omega_c)}{1+\epsilon^2 T_N^2(\Omega/\Omega_c)}$

MATLAB functions:

```
[N,Omegac]=cheb2ord(Omegap,Omegas,Ap,As,'s');
[C,D] = cheby2(N,As,Omegac,'s');
```



(a) Chebyshev I Lowpass (b) Chebyshev I Highpass (c) Chebyshev II Lowpass

Design of an Analog Lowpass Chebyshev-II Prototype

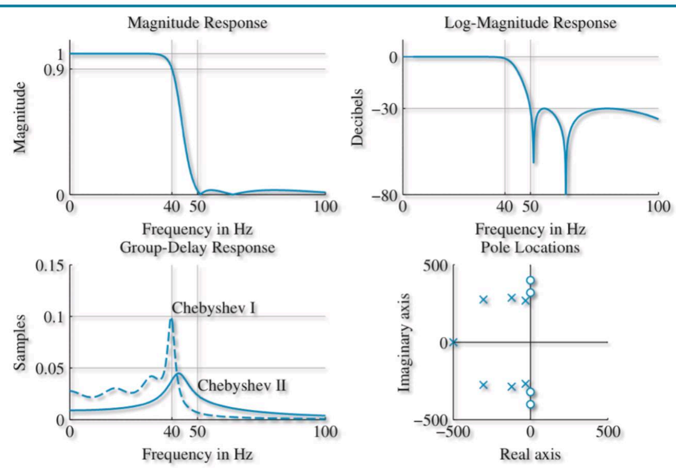
- Design an analog lowpass filter using the Chebyshev-II prototype that satisfies:
  - Passband edge: $F_p = 40$ Hz Passband ripple: $A_p = 1$ dB
  - Stopband edge: $F_s = 50$ Hz Passband ripple: $A_s = 30$ dB

  Using the following MATLAB script

```
[N,Omegac] = cheb2ord(2*pi*40,2*pi*50,1,30,'s'); N
N =
     7
Fc = Omegac/(2*pi)
Fc =
    49.8720
[C,D] = cheby2(N,30,Omegac,'s');
```

  The Chebyshev-II prototype order is $N = 7$, which is the same as for the Chebyshev-I prototype.
- The frequency response plots are shown in the next slide.

Design Plots of an Analog Lowpass Chebyshev-II Prototype



- At $F_c = F_p = 40$ Hz, the magnitude response is $1/\sqrt{1+\epsilon^2} < 0.89$.
- At $F_s = 50$ Hz, the log-magnitude response is $= 30$ dB.

EECE-5666: Digital Signal Processing IIR Filter Design: Analog Prototype 26/31

26

Elliptic (Cauer) Lowpass Prototype

- These filters exhibit equiripple behaviors in the passband and in the stopband
- They are similar in magnitude response characteristics to FIR equiripple filters
- Therefore, elliptic filters are optimum in that that they achieve minimum order N for the given specifications.
- For these filters, the $V(\Omega)$ is defined using a rational Chebyshev function $R_N(\Omega)$

$$R_N(\Omega) = \begin{cases} \nu^2 \dfrac{\Omega_1^2 - \Omega^2}{1-\Omega_1^2\Omega^2} \cdots \dfrac{\Omega_{2N-1}^2 - \Omega^2}{1-\Omega_{2N-1}^2\Omega^2}, & N \sim \text{even} \\ \nu^2 \dfrac{\Omega_2^2 - \Omega^2}{1-\Omega_2^2\Omega^2} \cdots \dfrac{\Omega_{2N}^2 - \Omega^2}{1-\Omega_{2N}^2\Omega^2}, & N \sim \text{odd} \end{cases}$$

- Analysis of this function requires theory of elliptic functions, which is complicated
- MATLAB functions: `[N,Omegac] = ellipord(Omegap,Omegas,Ap,As,'s');`
  `[C,D] = ellip(N,Ap,As,Omegac,'s');`

EECE-5666: Digital Signal Processing IIR Filter Design: Analog Prototype 27/31

27

Design of an Analog Lowpass Elliptic Prototype

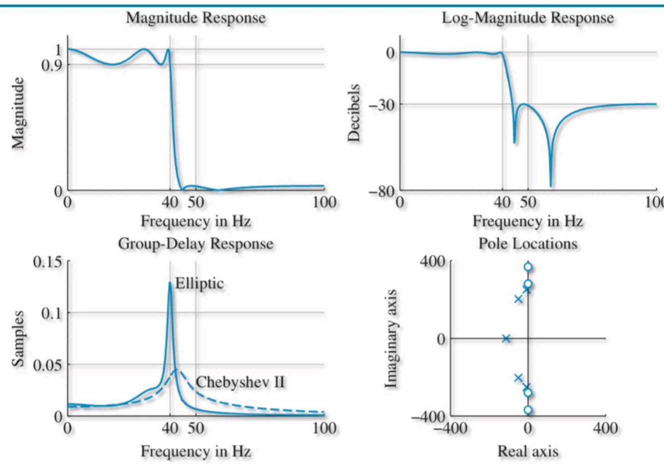
- Design an analog lowpass filter using the elliptic prototype that satisfies:
  – Passband edge: $F_p = 40$ Hz Passband ripple: $A_p = 1$ dB
  – Stopband edge: $F_s = 50$ Hz Passband ripple: $A_s = 30$ dB
  Using the following MATLAB script

```
[N,Omegac] = ellip2ord(2*pi*40,2*pi*50,1,30,'s'); N
N =
     5
Fc = Omegac/(2*pi)
Fc =
    40
[C,D] = ellip(N,1,30,Omegac,'s');
```

  The elliptic prototype order is $N = 5$, compared to $N = 7$ for both Chebyshev filters and $N = 19$ for the Butterworth filters.

- The frequency response plots are shown in the next slide.

Design Plots of an Analog Lowpass Elliptic Prototype



- At $F_c = F_p = 40$ Hz, the magnitude response is $1/\sqrt{1+\epsilon^2} = 0.89$.
- At $F_s = 50$ Hz, the log-magnitude response is $= 30$ dB.

Summary of MATLAB Functions

- Butterworth Lowpass Filter:
  - `[N,Omegac] = buttord(Omegap,Omegas,Ap,As,'s')`
  - `[b,a] = butter(N,Omegac,'s')`
- Chebyshev-1 Lowpass filter:
  - `[N,Omegac] = cheb1ord(Omegap,Omegas,Ap,As,'s')`
  - `[b,a] = cheby1(N,Rp,Omegac,'s')`
- Chebyshev-2 Lowpass filter:
  - `[N,Omegac] = cheb2ord(Omegap,Omegas,Ap,As,'s')`
  - `[b,a] = cheby2(N,As,Omegac,'s')`
- Elliptic Lowpass Filter:
  - `[N,Omegac] = ellip1ord(Omegap,Omegas,Ap,As,'s')`
  - `[b,a] = ellip(N,Rp,As,Omegac,'s')`

30

Comparison of Analog Lowpass Prototypes

- Given the four prototype analog filters, elliptic filters provide the best (optimal) performance in the magnitude-squared response.
- However, they have highly nonlinear phase response in the passband (which is undesirable in many applications).
- Even though we decided not to worry about phase response in our design, phase is still an important issue in the overall system.
- At other end of the performance scale are the Butterworth filters, which have maximally flat magnitude response and require a highest-order N (more poles) to achieve the same stopband specification.
- On the other hand, Butterworth filters exhibit very good linear phase response in their passband.

31