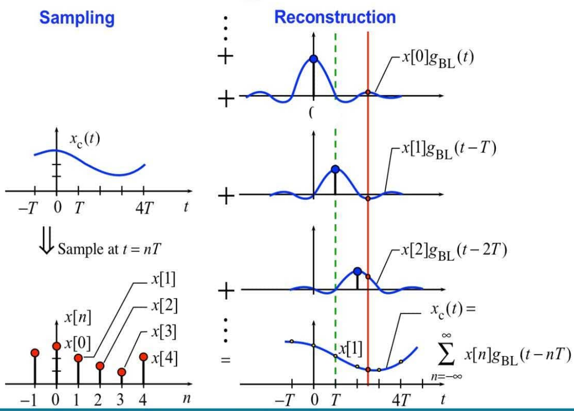
Sampling and Reconstruction



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Outline

- Periodic Sampling
- The Sampling Theorem
- Reconstruction of Bandlimited Signals
- Ideal A/D and D/A Converters
- Digital Processing of Analog Signals
- Summary

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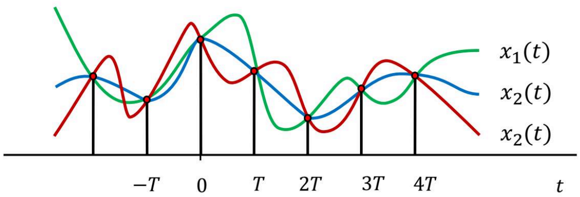
The Concept of Sampling



$$x_1(t) \neq x_2(t) \neq x_3(t) \Rightarrow x_1(nT) = x_2(nT) = x_3(nT)$$

- Problem: It is impossible to uniquely specify an arbitrary continuous–time signal from a sequence of equally spaced samples
- Solution: Impose constraints on some properties of the continuous–time signals

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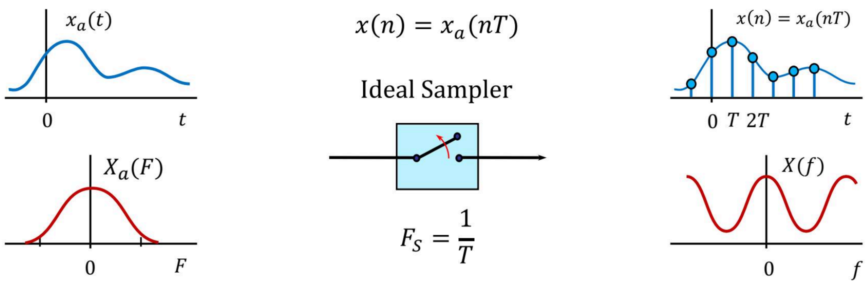
Periodic Sampling



$$x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi Ft} dF \quad \xleftrightarrow{\text{CTFT}} \quad X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi Ft} dt$$

$$x(n) = \int_{-1/2}^{1/2} X(f) e^{j2\pi fn} df \quad \xleftrightarrow{\text{DTFT}} \quad X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi fn}$$

If $x(n) = x_a(nT)$, what is the relationship between $X(f)$ and $X_a(F)$?

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Relationship Between $X(F)$ and $X_a(F)$

Since $t = nT$, the ICTFT can be written as

$$x_a(nT) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi F n T} dF$$

Using the IDTFT we have

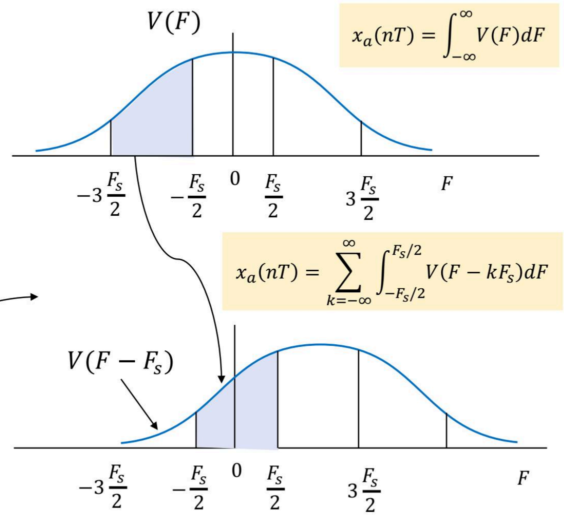
$$x(n) = \int_{-1/2}^{1/2} X(f) e^{j2\pi f n} df \Rightarrow f = FT$$

$$x(n) = \int_{-F_s/2}^{F_s/2} T X(F) e^{j2\pi (F/F_s) n} dF$$

The red integrals must be equal!

Let

$$V(F) = X_a(F) e^{j2\pi (F/F_s) n}$$



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Relationship Between $X(F)$ and $X_a(F)$

We first note that

$$V(F - kF_s) = X_a(F - kF_s) e^{j2\pi \frac{F - kF_s}{F_s} n} = X_a(F - kF_s) e^{j2\pi \left(\frac{F}{F_s}\right) n} \underbrace{e^{-j2\pi n k}}_{= 1}$$

Therefore, interchanging $\sum$ and $\int$ yields

$$x_a(nT) = \int_{-F_s/2}^{F_s/2} \left[\sum_{k=-\infty}^{\infty} X_a(F - kF_s) \right] e^{j2\pi (F/F_s) n} dF$$

We also recall

$$x(n) = \int_{-F_s/2}^{F_s/2} T X(F) e^{j2\pi (F/F_s) n} dF$$

$x(n) = x_a(nT)$

$$X(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$

The spectrum $X(F)$ of $x(n) = x_a(nT)$ is obtained by superimposing copies ("images") of the spectrum $X_a(F)$ of $x_a(t)$ at all integer multiples of the sampling frequency $F_s = 1/T$ and scaling the result by $1/T$

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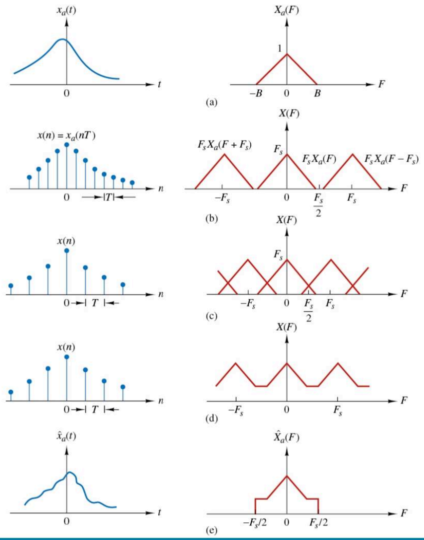
The Sampling Theorem



In theory, if $F_s \geq 2B$, then

$$X_a(F) = X(F)G_a(F)$$

where

$$G_a(F) = \begin{cases} T, & |F| \leq F_s/2 \\ 0, & |F| > F_s/2 \end{cases}$$

Therefore

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F)e^{j2\pi Ft}dF$$

A bandlimited signal $x_a(t)$ can be reconstructed from its samples $x(n) = x_a(nT)$ iff $F_s \geq 2B$

Reconstruction Using Ideal Bandlimited Interpolation

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F)e^{j2\pi Ft}dF \quad = \int_{-\infty}^{\infty} X(F)G_a(F)e^{j2\pi Ft}dF$$

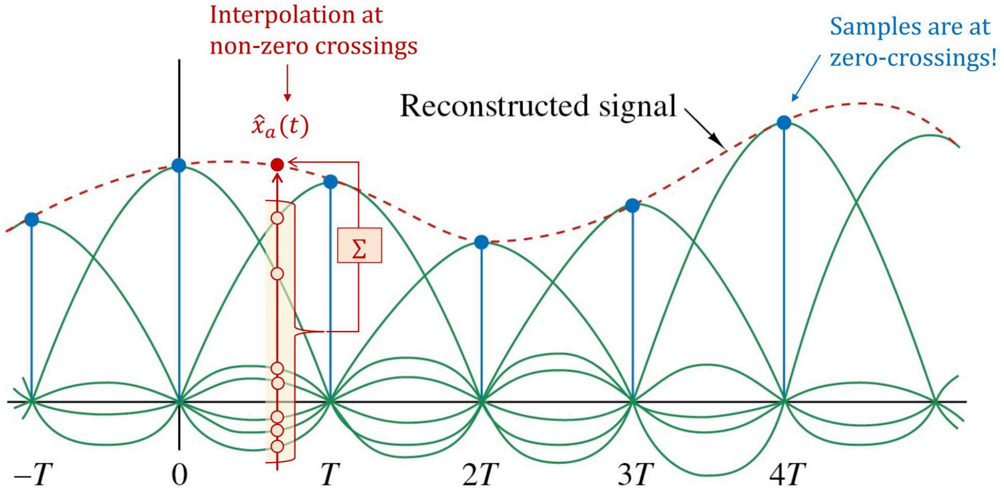
$$= \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} x_a(nT)e^{-j2\pi FnT}\right]G_a(F)e^{j2\pi Ft}dF \quad = \sum_{n=-\infty}^{\infty} x_a(nT)\left[\int_{-\infty}^{\infty} G_a(F)\, e^{j2\pi F(t-nT)}dF\right]$$

If we define $g_a(t) = \int_{-\infty}^{\infty} G_a(F)e^{j2\pi Ft}dF = \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} = g_{BL}(t) \Rightarrow$

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT)g_a(t-nT)$$

- This is not a convolution!
- The ideal D/A is a linear, time−varying system

Interpretation of Ideal Bandlimited Interpolation



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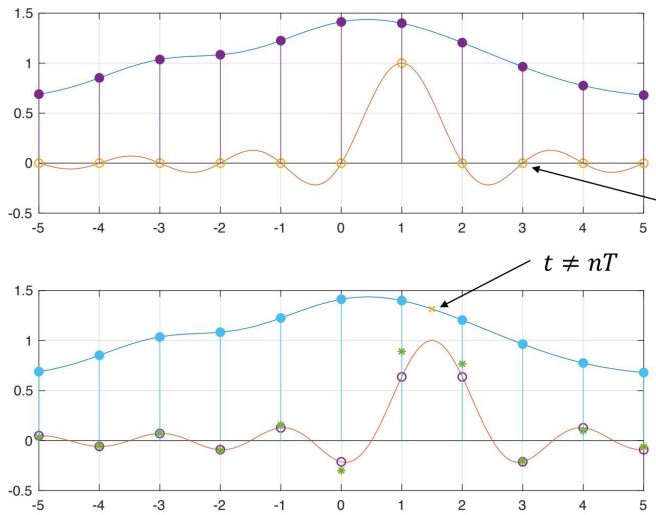
Implementation of Ideal Bandlimited Interpolation



$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT) \frac{\sin \frac{\pi}{T}(t - nT)}{\frac{\pi}{T}(t - nT)}$$

Zero-crossings at $t = nT$

The value of $x_a(t)$ is a linear combination of all other samples!

- The ideal BL interpolator is not practically realizable
- The ideal BL interpolator is not an ideal low pass filter

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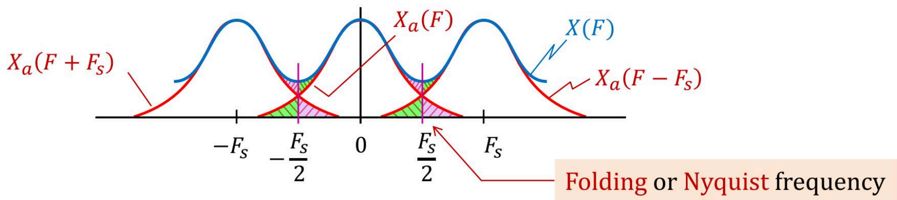
Aliasing Distortion

Sampling theorem requirements: $x_a(t)$ is BL to $B\ Hz$ and $F_s \geq 2B$

If $x_a(t)$ is non−BL or is BL but $F_s < 2B \Rightarrow$



- $X_a(F)$ cannot be recovered from $X(F) \Rightarrow$ perfect reconstruction of $x_a(t)$ from $x_a(nT)$ is impossible
- The resulting distortion is called "aliasing"
- Aliasing cannot be removed. It can only be avoided!
- Aliasing can be avoided by low−pass filtering $x_a(t)$ at $F_s/2$ (antialiasing filtering)

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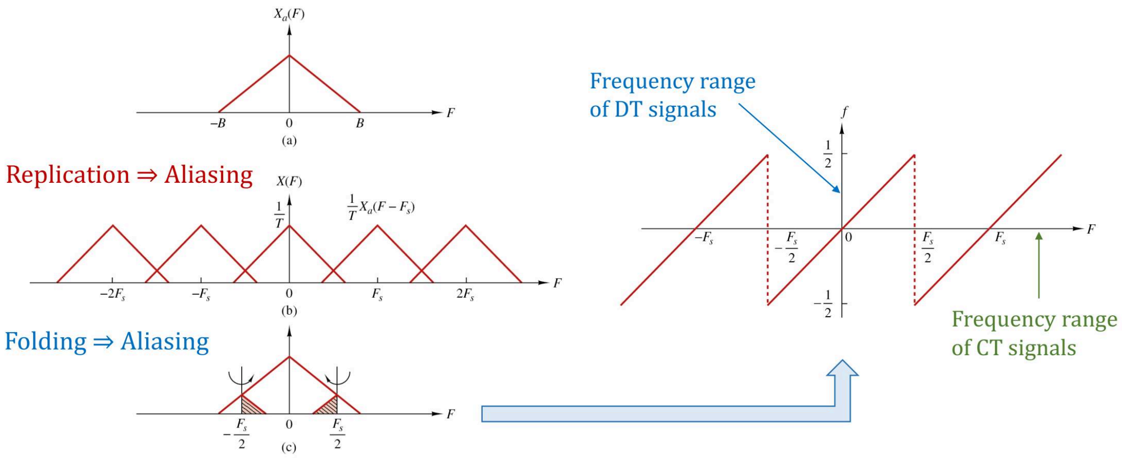
Implications of Folding on Frequency Range



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Sampling Theorem and its Implications

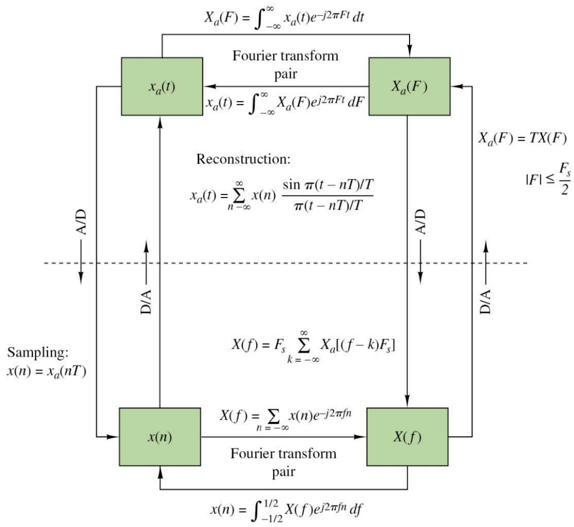
Sampling Theorem

Any real−valued continuous−time signal $x_a(t)$, which is BL to $B\ Hz$, can be recovered from the samples $x_a(nT), -\infty < n < \infty$ if the sampling frequency $F_s = \frac{1}{T} \geq 2B$

Claude Shannon (1916-2001)

Harry Nyquist 1889-1976

Vladimir A. Kotelnikov 1908-2005



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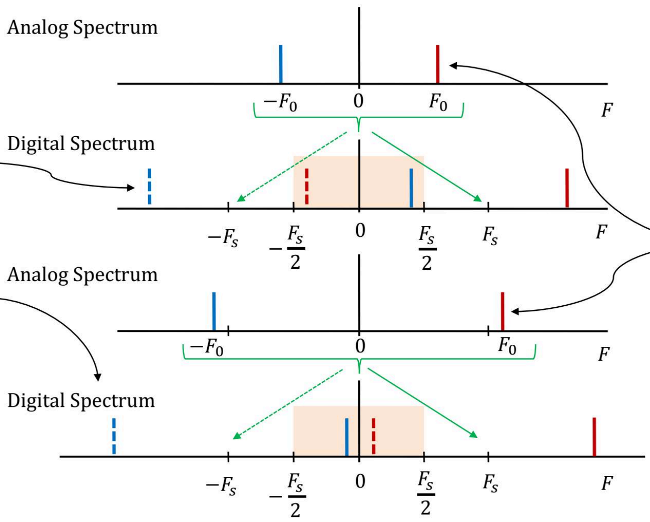
Undersampling: $F_s < 2F_0$

The images move into the fundamental



The baseband components are outside the fundamental range

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Sampling of a Non-Bandlimited Signal

$$x_a(t) = e^{-A|t|} \overset{\mathcal{F}}{\Leftrightarrow} X_a(F) = \frac{2A}{A^2 + (2\pi F)^2}, \; A > 0$$

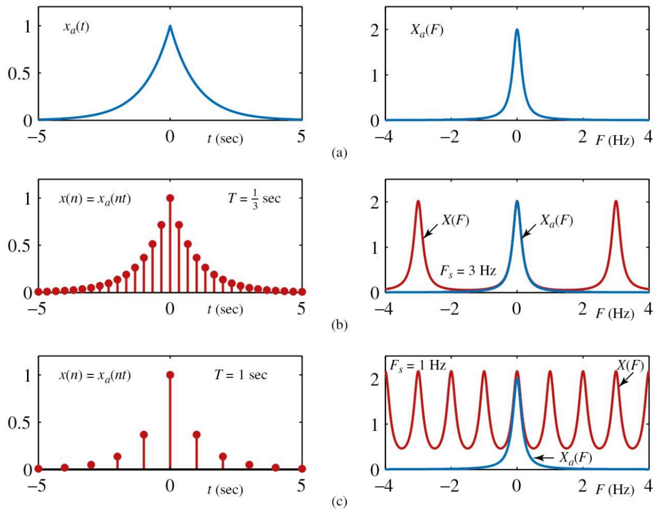
$$x(n) = x_a(nT) = e^{-AT|n|} = (e^{-AT})^{|n|} = a^{|n|}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

$$\omega = 2\pi \frac{F}{F_s} = 2\pi f$$

$$X(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$

$$\hat{x}_a(t) = \mathcal{F}^{-1}\{\hat{X}_a(F) = X(F) G_a(F)\}$$



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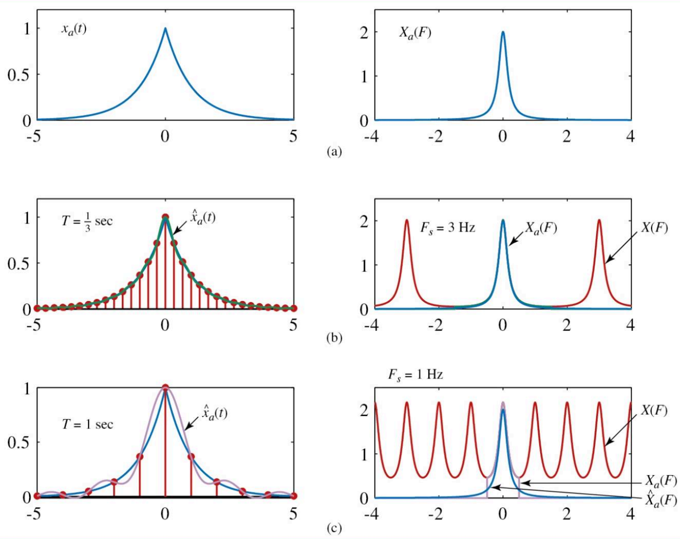
Reconstruction of a Non-Bandlimited Signal



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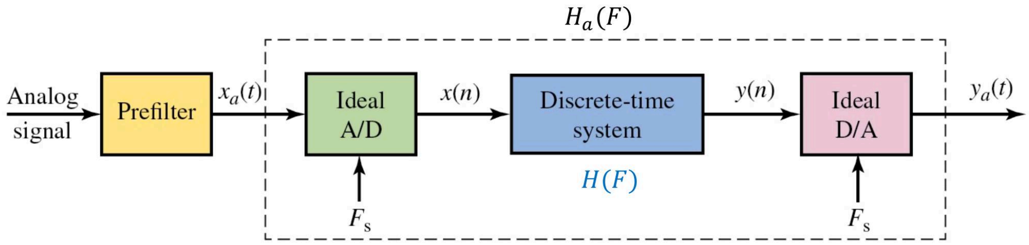
Discrete-time Processing of Continuous-time Signals



If $x_a(t)$ is bandlimited to B Hz and $F_s > 2B \Rightarrow X(F) = \frac{1}{T} X_a(F)$ for $|F| < \frac{F_s}{2}$

Discrete-time filter

$$Y(F) = H(F)X(F) \Rightarrow Y_a(F) = \begin{cases} H(F)X_a(F), & |F| \leq F_s/2 \\ 0, & |F| > F_s/2 \end{cases}$$

Equivalent analog filter

$$H_a(F) = \begin{cases} H(F), & |F| \leq F_s/2 \\ 0, & |F| > F_s/2 \end{cases}$$

Note: In this case the cascade connection of LTV-LTI-LTV = LTI!

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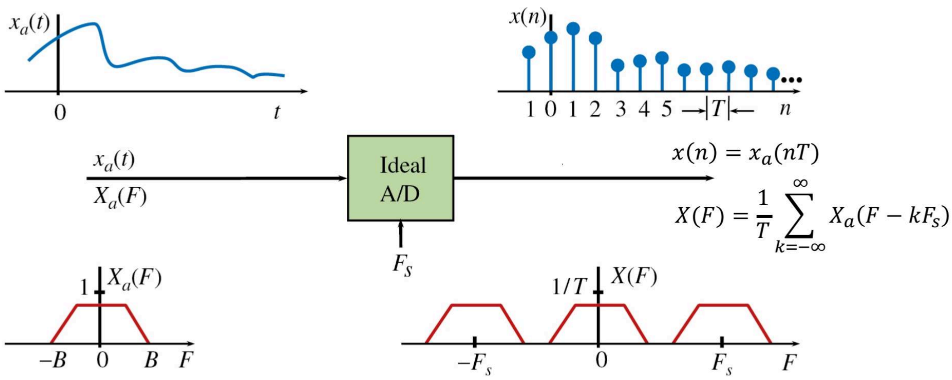
Ideal Analog-to-Digital Converter



- Scales the analog spectrum by the $F_s = 1/T$ factor
- Creates a periodic repetition of the scaled spectrum with fundamental period F_s

The ideal A/D converter is linear, time-varying system

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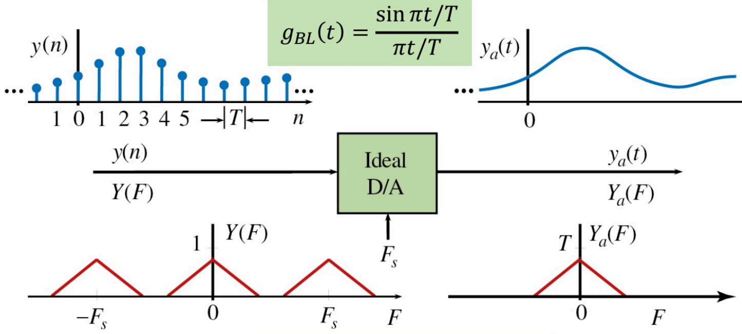
Ideal Digital-to-Analog Converter

$$g_{BL}(t) = \frac{\sin \pi t/T}{\pi t/T}$$



This is not a convolution!

$$y_a(t) = \sum_{n=-\infty}^{\infty} y(n) g_{BL}(t - nT)$$

$$Y_a(F) = G_{BL}(F) Y(F)$$

$$G_{BL}(F) = \begin{cases} T, & |F| \le F_s/2 \\ 0, & |F| > F_s/2 \end{cases}$$

The ideal D/A converter is a linear, time-varying system!

- Removes the frequency components for $|F| > F_s/2$
- Scales the digital spectrum by a factor of $T = 1/F_s$

The ideal D/A converter is not an ideal low-pass filter!

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Simulation of an Analog Integrator

$$RC \frac{dy_a(t)}{dt} + y_a(t) = x_a(t)$$

$$H_a(F) = \frac{Y_a(F)}{X_a(F)} = \frac{1}{1 + jF/F_c}$$

$$F_c = 1/(2\pi RC)$$

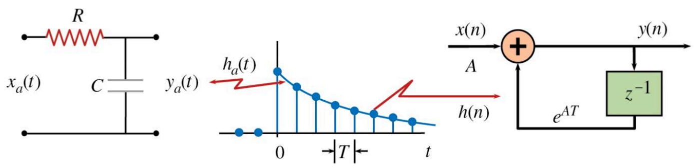
$$h_a(t) = Ae^{-At}u(t), \qquad A = \frac{1}{RC}$$

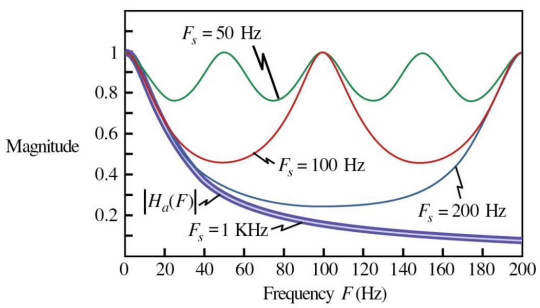
Impulse response sampling

$$h(n) = h_a(nT) = A(e^{-AT})^n u(n)$$

$$H(z) = \frac{A}{1 - e^{-AT} z^{-1}}$$

$$y(n) = e^{-AT} y(n-1) + Ax(n)$$





The approximation is satisfactory when the bandwidth of the input signal is much less than the sampling frequency

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Ideal Bandlimited Differentiator

Continuous-Time

$x_a(t) = \int X_a(F) e^{j2\pi Ft} dF \Rightarrow$

$$y_a(t) = \frac{dx_a(t)}{dt} = \int \underbrace{j2\pi F X_a(F)}_{Y_a(F)} e^{j2\pi Ft} dF$$

$$H_a(F) = \frac{Y_a(F)}{X_a(F)} = j2\pi F, \qquad -\infty < F < \infty$$

Ideal BL Differentiator

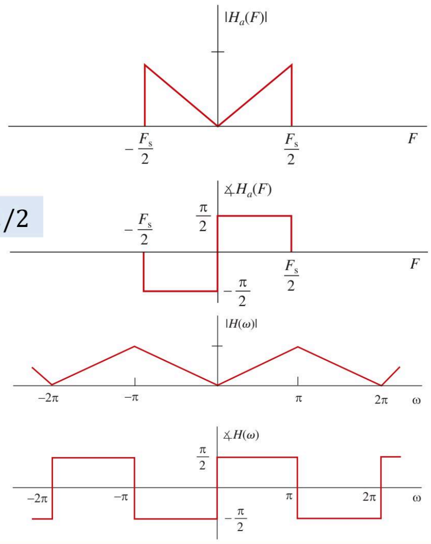
$H_a(F) = 0, |F| > F_s/2$

Discrete-Time

$h(n) = h_a(nT) \qquad H(\omega) = j\frac{\omega}{T}$

$$h(n) = \frac{\pi n \cos \pi n - \sin \pi n}{\pi n^2 T} = \begin{cases} 0, & n = 0 \\ \dfrac{\cos \pi n}{\pi T}, & n \neq 0 \end{cases}$$

Non-causal
Unstable



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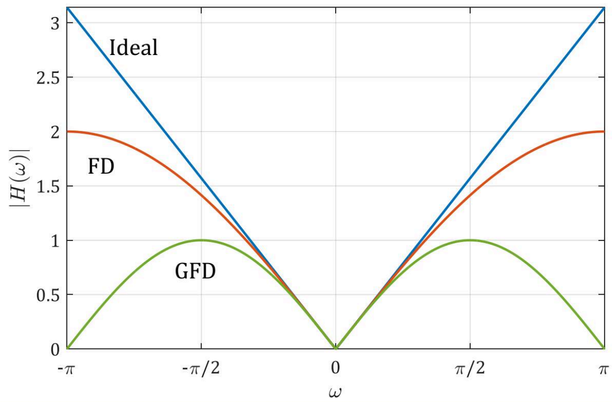
Simple Practical Differentiators

First Difference: $y(n) = x(n) - x(n-1) \Rightarrow H_d(\omega) = 1 - e^{-j\omega}$

Gapped FD: $y(n) = \frac{1}{2}[x(n+1) - x(n-1)] \Rightarrow H_d(\omega) = \frac{1}{2}\left(e^{j\omega} - e^{-j\omega}\right)$



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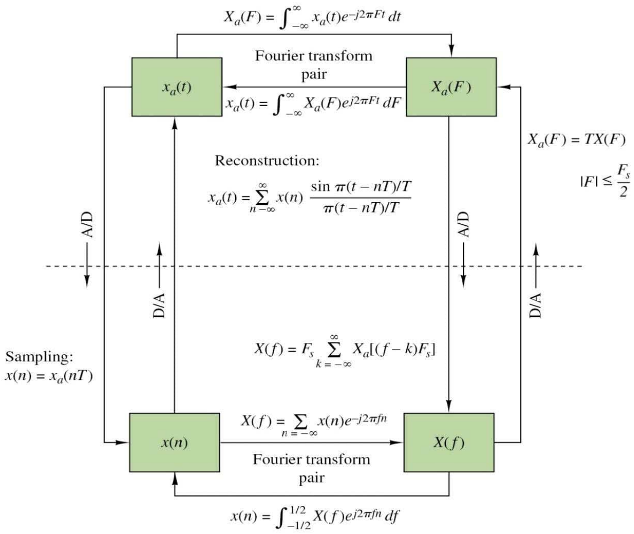
Summary

Sampling in Time with period T $\Rightarrow$ Periodization in Frequency with period $F_s = \frac{1}{T}$



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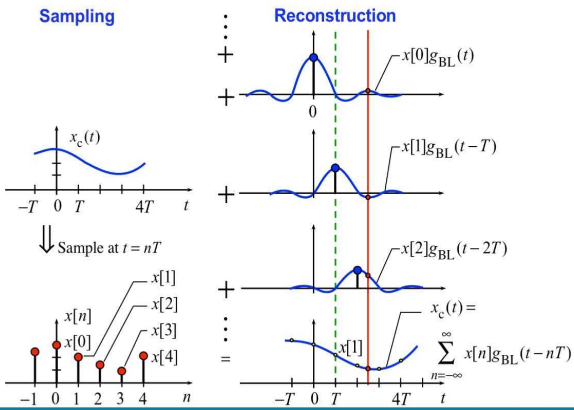
Sampling and Reconstruction



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