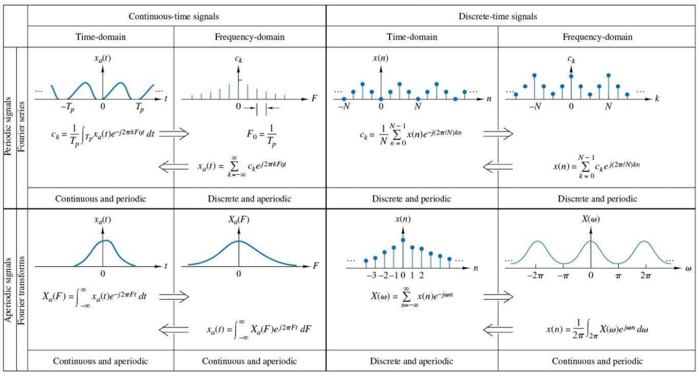
Fourier Analysis of Signals



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Frequency Analysis: Motivation

$$x(n) = \delta(n) \xrightarrow{\text{LTI}} y(n) = h(n)$$

Impulse response sequence
Input changes shape

$$x(n) = e^{j\omega_0 n} \xrightarrow{\text{LTI}} y(n) = ?$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \underbrace{\left(\sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega_0 k}\right)}_{H(\omega_0)} e^{j\omega_0 n}$$

$H(\omega_0)$ Frequency response function

$$x(n) = e^{j\omega_0 n} \xrightarrow{\text{LTI}} y(n) = H(\omega_0)e^{j\omega_0 n}$$

Input does not change shape!

Eigenfunctions Eigenvalues

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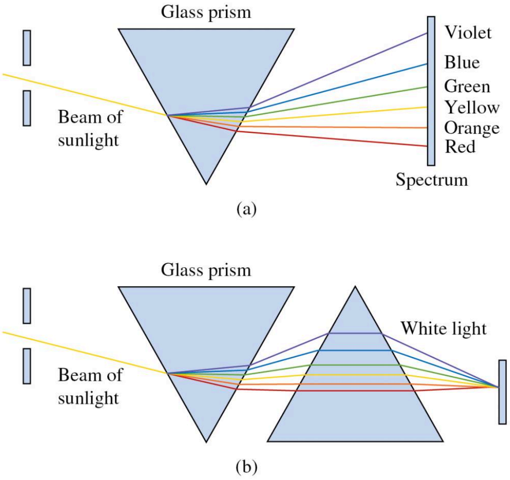
Frequency Analysis

Physics



Signal Processing

- Decomposition of a signal into "frequency components" i.e., complex exponential or sinusoidal signals
- The mathematical tools for frequency analysis depend upon:
  - The nature of time: Continuous or discrete
  - The existence of harmony: Periodic or aperiodic

In both cases we use the same mathematics: Fourier Analysis

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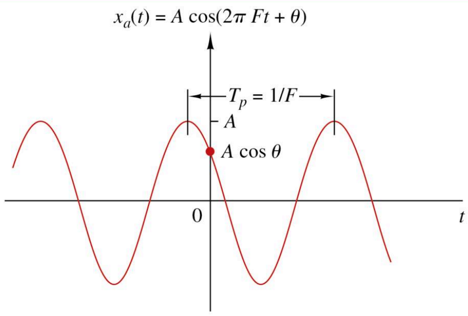
Continuous-Time Sinusoids



$$x_a(t) = A\cos(2\pi Ft + \theta)$$

Amplitude Phase (rads)

Frequency (Hz)

$\Omega = 2\pi F$ (Frequency in rads/sec)

- $x_a(t)$ is periodic for every value of F_0; the fundamental period is $T_p = \frac{1}{F_0}$
- $F_1 \neq F_2 \Rightarrow x_1(t) \neq x_2(t)$
- F increases $\Rightarrow$ rate of oscillation increases. This a consequence of the continuity of time, since as $T_p \to 0 \Rightarrow F_0 = \frac{1}{T_p} \to \infty$
- The same properties hold for the complex exponentials because $e^{\pm j\phi} = \cos\phi \pm j\sin\phi$

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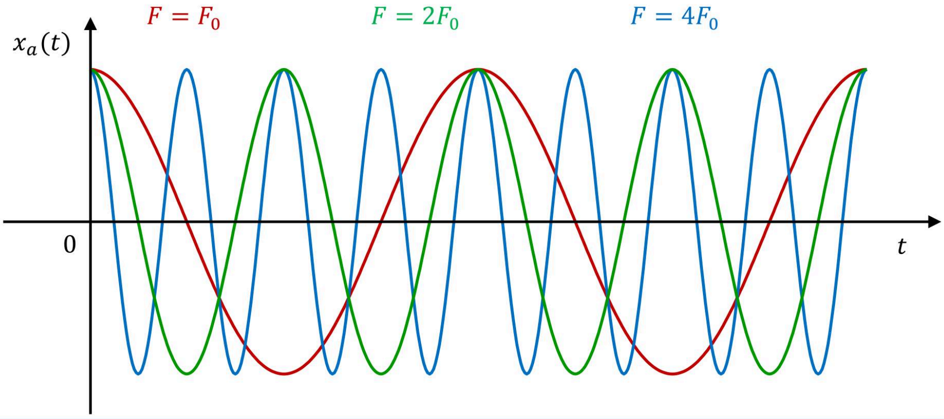
Frequency and Rate of Oscillation

$$x_a(t) = A\cos(2\pi F t + \theta)$$



In continuous-time the rate of oscillation increases with frequency always!

Harmonically Related Complex Exponentials

Consider the complex exponential signals

$$s_k(t) = e^{jk\Omega_0 t} = e^{j2\pi k F_0 t}, k = 0, \pm 1, \dots \quad T_p = \frac{1}{F_0} = \text{Fundamental period}$$

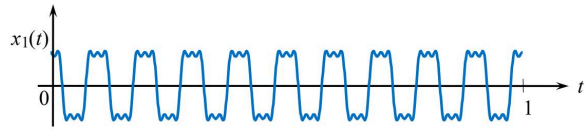
Properties

- All $s_k(t)$ are periodic with period T_p
- $k_1 \neq k_2 \Rightarrow s_{k_1}(t) \neq s_{k_2}(t)$
  - $\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$ is periodic with $T_p = \frac{1}{F_0}$
    - F_0 = fundamental frequency
    - $kF_0, k \neq 0 \Rightarrow$ harmonics
- Can we synthesize non-periodic signals?
- Yes! How? Break harmony! How?

Harmonic and Unharmonic Signals

Harmonic or periodic signals: $F_k = kF_0$

$$x_1(t) = \frac{1}{3}\cos(2\pi F_0 t) - \frac{1}{10}\cos(2\pi 3F_0 t) + \frac{1}{20}\cos(2\pi 5F_0 t) \qquad F_0 = 10 \text{ Hz}$$

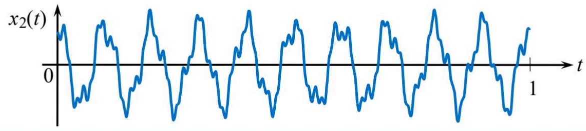


Periodic

Non-harmonic or quasi-periodic signals: $F_k \neq kF_0$

$$x_1(t) = \frac{1}{3}\cos(2\pi F_0 t) - \frac{1}{10}\cos\left(2\pi\sqrt{8}F_0 t\right) + \frac{1}{20}\cos\left(2\pi\sqrt{51}F_0 t\right)$$

Some frequencies are irrational numbers



Quasi-periodic

To synthesize aperiodic signals use frequencies on a continuous range!

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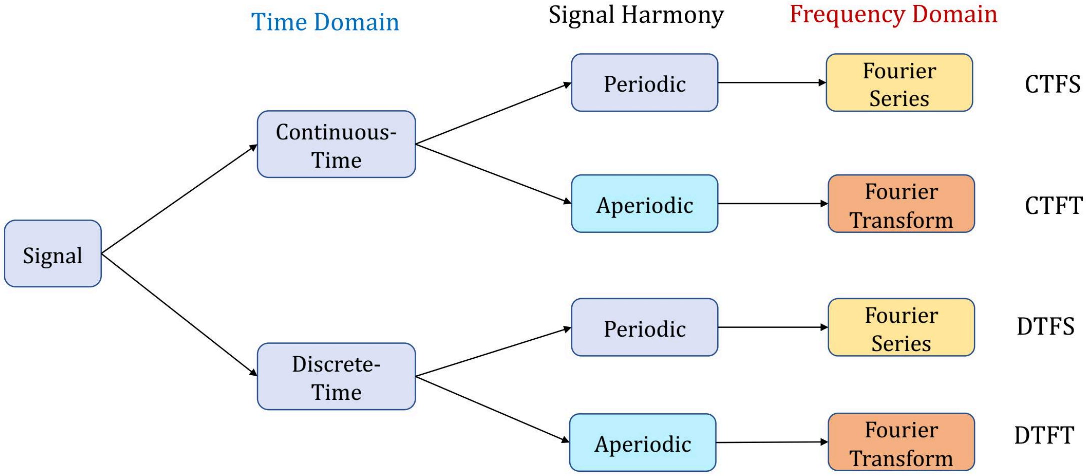
Taxonomy of Fourier Representations



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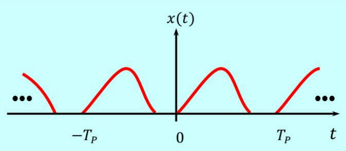
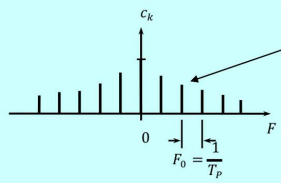
Continuous-Time Fourier Series

Time−Domain	Frequency−Domain
	
Continuous and Periodic	Discrete and Aperiodic

Discrete or line spectrum

Time ⇒ Frequency

Direct Transform or Analysis Equation

$$c_k = \frac{1}{T_p}\int_{T_p} x(t)\, e^{-j2\pi k F_0 t} dt \Rightarrow$$

We integrate on any interval of length T_p

Frequency ⇒ Time

Inverse Transform or Synthesis Equation

$$\Leftarrow \quad x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

$$P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad \text{(Parseval's Relation)}$$

$$\{|c_k|^2, -\infty < k < \infty\} \quad \text{(Power Spectrum)}$$

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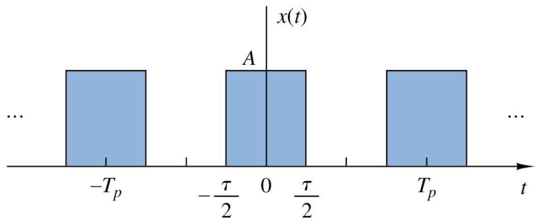
Rectangular Pulse Train

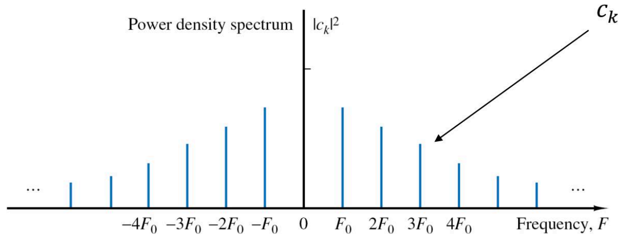


$$c_k = \frac{1}{T_p}\int_{-\tau/2}^{\tau/2} Ae^{-j2\pi k F_0 t} dt = \frac{A}{T_p}\left[\frac{e^{-j2\pi k F_0 t}}{-j2\pi k F_0}\right]_{\tau/2}^{\tau/2}$$

$$= \frac{A}{\pi F_0 k T_p}\frac{e^{j\pi k F_0 t} - e^{-j\pi k F_0 t}}{2j} \Rightarrow$$

$$c_k = \frac{A\tau}{T_p}\frac{\sin(\pi k F_0 \tau)}{(\pi k F_0 \tau)}, k = 0, \pm 1, \pm 2, \ldots$$

Samples of sinc function



Continuous-time periodic signals have discrete spectra

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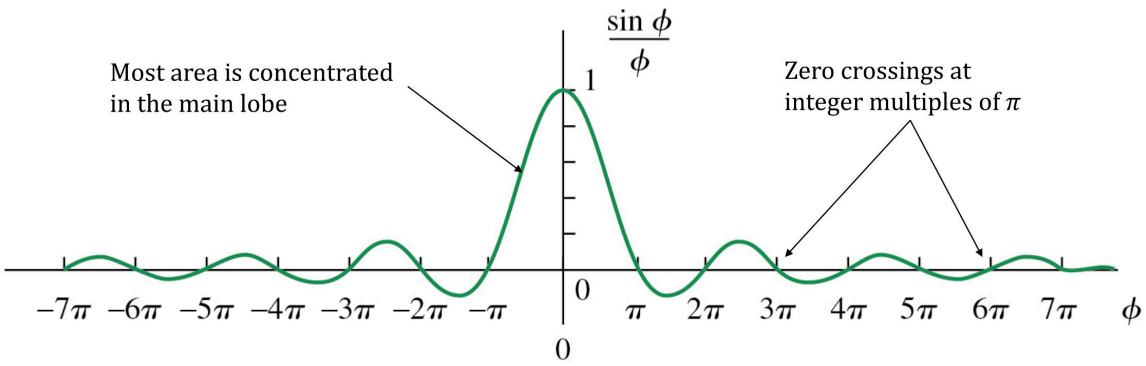
The Sinc Function



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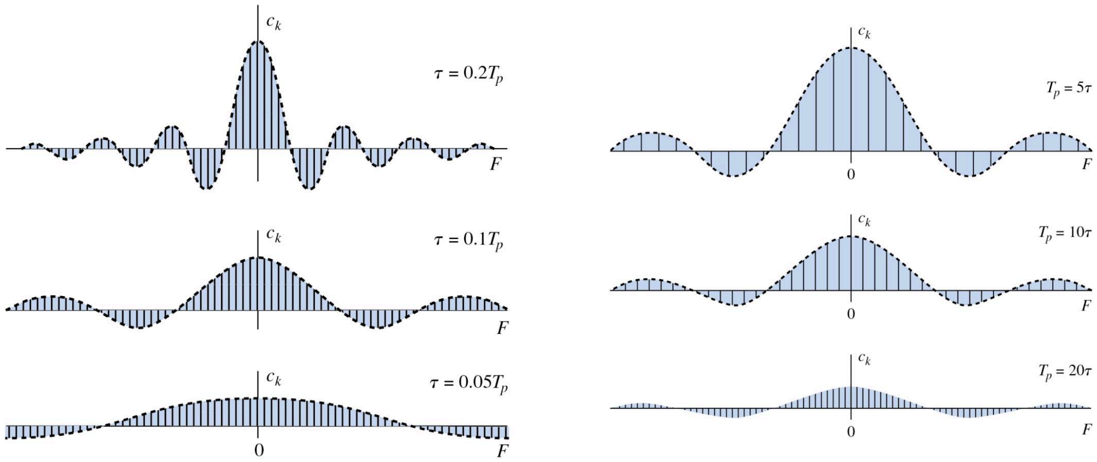
Spectrum of Rectangular Pulse Train

Fixed period – Varying Pulse Width

Fixed Pulse Width – Varying Period



"Time-width" × Bandwidth ≈ Constant

Spectral-line "packing"

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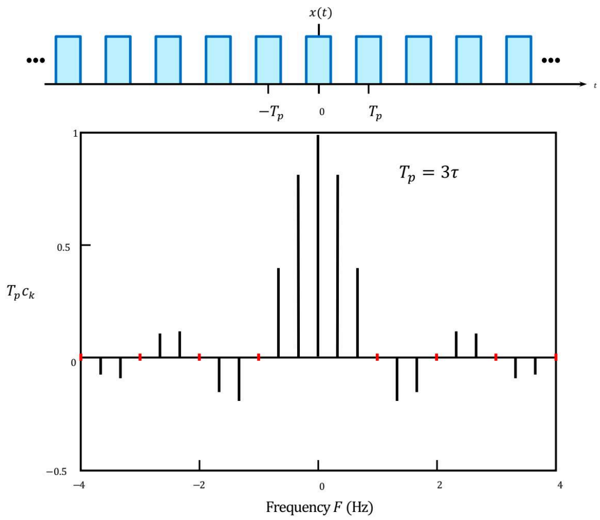
From CTFS to CTFT



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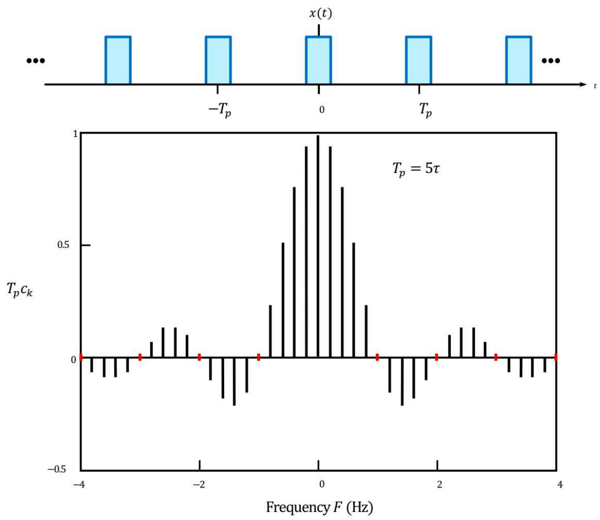
From CTFS to CTFT



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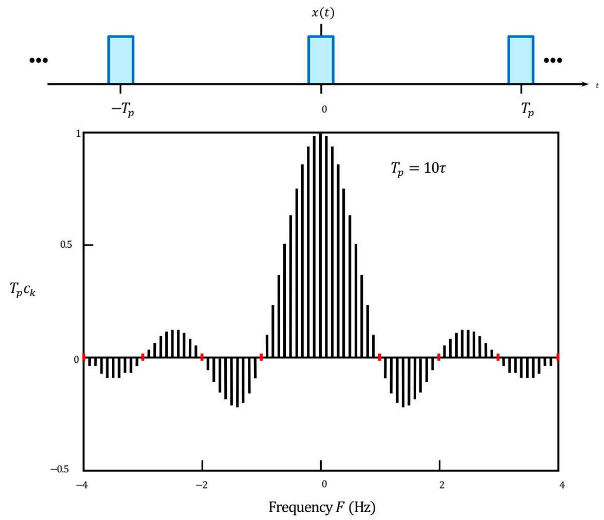
From CTFS to CTFT



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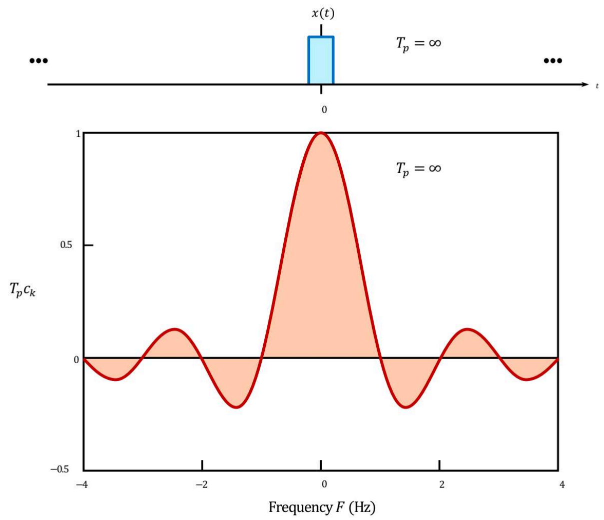
From CTFS to CTFT



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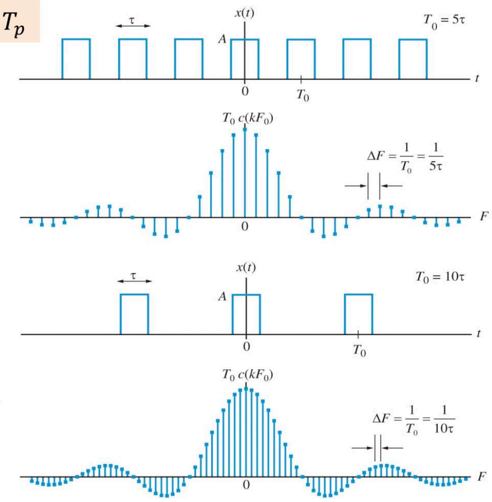
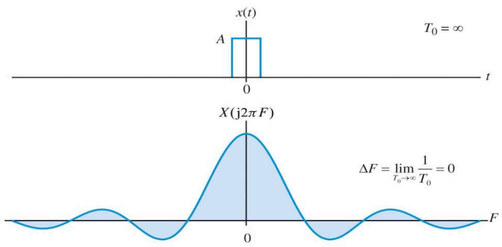
Fourier Series → Fourier Transform

$T_0 = T_p$





- If $x(t)$ is T_p-periodic we need $F_k = kF_0$ (harmonics) in $s(t) = e^{j2\pi Ft}$
- If $x(t)$ is aperiodic we need all frequencies F

$$T_p c_k = \int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} x(t) e^{-j2\pi k F_0 t} dt \Rightarrow X(F) = \int_{-\infty}^{\infty} x(t) e^{j2\pi Ft} dt$$

$$x(t) = \sum_{-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \Rightarrow x(t) = \int_{-\infty}^{\infty} [X(F) dF] e^{j2\pi Ft}$$

$$c_k \to X(F)\, dF$$

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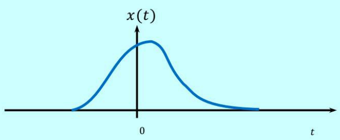
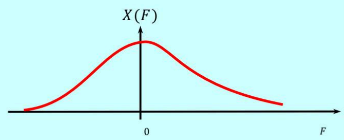
Continuous-Time Aperiodic Signals: Fourier Transform

Time−Domain	Frequency−Domain
	
$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi Ft} dt$ ⇒	⇐ $x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$
Continuous and Aperiodic	Continous and Aperiodic

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF \quad \text{(Parseval's Relation)}$$

$$S_x(F) = |X(F)|^2 \geq 0 \quad \text{(Energy Spectral Density)}$$

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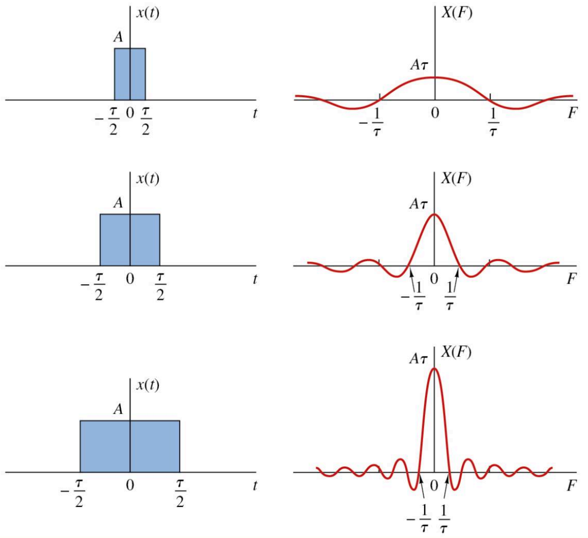
Fourier Transform of Square Pulse



$$x(t) = \begin{cases} A, & |t| \le \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

$$X(F) = A\tau \frac{\sin(\pi F \tau)}{\pi F \tau}$$

Zero crossings at multiples of $1/\tau$

Main lobe (width$\approx 2/\tau$) contains most of signal energy

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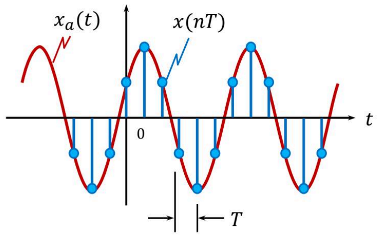
Discrete-Time Sinusoidal Signals



$$x_a(t) = A\cos(\Omega T + \theta) = A\cos(2\pi F t + \theta)$$

$$x(n) = x_a(nT), \qquad T = \text{Sampling period}$$

$$F_s = \frac{1}{T} = \text{Sampling frequency}$$

$$x(n) = x_a(nT) = A\cos(2\pi F n T + \theta) = A\cos\left(2\pi \frac{F}{F_s} n + \theta\right)$$

$f = \frac{F}{F_s}$ Normalized cyclic frequency

$\omega = 2\pi f$ Normalized angular frequency

$x(n) = A\cos(2\pi f n + \theta) = A\cos(\omega n + \theta)$ Discrete-time sinusoids

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Time and Frequency Variables and Units

	Time		Frequency			
	Symbol	Unit	Symbol	Unit	Symbol	Unit
Continuous	t	sec	F	Hz or $\frac{\text{cycles}}{\text{sec}}$	$\Omega = 2\pi F$	$\frac{\text{rads}}{\text{sec}}$
Discrete	nT	sec	F	Hz or $\frac{\text{cycles}}{\text{sec}}$	$\Omega = 2\pi F$	$\frac{\text{rads}}{\text{sec}}$
Discrete	n	Sample	$f = \frac{F}{F_s}$	$\frac{\text{cycles}}{\text{sample}}$	$\omega = 2\pi f$	$\frac{\text{rads}}{\text{sample}}$

Important Note: Per “sample” is used to mean per “sampling interval.”

Periodicity in Time and Frequency

The sequence $x(n) = A\cos(2\pi f_0 n + \theta)$ is periodic in n (time) iff $f_0 = k/N$, i.e., f_0 is a rational number

Proof

$$x(n) = x(n+N), \text{for all } n$$
$$x(n+N) = A\cos(2\pi f_0 n + 2\pi f_0 N + \theta)$$
$$\text{If } 2\pi f_0 N = k2\pi, \quad k = \text{integer} \Rightarrow$$
$$x(n+N) = A\cos(2\pi f_0 n + \theta) = x(n)$$

Physical Meaning

$$f_0 = \frac{F_0}{F_s} = \frac{k}{N} = \frac{1/T_p}{1/T} = \frac{T}{T_p} \Rightarrow NT = kT_p$$

Period of DT sinusoid in seconds equals an integer number of periods of the CT sinusoid

If k, N are prime $\Rightarrow N =$ fundamental period

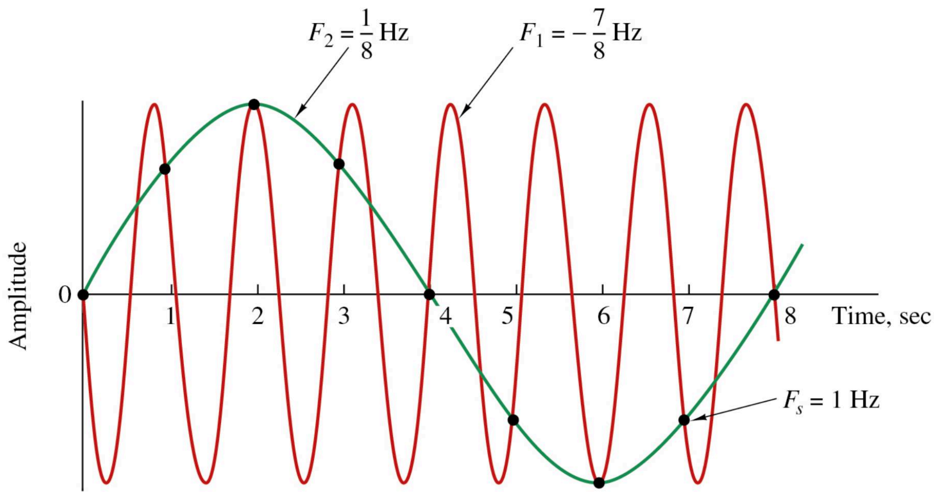
The sequence $x(n) = A\cos(\omega n + \theta)$ is always periodic in ω (frequency) with period 2π

Proof

$$\cos[(\omega + k2\pi)n + \theta] = \cos(\omega n + k2\pi n + \theta)$$
$$= \cos(\omega n + \theta)$$

- Sinusoidal sequences with frequencies separated by integer multiples of 2π are identical (aliasing)
- All distinct sinusoidal sequences have frequencies in an interval of 2π radians
- We use the ranges: $-\pi < \omega \le \pi$ or $0 \le \omega < 2\pi$
- Time shift $\Rightarrow$ phase change: $A\cos(\omega(n + n_0)) = A\cos(\omega n + \omega n_0)$

Illustration of Aliasing



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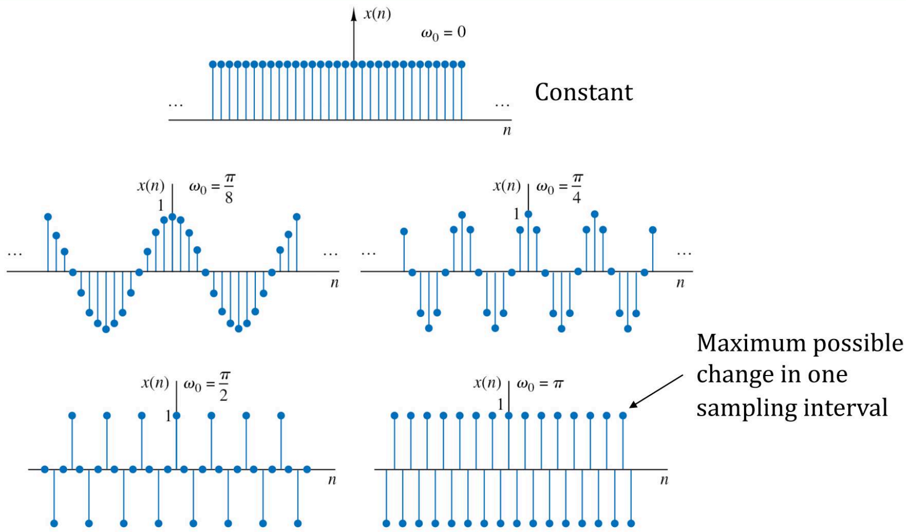
Frequency and Rate of Oscillation



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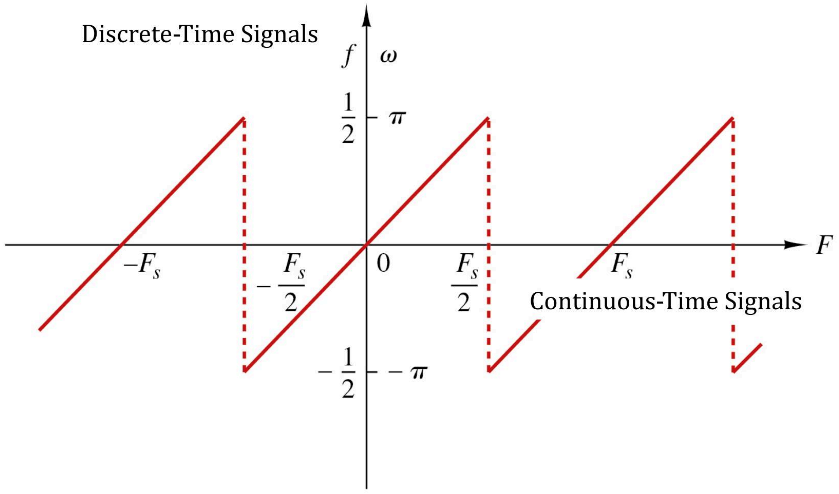
Frequency Ranges in Continuous-Time and Discrete-Time



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Relations Among Frequency Variables

Continuous-Time Signals		**Discrete-Time Signals**	
$\Omega = 2\pi F$		$\omega = 2\pi f$	
$\frac{\text{radians}}{\text{sec}}$	Hz	$\frac{\text{radians}}{\text{sample}}$	$\frac{\text{cycles}}{\text{sample}}$

$$\omega = \Omega T, f = F/F_s \quad \Longrightarrow$$

$$\Longleftarrow \quad \Omega = \omega/T, F = f \cdot F_s$$

Continuous-time:

$$-\infty < \Omega < \infty$$
$$-\infty < F < \infty$$

Discrete-time:

$$-\pi \le \omega \le \pi$$
$$-\frac{1}{2} \le f \le \frac{1}{2}$$
$$-\frac{\pi}{T} \le \Omega \le \frac{\pi}{T}$$
$$-\frac{F_2}{2} \le F \le \frac{F_s}{2}$$

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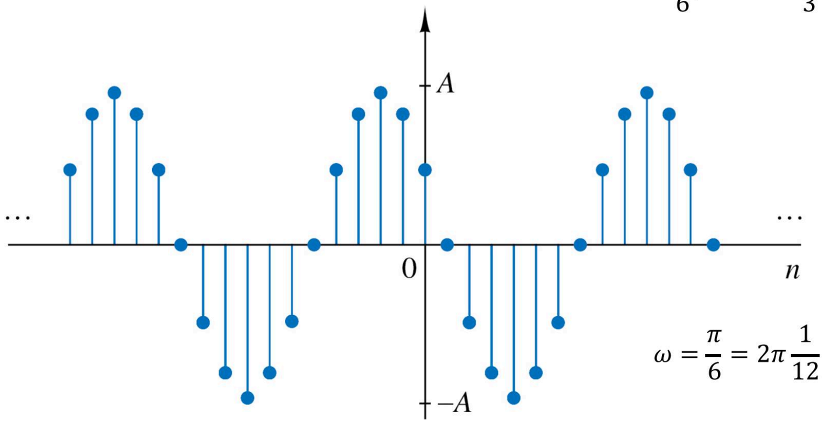
Example of Periodic Sinusoidal Sequence

$$x(n) = A\cos(\omega n + \theta) \qquad \omega = \frac{\pi}{6}, \qquad \theta = \frac{\pi}{3}$$



$$\omega = \frac{\pi}{6} = 2\pi\frac{1}{12} \Rightarrow f = \frac{1}{12} \Rightarrow N_p = 12$$

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Harmonically Related Complex Exponential Sequences

Consider the complex exponentials $s_k(n) = e^{j\omega_k n} = \cos\omega_0 n + j\sin\omega_0 n, \quad -\infty < n < \infty$

If $\omega_k = \omega_0 + k2\pi$, where $-\pi < \omega_0 \le \pi$ or $0 \le \omega_0 < 2\pi \Rightarrow s_k(n) = s_0(n)$ for all k

We need only complex exponentials in the fundamental ranges $-\pi < \omega_0 \le \pi$ or $0 \le \omega_0 < 2\pi$

If $f_k = \frac{k}{N}$ (rational), then $s_k(n)$ is periodic in k and n with fundamental period $N \Rightarrow$

$$s_k(n) = s_k(n+N) \text{ iff } f_k = \frac{k}{N} \Rightarrow s_k(n) = e^{j\frac{2\pi}{N}kn}$$

$$s_{k+N}(n) = e^{j\frac{2\pi}{N}(k+N)n} = e^{j\frac{2\pi}{N}kn} = s_k(n)$$

There are only N distinct harmonically related complex exponentials

$$s_k(n) = e^{j\frac{2\pi}{N}kn}, \qquad k = 0,1,\ldots,N-1$$

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Sum of Discrete Harmonic Complex Exponentials

Orthogonality property

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn} = \begin{cases} N, & k = 0, \pm 1, \pm 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

For any complex number a we have

$$S = \sum_{n=0}^{N-1} a^n = \begin{cases} N, & a = 1 \\ \dfrac{1 - a^N}{1 - a}, & a \neq 1 \end{cases}$$

Let $a = e^{j\frac{2\pi}{N}k}$. Then we have two cases:

If $k = \ell N$ then $a = 1$ and $S = N$

If $k \neq \ell N$ then $a \neq 1$, $a^N = e^{j2\pi k} = 1$, and $S = 0$

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$$

Periodic in n with fundamental period N

Multiply both sides by $e^{-j\frac{2\pi}{N}mn}$ and sum over $n \Rightarrow$

$$\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}mn} = \sum_{k=0}^{N-1} c_k \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}$$

Using the orthogonality property yields

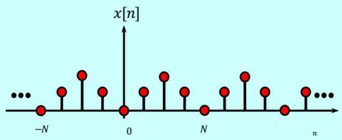
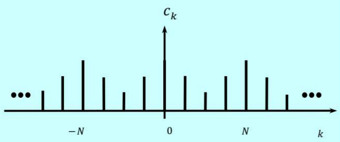
$$\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}mn} = c_m N$$

Replacing m by $k \Rightarrow$

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

Periodic in k with fundamental period N

Discrete-Time Periodic Signals: Fourier Series

Time−Domain	Frequency−Domain
	
$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$ $\Rightarrow$	$\Leftarrow$ $x(n) = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$
Discrete and Periodic	Discrete and Periodic

We can sum over any interval of N consecutive samples or coefficients

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |c_k|^2 \quad \text{(Parseval's Relation)}$$

$\{|c_k|^2, 0 \leq k \leq N-1 \text{ or } -\infty < k < \infty\}$ (Power−Spectral Density)

Numerical Computation of DTFS

Formula	MATLAB Function
$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$	`c = (1/N)*fft(x)`
$x(n) = \sum_{k=0}^{N-1} c_k\, e^{j\frac{2\pi}{N}kn}$	`x = N*ifft(c)`

- MATLAB assumes that: $x = [x(0)\ x(1) \cdots x(N-1)]$
- The `fft` or `ifft` functions are efficient implementations of the corresponding summations scaled by $(1/N)$ and N
- When c_k or $x(n)$ are real, we should use only the real part of $\boldsymbol{c}$ or $\boldsymbol{x}$. The imaginary parts, due to numerical accuracy limitations, are not zero; they take small values in the range $\pm 10^{-16}$

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Example: Sinusoidal Sequence

Compute the Fourier series of
$x(n) = \cos 2\pi f_0 n,$ all n

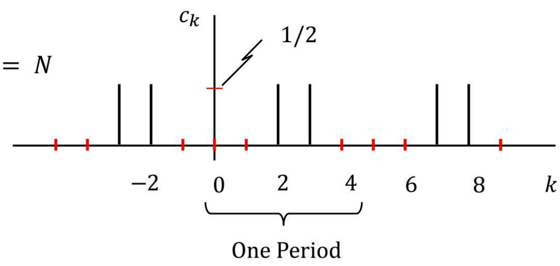
$x(n)$ is periodic iff $f_0 = \frac{k_0}{N}$

If k_0, N relative primes ⇒ fundamental period $= N$

$$x(n) = \frac{1}{2}e^{j\frac{2\pi}{N}k_0 n} + \frac{1}{2}e^{-j\frac{2\pi}{N}k_0 n} \Rightarrow$$

$$c_{-k_0} = \frac{1}{2},\ c_{k_0} = \frac{1}{2} \text{ or } c_{k_0} = \frac{1}{2},\ c_{N-k_0} = \frac{1}{2}$$

$k_0 = 2, N = 5 \Rightarrow$



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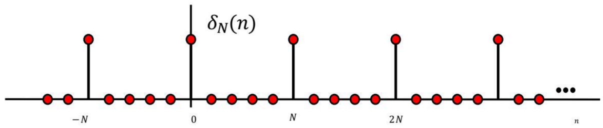
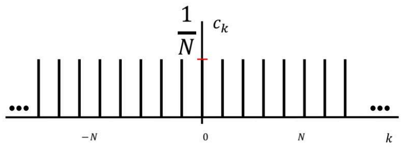
Example: Periodic Impulse Train

$$\delta_N(n) = \sum_{\ell=-\infty}^{\infty} \delta(n - \ell N)$$



$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} \delta_N(n) e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{n=0}^{N-1} \delta(n) e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}, \text{ all } k$$

$$\delta_N(n) = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn}, \text{ all } n$$



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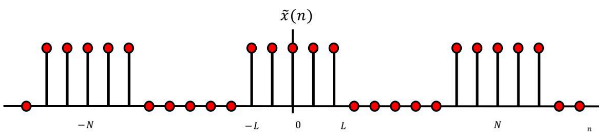
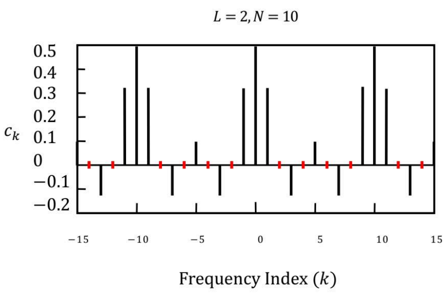
Example: Rectangular-Pulse Train



$$c_k = \frac{1}{N}\sum_{n=-L}^{L} \tilde{x}(n) e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} e^{j\frac{2\pi}{N}kL} \sum_{n=0}^{2L} e^{j\frac{2\pi}{N}kn}$$



$$c_k = \begin{cases} \dfrac{2L+1}{N}, & k = 0, \pm N, \pm 2N, \ldots \\ \dfrac{1}{N}\dfrac{\sin\left[\frac{2\pi}{N}k\left(L+\frac{1}{2}\right)\right]}{\sin\left(\frac{2\pi}{N}k\frac{1}{2}\right)}, & \text{otherwise} \end{cases}$$

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What Happens when L = fixed and $N \to \infty$?

Consider the scaled Fourier coefficients (otherwise $c_k \to 0$ as $N \to \infty$)

$$Nc_k = \begin{cases} 2L+1, & k = 0, \pm N, \pm 2N, \dots \\ \dfrac{\sin\left[\frac{2\pi}{N}k\left(L+\frac{1}{2}\right)\right]}{\sin\left(\frac{2\pi}{N}k\frac{1}{2}\right)}, & \text{otherwise} \end{cases}$$

Keep L fixed, increase N, and plot Nc_k as a function of the frequency variable

$$f = \frac{\omega}{2\pi} = \frac{k}{N}\frac{\text{cycles}}{\text{sampling interval}}$$

Note: As $N \to \infty$ the sequence becomes aperiodic!

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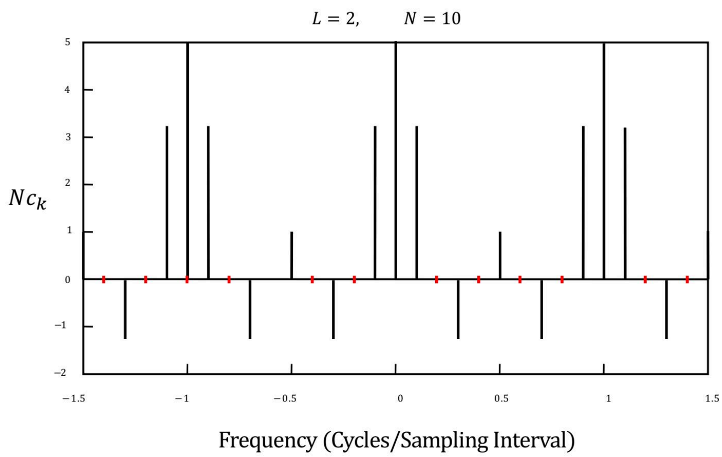
From DTFS to DTFT



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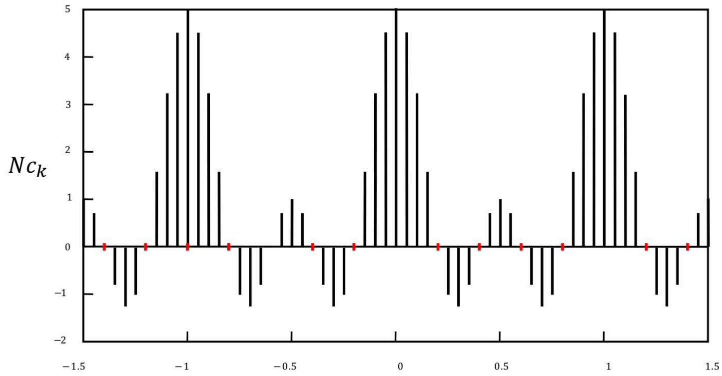
From DTFS to DTFT

$L = 2, \quad N = 20$



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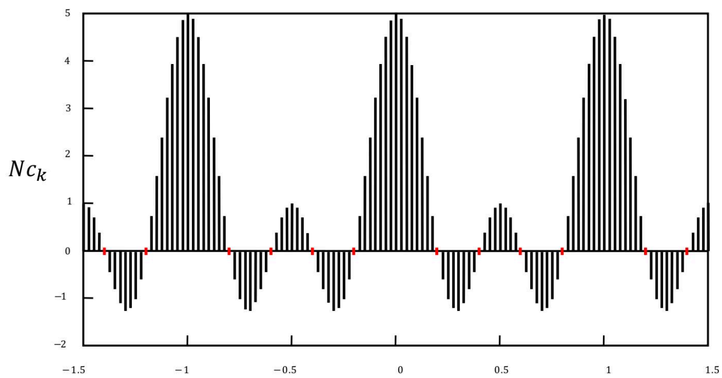
From DTFS to DTFT

$L = 2, \quad N = 40$



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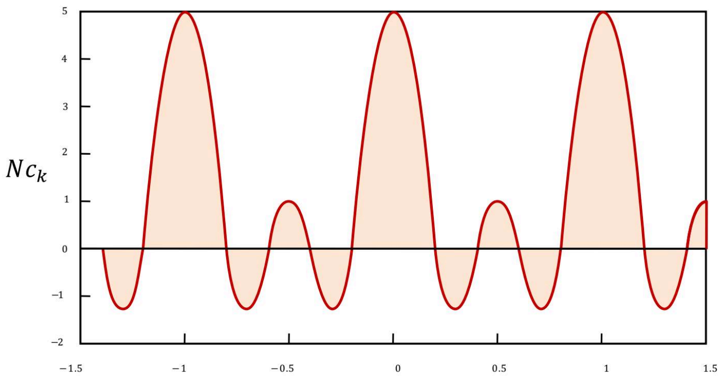
From DTFS to DTFT

$L = 2, \qquad N = \infty$



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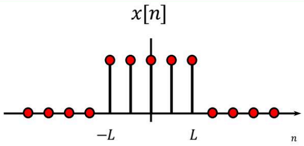
From DTFS to DTFT



$$x(n) = \begin{cases} 1, & -L \le n \le L \\ 0, & \text{otherwise} \end{cases}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-L}^{L} e^{-j\omega n} = \frac{\sin \omega \left(L + \frac{1}{2}\right)}{\sin(\omega/2)}$$

$$\Rightarrow X(\omega)\Big|_{\omega = \frac{2\pi}{N}k} = Nc_k$$

$X(\omega + 2\pi) = X(\omega)$ (Periodic function of ω) $\Rightarrow$

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

i.e., $x(n)$ are the Fourier coefficients of $X(\omega)$

If $x(n)$ is one period of the periodic sequence $\tilde{x}(n)$ $\Rightarrow$

$$c_k = \frac{1}{N} X(\omega)\Big|_{\omega = \frac{2\pi}{N}k}$$

The Fourier coefficients of $\tilde{x}(n)$ are samples of the Fourier transform of $x(n)$ at $\omega = \frac{2\pi}{N}k$

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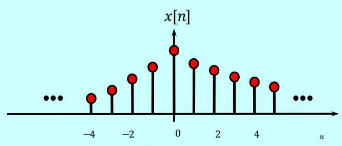
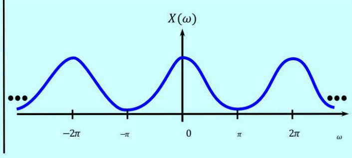
Discrete-Time Aperiodic Signals: Fourier Transform

Time−Domain	Frequency−Domain
	
Discrete and Aperiodic	Continous and Periodic

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \Rightarrow$$

$$\Leftarrow \quad x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega \quad \text{(Parseval's Relation)}$$

$$S_x(\omega) = |X(\omega)|^2 \geq 0 \quad \text{(Energy Spectral Density)}$$

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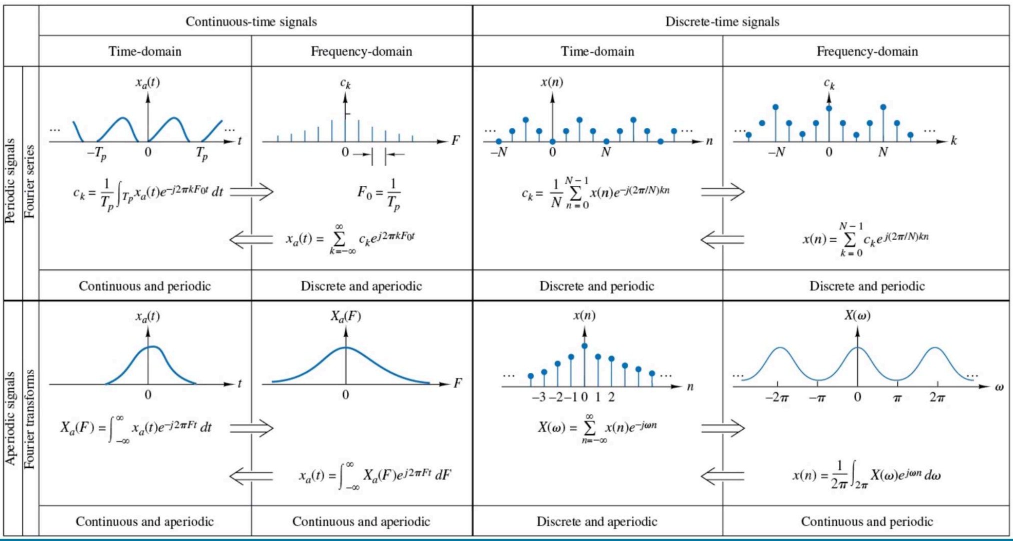
Summary of Fourier Analysis Tools



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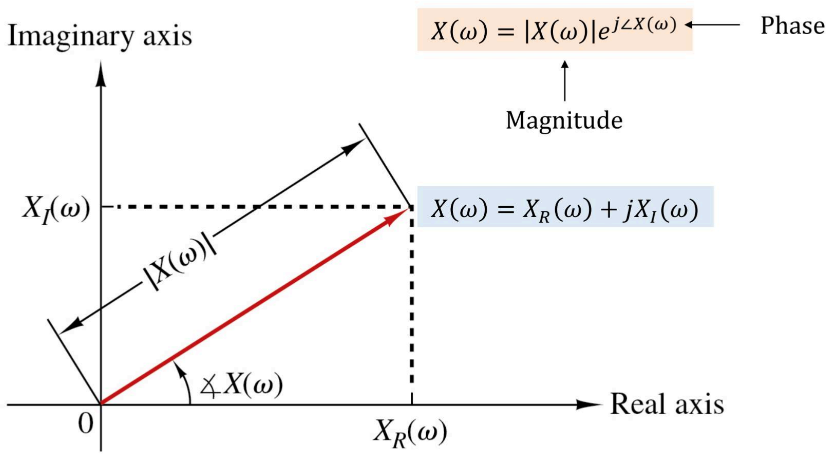
Rectangular and Polar Forms



Discrete Time Fourier Transform (DTFT)

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega \xleftrightarrow{\mathcal{F}} X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \mathcal{F}\{x(n)\}$$

Periodicity $X(\omega) = X(\omega + 2\pi) \Rightarrow$ Use any interval of length 2π for integration

Existence

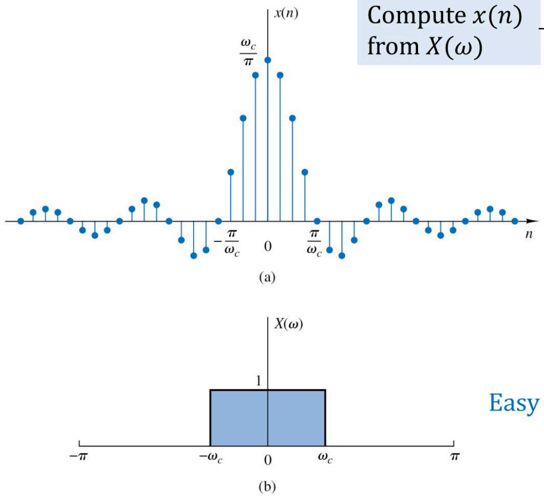
$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty \Rightarrow \text{Absolute Convergence}$$

$$E_x = \sum |x(n)|^2 < \infty \Rightarrow \text{Mean Square Convergence}$$

Differences and implications?

Gibb's Effect!

Ideal Low-Pass Sequence



Compute $x(n)$ from $X(\omega)$ → $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{\sin\omega_c n}{\pi n}, \qquad n \neq 0$$

For $n = 0$, direct integration yields $x(0) = \frac{\omega_c}{\pi}$

For convenience we use a single formula

$$x(n) = \frac{\omega_c}{\pi}\frac{\sin\omega_c n}{\omega_c n} = \frac{\sin\omega_c n}{\pi n}$$

Easy ?

$$X(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

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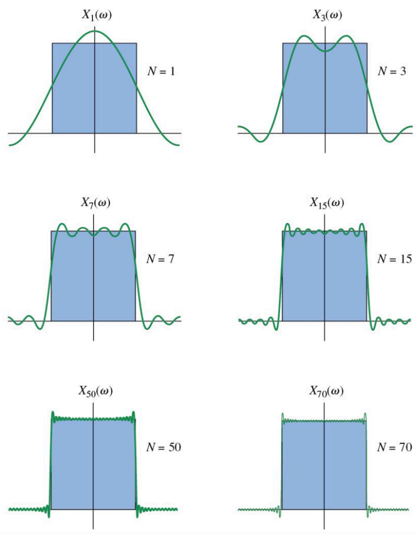
Gibbs Phenomenon

$$x(n) = \frac{\sin\omega_c n}{\pi n}$$

$$\sum_{n=-\infty}^{\infty} |x(n)| = \infty$$

$$\sum_{n=-\infty}^{\infty} |x(n)^2| = \frac{\omega_c}{\pi}$$

$$X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin\omega_c n}{\pi n} e^{-j\omega n}$$



Amplitude of oscillations (ripples) independent of N ⇒ non-uniform convergence

Area of oscillations (ripples) decreases with N ⇒ convergence in the mean-square sense

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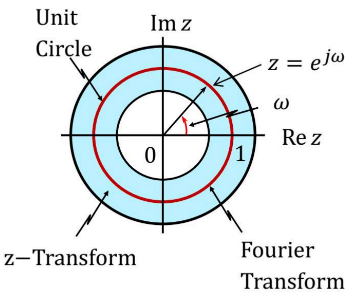
Relationship Between z-Transform and DTFT



$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = X(\omega)$$

$$x(n) = \frac{1}{2\pi j}\oint_{z=e^{j\omega}} X(z) z^{n-1} dz, \qquad dz = je^{j\omega} d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

All properties of z−transform hold for the DTFT (Simpler!)

$z = re^{j\omega} \Rightarrow \mathcal{Z}\{x[n]\} = \mathcal{F}\{x[n]r^n\}$ DTFT exists iff $z = e^{j\omega}$ is inside ROC

There are sequences with DTFT but not ZT and vice−versa

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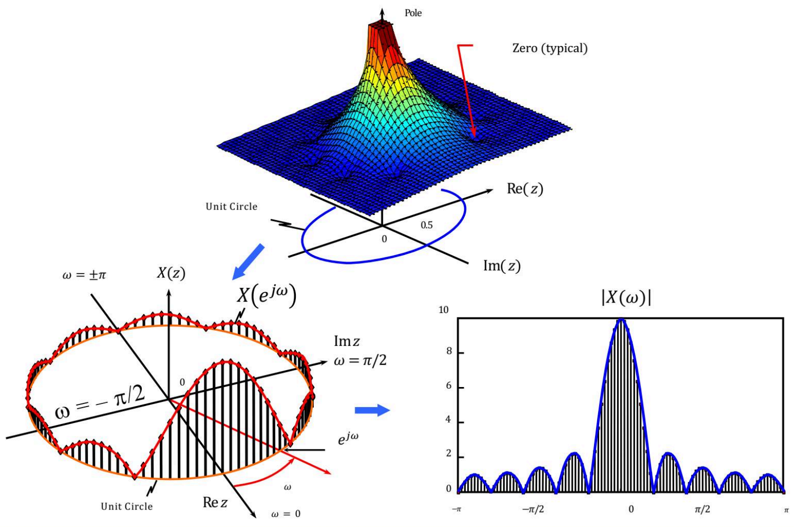
From ZT to DTFT in Pictures!



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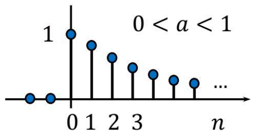
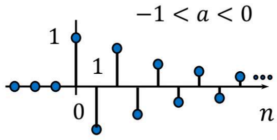
Exponential Sequence: FT

$$x(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$





$$\text{If } |a| < 1 \Rightarrow \sum_{n=-\infty}^{\infty} |x(n)| = \frac{1}{1-|a|} < \infty \Rightarrow \text{The FT exists}$$

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = \frac{1}{1-ae^{-j\omega}} \text{ since } \left|ae^{-j\omega}\right| = |a| < 1$$

Even $\quad X_R(\omega) = \dfrac{1 - a\cos\omega}{1 - 2a\cos\omega + a^2}, \qquad X_I(\omega) = -\dfrac{a\sin\omega}{1 - 2a\cos\omega + a^2}\quad$ Odd

Even $\quad |X(\omega)| = \dfrac{1}{\sqrt{1 - 2a\cos\omega + a^2}}, \qquad \arg X(\omega) = -\tan^{-1}\dfrac{a\sin\omega}{1 - a\cos\omega}\quad$ Odd

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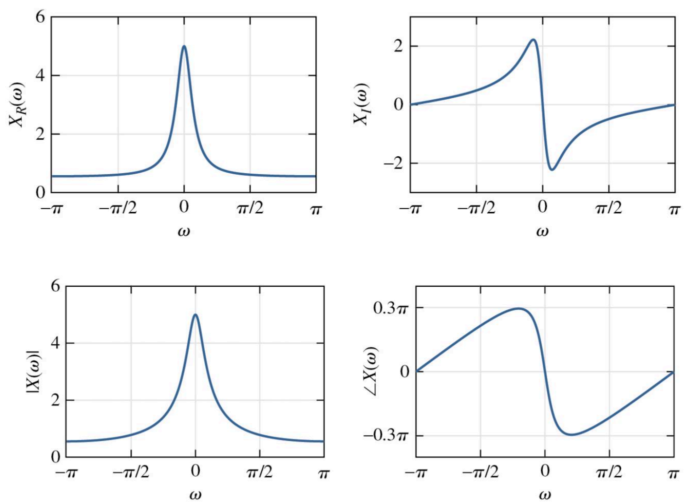
Exponential Sequence: FT Plots



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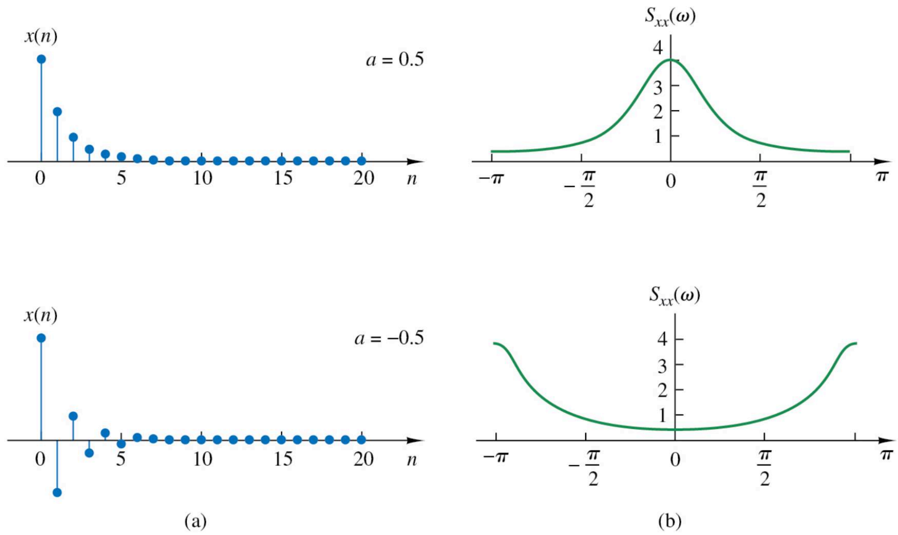
Exponential Sequence: Spectral Densities



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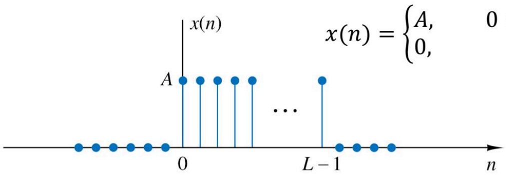
Rectangular Pulse Sequence

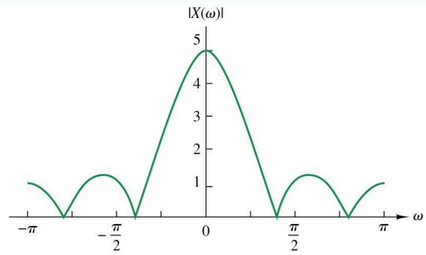


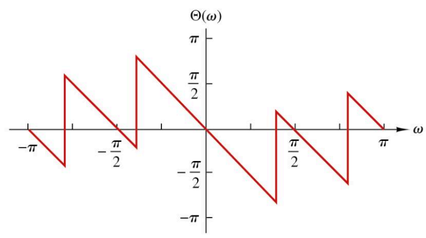
$$x(n) = \begin{cases} A, & 0 \le n \le L-1 \\ 0, & \text{elsewhere} \end{cases}$$

$$X(\omega) = \sum_{n=0}^{L-1} Ae^{-j\omega n} = A\frac{1-e^{-j\omega L}}{1-e^{-j\omega}} = Ae^{-j(\omega/2)(L-1)}\frac{\sin(\omega L/2)}{\sin(\omega/2)} \Rightarrow$$

$$|X(\omega)| = \begin{cases} |A|L, & \omega = 0 \\ |A|\left|\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right|, & \omega \neq 0 \end{cases}$$

$$\angle X(\omega) = \angle A - \frac{\omega}{2}(L-1) + \angle\frac{\sin(\omega L/2)}{\sin(\omega/2)} = \Theta(\omega)$$





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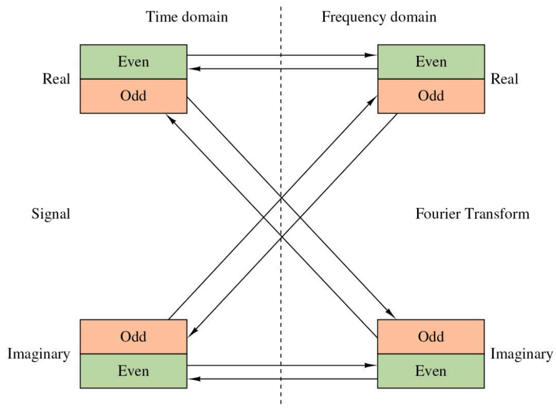
Symmetry Properties of DTFT



Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(\omega)]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}[X(\omega) - X^*(\omega)]$
$x_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_I(\omega)$
Real Signals	
Any real signal $x(n)$	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\|X(\omega)\| = \|X(-\omega)\|$ $\angle X(\omega) = -\angle X(-\omega)$
$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$ (real and even)	$X_R(\omega)$ (real and even)
$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$ (real and odd)	$jX_I(\omega)$ (imaginary and odd)

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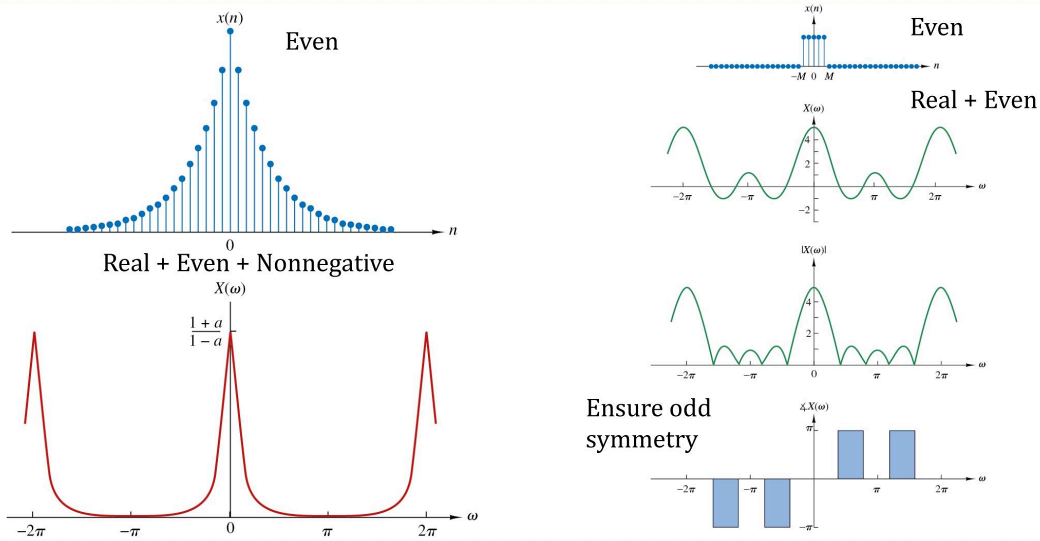
About the Phase of Even Sequences



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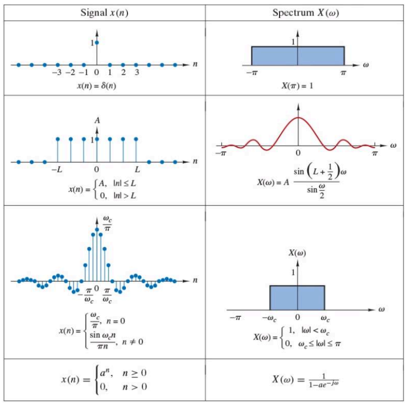
Properties of the DTFT

Property	Time Domain	Frequency Domain
Notation	$x(n), x_1(n), x_2(n)$	$X(\omega), X_1(\omega), X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time Shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time Reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega) = X_1(\omega)X_2^*(\omega)$
Wiener-Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency Shifting	$e^{j\omega_0 n}x(n)$	$X(\omega-\omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega+\omega_0)+\frac{1}{2}X(\omega-\omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in the frequency domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	



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Convolution Theorem

Theorem: $x(n) = x_1(n) * x_2(n) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = X_1(\omega)X_2(\omega)$

Example: $x_1[n] = x_2[n] = \{1 \quad \underset{\uparrow}{1} \quad 1\} \Rightarrow$

$$X_1(\omega) = X_2(\omega) = e^{-j\omega} + 1 + e^{j\omega} = 1 + 2\cos\omega$$

$$X(\omega) = X_1(\omega)X_2(\omega) = (1 + 2\cos\omega)^2$$

$$= 3 + 4\cos\omega + 2\cos 2\omega$$

$$= 3 + 2\left(e^{j\omega} + e^{-j\omega}\right) + \left(e^{j2\omega} + e^{-j2\omega}\right)$$

$$= e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega}$$

$$\Rightarrow \quad x(n) = \{1 \quad 2 \quad \underset{\uparrow}{3} \quad 2 \quad 1\}$$

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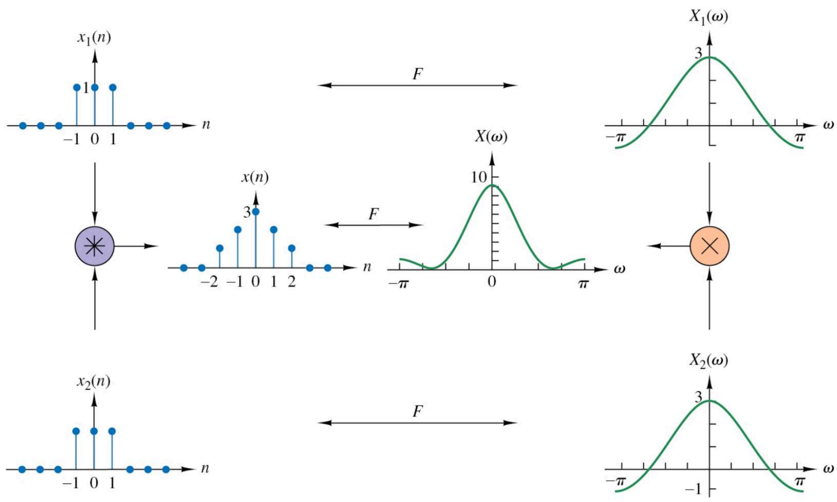
Graphical Illustration of Convolution Theorem



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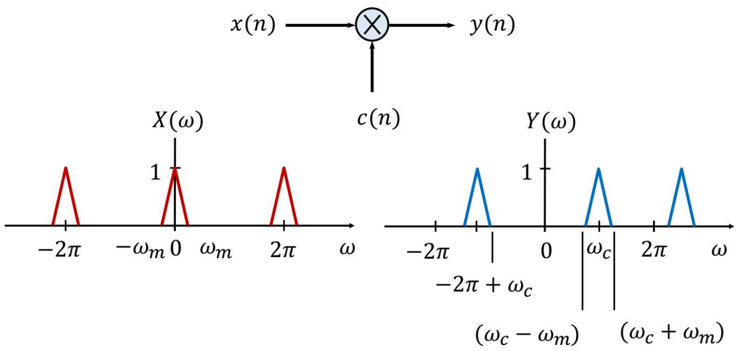
Amplitude Modulation with Exponential Carrier

$$x(n) \overset{F}{\longleftrightarrow} X(\omega) \Rightarrow y(n) = e^{j\omega_c n}x(n) \overset{F}{\longleftrightarrow} Y(\omega) = X(\omega - \omega_c)$$

$$\omega_c + \omega_m \leq 2\pi \Rightarrow \omega_c \leq 2\pi - \omega_m$$

Discrete−Time Amplitude Modulator



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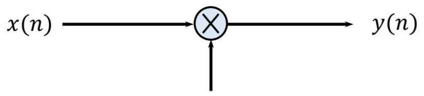
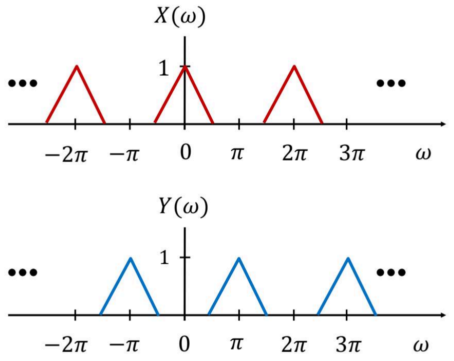
Modulation by "Odd-Sample Flipping"



$c(n) = e^{j\pi n} = (-1)^n$



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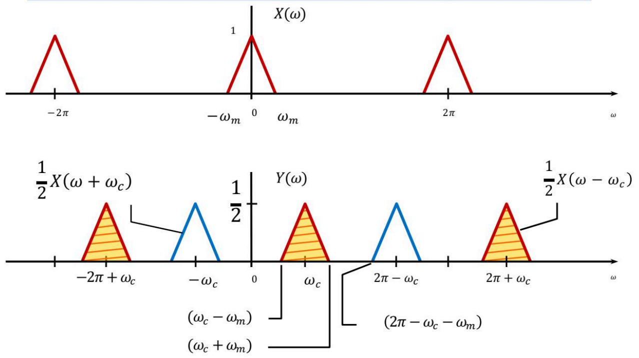
Amplitude Modulation with Sinusoidal Carrier

$$x(n)\cos\omega_c n \overset{F}{\longleftrightarrow} \frac{1}{2}X(\omega+\omega_c) + \frac{1}{2}X(\omega-\omega_c)$$



Requirements to Avoid Overlap

$\omega_c - \omega_m > 0$ and $(\omega_c + \omega_m) < (2\pi - \omega_c - \omega_m)$

$\Rightarrow \omega_m < \omega_c < (\pi - \omega_m)$

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Windowing Theorem

$$s[n] = x(n)w(n) \stackrel{F}{\longleftrightarrow} S(\omega) = X(\omega) * W(\omega) \triangleq \frac{1}{2\pi}\int_{2\pi} X(\theta)W(\omega-\theta)d\theta$$

Periodic convolution

Proof

$$S(\omega) = \sum_{n=-\infty}^{\infty} s(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(n)w(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{2\pi} X(\theta)e^{j\theta n}d\theta\right] w(n)\, e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{2\pi} X(\theta)\left[\sum_{n=-\infty}^{\infty} w(n)\, e^{-j(\omega-\theta)n}\right] d\theta$$

$$S(\omega) = \frac{1}{2\pi}\int_{\pi} X(\theta)\, W(\omega-\theta) d\theta$$

Applications:
- Spectral analysis
- Design of FIR filters

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Numerical Computation of the DTFT

Signal Length	Formula	Comments
Infinite	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$	Exact computation impossible
	$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)\, e^{j\omega n} d\omega$	Exact computation impossible
Finite	$X(\omega_k) = \sum_{n=0}^{N-1} x(n)e^{-j\omega_k n},\ k = 1,2,\dots,K$	`X = freqz(x,1,om)` x = $[x(0)\ x(1)\ \cdots\ x(N-1)]$ om = $[\omega_1\ \omega_2 \cdots \omega_K]$
	If $K = N$ we can recover $x(n)$ from $X(\omega_k)$ solving a linear system of equations	More details: See DFT and FFT lectures

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Important Note

Compute the DTFT of $x(n) = 0$ for $n < N_1$, and $n > N_2$, $N_1 < N_2$

$$X(\omega_k) = \sum_{n=N_1}^{N_2} x(n)\, e^{-j\omega_k n}$$

$$= x(N_1)\, e^{-j\omega_k N_1} + x(N_1+1)\, e^{-j\omega_k N_1} e^{-j\omega_k} + \cdots + x(N_2)\, e^{-j\omega_k N_1} e^{-j\omega_k (N_2-N_1)}$$

$$= e^{-j\omega_k N_1} \sum_{n=0}^{N_2-N_1} x(N_1+n)\, e^{-j\omega_k n}$$

$$X(\omega_k) = e^{-j\omega_k N_1} \sum_{n=0}^{N-1} x[N_1+n] e^{-j\omega_k n}, \quad N = N_2 - N_1 + 1$$

```
function X=dtft12(x,Nstart,om)
X=freqz(x,1,om);
X=exp(-j*om*Nstart).*X;
```


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Summary

- The DTFT expresses an aperiodic sequence as a superposition of complex exponential sequences
- The DTFT is a periodic function of the angular frequency ω with period 2π
- The DTFT is equal to the z-transform evaluated on the unit circle
- The DTFT can be numerically evaluated, at a discrete set of frequencies, for finite length sequences

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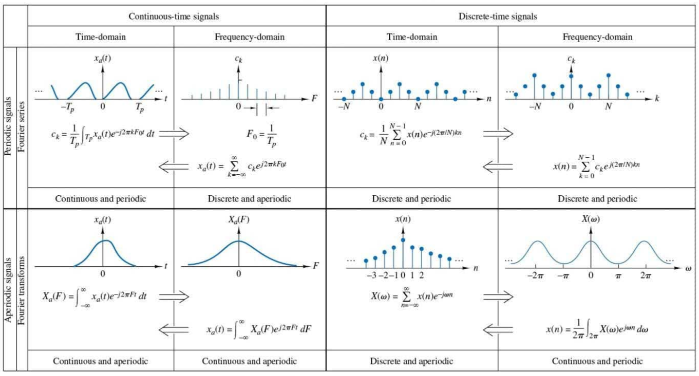
Fourier Analysis of Signals



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