APPLIED KALMAN FILTERING

Intuitive Concepts

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- To gain insight into the Kalman filter, it helps to consider the continuous-time implementation to improve our intuition regarding the impact of process noise strength relative to measure noise strength and their respective influence on the state estimation error covariance
- As before, consider the system model

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{w}(t)} \tag{1}$$

  where $\mathbf{x}(t) \in \Re^n$ is the state vector, $\mathbf{u}(t) \in \Re^p$ is a vector of known external inputs, $\mathbf{w}(t) \in \Re^n$ is the process noise, and $\mathbf{F}(t) \in \Re^{n\times n}$ and $\mathbf{G}(t) \in \Re^{n\times p}$ are matrices of time-varying smooth functions representing the system dynamics and input mapping, respectively, and and $E\{\mathbf{w}(t)\} = \mathbf{0}$ with $E\{\mathbf{w}(t)\mathbf{w}^{\mathsf{T}}(\tau)\} = \mathbf{Q}_s(t)\delta(t,\tau)$ and $\mathbf{Q}_s(t) = \mathbf{Q}_s^{\mathsf{T}}(t) \geq \mathbf{0}$
- Now consider that our measurements are available in continuous-time and modeled via

$$\boxed{\mathbf{y}(t) = \mathbf{H}(t)\mathbf{x}(t) + \mathbf{v}(t)} \tag{2}$$

  where $\mathbf{y}(t) \in \Re^m$ is the measurement vector, $\mathbf{v}(t) \in \Re^m$ is the measurement noise, $\mathbf{H}(t) \in \Re^{m\times n}$ is a matrix of time-varying smooth functions representing the sensor dynamics, and $E\{\mathbf{v}(t)\} = \mathbf{0}$ with $E\{\mathbf{v}(t)\mathbf{v}^{\mathsf{T}}(\tau)\} = \mathbf{R}(t)\delta(t,\tau)$ and $\mathbf{R}(t) = \mathbf{R}^{\mathsf{T}}(t) > \mathbf{0}$

- The (non-rigorous) derivation of the continuous-time Kalman filter presented here utilizes the well-known Luenberger observer
- The Luenberger observer is given by

$$\boxed{\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\hat{\mathbf{x}}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{K}(t)\Big(\mathbf{y}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t)\Big)} \tag{3}$$

where $\hat{\mathbf{x}}(t) \in \Re^n$ is the state estimate, and $\mathbf{K}(t)$ is the observer gain matrix
- Note that in the absence of process noise and measurement noise and with the estimation error defined as $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ we have

$$\dot{\mathbf{e}}(t) = \Big[\mathbf{F}(t) - \mathbf{K}(t)\mathbf{H}(t)\Big]\mathbf{e}(t)$$

For any $\mathbf{e}(t_0)$ it follows that $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$ when $\mathbf{F}(t) - \mathbf{K}(t)\mathbf{H}(t)$ is stable. When $\mathbf{F}$, $\mathbf{K}$, and $\mathbf{H}$ are constant matrices, then we design $\mathbf{K}$ so that the eigenvalues of $\mathbf{F} - \mathbf{KH}$ lie in the left-half complex plane which is always possible when the pair $(\mathbf{F}, \mathbf{H})$ is completely observable (or reconstructible).

- With $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ it follows that $\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t)$ and using Eq 1 and Eq. 3 we compute

$$\boxed{\dot{\mathbf{e}}(t) = \Big[\mathbf{F}(t) - \mathbf{K}(t)\mathbf{H}(t)\Big]\mathbf{e}(t) + \mathbf{w}(t) + \mathbf{K}(t)\mathbf{v}(t)} \tag{4}$$

- Define the state transition matrix, denoted by $\bar{\Phi}(t,\tau) \in \Re^{n\times n}$, $\forall t,\tau$, such that

$$\boxed{\dot{\bar{\Phi}}(t,\tau) = \Big[\mathbf{F}(t) - \mathbf{K}(t)\mathbf{H}(t)\Big]\bar{\Phi}(t,\tau)} \text{ where } \bar{\Phi}(t,t) = \mathbf{I} \tag{5}$$

- The solution $\mathbf{e}(t)$ in Eq 4 is

$$\boxed{\mathbf{e}(t) = \bar{\Phi}(t,t_0)\mathbf{e}_0 + \int_{t_0}^{t} \bar{\Phi}(t,\tau)\mathbf{w}(\tau)d\tau + \int_{t_0}^{t} \bar{\Phi}(t,\tau)\mathbf{K}(\tau)\mathbf{v}(\tau)d\tau} \tag{6}$$

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- As before, the covariance $\mathbf{P}(t)$ is given by

$$\mathbf{P}(t) = E\{\mathbf{e}(t)\mathbf{e}^{\mathsf{T}}(t)\} \tag{7}$$

- Computing $\mathbf{P}(t)$ yields

$$\mathbf{P}(t) = E\{\mathbf{e}(t)\mathbf{e}^{\mathsf{T}}(t)\} = \bar{\Phi}(t,t_0)E\{\mathbf{e}(t_0)\mathbf{e}^{\mathsf{T}}(t_0)\}\bar{\Phi}^{\mathsf{T}}(t,t_0) + \int_{t_0}^{t}\int_{t_0}^{t}\bar{\Phi}(t,\tau)E\{\mathbf{w}(\tau)\mathbf{w}^{\mathsf{T}}(\sigma)\}\bar{\Phi}^{\mathsf{T}}(t,\sigma)d\sigma d\tau$$
$$+ \int_{t_0}^{t}\int_{t_0}^{t}\bar{\Phi}(t,\tau)\mathbf{K}(\tau)E\{\mathbf{v}(\tau)\mathbf{v}^{\mathsf{T}}(\sigma)\}\mathbf{K}^{\mathsf{T}}(\sigma)\bar{\Phi}^{\mathsf{T}}(t,\sigma)d\sigma d\tau$$

where we employ the assumptions that $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are white noise processes uncorrelated with each other and with $\mathbf{e}(t_0)$

- With $\mathrm{E}\left\{\mathbf{w}(t)\mathbf{w}^{\mathsf{T}}(\tau)\right\} = \mathbf{Q}_s(t)\delta(t-\tau)$ and $\mathrm{E}\left\{\mathbf{v}(t)\mathbf{v}^{\mathsf{T}}(\tau)\right\} = \mathbf{R}(t)\delta(t-\tau)$, $\forall t, \tau$, it follows that

$$\mathbf{P}(t) = \Phi(t,t_0)E\{\mathbf{e}(t_0)\mathbf{e}^{\mathsf{T}}(t_0)\}\Phi^{\mathsf{T}}(t,t_0) + \int_{t_0}^{t}\int_{t_0}^{t}\Phi(t,\tau)\mathbf{Q}_s(\tau)\delta(\tau-\sigma)\Phi^{\mathsf{T}}(t,\sigma)d\sigma d\tau$$
$$+ \int_{t_0}^{t}\int_{t_0}^{t}\Phi(t,\tau)\mathbf{K}(\tau)\mathbf{R}(\tau)\delta(\tau-\sigma)\mathbf{K}^{\mathsf{T}}(\tau)\Phi^{\mathsf{T}}(t,\sigma)d\sigma d\tau$$

- Let $\mathbf{P}_0 = E\{\mathbf{e}(t_0)\mathbf{e}^\mathsf{T}(t_0)\}$ and by the property of the Dirac delta function we have

$$\mathbf{P}(t) = \bar{\Phi}(t, t_0)\mathbf{P}_0\bar{\Phi}^\mathsf{T}(t, t_0) + \int_{t_0}^{t} \bar{\Phi}(t, \tau)\mathbf{Q}_s(\tau)\bar{\Phi}^\mathsf{T}(t, \tau)d\tau + \int_{t_0}^{t} \bar{\Phi}(t, \tau)\mathbf{K}(\tau)\mathbf{R}(\tau)\mathbf{K}^\mathsf{T}(\tau)\bar{\Phi}^\mathsf{T}(t, \tau)d\tau \tag{8}$$

- Since $\mathbf{P}_0$ is constant we compute the time derivative of $\mathbf{P}(t)$ as

$$\begin{aligned}\dot{\mathbf{P}}(t) = \dot{\bar{\Phi}}(t, t_0)\mathbf{P}_0\bar{\Phi}^\mathsf{T}(t, t_0) + \bar{\Phi}(t, t_0)\mathbf{P}_0\dot{\bar{\Phi}}^\mathsf{T}(t, t_0) + \frac{d}{dt}\int_{t_0}^{t} \bar{\Phi}(t, \tau)\mathbf{Q}_s(\tau)\bar{\Phi}^\mathsf{T}(t, \tau)d\tau \\ + \frac{d}{dt}\int_{t_0}^{t} \bar{\Phi}(t, \tau)\mathbf{K}(\tau)\mathbf{R}(\tau)\mathbf{K}^\mathsf{T}(\tau)\bar{\Phi}^\mathsf{T}(t, \tau)d\tau\end{aligned}$$

- Recalling that

$$\dot{\bar{\Phi}}(t, \tau) = \Big[\mathbf{F}(t) - \mathbf{K}(t)\mathbf{H}(t)\Big]\bar{\Phi}(t, \tau),$$

$\mathbf{P}_0$ is constant and and using Leibniz's Rule, yields the (complex) result for $\dot{\mathbf{P}}(t)$

- Taking the time-derivative yields

$$\dot{\mathbf{P}}(t) = \Big[\mathbf{F}(t) - \mathbf{K}(t)\mathbf{H}(t)\Big]\bar{\Phi}(t,t_0)\mathbf{P}_0\bar{\Phi}^\mathsf{T}(t,t_0) + \bar{\Phi}(t,t_0)\mathbf{P}_0\bar{\Phi}^\mathsf{T}(t,t_0)\Big[\mathbf{F}(t) - \mathbf{K}(t)\mathbf{H}(t)\Big]^\mathsf{T}$$

$$+\int_{t_0}^{t}\Big[\mathbf{F}(t) - \mathbf{K}(t)\mathbf{H}(t)\Big]\bar{\Phi}(t,\tau)\mathbf{Q}_s(\tau)\bar{\Phi}^\mathsf{T}(t,\tau)d\tau + \int_{t_0}^{t}\bar{\Phi}(t,\tau)\mathbf{Q}_s(\tau)\bar{\Phi}^\mathsf{T}(t,\tau)\Big[\mathbf{F}(t) - \mathbf{K}(t)\mathbf{H}(t)\Big]^\mathsf{T}d\tau$$

$$+\int_{t_0}^{t}\Big[\mathbf{F}(t)-\mathbf{K}(t)\mathbf{H}(t)\Big]\bar{\Phi}(t,\tau)\mathbf{K}(\tau)\mathbf{R}(\tau)\mathbf{K}^\mathsf{T}(\tau)\bar{\Phi}^\mathsf{T}(t,\tau)d\tau+\int_{t_0}^{t}\bar{\Phi}(t,\tau)\mathbf{K}(\tau)\mathbf{R}(\tau)\mathbf{K}^\mathsf{T}(\tau)\bar{\Phi}^\mathsf{T}(t,\tau)\Big[\mathbf{F}(t)-\mathbf{K}(t)\mathbf{H}(t)\Big]^\mathsf{T}d\tau$$

$$+\bar{\Phi}(t,\tau)\mathbf{Q}_s(\tau)\bar{\Phi}^\mathsf{T}(t,\tau)\Big|_{\tau=t}+\bar{\Phi}(t,\tau)\mathbf{K}(\tau)\mathbf{R}(\tau)\mathbf{K}^\mathsf{T}(\tau)\bar{\Phi}^\mathsf{T}(t,\tau)\Big|_{\tau=t}$$

- Since $\mathbf{F}(t) - \mathbf{H}(t)\mathbf{K}(t)$ is not a function of τ, we can take it outside the integration, collect terms, and using $\mathbf{P}(t)$ in Eq. 8 we have the relationship

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^\mathsf{T}(t) + \mathbf{Q}_s(t) + \mathbf{K}(t)\mathbf{R}(t)\mathbf{K}^\mathsf{T}(t) - \mathbf{K}(t)\mathbf{H}(t)\mathbf{P}(t) - \mathbf{P}(t)\mathbf{H}^\mathsf{T}(t)\mathbf{K}^\mathsf{T}(t) \quad (9)$$

- Completing the square in Eq. 9 yields

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^{\mathsf{T}}(t) + \mathbf{Q}_s(t) - \mathbf{P}(t)\mathbf{H}^{\mathsf{T}}(t)\mathbf{R}^{-1}(t)\mathbf{H}(t)\mathbf{P}(t)$$
$$+ \left[\mathbf{K}(t)\mathbf{R}(t) - \mathbf{P}(t)\mathbf{H}^{\mathsf{T}}(t)\right]\mathbf{R}^{-1}(t)\left[\mathbf{K}(t)\mathbf{R}(t) - \mathbf{P}(t)\mathbf{H}^{\mathsf{T}}(t)\right]^{\mathsf{T}}$$

- We need to find $\mathbf{K}(t)$ such that $\mathbf{P}(t)$ is as small as possible. For a given $\mathbf{P}_0 > \mathbf{0}$, we want to choose $\mathbf{K}(t)$ so that $\mathbf{P}(t)$ increases as little as possible at each time.
- This is accomplished by setting

$$\boxed{\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^{\mathsf{T}}(t)\mathbf{R}^{-1}(t)} \tag{10}$$

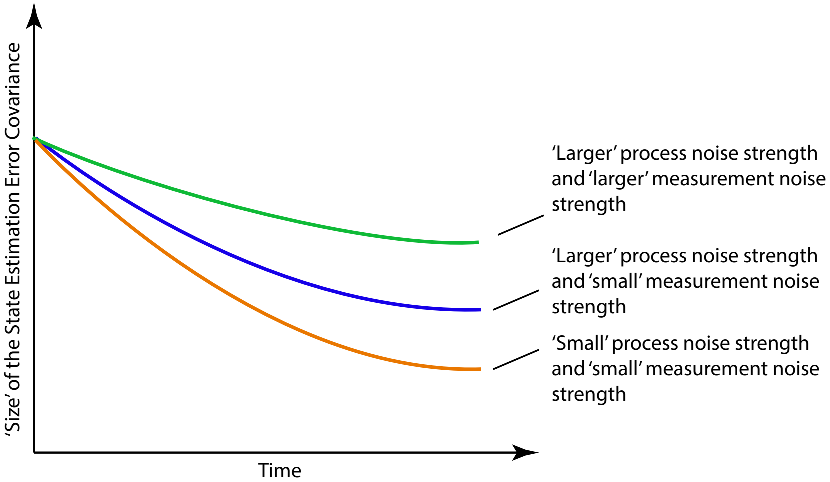
- This is the optimal Kalman gain for continuous-time dynamics and measurements. It can be more rigorously derived ... but that rigor is not needed here since we will not actually utilize this form in the course.
- The associated state estimation error covariance matrix is given by the solution to

$$\boxed{\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^{\mathsf{T}}(t) + \mathbf{Q}_s(t) - \mathbf{P}(t)\mathbf{H}^{\mathsf{T}}(t)\mathbf{R}^{-1}(t)\mathbf{H}(t)\mathbf{P}(t)}$$

where $\mathbf{P}_0 = \mathbf{P}_0^{\mathsf{T}} > \mathbf{0}$, $\mathbf{R}(t) = \mathbf{R}^{\mathsf{T}}(t) > \mathbf{0}$, and $\mathbf{Q}_s(t) = \mathbf{Q}_s^{\mathsf{T}}(t) \geq \mathbf{0}$ are given

$$\boxed{\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^{\mathsf{T}}(t) + \mathbf{Q}_s(t) - \mathbf{P}(t)\mathbf{H}^{\mathsf{T}}(t)\mathbf{R}^{-1}(t)\mathbf{H}(t)\mathbf{P}(t)} \qquad \mathbf{P}(t_0) = \mathbf{P}_0 = \mathbf{P}_0^{\mathsf{T}} > \mathbf{0}$$

- The combined terms $\mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^{\mathsf{T}}(t)$ are symmetric $\rightarrow$ the symmetry of $\mathbf{P}(t)$ is preserved
- Consider the process noise $\mathbf{Q}_s(t)$. With $\mathbf{Q}_s(t) = \mathbf{Q}_s^{\mathsf{T}}(t) \geq \mathbf{0}$ two things happen
  - $\mathbf{Q}_s(t) = \mathbf{Q}_s^{\mathsf{T}}(t) \rightarrow$ the symmetry of $\mathbf{P}(t)$ is preserved
  - $\mathbf{Q}_s(t) \geq \mathbf{0} \rightarrow$ the contribution to $\dot{\mathbf{P}}(t)$ is always greater than or equal to zero, hence the process noise tends to increase the state estimation error covariance
- Consider the measurement noise $\mathbf{R}(t)$. With $\mathbf{R}(t) = \mathbf{R}^{\mathsf{T}}(t) > \mathbf{0}$ three things happen
  - $\mathbf{R}(t) = \mathbf{R}^{\mathsf{T}}(t) > \mathbf{0} \rightarrow \mathbf{R}^{-1}(t)$ exists and is symmetric. The inverse of a positive definite matrix is also positive definite $\rightarrow \mathbf{R}^{-1}(t) > \mathbf{0}$
  - $\mathbf{P}(t)\mathbf{H}^{\mathsf{T}}(t)\mathbf{R}^{-1}(t)\mathbf{H}(t)\mathbf{P}(t)$ is symmetric $\rightarrow$ the symmetry of $\mathbf{P}(t)$ is preserved
  - $\mathbf{P}(t)\mathbf{H}^{\mathsf{T}}(t)\mathbf{R}^{-1}(t)\mathbf{H}(t)\mathbf{P}(t) > \mathbf{0} \rightarrow -\mathbf{P}(t)\mathbf{H}^{\mathsf{T}}(t)\mathbf{R}^{-1}(t)\mathbf{H}(t)\mathbf{P}(t) < \mathbf{0} \rightarrow$ the contribution to $\dot{\mathbf{P}}(t)$ is always less than zero, hence the measurements (represented by the measurement noise) tend to decrease the state estimation error covariance


'Size' of the State Estimation Error Covariance
Time
'Larger' process noise strength and 'larger' measurement noise strength
'Larger' process noise strength and 'small' measurement noise strength
'Small' process noise strength and 'small' measurement noise strength

- There is an intuitive logic behind the Kalman gain matrix in Eq. 10

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^{\mathsf{T}}(t)\mathbf{R}^{-1}(t)$$

- As can be seen, the Kalman gain is proportional to the uncertainty in the state estimate represented by the state estimation error covariance and inversely proportional to the measurement noise.
- Consider the state estimate update in Eq. 3

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\hat{\mathbf{x}}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{K}(t)\Big(\mathbf{y}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t)\Big)$$

  Note that the difference between the measurement and the expected measurement, represented by $\mathbf{y}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t)$ drives the state estimate update
- If measurements are noisy and the uncertainty in the state estimate errors is low, then $\mathbf{K}(t)$ should lead to relatively minimal updates to the state estimate. This is consistent with our intuition: if we have a highly certain state estimate (represented by "small" $\mathbf{P}(t)$) and the measurements are noisy (represented by "large" $\mathbf{R}(t)$), we should make minimal updates to the state estimate since the situation implies that $\mathbf{y}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t)$ contains mostly noise. We do not want to chase the noise.

- If the measurements are accurate and the state estimate errors are highly uncertain, then $\mathbf{K}(t)$ should lead to significant updates to the state estimate. This is consistent with our intuition: if we have a highly uncertain state estimate (represented by "large" $\mathbf{P}(t)$) and the measurements are accurate (represented by "small" $\mathbf{R}(t)$), we should make significant updates to the state estimate since the situation implies that $\mathbf{y}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t)$ contains considerable information about errors in the state estimates.
- The Kalman gain formula coincides with our intuitive approach to improving the estimate. It is essentially the ratio between statistical measures of the uncertainty in the state estimate and the uncertainty in a measurement.
- $\mathbf{K}(t)$ "small" $\rightarrow$ trust the model
  $\mathbf{K}(t)$ "large" $\rightarrow$ trust the measurements

- Optimality of the Kalman filter does not guarantee stability where stability describes the trajectory of the state estimate when measurements and known inputs are suppressed

$$\dot{\hat{\mathbf{x}}}(t) = \Big[\mathbf{F}(t) - \mathbf{K}(t)\mathbf{H}(t)\Big]\hat{\mathbf{x}}(t)$$

- Both stability and uniqueness of the behavior of $\mathbf{P}(t)$ as $t \to \infty$ independently of $\mathbf{P}(t_0)$ requires stochastic uniform complete observability, stochastic uniform complete controllability, bounded $\mathbf{Q}_s(t)$ and $\mathbf{R}(t)$ (from above and below), and bounded $\mathbf{F}(t)$ (from above)
- These conditions are not satisfied in many practical situations, yet the Kalman filter will operate satisfactorily when designed, implemented, and analyzed as discussed in this class
- In practice, the key issues pertaining to various forms of instability are those associated with modeling errors and implementation considerations (discussed later)

- The optimal filtering covariance equations can only be solved analytically for simple problems
- We will utilize numerical integration methods as the direct method for our practical applications
- In some cases, to enhance our understanding and gain more intuition about Kalman filters, we might utilize simple problems that demonstrate concepts
- Consider a situation where $\mathbf{F}$, $\mathbf{H}$, $\mathbf{R}$ and $\mathbf{Q}_s$ are all constant matrices. Let $\boldsymbol{\lambda} = \mathbf{P}\mathbf{y}$ where $\dot{\mathbf{y}} = -\mathbf{F}^{\mathsf{T}}\mathbf{y} + \mathbf{H}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{H}\mathbf{P}\mathbf{y}$. Then computing $\dot{\boldsymbol{\lambda}} = \dot{\mathbf{P}}\mathbf{y} + \mathbf{P}\dot{\mathbf{y}}$ yields

$$\boxed{\dot{\mathbf{z}} = \mathbf{M}\mathbf{z}} \tag{11}$$

where

$$\mathbf{z} = \begin{pmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \end{pmatrix} \in \Re^{2n} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} -\mathbf{F}^{\mathsf{T}} & \mathbf{H}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{H} \\ \mathbf{Q}_s & \mathbf{F} \end{bmatrix} \in \Re^{2n \times 2n}$$

Then $\mathbf{z}(t) = \Phi(t, t_0)\mathbf{z}(t_0)$ where $\Phi(t, t_0) = \mathbf{e}^{\mathbf{M}(t - t_0)}$

- Partition the state transition matrix as

$$\boldsymbol{\Phi}(t, t_0) = \left[\begin{array}{c|c} \boldsymbol{\Phi}_{yy}(t, t_0) & \boldsymbol{\Phi}_{y\lambda}(t, t_0) \\ \hline \boldsymbol{\Phi}_{\lambda y}(t, t_0) & \boldsymbol{\Phi}_{\lambda\lambda}(t, t_0) \end{array}\right]$$

  where $\boldsymbol{\Phi}_{yy}(t, t_0) \in \Re^{n\times n}$, $\boldsymbol{\Phi}_{y\lambda}(t, t_0) \in \Re^{n\times n}$, $\boldsymbol{\Phi}_{\lambda y}(t, t_0) \in \Re^{n\times n}$, and $\boldsymbol{\Phi}_{\lambda\lambda}(t, t_0) \in \Re^{n\times n}$
- Using Eq. 11 and the fact that $\boldsymbol{\lambda}_0 = \mathbf{P}_0\mathbf{y}_0$, we can obtain the relationship

$$\boxed{\mathbf{P}(t) = \left[\boldsymbol{\Phi}_{\lambda y}(t, t_0) + \boldsymbol{\Phi}_{\lambda\lambda}(t, t_0)\mathbf{P}_0\right]\left[\boldsymbol{\Phi}_{yy}(t, t_0) + \boldsymbol{\Phi}_{y\lambda}(t, t_0)\mathbf{P}_0\right]^{-1}} \tag{12}$$

- Consider the simple 1^{st}-order system

$$\begin{aligned} \dot{x}(t) &= -x(t) + w(t) \\ y(t) &= x(t) + v(t) \end{aligned}$$

In terms of the models in Eqs. 1 and 2, $F = -1$, $G = 0$, $H = 1$, $E\{w(t)w^{\mathsf{T}}(\tau)\} = q\delta(t,\tau)$ and $E\{v(t)v^{\mathsf{T}}(\tau)\} = r\delta(t,\tau)$

- Let $t_0 = 0$, $p_0 = p(0)$, and $\beta := \sqrt{q/r+1}$. Then

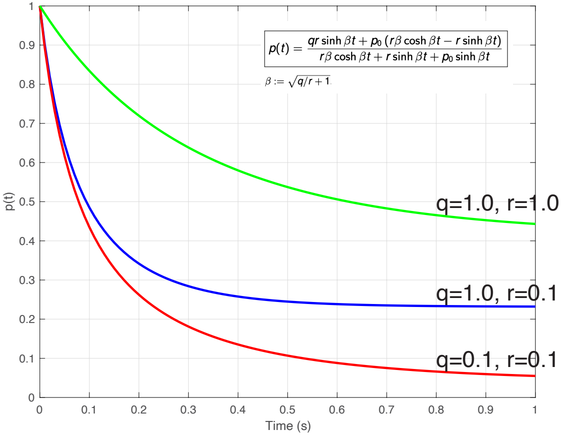
$$\boldsymbol{\Phi}(t,0) = \mathbf{e}^{\mathbf{M}t} = \frac{1}{r\beta}\begin{bmatrix} r\beta\cosh\beta t + r\sinh\beta t & \sinh\beta t \\ qr\sinh\beta t & r\beta\cosh\beta t - r\sinh\beta t \end{bmatrix}$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & 1/r \\ q & -1 \end{bmatrix}$$

- From Eq. 12 it follows that

$$\boxed{p(t) = \frac{qr\sinh\beta t + p_0\left(r\beta\cosh\beta t - r\sinh\beta t\right)}{r\beta\cosh\beta t + r\sinh\beta t + p_0\sinh\beta t}}$$


$p(t) = \dfrac{qr\sinh\beta t + p_0\,(r\beta\cosh\beta t - r\sinh\beta t)}{r\beta\cosh\beta t + r\sinh\beta t + p_0\sinh\beta t}$
$\beta := \sqrt{q/r+1}$
q=1.0, r=1.0
q=1.0, r=0.1
q=0.1, r=0.1
p(t)
Time (s)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

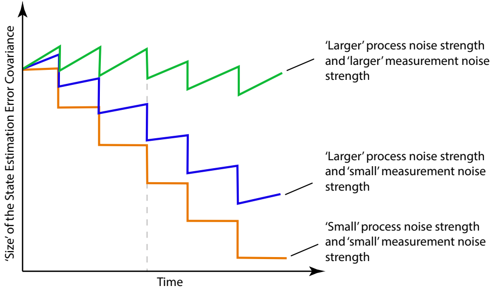
- The steady-state values of the optimal filtering covariance equations generally can only be solved for analytically for simple problems
- A sufficient condition for the existence of a steady-state solution is complete observability and complete controllability guarantees that the steady-state solution is unique
- We will utilize numerical integration for our practical applications and observe the steady-state values in our simulations
- Let's revisit the simple 1^{st}-order system with $F = -1$, $G = 0$, $H = 1$, $E\{w(t)w^{\mathsf{T}}(\tau)\} = q\delta(t,\tau)$ and $E\{v(t)v^{\mathsf{T}}(\tau)\} = r\delta(t,\tau)$
- Then the steady-state performance of the state estimation error covariance is found by setting $\dot{p}(t) = 0$ and solving for p_∞ yielding

$$\dot{p}(t) = -2p_\infty + q - p_\infty^2/r = 0 \qquad \rightarrow p_\infty = -r\left(1 - \sqrt{(q+r)/r}\right)$$

and the steady-values (q, r) are

$$(0.1, 0.1) \rightarrow p_\infty = 0.0414, \quad (1, 0.1) \rightarrow p_\infty = 0.2317, \quad (1, 1) \rightarrow p_\infty = 0.4142$$

- When considering the continuous-discrete Kalman filter algorithm that serves as the basis for this class, all the previous intuitions apply



- $\mathbf{K}_k$ "small" $\rightarrow$ trust the model and $\mathbf{K}_k$ "large" $\rightarrow$ trust the measurements
- In terms of stability, the continuous-discrete Kalman filter will operate satisfactorily when designed, implemented, and analyzed as discussed in this class

END MODULE