Applied Kalman Filtering
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Part 3: Kalman Filter Development – Solution

Objective: Begin the development of the Kalman filter of the continuous-time dynamics of the falling body with discrete-time measurements of range

Background: Consider the tracking problem from Parts 1 and 2.

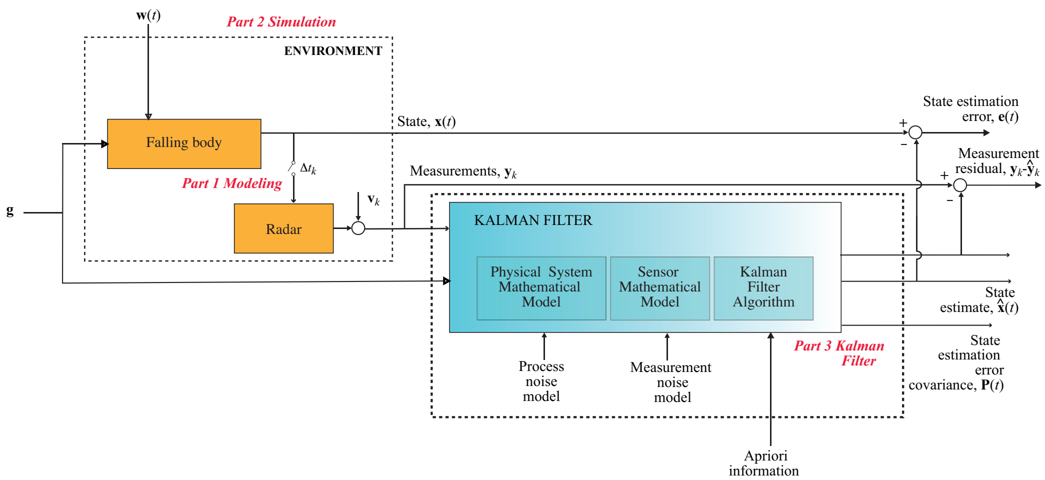

Figure 1: Simulation functional diagram

Truth

In the simulation, the true motion of the falling body was simulated from the initial time $t_0 = 0$ to the final time $t_f = 14$ s with constant $\Delta t = 0.1$ s. The state vector is

$$\mathbf{x}(t) = \begin{pmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \end{pmatrix},$$

where $\mathbf{r}(t)$ is the position of the target in the fixed reference frame and $\mathbf{v}(t)$ is the velocity. The true initial position and velocity are

$$\mathbf{r}_0 = \begin{pmatrix} 0 \\ 1000 \end{pmatrix} \text{ m} \quad \text{and} \quad \mathbf{v}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ m/s}^2,$$

respectively, and we denote

$$\mathbf{x}_0 = \begin{pmatrix} \mathbf{r}_0 \\ \mathbf{v}_0 \end{pmatrix}.$$

Let $x_k := x(t_k)$. Then we can utilize the discrete-time model given by

$$\boxed{\mathbf{x}_k = \Phi_{k-1}\mathbf{x}_{k-1} + \mathbf{u}_{k-1} + \mathbf{w}_{k-1}}$$

where $\Delta t = t_k - t_{k-1} = 0.1$ s and

$$\Phi_{k-1} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{u}_{k-1} = -g \begin{pmatrix} 0 \\ \Delta t^2/2 \\ 0 \\ \Delta t \end{pmatrix} = -9.806 \begin{pmatrix} 0 \\ 0.005 \\ 0 \\ 0.1 \end{pmatrix},$$

and $E\{\mathbf{w}_{k-1}\} = \mathbf{0}$ with

$$\mathbf{Q}_{k-1} = q_s \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \Delta t^3/3 & 0 & \Delta t^2/2 \\ 0 & 0 & 0 & 0 \\ 0 & \Delta t^2/2 & 0 & \Delta t \end{bmatrix} = 0.01 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.0003 & 0 & 0.005 \\ 0 & 0 & 0 & 0 \\ 0 & 0.005 & 0 & 0.1000 \end{bmatrix}.$$

A typical true trajectory is illustrated in Figures 2 and 3.

Propagation

The Kalman filter timeline is illustrated in Figure 4. We need to address both major phases: propagation and update. The sensor frequency is 2 Hz. Therefore, the measurement $\Delta t_m = 1/f$. So, in the Kalman filter propagation stage, the time step is $\Delta t_m = 0.5$ s. In other words, $t_k - t_{k-1} = 0.5$ s. In the simulation, since the simulation time step is $\Delta t = 0.1$ s, we will sample the true range every 5th time step and use the mathematical model of the measurement including the bias and noise to generate the measurement to be processed by the Kalman filter at the update.

The estimated state at any time t_k s given by

$$\hat{\mathbf{x}}_k = \begin{pmatrix} \hat{\mathbf{r}}_k \\ \hat{\mathbf{v}}_k \end{pmatrix},$$

where $\hat{\mathbf{r}}_k$ is the estimated position of the target in the fixed reference frame and $\hat{\mathbf{v}}_k$ is the estimated velocity. In the simulation, the initial estimated position and velocity are computed using

$$\hat{\mathbf{r}}_0 = \mathbf{r}_0 + \mathbf{e}_{r_0} \text{ m} \quad \text{and} \quad \hat{\mathbf{v}}_0 = \mathbf{v}_0 + \mathbf{e}_{v_0} \text{ m/s}^2,$$

where $\mathbf{e}_{r_0}$ and $\mathbf{e}_{v_0}$ are the initial state estimation errors in position and velocity, respectively. With the initial state estimation error vector represented in vector format

$$\mathbf{e}_0 = \begin{pmatrix} \mathbf{e}_{r_0} \\ \mathbf{e}_{v_0} \end{pmatrix},$$

we have

$$\hat{\mathbf{x}}_0 = \mathbf{x}_0 + \mathbf{e}_0,$$

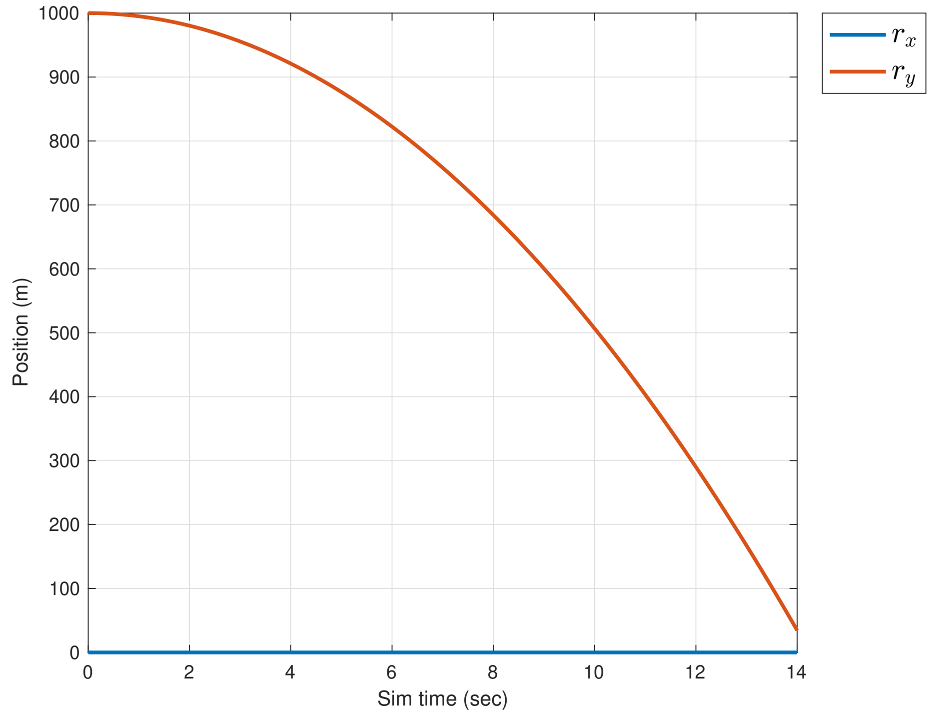

Figure 2: True position

where $E\{\mathbf{e}_0\} = \mathbf{0}$ and

$$E\{\mathbf{e}_0\mathbf{e}_0^{\mathrm{T}}\} = \mathbf{P}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}.$$

where $\mathbf{P}_0$ is the initial state estimation error covariance matrix. Note that typically we require $\mathbf{P}_0 > \mathbf{0}$. But looking forward to a future extension of the tracking problem to two dimensions, I included the additional states in the x direction even though they would be zero the entire trajectory.

Between measurements the state estimate is propagated via

$$\boxed{\hat{\mathbf{x}}_k^- = \Phi_{k-1}\hat{\mathbf{x}}_{k-1}^+ + \mathbf{u}_{k-1}}$$

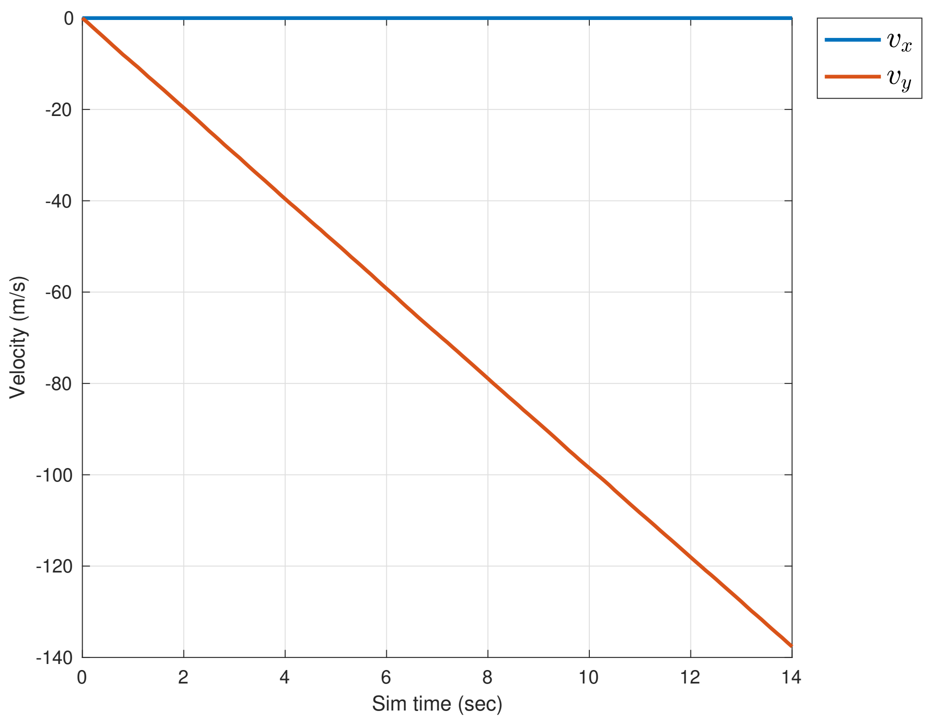

Figure 3: True velocity

where $\Delta t_m = t_k - t_{k-1} = 0.5$ s and

$$\Phi_{k-1} = \begin{bmatrix} 1 & 0 & \Delta t_m & 0 \\ 0 & 1 & 0 & \Delta t_m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{u}_{k-1} = -g \begin{bmatrix} 0 \\ \Delta t_m^2/2 \\ 0 \\ \Delta t_m \end{bmatrix} = -9.806 \begin{bmatrix} 0 \\ 0.125 \\ 0 \\ 0.5 \end{bmatrix}.$$

The state estimation error covariance is mapped forward via

$$\boxed{\mathbf{P}_k^- = \Phi_{k-1}\mathbf{P}_{k-1}^+\Phi_{k-1}^{\mathrm{T}} + \mathbf{Q}_{k-1}}$$

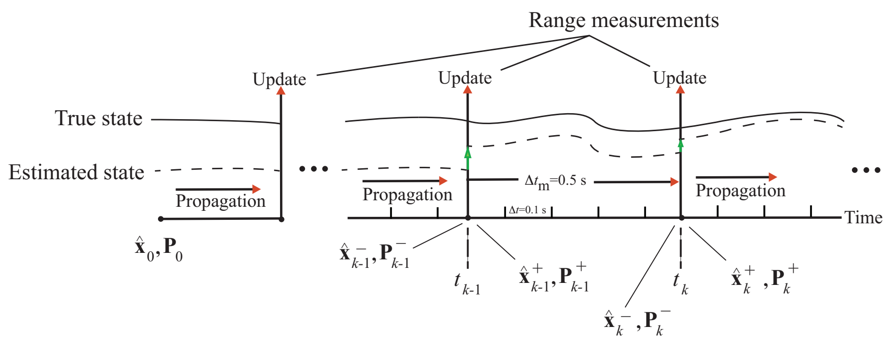

Figure 4: Kalman filter timeline

with

$$\mathbf{Q}_{k-1} = q_s \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \Delta t_m^3/3 & 0 & \Delta t_m^2/2 \\ 0 & 0 & 0 & 0 \\ 0 & \Delta t_m^2/2 & 0 & \Delta t_m \end{bmatrix} = 0.01 \begin{bmatrix} 0. & 0 & 0 & 0 \\ 0 & 0.0417 & 0 & 0.125 \\ 0 & 0 & 0 & 0 \\ 0 & 0.125 & 0 & 0.5 \end{bmatrix}.$$

Radar
The true measurement is the range to the target given by the discrete-time measurement model

$$\boxed{\mathbf{y}_k = \mathbf{H}\mathbf{x}_k + b + v_k}$$

where

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \quad b = -h$$

where

$$h = 2 \text{ m}$$

is the known height of the radar antenna above the ground and v_k is a zero-mean white noise sequence with $E\{v_k^2\} = R$ where

$$R = 4 \text{ m}^2.$$

The measurement update rate is 2 Hz. The estimated measurement is given by

$$\boxed{\hat{\mathbf{y}}_k = \mathbf{H}\hat{\mathbf{x}}_k + b}$$

where

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \quad b = -2.$$

Update The Kalman gain is computed via

$$\boxed{\mathbf{K}_k = \frac{\mathbf{P}_k^- \mathbf{H}^{\mathrm{T}}}{\mathbf{H}\mathbf{P}_k^- \mathbf{H}^{\mathrm{T}} + R}}$$

where $\mathbf{H}$ and R are given above. Then we can update the state estimate via

$$\boxed{\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_k \left(\mathbf{y}_k - \hat{\mathbf{y}}_k^-\right)}$$

where

$$\hat{\mathbf{y}}_k^- = \mathbf{H}\hat{\mathbf{x}}_k^- + b.$$

The state estimation error covariance is updated via

$$\boxed{\mathbf{P}_k^+ = \left[\mathbf{I} - \mathbf{K}_k\mathbf{H}\right]\mathbf{P}_k^-}.$$

The updated $\hat{\mathbf{x}}_k^+$ and $\mathbf{P}_k^+$ are then used to initialize the next propagation stage.

Simulation results An example of the simulation results are presented in Figures 5-8. Note that your simulation results will differ due to the fact that each run likely used a different random process noise and measurement noise sequence.

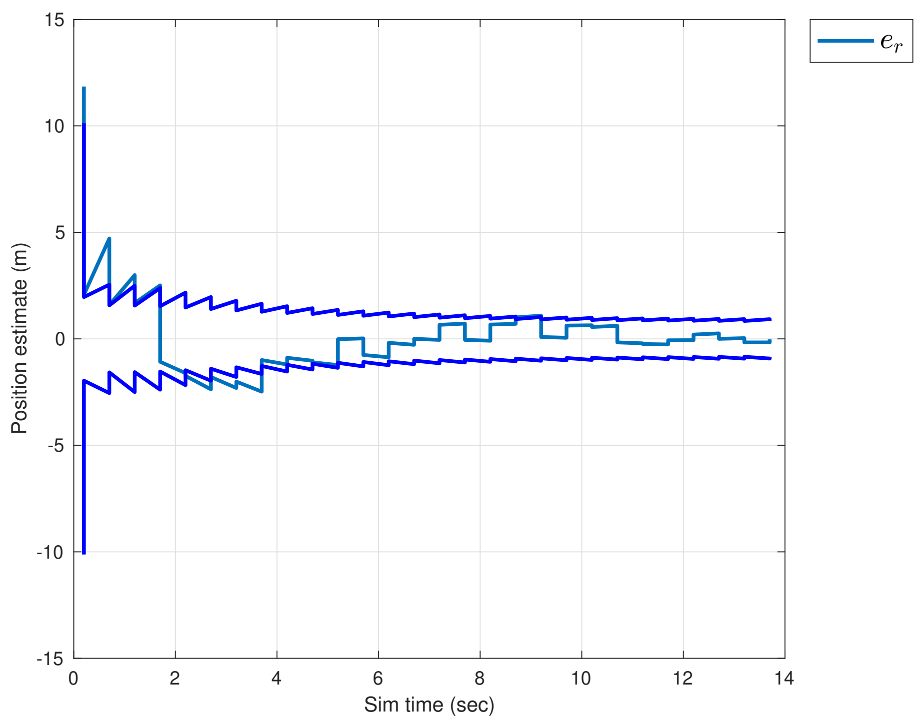

Figure 5: Position estimation

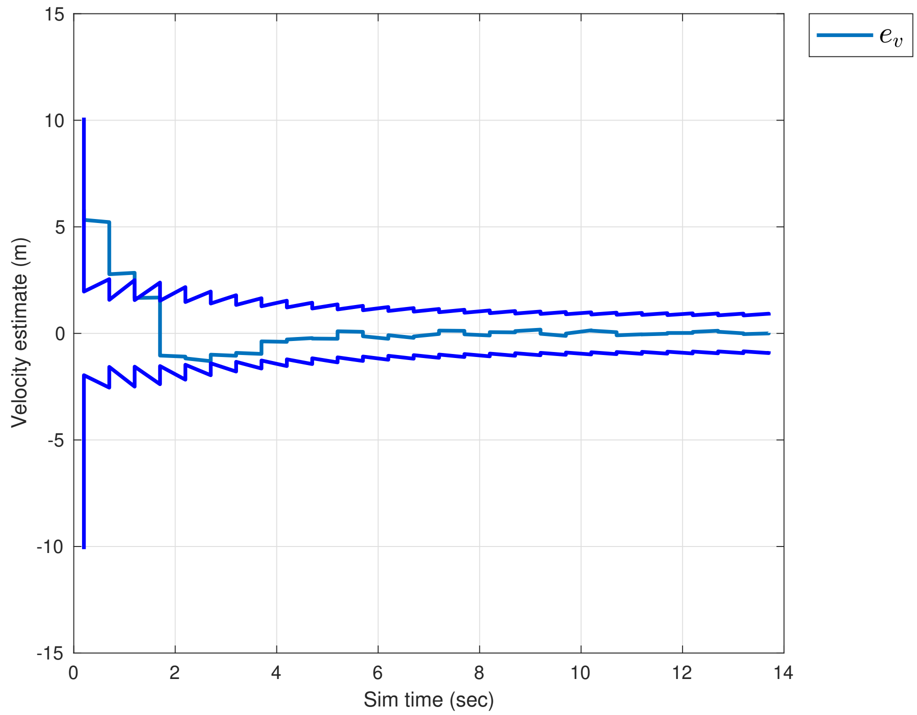

Figure 6: Velocity estimation

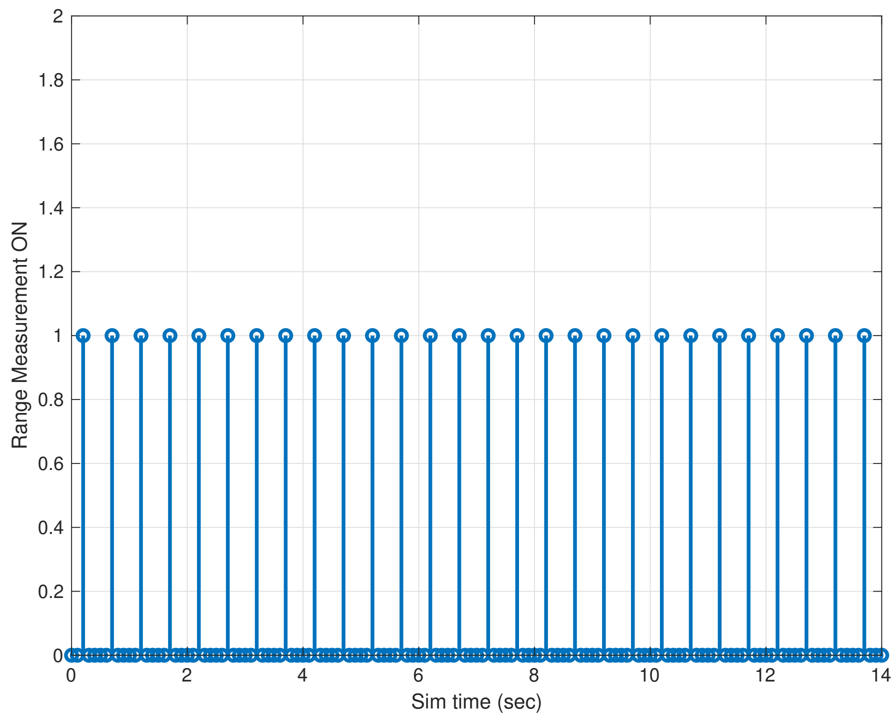

Figure 7: Measurement on times

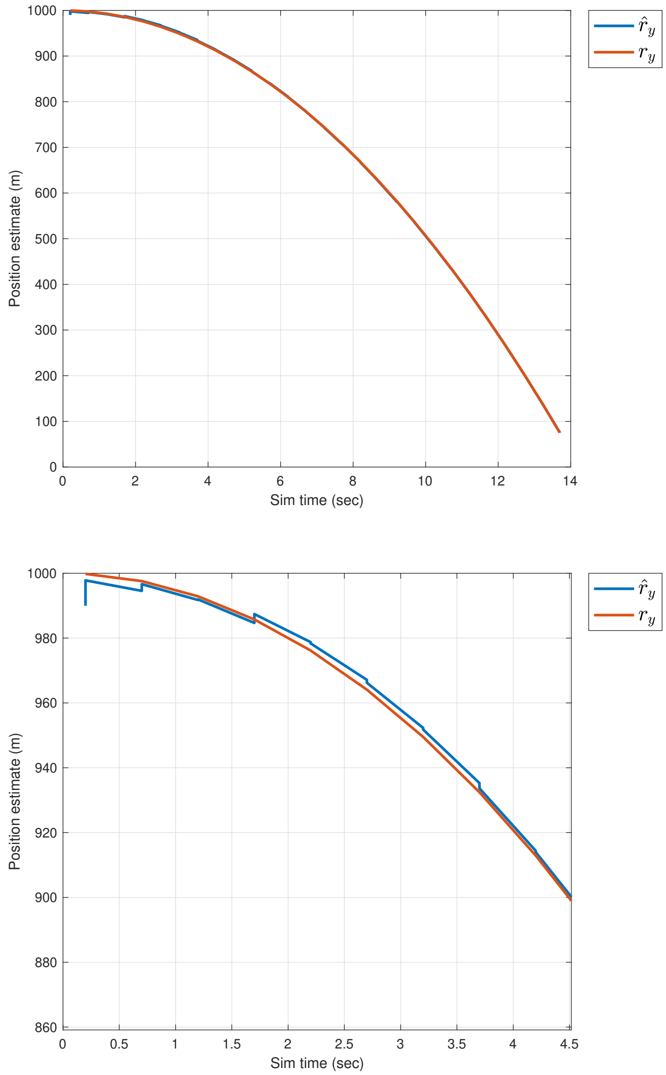

Figure 8: (a) Position estimation; (b) Position estimation close up during early tracking