Applied Kalman Filtering
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Part 1: One-Dimensional Tracking Model Development – Solution

Objective: Develop state variable mathematical models for the one-dimensional tracking problem

Background: Consider the tracking problem illustrated in Figure 1.

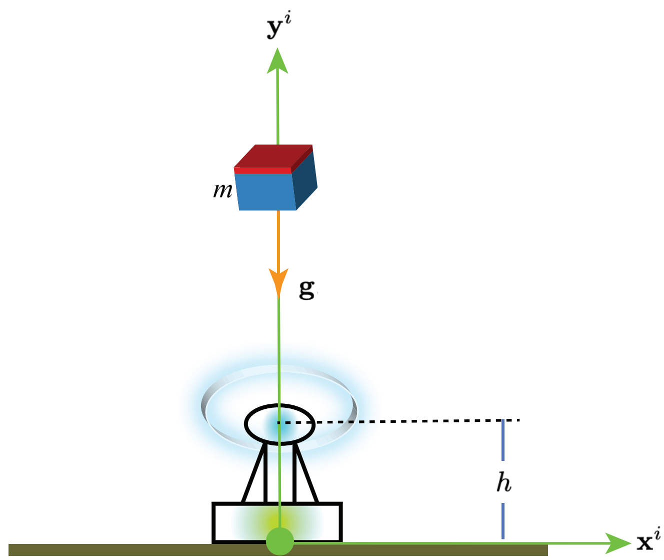

Figure 1: One-dimensional tracking problem scenario

Assumptions:

1. Gravity, $\mathbf{g}$, and height, h, are known constants
2. Neglect atmospheric drag
3. Motion is one-dimensional (along the $\mathbf{y}^i$ axis)
4. Range measurements are available at discrete times

Consider the fixed reference frame $(\mathbf{x}^i, \mathbf{y}^i)$ in Figure 1. Define the state vector to be

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \end{bmatrix} \in \Re^4$$

where $\mathbf{r}(t)$ is the position of the target in the fixed reference frame and $\mathbf{v}(t)$ is the velocity. The known input, $\mathbf{u}(t)$, is given by

$$\mathbf{u}(t) = \mathbf{g} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} g$$

where g=9.8 m/s The continuous-time mathematical model of the motion is given by

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{w}(t)} \tag{1}$$

where

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \Re^{4\times 4}, \quad \mathbf{G} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \in \Re^{4\times 2},$$

and $\mathbf{w}(t) \in \Re^4$ is the process noise with $E\{\mathbf{w}(t)\} = \mathbf{0}$ for all t and $E\{\mathbf{w}(t)\mathbf{w}^T(\tau)\} = \mathbf{Q}_s\delta(t-\tau)$ for all t, τ where

$$\mathbf{Q}_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_s \end{bmatrix},$$

and $q_s \geq 0$ is given. Since we assume (in this case) that the motion is one-dimensional, we have

$$\mathbf{r}(t) = \begin{bmatrix} 0 \\ r(t) \end{bmatrix}, \quad \mathbf{v}(t) = \begin{bmatrix} 0 \\ v(t) \end{bmatrix}$$

where $r(t) = ||\mathbf{r}(t)||_2$ is the distance above the ground from the origin of the fixed reference frame. Since the measurement itself is the distance above the radar (known as the range), we have the discrete-time measurement model

$$\boxed{\mathbf{y}_k = \mathbf{H}\mathbf{x}_k + b + v_k} \tag{2}$$

where

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \in \Re^{1\times 4}, \quad b = -h \in \Re,$$

$\mathbf{x}_k := \mathbf{x}(t_k)$, and v_k is the measurement noise sequence with $E\{v_k\} = 0$ for all k, $E\{v_k v_j\} = R\delta_{kj}$ for all k, j where $R > 0$ is given and b is a *known* bias with $h \geq 0$. The time t_k is the measurement update time and $\Delta t = t_k - t_{k-1}$ is the measurement update rate. We will assume that the measurement update rate is constant across the entirety of the tracking trajectory, that is, Δt is constant.

The discrete-time mathematical model of the motion is given by

$$\boxed{\mathbf{x}_k = \Phi_{k-1}\mathbf{x}_{k-1} + \mathbf{u}_{k-1} + \mathbf{w}_{k-1}}$$

where $\Phi_{k-1} := \Phi(t_k, t_{k-1})$ is the state transition matrix with

$$\Phi(t_k, t_{k-1}) = \mathbf{e}^{\mathbf{F}(t_k - t_{k-1})} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \Re^{4\times 4}$$

where $\Delta t = t_k - t_{k-1}$ and

$$\mathbf{u}_{k-1} = \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau)\mathbf{G}(\tau)\mathbf{u}(\tau)d\tau = -g \begin{bmatrix} 0 \\ \Delta t^2/2 \\ 0 \\ \Delta t \end{bmatrix} \in \Re^{4\times 1}$$

Also, for the process noise $\mathbf{w}_k \in \Re^4$ we have

$$E\{\mathbf{w}_{k-1}\} = \mathbf{0}$$

and $\mathbf{Q}_{k-1} := E\{\mathbf{w}_{k-1}\mathbf{w}_{k-1}^{\mathrm{T}}\} = \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau)\mathbf{Q}_s(\tau)\Phi^{\mathrm{T}}(t_k, \tau)d\tau$ is given by

$$\mathbf{Q}_{k-1} = q_s \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \Delta t^3/3 & 0 & \Delta t^2/2 \\ 0 & 0 & 0 & 0 \\ 0 & \Delta t^2/2 & 0 & \Delta t \end{bmatrix} \in \Re^{4\times 4}$$