165 Math & Computers

Antonio J. Torres

M,W,F 4:10 - 5:00 PM

Hart Hall 1150

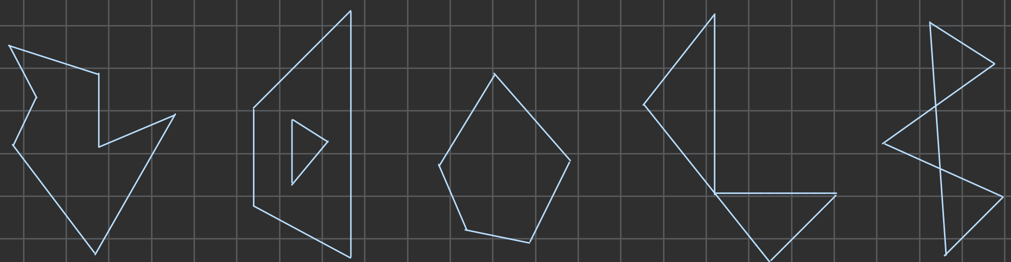
Office h: 1004, PSEL

T,T 11:00 - 12:00

Polygons

Computational geometry is fundamentally discrete. Computation with curves and smooth surfaces are generally considered part of another field, often called "geometric modeling".

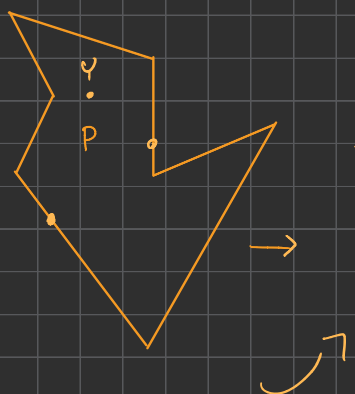
A polygon P is the closed region of the plane bounded by a finite collection of line segments forming a closed curve that does not intersect itself.



(a) (b) (c) (d) (e)

Theorem (Polygonal Jordan Curve). The boundary ∂P of a polygon P partitions the plane into two parts. In particular, the two components of $\mathbb{R}^2/\partial P$ are the bounded interior and the unbounded exterior.

Sketch of the proof.



Choose a fixed direction that is not parallel to any edge of P.

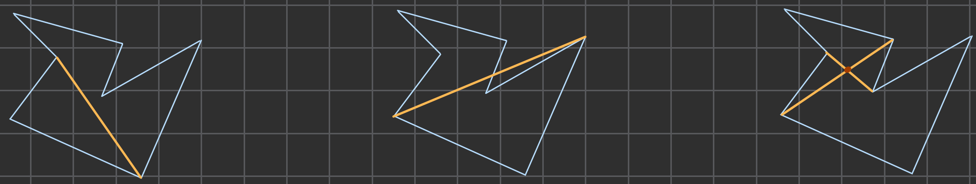
Then any point $x \in \mathbb{R}^2 \setminus \{\partial P\}$ lies in one of the following two sets:

A 1) The ray through x in direction u crosses ∂P in an even number of times.

B 2) The ray through x in direction u crosses ∂P in an odd number of times.

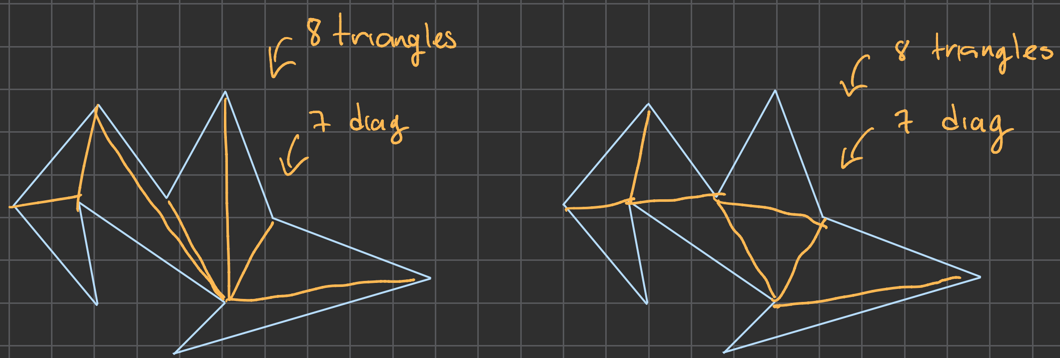
Exercise. Prove that a)- Every path between points lying in different sets must cross ∂P.
b)- There is a path between points in the same set that doesn't contains points of ∂P.

A diagonal of a polygon P is a line segment connecting two vertices of P and lying in the interior of P, not touching ∂P except at its endpoints.



(a) (b) (c)

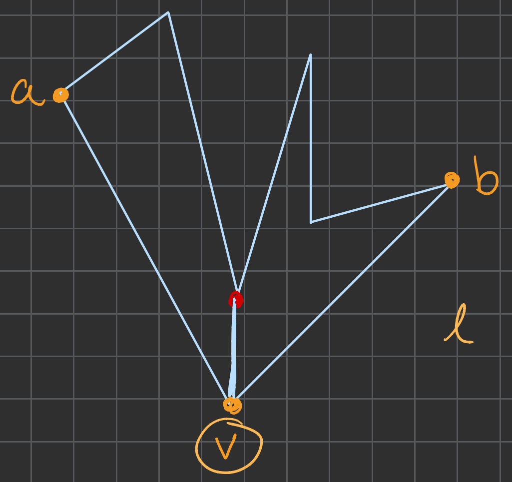
Definition. A triangulation of a polygon P is a decomposition of P into triangles by a maximal set of non crossing diagonals.



Some questions:

- How many different triangulations does a given polygon have?
- How many triangles are in each triangulation of a given polygon?
- Must every polygon always have at least one diagonal?

Lemma: Every polygon P with more 3 vertices has a diagonal.



Proof: Let be v the lowest vertex of P; If there are several take the rightmost. Let a and b the neighbors of v.

If the segment $\overline{ab}$ is contained in P and $\overline{ab} \cap \partial P = \{a, b\}$ then $\overline{ab}$ is a diagonal.

Otherwise, since P has more than three vertices, the closed triangle Δabv contains at least one vertex of P.

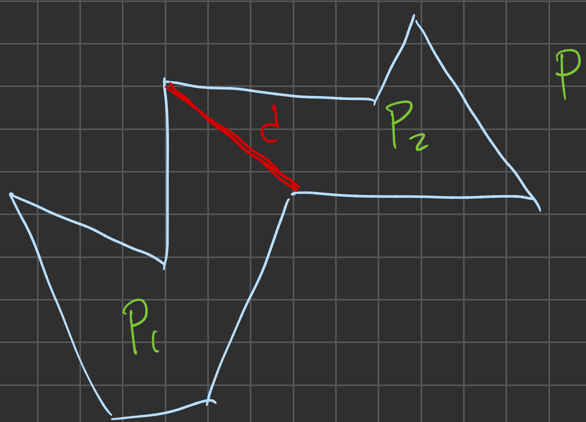
Let $\perp$ be a line parallel to $\overline{ab}$ through v. Sweep this line parallel to itself upward toward $\overline{ab}$.

Let x the first vertex different to a, b or v. The (shaded) triangular region of the polygon below line $\perp$ and above v is empty of vertices.

Because vx cannot int ∂P except at v and x vx is our diagonal. $\blacksquare$

Theorem: Every polygon has a triangulation.

Proof:
- If P have 3 vertices ✓
- Suppose $|V| > 3$ and the thm is valid for polygons with fewer vertices

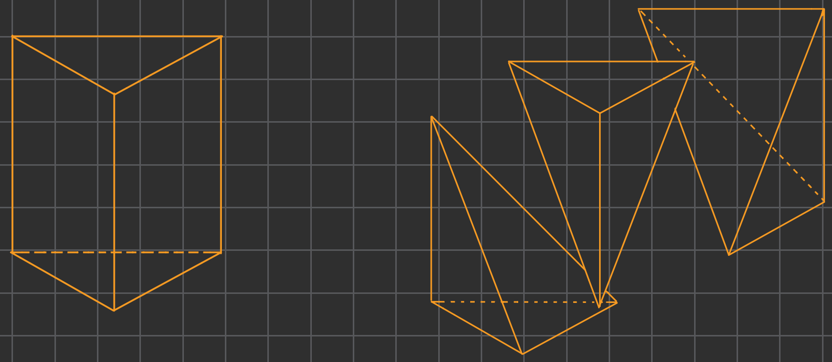


- There is a diag d s.t. d divides P into P_1 and P_2

Tetrahedralization

For a 3-dimensional polytope (polyhedron) P, we can "triangulate" P using tetrahedrons.

¿How many tetrahedrons?



Can all polyhedra be tetrahedralized?

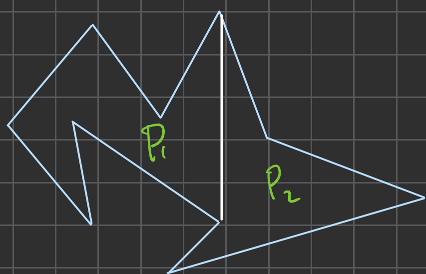
Open problem:

Find a characterization for tetrahedralizable polyhedra.

In 1992 Jim Ruppert and Raimund Seidel proved that determining whether a polyhedron is tetrahedralizable is NP-complete.

Theorem. Every triangulation of a polygon P with n vertices has $n-2$ triangles and $n-3$ diagonals.

Proof:



$|V(P_1)| = n_1$ $|V(P_2)| = n_2$

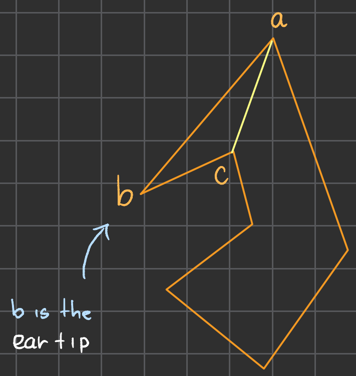
P_1 have
n_1-2 triang
n_1-3 edges

P_2 have
n_2-2 trian
n_2-3 diag

$$n_1-2+n_2-2 = (n_1+n_2)-4$$
$$= n+2-4$$
$$= n-2$$

$$n_1-3+n_2-3+1 =$$
$$n_1+n_2-5 = n+2-5 = n-3$$

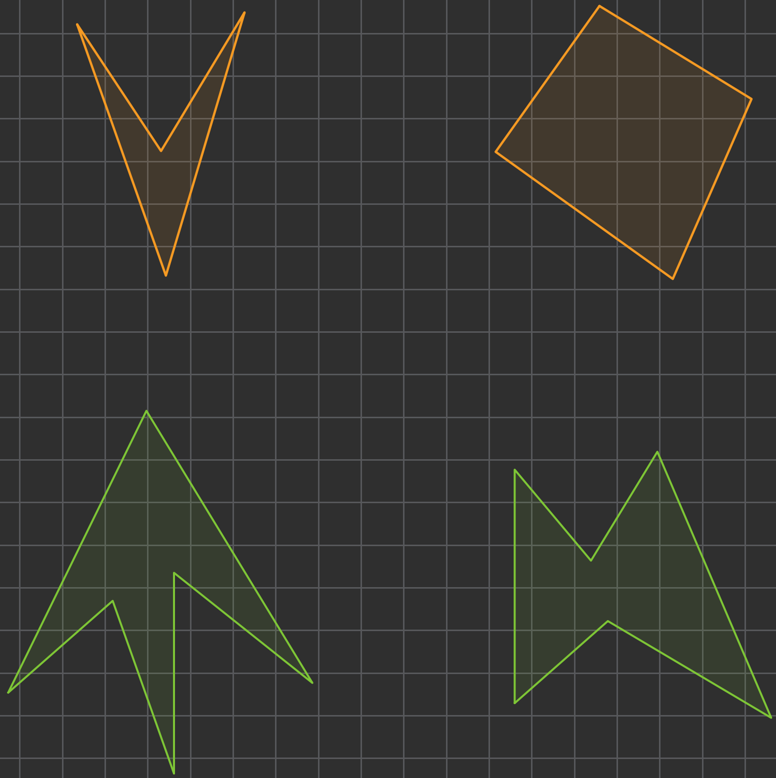
We sometimes call $\boxed{\text{ears}}$ three consecutive vertices a, b, c & ac is a diagonal.



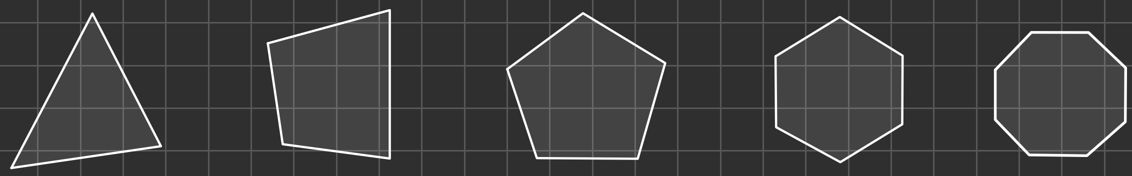
Corollary. Every polygon with $|V(P)| \geq 3$ has at least two ears.

Proof: Exercise

The number of triangulations of a fixed polygon P has much to do with the "shape".



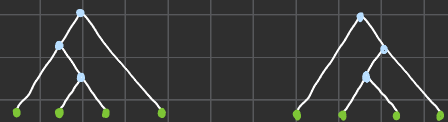
How many triangulations there are in a convex n-gon?

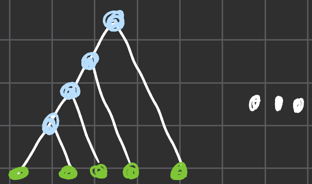


Binary Trees

A binary tree is a graph where each vertex has a maximum degree equal two.

The order of a binary tree is the number of vertices with degree 1 different to the root.





Dyck words

A word with alphabet consisting in only two letters, say $\{x, y\}$ is called Dyck-word if have the same number of x's and y's and in every "step" $\# x \geq \# y$

⓪ $\emptyset$ 1

① xy 1

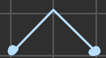
② $xxyy$, $xyxy$ 2

③ $xxxyyy$, $xxyxyy$, $xyxxyy$, $xyxyxy$, $xxyyxy$ 5

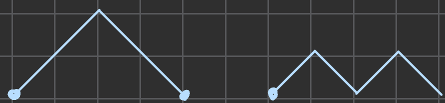
④ $xxxxyyyy$, ...

A Dyck path is a lattice path in the plane that starts at the origin $(0,0)$, consist of steps $(1,1)$ (up) $(1,-1)$ (down), stays on above the x-axis, and ends at the point $(2n,0)$ for a non-negative integer n.

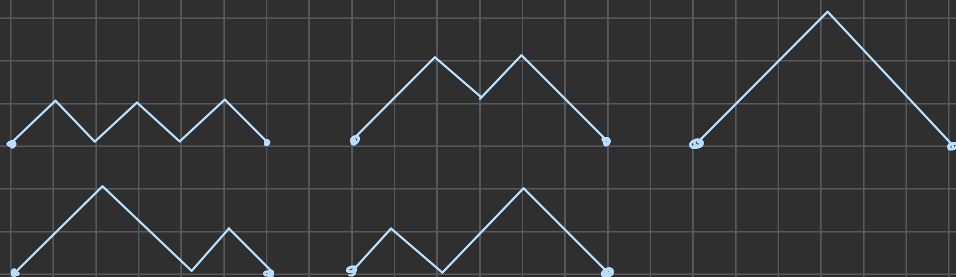
$n=1$



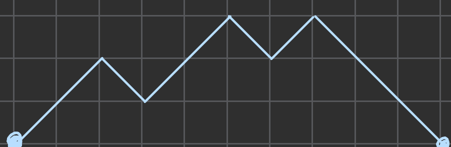
$n=2$



$n=3$

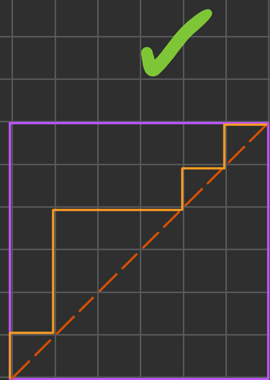


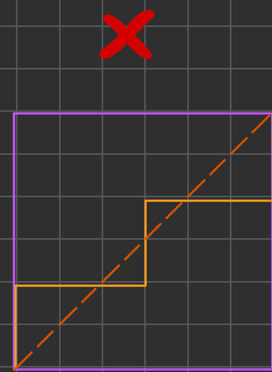
$n=5$



Lattice paths

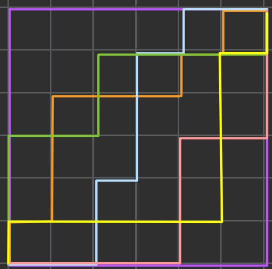
How many northeast lattice paths from $(0,0)$ to (n,n) don't pass below the $x=y$ diagonal?





Let's count it!!

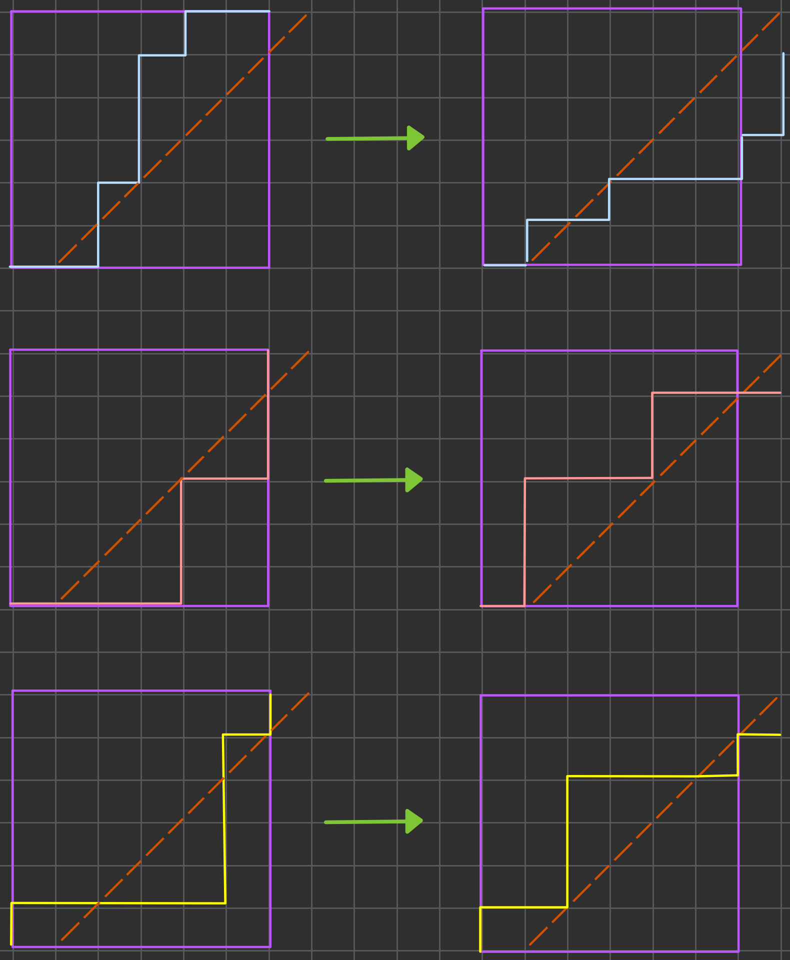
①



First let's count all the possible paths.

There are ____ to many paths goin from (0,0) to (n,n)

Bad paths reflections

② 

Observe that there is a bijection between every reflected bad path and the set of all possible paths going from (0,0) to (n+1, n-1).

There are ____ such paths.

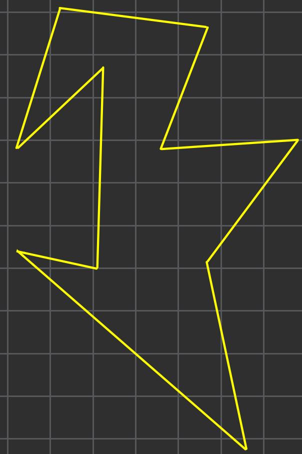
Finally by the inclusion-exclusion principle

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

Then: ① A convex n-gon admit C_{n-2} triangulations.

② There are C_n B.trees / D. words / D. paths / R lattice paths of order n.

Art gallery problem. (by Klee)



Our gallery (in $\mathbb{R}^2$) is:

- A simple polygon P (no holes, no autointersections)

Our guards are:

- A set of points $S \subset P$

We said that our gallery is safe if

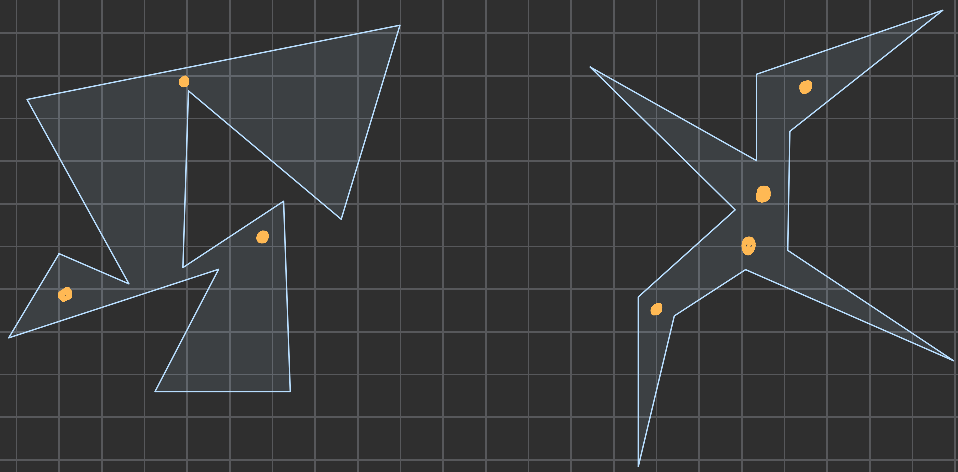
- Every point $p \in P$ can be "seen" by a point in S.

How many guards do we needs our gallery to be safe?

Can one guard keep safe the gallery?

If the guards are located in the corners (vertices) what is the small size of the set S?

We said that point x can see point y (or y is visible to x) iff the closed segment xy is nowhere exterior to the polygon P.



Two polygons of n=12 vertices: (a) requires 3 guards; (b) requires 4.

More formally:

Express as a function of n, the smallest number of guards that suffice to cover any polygon of n vertices.

Let $g(P)$ be the smallest number of guards needed to cover P.

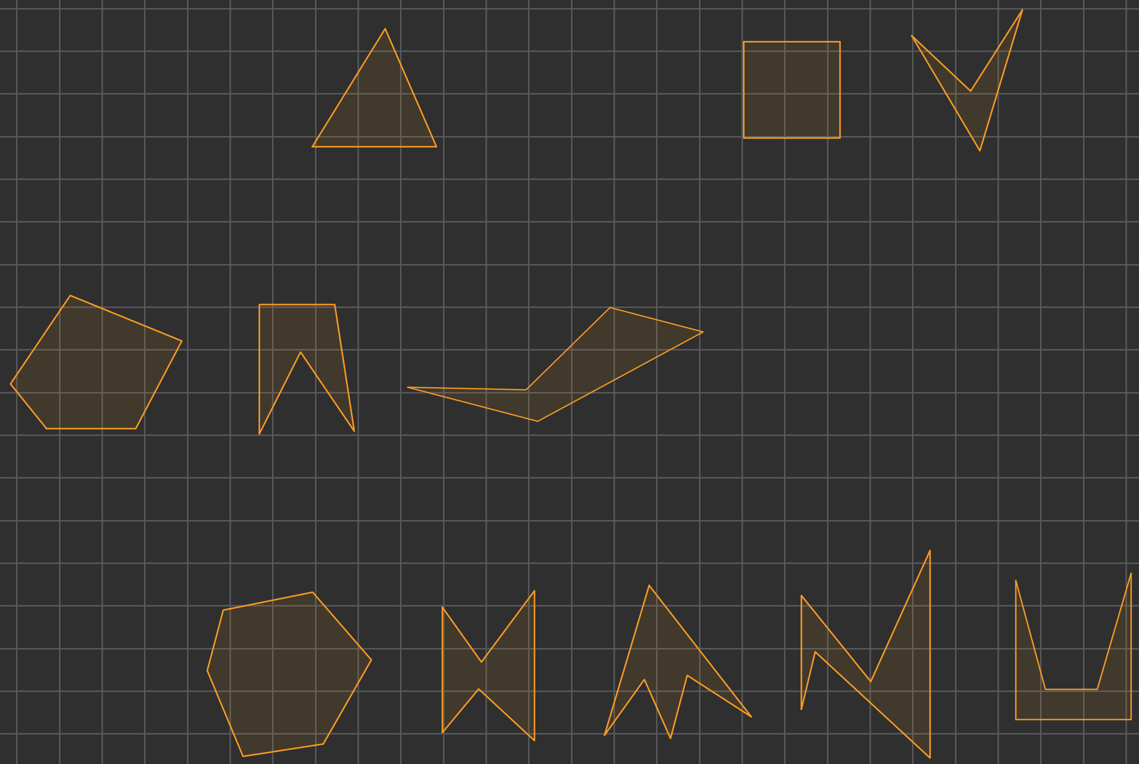
i.e., $$g(P) = \min_S |\{S : S \text{ covers } P\}|,$$

Let P_n be a polygon of n vertices, then we define

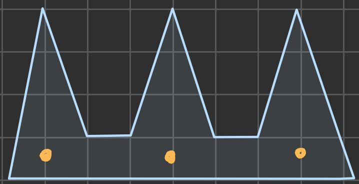
$$G(n) = \max_{P_n} g(P_n).$$

Then we are looking for $G(n)$.

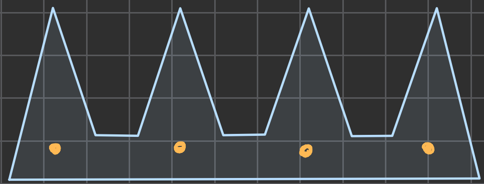
For a "small" n



We need at least $\lfloor n/3 \rfloor$



$n=9$



$n=12$

Chvátal construction

Then, is it true that $G(n) = \lfloor n/3 \rfloor$?

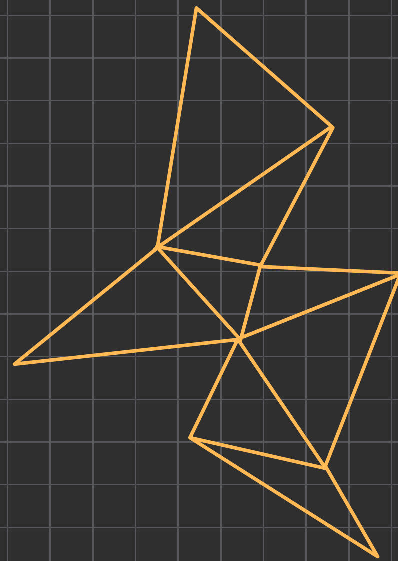
lemma: Every triangulation of a polygon is 3-colorable.

Proof: By induction on the number of vertices of P.

Base case: Consider the simple triangulation, a single triangle. Coloring each vertex with different colors there are no two adjacent with the same color.

Inductive hypothesis: Assume the lemma is valid for any triangulation of a polygon P with n vertices.

Inductive step: Now consider a polygon P with n+1 vertices. Choose a diagonal d that divides P into two smaller polygons P_1 and P_2. By inductive hypothesis these polygons can be 3-colored. Considering the colors assigned to the diagonal d in P_1 and perhaps after a possible permutation of the colors assigned to P_2, we obtain a 3-coloring of P.

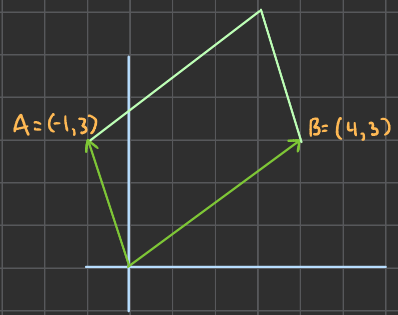


Thm [Fisk 1978]: $G(n) = \lfloor \frac{n}{3} \rfloor$.

Proof: Chvátal construction give us $G(n) \geq \lfloor \frac{n}{3} \rfloor$.
By the lemma every triangulation T of a polygon P is 3-colorable. Since every point in P lies in a triangle $t \in T$ and every point in a triangle is visible for all its vertices, choosing one chromatic class we can see all the points of P.

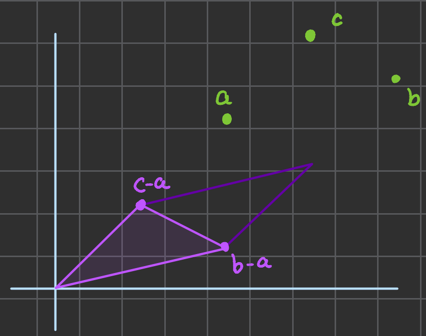
Area of a Triangle.

From linear algebra we know that if A and B are vectors, then the cross product $|A \times B|$ determine the area of the parallelogram with sides A and B.



$$|A \times B| = \left|\det\begin{pmatrix} -1 & 3 \\ 4 & 3 \end{pmatrix}\right| = |-3-12| = 15$$

Then for a,b,c points in $\mathbb{R}^2$ we have $\text{Area}\,\Delta abc = \frac{1}{2}|(b-a)\times(c-a)|$

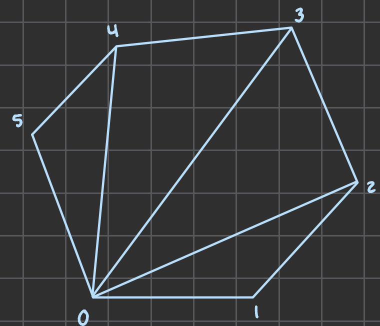


Lemma: Twice the area of a triangle $T=(a,b,c)$ is given by

$$2A(T) = \begin{vmatrix} a_0 & a_1 & 1 \\ b_0 & b_1 & 1 \\ c_0 & c_1 & 1 \end{vmatrix} = (b_0-a_0)(c_1-a_1)-(c_0-a_0)(b_1-a_1)$$

Area of a Polygon.

$$A(P) = A(v_0,v_1,v_2)+A(v_0,v_2,v_3)+\ldots+A(v_0,v_{n-2},v_{n-1}).$$



Area of a quadrilateral

$$A(Q) = A(a,b,c) + A(a,c,d) = A(d,a,b) + A(d,b,c)$$

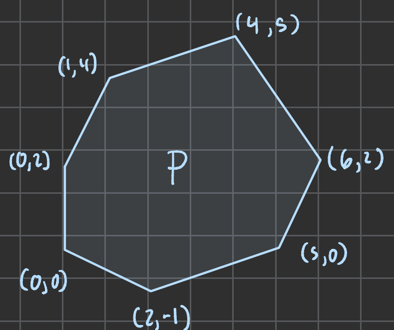
$$\Rightarrow \quad 2A(Q) = a_0b_1 - a_1b_0 + a_1c_0 - a_0c_1 + b_0c_1 - c_0b_1$$
$$+ a_0c_1 - a_1c_0 + a_1d_0 - a_0d_1 + c_0d_1 - d_0c_1$$

$$= a_0b_1 - a_1b_0 + b_0c_1 - c_0b_1 + a_1d_0 - a_0d_1 + c_0d_1 - d_0c_1$$
$$= a_0b_1 - a_1b_0 + b_0c_1 - b_1c_0 + c_0d_1 - c_1d_0 + d_0a_1 - d_1a_0$$

In general for a convex polygon P

$$2A(P) = \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i)$$

$A(P)$

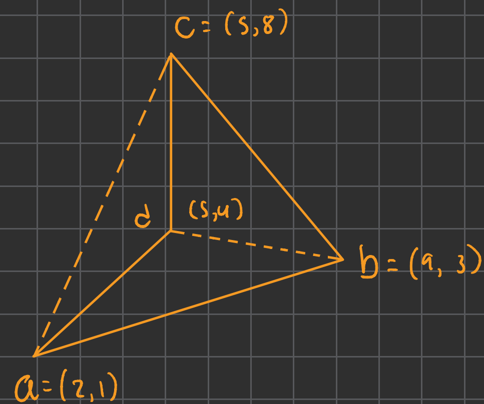


$$\begin{vmatrix} 0 & 0 \\ 2 & -1 \\ 5 & 0 \\ 6 & 2 \\ 4 & 5 \\ 1 & 4 \\ 0 & 2 \end{vmatrix} = \frac{1}{2}\left|\left[0(-1) + 2(0) + 5(2) + 6(5) + 4(4) + 1(2)\right] - \left[0(2) + (-1)(5) + 0(6) + 2(4) + 5(1) + 4(0)\right]\right|$$

$$= \frac{1}{2}\left|(10+30+16+2) - (-5+8+5)\right|$$

$$= 25$$

Area of a Nonconvex Quadrilateral.



$$\mathcal{A}(Q) = \mathcal{A}(a,b,c) + \mathcal{A}(a,c,d)$$

$$\mathcal{A}(Q) = \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 9 & 3 & 1 \\ 5 & 8 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 5 & 8 & 1 \\ 5 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[(6+72+5)-(15+16+9) + (16+5+20)-(40+8+5)\right]$$

$$= \frac{1}{2}\left[(83-40)+(41-53)\right]$$

$$= \frac{1}{2}\left[43-13\right]$$

$$= 15$$

Theorem [Area of Polygon]. Let a polygon (convex or nonconvex) P have vertices $v_0, \dots, v_{n-1}$ labeled counterclockwise, and let p be any point in the plane. Then

$$A(P) = A(p, v_{n-2}, v_{n-1}) + A(p, v_1, v_2) + A(p, v_2, v_3) + \dots + A(p, v_{n-2}, v_{n-1}) + A(p, v_{n-1}, v_0)$$

If $v_i = (x_i, y_i)$ $\Rightarrow$

$$2A(P) = \sum_{i=0}^{n-1} (x_i y_{i+1} - y_i x_{i+1})$$

$$= \sum_{i=0}^{n-1} (x_i + x_{i+1})(y_{i+1} - y_i)$$