

Project Outline: A Brief Introduction To Linear Transformations

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Contents

1	Idea of Transformations	1
1.1	Linear Transformation	2
1.2	Non-Linear Transformation	3
1.3	Examples	4
2	The Matrix of a Linear Transformation	8
2.1	Rule of Linearity	8
2.2	Integrals	9
2.3	Matrices for the Derivative and Integral	9
2.4	Construction of the Matrices	10
2.5	Productivity of AB Matrix Transformations	10
3	Arnold's Cat Map	11
4	Fractals	11
4.1	FRACTALS PICTURES	12
5	Conclusion	14

Abstract

A Linear Transformation will take a vector V from one vector space to another. This simple idea can be much more than one vector transformed to another vector. Here will be the exploration of Linear Transformations.

1 Idea of Transformations

You might be most familiar with the linear transformation of

$$A\vec{x} = b$$

Matrix A is our function matrix, \vec{x} is our input, and b is our output. Our function, A , takes an input, \vec{x} , and gives an output, b . This is also written as

$$A(\vec{v}) = b$$

The transformation function will change \vec{v} to $A\vec{v}$. We can take a stick house and transpose the end points of the stick to get a warped house. If we take this a step further we can take a full picture and warp its shape to get a new picture.

1.1 Linear Transformation

It is very important to find if a transformation matrix is linear or non-linear because we are looking in to the linear aspect of transformations. The laws of a linear transformation are

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$$

$$T(c\vec{v}) = cT(\vec{v})$$

$$T(\vec{0}) = \vec{0}$$

Note that $T(\vec{0}) = \vec{0}$ comes from $T(0\vec{v}) = \vec{0}$. We can combined two of these equations to reduce the checks for a linear transformation of

$$T(c\vec{v} + d\vec{w}) = cT(\vec{v}) + dT(\vec{w})$$

First we test this where the equation is

$$A\vec{0} = \vec{0}$$

Any matrix A multiplied by the $\vec{0}$ and still will receive $\vec{0}$. Now to test

$$T(c\vec{v} + d\vec{w}) = cT(\vec{v}) + dT(\vec{w})$$

Fist sub into the equation $T(c\vec{v} + d\vec{w})$

$$A(c\vec{v} + d\vec{w}) = cA\vec{v} + dA\vec{w}$$

Working from the left the A matrix can be factored into $c\vec{v}$ and $d\vec{w}$.

$$Ac\vec{v} + Ad\vec{w} = cA\vec{v} + dA\vec{w}$$

The final step is to bring c and d to the front because they are constants.

$$cA\vec{v} + dA\vec{w} = cA\vec{v} + dA\vec{w}$$

This has successfully proved that any matrix A is a linear transformation of \vec{x}

1.2 Non-Linear Transformation

A Non-Linear transformation is any transformation that fails the two equations

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$$

$$T(c\vec{v}) = cT(\vec{v})$$

A non-linear transformation includes adding, subtracting, powers, and magnitude. Looking back at the original laws of linear transformations we will test these. Adding and subtracting fail the first law, $A + \vec{x}$.

$$A + (\vec{w} + \vec{v}) = (A + (\vec{w})) + (A + (\vec{v}))$$

$$A + \vec{w} + \vec{v} = 2A + \vec{w} + \vec{v}$$

The right side will be A bigger than the left. Powers don't work because you can't square a vector due to improper dimension, an $N \times 1 * N \times 1$, and therefore cannot be put to a higher power either, $(\vec{x})^2$. Magnitude fails both laws of linear transformations, $\|\vec{x}\|$.

$$\|\vec{w} + \vec{v}\| = \|\vec{w}\| + \|\vec{v}\|$$

But we know,

$$\|\vec{w} + \vec{v}\| \leq \|\vec{w}\| + \|\vec{v}\|$$

and by substitution we get

$$\|\vec{w}\| + \|\vec{v}\| \geq \|\vec{w}\| + \|\vec{v}\|$$

You can fail the first test by picking two vectors that are not scalars of each other. Now will be the test for the second linear test, we use a scalar which is $-c$ where c is a positive integer.

$$\| -c\vec{v} \| = -c\|\vec{v}\|$$

The magnitude of \vec{v} is the same as the magnitude of $-\vec{v}$ because magnitude removes direction and leaves only the distance.

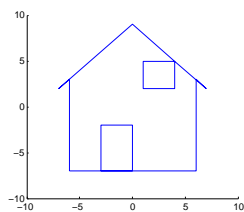
$$c\|\vec{v}\| = -c\|\vec{v}\|$$

When a transformation is non-linear it does not mean that you cannot do the transformation, it means that you will not get to the goal of your transformation. The goal is to transform some vectors to get a new set of vectors, thus outputting a new transformed image.

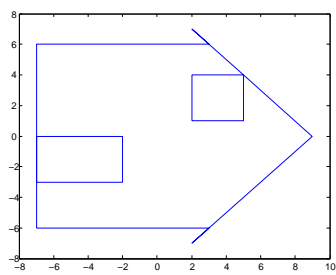
1.3 Examples

First we will go simple with just a few points that we will transpose. The matrix H is a series of points of a house. Put this matrix H into MATLAB and the following image results.

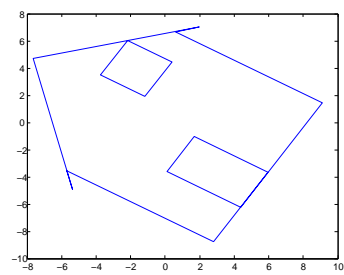
$$H = \begin{bmatrix} -6 & -6 & -7 & 0 & 4 & 1 & 1 & 4 & 4 & \dots \\ -7 & 3 & 2 & 9 & 5 & 5 & 2 & 2 & 5 & \dots \end{bmatrix}$$
$$H = \begin{bmatrix} \dots & 7 & 6 & 6 & -3 & -3 & 0 & 0 & -6 \\ \dots & 2 & 3 & -7 & -7 & -2 & -2 & -7 & -7 \end{bmatrix}$$



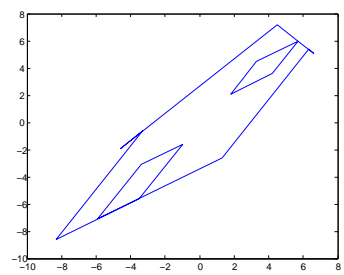
Hit this house with matrix A to get these various houses.



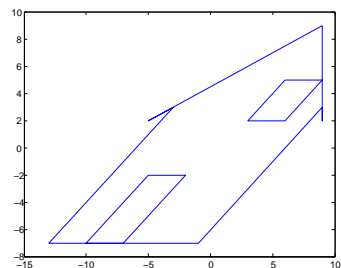
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$$A = \begin{bmatrix} \cos(45) & -\sin(45) \\ \sin(45) & \cos(45) \end{bmatrix}$$



$$A = \begin{bmatrix} .8 & .5 \\ .5 & .8 \end{bmatrix}$$



$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Similarly you can hit any picture with these same matrices and the whole picture will warp. The matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Takes a coordinate (x,y) and changes it such that the top line of your matrix is the ratio of your current x and y that will be used to make the new x. The bottom row determines what will be the new y coordinate.

2 The Matrix of a Linear Transformation

2.1 Rule of Linearity

Suppose we have the equation:

$$v = c_1 v_1 + \dots + c_n v_n.$$

Then linearity requires that

$$T(v) = c_1 T(v_1) + \dots + c_n T(v_n).$$

This basic rule of linearity extends from the linear combinations of $cv + dw$ to all combinations $c_1v_1 + \dots + c_nv_n$. An Example of Linearity with transformation matrices. Suppose T transforms $v_1 = (1, 0)$ to $T(v_1) = (5, 8, 10)$. Suppose the second basis vector $v_2 = (0, 1)$ goes to $T(v_2) = (4, 5, 10)$. If T is linear from \mathbb{R}^2 to \mathbb{R}^3 then its "standard matrix" is 3 by 2. Those outputs go into its columns:

$$A = \begin{bmatrix} 5 & 4 \\ 8 & 5 \\ 10 & 10 \end{bmatrix} T(v_1 + v_2) = T(v_1) + T(v_2) \text{ is } \begin{bmatrix} 5 & 4 \\ 8 & 5 \\ 10 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 20 \end{bmatrix}.$$

2.2 Integrals

We know that the **integral** is the inverse of the derivative. This is the Fundamental Theorem of Calculus, when applied in linear algebra, the transformation T^{-1} "takes the integral from 0 to x " is linear! If we apply T^{-1} to $1, x, x^2$, which will be defined as w_1, w_2, w_3 :

$$\begin{aligned} x &= \int_0^x 1 \, dx \\ \frac{1}{2}x^2 &= \int_0^x x \, dx \\ \frac{1}{3}x^3 &= \int_0^x x^2 \, dx \end{aligned}$$

By the rule of linearity, the integral of the function $w = B + Cx + Dx^2$ is $T^{-1}(w) = Bx + \frac{1}{2}Cx^2 + \frac{1}{3}Dx^3$. The integral of a quadratic is a cubic. The integral of a cubic is a quartic. The input space is the quadratics, the output space is the cubics. **Integration take W back to V.**

2.3 Matrices for the Derivative and Integral

The derivative transforms the space V of cubics to the space W of quadratics. The basis for V is $(1, x, x^2, x^3)$. The basis for W is $(1, x, x^2)$. THE MATRIX THAT "TAKES THE DERIVATIVE" IS 3 BY 4:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \text{matrix form of derivative } T$$

Why is A the correct matrix your ask? Because multiplying by A agrees with transforming by T . The derivative of $v = a + bx + cx^2 + dx^3$ is $T(v) = b + 2cx + 3dx^2$. The same b and $2c$ and $3d$ appear when we multiply by the matrix.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$$

Look also at T^{-1} . The integration matrix is 4 by 3. Watch how the following matrix starts with $w = B + Cx + Dx^2$ and produces its integral $Bx + \frac{1}{2}Cx^2 + \frac{1}{3}Dx^3$:

$$\text{Integration : } \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ B \\ \frac{1}{2}C \\ \frac{1}{3}D \end{bmatrix}$$

2.4 Construction of the Matrices

Each linear transformation T from V to W is represented by a matrix A (after the bases are chosen for V and W). The j th column of A is found by applying T to the j th basis vector v_j :

$$T_{vj} = \text{combination of basis vectors of } W = a_{1j}w_1 + \dots + a_{mj}w_m.$$

These numbers a_{1j}, \dots, a_{mj} go into column j of A . The matrix is constructed to get the basis vectors right. Then linearity gets all the other vectors right. Every v is a combination $c_1v_1 + \dots + c_nv_n$, and $T(v)$ is a combination of the w 's. When A multiplies the coefficient vector $c = (c_1, \dots, c_n)$ in the v combination, Ac produces the coefficients in the $T(v)$ combination. This is because matrix multiplication is linear like T .

2.5 Productivity of AB Matrix Transformations

With the examples of integration matrices and derivative matrices, we can see that T comes down to a matrix A as well. with the projection matrix A , if you square A it doesn't change. Projecting twice is the same as projecting once. We can say that $T^2 = T$ or $A^2 = A$. This is something very important, the reason for the way matrices are multiplied. Two transformations T and S are represented by two matrices B and A . The transformation T is from a space U to V . Its matrix B uses a basis in u_1, \dots, u_p defined as U and a basis v_1, \dots, v_n defined as V . The matrix B is then a n by p matrix. Then, the transformation S is from V to W . Its matrix has to use the same basis of v_1, \dots, v_n for V . We can then say that the linear transformation TS starts with any vector in u in U , goes to $S(u)$ in V and then to $T(S(u))$ in W . Since linear transformations boil down to matrices, the matrix AB starts with any x in \mathbb{R}^p , goes to Bx in \mathbb{R}^n and then to ABx in \mathbb{R}^m . The matrix AB **represents** the two transformations.

$$TS : U \rightarrow V \rightarrow W$$

$$AB : (m \times n)(n \times p) = (m \times p)$$

The input is $\mathbf{u} = x_1\mathbf{u}_1 + \dots + x_p\mathbf{u}_p$. The output $T(S(\mathbf{u}))$ is the same as the output of $AB\mathbf{x}$. **The product of transformations is the same as the product of matrices.**

3 Arnold's Cat Map

Arnold's Cat Map is a famous example of matrix transformations and chaos.

The cat map is called Arnolds cat map in recognition of Russian mathematician Vladimir I. Arnold, who discovered it using an image of a cat. The essence of the cat map is that there is a $n \times n$ image of pixels which can be the be represented by a $n \times n$ matrix. Where the entries in the matrix are certain numerical values of colors. When that matrix is then "wacked" with a transformation matrix, the pixels or entries in the matrix appear to be randomized, creating an entirely new image. But after wacking each matrix created with the transformation matrix, the original image will once again appear!

So let $A = \begin{bmatrix} x \\ y \end{bmatrix}$ is some $n \times n$ image. Arnolds transformation is defined as

$$\Gamma \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x + y \\ x + 2y \end{bmatrix} \bmod n$$

If we break down what Arnolds transformation is really doing to our image we would see: (insert original image) that it alters our x value by a taking the x value, adding y to it, then y value changes by taking this new x value and adding to y.(insert altered image) We then have a total new number in each original x and y place, giving us the look of a parallelogram. Then the mod n is evaluated, reforming the parallelograms parts into the $n \times n$ square.

If we look at this in actual program, this it what it would look like.

4 Fractals

Fractals are the most intriguing and mysterious part of mathematics in my opinion. Fractals are also an application of liner transformations because you transform your original set into a new set **using** transformations. The study of fractals generated by *Iterated Function Systems or (IFS)* IFS fractals can be of any number of dimensions. The fractal is made up of the union of several copies of itself, each copy being transformed by a function, hence "function system". The functions are normally contractive, In non-technical terms, a contraction mapping brings every two points x and y in some space M closer together and makes the shapes of itself smaller. Hence the shape of an IFS fractal is made up of several smaller copies of itself, each of which is also made up of copies

of itself, do this infinitely many times. After we do that we can then see the nature of a fractal.

The simplest IFS consists of iteration of a single map $(F) : Y \Rightarrow Y$, where Y is a real line, a plane, or a set of images of closed bounded subsets of \mathbb{R}^k .

If $x \in Y$ the **orbit** of x is said to be:

$$x, F(x), F(F(x)) = F^2(x), F(F(F(x))) = F^3(x), F(F(F(F(x)))) = F^4(x), \dots$$

and F^n is the n th iteration of F . Now what happens as $n \rightarrow \infty$? Let consider the transformation $F : \mathbb{R} \rightarrow \mathbb{R}$ and the transformation is defined as $x \rightarrow ax + b, a \neq 1$. Then

$$F^n(x) = a^n x + b \left(\frac{1 - a^n}{1 - a} \right)$$

for an $x \in \mathbb{R}$ and any $N \geq 1$. Then if $|a| < 1$ ALL orbits are contractive to the number $z = \frac{b}{(1-a)}$. This z number is a fixed point such that $F(z) = z$. As $|a|$ is greater than 1 though, all orbits go to ∞ so we restrict our look at fractals to $|a| < 1$. With this restriction, the equation of our contraction looks like:

$$|F(x) - F(y)| = |a||x - y|$$

This makes so that the transformation applied shrinks its length by a factor of $|a|$. Now with this in mind, consider the transformation $F : \mathbb{R}^k \rightarrow \mathbb{R}^k$, if F is a contraction transformation; applying F to a set shrinks the area of that set. And if $x \in \mathbb{R}^k$, what would happen to the orbit of x when the transformation? If F is linear, it has a unique fixed point z ! Then ALL orbits converge to z !

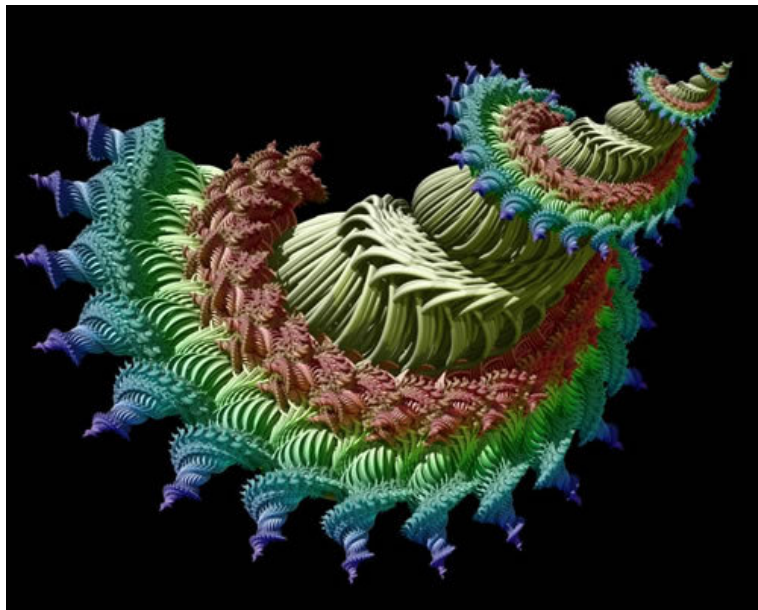
Now the imagining of a single contracting function in either \mathbb{R} or \mathbb{R}^k isn't so much of a reach, but when we start to look at iterated function systems with more than one function, the fixed point to which the orbits converge to can be extremely difficult to see. But if we look at the the functions united, the iterated function starting at an initial point, they will generate subsets of \mathbb{R} . With a closer look at the orbits around the initial point and limiting the values which go into our union of functions, we would find that there is still a fixed point, but now there is an image that the function begins to resemble constantly, known as the **attractor** to the IFS. Let us say that fixed point is C , such that $F(C) = C$. C is then on the attractor.

To wrap up! The question you keep asking yourself though is "where does linear transformations come into play with fractals?!" Well thus far we've said little about what is actually happening when we create a fractal; what we are simply doing is a linear transformation along with a translation, or creating a **affine map** of the plane. We also now know that for any finite collection of contracting affine maps of a plane, all orbits converge to a unique fixed point while also converging to an attractor, or image that repeats itself within the fractal.

4.1 FRACTALS PICTURES



A Fractal Cow!



A Really Cool Fractal Spiral.



Romanesque cauliflower - a perfect fractal.

5 Conclusion

A linear transformation is simply $A\vec{x} = \vec{b}$. Any other form, such as $A+x=b$ or $\|\vec{v}\| = \vec{b}$, is a non-linear transformation. A linear transformation models derivatives and integrals of a polynomial. With the idea that you can use a matrix that brings a vector from \mathbb{R}^2 to \mathbb{R}^2 you can put the output of a \vec{b} and put it back into the input \vec{x} . With Arnold's Cat Map you can do enough iterations so that the image will restore itself. A Fractal is different though. With inf iterations you start to approach a shape called the attraction. All done by $A\vec{x} = \vec{b}$ we can do inf iterations and create a new image or the same.

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