Seminar 3 Homework Suciu Antonio, 937

22.10.2022

1. Given the grammar

$G = (\{S, H\}, \{b, c, d, e\}, \{S \to b^2Se \mid H, H \to cHd^2 \mid cd\}, S)$

find $L(G)$. (+proof)

$S \to b^2Se$ (1)

$S \to H$ (2)

$H \to cHd^2$ (3)

$H \to cd$ (4)

• Let $L = \{ b^{2n} c^{m+1} d^{2m+1} e^n \mid m, n \in \mathbb{N} \}$

• We prove that $L = L(G) \iff \begin{cases} 1) \ L \subseteq L(G) \\ 2) \ L(G) \subseteq L \end{cases}$

1) $\forall n, m \in \mathbb{N}$, $b^{2n} c^{m+1} d^{2m+1} e^n \in L(G)$

• Take $P(n, m)$: $b^{2n} c^{m+1} d^{2m+1} e^n \in L(G)$

• We have to prove $P(n, m)$ - true, $\forall n, m \in \mathbb{N}$

I. Base case:

Let $n = m = 0$.

$P(0,0)$: $b^0 c d e^0 \in L(G)$ - true:

$S \underset{(2)}{\Rightarrow} H \underset{(4)}{\Rightarrow} cd \in L(G)$ - true

II. Induction over m:

We assume that $P(0, k)$ is true. We need to prove that

$P(0,k) \to P(0, k+1)$ - true

$P(0, k+1)$: $c^{k+1} d^{2k+1} \in L(G)$

~~$c^{k+1}d$~~

$c^{k+1} d^{2k+1} \in L(G) \underset{\text{ind.hyp}}{\Rightarrow} \cancel{S \overset{*}{\Rightarrow}}\; S \underset{(2)}{\Rightarrow} H \underset{(4)}{\overset{*}{\Rightarrow}} c^{k+1} d^{2k+1}$

$S \Rightarrow H \underset{(3)}{\Rightarrow} cHd^2$ $\Big\} \Rightarrow$

$\Rightarrow S \Rightarrow H \underset{\text{ind.hyp}}{\Rightarrow} c\, c^{k+1} d^{2k+1} d^2 = c^{(k+1)+1} d^{2(k+1)+1} \in L(G)$

$\Rightarrow P(0,k) \rightarrow P(0,k+1)$ - TRUE

III Induction over n

We assume that $P(k,0)$ - true. We prove $p(k,0) \Rightarrow p(k+1,0)$ - t

$P(k,0)$: $b^{2k} c d e^k \in L(G)$

$b^{2k} cde^k \in L(G) \underset{\text{ind.hyp}}{\Leftrightarrow} S \underset{(1)}{\overset{*}{\Rightarrow}} b^{2k} S e^k \underset{(3,4)}{\Rightarrow} b^{2k} cd e^k$

$S \underset{(1)}{\Rightarrow} b^2 S e$ $\Big\} \Rightarrow$

$\Rightarrow S \underset{\text{ind. hyp}}{\Rightarrow} b^2 b^{2k} S e e^k \underset{(3,4)}{\Rightarrow} b^{2k+2} cd e^{k+1} \in L(G)$

$\Rightarrow P(k,0) \rightarrow P(k+1,0)$ - True

$\underline{I}, \underline{II}, \underline{III} \rightarrow L \subseteq L(G)$

2) $L(G) \subseteq L$; $L = \{ b^{2n} c^{m+1} d^{2m+1} e^n \mid m,n \in N \}$

a) $H \underset{(4)}{\Rightarrow} cd$

$(3) \Downarrow$ $cHd^2 \underset{(4)}{\Rightarrow} c^2 d^3$

$\Downarrow$ $c^2 H d^4 \underset{(4)}{\Rightarrow} \dots$

$(3) \Downarrow$ $\dots$

$\Rightarrow$ H can only generate sequences of the shape $c^{m+1} d^{2m+1}$, $m \in N$

b) $S \underset{2,a)}{\Rightarrow} cd$ $\quad S \underset{2),a)}{\Rightarrow} c^{m+1} d^{2m+1}$

$1) \Downarrow$ $b^2 S e \underset{2),a)}{\Rightarrow} b^2 c^{m+1} d^{2m+1} e$

$1) \Downarrow$ $b^4 S e^2 \underset{2,a)}{\Rightarrow} \dots$

$1) \Downarrow$

$\Rightarrow$ S can only generate sequences of the shape $b^{2n} c^{m+1} d^{2m+1} e^n$, $m,n \in N$

$\Rightarrow \underline{a), b)} \Rightarrow$ 2) - true

1), 2) $\Rightarrow L = L(G)$ Q.E.D.

Seminar 3 — Homework — Suciu Antonio, gr. 937, 22.10.2022

2). Find all the grammars that generate the following languages

a) $L_1: \{x^n y^n \mid n \in \mathbb{N}\}$ + proof

b) $L_2: \{a^n b^{2n} \mid n \in \mathbb{N}^*\}$ + proof

c) $L_3: \{a^n b^m \mid n, m \in \mathbb{N}^*\}$ - regular grammar required + proof

d) $L_4: \{x^{2n} \mid n \in \mathbb{N}\}$, $L_4' = \{x^{2n} \mid n \in \mathbb{N}^*\}$ - regular grammars required + proof

e) $\mathbb{N}$

f) All arithmetic expressions containing $\boxed{a}$ as operand, $+$ $*$ as operators, and ( ).

a) $L_1: \{x^n y^n \mid n \in \mathbb{N}\}$

$?G$, s.t. $L(G) = L$; $G = (\{S\}, \{\varepsilon, x, y\}, \{S \to xSy \mid \varepsilon\}, S)$

P: $S \Rightarrow xSy \mid \varepsilon$; $S \to \varepsilon$ (1)
$S \to xSy$ (2)

$L(G) = L \iff \begin{cases} 1) \ L(G) \subseteq L \\ 2) \ L \subseteq L(G) \end{cases}$

1) $L(G) \subseteq L$

$S \xRightarrow{(1)} \varepsilon$

$\xRightarrow{(2)} xSy \xRightarrow{(1)} xy$

$\xRightarrow{(2)} x^2Sy^2 \xRightarrow{(1)} x^2y^2$

$\xRightarrow{(2)} x^3Sy^3 \xRightarrow{(1)} \dots$

$\xRightarrow{(2)} \dots$

$\Rightarrow$ S can only generate sequences of the shape $x^n y^n$, $n \in \mathbb{N}$

Scanned with CamScanner

2) $L \subseteq L(G)$

$\forall n \in \mathbb{N}, x^n y^n \in L(G)$

Take $P(n): x^n y^n \in L(G)$, $\cancel{\forall n \in \mathbb{N}}$

Prove that $P(n)$ holds, $\forall n \in \mathbb{N}$.

I. Verification

$n = 0 \to P(0): x^0 y^0 \in L(G) \Rightarrow \varepsilon \in L(G)$ - true

II. Assume $p(k)$ - true. Prove that $p(k) \to p(k+1)$ - true

$p(k): x^k y^k \in L(G)$, $k \in \mathbb{N}$ ~~(hyp.ind~~ (ind. hyp)

$$\left.\begin{array}{l} S \overset{*}{\Rightarrow} x^k y^k \\ S \underset{(2)}{\Rightarrow} xSy \end{array}\right\} \Rightarrow S \underset{\text{ind. hyp}}{\Rightarrow} x x^k y^k y \Rightarrow S = x^{k+1} y^{k+1} \Rightarrow P(k) \to P(k+1) \text{ - true}$$

$\overset{I, II}{\Longrightarrow} P(n)$ - true, $\forall n \in \mathbb{N}$

$\overset{1), 2)}{\Longrightarrow} L = L(G)$ Q.E.D.

b) $L_2: \{a^n b^{2n} \mid n \in \mathbb{N}^*\}$ + proof

? G, s.t. $L = L(G)$; $G = (\{S\}, \{a, b\}, \{S \to aSb^2 \mid ab^2\}, S)$

$$P: \begin{array}{l} S \to ab^2 \ (1) \\ S \to aSb^2 \ (2) \end{array} ; \quad L = L(G) \Leftrightarrow \left\{ \begin{array}{l} 1) L \subseteq L(G) \\ 2) L(G) \subseteq L \end{array} \right.$$

1) $L \subseteq L(G)$

Take $P(n): a^n b^{2n} \in L(G)$. Prove that $P(n)$ holds, $\forall n \in \mathbb{N}^*$

I. Verification

$n = 1 \to P(1): ab^2 \in L(G) \Leftrightarrow S \to ab^2$ - true $\Rightarrow P(1)$ - true

2)

II. Assume $P(k)$-true. Prove $P(k) \Rightarrow P(k+1)$-true

$P(k): a^k b^{2k} \in L(G), k \in \mathbb{N}^*$ (ind. hyp)

$$\left.\begin{array}{l} S \stackrel{*}{\Rightarrow} a^k b^{2k} \\ S \underset{(2)}{\Rightarrow} a S b^2 \end{array}\right\} \Rightarrow S \underset{\substack{\text{ind.}\\ \text{hyp}}}{\Rightarrow} a a^k b^{2k} b^2 = a^{k+1} b^{2(k+1)} \Rightarrow P(k) \Rightarrow P(k+1) \text{ true}$$

$\overset{I, II}{\Longrightarrow} P(n)$-true, $\forall n \in \mathbb{N}^*$

2) $L(G) \subseteq L$

$$\begin{array}{l} S \underset{(1)}{\Rightarrow} ab^2 \\ \Downarrow (2) \\ aSb^2 \underset{(1)}{\Rightarrow} a^2b^4 \\ \quad \Downarrow \\ \quad a^2Sb^4 \underset{(1)}{\Rightarrow} \dots \\ \qquad \Downarrow (2) \\ \qquad \dots \end{array}$$

$\Rightarrow$ S can only generate sequences of the shape $a^n b^{2n}$, $n \in \mathbb{N}^*$

$\underset{1,2)}{\Rightarrow}$ $L = L(G)$, Q.E.D.

3)