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Wiley's CFA® Program Exam Review
Study Guide for 2020
Level II CFA Exam

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Wiley's Study Guides are written by a team of highly qualified CFA charterholders and leading CFA instructors from around the globe. Our team of CFA experts work collaboratively to produce the best study materials for CFA candidates available today.

Wiley's expert team of contributing authors and instructors is led by Content Director Basit Shajani, CFA. Basit founded online education start-up Élan Guides in 2009 to help address CFA candidates' need for better study materials. As lead writer, lecturer, and curriculum developer, Basit's unique ability to break down complex topics helped the company grow organically to be a leading global provider of CFA Exam prep materials. In January 2014, Élan Guides was acquired by John Wiley & Sons, Inc., where Basit continues his work as Director of CFA Content. Basit graduated magna cum laude from the Wharton School of Business at the University of Pennsylvania with majors in finance and legal studies. He went on to obtain his CFA charter in 2006, passing all three levels on the first attempt. Prior to Élan Guides, Basit ran his own private wealth management business. He is a past president of the Pakistani CFA Society.

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Study Session 12: Fixed Income (1)

Reading 32: The Term Structure and Interest Rate Dynamics

LESSON 1: SWAP RATES AND FORWARD RATES

LOS 32a: Describe relationships among spot rates, forward rates, yield to maturity, expected and realized returns on bonds, and the shape of the yield curve. **Vol 5, pp 6–18**

LOS 32b: Describe the forward pricing and forward rate models and calculate forward and spot prices and rates using those models. **Vol 5, pp 8–16**

LOS 32c: Describe how zero-coupon rates (spot rates) may be obtained from the par curve by bootstrapping. **Vol 5, pp 15–16**

LOS 32d: Describe the assumptions concerning the evolution of spot rates in relation to forward rates implicit in active bond portfolio management. **Vol 5, pp 21–23**

LOS 32e: Describe the strategy of riding the yield curve. **Vol 5, pp 18–23**

Spot Rates and Forward Rates

Some introductory terminology and a review of Level I material are presented in the following bullets:

- The **discount factor** with maturity T, denoted by P(T), refers to the value of a risk-free payment of $1 that will be made at time T.
- The discount factor for a range of maturities in years is called the discount function.
- The **spot rate** for maturity T, denoted by ${}_{T}s_{0}$ (the T-year spot rate as of today, t = 0), is the yield to maturity (YTM) of this $1 payment that will be made at time T.
  - Note that the curriculum denotes the T-year spot rate as of today as r(T).
- The spot rate for a range of maturities in years is called the spot yield curve (or spot curve).
- The spot curve shows, for a range of maturities, the annualized return on an option-free and default-risk-free zero-coupon bond with a single payment of principal at maturity.
- Spot rates are based on market prices of these option-free zero-coupon bonds at any point in time. As the prices of these bonds fluctuate, the shape and level of the spot yield curve also change.
- The advantage of using spot rates is that because they represent the yield on bonds that make no payments during their terms, there is no element of reinvestment risk. The stated yield equals the realized yield if the bond is held until maturity.
- A **forward rate** is an interest rate that is determined today for a loan that will be initiated in the future. We denote the x-year forward rate y years from today as ${}_{x}f_{y}$. Here, a loan with a term of x years will be initiated y years from today.
  - For example, the three-year forward rate two years from today will be denoted as ${}_{3}f_{2}$.
- The curriculum denotes x-year forward rate y years from today as f(T*,T), where T* represents the point in time when the underlying loan will be initiated, and T represents the term of the underlying loan.
  - For example, the three-year forward rate two years from today will be denoted as f(2,3).

Note that the discount function and the spot curve contain the same set of information about the time value of money.

- The term structure of forward rates for a loan made on a specific initiation date is called the **forward curve**.
- Forward rates and forward curves are mathematically derived from the current spot curve.

The Forward Pricing Model

The **forward pricing model** describes the valuation of forward contracts. It is based on the no-arbitrage argument.

$$\boxed{\text{Forward pricing model : } P(T^* + T) = P(T^*)\, F(T^*, T)}$$

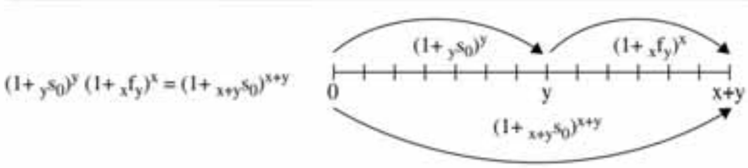
To understand the reasoning behind the forward pricing model, consider two alternative investments:

- Investment 1: Buy a zero-coupon bond today with a par value of $1 that matures in $T^* + T$ years.
  - This bond will cost $P(T^* + T)\ (1) = (T^* + T)$ today.
  - $P(T^* + T)$ is the discount factor for a payment that will be received at time $T^* + T$.
- Investment 2: Enter into a forward contract, valued at $F(T^* + T)$, to buy at T^* a zero-coupon bond with a par value of $1 and with maturity T.
  - This bond would cost $P(T^*)F(T^*,T)$ today.

The payoffs for the two investments at time $T^* + T$ are the same (i.e., the par value of $1). For this reason, the initial costs of the investments must be the same, and, therefore, the forward pricing model must hold to prevent arbitrage.

I understand that the discussion relating to the forward pricing model might have been too much to digest given that we did not work with discount factors to derive forward rates from spot rates at Level I. Try to understand that all we are really doing here is working with discount factors and (forward) bond prices instead of working with spot rates and forward rates (as we did at Level I). Recall Figure 1-1 from Level I.

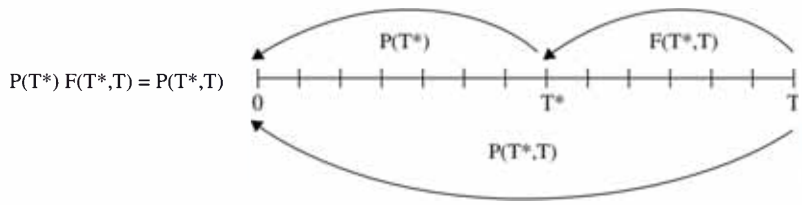
Figure 1-1: Determining Forward Rates from Spot Rates



Now going back to the forward pricing model, we are just working backward (see Figure 1-2).

- Investment 1: Discount factor of $P(T^* + T)$ is applied to the $1 par value to determine its present value today ($t = 0$).
- Investment 2: The forward price, $F(T^*,T)$, represents the value of T^* of a zero-coupon bond with a par value of $1 and with maturity T. A discount factor of $P(T^*)$ is applied to this forward price, to bring it to the present ($t = 0$).

Figure 1-2: Forward Pricing Model



Example 1-1: Spot and Forward Prices and Rates (1)

Consider a two-year loan beginning in one year. The one-year spot rate is 7% and the three-year spot rate is 9%.

1. Calculate the one-year discount factor.
2. Calculate the three-year discount factor.
3. Calculate the forward price of a two-year bond to be issued in one year.

Solution:

Notice that the loan is initiated in one year so T* = 1. Because it is a two-year loan, T = 2.

1. The one-year discount factor is calculated as:

   $P(T^*) = P(1) = 1/(1 + 0.07) = 0.9346$

2. The three-year discount factor is calculated as:

   $P(T^* + T) = P(1 + 2) = P(3) = 1/(1 + 0.09)^3 = 0.7722$

3. Based on the forward pricing model, the forward price of a two-year bond to be issued in one year is calculated as:

   $F(T^*, T) = P(T^* + T)/P(T^*)$

   $F(1, 2) = P(1 + 2)/P(1) = P(3)/P(1) = 0.7722/0.9346 = 0.8262$

Notice that F(1,2) simply represents the discount factor that would be applied over a two-year period to compute the value, as of the end of Year 1, of a $1 payment that will be received at the end of Year 3.

In order to relate the forward pricing model to the forward rate model (that you must be familiar with from Level I), let's work with spot rates and forward rates (instead of with discount factors and forward prices).

Based on the one- and three-year spot rates provided, we can compute the two-year forward rate one year from today (${}_2f_1$), as follows:

$$(1 + {}_1s_0)(1 + {}_2f_1)^2 = (1 + {}_3s_0)^3$$

$$(1 + 0.07)(1 + {}_2f_1)^2 = (1 + 0.09)^3$$

$${}_2f_1 = 10.014\%$$

Based on this two-year forward rate one year from today (${}_2f_1$ = 10.014%), we can compute the forward price (which is also essentially simply a discount factor) as $1/(1 + 0.10014)^2 = 0.8262$.

Important: The point of explicitly mentioning the (rather confusing) curriculum terminology and illustrating this example was just to introduce you to the forward **pricing** model. The forward price will be mentioned again when we draw conclusions from the section on rolling down the yield curve. Until we reach that stage, we will just work with forward rates and spot rates, and continue to use our notations (not the curriculum's).

The Forward Rate Model (Recap from Level I)

Example 1-2: Spot Rates and Forward Rates (2)

Consider the following spot rates:

1-year = 9%

2-year = 10%

3-year = 11%

Calculate the following:

1. The one-year forward one year from today (${}_1f_1$). The curriculum refers to this as f(1,1).
2. The one-year forward two years from today (${}_1f_2$). The curriculum refers to this as f(2,1).
3. The two-year forward one year from today (${}_2f_1$). The curriculum refers to this as f(1,2).

Solution:

Note that the curriculum's notations are different from ours. Get used to working with one particular style and then stick with it. I'd recommend ours (of course).

1. $(1 + {}_1s_0)(1 + {}_1f_1) = (1 + {}_2s_0)^2$
   $(1 + 0.09)(1 + {}_1f_1) = (1 + 0.10)^2$
   ${}_1f_1 = 11.01\%$
2. $(1 + {}_2s_0)^2(1 + {}_1f_2) = (1 + {}_3s_0)^3$
   $(1 + 0.10)^2(1 + {}_1f_2) = (1 + 0.11)^3$
   $f_2 = 13.03\%$
3. $(1 + {}_1s_0)(1 + {}_2f_1)^2 = (1 + {}_3s_0)^3$
   $(1 + 0.09)(1 + {}_2f_1)^2 = (1 + 0.11)^3$
   ${}_2f_1 = 12.01\%$

Notice that the upward-sloping spot rate curve is associated with an upward-sloping forward curve. The one-year forward rate two years from today is greater than the one-year forward rate one year from today (13.03% > 11.01%). We will look into this relationship in detail in subsequent sections.

Takeaways:

- Forward rates can be extrapolated from spot rates.
- A forward rate can be looked upon as a type of breakeven interest rate. For example, the one-year forward rate seven years from today is the rate that would make an investor indifferent between (1) buying a seven-year zero-coupon bond today and then reinvesting the proceeds, after seven years, in a one-year zero-coupon bond and (2) buying an eight-year zero-coupon bond today.
- A forward rate can also be looked upon as the rate that can be locked in by extending maturity by one year. For example, the one-year forward rate seven years from today would be the one-year rate that an investor can lock in by purchasing an eight-year zero-coupon bond today instead of a seven-year zero-coupon bond.

An Important Relationship between Spot and Forward Rates

Our next task is to show that the x-year spot rate today can be expressed as a geometric mean of the one-year spot rate today, and a series of one-year forward rates (where the number of one-year forward rates equals x – 1).

$$(1 + {}_{x}s_{0})^{x} = (1 + {}_{1}s_{0})(1 + {}_{1}f_{1})(1 + {}_{1}f_{2})(1 + {}_{1}f_{3}) \ldots (1 + {}_{1}f_{x-1})$$

Therefore:

$$(1 + {}_{x}s_{0}) = [(1 + {}_{1}s_{0})(1 + {}_{1}f_{1})(1 + {}_{1}f_{2})(1 + {}_{1}f_{3}) \ldots (1 + {}_{1}f_{x-1})]^{1/x}$$

And:

$${}_{x}s_{0} = [(1 + {}_{1}s_{0})(1 + {}_{1}f_{1})(1 + {}_{1}f_{2})(1 + {}_{1}f_{3}) \ldots (1 + {}_{1}f_{x-1})]^{1/x} - 1$$

If your understanding of spot rates and forward rates at Level I was adequate, the algebra above should suffice to prove this. Otherwise, refer to Example 1-3.

Example 1-3: Spot and Forward Rates

Consider the following information (based on the conclusions from Example 1-2):

$${}_{1}s_{0} = 9\%$$

$${}_{1}f_{1} = 11.01\%$$

$${}_{1}f_{2} = 13.03\%$$

Show that the two-year spot rate (10%) and the three-year spot rate (11%) are geometric averages of the one-year spot and forward rates.

Solution:

1. $(1 + {}_2s_0)^2 = (1 + {}_1s_0)\,(1 + {}_1f_1)$
   ${}_2s_0 = [(1.09)\,(1.1101)\,]^{1/2} - 1 = 10\%$

2. $(1 + {}_3s_0)^3 = (1 + {}_1s_0)\,(1 + {}_1f_1)\,(1 + {}_1f_2)$
   ${}_3s_0 = [(1.09)\,(1.1101)\,(1.1303)]^{1/3} - 1 = 11\%$

Other Important Relationships between Spot and Forward Rates

The slope of the yield curve dictates whether the forward rates are greater than or lower than spot rates.

- If the yield curve is upward sloping (i.e., long-term spot rates are greater than short-term spot rates), then the x-year forward rate y years from today, ${}_xf_y$, will be greater than the long-term spot rate, ${}_{x+y}s_0$.
  - Example: If ${}_3s_0 > {}_2s_0 > {}_1s_0$ then ${}_2f_1 > {}_3s_0$.
- Further, for an upward-sloping yield curve, the forward rate rises as y (the initiation date) increases.
  - Example: If ${}_3s_0 > {}_2s_0 > {}_1s_0$ then ${}_1f_2 > {}_1f_1$.
- If the yield curve is downward sloping (i.e., long-term spot rates are lower than short-term spot rates), then the x-year forward rate y years from today, ${}_xf_y$, will be lower than the long-term spot rate, ${}_{x+y}s_0$.
  - Example: If ${}_3s_0 < {}_2s_0 < {}_1s_0$ then ${}_2f_1 < {}_3s_0$.
- Further, for a downward-sloping yield curve, the forward rate declines as y increases.
  - Example: If ${}_3s_0 < {}_2s_0 < {}_1s_0$ then ${}_1f_2 < {}_1f_1$.
- If the yield curve is flat, all one-year forward rates are equal to the spot rate.

Look at the information provided and calculations performed in Example 1-4 to confirm these conclusions.

An easy way to remember these relationships is to look at the spot curve as the average over a whole time period, and at forward rates as marginal changes between future time periods.

- If spot rates increase with maturity (the average is rising), forward rates (the marginal data point) must lie above the average.
- If spot rates decrease with maturity (the average is falling), forward rates (the marginal data point) must lie below the average.

The Government Par Curve

The par curve represents the yield to maturity on coupon-paying government bonds, priced at par, over a range of maturities. It is important for valuation because it can be used to construct the zero-coupon yield curve through a process known as bootstrapping. Bootstrapping makes use of the fact that a coupon-paying bond can be viewed as a portfolio of zero-coupon bonds. It is illustrated in Example 1-4.

Example 1-4: Bootstrapping

The following yields are observed for annual-coupon government bonds assuming they are trading at par:

1-year par rate = 5%
2-year par rate = 5.35%
3-year par rate = 5.75%

Bootstrap zero-coupon rates from the par rates provided.

Solution:

The one-year par rate equals the one-year zero coupon rate because, with the assumption of annual coupons, the bond is a pure discount instrument. The bond will make a bullet payment of $105 (including coupon) at maturity, so the yield on this bond would be the one-year spot rate.

The two- and three-year bonds in this example are not pure discount instruments, as they will make interim coupon payments through their terms. However, using the one-year spot rate and the yields and prices (par) of the longer-term bonds, we can solve for longer-term spot rates one by one from earliest to latest maturities.

The two-year bond makes payments at two points in time: a $5.35 payment at the end of Year 1, and a $105.35 payment at the end of Year 2. If we were to discount the first payment at the one-year spot rate, and the second one at the two-year spot rate, we should expect to attain a value of $100 (par) for the bond.

$$\$100 = \frac{5.35}{(1.05)} + \frac{100 + 5.35}{(1 + {}_2s_0)^2}$$

$${}_2s_0 = 5.3594\%$$

Similarly, we can determine the three-year spot rate using the price of the three-year bond (par) and the one- and two-year spot rates that we just computed.

$$\$100 = \frac{5.75}{(1 + 0.05)} + \frac{5.75}{(1 + 0.053594)^2} + \frac{100 + 5.75}{(1 + {}_3s_0)^3}$$

$${}_3s_0 = 5.7804\%$$

More on Yield Curves

In developed markets, yield curves are typically upward sloping. The curve tends to flatten at longer maturities; that is, as maturity increases, the increase in yields gets smaller. An upward-sloping yield curve is generally interpreted as reflecting increasing or at least level inflation, which is associated with relatively strong economic growth. Premiums for risks (such as the greater interest rate risk or duration of long-term bonds) also contributes to the slope. Inverted yield curves, in contrast, are relatively uncommon. They indicate a market expectation of declining future inflation, which is associated with declining economic activity.

Yield to Maturity in Relation to Spot Rates and Expected and Realized Returns on Bonds

Let's begin with an example to illustrate how the yield to maturity (YTM) is calculated (see Example 1-5).

Example 1-5: Spot Rates and Yield to Maturity

Consider the following spot rates:

1-year = 9%
2-year = 10%
3-year = 11%

Calculate the price of a three-year annual-pay bond with a coupon rate of 5% and a face value of \$100. Then compute the bond's yield to maturity.

Solution:

The price of the bond based on spot rates is calculated as:

$$\text{Bond price} = 5/(1.09) + 5/(1.10)^2 + 105/(1.11)^3 = \$85.49$$

The YTM on the bond is computed as:

$$\$85.49 = 5/(1 + \text{YTM}) + 5/(1 + \text{YTM})^2 + 105/(1 + \text{YTM})^3$$

PV = 85.49, N = 3, PMT = –5, FV = –100, CPT I/Y → I/Y = 10.93%

Notice how the YTM is closest to the three-year spot rate. The YTM can be thought of as some weighted average of the spot rates used to value a bond. In this example, the bond's largest cash flow occurs in Year 3, which gives the three-year spot rate a much greater weight than the one- and two-year spot rates in determining the bond's YTM.

The **yield to maturity (YTM)** is calculated as the internal rate of return on a bond's cash flows. It equals the **expected rate of return** on a bond only if (1) the bond is held until maturity, (2) all coupon and principal payments are made in full when due, and (3) all coupons are reinvested at the original YTM. This last condition typically does not hold.

The YTM is a poor estimate of the expected return on a bond if (1) interest rates are volatile, (2) the yield curve has a steep slope, (3) there is significant risk of default, and (4) the bond contains embedded options.

- In Cases 1 and 2, reinvestment of coupons would not be possible at the original YTM.
- In Case 3, actual cash flows may turn out to be different from those used in the YTM calculation.
- In Case 4, any exercise of an embedded option would result in a holding period shorter than the bond's original maturity.

Even if we make the assumption that actual future spot rates will turn out to be the same as forward rates (that are based on current spot rates), the expected return on a bond still will not equal its YTM. This is because the YTM implicitly assumes that the yield curve is flat (the same discount rate is applied to each cash flow regardless of its maturity).

The **realized rate of return** is the actual return earned by a bond investor over the holding period. It is based on (1) actual reinvestment rates and (2) the yield curve at the end of the holding period.

Yield Curve Movement and the Forward Curve

Earlier in the reading, we showed you how one-year forward rates can be computed based on the current term structure of spot rates. In this section, we will build on this and establish several important results concerning forward rates/forward prices and the spot yield curve before moving on to discussing relevance of the forward curve to active bond investors (in the next section).

We start with a numerical example. Suppose the current yield curve (at t = 0) gives us the following spot rates:

1-year = 4%
2-year = 5%
3-year = 6%

Based on these spot rates, we can compute ${}_1f_1$, ${}_1f_2$, and ${}_2f_1$ as follows:

$$(1 + {}_1s_0)(1 + {}_1f_1) = (1 + {}_2s_0)^2$$
$$(1 + 0.04)(1 + {}_1f_1) = (1 + 0.05)^2$$
$${}_1f_1 = 6.0096\%$$

$$(1 + {}_2s_0)^2(1 + {}_1f_2) = (1 + {}_3s_0)^3$$
$$(1 + 0.05)^2(1 + {}_1f_2) = (1 + 0.06)^3$$
$${}_1f_2 = 8.0287\%$$

$$(1 + {}_1s_0)(1 + {}_2f_1)^2 = (1 + {}_3s_0)^3$$
$$(1 + 0.04)(1 + {}_2f_1)^2 = (1 + 0.06)^3$$
$${}_2f_1 = 7.0143\%$$

Now we assume that future spot rates equal the forward rates predicted by today's forward curve. What this means is that the forward rates that we computed at t = 0 based on the yield curve at t = 0 turn out to be the actual spot rates at t = 1.

- ${}_1f_1$ (computed at t = 0) turns out to be ${}_1s_0$ (as of t = 1).
- ${}_2f_1$ turns out to be ${}_2s_0$.
- ${}_1f_2$ turns out to be ${}_1f_1$.

Therefore, as of today (t = 1):

- ${}_1s_0 = 6.0096\%$
- ${}_2s_0 = 7.0143\%$
- ${}_1f_1 = 8.0287\%$

The thing to observe here is that if we were to use ${}_1s_0$ and ${}_2s_0$ to compute ${}_1f_1$ (all of them as of t = 1), we would obtain the same value that we already have (${}_1f_1 = 8.0287\%$).

$$(1 + {}_1s_0)(1 + {}_1f_1) = (1 + {}_2s_0)^2$$
$$(1 + 0.060096)(1 + {}_1f_1) = (1 + 0.070143)^2$$
$${}_1f_1 = 8.0285\% \text{ (slight difference due to rounding)}$$

You need to understand the following before moving ahead:

- If future spot rates evolve as predicted by today's forward curve, forward rates remain unchanged, and because forward rates remain unchanged so does the forward contract price. (We illustrated earlier how the forward contract price is simply the discount factor based on the forward rate.)
- A change in the forward rate (and the forward price) reflects a deviation of the spot curve from that predicted by today's forward curve.
- If a trader expects that the future spot rate will be lower than what is predicted by the prevailing forward rate, he would buy the forward contract because he expects its value to increase. Stated differently, if the trader expects that interest rates in the future will actually be lower than the rates anticipated by the market, she would buy a bond.
- If a trader expects that the future spot rate will be greater than what is predicted by the prevailing forward rate, he would sell the forward contract because he expects its value to decrease. Stated differently, if the trader expects that interest rates in the future will actually be greater than the rates anticipated by the market, she would sell a bond.

The last two bullets will become clearer in the next section.

Active Bond Portfolio Management

The first thing that you need to understand in this section is that the return on bonds of varying maturities over a one-year period is always the one-year rate (the risk-free rate over the one-year period) if spot rates evolve (at the end of the first year) as implied by the current forward curve. Example 1-6 illustrates this.

Example 1-6: When Spot Rates Evolve as Implied by the Current Forward Curve

Consider the following information:

$_1S_0 = 9\%$

$_2S_0 = 10\%$

$_3S_0 = 11\%$

$_1f_1 = 11.01\%$

$_2f_1 = 12.01\%$

Compute the return on the following bonds over a one-year period assuming that the spot curve one year from today reflects the current forward curve.

1. One-year zero-coupon bond.
2. Two-year zero-coupon bond.
3. Three-year zero-coupon bond.

Solution:

1. At t = 0, the one-year zero-coupon bond will be valued at 100/(1.09) = \$91.74. In one year (at maturity) this bond will be worth \$100, so the total return over the year can be calculated as:

$$(100/91.74) - 1 = 9\%$$

2. At t = 0, the two-year zero-coupon bond will be valued at $100/(1.10)^2$ = \$82.64. At t = 1, the one-year spot rate equals the one-year forward rate one year from today ($_1f_1$). Therefore, this bond will be worth 100/(1.1101) = \$90.08. The total return over the year can be calculated as:

   (90.08/82.64) – 1 = 9%

3. At t = 0, the three-year zero-coupon bond will be valued at $100/(1.11)^3$ = \$73.12. At t = 1, the two-year spot rate equals the two-year forward rate one year from today ($_2f_1$). Therefore, this bond will be worth $100/(1.1201)^2$ = \$79.71. The total return over the year can be calculated as:

   (79.71/73.12) – 1 = 9%

Takeaway: Even though the bonds have different maturities, the one-year return on all three bonds is the same (9%). This will be the case as long as spot rates evolve as implied by the current forward curve.

Now let's see what happens if the spot curve one year from today differs from today's forward curve (see Example 1-7).

Example 1-7: When Spot Rates Are Lower Than Those Implied by the Current Forward Curve

Using the same information as in Example 1-6, compute the one-year returns on the three bonds assuming that the spot rate curve at Year 1 is flat with a yield of 10% for all maturities.

1. At t = 0, the one-year zero-coupon bond will be valued at 100/(1.09) = \$91.74. In one year (at maturity) this bond will be worth \$100, so the total return over the year can be calculated as:

   (100/91.74) – 1 = 9%

2. At t = 0, the two-year zero-coupon bond will be valued at $100/(1.10)^2$ = \$82.64. At t = 1, if the one-year spot rate equals 10%, this bond will be worth 100/(1.10) = \$90.91. The total return over the year can be calculated as:

   (90.91/82.64) – 1 = 10%

3. At t = 0, the three-year zero-coupon bond will be valued at $100/(1.11)^3$ = \$73.12. At t = 1, if the two-year spot rate equals 10%, this bond will be worth $100/(1.1)^2$ = \$82.64. The total return over the year can be calculated as:

   (82.64/73.12) – 1 = 13.03%

Takeaways:

- If the spot curve one year from today differs from today's forward curve, the returns on the three bonds over the one-year period will not all be 9%. The returns on the two- and three-year bonds are higher than 9%.
- If any one of the investor's expected future spot rates is lower than the implied forward rate for the same maturity, then (all else being equal) the investor would perceive the bond to be undervalued in the sense that the market is effectively discounting the bond's payments at a higher rate than the investor is, and the bond's market price is below the intrinsic value perceived by the investor.

- If a portfolio manager's projected spot curve is above (below) the forward curve and his or her expectation turns out to be true, the return will be less (more) than the one-period risk-free interest rate because the market would have overpriced (underpriced) the forward contract.
    - In Example 1-7, the spot curve turns out to be lower than the current forward curve, so the returns on the two- and three-year bonds are greater than the current risk-free rate (9%) because the market had underpriced the forward contract.

Riding the Yield Curve

We now move into describing a popular yield curve trade known as **riding the yield curve** or **rolling down the yield curve**. We already know that if the yield curve is upward sloping, the forward curve will be above the current spot curve. In such a situation, if a trader is confident that the yield curve will not change its level and shape over her investment horizon, she would buy bonds with a maturity greater than her investment horizon (instead of bonds with maturities that exactly match her investment horizon). As time passes and these bonds near maturity, they will automatically trade at lower yields as they roll down the yield curve (if the yield curve retains its initial shape and level, then, as the number of years to maturity on these bonds decreases, the applicable yield will be lower). The lower yields will result in a higher price, so effectively the trader can hold the bonds for a period of time as they appreciate in value and then sell them before maturity to realize a higher return. If she were to buy bonds that mature right at the end of her investment horizon, the bonds would be worth only their par value (at maturity), so her total return would be lower.

LESSON 2: THE SWAP RATE CURVE

LOS 32f: Explain the swap rate curve and why and how market participants use it in valuation. **Vol 5, pp 24–27**

LOS 32g: Calculate and interpret the swap spread for a default-free bond. **Vol 5, pp 29–31**

LOS 32h: Describe the Z-spread. **Vol 5, pg 30**

LOS 32i: Describe the TED and Libor-OIS spreads. **Vol 5, pp 32–33**

THE SWAP RATE CURVE

Just like the spot rate curve of risk-free bonds, the **swap rate curve** (or **swap curve**) is also used in international fixed-income markets as a representation of the time value of money.

A **plain-vanilla interest rate swap** entails an exchange of periodic fixed- and floating-rate payments based on the **notional principal** over the tenor of the swap. The interest rate on the fixed-rate leg of the swap is known as the **swap rate**. This swap (fixed) rate is calculated such that the swap has zero value to either party at swap initiation. Floating rates are based on a short-term reference rate such as LIBOR.

The yield curve of swap fixed rates is known as the **swap rate curve** or **swap curve**. Because it is based on par swaps (the present values of the fixed- and floating-rate legs are equal at swap initiation) the swap curve is a type of par curve.

The swap market is a highly liquid market for two main reasons:

- Swap arrangements offer significant flexibility and customization in contract design.
- Swaps provide a very efficient way to hedge interest rate risk.

The swap curve serves as the primary benchmark for interest rates in countries that do not have a liquid government bond market for maturities longer than one year. Further, in countries where the private sector is larger than the public sector, the swap curve is a more relevant measure of the time value of money than the government's cost of borrowing.

Why Do Market Participants Use Swap Rates When Valuing Bonds?

Aside from the relative liquidity of the two markets (bond market versus swap market), the choice between the government spot rate curve and swap rate curve as the reference curve in fixed-income valuation also depends on the business operation of the institution using the benchmark.

- Wholesale banks frequently use the swap curve to value assets and liabilities because these organizations hedge many items on their balance sheet with swaps.
- On the other hand, retail banks with little exposure to the swap market are more likely to use the government spot curve as their benchmark.

Example 2-1 illustrates the use of a swap by a financial institution.

Example 2-1: Application of Swaps

ABC Bank wants to raise funds by issuing certificates of deposit (CDs).

- It can raise \$10m through a 2.5% CD that has a three-year term.
- It can raise another \$10m through a 3.5% CD that has a five-year term.

The bank issues \$20m worth of CDs and takes the pay-floating side of two plain-vanilla interest rate swaps.

- On Swap 1, the bank receives 2.5% fixed and pays three-month LIBOR minus 10 bp with a three-year term and a notional amount of \$10m.
- On Swap 2, the bank receives 3.5% fixed and pays three-month LIBOR minus 20 bp with a five-year term and a notional amount of \$10m.

By getting into these swap contracts, the bank has effectively converted its fixed-rate liabilities to floating-rate liabilities. The fixed-rate payments received from the swaps will be passed on to the CD investors, leaving the bank with only floating-rate obligations. The margins on the floating rates thereby become the standard by which value is measured in assessing the bank's cost of funding. In this case, the bank's cost of funding is three-month LIBOR minus 15 bp for the next three years.

How Do Market Participants Use the Swap Curve in Valuation?

I don't understand why the CFA Program curriculum has inserted this section over here. One can only learn how to determine the swap curve once one has understood how plain-vanilla interest rate swaps are priced. This process is described in great detail in the Derivatives section. If you haven't studied the Level II Derivatives materials yet, here's a brief overview of swap pricing.

Plain-Vanilla Interest Rate Swaps as a Combination of Bonds

Taking a position as a fixed-rate payer (floating-rate receiver) on a plain-vanilla interest rate swap is equivalent to issuing a fixed-rate bond (on which fixed payments must be made) and using the proceeds to purchase a floating-rate bond (on which floating payments will be received). In other words, the fixed-rate payer can be viewed as being long on a floating-rate bond and short on a fixed-rate bond.

Pricing a Swap: Determining the Swap Fixed Rate

To price an interest rate swap, we compute the swap fixed rate that results in the swap having zero value to either party at initiation.

At issuance, a floating-rate bond is priced at par, as the coupon rate in effect over the next settlement period equals the market interest rate. Our task is to compute the coupon rate on the fixed-rate bond (which represents the swap fixed rate) that would result in it having the same value as the floating-rate bond (i.e., par).

Example 2-2 illustrates how the swap rate curve is determined.

Example 2-2: Determining the Swap Rate Curve

Consider the following spot rates:

$_1s_0 = 5\%$
$_2s_0 = 6\%$
$_3s_0 = 7\%$
$_4s_0 = 8\%$

Based on this information, determine the swap fixed rate for the following:

1. One-year plain-vanilla interest rate swap.
2. Two-year plain-vanilla interest rate swap.
3. Three-year plain-vanilla interest rate swap.
4. Four-year plain-vanilla interest rate swap.

Solution:

The key is to remember that the swap fixed rate represents the coupon rate that would result in the fixed leg of the swap having the same value as the floating leg at swap initiation given the current term structure of spot rates. The floating leg is worth $1 (par value) at swap initiation (as the floating rate equals the market rate) and at every coupon reset date.

To understand the calculations that follow, note that:

- In the equations used, CR(T) represents the coupon rate at time T. This coupon rate represents the swap rate.
- The left side of the equations represents the value of the fixed-rate bond. For example, in the solution to Question 1, the one-year fixed rate bond will make (1) a coupon payment and (2) principal repayment at the end of Year 1. We discount these two payments at the one-year spot rate to compute its value.

- The right side of the equations represents the value of the floating-rate bond (which is par at swap initiation).
- The two bonds must have the same value for the swap to hold zero value at initiation.

1. $CR(1)/(1 + {}_1s_0) + 1/(1 + {}_1s_0) = 1$

   $CR(1) = 5\%$

2. $CR(2)/(1 + {}_1s_0) + CR(2)/(1 + {}_2s_0)^2 + 1/(1 + {}_2s_0)^2 = 1$

   $CR(2) = 5.97\%$

3. $CR(3)/(1 + {}_1s_0) + CR(3)/(1 + {}_2s_0)^2 + CR(3)/(1 + {}_3s_0)^3 + 1/(1 + {}_3s_0)^3 = 1$

   $CR(3) = 6.91\%$

4. $CR(4)/(1 + {}_1s_0) + CR(4)/(1 + {}_2s_0)^2 + CR(4)/(1 + {}_3s_0)^3 + CR(4)/(1 + {}_4s_0)^4 + 1/(1 + {}_4s_0)^4 = 1$

   $CR(4) = 7.81\%$

We have just determined that the one-year swap rate equals 5%, the two-year swap rate equals 5.97%, the three-year swap rate equals 6.91%, and the four-year swap rate equals 7.81%. These four rates represent our swap curve, which is computed from current spot rates.

The Swap Spread

The **swap spread** is defined as the difference between the swap fixed rate on a swap and the rate on the "on-the-run" government security with the same maturity/tenor as the swap. For example, if the swap fixed rate on a three-year plain-vanilla interest rate swap is 4% and the three-year Treasury is yielding 3.65%, the swap spread equals 4% – 3.65% = 0.35% or 35 bps.

The most widely used interest rate curve is the Libor/swap curve. LIBOR is used for short-maturity yields, rates derived from Eurodollar futures are used for mid-maturity yields, and swap rates are used for maturities greater than one year. The reasons for the popularity of the Libor/swap curve are that (1) it reflects the default risk of private entities with a rating of A1/A+, which is roughly what most commercial banks are rated, (2) the swap market is unregulated by the government so swap rates are more comparable across countries, and (3) the swap market has more maturities to construct a yield curve than government bond markets.

By using the swap curve as a benchmark for the time value of money, the investor can determine the swap spread so that the swap would be fairly priced given the spread. Conversely, given a swap spread, the investor can determine a fair price for the bond.

The swap spread is used to measure credit risk and liquidity risk. The higher the swap spread, the higher the compensation demanded by investors for accepting credit and/or liquidity risk. Note, however, that if the bond entails no default risk, then the swap spread could indicate liquidity risk, or it could suggest that the bond is mispriced.

A more accurate measure (than the swap spread) of credit and liquidity risk is the **zero-spread** (or **Z-spread**). The Z-spread is the constant spread that would be added to the implied spot curve such that the present value of the cash flows of a bond, when discounted at relevant spot rates plus the Z-spread, equals its current market price. An example of valuing a bond using the Z-spread was covered at Level I, so we will not get into it here.

Spreads as a Price Quote Convention

For reasons that we do not need to get into, the Treasury rate for a given maturity can differ from the swap rate for the same tenor. As a result, when we quote the price of a bond in terms of spreads on top of one of these benchmarks, the actual spread would differ depending on which benchmark is used.

The **swap spread** represents the difference between the swap rate and the government bond yield (of the same maturity). Note that there is one problem with this definition. A five-year swap initiated today would mature in exactly five years, but there may be no government bond with exactly five years to maturity. Therefore, convention dictates that the swap spread would be calculated as the difference between the five-year swap rate and the five-year on-the-run government bond. Finally, note that the swap rate does reflect some counterparty credit risk, while the U.S. Treasuries are considered free from default risk, so the swap rate is typically greater than the corresponding Treasury note rate.

The **TED spread** is calculated as the difference between LIBOR and the yield on a T-bill with the same maturity. The TED spread indicates the perceived level of credit risk in the overall economy. An increase (decrease) in the spread suggests that lenders believe that the risk of default on interbank loans is increasing (decreasing). The TED spread can also be thought of as a measure of counterparty credit risk in swap contracts. Compared with the 10-year swap spread, the TED spread is a better reflection of risk in the banking system, while the 10-year swap spread is more of a reflection of differing demand and supply conditions.

The **Libor–OIS spread** is calculated as the difference between LIBOR and the overnight indexed swap (OIS) rate. An OIS is an interest rate swap in which the periodic floating rate is equal to the geometric average of an overnight rate (or overnight index rate) over every day of the payment period. The index rate is typically the rate for overnight unsecured lending between banks (e.g., the federal funds rate for U.S. dollars). The Libor–OIS spread is an indicator of the risk and liquidity of money-market securities.

The **I-spread** is calculated as the difference between the yield on a corporate bond and the swap rate on a swap with the same tenor as the bond.

LESSON 3: TRADITIONAL TERM STRUCTURE THEORIES

LOS 32j: Explain traditional theories of the term structure of interest rates and describe the implications of each theory for forward rates and the shape of the yield curve. Vol 5, pp 33–38

TRADITIONAL THEORIES OF THE TERM STRUCTURE OF INTEREST RATES

Unbiased Expectations Theory

The unbiased expectations theory or pure expectations theory states that the forward rate is an unbiased predictor of future spot rates. This implies that bonds of **any** maturity are perfect substitutes for each other. Basically, this theory asserts that an investor with a four-year horizon would attain the same return from the following three investments:

- Buying a six-year bond and holding it for four years.
- Buying a four-year bond.
- Buying a series of one-year bonds, one after the other.

The unbiased expectations theory assumes that investors are risk-neutral. The obvious argument against this theory is that investors tend to be risk-averse.

Local Expectations Theory

The **local expectations theory** is similar but narrower than the unbiased expectations theory. Instead of asserting that every maturity strategy has the same expected return over a given investment horizon, this theory states that the return over a short-term investment horizon that starts today will be the same (i.e., the risk-free rate) regardless of maturity of the chosen security as long as forward rates are actually realized. Recall that we came to this conclusion earlier in Example 1–5.

The main difference between the local expectations theory and the unbiased expectations theory is that the local expectations theory can be extended to a world where there is risk. While the theory requires that there be no risk premiums for very short holding periods, it does allow for risk premiums for longer-term investments. Thus, the theory is applicable to both risk-free as well as risky bonds.

The local expectations theory is economically appealing, but it is often observed that short-term returns on longer-maturity bonds exceed those on shorter-maturity bonds. Demand for short-term securities typically exceeds demand for longer-term securities, which results in lower yields than those of long-term bonds.

Liquidity Preference Theory

The **liquidity preference theory** attempts to account for risk aversion. (Recall that the unbiased expectations theory made no room for risk aversion.) The liquidity preference theory accounts for the fact that investors in long-term securities require some compensation for taking higher interest rate risk. This theory asserts that the longer the term to maturity, the greater the price sensitivity (duration) of a bond to changes in interest rates, and the greater the compensation (risk premium) required by investors. Long-term investors are rewarded for investing over a longer horizon in the form of higher yields or a liquidity premium on long-term instruments.

Under the liquidity preference theory, yields are determined by two factors:

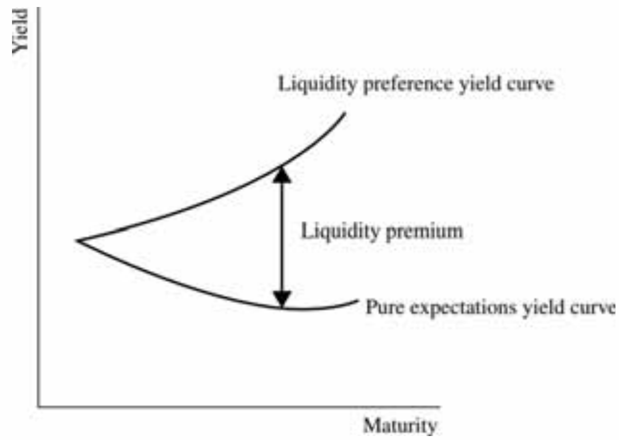
- Expected future short-term rates (as in the pure expectations theory).
- A yield premium for taking greater interest rate risk by investing in a longer-term instrument. The greater the interest rate risk of a bond, the lower its liquidity, hence the liquidity preference for short-term instruments, and the higher liquidity (or risk) premium on long-term investments.

Under this theory, an upward-sloping yield curve can be explained by:

- Higher expected future short-term rates, or
- Unchanged or even falling expected future short-term rates, but with a yield premium that increases with maturity to produce an overall upward-sloping yield curve.

Under the liquidity preference theory, if the yield curve is flat or downward sloping, we can conclude that short-term interest rates are expected to fall. Bear in mind, however, that an upward-sloping yield curve does not necessarily mean that short-term rates are expected to rise in the future. Figure 3-1 illustrates a situation where an increasing liquidity premium with maturity results in an upward-sloping yield curve even though short-term spot rates are expected to fall in the future.

Figure 3-1: Liquidity Premium Added to Decreasing Expected Rates



Also note that:

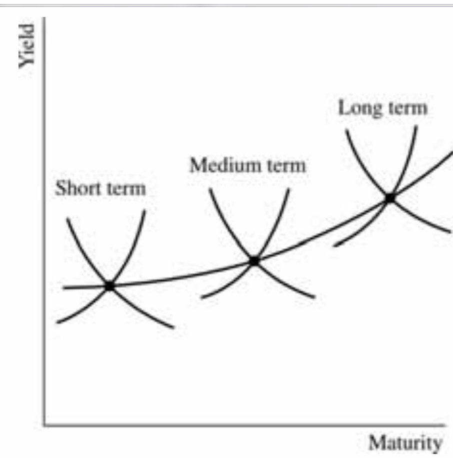
- The premium here should not be confused with a yield premium for lack of liquidity associated with thinly traded bonds. The premium here applies to all long-term bonds, even those that are very actively traded.
- Under this theory, forward rates derived from spot rates will be biased upward by the amount of the liquidity premium. Market evidence does support this claim.

Segmented Markets Theory

The **segmented markets theory** asserts that the yield for each maturity along the yield curve is determined independently, solely by the supply and demand of funds for the particular maturity, which means that yields do not reflect expected spot rates or liquidity premiums.

Financial institutions try to match the maturity of their assets and liabilities (e.g., pension funds have long-term liabilities so they predominantly invest in long-term assets, while commercial banks with short-term obligations usually invest heavily in short-term assets). This theory asserts that market participants are either unable or unwilling to invest in anything other than the securities of their preferred segments, so the yield for a maturity is determined entirely by supply and demand of funds of that particular maturity. Figure 3-2 illustrates how yields are determined under the market segmentation theory.

Figure 3-2: Market Segmentation Theory and the Yield Curve



The Preferred Habitat Theory

The **preferred habitat theory** is similar to the segmented markets theory in proposing that many borrowers and lenders have strong preferences for particular maturities, but it does not assert that yields at different maturities are determined independently of each other. Further, instead of asserting that institutions are either unable or unwilling to invest in maturities other than their preferred ones, the theory contends that if the expected additional returns are large enough, institutions will be willing to purchase bonds outside their preferred maturity or habitat. Therefore, in this theory, both market expectations and the institutional factors emphasized in the segmented markets theory influence the term structure of interest rates.

LESSON 4: MODERN TERM STRUCTURE THEORIES

LOS 32k: Describe modern term structure models and how they are used.

Vol 5, pp 38–44

MODERN TERM STRUCTURE MODELS

Equilibrium Term Structure Models

- Equilibrium term structure models look to describe the dynamics of the term structure using fundamental economic variables that affect interest rates.
- They require the specification of a drift term (described later) and the assumption of a functional form for interest rate volatility.
- They can be structured as one-factor or multifactor models.
    - One-factor models assume that a single observable factor (sometimes called a state variable) drives all yield curve movements, while multifactor models use more than a single factor.
        - Both the Vasicek and CIR models (described next) are one-factor models where the single factor that is used to explain the yield curve is the short-term interest rate. This approach makes sense because studies have shown that parallel shifts can be used to explain a significant portion of yield curve changes (more on this later).
        - Multifactor models may be able to model the curvature of a yield curve more accurately (for example, by adding changes in the slope of the yield curve as a factor in the model), but they come with added complexity.
- They make assumptions about the behavior of factors. For example, the model must make an assumption about whether the short-term interest rate should be modeled as mean-reverting or whether it should be modeled to exhibit jumps.
- They typically require estimation of fewer parameters than arbitrage-free term structure models (described later), but this comes at the cost of less precision in modeling the observed yield curve.
- They typically make an assumption regarding the term premium (i.e., the additional return required by lenders to invest in long-term securities). Arbitrage-free models do not make a term premium assumption.

The Cox-Ingersoll-Ross (CIR) Model

- The CIR model uses one factor (i.e., the short-term rate) to determine the entire term structure of interest rates.

- The model has (1) a deterministic component and (2) a stochastic (or random) component.
  - The **deterministic component** is also called the **drift term**.
    - Within this deterministic component, the interest rate is modeled as mean-reverting (toward a level, *b*).
    - Another parameter (*a*) represents the speed at which interest rates revert to the mean. The higher the value of this parameter, the quicker the mean reversion.
  - The **stochastic term** models risk (volatility).
    - This term makes volatility proportional to the short-term rate (i.e., interest rate volatility increases with the level of interest rates).
    - It also precludes the possibility of negative interest rates.

The Vasicek Model

- The Vasicek model also assumes that interest rates are mean-reverting.
- The model also has the same drift term as the CIR model.
- However, the stochastic component assumes that interest rate volatility is constant.
- The main disadvantage of the model is that it is theoretically possible for interest rates to be negative.

Note that the yield curve estimated with the Vasicek or CIR model may not match the observed yield curve. However, if model parameters are believed to be correct, these models can be used to identify mispricings.

Arbitrage-Free Models

- Arbitrage-free models start with market prices of a reference set of financial instruments. Under the assumption that these instruments are correctly priced, a random process with a drift term and volatility factor is used to generate the yield curve.
- The advantage of arbitrage-free models is the ability to calibrate the model to market data. Equilibrium models such as the CIR and Vasicek models have only a finite number of free parameters, so it is not possible to specify these parameter values in a manner that matches model prices with observed market prices. Arbitrage-free models exhibit greater accuracy in modeling the market yield curve.
- Arbitrage-free models are also known as **partial equilibrium models** because they do not attempt to explain the yield curve; instead they take the yield curve as given.

The Ho-Lee Model

- The **Ho-Lee model** uses the relative valuation concepts of the Black-Scholes-Merton option-pricing model.
- The time-dependent drift term of the model is inferred from market prices, so the model can accurately generate the current term structure.
- This exercise is typically performed via a binomial lattice–based model, which makes use of risk-neutral probabilities (that are discussed in detail in the Derivatives section).
- The advantage of the model lies in its simplicity, and it can be used to illustrate most of the salient features of arbitrage-free interest rate models.
- Interest rate volatility can be modeled as a function of time in the model.
- The downside is that negative interest rates are theoretically possible.

LESSON 5: YIELD CURVE FACTOR MODELS

LOS 32l: Explain how a bond's exposure to each of the factors driving the yield curve can be measured and how these exposures can be used to manage yield curve risks. Vol 5, pp 45–53

LOS 32m: Explain the maturity structure of yield volatilities and their effect on price volatility. Vol 5, pp 50–51

YIELD CURVE FACTOR MODELS

A Bond's Exposure to Yield Curve Movement

The sensitivity of a bond's price to changes in the shape of the yield curve is known as **shaping risk**. Duration measures the interest rate risk of a bond in response to a parallel shift in the yield curve, but yield curve changes are rarely parallel. Therefore, the challenge for a fixed-income manager is to manage the yield curve shape risk of her portfolio.

Factors Affecting the Shape of the Yield Curve

Studies that have made use of **yield curve factor models** have found that yield curve movements are historically well described by three independent movements: (1) level, (2) steepness, and (3) curvature. For example, these three movements combined explain 97% of U.S. yield curve changes from August 2005 to July 2013.

- The **level movement** refers to upward or downward (parallel) shifts in the yield curve.
    - This component explains about 77% of the total variance/covariance in the U.S. yield curve from August 2005 to July 2013.
- The **steepness movement** refers to changes in the slope of the yield curve (when either short-term rates change more than long-term rates or short-term rates change less than long-term rates). These movements are also known as yield curve **twists**.
    - This component explains about 17% of the total variance/covariance in the U.S. yield curve from August 2005 to July 2013.
- The **curvature movement** refers to movements in the three segments of the yield curve (the short- and long-term segments rise while the middle segment falls, or vice versa). These movements are also known as **butterfly shifts**.
    - This component explains about 3% of the total variance/covariance in the U.S. yield curve from August 2005 to July 2013.

Managing Yield Curve Risks

> Management of yield curve risk involves changing certain identified exposures to desired levels through security and/or derivative trades.

Yield curve risk refers to the risk to portfolio value from unanticipated changes in the yield curve. It can be managed based on one of several measures of sensitivity to yield curve movements. We shall be describing three such measures in this section.

- **Effective duration,** which measures the change in the value of a portfolio to a small parallel change in the yield curve.
- **Key rate duration**, which measures the change in the value of a portfolio in response to a small change in the yield for a specific maturity.
- **A measure based on the factor model** (that we mentioned in the previous section), which uses three types of movements (level, steepness, and curvature) to explain changes in the yield curve.

Using one of these last two measures allows identification and management of shaping risk (sensitivity to changes in the shape of the yield curve) as well as the risk associated with parallel shift in the yield curve. Effective duration addresses only the risk associated with parallel shifts.

Let's work with some numbers to illustrate these three measures:

Consider a portfolio consisting of one-year, five-year, and 10-year zero-coupon bonds, with \$100 invested in each of them (total portfolio value = \$300). A hypothetical set of factor movements is presented here:

Year	1	5	10
Level	1	1	1
Steepness	–1	0	1
Curvature	1	0	1

- A level movement means that all the rates change by an equal amount (parallel shift)—in this case, by a unit of 1.
- A steepness movement means that the slope of the yield curve steepens with the long-term rate shifting up by one unit and the short-term rate shifting down by one unit.
- A curvature movement means that both the short- and the long-term rates shift up by one unit, whereas the medium-term rate remains unchanged.
    - Notice that this definition of a curvature movement is different from the earlier definition, as medium-term rates are unchanged here. We have just assumed no change in the medium-term rate in this illustration for simplicity.
- Because these bonds are zero-coupon bonds, their effective durations equal their terms.

Calculating Effective Duration
The effective duration of the portfolio is calculated as the weighted average of the effective durations of the individual bonds.

$$\text{Effective duration} = 0.333(1) + 0.333(5) + 0.333(10) = 5.333$$

Calculating Key Rate Duration
If the one-year rate changes by 100 bps while others remain unchanged, the sensitivity of the portfolio's value to the shift can be calculated as:

$$\text{Sensitivity to 100 bps change in 1-year rate} = D_1 = 1/[(300)(0.01)] = 0.3333$$

To understand this calculation, note that:

- The value of the portfolio will change by \$1 in response to a one-unit change in the one-year rate. Here the change in the one-year rate is 0.01 (100 bps).
- The initial value of the portfolio is \$300.

If the five-year rate changes by 100 bps while others remain unchanged, the sensitivity of the portfolio's value to the shift can be calculated as:

$$\text{Sensitivity to 100 bps change in 5-year rate} = D_5 = 5/[(300)(0.01)] = 1.6667$$

- The value of the portfolio will change by \$5 (because the effective duration of the five-year bond equals 5) in response to a one-unit change in the one-year rate. Here the change in the one-year rate is 0.01 (100 bps).
- The initial value of the portfolio is \$300.

If the 10-year rate changes by 100 bps while others remain unchanged, the sensitivity of the portfolio's value to the shift can be calculated as:

$$\text{Sensitivity to 100 bps change in 10-year rate} = D_{10} = 10/[(300)(0.01)] = 3.3333$$

- The value of the portfolio will change by \$10 (because the effective duration of the 10-year bond equals 10) in response to a one-unit change in the one-year rate. Here the change in the one-year rate is 0.01 (100 bps).
- The initial value of the portfolio is \$300.

Duration is always negative, hence the negative signs next to the coefficients.

Note that the sum of the key rate durations equals the effective duration of the portfolio (0.3333 + 1.6667 + 3.3333 = 5.333).

Now we have all the inputs (key rate durations) to derive the key rate duration–based model for yield curve risk:

$$\begin{aligned}(\Delta P/P) &\approx -D_1\Delta s_1 - D_5\Delta s_5 - D_{10}\Delta s_{10} \\ &= -0.3333\Delta s_1 - 1.6667\Delta s_5 - 3.3333\Delta s_{10}\end{aligned}$$

Note that Δs_x represents the change in the current x-year spot rate.

Working with the Yield Curve Factor Model

We will work with the following model:

$$(\Delta P/P) \approx -D_L\Delta\chi_L - D_S\Delta\chi_S - D_C\Delta\chi_C$$

- D_L measures the sensitivity of portfolio value to a parallel shift in the yield curve.
- D_S measures the sensitivity of portfolio value to a change in slope.
- D_C measures the sensitivity of portfolio value to a change in curvature.

Now let's determine these factor sensitivities for our example:

- D_L measures sensitivity to a parallel shift, which is measured by effective duration. We have already calculated this to be 5.333, but it can also be calculated as:

$$D_L = (1 + 5 + 10)/[(300)(0.01)] = 5.3333$$

The value of the portfolio will change by \$16. The value of the one-year bond would change by \$1, that of the five-year bond would change by \$5, and that of the 10-year bond would change by \$10 (based on their respective durations) in response to a one-unit change in yields.

- D_S is calculated as:

  $$D_S = (1 - 10)/[(300)(0.01)] = 3.0$$

  A downward shift in the one-year rate (notice the –1 in the table and read the definition of a steepness movement provided following the table very carefully) would result in a gain of $1, but the associated upward shift in the 10-year rate would result in a loss of $10. Overall the change in portfolio value would be $9.
- D_C is calculated as:

  $$D_C = (1 + 10)/[(300)(0.01)] = 3.6667$$

  The value of the portfolio will change by $11. The value of the one-year bond would change by $1, and that of the 10-year bond would change by $10 (based on their respective durations).

Now, we have all the inputs (factor sensitivities) to derive the model for yield curve risk:

$$(\Delta P/P) \approx -5.3333\Delta\chi_L - 3.0\Delta\chi_S - 3.6667\Delta\chi_C$$

So, for example, if the change in the level factor, $\Delta\chi_L = -0.005$, the change in the steepness factor, $\Delta\chi_S = 0.002$ and the change in the curvature factor, $\Delta\chi_C = 0.001$, the predicted change in portfolio value would be $-5.3333(-0.005) - 3.0(0.002) - 3.6667(0.001) = 0.01699$ or 1.7%.

The Maturity Structure of Yield Curve Volatilities

- Interest rate volatility is important for at least two reasons:
  - Recall from Level I that the change in the price of a bond depends on (1) its duration and convexity (i.e., the impact per basis point change in the YTM) **and** (2) the magnitude of the change in the YTM (i.e., yield volatility). Therefore, controlling the impact of interest rate volatility on a bond's price volatility is an important part of risk management.
  - Most fixed-income instruments and derivatives contain embedded options, whose values depend on the level of interest rate volatility.
- The **term structure of interest rate volatilities** is a representation of the yield volatility of a zero-coupon bond for a range of maturities. This **volatility curve** (also known as **vol** or **volatility term structure**) measures yield curve risk.
- Interest rate volatility varies across different maturities. Generally speaking, short-term rates tend to be more volatile than long-term rates.
  - Short-term volatility is primarily linked to uncertainty regarding monetary policy.
  - Long-term volatility is strongly linked with uncertainty regarding the real economy and inflation.
  - The co-movement of short-term and long-term volatility depends on the correlations among monetary policy, the real economy, and inflation.
- Interest rate volatility is typically expressed in terms of an annualized standard deviation. For example, if the monthly standard deviation of the three-month T-bill is 10.15%, then its yield volatility will be expressed as 0.3516 or 35.16% [$= 0.1015/(1/12)^{0.5}$].

READING 33: ARBITRAGE-FREE VALUATION FRAMEWORK

LESSON 1: THE MEANING OF ARBITRAGE-FREE VALUATION

LOS 33a: Explain what is meant by arbitrage-free valuation of a fixed-income instrument. Vol 5, pp 76–79

LOS 33b: Calculate the arbitrage-free value of an option-free, fixed-rate coupon bond. Vol 5, pp 79–81

The Meaning of Arbitrage-Free Valuation

The traditional approach to valuing a financial asset requires the following three steps:

1. Estimate the future cash flows.
2. Determine the appropriate discount rate.
3. Compute the present value of the cash flows.

In the fixed-income arena, the traditional approach to computing the value of a bond is to discount all the cash flows at the **yield to maturity (YTM).** However, as we learned in an earlier reading, one of the (several) problems with the YTM is that in discounting each cash flow at the same rate regardless of time of receipt, it assumes that the yield curve is flat.

Our focus in this reading is on the **arbitrage-free valuation approach.** Before learning how to apply this approach to bond valuation, let's cover a few fundamental concepts.

The Law of One Price

The **law of one price** states that two securities or portfolios that will generate identical cash flows in the future, regardless of future events, should have the same price today (in the absence of transaction costs). If this was not the case, individuals would simultaneously buy the underpriced asset and sell the overpriced asset to generate risk-free arbitrage profits until the two prices converged.

Arbitrage Opportunity

An **arbitrage opportunity** can be defined as a transaction that involves no cash outlay and results in a risk less profit. There are two types of arbitrage opportunities. To describe both these types, we will use the securities listed in Table 1-1. Note that all these assets are risk-free.

Table 1-1: Price Today and Payoff in One Year for Four Hypothetical Assets

Asset	Price Today	Payoff in One Year
A	$0.9434	$1
B	$95	$106
C	$100	$110
D	$150	$170

1. **Value additivity,** where the value of the whole equals the sum of the values of the parts. Consider Assets A and B.
   - The payoff of Asset B can be replicated by purchasing 106 units of Asset A.
   - It would cost \$100 (= 106 × 0.9434) to buy 106 units of Asset A, but only \$95 to purchase Asset B.
   - An astute investor would purchase Asset B and simultaneously sell 106 units of Asset A.
   - The combined position generates a certain \$5 today with no net obligation in one year.
2. **Dominance**, where one asset offers a greater return for the same level of risk. Consider Assets C and D.
   - Because both assets are risk-free they should offer the same return over the one-year horizon.
   - An investor could make money by selling 1.5 units of Asset C and using the proceeds to purchase 1 unit of Asset D.
   - The combined position generates \$0 today, but will generate a net inflow of \$5 (= \$170 – \$165) in one year.

Implications of Arbitrage-Free Valuation for Fixed-Income Securities

Under the arbitrage-free approach, any fixed-income security should be thought of as a package or portfolio of zero-coupon bonds. For example, a six-year coupon-bearing Treasury bond should be viewed as a package containing 13 zero-coupon instruments (12 semiannual coupon payments, one of which is made at maturity, and the principal repayment at maturity). The arbitrage-free approach uses the relevant **spot rate** (i.e., YTM on the appropriate zero-coupon bond) to discount each individual cash flow from a bond. The sum of the present values of cash flows from a bond, individually discounted at the relevant spot rate, should equal the price of the bond. If the market price of the bond is different from the value determined by discounting cash flows at individual spot rates, arbitrage opportunities arise. In Example 1-1, we illustrate how bonds are priced using the two methods: discounting at Treasury YTMs and discounting at Treasury spot rates.

Example 1-1: Valuing Bonds Using YTMs and Spot Rates

Suppose a \$1,000 par, annual-pay bond matures in three years and has a coupon rate of 8%. The bond is currently selling for \$1,001.34 at a yield to maturity of 7.948%. The relevant spot rates are given:

- 1-year = 7%
- 2-year = 7.5%
- 3-year = 8%

Verify that the value of the bond computed by discounting each cash flow at the relevant spot rate is the same as its current market price.

Solution:

If we discount each cash flow at the relevant spot rate (arbitrage-free approach), the value of the bond is calculated as:

$$80/(1.07)^1 + 80/(1.075)^2 + 1{,}080/(1.08)^3 = \$1{,}001.34$$

If we discount every cash flow at the yield to maturity, the price of the bond will equal:

$$N = 3;\ PMT = -\$80;\ I/Y = 7.948;\ FV = -\$1{,}000;\ CPT\ PV;\ PV \rightarrow \$1{,}001.34$$

In an arbitrage-free environment (in a perfectly efficient market), the two methods should result in the same price for the bond.

- If the market price of the bond is greater than the sum of the present values of the individual cash flows discounted at relevant spot rates, arbitrageurs would purchase the individual zero-coupon bonds in the strip market, "reconstitute" the cash flows, and sell the bond. The obligations arising from selling the bond would be met by the cash flows received from the strips package as individual components mature, and the trade would generate an immediate profit.
- If the market price of the bond is less than the sum of the present values of the individual cash flows discounted at relevant spot rates, arbitrageurs would sell the individual components in the Treasury strip market and purchase the bond. The obligations that arise as the individual strips mature would be met by periodic cash flows received from the bond, and the trade would generate an immediate profit.

For example, if the bond in Example 1-1 was selling in the market for $1,000.50 (less than its arbitrage-free value), an arbitrageur could make a profit by purchasing the bond and selling the individual strips. Her cash receipts and obligations are illustrated in the following table:

Time	Long Bond	Short Strips	Net Cash Flow
t = 0	Outflow of $1,000.50	Inflow of $1,001.34	Inflow of $0.84
t = 1	Inflow of $80	Outflow of $80	Nil
t = 2	Inflow of $80	Outflow of $80	Nil
t = 3	Inflow of $1,080	Outflow of $1,080	Nil

LESSON 2: INTEREST RATE TREES AND ARBITRAGE-FREE VALUATION

LOS 33c: Describe a binomial interest rate tree framework. Vol 5, pp 81–84

LOS 33d: Describe the backward induction valuation methodology and calculate the value of a fixed-income instrument given its cash flow at each node. Vol 5, pp 85–87

LOS 33e: Describe the process of calibrating a binomial interest rate tree to match a specific term structure. Vol 5, pp 92–94

LOS 33f: Compare pricing using the zero-coupon yield curve with pricing using an arbitrage-free binomial lattice. Vol 5, pp 94–96

LOS 33g: Describe pathwise valuation in a binomial interest rate framework and calculate the value of a fixed-income instrument given its cash flows along each path. Vol 5, pp 96–100

Interest Rate Trees and Arbitrage-Free Valuation

For option-free bonds, the simplest approach to arbitrage-free valuation involves determining the sum of the present values of expected future values using benchmark spot rates (as illustrated in Example 1-1). Benchmark spot rates can be derived from the government bond yield curve or the swap curve in the country.

For bonds that do contain embedded options, the challenge in developing an arbitrage-free valuation framework is to account for the fact that their expected cash flows are **interest rate dependent** (i.e., changes in interest rates in the future affect the likelihood of the embedded option being exercised, thereby affecting the bond's cash flows). For option-free bonds, the path taken by interest rates in the future has no bearing on size and timing of cash flows.

Our purpose in this reading now is to develop a framework that allows future interest rates to fluctuate based on an assumed level of **interest rate volatility**. We will use this framework to value an option-free bond in this reading (note the resulting value will be the same as that obtained by simply discounting the bond's cash flows at current spot rates), and then in subsequent readings we will use it to value different types of bonds with embedded options.

Since the interest rate tree resembles a lattice, these models are also known as **lattice models**.

We start by developing an **interest rate tree**, which represents the possible future interest rates consistent with the assumed level of volatility. The interest rate tree serves two purposes in the valuation process: (1) it is used to generate cash flows that are dependent on interest rates and (2) it supplies the interest rates that are used to compute present values of cash flows.

We begin with a description of the **binomial interest rate tree framework**, which assumes that the one-year interest rate can take on one of two possible values in the next period. These two possible interest rates in the next period must be consistent with three conditions:

1. The interest rate model that dictates how interest rates move.
2. The assumed level of interest rate volatility.
3. The current benchmark yield curve.

We will be taking the price of the benchmark bond as given such that when this bond is valued using our model, we should obtain the market value for the bond. This results in the model being tied to the current yield curve, thereby reflecting current economic reality.

The Binomial Interest Rate Tree

The first step in determining the interest rate tree is to determine the benchmark par curve. This is presented in Table 2-1. We will assume that all bonds make coupon payments annually during this illustration.

Table 2-1: Benchmark Par Curve

Maturity (Years)	Par Rate	Bond Price
1	1.00%	100
2	1.20%	100
3	1.25%	100
4	1.40%	100
5	1.80%	100

The **par curve** assumes that benchmark bonds are priced at par, which implies that for each bond, its YTM equals its coupon rate. From this par curve, the **bootstrapping** methodology is used to derive the underlying **spot rate curve** (Table 2-2).

Table 2-2: Underlying Spot Rates

Maturity (Years)	One-Year Spot Rate
1	1.000%
2	1.201%
3	1.251%
4	1.404%
5	1.819%

To illustrate the bootstrapping process, we derive the two-year spot rate using the par curve, below.

In order to determine the two-year spot rate, we first need to know the one-year spot rate. The one-year spot rate can be determined based on the price and coupon-rate on the one-year benchmark bond. From Table 2-1, we know that the price of this bond is \$100, and its coupon rate is the same as its YTM (1%) because it is trading at par.

$$100 = 1/(1 + {}_1S_0)^1 + 100/(1 + {}_1S_0)^1 \rightarrow {}_1S_0 = 0.01 \text{ or } 1\%$$

Now we can determine the two-year spot rate using the price of the two-year benchmark bond (\$100), its coupon rate (which equals its YTM, 1.20%), and the one-year spot rate (1%):

$$100 = 1.20/(1 + 0.01)^1 + 1.20/(1 + {}_2S_0)^2 + 100/(1 + {}_2S_0)^2 \rightarrow {}_2S_0 = 0.012012 \text{ or } 1.201\%$$

The same procedure is used to "bootstrap" the other spot rates presented in Table 2-2.

Once we have derived the spot rate curve, we derive one-year forward rates using the method illustrated in an earlier reading (and at Level I). These one-year forward rates are presented in Table 2-3.

Table 2-3: One-Year Implied Forward Rates

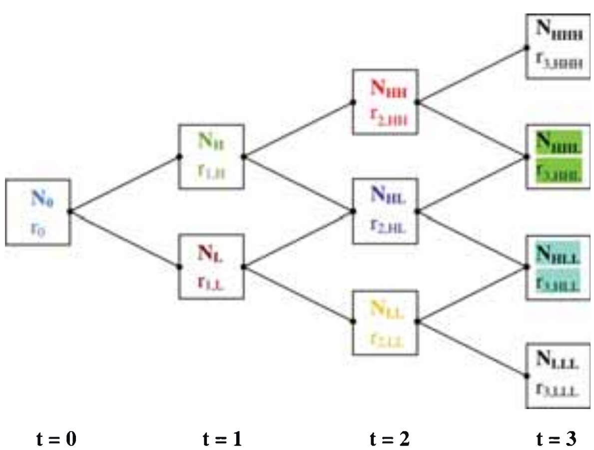
Rate	Forward Rate
$_1s_0$	1.000%
$_1f_1$	1.400%
$_1f_2$	1.350%
$_1f_3$	1.860%
$_1f_4$	3.500%

For example, the one-year forward rate two years from today, $_1f_2$, is calculated as:

$$\boxed{(1 + {}_2s_0)^2(1 + {}_1f_2)^1 = (1 + {}_3s_0)^3 \rightarrow {}_1f_2 = 0.0135 \text{ or } 1.35\%}$$

A **binomial interest rate tree** is a visual representation of possible interest rates based on the interest rate model used and the interest rate volatility assumption. Figure 2-1 presents a binomial interest rate tree. Our purpose now is to populate the tree:

Figure 2-1: Binomial Interest Rate Tree



Note the following:

- At the bottom of the figure, there is a one-year "time step" between each possible interest rate, which matches the frequency of annual cash flows.

We start at the "root" of the tree, N_0, which corresponds to t = 0 (today). The interest rate at the root of the tree is the one-year spot rate today, denoted by r_0. Moving toward the right from its root, the tree branches out into two nodes at t = 1:

1. N_H (if interest rates move higher), and
2. N_L (if interest rates move lower).

Note that the one-year interest rates at N_H and N_L ($r_{1,H}$ and $r_{1,L}$) are possible one-year forward rates one year from today.

Let's focus on the nodes at t = 1 for now. We know that the implied one-year forward rate one year from today, ${}_1f_1$, equals 1.4% (see Table 2-3). The two possible rates that we will calculate for the interest rate tree corresponding to t = 1 are going to be higher and lower than this implied forward rate. We will work with the following **interest rate model**, which is based on a lognormal random walk, in this illustration. It posits the following relationship between the lower forward rate (denoted by r_L) and the higher forward rate (denoted by r_H):

$$r_{1,H} = r_{1,L} \times e^{2(\sigma)} \quad \text{where } \sigma \text{ is the standard deviation (a measure of volatility)}$$

For example, suppose that $r_{1,L}$ is 1.194% and that σ is 15% per year. $r_{1,H}$ will therefore equal 1.612% (= 1.194% × $e^{2*0.15}$). Notice that these two possible future rates are (nearly) centered on the forward rate calculated from the benchmark curve (${}_1f_1$ = 1.40%). Think of the one-year forward rate implied by the benchmark curve as the average of possible values for the one-year rate at t = 1. The lower of the two rates ($r_{1,L}$) is one standard deviation below the mean (${}_1f_1$), while the higher of the two ($r_{1,H}$) is one standard deviation above the mean. As a result, the two possible future rates are multiples of each other with the multiplier being $e^{2(\sigma)}$. Note that as volatility increases, the two possible rates will drift farther apart, but still be (nearly) centered on the forward rate implied by the benchmark curve.

Now let's move farther to the right to the following three nodes that correspond to t = 2:

1. N_{HH}, if interest rates move higher in Year 1 and higher again in Year 2 to a level denoted by $r_{2,HH}$.
2. N_{HL} (N_{LH}), if interest rates move higher (lower) in Year 1 and lower (higher) in Year 2 to a level denoted by $r_{2,HL}$ ($r_{2,LH}$).
3. N_{LL}, if interest rates move lower in Year 1 and lower again in Year 2 to a level denoted by $r_{2,LL}$.

The one-year interest rates at N_{HH}, N_{HL}, and N_{LL} ($r_{2,HH}$, $r_{2,HL}$ and $r_{2,LL}$) are all possible one-year forward rates two years from today. If interest rates move higher in Year 1 and then lower in Year 2, we would get to the same node as if interest rates had moved lower in Year 1 and then higher in Year 2. This middle rate will be close to the implied one-year forward rate two years from today, ${}_1f_2$, and the multiplier for adjacent rates on the tree will still be $e^{2\sigma}$.

This type of a tree is known as a **recombining tree** because there are two paths to get to the middle rate. This results in faster computation, as the number of possible outcomes grows linearly instead of exponentially.

The relationship among $r_{2,LL}$, $r_{2,HL}$, and $r_{2,HH}$ is as follows:

$$r_{2,HL} = r_{2,LL}(e^{2\sigma}) \text{ and } r_{2,HH} = r_{2,LL}(e^{4\sigma})$$

Notice that adjacent outcomes are still two standard deviations apart. So, for example, if $r_{2,LL}$ is 0.980%, then $r_{2,HL}$ equals 0.980%($e^{2\times0.15}$) = 1.323% and $r_{2,HH}$ equals 0.980%($e^{4\times0.15}$) = 1.786% (assuming again that σ = 0.15).

Moving to t = 3, the possible one-period forward rates at four nodes are each related as follows:

$$r_{3,LLH} = r_{3,LLL}(e^{2\sigma});\ r_{3,HHL} = r_{3,LLL}(e^{4\sigma});\ \text{and } r_{3,HHH} = r_{3,LLL}(e^{6\sigma})$$

So far we have only illustrated how the interest rates in the tree are interlinked (based on the interest rate model used and the volatility assumption). Before actually building the interest rate tree based on the rates in Tables 2-1, 2-2, and 2-3 we need to (1) discuss the interest rate volatility assumptions and (2) learn to determine the value of a bond at a node via backward induction.

What Is Volatility and How Is It Estimated?

With a simple lognormal distribution, changes in interest rates are proportional to the level of the one-period rate in each period. For a lognormal distribution, the standard deviation of the one-year rate can be calculated as $r_0\sigma$. For example, if σ is 15% and the current one-year rate is 1%, then the standard deviation of the one-year rate equals 1% × 15% = 0.15% or 15 bps. Therefore, interest rate changes get larger (smaller) as interest rates are high (low).

There are two methods of estimating interest rate volatility:

- **Historical interest rate volatility** can be computed from past data. This approach assumes that the past is representative of what might happen in the future.
- **Implied volatility** can be computed from observed market prices of interest rate derivatives. This approach assumes that those instruments are properly priced.

Determining the Value of a Bond at a Node

- To determine the value of a bond using the binomial interest rate tree, we must first determine the value of the bond at each node in the tree, starting from the right and moving left.
- In order to calculate the value of a bond at a particular node (e.g., at N_{HL}):
    - Calculate the value of the bond at the two nodes (N_{HHL} and N_{HLL}) that branch out (to the right) from the node.
    - Add the coupon payment (C) that will be received at those two nodes.
    - Discount the sum of the value at the bond and the coupon payment at each node at the current one-year interest rate ($r_{2,HL}$).
    - Compute the average of those two values.

Because the procedure for calculating the value of the bond involves moving backward (from right to left) along the binomial tree, this methodology is known as **backward induction**.

What all this effectively means is that the value of the bond at any given node depends on:

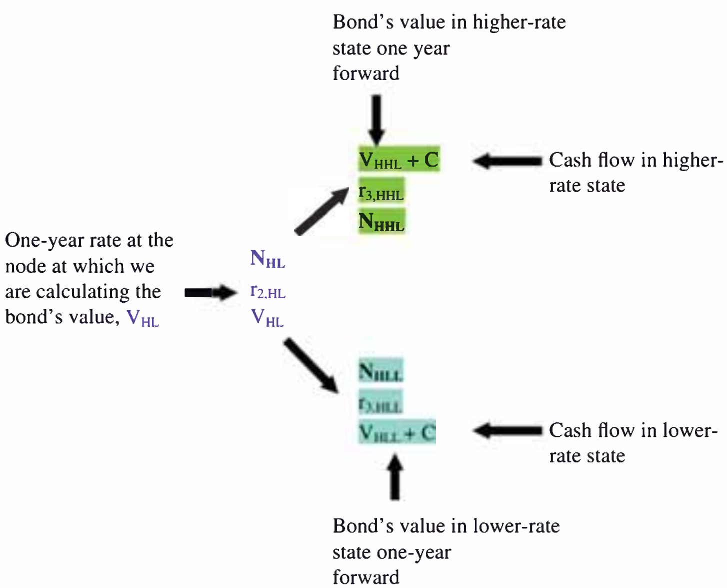
1. The possible values of the bond one year from that node (where the value of the bond at each future node represents the present values of all cash flows expected to be received in future time periods),
2. The coupon payment one year from that node, and
3. The current one-year interest rate at that node.

For example, the value of the bond at a node corresponding to t = 2 equals the present value of (1) cash flows expected to be received at t = 3 and beyond (which equals the value of the bond at t = 3) and (2) the cash flow (coupon) to be received at t = 3, discounted at the one-year rate at that node (t = 2).

While the coupon payment for the next period is known, the bond's value depends on the path taken by interest rates (whether they move higher or lower). Because the probability of an increase or decrease in interest rates at each node equals 50%, we compute the average of the two present values. For example (refer to Figure 2-2, which is an extract from Figure 2-1), at Node N_{HL}, the expected future cash flows are represented by:

1. The sum of V_{HHL}, the bond's value at Node N_{HHL} (if interest rates move higher) and C, the coupon payment; and
2. The sum of V_{HLL}, the bond's value at Node N_{HLL} (if interest rates move lower) and C, the coupon payment.

Figure 2-2: Determining Bond Value at a Node Applying Backward Induction



The present values of these sets of cash flows, discounted at the one-year rate ($r_{2,HL}$) at Node N_{HL}, are:

1. $\left(\dfrac{V_{HHL} + C}{(1 + r_{2,HL})}\right) \rightarrow$ Present value of future cash flows in the higher one-year rate scenario
2. $\left(\dfrac{V_{HLL} + C}{(1 + r_{2,HL})}\right) \rightarrow$ Present value of future cash flows in the lower one-year rate scenario

Finally, the expected value of the bond, V_{HHL}, at Node N_{HL} is calculated as:

$$\boxed{V = \frac{1}{2}\left(\frac{V_{HHL} + C}{(1 + r_{2,HL})} + \frac{V_{HLL} + C}{1 + r_{2,HL}}\right)}$$

Note that CFAI uses a slightly different looking but mathematically identical formula for computing the value of the bond at a node.

Note that at maturity of a bond, the par value and the final coupon will be paid out. So we start the valuation process at the right-most nodes of the tree and work backward (leftward) from there to determine the value of a bond. We will illustrate this process with numbers after first populating our interest rate tree with interest rates (in the next section).

Constructing the Binomial Interest Rate Tree

Three conditions must be satisfied in populating the binomial interest rate tree:

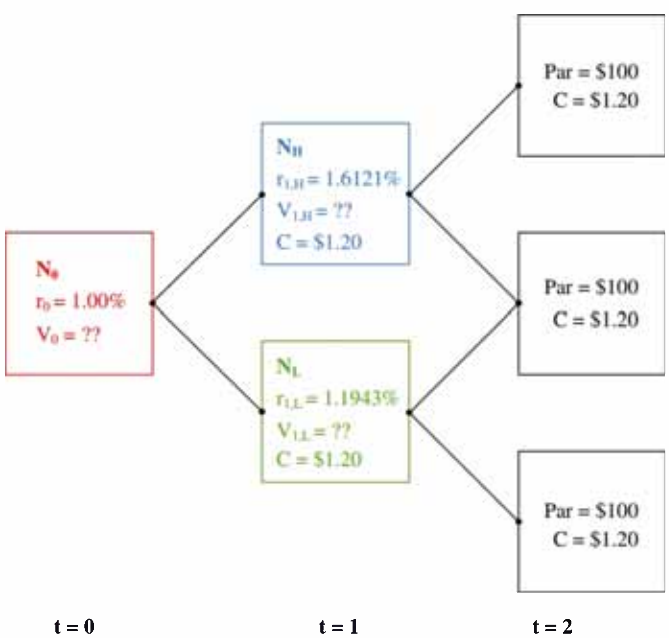
- The rates must adhere to the interest rate model. In our example, the model is given by $r_H = r_L \times e^{2\sigma}$.
- The rates must adhere to the interest rate volatility assumption. In our example, $\sigma = 0.15$.
- The model must be arbitrage-free in that it must fit the current yield curve (in Table 2-2) to produce the benchmark bond values (in Table 2-1).

Let's start with the two-year bond in Table 2-1, which carries a coupon rate of 1.20% and is trading at \$100. We already know that the current one-year rate is 1%. Finding the two possible rates at t = 1 is an iterative process. Given the interest rate model, the interest rate volatility assumption, the coupon rate and par value of the two-year bond, and the current one-year rate, for the rates in our tree to be arbitrage-free, the application of backward induction must result in a value for the bond that equals its current market price (\$100).

An analytical tool, such as Solver in Excel, can be used to determine values for $r_{1,L}$ and $r_{1,H}$ given the above information. Let's suppose that Solver offers us a value of 1.1943% for $r_{1,L}$. Let's determine if this value for $r_{1,L}$ is indeed arbitrage-free.

So we start with a value of 1.1943% for $r_{1,L}$. Given our interest rate model ($r_H = r_L e^{2\sigma}$) and interest rate volatility assumption (σ = 15%), we obtain a value of 1.6121% (= 1.1943% × $e^{2\times0.15}$) for $r_{1,H}$. See Figure 2-3.

Figure 2-3: Deriving One-Year Forward Rates One Year from Today



Using these rates, the values of the bond at N_H and N_L are calculated as:

$$V_{1,H} = \frac{1}{2}\left(\frac{100 + 1.20}{(1 + 0.016121)^1} + \frac{100 + 1.20}{(1 + 0.016121)^1}\right) = \$99.59444$$

$$V_{1,L} = \frac{1}{2}\left(\frac{100 + 1.20}{(1 + 0.011943)^1} + \frac{100 + 1.20}{(1 + 0.011943)^1}\right) = \$100.0056$$

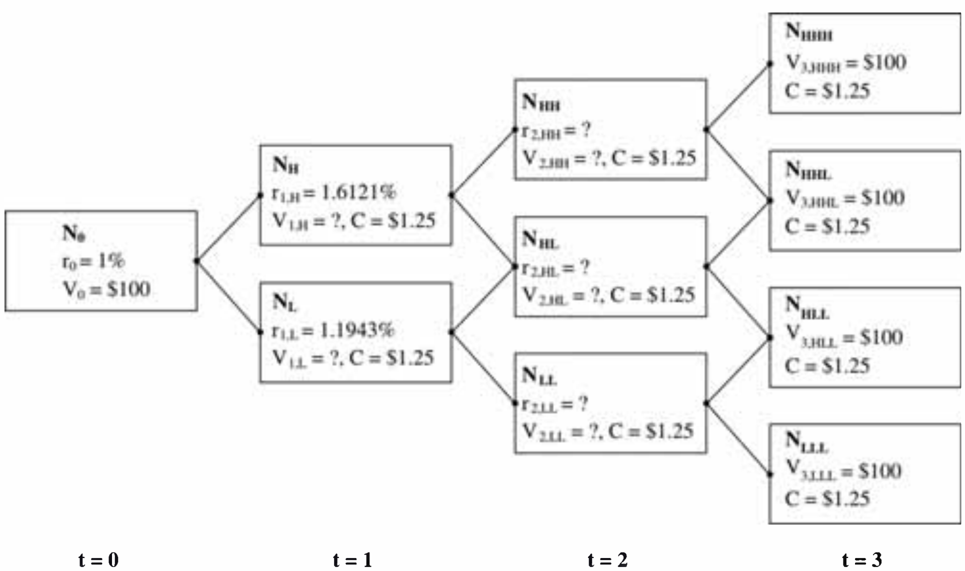
Based on these expected values of the bond at t = 1 plus the coupon payment and the one-year rate today, we can now compute V_0 as:

$$V_0 = \frac{1}{2}\left(\frac{99.59444 + 1.20}{(1 + 0.01)^1} + \frac{100.0056 + 1.20}{(1 + 0.01)^1}\right) = \$100$$

The expected value of the bond determined by discounting its expected future cash flows at the interest rates in the binomial tree equals its observed market price, so we can conclude that our possible one-year forward rates at t = 1 are arbitrage-free.

Now let's determine the one-year forward rates at t = 2 (see Figure 2-4). We use the same process that we just described, but we use the three-year benchmark bond that carries a coupon rate of 1.25% (see Table 2-1), the same interest rate model, the same interest rate volatility assumption, a current one-year rate of 1%, and possible one-year forward rates at t = 1 of 1.6121% (higher rate) and 1.1943% (lower rate).

Figure 2-4: Deriving One-Year Forward Rates Two Years from Today



Let's say that Solver offers us a value of 0.9803 for $r_{2,LL}$. Let's see whether this value is correct (i.e., arbitrage-free). Based on the value for $r_{2,LL}$, we can compute $r_{2,HL}$ and $r_{2,HH}$ as:

$$r_{2,HL} = 0.9803(e^{2\times0.15}) = 1.3233\%$$

$$r_{2,HH} = 0.9803(e^{4\times0.15}) = 1.7862\%$$

Based on these (three) possible one-year forward rates at t = 2, we apply the backward induction methodology to determine the possible values of the bond at t = 2.

$$V_{2,HH} = \frac{1}{2}\left(\frac{100 + 1.25}{(1 + 0.01786)^1} + \frac{100 + 1.25}{(1 + 0.01786)^1}\right) = \$99.4734$$

$$V_{2,HL} = \frac{1}{2}\left(\frac{100 + 1.25}{(1 + 0.01323)^1} + \frac{100 + 1.25}{(1 + 0.01323)^1}\right) = \$99.928$$

$$V_{2,LL} = \frac{1}{2}\left(\frac{100 + 1.25}{(1 + 0.0098)^1} + \frac{100 + 1.25}{(1 + 0.0098)^1}\right) = \$100.2674$$

Based on these values, the bond's coupon rate, and the (two) possible one-year forward rates at t = 1 ($r_{1,H}$ and $r_{1,L}$), we can compute the (two) possible values of the bond at t = 1.

$$V_{1,H} = \frac{1}{2}\left(\frac{99.4734 + 1.25}{(1 + 0.016121)^1} + \frac{99.928 + 1.25}{(1 + 0.016121)^1}\right) = \$99.3492$$

$$V_{1,L} = \frac{1}{2}\left(\frac{99.928 + 1.25}{(1 + 0.011943)^1} + \frac{100.2674 + 1.25}{(1 + 0.011943)^1}\right) = \$100.1519$$

Finally, based on these possible bond values at t = 1, the bond's coupon rate, and the one-year rate at t = 0, we can compute the current value of the bond.

$$V_0 = \frac{1}{2}\left(\frac{99.3492 + 1.25}{(1 + 0.01)^1} + \frac{100.1519 + 1.25}{(1 + 0.01)^1}\right) = \$100$$

Now we have proved that the one-year rates in our interest rate tree are arbitrage-free (as the resulting value of the three-year 1.25% coupon bond equals its price, \$100). Figure 2-5 presents the same information as Figure 2-4, but with all the values computed above filled in.

Figure 2-5: Deriving One-Year Forward Rates Two Years from Today

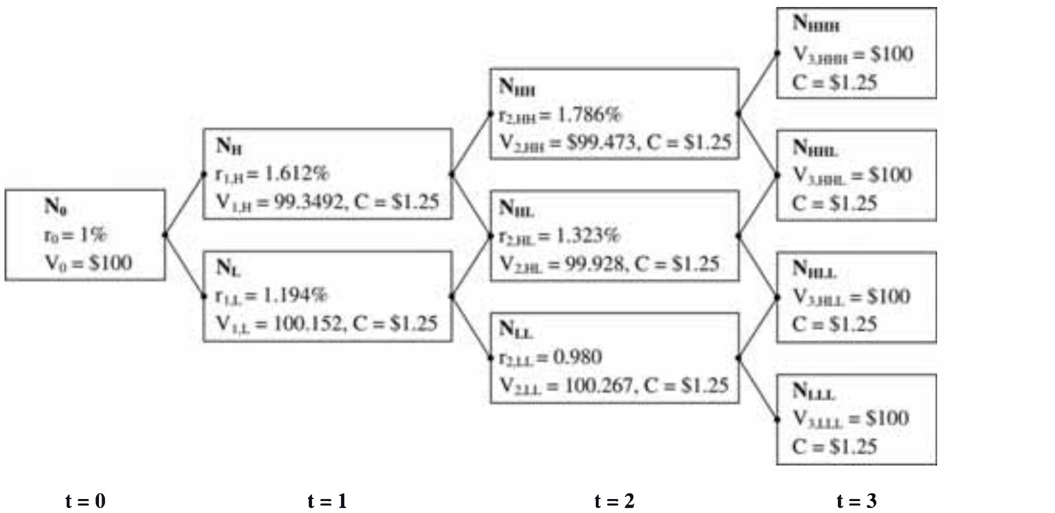
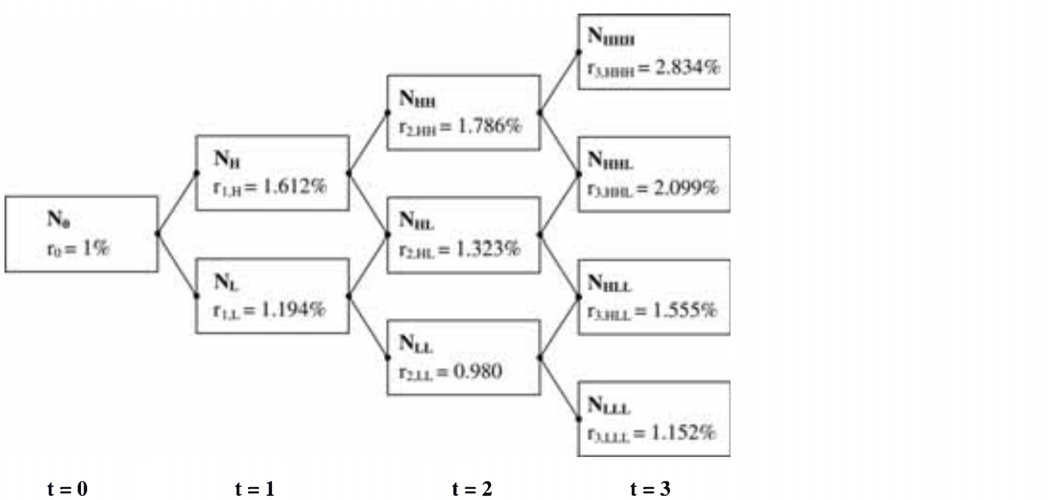


Figure 2-6 presents the binomial interest rate tree that includes possible one-year forward rates at t = 3 based on the information in Table 2-1. Note that we have only illustrated the computation of possible one-year forward rates at t = 1 and t = 2. Determining the rates at t = 3 requires us to solve for the possible one-year forward rates that would result in a $100 (market price) value for the four-year 1.4% coupon bond (see Table 2-1) while adhering to our interest rate model and volatility assumption and using the computed possible one-year forward rates at t = 1 and t = 2.

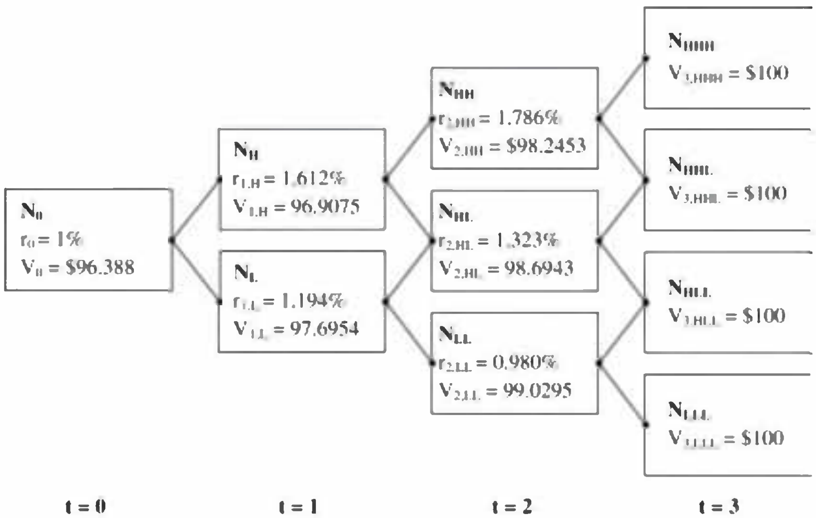
Figure 2-6: Three-Year Binomial Interest Rate Tree



Valuing an Option-Free Bond with the Tree

In the previous section, even though the aim was to learn how the binomial interest rate tree is populated (with arbitrage-free one-year forward rates), we got enough practice in valuing coupon-bearing bonds as well (when we proved that the values of the two- and three-year bonds derived from rates in the tree were the same as their market prices, $100). In Figure 2-7, we'll value a three-year zero-coupon bond based on the interest rate tree in Figure 2-5. The bond makes no coupon payments and is worth par in three years (at maturity).

Figure 2-7: Pricing a Three-Year Zero-Coupon Bond



To illustrate how the values in Figure 2-7 have been derived, we have provided the computation of $V_{2,HL}$ below:

$$V_{2,HL} = \frac{1}{2}\left(\frac{100}{(1 + 0.01323)^1} + \frac{100}{(1 + 0.01323)^1}\right) = \$98.6943$$

We shall reverify that the interest rates used in our model are indeed arbitrage-free in the next section.

Path-Wise Valuation

Path-wise valuation is an alternative approach to backward induction. It computes the present value of a bond for each possible interest rate path and then computes the average of those values. The resulting price of an option-free bond is the same as the price determined through backward induction. Path-wise valuation entails the following steps:

1. Specify a list of possible paths through a tree.
2. Determine the present value of the bond through each potential path.
3. Compute the average present value across all possible paths.

The number of interest rate paths in the binomial model can be computed using **Pascal's Triangle**, which is built as follows (see Figure 2-8):

- We start with the number 1 at the top of the triangle.
- The number in the boxes below are the sum of the two numbers above them, except that the edges on each side are all 1.

Figure 2-8: Pascal's Triangle

```
          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5  10  10   5   1
```

Figure 2-9 shows the possible outcomes for interest rates:

Figure 2-9: Possible Outcomes at t = 1, t = 2, and t = 3

Number of Years	Interest Rate Movement U = Up, D = Down	Pascal's Triangle
1	U D	1, 1
2	UU UD DU DD	1,2,1
3	UUU UUD UDU DUU UDD DUD DDU DDD	1, 3, 3, 1

As you can see from Figure 2-9, the total number of possible paths can easily be worked out using Pascal's Triangle. For example, at t = 3 there are eight possible interest rate paths.

Let's illustrate path-wise valuation by valuing the three-year zero coupon bond (which we valued at $96.388 in the previous section using the interest rate tree).

From Figure 2-9, we see that there are four possible paths at t = 2 to arrive at Year 3: UU, UD, DU, and DD. In Table 2-4, we specify these four paths as well as the one-year interest rates along each of these paths (which are the same as in Figure 2-7). The last column in the table shows the present value for each path. The average of these present values represents the current price of the bond.

Table 2-4: Valuing a Three-Year Zero-Coupon Bond Using Path-Wise Valuation

Path	$_1s_1$	$_1f_1$	$_1f_2$	Present Value
1	1.0000%	1.612%	1.786%	95.729
2	1.0000%	1.612%	1.323%	96.167
3	1.0000%	1.194%	1.323%	96.564
4	1.0000%	1.194%	0.980%	96.892
				96.338

To illustrate how the present value along each path is computed, we show the calculation of the present value for Path 2 below:

$$\text{Present Value}_2 = 100 / (1.01)(1.01612)(1.01323) = 96.167$$

Notice that the value of the bond obtained here, $96.338, is identical to the value obtained in Figure 2-7. Both the methods result in the same value for the three-year zero-coupon bond.

In one final proof that the interest rates in our tree are indeed arbitrage-free, we can determine the value of the three-year zero-coupon bond using (1) the three-year spot rate in Table 2-2 and (2) the one-year forward rates in Table 2-3.

1. Value of three-year zero = 100 / $(1.01251)^3$ = $96.34
2. Value of three-year zero = 100 / (1.01)(1.014)(1.0135) = $96.34

LESSON 3: MONTE CARLO METHOD

LOS 33h: Describe a Monte Carlo forward-rate simulation and its application. **Vol 5, pp 100–101**

Monte Carlo Method

The **Monte Carlo method** is a method for simulating a sufficiently large number of potential interest rate paths in an effort to discover how a value of a fixed-income security is affected. It is often used when a security's cash flows are **path-dependent** (i.e., when the cash flow to be received in a particular period depends not only on the current level of interest rates [as in the valuation exercises that we have performed so far in this reading] but also on the path interest rates have taken to reach their current level).

For example, consider mortgage-backed securities, where cash flows in any particular period include prepayments. Interest rates may currently be relatively low, but if they have been lower in the recent past, the current prepayment speed will not be as high as it would be had interest rates fallen (from higher levels) to the current low level.

Interest rate paths in a Monte Carlo simulation are generated based on (1) a probability distribution, (2) an assumption about volatility, and (3) the current benchmark term structure of interest rates. The benchmark term structure is represented by the current spot rate curve such that the average present value across all the interest rate paths equals the benchmark bond's actual market value. By using this approach, the model is rendered arbitrage-free.

Suppose we intend to value a 30-year mortgage-backed security (that makes monthly coupon payments) with the Monte Carlo method. We will have to undertake the following steps:

1. Simulate numerous paths of one-month interest rates under (1) an interest rate volatility assumption and (2) probability distribution.
2. Generate spot rates from the simulated future one-month interest rates.
3. Determine the periodic cash flows along each interest rate path.
4. Compute the present value of cash flows for each path.
5. Calculate the average present value across all interest rate paths.

The value of the benchmark bond generated from the procedure described may or may not equal the bond's actual market value. Therefore, we must check that the value generated equals the observed market price. If it does not, then a constant (known as the **drift term**) must be added to all interest rates on all paths such that the average present value equals the observed market price. Such a model is said to be **drift-adjusted**.

The model can also incorporate mean reversion in interest rates. This has the effect of moving the interest rate toward the forward rates implied by the yield curve.

Finally, note that increasing the number of paths in the model will not necessarily bring the resulting value closer to the true fundamental value of the security. After all, results are only as good as the valuation model and inputs used.

Example 3-1: Application of the Monte Carlo Method to Bond Pricing

Consider a three-year annual-pay 5% coupon bond that is currently trading at $102.81. This market price has been verified as the arbitrage-free price for this bond given the current term structure of interest rates. An analyst uses the Monte Carlo method to generate the eight interest rate paths given (presented in the Table 3-1). Determine whether the Monte Carlo simulation has been calibrated correctly.

Table 3-1: Discount Rates

Path	Time 0 (%)	Time 1 (%)	Time 2 (%)
1	2	2.500	4.548
2	2	3.600	6.116
3	2	4.600	7.766
4	2	5.500	3.466
5	2	3.100	8.233
6	2	4.500	6.116
7	2	3.800	5.866
8	2	4.000	8.233

Given the bond's cash flows, their present value along each interest rate path, as well as the average present value, is presented in Table 3-2.

Table 3-2: Present Values

Path	PV_0
1	105.7459
2	103.2708
3	100.91064
4	103.8543
5	101.9075
6	102.4236
7	103.3020
8	101.0680
Average	**102.8103**

To illustrate the calculation of the present value along the interest rate paths, we present the calculation of the present value of cash flows along Path 5 below. Note that the annual coupon is \$5 and the par value of the bond is \$100.

$$PV_0 = 5/(1 + 0.02) + 5/(1 + 0.02)(1 + 0.031) + 105/(1 + 0.02)(1 + 0.031)(1 + 0.08233)$$
$$= \mathbf{\$101.9075}$$

Because the average of the present values along each interest rate path used in the simulation equals the benchmark bond's observed market price, we can conclude that the model has been calibrated correctly. There are enough representative paths for us to now value path-dependent securities.

Study Session 13: Fixed Income (2)

Reading 34: Valuation and Analysis of Bonds with Embedded Options

LESSON 1: OVERVIEW OF EMBEDDED OPTIONS

LOS 34a: Describe fixed-income securities with embedded options. Vol 5, pp 122–127

OVERVIEW OF EMBEDDED OPTIONS

Embedded options refer to options found in a bond's indenture, where these options cannot be traded separately (hence the adjective "embedded").

The bond that these options are embedded in is referred to as a **straight bond**. The straight bond may carry a fixed rate or a floating rate. In most of this Reading (except for the discussion on caps and floors) we will be working with fixed-rate bonds.

Callable Bonds

Callable bonds give the **issuer** the right to redeem (or call) all or part of the bond before maturity. This embedded option offers the issuer the ability to take advantage of (1) a decline in market interest rates and/or (2) an improvement in its creditworthiness. If interest rates decline and/or the issuer's credit rating improves, the issuer would call the outstanding issue and replace this old (expensive in terms of paying coupon) issue with a new issue that carries a lower interest rate.

- Most callable bonds come with **lockout periods** during which they cannot be called.
- Callable bonds include different types of call features:
  - An **American-style** callable bond can be called by the issuer at any time, starting with the first call date until maturity.
  - A **European-style** callable bond can only be called by the issuer at a single date at the end of the lockout period.
  - A **Bermudan-style** callable bond can be called by the issuer on specified dates following the lockout period.

Putable Bonds and Extendible Bonds

Putable bonds give **bondholders** the right to sell (or put) the bond back to the issuer at a predetermined price on specified dates. The embedded put option offers bondholders protection against an increase in interest rates (i.e., if interest rates increase [decreasing the value of the straight bond] they can sell the putable bond back to the issuer at the pre-specified price, and then reinvest the principal at [higher] newer interest rates).

- As is the case with callable bonds, putable bonds also typically include lockout periods.
- Most of the time they tend to be European-style and are rarely Bermudan-style, but there are no American-style putable bonds.

An **extendible bond** is similar to a putable bond. It contains an **extension option** that offers the bondholder the right to keep the bond a number of years after maturity, possibly with a different coupon rate. We will discuss the similarities between putable and extendible bonds later in the Reading.

Bonds with Other Types of (Complex) Embedded Options

Convertible bonds offer investors the option to convert their bonds into the issuer's common stock. Typically, they are also callable, which enables the issuer (1) to take advantage of lower interest rates or (2) to force conversion. These are discussed in more detail later in the reading.

Sometimes the option embedded in a bond may be contingent upon some future event. For example, **death-put bonds** (which come with an **estate put** or **survivor's option**) can be redeemed at par by the heirs of the deceased bondholder. The important thing to note here is that the value of such bonds depends not only on interest rate movements, but also on the holder's life expectancy.

Sinking fund bonds (or **sinkers**) require the issuer to retire a certain proportion of the issue each year following the lockout period (thereby reducing credit risk). These bonds may also contain additional options, such as:

- **Standard call options,** which allow the issuer to redeem the entire issue at any point in time following the lockout period.
- **Acceleration provisions,** which allow the issuer to repurchase more than the mandatory amount of bonds.
- **Delivery options,** which allow the issuer to satisfy a sinking fund payment by delivering bonds to the trustee instead of cash. If the bonds are trading at less than par, then it is more cost-effective for the issuer to buy back bonds from investors to meet sinking fund requirements than to pay par to redeem them. The delivery options benefit the issuer when interest rates rise.
- A combination of a call option and a delivery option is effectively a **long straddle,** where the sinking fund benefits the issuer not only if interest rates decline, but also if they rise.
  - A long straddle is an option strategy where a call and a put are both purchased with the same exercise price and exercise date. This strategy benefits the investor when the underlying moves up or down. The greater the extent of the up or down move, the greater the benefit to the investor.

LESSON 2: VALUATION AND ANALYSIS OF CALLABLE AND PUTABLE BONDS PART I

LOS 34b: Explain the relationships between the values of a callable or putable bond, the underlying option-free (straight) bond, and the embedded option. **Vol 5, pp 127–129**

VALUATION AND ANALYSIS OF CALLABLE AND PUTABLE BONDS

Relationships between Values of a Callable or Putable Bond, Straight Bond, and Embedded Option

The option embedded in a callable bond favors the issuer. The investor in a callable bond is effectively long on a straight bond, but short on the call option. Therefore, the value of the call option decreases the value of a callable bond relative to the value of a straight bond.

Value of callable bond = Value of straight bond – Value of embedded call option

Determining the value of a straight bond is relatively straightforward (discount its cash flows at appropriate rates). The challenge, however, lies in valuing the embedded option. In practice, the value of the embedded option is often calculated as the difference between the value of a straight bond and that of a callable bond.

Value of embedded call option = Value of straight bond – Value of callable bond

Recall that in an earlier reading, we populated our binomial interest rate tree with possible one-year forward rates. This is exactly what we shall be doing in this reading as well.

For a putable bond, the embedded option favors the investor. The investor is effectively long on a straight bond and long on the put option. Therefore, the value of the put option increases the value of a putable bond relative to the value of a straight bond.

Value of putable bond = Value of straight bond + Value of embedded put option

Value of embedded put option = Value of putable bond – Value of straight bond

Valuation of Default-Free Option-Free Bonds

We learned at Level I (and in an earlier reading) that a straight bond can be valued by discounting its expected future cash flows at either:

- Current spot rates bootstrapped from par rates on bonds with varying maturities, or
- Forward rates implied by the current spot rate curve.

When it comes to valuing bonds with embedded options, we rely on one-period forward rates rather than on spot rates. This is because we need to know the possible values of the bond at different points in time in the future in order to determine whether the embedded options will be exercised at those points in time.

For the remainder of this Reading, we will be working with the par rates, (bootstrapped) spot rates, and (implied) forward rates presented in Table 2-1, Table 2-2, and Table 2-3.

Table 2-1: Benchmark Par Curve

Maturity	Yield to Maturity (%)	Market Price
1	3.5	100
2	4.2	100
3	4.7	100
4	5.2	100

Table 2-2: Spot Rates

Maturity	Spot Rate
1	${}_1S_0 = 3.5\%$
2	${}_2S_0 = 4.2148\%$
3	${}_3S_0 = 4.7352\%$
4	${}_4S_0 = 5.2706\%$

Table 2-3: Implied Forward Rates

	Forward Rates
${}_1f_0$ (or ${}_1s_0$)	3.5%
${}_1f_1$	4.935%
${}_1f_2$	5.784%
${}_1f_3$	6.893%

The "bootstrapping" of spot rates and computation of implied forward rates was illustrated in an earlier Reading.

LOS 34c: Describe how the arbitrage-free framework can be used to value a bond with embedded options. Vol 5, pp 129–150

Valuation of Default-Free Callable and Putable Bonds in the Absence of Interest Rate Volatility

Valuing a Callable Bond at Zero Volatility

Let's start with valuing a **default-free**, Bermudan-style, four-year, 7% annual-pay bond that is callable at par two years and three years from today. The call option lies with the issuer, and it will choose to exercise the option if the value of the bond's future cash flows is higher than the exercise price ($100). Table 2-4 illustrates how the value of this callable bond is computed using the one-year forward rates from Table 2-3.

Table 2-4: Valuing a Callable Bond with Zero Volatility

	Today	t = 1	t = 2	t = 3	t = 4
Cash flow		$7	$7	$7	$107
One-year rate	3.500%	4.935%	5.784%	6.893%	
Value of bond	(101.97 + 7)/1.035 = $105.29	(100 + 7)/1.04935 = $101.97 *Cannot be called*	(100 + 7)/1.05784 = $101.15 *Called at $100*	107/1.06893 = $100.10 *Called at $100*	

We are using backward induction (introduced in an earlier reading) here. We start by discounting the cash flow at t = 4 ($100 principal plus $7 final coupon) at the one-year forward rate three years from today (${}_1f_3$ = 6.893%). This present value of the bond's future cash flows as of t = 3 ($100.10) is greater than the call price ($100). A rational borrower would call the bond at this point in time, as leaving it outstanding would be more expensive than redeeming it, so we consider the $100 call price the true/correct present value of the bond's cash flows as of t = 3.

Next, we add the cash flow at t = 3 ($7) to the present value of the bond's cash flows as of t = 3 ($100) and discount the sum at the one-year forward rate two years from today (${}_1f_2$ = 5.784%). This present value of the bond's future cash flows as of t = 2 ($101.15) is greater than the call price ($100). A rational borrower would call this bond, so we consider the $100 call price the true present value of the bond's future cash flows as of t = 2.

Then we add the cash flow at t = 2 ($7) to the present value of the bond's cash flows as of t = 2 ($100) and discount the sum at the one-year forward rate one year from today (${}_1f_1$ = 4.935%). The present value of the bond's future cash flows as of t = 1 is $101.97. (Note that the bond is not callable at t = 1.)

Finally, we add the cash flow at t = 1 ($7) to the present value of the bond's cash flows as of t = 1 ($101.97) and discount the sum at the one-year spot rate today ($_1s_0$ = 3.5%) to determine the current price of the bond, $105.29.

Now we can compute the value of the embedded call option as the difference between the value of the straight bond and the value of the callable bond. The value of a default-free, four-year, 7% annual-pay straight bond can be calculated as:

Using Spot Rates (in Table 2-2)

$$\frac{7}{(1.035)^1} + \frac{7}{(1.042148)^2} + \frac{7}{(1.047352)^3} + \frac{107}{(1.052706)^4} = \$106.43$$

Using 1-Year Forward Rates (in Table 2-3)

$$\frac{7}{(1.035)} + \frac{7}{(1.035)(1.04935)} + \frac{7}{(1.035)(1.04935)(1.05784)} +$$
$$\frac{107}{(1.035)(1.04935)(1.05784)(1.06893)} = \$106.43$$

Therefore, the value of the embedded call option can be calculated as:

Value of embedded call option = Value of straight bond – Value of callable bond
= $106.43 – $105.29 = $1.14

Valuing a Putable Bond at Zero Volatility

When valuing a putable bond with zero volatility, we need to determine whether the bond will be put at any put date. Let's work with a **default-free**, Bermudan-style, four-year, 7% annual-pay bond that is putable at par, two years and three years from today. Table 2-5 illustrates how the value of this putable bond is determined.

Table 2-5: Valuing a Putable Bond with Zero Volatility

	Today	t = 1	t = 2	t = 3	t = 4
Cash flow		$7	$7	$7	$107
One-year rate	3.500%	4.935%	5.784%	6.893%	
Value of bond	(103.15 + 7)/1.035 = $106.43	(101.24 + 7)/1.04935 = $103.15 *Not putable*	(100.10 + 7)/1.05784 = $101.24 *Not put*	107/1.06893 = $100.10 *Not put*	

Notice that the present value of the bond's future cash flows as of t = 3 ($100.10), t = 2 ($101.24), and t = 1 ($103.15) is always greater than the put price ($100). A rational investor would never put this bond back to the issuer. Notice that the putable bond's current price ($106.43) is the same as that of the straight bond (computed earlier as $106.43), which means that the put option is worth zero. Given (1) one-year forward rates that are all lower than the bond's coupon rate (7%), which means that the straight bond will always trade at a premium, and (2) zero interest rate volatility, which means that there is no chance of interest rates changing, the value of the embedded put option will be zero. The option is currently out-of-the-money (so there is zero exercise value) and there is no speculative value because interest rates are assumed constant.

We will go over another example in the next section where the bond will be put back to the issuer at certain points in time, and therefore the put option will hold a positive value for the investor.

LOS 34d: Explain how interest rate volatility affects the value of a callable or putable bond. **Vol 5, pp 132–134**

Effect of Interest Rate Volatility on the Value of Callable and Putable Bonds

In real life, interest rates fluctuate, so assuming that interest rate volatility was zero (in the previous section) was a very unrealistic assumption to make. However, the previous section did introduce you to applying the call and put rules at each call/put date in order to determine the values of callable/putable bonds. In this section we will discuss the effects of (1) interest rate volatility and (2) the shape of the yield curve on the value of bonds with embedded options.

The effect of interest rate volatility in the bond valuation exercise is captured through a binomial interest rate tree (described in an earlier reading). All else remaining the same:

- The embedded call option increases in value with higher interest rate volatility. Therefore, the value of a callable bond decreases with higher volatility.
- The embedded put option also increases in value with higher interest rate volatility. Therefore, the value of a putable bond increases with higher volatility.

LESSON 3: VALUATION AND ANALYSIS OF CALLABLE AND PUTABLE BONDS PART II

LOS 34e: Explain how changes in the level and shape of the yield curve affect the value of a callable or putable bond. **Vol 5, pp 134–137**

Effect of Level and Shape of the Yield Curve on the Value of Callable and Putable Bonds

Callable Bonds

All else remaining the same:

- As interest rates decrease, the value of a straight bond increases, and the value of the embedded call option in a callable bond also increases. Because the investor is effectively short on the embedded call option, the value of the callable bond increases less than that of the straight bond, limiting the upside for the investor.
- The value of the embedded call option increases as the yield curve goes from being upward sloping, to being flat, to being downward sloping. When the yield curve is upward sloping, forward rates are high, resulting in lower present values of expected future cash flows and fewer opportunities for the issuer to call the bond. As the yield curve flattens or inverts, forward rates fall, resulting in higher present values of future cash flows and more opportunities to call the bond.
- Finally, note that if the yield curve is upward-sloping at the time of issue, if a callable bond is issued at par, it implies that the embedded call option is out-of-the-money. It would not be called if the arbitrage-free forward rates at zero volatility were actually realized. Typically, callable bonds are issued at a large premium. Embedded call options are in-the-money and would be exercised if arbitrage-free forward rates actually prevailed in the future.

Putable Bonds

All else remaining the same:

- As interest rates increase, the value of a straight bond decreases, but the value of the embedded put option in a putable bond increases. Because the investor is effectively long on the embedded put option, the value of the putable bond decreases less than that of the straight bond, offering the investor protection on the downside.
- The value of the embedded put option decreases as the yield curve goes from being upward sloping, to being flat, to being downward sloping. When the yield curve is upward sloping, forward rates are high, resulting in lower present values of expected cash flows and greater opportunities for investors to put the bond. As the yield curve flattens or inverts, forward rates fall, resulting in higher present values of future cash flows and fewer opportunities to put the bond.

LOS 34f: Calculate the value of a callable or putable bond from an interest rate tree. Vol 5, pp 129–145

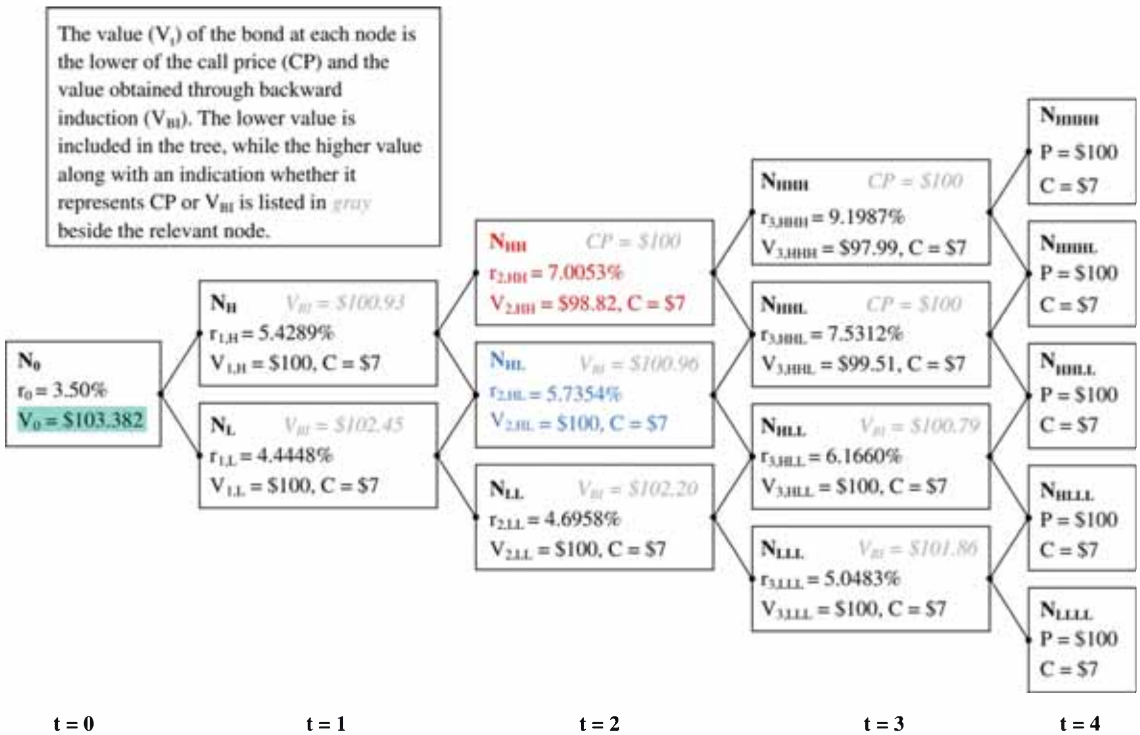
Valuation of Default-Free Callable and Putable Bonds in the Present of Interest Rate Volatility

Valuing and Analyzing a Callable Bond

The procedure for determining the value of a callable bond in the presence of interest rate volatility (with a binomial interest rate tree) is similar to the procedure we used for valuing an option-free bond with a binomial interest rate tree in an earlier reading. However, there is one important exception: At each node during the call period, the value of the bond must equal the lower of (1) the value if the bond is not called (using the backward induction methodology described previously) and (2) the call price. This is because at each node during the call period, the issuer must decide whether it wants to call the bond. For simplicity, we assume that if the call price is lower than the value of the bond based on its expected future cash flows (backward induction) the bond will be called. Note that when comparing the call price to the value obtained through backward induction (to determine which is lower and, therefore, must be included in the tree) we move from right to left.

In Figure 3-1, we calculate the value of a **default-free**, Bermudan-style, four-year, 7% annual-pay callable bond that can be called at par one year, two years, and three years from today. Note that we generated the arbitrage-free one-year rates in Figure 3-1 based on (1) the same interest rate model that we used in an earlier reading ($r_H = r_L \times e^{2\sigma}$), (2) an interest rate volatility assumption of 10%, and (3) the par rates and spot rates presented in Table 2-1 and Table 2-2.

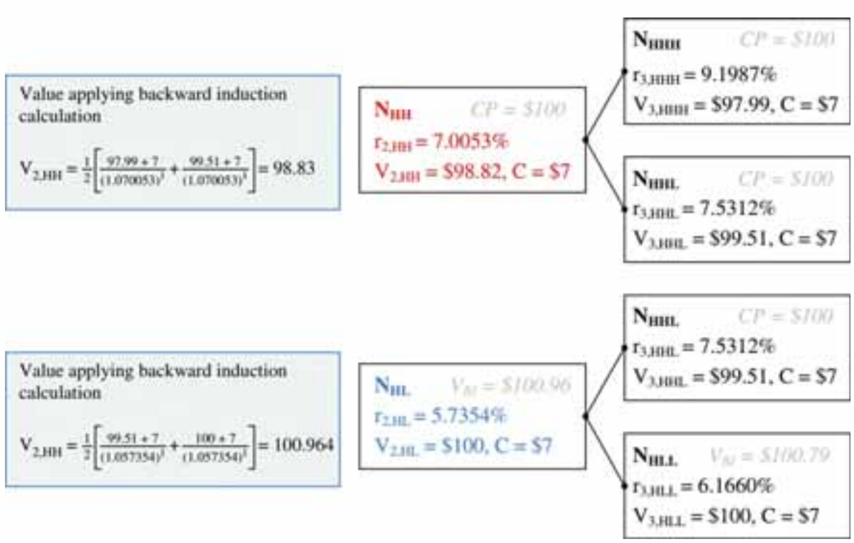
Figure 3-1: Determining the Value of a Default-Free, 4-Year, 7% Bermudan-Style Callable Bond



Just one minor point before moving on: Note that the lockout period here is different from the callable bond valued under the assumption of zero interest rate volatility earlier in the Reading, where the bond could only be called two or three years from today.

What you should notice in Figure 3-1 is that for all the nodes that are boxed in green, the call price ($100) has replaced the value computed using backward induction. For all the nodes that are not boxed in green, the values used in the analysis are the ones obtained through backward induction. Let's study nodes N_{HH} and N_{HL} in more detail to understand what we have done. Both the nodes along with their branches are reproduced in Figure 3-2.

Figure 3-2: Determining the Value of a Callable Bond at a Node during the Call Period



- Node N_{HH} occurs on a call date. Therefore, the value of the bond at this node equals the lower of the value computed by applying the backward induction methodology ($98.82) and the call price ($100).
- Node N_{HL} occurs on a call date. Therefore, the value of the bond at this node equals the lower of the value computed by applying the backward induction methodology ($100.96) and the call price ($100).

To summarize, the value of a callable bond is calculated applying the same backward induction methodology that we used in an earlier reading, but at each node during the call period, the value of the bond equals the **lower** of the value calculated through backward induction and the call price.

Other Variations of the Call Rule

- The bond may be callable at different prices each year during the call period. When this is the case, to compute the value of a bond at a particular node, compare the value obtained through backward induction to the call price applicable for that particular year, and use the lower of the two values in the analysis. Remember to work from right to left when applying the call rule.
- In other cases, the lockout period may be longer than one year. For example, if the call period starts in Year 2, then the values at the nodes corresponding to Year 1 will be the ones calculated through backward induction, even if those values are greater than the call price. This is because the bond is not callable in Year 1.

Determining Call Option Value

We have obtained a value of V_0 = $106.43 for the straight bond and a value of V_0 = $103.382 for the callable bond. Therefore, the value of the embedded call option in our example (given the interest rate model and the interest rate volatility assumption) can be calculated as:

Value of embedded call option = Value of straight bond – Value of callable bond
= $106.43 – $103.38 = $3.05

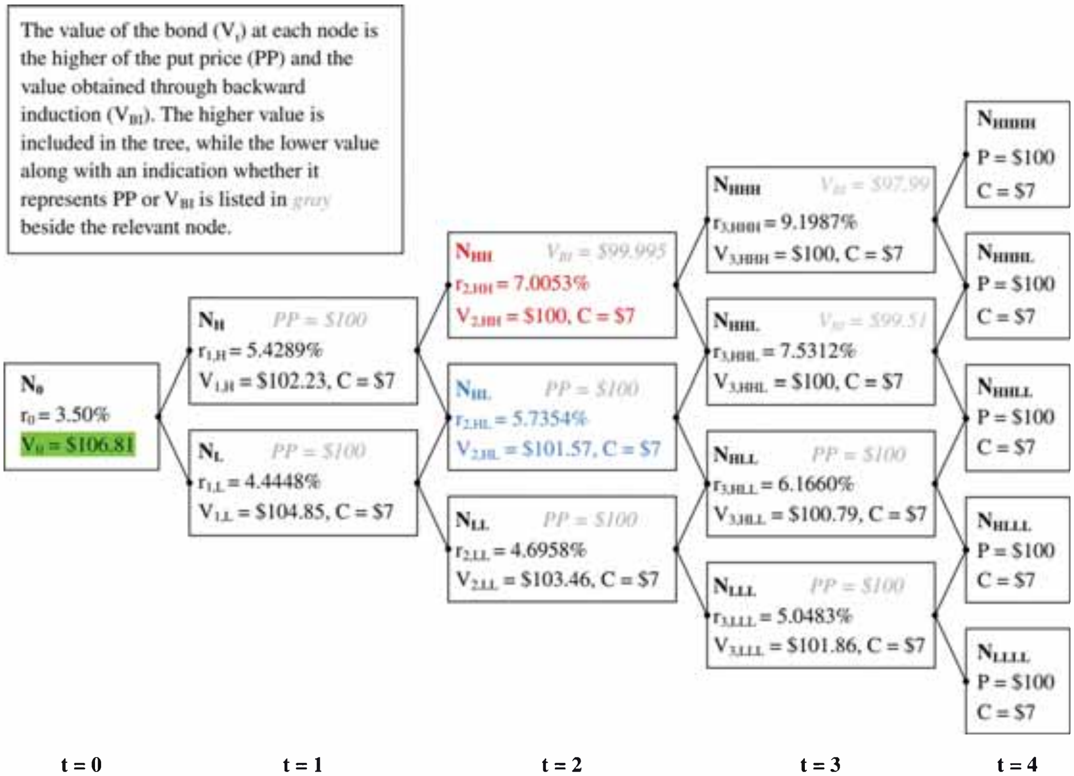
Finally, note that:

- If assumed interest rate volatility is increased (decreased), the value of the callable bond will fall (rise).
- If the call price increases (decreases), the value of the **putable** bond will increase (decrease).

Valuing a Putable Bond

We will now value a **default-free**, Bermudan-style, four-year, 7% annual-pay bond that can be put back to the issuer at par, one year, two years, and three years from today. We will use the same binomial interest rate tree that we used to value the callable bond. The assumption is that the investor will put the bond back to the issuer if the put price exceeds the value of the bond at any put date. (See Figure 3-3.)

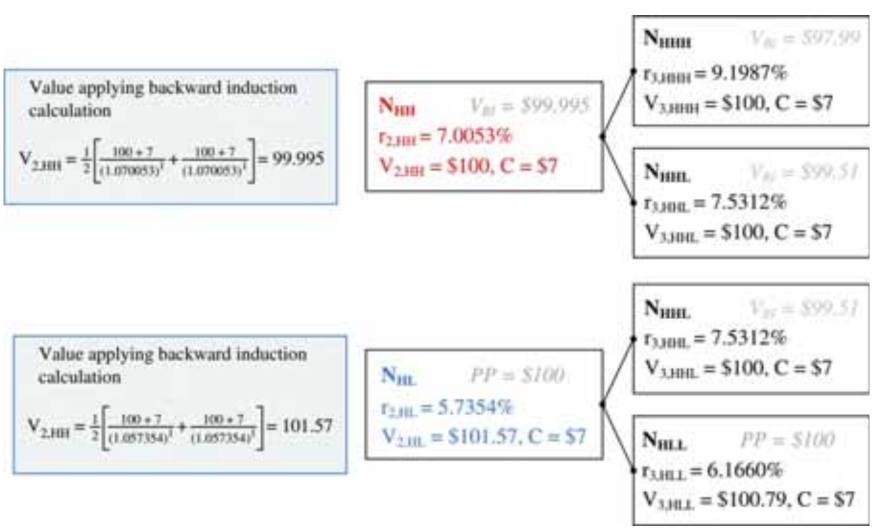
Figure 3-3: Valuing a Putable Bond



The difference when valuing a putable bond (compared to a callable bond) is that at every node we use the **higher** of (1) the value determined through backward induction and (2) the put price, in the analysis.

What you should notice in Figure 3-1 is that for all the nodes that are boxed in green, the put price (\$100) has replaced the value computed using backward induction in the tree. For all nodes that are not boxed in green, the values used in the analysis are obtained through backward induction. Let's study nodes **N_{HH}** and **N_{HL}** in more detail to understand the process of valuing a putable bond. Both the nodes along with their branches are reproduced in Figure 3-4.

Figure 3-4: Determining the Value of a Putable Bond at a Node during the Put Period



- Node N_{HH} occurs on a put date. Therefore, the value of the bond at this node is calculated as the higher of the value computed by applying the backward induction methodology (\$99.995) and the put price (\$100).
- Node N_{HL} also occurs on a put date. Therefore, the value of the bond is calculated as the higher of the value computed by applying the backward induction methodology (\$101.57) and the put price (\$100).

To summarize, the value of a putable bond is calculated by applying the same backward induction methodology that we used earlier, but at each node during the put period, the value of the bond equals the **higher** of the value calculated through backward induction and the put price.

Determining Put Option Value

We have obtained values of $V_0 = \$106.43$ for the option-free bond and $V_0 = \$106.81$ for the putable bond. Therefore, the value of the embedded put option in our example (given the interest rate model and the interest rate volatility assumption) can be calculated as:

$$\begin{aligned}\text{Value of embedded put option} &= \text{Value of putable bond} - \text{Value of straight bond}\\ &= \$106.81 - \$106.43 = \$0.38\end{aligned}$$

Finally, note that:

- If assumed interest rate volatility is increased (decreased), the value of the putable bond will rise (fall).
- If the put price is increased (decreased), the value of the putable bond will increase (decrease).

Putable versus Extendible Bonds

Putable and extendible bonds are equivalent, except that their underlying option-free bonds are different. Consider (1) a four-year 3.50% bond that is putable at t = 3 and (2) a three-year 3.50% bond that is extendible by one year.

The cash flows of the two bonds are exactly the same up to t = 3. The cash flows at t = 4 depend on the one-year forward rate three years from today. For both these bonds, the t = 4 cash flows will be the same regardless of the level of interest rates at t = 3.

If the one-year forward rate at t = 3 is higher than 3.50%:

- The putable bond will be put because the bondholder can reinvest the proceeds of the retired bond at a higher yield.
- The extendible bond will not be extended because the bondholder can reinvest the proceeds of the bond at a higher yield.
- Thus, both bonds pay 3.5% for three years and are then redeemed.

If the one-year forward rate at t = 3 is lower than 3.50%:

- The putable bond will not be put because the bondholder would not want to reinvest at a lower yield.
- The extendible bond will be extended so that the investor can continue to earn the higher interest rate.

- Thus, both bonds pay 3.50% for four years and are then redeemed.

Because the cash flows on both the bonds are identical, their current values should be the same. Otherwise, an arbitrage opportunity would arise.

LOS 34g: Explain the calculation and use of option-adjusted spreads. **Vol 5, pp 145–150**

LOS 34h: Explain how interest rate volatility affects option-adjusted spreads. **Vol 5, pp 147–150**

Valuing Risky Callable and Putable Bonds

So far in the Reading, we have been working with default-free bonds. When it comes to valuing bonds that do carry default risk, there are two approaches:

1. Increase the discount rates used in the valuation above default-free rates to reflect the greater default risk. Higher discount rates result in lower values for risky bonds relative to default-free bonds. This approach is discussed in detail in the next section.
2. Assign a probability of default to each time period in the future. For example, if the probability of default in Year 1 is 0.5%, then the probability of default in Year 2 may be 0.45% given that the company has survived Year 1. Under this approach, recovery rates given default must also be specified. Information regarding default probabilities and recovery rates are obtained from credit default swaps. This approach is discussed in another reading.

Option-Adjusted Spread

There are two approaches to construct the yield curve for a risky bond:

1. Construct an issuer-specific curve that represents the issuer's borrowing rates over different horizons.
2. Uniformly raise all the one-year forward rates derived from the default-free benchmark yield curve by a fixed spread. This spread is known as the **z-spread** and is estimated from market prices of bonds of similar credit quality. Even though this approach is less satisfactory than the first one, it is more convenient.

To illustrate this approach, consider the default-free, four-year, 7% annual-pay straight bond whose value we computed as $106.43 earlier in the reading. If we now assume that this bond is actually a risky bond and that the appropriate z-spread is 100 bps, we can compute its arbitrage-free value by raising all the forward rates in Table 2-3 by 100 bps:

Value (Risky bond)

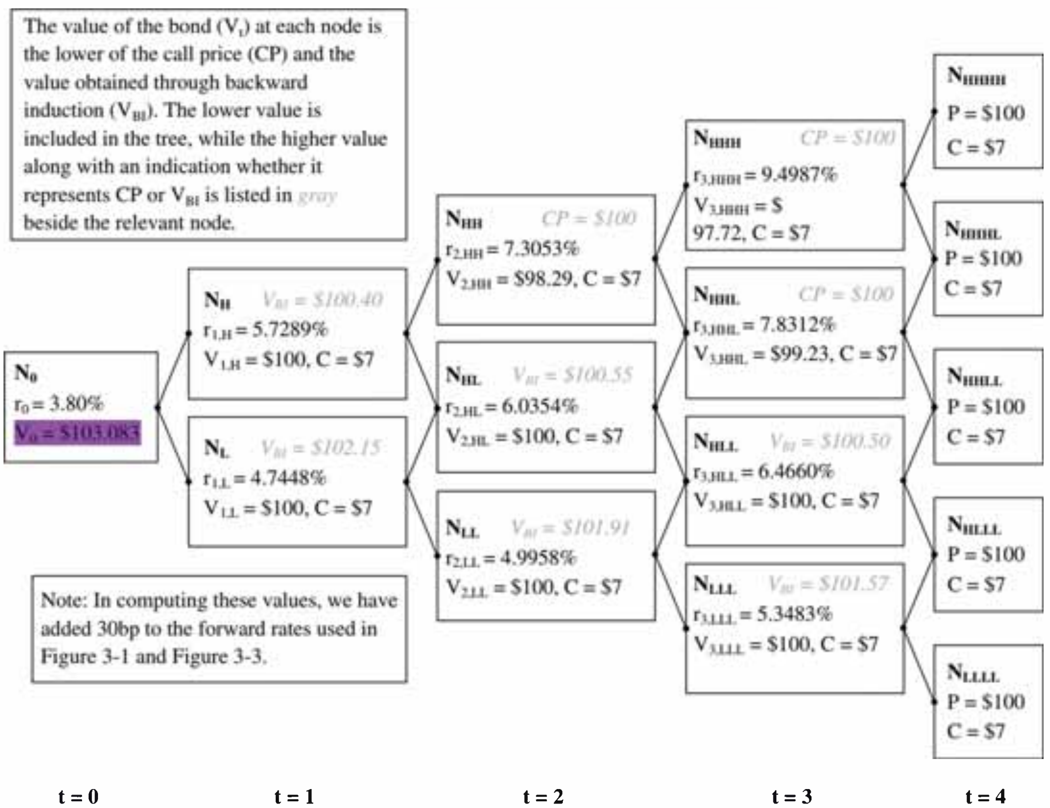
$$= \frac{7}{(1+0.045)^1} + \frac{7}{(1+0.045)(1+0.05935)} + \frac{7}{(1+0.045)(1+0.05935)(1+0.06784)}$$
$$+ \frac{107}{(1+0.045)(0.05935)(1+0.06784)(1+0.07893)} = \$102.83$$

As expected, the value of this risky bond ($102.83) is lower than that of the default-free bond ($106.43).

When it comes to valuing bonds with embedded options using an interest rate tree, the **option-adjusted spread (OAS)** is used instead of the z-spread. The OAS is the constant spread that, when added to all the one-year forward rates in the interest rate tree, makes the arbitrage-free value of the bond equal to its current market price. Let's get into an illustration.

We have already determined that the theoretical value of the **default-free**, four-year, 7% callable bond assuming 10% interest rate volatility is \$103.382. Now suppose that a risky callable bond with otherwise-identical features is trading in the market for \$103.083. Figure 3-5 illustrates that when a spread of 30 bp is added to each of the rates in our binomial interest rate tree, we obtain a value of \$103.083 (its observed market price) for the risky callable bond. Therefore, the OAS for this bond equals 30 bp.

Figure 3-5: Demonstrating That the OAS for Our Risky Callable Bond Is 30 Basis Points



The OAS essentially removes option risk from the z-spread. When determining the value of the bond, it adjusts the cash flows at each node for the option (in our example, by applying the call rule). Therefore, the resulting spread is "option-adjusted." The OAS is often used as a measure of value relative to the benchmark.

- If the OAS for a bond is lower than that for a bond with similar characteristics and credit quality, it suggests that the bond is relatively overpriced (rich).
- If the OAS for a bond is greater than that for a bond with similar characteristics and credit quality, it suggests that the bond is relatively underpriced (cheap).
- If the OAS for a bond is close to that of a bond with similar characteristics and credit quality, the bond looks fairly priced.

Effect of Interest Rate Volatility in Option-Adjusted Spread

Just like the value of the call option embedded in a callable bond, the OAS also depends on the interest rate volatility assumption. For a given bond price, the lower the interest rate volatility assumed, the lower the option cost and, therefore, the higher the OAS for the callable bond given the z-spread. Note that the interest rate volatility assumption is very important here. A callable bond may appear underpriced assuming an interest rate volatility of 10%, but if actual volatility were higher, the bond's OAS would be overstated, meaning that the bond would not be as underpriced as initially thought.

LESSON 4: INTEREST RATE RISK OF BONDS WITH EMBEDDED OPTIONS

LOS 34i: Calculate and interpret effective duration of a callable or putable bond. **Vol 5, pp 150–152**

INTEREST RATE RISK OF BONDS WITH EMBEDDED OPTIONS

Duration

Duration measures the sensitivity of a bond's full price (including accrued interest) (1) to a change in the bond's yield-to-maturity (in the case of yield duration measures) or (2) to changes in benchmark interest rates (in the case of curve duration measures).

Yield duration measures can only be used for option-free bonds because they assume that the bond's expected cash flows do not change when yields change. For bonds with embedded options, cash flows can change if the embedded options (which are typically contingent on interest rates) are exercised. Therefore, the appropriate duration measure for bonds with embedded options is a curve duration measure known as **effective** (or **option-adjusted**) **duration**. Note that effective duration also works for straight bonds.

Effective Duration

Effective duration measures the sensitivity of a bond's price to a 100 bps parallel shift in the benchmark yield curve, assuming no change in the bond's credit spread. It is calculated as:

$$\text{Effective Duration} = \frac{[[(\text{PV}]]_{-}) - [[(\text{PV}]]_{+})}{2 \times (\Delta\text{Curve}) \times \text{PV}_0}$$

ΔCurve = The magnitude of the parallel shift in the benchmark yield curve (in decimal)

PV_{-} = Full price of the bond when the benchmark yield curve is shifted down by ΔCurve

PV_{+} = Full price of the bond when the benchmark yield curve is shifted up by ΔCurve

PV_0 = Current full price of the bond (i.e., with no shift)

Effective duration for a bond with an embedded option is computed through the following procedure:

1. Given the price of the bond, compute its OAS (using the methodology described earlier in the Reading) to the benchmark yield curve at an appropriate interest rate volatility. We have already calculated the OAS for our risky, Bermudan style, Four-year, 7% annual-pay callable bond as **30 bp** based on a market price of $103.083 (Figure 3-5).

2. Shift the benchmark yield curve up by a small number of basis points (ΔCurve). Based on these benchmark rates, generate a new arbitrage-free interest rate tree. To each of the one-year rates in the tree add the same OAS that was computed earlier (30 bp), and revalue the bond. The resulting value is PV_+. It is important for you to remember that the effective duration and effective convexity calculations assume that the OAS remains unchanged when interest rates change.
3. Shift the benchmark yield curve down by the same number of basis points (ΔCurve) as in Step 2. Based on these benchmark rates, generate a new arbitrage-free interest rate tree. To each of the one-year rates in the tree add the same OAS that was computed earlier (30 bp), and revalue the bond. The resulting value is PV_-.
   - Note that we have used ΔCurve = 25 bp to derive the interest rate tree in Figure 3-5. We have not simply added 25 bps to the forward rates used in Figure 3-1 and Figure 3-3. The process outlined above requires adding 25 bps to the benchmark yield curve, then recalculating all the forward rates for the interest rate tree (given the interest rate model used and the volatility assumption), then adding the OAS before generating the value of a bond.
4. Calculate the bond's effective duration.

In Figure 4-1, we compute the value of our **risky**, Bermudan style, four-year, 7% annual-pay, callable bond (that offered an OAS of 30 bp) assuming that the yield curve has moved up by 25 bp.

Figure 4-1: Computing the Value of V_+ (To Be Used in the Duration and Convexity Calculation)

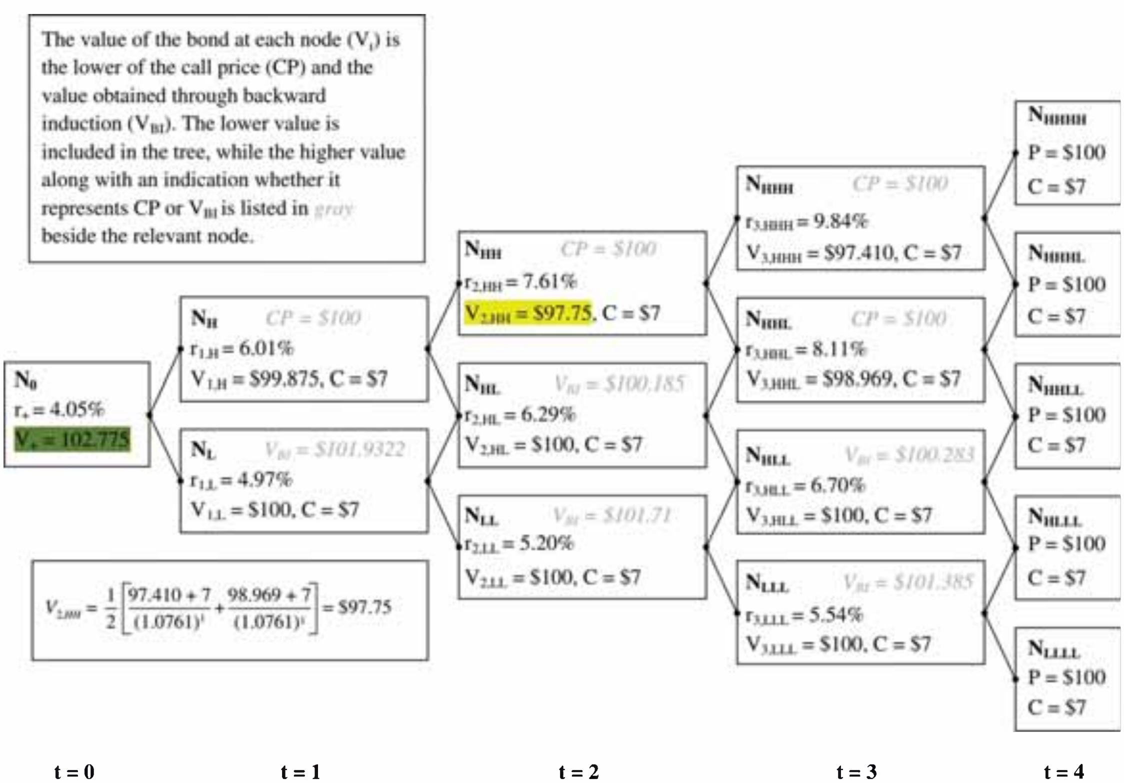


Figure 4-1 gives us a value of $102.775 for V_+ for assuming a yield volatility of 10%, an OAS of 30 bp, and a ΔCurve of 25 bp. Given a V_- of $103.332 (we have not illustrated the calculation here), the effective duration for this bond can be calculated as:

$$\text{Effective Duration} = \frac{103.332 - 102.775}{2(103.083)(0.0025)} = 1.0807$$

An effective duration of 1.0807 indicates that a 100 bps increase in interest rates would result in a decrease of 1.0807% in the value of our callable bond.

LOS 34j: Compare effective durations of callable, putable, and straight bonds. **Vol 5, pp 153–155**

The effective duration of a callable bond cannot exceed that of the straight bond.

- When interest rates are high relative to the bond's coupon, the embedded call option is out-of-the-money, so the bond is unlikely to be called. In this case, the price-yield profile of the callable bond is very similar to that of a straight bond, implying that the effect of an interest rate change on the price of a callable bond is very similar to that on the price of an otherwise-identical option-free bond. At high interest rates, the callable and straight bonds have very similar effective durations.
- When interest rates fall, the embedded call option moves toward the money and it becomes more likely that the bond will be called. The price of the callable bond is effectively capped at the call price, so its price-yield profile exhibits price compression at low interest rates. The limit on the upside due to the presence of the call option reduces the effective duration of the callable bond relative to that of the straight bond.

The effective duration of a putable bond also cannot exceed that of the straight bond.

- When interest rates are low relative to the bond's coupon, the embedded put option is out-of-the-money, so the bond is unlikely to be put. In this case, the price-yield profile of the putable bond is very similar to that of a straight bond, resulting in the putable bond and straight bond having very similar effective durations.
- When interest rates rise, the embedded put option moves toward the money and effectively places a floor on the value of a putable bond. The price of the putable bond does not fall as much as that of a straight bond as interest rates rise, meaning that its effective duration is lower than that of the straight bond.

It may help you to remember the relationships described here by looking at the slope of the price-yield profiles of callable and putable bonds relative to straight bonds (presented in Figure 4-2 and Figure 4-3). The steeper (flatter) the slope, the higher (lower) the duration (price sensitivity of the bond to changes in interest rates).

Figure 4-2: Callable Bonds versus Straight Bonds

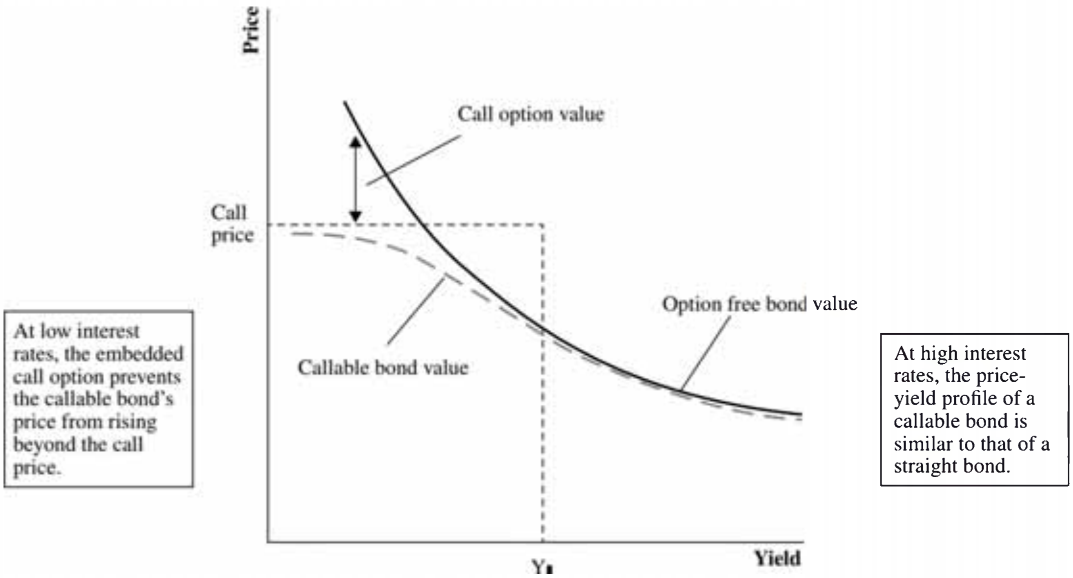
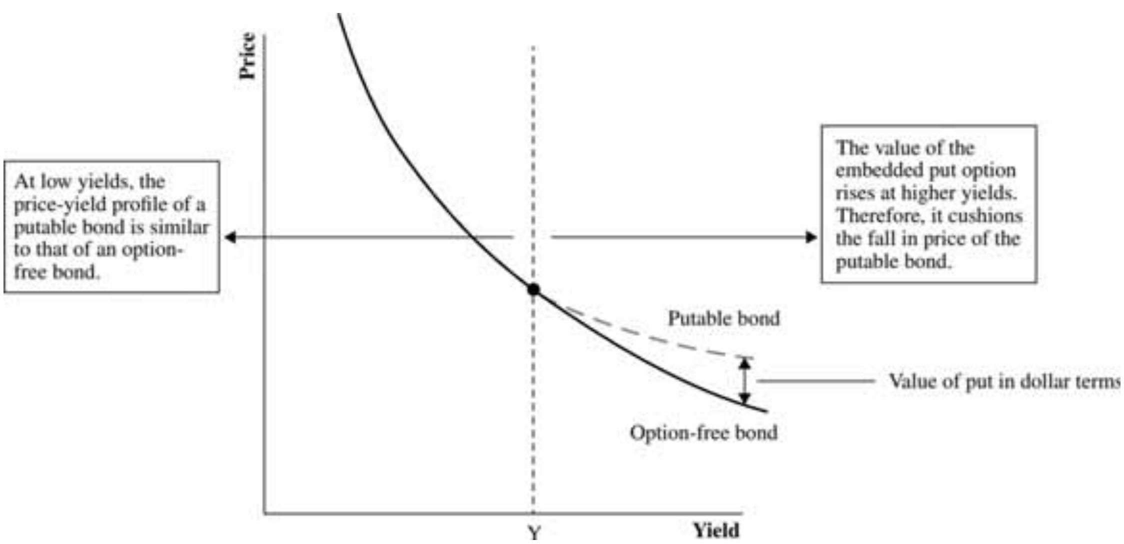


Figure 4-3: Putable Bonds versus Straight Bonds



When the embedded option (call or put) is deep in the money, the effective duration of the bond with an embedded option resembles that of the straight bond maturing on the upcoming exercise date, reflecting the fact that the bond is highly likely to be called or put on that date.

Also note that:

- The effective duration of an option-free bond changes very little in response to interest rate movements.
- For a putable bond, its effective duration falls when interest rates rise, as the put moves into the money and the bond's price-yield profile flattens out.
- For a callable bond, its effective duration falls when interest rates fall, as the call moves into the money and the bond's price-yield profile flattens out.

Practically speaking, effective duration is often applied in the context of portfolio management. Understanding the effective durations of various types of instruments helps manage portfolio duration. Table 4-1 presents some properties of the effective durations of cash and common types of bonds.

Table 4-1: Properties of Effective Durations of Cash and Common Types of Bonds

Type of Bond	Effective Duration
Cash	0
Zero-coupon bond	≈ Maturity
Fixed-rate bond	< Maturity
Callable bond	≤ Duration of straight bond
Putable bond	≤ Duration of straight bond
Floater (LIBOR flat)	≈ Time (in years) to next reset

- Generally speaking, a bond's effective duration does not exceed its maturity. However, there are a few exceptions (e.g., tax-exempt bonds when analyzed on an after-tax basis).
- If a portfolio manager wants to shorten the effective duration of a portfolio of fixed-rate bonds, she could do so by adding floating-rate bonds to the portfolio.
- A company can reduce the effective duration of its liabilities by issuing callable bonds instead of straight bonds.

LOS 34k: Describe the use of one-sided durations and key rate durations to evaluate the interest rate sensitivity of bonds with embedded options. **Vol 5, pp 155–158**

One-Sided Durations

The calculation of effective duration works well for option-free bonds, but can be misleading when it comes to bonds with embedded options. This is because effective duration is calculated by averaging the changes in a bond's price resulting from shifting the benchmark curve up and down by a specified number of basis points. The problem with this measure is that:

- For callable bonds, when the embedded call option is in the money, the upside potential from a decline in interest rates is limited to the call price, while the downside from an increase in interest rates is much larger.
    - In short, callable bonds are much more sensitive to interest rate increases than to interest rate declines.
- For putable bonds, when the embedded put option is in the money, the downside potential from an increase in interest rates is limited to the put price, while the upside from a decrease in interest rates is much larger.
    - In short, putable bonds are much more sensitive to interest rate declines than to interest rate increases.

Therefore, the average price response to up-and down-movements in interest rates (i.e., effective duration) is not as effective in capturing the interest rate sensitivity of callable and putable bonds as (1) the price response to up-movements (i.e., **one-sided up-duration**) and (2) the price response to down-movements (i.e., **one-sided down-duration**). This is especially true when the embedded option is near-the-money.

Key Rate Durations

Key rate durations (or **partial durations**) measure the sensitivity of a bond's price to changes in specific maturities on the benchmark yield curve. Key rate durations are used to identify the **shaping risk** for bonds (i.e., the bond's sensitivity to changes in the shape of the yield curve: steepening and flattening). The calculation of key rate duration is similar to the calculation of effective duration, but instead of shifting the entire benchmark yield curve, only key points are shifted, one at a time.

Table 4-2 presents the key rate durations for 10-year bonds with various coupon rates, assuming a flat 4% yield curve.

Table 4-2: Key Rate Durations of 10-Year Option-Free Bonds Valued at a 4% Flat Yield Curve

		Key Rate Durations				
Coupon (%)	Price (% of par)	Total	2-Year	3-Year	5-Year	10-Year
0	67.30	9.81	–0.07	–0.34	–0.93	11.15
2	83.65	8.83	–0.03	–0.13	–0.37	9.37
4	100.00	8.18	0.00	0.00	0.00	8.18
6	116.35	7.71	0.02	0.10	0.27	7.32
8	132.70	7.35	0.04	0.17	0.47	6.68
10	149.05	7.07	0.05	0.22	0.62	6.18

Notice the following from Table 4-2:

- For option-free bonds that are not trading at par (the white rows), a change in any of the key rate durations has an impact on the price of the bond.
    - For example, for the 2% bond, a 1% change in the two-year results in a 0.03% change in its value.
- For option-free bonds that are not trading at par (the white rows), a change in the key/par rate corresponding to the bond's maturity (10 years) has the greatest impact

on price. This is because duration is some sort of weighted average time to receipt of the bond's cash flows, and the largest cash flow for fixed-rate bonds occurs at maturity, when both the final coupon and principal are paid.
    - For example, for the 2% bond, a 1% change in the 10-year rate results in a 9.37%. change in its value.
- For option-free bonds trading at par (the row shaded in green), the par rate corresponding to the bond's maturity is the only one that affects its price. Given that the yield curve is flat at 4%, the 4% coupon bond trades at par. This bond's price is not affected by a change in any key rates other than the 10-year par rate.
    - For example, if there is a change in the two-year or six-year par rate, the value of the 4% coupon bond that is trading at par will not change.
- Notice (from the top two rows in the table) that key rate durations can sometimes be negative at maturity points that are shorter than the maturity of the bond if the bond has a very low coupon rate or is a zero-coupon bond. Recall that bond prices and interest rates are negatively related, so, mathematically speaking, duration is a negative number. However, convention dictates that the inverse relationship between bond prices and interest rates be captured by a positive duration measure. **Therefore, a negative duration value in this table implies that the bond price is positively related to the relevant par rate.** We explain this relationship using the zero-coupon bond in Table 4-2 (top row).
    - If there is an increase in the five-year par rate, there will be no impact on the price of a 10-year 4% coupon-bearing bond trading at par. Note that we just mentioned (in the previous bullet) that a bond that trades at par is only sensitive to changes in the par rate corresponding to its maturity.
    - However, the increase in the five-year par rate would result in an increase in the five-year zero-coupon rate (i.e., the five-year spot rate). Therefore, the discount factor for the coupon paid at Year 5 on the 10-year coupon-bearing bond will be lower, and the present value of that payment will also be lower.
    - To make up for this reduction in the present value of the fifth coupon (in order to ensure that the overall value of the 10-year coupon bearing bond remains at par), all other cash flows on the bond, including the cash flow occurring at Year 10, must be discounted at slightly lower rates.
    - This results in a slightly lower 10-year zero-coupon rate (which would be used to discount the Year 10 cash flow on the coupon-bearing bond).
    - This slight decline in the 10-year zero-coupon rate makes the value of a 10-year zero-coupon bond rise.
    - Connecting the dots, an **increase** in the five-year par rate leads to a slightly lower 10-year zero-coupon rate, which in turn results in an **increase** in the 10-year zero-coupon bond. Therefore, the five-year key rate duration for the 10-year zero-coupon bond is negative (–0.93).

We know that for option-free bonds, duration is (among other factors) a function of time to maturity. For bonds with embedded options, duration depends on time to maturity and on time to exercise. We illustrate this below.

Table 4-3 presents the key rate durations for 30-year bonds that are callable in 10 years (European-style) with various coupon rates, assuming a flat 4% yield curve and volatility of 15%.

Table 4-3: Key Rate Durations of 30-Year Bonds Callable in 10 Years Valued at a 4% Flat Yield Curve with 15% Interest Rate Volatility

		Key Rate Durations					
Coupon (%)	Price (% of par)	Total	2-Year	3-Year	5-Year	10-Year	30-Year
2	64.99	19.73	–0.02	–0.08	–0.21	–1.97	22.01
4	94.03	13.18	0.00	0.02	0.05	3.57	9.54
6	114.67	9.11	0.02	0.10	0.29	6.00	2.70
8	132.27	7.74	0.04	0.17	0.48	6.40	0.66
10	148.95	7.14	0.05	0.22	0.62	6.06	0.19

Notice the following from Table 4-3:

- When the bond's coupon rate is significantly less than market interest rates, it is highly unlikely to be called. In such a case it is more likely to behave like a straight bond, and the par rate that will have the largest impact on its price will be the par rate that has the greatest impact on the straight bond's price.
    - For example, the bond with a 2% coupon rate is unlikely to be called, as its coupon rate is much lower than the market yield of 4%. Therefore, the 30-year (maturity matched) par rate has the greatest impact on its price (i.e., has the highest key rate duration; 22.01).
- As the bond's coupon increases, the likelihood of the bond being called also increases. As a result, the bond's effective duration decreases and, gradually, the rate that has the largest impact on the bond's price moves from the 30-year rate to the 10-year rate.
- For example, the bond with a 10% coupon rate is almost certain to be called, so it behaves more like a 10-year straight bond. The 10-year key rate duration is much larger than the 30-year key rate duration (6.06 versus 0.19).

Table 4-4 presents the key rate durations for 30-year bonds that are putable in 10 years (European-style) with various coupon rates, assuming a flat 4% yield curve and volatility of 15%.

Table 4-4: Key Rate Durations of 30-Year Bonds Putable in 10 Years Valued at a 4% Flat Yield Curve with 15% Interest Rate Volatility

		Key Rate Durations					
Coupon (%)	Price (% of par)	Total	2-Year	3-Year	5-Year	10-Year	30-Year
2	83.89	9.24	–0.03	–0.14	–0.38	8.98	0.81
4	105.97	12.44	0.00	–0.01	–0.05	4.53	7.97
6	136.44	14.75	0.01	0.03	0.08	2.27	12.37
8	169.96	14.90	0.01	0.06	0.16	2.12	12.56
10	204.38	14.65	0.02	0.07	0.21	2.39	11.96

Notice the following from Table 4-4:

- When the bond's coupon rate is significantly higher than market interest rates, it is highly unlikely to be put. In such a case it is more likely to behave like a straight

bond, and the par rate that will have the largest impact on its price will be the par rate that has the greatest impact on the straight bond's price.
  - For example, the bond with a 10% coupon rate is unlikely to be put, as its coupon rate is much higher than the market yield of 4%. Therefore, the 30-year (maturity matched) par rate has the greatest impact on its price (i.e., has the highest key rate duration; 11.96).
- As the bond's coupon decreases, the likelihood of the bond being put increases. As a result, the bond's effective duration decreases and, gradually, the rate that has the largest impact on the bond's price moves from the 30-year rate to the 10-year rate.
  - For example, the bond with a 2% coupon rate is highly likely to be put, so it behaves more like a 10-year straight bond. The 10-year key rate duration is much larger than the 30-year key rate duration (8.98 versus 0.81).

LOS 34l: Compare effective convexities of callable, putable, and straight bonds. **Vol 5, pp 158–161**

Effective Convexity

Recall from Level I that duration is only an approximate measure of the sensitivity of a bond's price to changes in interest rates. This is because a bond's price-yield profile is not linear. Bond price sensitivity estimates based on duration can be improved upon via the convexity adjustment. A bond's effective convexity can be calculated as:

$$\text{Effective convexity} = \frac{[[(\text{PV}]]_{-}) + [[(\text{PV}]]_{+}) - 2\,[[(\text{PV}]]_{0})}{(\Delta\text{Curve})^2 \times \text{PV}_0}$$

Let's compute effective convexity for the same risky, Bermudan style, four-year, 7% annual-pay callable bond that we computed effective duration for earlier in the Reading (PV_+ = \$102.775; PV_- = \$103.332; PV_0 = \$103.083; OAS = 30 bps; σ = 10%; ΔCurve = 25 bps).

$$\text{Effective convexity} = \frac{102.775 + 103.332 - 2(103.083)}{(103.083)(0.0025)^2} = -91.577$$

It is important for you to understand the following:

- Option-free bonds exhibit positive effective convexity. Their prices rise slightly more when interest rates fall than they fall when interest rates rise by the same magnitude.
- For callable bonds:
  - When interest rates are high, they behave like straight bonds.
  - When interest rates fall and the embedded call option is at or near the money, their effective convexity turns negative. This is because their price is effectively capped at the call price.
- For putable bonds:
  - When interest rates are low, they behave like straight bonds.

- When interest rates increase and the embedded put option is at or near the money, their effective convexity remains positive. At this point, the upside from a decrease in interest rates is far greater than the downside from an increase in interest rates, as the put price effectively serves as a floor on the putable bond's value.

- Therefore, putable bonds have more upside potential than otherwise-identical callable bonds when interest rates fall, while callable bonds have more downside potential than otherwise-identical putable bonds when interest rates rise.

LESSON 5: VALUATION AND ANALYSIS OF CAPPED AND FLOORED FLOATING-RATE BONDS

LOS 34m: Calculate the value of a capped or floored floating-rate bond.

Vol 5, pp 161–166

VALUATION AND ANALYSIS OF CAPPED AND FLOORED FLOATING-RATE BONDS

Valuation of a Capped Floater

Capped floaters are floating-rate securities whose coupon rate is capped at a pre-specified level. Valuing a capped floater using a binomial interest rate tree is different from valuing an option-free bond in the following two ways:

- Because it is a floating-rate security, the coupon payment on the capped floater one year from today is based on the interest rate today. What this means is that the coupon for the period is determined at the beginning of the period, but paid (in arrears) at the end of the period.
- Because the coupon rate is capped, the coupon at each node must be adjusted to reflect the coupon characteristics of the cap.

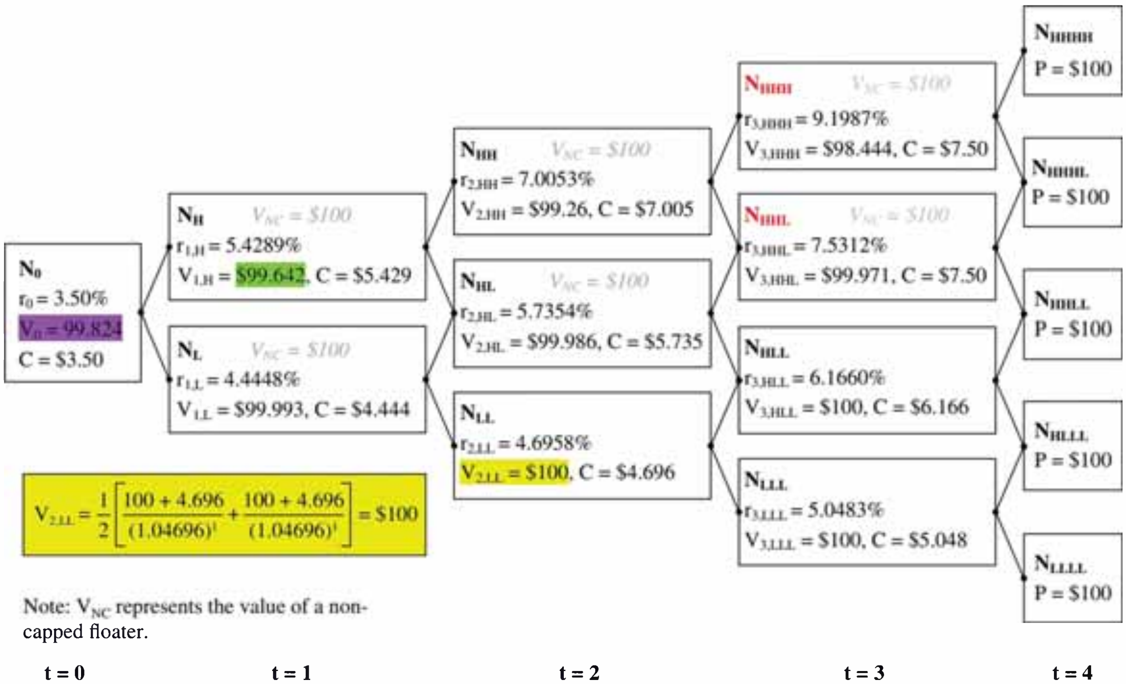
Because an investor in a capped floater is long on an uncapped bond, but effectively short on the embedded cap (the option favors the issuer), the value of a capped floater can be calculated as:

Value of capped floater = Value of uncapped floater – Value of embedded cap

Value of embedded cap = Value of uncapped floater – Value of capped floater

Figure 5-1 illustrates the valuation of a capped floater with a cap of 7.5%. The interest rates in the tree are the same as in Figure 2-1 and Figure 2-3. Note that (in the interests of simplicity) we have assumed that (1) the issuers' credit quality is really high, such that there is no need for a credit spread and (2) the LIBOR swap curve is the same as the par yield curve provided in Table 2-1.

Figure 5-1: Valuing a Floater with a Cap of 7.5%



Important: At each node, the value listed for the coupon payment is the lower of (1) the coupon based on the cap rate and (2) the coupon based on the current one-year rate. Note that the coupon will actually be paid in the next period. For example, the coupon payment ($7.50) listed at node N_{HHH} is based on the cap rate (7.50%), as it is lower than the current one-year rate ($r_{3,HHH}$ = 9.1987%). **This coupon will actually be paid at t = 4 even though it is listed at a node corresponding to t = 3 in the tree.**

To illustrate the computation of the values of the bond at various nodes focus on Node N_H. The effective coupon rate for this node is 5.4289%, and the coupon will actually be paid at the nodes corresponding to t = 2 (N_{HH} and N_{HL}). The value of the bond at Node N_H is calculated as:

$$V_H = \frac{1}{2}\left(\frac{99.26 + 5.4289}{(1 + 0.054289)^1} + \frac{99.986 + 5.4289}{(1 + 0.054289)^1}\right) = \$99.6424$$

If the coupon rate on this issue were not capped, the value of the bond at each node would equal $100. This is because the coupon rate and the discount rate would be equal. Recall from Level I that a floating-rate security trades at par at each coupon-reset date as its coupon rate is brought in line with market interest rates.

With the cap, however, the issuer will exercise the caplet at each node where the coupon rate exceeds the cap rate. In this example, the caplet is exercised at the nodes boxed in green in Figure 5-1. The value of the bond at any node at which the caplet is exercised will be lower than par, as the coupon rate is lower than market interest rates. The value of the cap in our example can be calculated as:

Value of embedded cap = Value of uncapped floater – Value of capped floater

= $100 – $99.824 = $0.176

Also know that the higher the cap rate, the closer the instrument will trade to its par value (as the embedded interest rate call options carry a lower likelihood of exercise, and are therefore less valuable to the issuer).

Finally, note that the curriculum has used a non-recombining tree to compute the value of the capped floater (and cap). In Figure 5-1 we have used a recombining tree (as in the earlier figures in this reading and an earlier reading). You will obtain the same value for the capped floater and the cap using either method, but there are substantially fewer calculations in the recombining tree. If you are using our method (recombining tree) to value a floating-rate security with a cap/floor, remember to list the coupon amount on the same node as the effective coupon rate for the upcoming period (even though the coupon will actually be paid out in the next period). This will keep things quick and simple.

Valuation of a Floored Floater

A floor provision in a floater prevents the effective coupon rate on the bond from declining below a specified minimum rate. Therefore, it offers investors protection against declining interest rates. Because the investor is long on the underlying floating bond and long on the embedded option, the value of a floored floater can be calculated as:

Value of floored floater = Value of non-floored floater + Value of embedded floor

Value of embedded floor = Value of floored floater – Value of non-floored floater

The procedure valuing a floored floater through a binomial interest rate tree is similar to that of valuing a capped floater, except that at each node, the effective coupon rate for the upcoming period is the **higher** of (1) the current one-year rate and (2) the floor rate. As a result, the value of the floored floater will exceed that of a non-floored floater.

Ratchet Bonds

Ratchet bonds are floating-rate bonds with both investor and issuer options.

- Just like conventional floaters, the coupon rate is periodically reset according to a formula based on a reference rate and credit spread.
- However, the bonds are structured to ensure that at any reset date, the effective coupon rate can only decline; it can never exceed the existing level. As a result, the coupon rate “ratchets down” over time.
- In order to compensate investors for this, the coupon rate on ratchet bonds at the time of issuance is set at a level much higher than that of a standard floater.
- This makes a ratchet bond similar to a conventional callable bond in that when a bond is called, the investor can only purchase a replacement bond carrying a lower (prevailing) coupon rate. A ratchet bond can be thought of as the life cycle of a callable bond, with one callable bond being replaced by another until the bond’s eventual maturity is reached. The appeal for the issuer is that the call decision is on autopilot and there are no transaction costs.
- Ratchet bonds also contain investor options. At any coupon reset date, the investor has the option to put the bonds back to the issuer at par. Note that this put is actually a “contingent put,” as it can only be exercised if the coupon is actually reset. Generally speaking, the market price of ratchet bonds remains above par at time of reset unless there has been a deterioration in the credit quality of the issuer.

LESSON 6: VALUATION AND ANALYSIS OF CONVERTIBLE BONDS

LOS 34n: Describe defining features of a convertible bond. Vol 5, pp 166–169

VALUATION AND ANALYSIS OF CONVERTIBLE BONDS

A **convertible bond** is a hybrid security that grants the holder the option to convert the security into a predetermined number of common shares of the issuer (so it is essentially a call option on the stock) during a predetermined **conversion period** at a predetermined **conversion price**.

Convertible bonds offer benefits to the issuer and the investor:

- Investors accept lower coupons on convertible bonds compared to otherwise identical non-convertible bonds because they can participate in the upside of the issuer's equity through the conversion mechanism.
- Issuers benefit from lower coupon rates and from no longer having to repay the debt (if the bonds are converted into equity).

However, note that:

- The issuer's current shareholders face dilution if these securities are converted.
- If the bond is not converted (if the share price remains below the conversion price), the issuer may have to refinance the debt at a higher cost.
- If conversion is not achieved, bondholders would have lost out on interest income relative to an otherwise-identical non-convertible bond.

We will use the information provided in Exhibit 6-1 and Exhibit 6-2 to describe the features of a convertible bond and to illustrate how to analyze it.

Exhibit 6-1: Excerpt from ABC Company's Callable Convertible Bond Offering Circular

Issue Date: March 3, 2014
Due Date: March 3, 2019
Status: Senior unsecured, unsubordinated
Interest: 3.50% of nominal value (par) per annum payable annually in arrears, with the first interest payment date on March 3, 2015, unless prior redeemed or converted
Issue Price: 100% of par denominated into bonds of $100,000 each and integral multiples of $1,000 each thereafter
Conversion Period: April 3, 2014, to February 5, 2019
Initial Conversion Price: $5.00 per share
Conversion Ratio: Each bond of par value of $100,000 is convertible to 20,000 ordinary shares
Threshold Dividend: $0.25 per share
Change of Control Conversion Price: $3.50 per share
Issuer Call Price: From the third anniversary of issuance: 110%; from the fourth anniversary of issuance: 105%

Exhibit 6-2: Market Information

Convertible Bond Price on September 9, 2014: $125,628
Share Price on Issue Date: $4.50
Share Price on September 9, 2014: $5.85
Dividend per Share: $0.13
Share Price Volatility per annum as of September 9, 2014: 20%

Further, note that the straight value of the bond at September 9, 2014, is estimated at $106,657.87.

The share price at which the investor can convert the convertible bond into ordinary shares is known as the **conversion price**. For this bond, the conversion price was set at $5 per share.

The number of shares of common stock that the holder of a convertible bond receives from converting the bonds into shares is dictated by the **conversion ratio**. In this example, if a bondholder invests the minimum stipulated amount of $100,000, and converts her bonds into shares, she would receive 20,000 shares (= $100,000/$5). Conversion may be exercised during a particular period or at set intervals during the life of the bond. In this example, the conversion period begins shortly after issuance and ends shortly before maturity. The conversion price listed in Exhibit 6-1 is the initial conversion price. The conversion ratio is adjusted for stock splits and stock dividends. So if ABC were to undertake a 3:1 stock split, the conversion price would be adjusted to $1.67 per share, and the conversion ratio would be adjusted to 60,000 shares per $100,000 of nominal value.

Until they convert their bonds into shares, holders of convertible bonds only receive coupon payments, while common shareholders receive dividend payments. At one extreme, the terms of a convertible bond issue can be structured to provide no compensation to convertible bondholders for dividend payments made during the life of the bond, or, on the other extreme, they may offer full protection by adjusting the conversion price downward for dividend payments. Typically, a threshold dividend is defined ($0.25/share in this case). Annual dividends below this level have no impact on the conversion price. However, if annual dividends exceed this level, the conversion price is adjusted downward for annual dividend payments to compensate convertible bondholders.

Change-of-control events (e.g., mergers) are defined in the prospectus. If such an event occurs, convertible bondholders have a choice between:

- A **put option** (that can be exercised during a specified term following the change-of-control event), which provides full redemption of the nominal value of the bond, and
- An **adjusted conversion price** that is lower than the initial conversion price. This offers convertible bondholders an opportunity to convert their bonds into shares earlier and at more advantageous terms.

Almost all convertible bonds are **callable** and some are also **putable**.

- The issuer may call the bond (1) if interest rates have fallen, (2) if its credit rating has improved, or (3) if it believes that its share price will improve significantly in the future.
- There may be a **lockout period** during which the security cannot be called. After the lockout period ends, the bonds can be called at a premium, which declines with time.

In our example, the bond cannot be called until its third anniversary (at a 10% premium). The premium declines to 5% at its fourth anniversary.

- If a convertible bond is callable, the issuer has an incentive to call the bond if the underlying share price rises beyond the conversion price in order to avoid making interest payments. Such an event is known as a **forced conversion**, as bondholders would be forced to convert their bonds into shares (as the value of the shares received from conversion would exceed the amount received from redemption of the bond). A forced conversion allows the issuer to take advantage of favorable capital market conditions and strengthen its capital structure, and eliminates the risk that a fall in the equity market prevents conversion (thereby requiring the issuer to redeem the bonds at maturity).
- The put option may be classified as a **hard put** (which can only be redeemed for cash) or a **soft put** (which may be redeemed for cash, common stock, subordinated notes, or a combination of the three).

LOS 34o: Calculate and interpret the components of a convertible bond's value. Vol 5, pp 169–172

Analysis of a Convertible Bond

Conversion Value

The **conversion** (or **parity**) **value** of a convertible bond indicates the value of the bond if converted at the market price of the shares.

$$\boxed{\text{Conversion value} = \text{Market price of common stock} \times \text{Conversion ratio}}$$

For ABC's convertible bond:

Conversion value at issuance = \$4.50 × 20,000 = \$90,000

Conversion value on September 9 2014 = \$5.85 × 20,000 = \$117,000

Minimum Value of a Convertible Bond

The **minimum value** of a convertible bond equals the greater of (1) its **conversion value** (the value if the security is converted into shares immediately) and (2) its **straight value** (its value without the conversion option). The straight value is computed through the arbitrage-free valuation framework by discounting the bond's future cash flows at appropriate rates.

The minimum value of the convertible bond can be described as a floor. However, note that it is actually a moving floor, because the straight value is not fixed; it changes with fluctuations in interest rates and the issuer's credit quality.

For ABC's convertible bond, the minimum value can be computed as:

Minimum value at issuance = Maximum (\$90,000; \$100,000) = \$100,000

Note that the straight value at issuance date is \$100,000
because the issue price is set to 100% of par.

Minimum value on September 9, 2014 = Maximum (\$117,000; \$106,657.87) = \$117,000

If the value of the convertible bond were lower than the greater of the conversion value and the straight value (i.e., its minimum value), an arbitrage opportunity would arise.

- Suppose that the minimum value of the convertible bond equals its straight value. If the convertible bond is selling for less than its straight value, it implies that the convertible bond offers a higher yield than an otherwise-identical non-convertible bond. Investors will, therefore, buy the convertible bond until its price equals its straight value.
- Now suppose that the minimum value of the convertible bond equals its conversion value. If the convertible bond is selling for less than its conversion value, an arbitrageur can buy the convertible bond, convert it into shares based on the conversion ratio, and sell the shares in return for the conversion value. Eventually, as more and more arbitrageurs undertake this trade, the convertible bond's price will rise to its conversion value.

Market Conversion Price, Market Conversion Premium per Share, and Market Conversion Premium Ratio

The price that an investor effectively pays for common stock if a convertible bond is purchased and then converted into common stock is known as the **market conversion price** or **conversion parity price**. It basically reflects the "breakeven" price for the investor, as once this price has been exceeded, any increase in the stock price will increase the value of the convertible bond by at least the same percentage.

$$\text{Market conversion price} = \frac{\text{Market price of convertible security}}{\text{Conversion ratio}}$$

For ABC's convertible bond, the market conversion price on September 9, 2014, can be calculated as:

Market conversion price = \$125,628 / 20,000 = \$6.28

The **market conversion premium per share** is the excess amount that an investor pays for acquiring the company's shares by purchasing a convertible bond instead of purchasing them in the open market.

$$\text{Market conversion premium per share} = \text{Market conversion price} - \text{Current market price}$$

For ABC's convertible bond, the market conversion premium per share on September 9, 2014, can be calculated as:

Market conversion premium per share = \$6.28 – \$5.85 = \$0.43

The **market conversion premium ratio** equals the market conversion premium per share divided by the current market price of the company's common stock.

$$\text{Market conversion premium ratio} = \frac{\text{Market conversion premium per share}}{\text{Market price of common stock}}$$

For ABC's convertible bond, the market conversion premium ratio on September 9, 2014, can be calculated as:

Market conversion premium ratio = \$0.43 / \$5.85 = 7.35%

The reason that an investor pays a "premium" when purchasing common stock through a convertible bond (as opposed to purchasing shares directly from the open market) is that the straight value of the bond acts as a moving floor for the price of the convertible security. In this context, the market conversion premium per share can be viewed as the price of the conversion call option. However, there is a difference between (1) buying a stand-alone call option on the issuer's stock and (2) purchasing the call option on the issuer's stock embedded in a convertible bond. The buyer of the stand-alone call option knows the exact dollar amount of downside risk (the call premium), while the buyer of the convertible bond only knows that her maximum loss equals the difference between the price paid for the convertible bond and its straight value. The straight value at any future date is unknown.

Downside Risk of a Convertible Bond

The **premium over straight value** is sometimes used as a measure for downside risk in a convertible bond. This interpretation, however, is theoretically flawed because the straight value (which serves as a floor on the convertible's price) changes as interest rates/credit spreads change. Downside risk is measured as a percentage of straight value. It is calculated as:

$$\text{Premium over straight value} = \frac{\text{Market price of convertible bond}}{\text{Straight value}} - 1$$

For ABC's convertible bond, the premium over straight value is calculated as:

Premium over straight value = (\$125,628 / \$106,657.87) – 1 = 17.79%

Note that, all other factors remaining the same, the higher the premium over straight value, the less attractive the convertible bond.

Upside Potential of a Convertible Bond

The upside potential for a convertible security depends on the prospects for the underlying stock, which are evaluated using techniques for analyzing common stocks. These techniques are discussed in the equity analysis section.

LOS 34p: Describe how a convertible bond is valued in an arbitrage-free framework. Vol 5, pp 172–173

An investor should consider the following when investing in a convertible bond:

- The ability of the issuer to service the bonds and repay the principal amount.
- The bond's terms of issuance (collateral, credit enhancements, covenants, and contingent provisions).
- Interest rate forecasts.

- Factors that may affect the issuer's common stock, including dividend payments and potential acquisitions, disposals, or rights issues.
- Any exogenous factors that could have a negative impact on the value of the bonds.

Valuation of a Convertible Bond

The arbitrage-free framework remains the most commonly used model for valuing convertible bonds.

An investor in a **convertible bond that is not callable or putable** effectively (1) invests in a straight bond and (2) is long on a call option on the company's stock, where the number of shares that can be purchased with the call equals the conversion ratio.

Convertible security value = Straight value + Value of the call option on the stock

Note:

- The Black-Scholes-Merton option pricing model can be used to determine the value of the call option on the stock.
- The higher the stock price volatility (a key input into the BSM model), the higher the value of the call option, and the higher the value of the convertible bond.

An investor in a **convertible bond that is callable but not putable** effectively (1) invests in a straight bond, (2) is long on a call option on the company's stock, and (3) writes (is short on) a call option on the bond.

Convertible callable bond value = Straight value + Value of the call option on the stock – Value of the call option on the bond

Note:

- The valuation of the call option on the bond that is embedded in the callable convertible bond requires an analysis of (1) future interest rates and (2) economic factors that determine whether the issuer will call the security, so the arbitrage-free framework is used. The BSM model (which incorporates stock price volatility but not interest rate volatility) cannot be used to price the call option on the bond. It can only be used to price the call option on the stock.
- An increase in interest rate volatility would increase the value of the call option on the bond and reduce the value of a callable convertible bond.

An investor in a **convertible bond that is callable and putable** effectively (1) invests in a straight bond, (2) is long on a call option on the company's stock, (3) writes (is short on) a call option on the bond, and (4) is long on a put option on the bond.

Convertible callable and putable bond value = Straight value + Value of the call option on the stock – Value of the call option on the bond + Value of the put option on the bond

LOS 34q: Compare the risk–return characteristics of a convertible bond with the risk–return characteristics of a straight bond and of the underlying common stock. Vol 5, pp 173–178

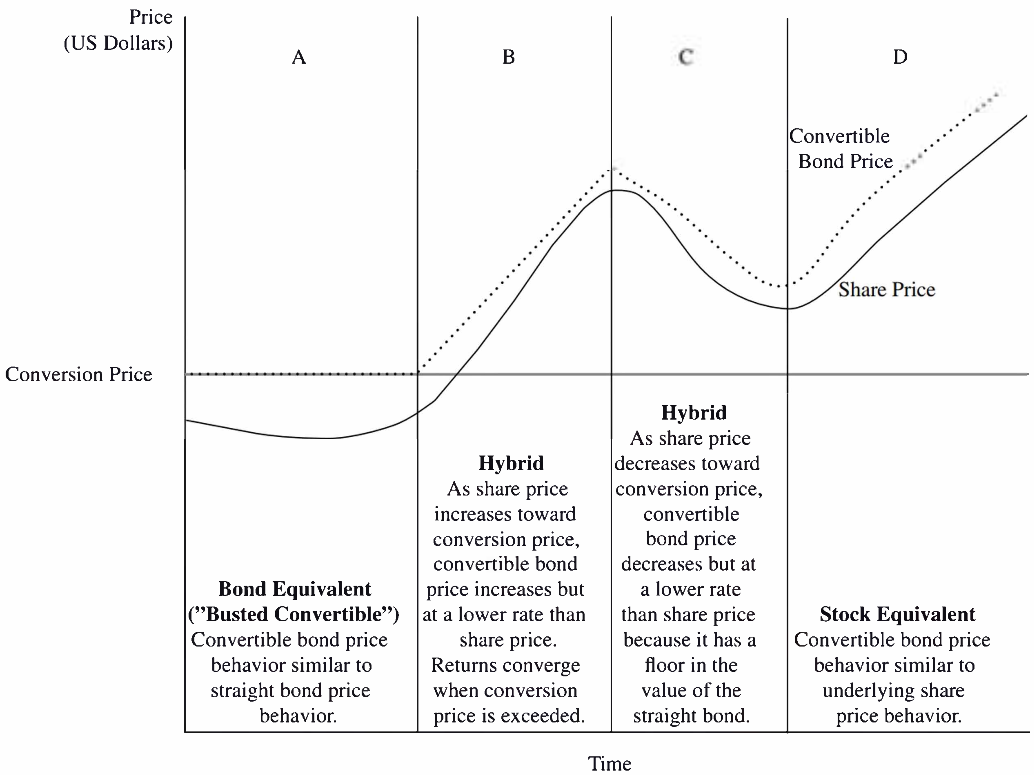
Investment Characteristics of a Convertible Security

- If the price of the company's common stock is relatively low, such that the straight value of a convertible bond is significantly greater than the conversion value, the security will trade like a fixed-income security, with factors such as interest rates and credit spreads having a significant impact on its price. In such a situation, with the call option on the stock out-of-the-money, the security may be referred to as a **fixed-income equivalent** or a **busted convertible**. Further, the closer the security is to the end of the conversion period, the more the convertible bond will behave like a fixed-income security, as it becomes increasingly likely that the call option on the stock will expire out-of-the-money.
- If the price of the company's common stock is relatively high, such that the conversion value is significantly greater than the straight value, the security will trade like an equity instrument, with its value heavily influenced by share price movements and relatively immune to interest rate movements. In this case, the security may be referred to as a **common stock equivalent**. Note that when the embedded call option on the stock is in-the-money, it is more likely to be exercised when the conversion value exceeds the redemption value of the bond.
- In between these two scenarios, the convertible will trade as a **hybrid security** with characteristics of both fixed-income and equity instruments. However, it is important to note the risk-return characteristics of convertible bonds when (1) the underlying share price is below the conversion price and increases toward it and (2) the underlying share price is above the conversion price and decreases toward it.

    - In the first case, the return on the convertible bond increases significantly, but at a lower rate than the underlying share price. Once the share price exceeds the conversion price, the change in the convertible bond's price increases at the same rate as the underlying share price.
    - In the second case, the relative change in the convertible bond's price is less than the change in the share price because the convertible bond has a floor on its value.

Figure 6-1 illustrates the price behavior of a convertible bond and the underlying stock.

Figure 6-1: Price Behavior of Convertible Bond and Underlying Common Stock[1]



Notice in Figure 6-1 that the price of the underlying share is above the conversion price in Area B, Area C, and Area D. In these scenarios, it is still possible that the investor may not exercise the conversion option. This could be for any of the following reasons:

- The call option on the stock embedded in the convertible bond may be European-style and cannot be exercised currently.
- The investor may choose to wait to exercise the in-the-money call option on the stock until it is even more in-the-money.
- The investor may wish to sell the convertible bond instead of converting her bond into shares.

Finally, note that, except in the case of busted convertibles, the primary driver of a convertible bond's value is the underlying share price. However, large movements in interest rates or the issuer's credit spread continue to affect the value of a convertible bond. For a convertible bond with a fixed coupon, all else being equal:

- If interest rates rise (fall) significantly, its value would decrease (increase).
- If the issuer's credit rating were to improve (decline), its value would increase (decrease).

[1] Exhibit 30, Vol 5, CFA Program Curriculum 2020

READING 35: CREDIT ANALYSIS MODELS

LESSON 1: THE IMPACT OF CREDIT EXPOSURE ON BOND VALUATION

LOS 35a: Explain expected exposure, the loss given default, the probability of default, and the credit valuation adjustment. Vol 5, pp 202–208

Expected exposure is the amount of money an investor could lose if a bond goes into default, excluding any possible recovery of assets from bankruptcy proceedings. For example, if we have a one-year bond trading at par value with a 10% annual coupon payment, the expected exposure is 110 (or par value plus the 10% coupon). Expected exposure calculations can get more complicated when you take into account multiple time periods and interest rate volatility that will change the price of a bond over time. This will be shown later in the study materials.

The **recovery rate** is the amount/percent of any monies recovered after a bond goes into default. Recovery rates vary from bond to bond and industry to industry based on seniority in the capital structure, leverage in the capital structure, and whether a bond is secured or collateralized. Recovery rates are used in the calculation of **loss given default** (LGD), or the amount of money lost in case of a default, and **loss severity**, which is simply the opposite of the recovery rate (i.e., a 30% recovery rates equals a 70% loss severity).

We can illustrate these concepts using the previous bond example and an assumed recovery rate of 30%.

Expected exposure	= 110
Expected recovery rate	= 30%
Amount recovered	= 33 (or 110 × 0.30)
Loss given default	= 77 (or 110 – 33)

The **probability of default (POD)** is the chance of a bond issuer defaulting, or failing to make principal and interest rate payments as scheduled. When dealing with credit risk modeling, it is important to note the difference between risk-neutral probabilities and historical default probabilities in valuing bonds.

For example, assume a credit rating agency has historical default data that says that 95% of one-year corporate bonds meet their obligations. Based on these historical data, the probability of default is simply 5% (or 100% – 95%). Using the same bond example as earlier, we can calculate the bond's expected future value.

Expected exposure	= 110
Expected recovery rate	= 30%
Amount recovered	= 33 (or 110 × 0.30)
Bond value	= (expected exposure × (1 – default rate)) + (amount recovered × default rate)
Bond value	= (110 × 0.95) + (33 × 0.05)
Bond value	= 104.5 + 1.65 = 106.15

We can then discount this bond value by an assumed risk-free rate of 5%, rather than the bond's yield to maturity, to get its present value.

$$\frac{106.16}{1.05} = 101.10$$

Note that the historical default probability method actually overstates the value of the bond, given it is trading above par value (i.e., 101.10 vs. 100).

To determine the risk-neutral default probability of this bond, we need to assume that it is trading at PAR value. The equation for this assumes that P* is the risk-neutral default probability and that 1 – P* is the probability of survival (or no default). This equation is the same used earlier in the historical default example and can be expressed as:

$$100 = \frac{[(110 \times (1 - P^*)) + (33 \times P^*)]}{1.05}$$
$$105 = 110 - 110P^* + 33P^*$$
$$-5 = -77P^*$$
$$P^* = \frac{5}{77} = 6.50\%$$

You should take note that the risk-neutral default probability is higher than the historic default probability. This is because risk-neutral default probabilities incorporate the uncertainty of when defaults could occur, a credit premium over risk-free government bonds, liquidity issues, and tax considerations.

Credit valuation adjustment (CVA) is the present value of credit risk for a bond, which can then be used to determine the fair value of a risky bond. To illustrate the derivation and use of the CVA, we will work with a four-year, zero-coupon corporate bond. There are several calculations in this process, which we will sequentially illustrate using Table 1-1.

Table 1-1: Four-Year Zero-Coupon Corporate Bond

Year	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Discount Factor	PV of Expected Loss
1								
2								
3								
4								
							CVA =	

The first step in the process is to fill in the values for the exposure to default loss (Table 1-2). Our assumptions for this step are that defaults only occur at the end of each year, the risk-free yield curve is flat at 5% for all maturities, and the exposure at year 4 is 100. Therefore, we need to discount the values of exposure to default loss by the risk-free rate.

$$\frac{100}{1.05^3} = 86.3838$$

$$\frac{100}{1.05^2} = 90.7029$$

$$\frac{100}{1.05^1} = 95.2381$$

Table 1-2: Four-Year Zero-Coupon Corporate Bond: Step 1

Year	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Discount Factor	PV of Expected Loss
1	86.3838							
2	90.7029							
3	95.2381							
4	100.0000							
							CVA =	

The second step is to calculate the amount that can be recovered should a default occur (i.e., exposure × recovery rate) (Table 1-3). For this example, we will assume a recovery rate of 30% (Table 1-4).

Table 1-3: Recovery Calculation

Exposure		Recovery Rate		Recovery
86.3838	×	30%	=	25.9151
90.7029	×	30%	=	27.2109
95.2381	×	30%	=	28.5714
100.0000	×	30%	=	30.0000

Table 1-4: Four-Year Zero-Coupon Corporate Bond: Step 2

Year	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Discount Factor	PV of Expected Loss
1	86.3838	25.9151						
2	90.7029	27.2109						
3	95.2381	28.5714						
4	100.0000	30.0000						
							CVA =	

The third step is to determine the loss given default (LGD) (Table 1-5), which is the exposure minus the assumed recovery (Table 1-6).

Table 1-5: Loss Given Default Calculation

Exposure		Recovery		Loss Given Default
86.3838	–	25.9151	=	60.4687
90.7029	–	27.2109	=	63.4920
95.2381	–	28.5714	=	66.6667
100.0000	–	30.0000	=	70.0000

Table 1-6: Four-Year Zero-Coupon Corporate Bond: Step 3

Year	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Discount Factor	PV of Expected Loss
1	86.3838	25.9151	60.4687					
2	90.7029	27.2109	63.4920					
3	95.2381	28.5714	66.6667					
4	100.0000	30.0000	70.0000					
							CVA =	

The fourth step is to determine the risk-neutral POD and probability of survival (POS). To do this we assume conditional probabilities of default (i.e., each year's POD assumes no prior default) and the previously calculated risk-neutral POD of 6.5%. For the first year, the POD is 6.5% (or the hazard rate) and the POS is 93.5% (100% – 6.5%). For all following years, the POD is calculated as the prior year's POS times the hazard rate (93.5% × 6.5% = 6.0775%) and the new POS is calculated as the prior year's POS minus the new POD, as in the calculations in Tables 1-7 and 1-8.

Table 1-7: POD and POS Calculation Example 1

			POD		POS
Year 1	100.0000%	–	6.5000%	=	93.5000%

Table 1-8: POD and POS Calculation Example 2

Prior Year POS		Hazard Rate		New POD		Prior Year POS		New POD		New POS
93.5000%	×	6.5000%	=	6.077500%	⇒	93.5000%	–	6.0775%	=	87.4225%
87.4225%	×	6.5000%	=	5.682463%	⇒	87.4225%	–	5.6825%	=	81.7400%
81.7400%	×	6.5000%	=	5.313102%	⇒	81.7400%	–	5.3131%	=	76.4269%

An alternative way to derive the POS is to take 100% and subtract the first year's POD raised to the power of the number of years (Table 1-9).

Table 1-9: POD and POS Alternative Calculation

(100%	–	6.5%)	=	93.5000%
(100%	–	$6.5\%)^2$	=	87.4225%
(100%	–	$6.5\%)^3$	=	81.7400%
(100%	–	$6.5\%)^4$	=	76.4269%

As a result of these calculations, we can see that the cumulative POD over the four-year lifetime of the bond is the sum of each year's POD (or 23.5731%) and the POS for the bond until maturity is 76.4269%, which together equal 100% (Table 1-10).

Table 1-10: Four-Year Zero-Coupon Corporate Bond: Step 4

Year	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Discount Factor	PV of Expected Loss
1	86.3838	25.9151	60.4687	6.5000%	93.5000%			
2	90.7029	27.2109	63.4920	6.0775%	87.4225%			
3	95.2381	28.5714	66.6667	5.6825%	81.7400%			
4	100.0000	30.0000	70.0000	5.3131%	76.4269%			
				23.5731%			CVA =	

The fifth step is to calculate expected losses for each year, assuming no defaults in prior years, by multiplying the LGD by the POD (Tables 1-11 and 1-12).

Table 1-11: Expected Loss Calculation

LGD		POD		Expected Loss
60.4687	×	6.5000%	=	3.9305
63.4920	×	6.0775%	=	3.8587
66.6667	×	5.6825%	=	3.7883
70.0000	×	5.3131%	=	3.7192

Table 1-12: Four-Year Zero-Coupon Corporate Bond: Step 5

Year	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Discount Factor	PV of Expected Loss
1	86.3838	25.9151	60.4687	6.5000%	93.5000%	3.9305		
2	90.7029	27.2109	63.4920	6.0775%	87.4225%	3.8587		
3	95.2381	28.5714	66.6667	5.6825%	81.7400%	3.7883		
4	100.0000	30.0000	70.0000	5.3131%	76.4269%	3.7192		
				23.5731%			CVA =	

The sixth step is to calculate the discount factor based on the aforementioned flat risk-free government bond yield of 5.0% (Tables 1-13 and 1-14).

Table 1-13: Discount Factor Calculation

1	/	1.05	=	0.9524
1	/	1.05^2	=	0.9070
1	/	1.05^3	=	0.8638
1	/	1.05^4	=	0.8227

Table 1-14: Four-Year Zero-Coupon Corporate Bond: Step 6

Year	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Discount Factor	PV of Expected Loss
1	86.3838	25.9151	60.4687	6.5000%	93.5000%	3.9305	0.9524	
2	90.7029	27.2109	63.4920	6.0775%	87.4225%	3.8587	0.9070	
3	95.2381	28.5714	66.6667	5.6825%	81.7400%	3.7883	0.8638	
4	100.0000	30.0000	70.0000	5.3131%	76.4269%	3.7192	0.8227	
				23.5731%			CVA =	

Finally, we calculate the present value (PV) of expected loss, which is the expected loss times the discount factor (Table 1-15). The sum of these values is the CVA.

Table 1-15: Present Value Calculation

Expected Loss	Discount Factor	PV of Expected Loss
3.9305	0.9524	3.7434
3.8587	0.9070	3.4998
3.7883	0.8638	3.2723
3.7192	0.8227	3.0598

Table 1-16: Four-Year Zero-Coupon Corporate Bond: Step 7

Year	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Discount Factor	PV of Expected Loss
1	86.3838	25.9151	60.4687	6.5000%	93.5000%	3.9305	0.9524	3.7433
2	90.7029	27.2109	63.4920	6.0775%	87.4225%	3.8587	0.9070	3.4998
3	95.2381	28.5714	66.6667	5.6825%	81.7400%	3.7883	0.8638	3.2725
4	100.0000	30.0000	70.0000	5.3131%	76.4269%	3.7192	0.8227	3.0598
				23.5731%			CVA =	13.5755

Now that we have the CVA, we can calculate the fair value of the four-year, zero-coupon bond. To do this, we first need to calculate the price of a similar risk-free government bond, which is done by multiplying the par value of 100 by the four-year discount factor (100 × 0.822702 = 82.2702). Then we subtract the CVA from what would be the price of a risk-free bond to determine the fair value of the corporate bond (82.2702 – 13.5755 = 68.6947).

Now with the fair value of 68.69, we can compute the yield and credit spread of this bond:

$$\frac{100}{(1 + \text{yield})^4} = 68.6947$$

$$\text{Yield} = 9.8422\%$$

Therefore, this bond's credit spread is:

$$9.8422\% - 5.0\% = 4.8422\%$$

LESSON 2: CREDIT SCORES AND CREDIT RATINGS

LOS 35b: Explain credit scores and credit ratings. Vol 5, pp 210–215

Credit scores are a rating system used by lenders to evaluate the creditworthiness of retail borrowers (individuals and small businesses) and establish terms of a lending contract. These methodologies vary by country given differing privacy and legal considerations.

In the United States, 90% of retail lenders use FICO scores produced by the Fair Isaac Corporation, which are derived from algorithms that use data from three major credit bureaus—Experian, Equifax, and TransUnion. FICO scores can range from very poor (300) to exceptional (850). Five primary factors are used to determine a retail borrower's FICO score:

1. Payment history has a 35% weighting, which consists of information on delinquencies, bankruptcies, court judgments, repossessions, and foreclosures.
2. Debt burden has a 30% weighting, which includes credit card debt-to-limit ratios, the number of accounts with balances, and the total amount of debt.
3. The length of a retail borrower's credit history has a 15% weighting, which includes the age of the oldest credit account and the average age of all credit accounts.
4. The type of accounts used has a 10% weighting, which includes installment payments, credit cards, and mortgages.
5. Recent credit searches have a 10% weighting, which includes so-called "hard" searches for new loans, but does not include "soft searches" that can involve someone checking their own credit score or employee verification.

Credit ratings are used to evaluate bonds and asset-backed securities (ABSs) issued by corporations and governments. Similar to the retail lending market, there are three major global credit ratings agencies used by investors (or the lenders). These agencies are Moody's Investors Service, Standard & Poor's, and Fitch Ratings, and they issue debt quality ratings (letter grades) that focus on the probability of default for issuers and specific debt instruments.

Credit agencies also consider LGD through a process known as "notching" various debt instruments from a certain issuer. For example, a company's credit rating is typically based on senior unsecured debt, but lower-quality (or subordinated) debt will be "notched" down given a higher LGD because it is lower in the firm's capital stack in the case of default. The way this works is that a company's senior unsecured debt may have a AA rating, but a rating agency will notch any subordinated debt down to AA– or A given a higher LGD. It is through the consideration of both default and LGD that ratings agencies produce overall credit ratings, as opposed to just default ratings.

Nonetheless, it is interesting to see how well credit agencies and their ratings work in predicting defaults over time. This can be seen in Table 2-1, which shows Standard & Poor's annual corporate default rates by rating category from 1996 to 2016 and how highly rated debt issues (i.e., AAA–A) experience much lower default rates than poorly rated debt issues (CCC/C).

Table 2-1: Global Corporate Annual Default Rates by Rating Category (%)

	AAA	AA	A	BBB	BB	B	CCC/C
1996	0.00	0.00	0.00	0.00	0.45	2.91	8.00
1997	0.00	0.00	0.00	0.25	0.19	3.51	12.00
1998	0.00	0.00	0.00	0.41	0.82	4.63	42.86
1999	0.00	0.17	0.18	0.20	0.95	7.29	33.33
2000	0.00	0.00	0.27	0.37	1.16	7.70	35.96
2001	0.00	0.00	0.27	0.34	2.96	11.53	45.45
2002	0.00	0.00	0.00	1.01	2.89	8.21	44.44
2003	0.00	0.00	0.00	0.23	0.58	4.07	34.73
2004	0.00	0.00	0.08	0.00	0.44	1.45	16.18
2005	0.00	0.00	0.00	0.07	0.31	1.74	9.09
2006	0.00	0.00	0.00	0.00	0.30	0.82	13.33
2007	0.00	0.00	0.00	0.00	0.20	0.25	15.24
2008	0.00	0.38	0.39	0.49	0.81	4.08	27.27
2009	0.00	0.00	0.22	0.55	0.75	10.92	49.46
2010	0.00	0.00	0.00	0.00	0.58	0.86	22.62
2011	0.00	0.00	0.00	0.07	0.00	1.67	16.30
2012	0.00	0.00	0.00	0.00	0.30	1.56	27.52
2013	0.00	0.00	0.00	0.00	0.10	1.63	24.34
2014	0.00	0.00	0.00	0.00	0.00	0.78	17.13
2015	0.00	0.00	0.00	0.00	0.16	2.39	25.88
2016	0.00	0.00	0.00	0.00	0.47	3.68	32.67

https://www.spratings.com/documents/20184/774196/2016+Annual+Global+Corporate+Default+Study+And+Rating+Transitions.pdf/2ddcf9dd-3b82-4151-9dab-8e3fc70a7035

In addition to assigning letter grades to companies, ratings agencies provide other metrics such as having a positive, stable, or negative outlook for a company. Ratings agencies also put companies under watch for potential changes in credit ratings. This process is best illustrated in Table 2-2, which shows how a company's credit rating changes over time.

LOS 35c: Calculate the expected return on a bond given transition in its credit rating. Vol 5, pp 210–215

Table 2-2: Pacific Exploration and Production Corp./Issuer Credit Rating History

Date	To
November 3, 2009	B + /Positive/—
November 3, 2010	BB–/Stable/—
July 5, 2011	BB/Stable/—
July 13, 2012	BB/Positive/—
March 15, 2013	BB + /Stable/—
January 16, 2015	BB + /Watch Neg/—
April 1, 2015	BB/Negative/—
October 14, 2015	BB–/Stable/—
December 28, 2015	CCC + Watch Neg/—
January 15, 2016	CC/Negative/—
January 20, 2016	D/—/—
November 3, 2016	B + /Stable/—

http://media.spglobal.com/documents/SPGlobal_Ratings_Article_13+April+2017_Annual+Corporate+Default+Study+and+Rating+Transitions.pdf

Rating agencies create transition matrices that track the probabilities of an issuer's ratings change over time, as shown in Table 2-3. For an example of how to read this table, consider the case of an A-rated company that has the following:

- 87.79% chance of staying an A-rated bond
- 0.03% chance of being upgraded to a AAA rating
- 1.77% chance of being upgraded to a AA rating
- 5.33% chance of being downgraded to a BBB rating
- 0.32% chance of being downgraded to a BB rating
- 0.13% chance of being downgraded to a B rating
- 0.02% chance of being downgraded to a CCC/C rating
- 0.06% chance of going into default

Table 2-3: Global Corporate Average Transition Rates (1981–2016) (%)

From/to	AAA	AA	A	BBB	BB	B	CCC/C	D
AAA	**87.05**	9.03	0.53	0.05	0.08	0.03	0.05	0.00
AA	0.52	**86.82**	8.00	0.51	0.05	0.07	0.02	0.02
A	0.03	1.77	**87.79**	5.33	0.32	0.13	0.02	0.06
BBB	0.01	0.10	3.51	**85.56**	3.79	0.51	0.12	0.18
BB	0.01	0.03	0.12	4.97	**76.98**	6.92	0.61	0.72
B	0.00	0.03	0.09	0.19	5.15	**74.26**	4.46	3.76
CCC/C	0.00	0.00	0.13	0.19	0.63	12.91	**43.97**	26.78

http://media.spglobal.com/documents/SPGlobal_Ratings_Article_13+April+2017_Annual+Corporate+Default+Study+and+Rating+Transitions.pdf

To use a transition matrix to calculate the expected percent change in the price of a bond, we first need to compute the change in the price based on changes in credit spreads associated with various ratings. For the purposes of these calculations, let's consider an

A-rated, 10-year corporate bond with a modified duration of 6.0 and a series of assumed credit spreads, as shown in Table 2-4.

Table 2-4: Assumed Credit Spreads to a One-Year Risk-Free Bond by Bond Rating

	AAA	AA	A	BBB	BB	B	CCC/C	D
Assumed Credit Spreads	0.50%	0.70%	1.00%	1.75%	2.75%	4.00%	5.50%	7.00%

Recall the basic formula for calculating changes in bond prices:

$$-\text{Modified duration} \times (\text{Change in yield})$$

Or in this instance:

$$-\text{Modified duration} \times (\text{New credit rating credit spread} - \text{Original credit rating credit spread})$$

The next step in the process is to multiply price changes given a possible credit transition (calculations shown in Table 2-5) by the respective transition probability and summing the products (Table 2-6).

Table 2-5: Bond Price Calculation

	Duration		Difference in Spreads		Price Change
From A to AAA	–6.0	×	(0.50 – 1.00)	=	3.00%
From A to AA	–6.0	×	(0.70 – 1.00)	=	1.80%
From A to BBB	–6.0	×	(1.75 – 1.00)	=	–4.50%
From A to BB	–6.0	×	(2.75 – 1.00)	=	–10.50%
From A to B	–6.0	×	(4.00 – 1.00)	=	–18.00%
From A to CCC/C	–6.0	×	(5.50 – 1.00)	=	–27.00%

Table 2-6: Expected Price Changes in a Bond Calculation

	Transition Probability	Change in Bond Price	
From A to AAA	0.0300%	3.000%	0.001%
From A to AA	1.7700%	1.800%	0.032%
No change	87.7900%	0.000%	0.000%
From A to BBB	5.3300%	–4.500%	–0.240%
From A to BB	0.3200%	–10.500%	–0.034%
From A to B	0.1300%	–18.000%	–0.023%
From A to CCC/C	0.0200%	–27.000%	–0.005%
		Expected price change in a bond	–0.2695%

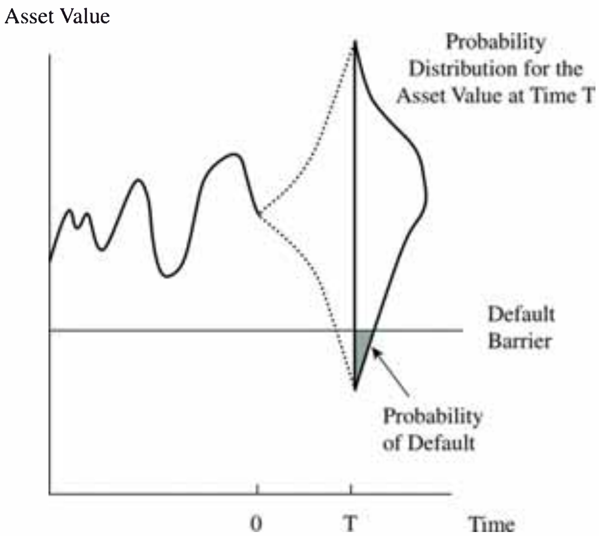
The end result of this process is that the expected return for a bond over the next year is its yield to maturity (YTM) minus 0.2695%. It is important to note that taking credit rating transitions into account typically reduces a bond's expected return. This is because transition probabilities are skewed toward downgrades rather than upgrades, and changes in credit spreads are much larger for lower ratings than for higher ratings.

LESSON 3: STRUCTURAL AND REDUCED-FORM MODELS OF CREDIT RISK

LOS 35d: Explain structural and reduced-form models of corporate credit risk, including assumptions, strengths, and weaknesses. Vol 5, pp 216–219

Structural credit risk models are based on the idea that a company will default on its debt when the value of liabilities exceeds the value of assets. The basic relationship of a structural default model can be seen in Figure 3-1, which shows that as long as asset values remain above the "default barrier" (the value of liabilities), even if they are volatile, no default will occur. However, once the value of assets drops below the default barrier, the company will default on its debt.

Figure 3-1: The Structural Credit Default Model



Source: This exhibit is adapted from Duffie and Singleton, 2003, page 54.

As can be seen from Figure 3-1, this model assumes that there is a probability distribution of asset values between time 0 (a current observation) and time T (some future date). This model assumes that the probability of default is an endogenous (internal to a company) variable and represents the probability distribution that falls below the default barrier. The probability of default increases due to higher asset volatility, longer time periods, and higher debt levels. Conversely, the probability of default and the default barrier will drop as a company lowers the amount of debt in its capital structure.

Another aspect of structural credit models is that they allow debt and equity levels to be interpreted in terms of options contracts. To demonstrate, let's make the following assumptions:

1. A(T) is the random asset value as of time T.
2. Debt is comprised of zero-coupon bonds that mature at time T.
3. Bonds have a face value of K, representing the default barrier.
4. The value for debt and equity at time T are denoted as D(T) and E(T) and depend on the relationship between A(T) and K.

These assumptions mean we can create the following three equations:

1. D(T) + E(T) = A(T)
    - This equation indicates that the values of debt and equity at time T equal the asset value.
2. E(T) = Max[A(T) – K,0]
    - This equation indicates that equity is basically a call option on assets held by shareholders, with a strike price equal to the value of debt.
    - More specifically, this is a long call option because as asset values go up, equity goes up. Also, like an option, equity cannot be a negative value based on the assumption of limited liability.
3. D(T) = A(T) – Max[A(T) – K,0] or D(T) = A(T) – E(T)
    - This equation shows that debt holders own company assets by writing the call option (equation 2) held by shareholders.
    - The premium debt holders receive for writing this call is having first priority on claiming assets. However, if equity falls to zero, they own the remaining assets.

To better understand these relationships, assume the following:

- If at time T, A(T) > K, the call is in the money.
    - Then, E(T) = A(T) – K and D(T) = A(T) – (A(T) – K) = K
- If at time T, A(T) < K, the call option is out of the money and the company defaults.
    - Then, E(T) = 0 and D(T) = A(T) – 0 = A(T)

An advantage of using structural models is that they require information that is well known to company management and their bankers and credit ratings agencies. This makes them a good tool for internal risk management, a banker's assessment of credit risk, and the production of credit agency ratings. Another advantage is that structural models can link credit risk to options models, which provide practical value to ratings agencies that can use option-pricing methodologies to estimate POD and LGD using a company's historic equity prices as volatility estimates.

However, a major disadvantage of structural models is that a company's asset values do not trade on public markets, which makes estimating asset volatility difficult in practice. Furthermore, establishing default barriers can be impractical when company debt is not fully reflected on balance sheets, such as when Enron, WorldCom, and Lehman hid debt.

Reduced-form models treat default as an exogenous variable that randomly occurs and try to statistically explain it using a Poisson stochastic process. This contrasts with structural models that try to explain why a default occurs (i.e., asset values fall beneath liabilities). The key parameter for reduced-form models is default intensity, or the probability of default over a certain period of time. This is why these models are referred to as intensity-based or stochastic default rate models.

The advantage of reduced-form models is that they only need the use of publicly available data, such as company-specific and macroeconomic variables, to conduct regression analysis. This makes them well suited for valuing public debt and associated derivatives. However, the disadvantage of reduced-form models is that they do not explain the reasons for default and assume they randomly/surprisingly occur over time. This is unrealistic because defaults can usually be seen coming well in advance, as ratings agencies usually lower a company's credit rating prior to default.

LESSON 4: BOND VALUATION AND CREDIT RISK

LOS 35e: Calculate the value of a bond and its credit spread, given assumptions about the credit risk parameters. Vol 5, pp 219–234

To calculate the value of a bond and its credit spread, we need to use binomial interest rate trees assuming no arbitrage. Tables 4-1–4-4 show the annual payment benchmark for government bonds, spot rates, discount factors, and forward rates. As we will sequentially show in these tables, coupon rates are equal to yields to maturity and discount factors, and spot rates are bootstrapped using bond cash flows.

Table 4-1: Par Curve for Annual Payment Benchmark Government Bonds, Spot Rates, Discount Factors, and Forward Rates

Maturity	Coupon Rate	Price	Discount Factor	Spot Rate	Forward Rate
1	1.00%	100			
2	1.50%	100			
3	1.75%	100			
4	2.00%	100			
5	2.25%	100			

To determine discount factors (DF), the following equations are used for each year:

$$100 = (100 + 1) \times DF1$$

$$DF1 = 0.9901$$

$$100 = (1.5 \times 0.9901) + (101.5 \times DF2)$$

$$DF2 = 0.9706$$

$$100 = (1.75 \times 0.9901) + (1.75 \times 0.9706) + (101.75 \times DF3)$$

$$DF3 = 0.9491$$

$$100 = (2.0 \times 0.9901) + (2.0 \times 0.9706) + (2.0 \times 0.9491) + (102 \times DF4)$$

$$DF4 = 0.9233$$

$$100 = (2.25 \times 0.9901) + (2.25 \times 0.9706) + (2.25 \times 0.9491) + (2.25 \times 0.9233) + (102.25 \times DF5)$$

$$DF5 = 0.8936$$

Table 4-2: Par Curve for Annual Payment Benchmark Government Bonds, Spot Rates, Discount Factors, and Forward Rates: Discount Factor

Maturity	Coupon Rate	Price	Discount Factor	Spot Rate	Forward Rate
1	1.00%	100	0.9901		
2	1.50%	100	0.9706		
3	1.75%	100	0.9491		
4	2.00%	100	0.9233		
5	2.25%	100	0.8936		

Spot rates (i.e., implied zero-coupon rates) are calculated from the discount factors:

Spot rate Year 1 = 1/DF1 – 1 = 1.0000%

Spot rate Year 2 = $(1/DF2)^{(1/2)}$ – 1 = 1.5038%

Spot rate Year 3 = $(1/DF3)^{(1/3)}$ – 1 = 1.7574%

Spot rate Year 4 = $(1/DF4)^{(1/4)}$ – 1 = 2.0140%

Spot rate Year 5 = $(1/DF5)^{(1/5)}$ – 1 = 2.2743%

Table 4-3: Par Curve for Annual Payment Benchmark Government Bonds, Spot Rates, Discount Factors, and Forward Rates: Spot Rate

Maturity	Coupon Rate	Price	Discount Factor	Spot Rate	Forward Rate
1	1.00%	100	0.99010	1.000%	
2	1.50%	100	0.97059	1.504%	
3	1.75%	100	0.94908	1.757%	
4	2.00%	100	0.92334	2.014%	
5	2.25%	100	0.89365	2.274%	

Forward rates (FR) are calculated as a ratio of discount factors:

FR Year 2 = DF1/DF2 – 1 = 2.0101%

FR Year 3 = DF2/DF3 – 1 = 2.2665%

FR Year 4 = DF3/DF4 – 1 = 2.7878%

FR Year 5 = DF4/DF5 – 1 = 3.3223%

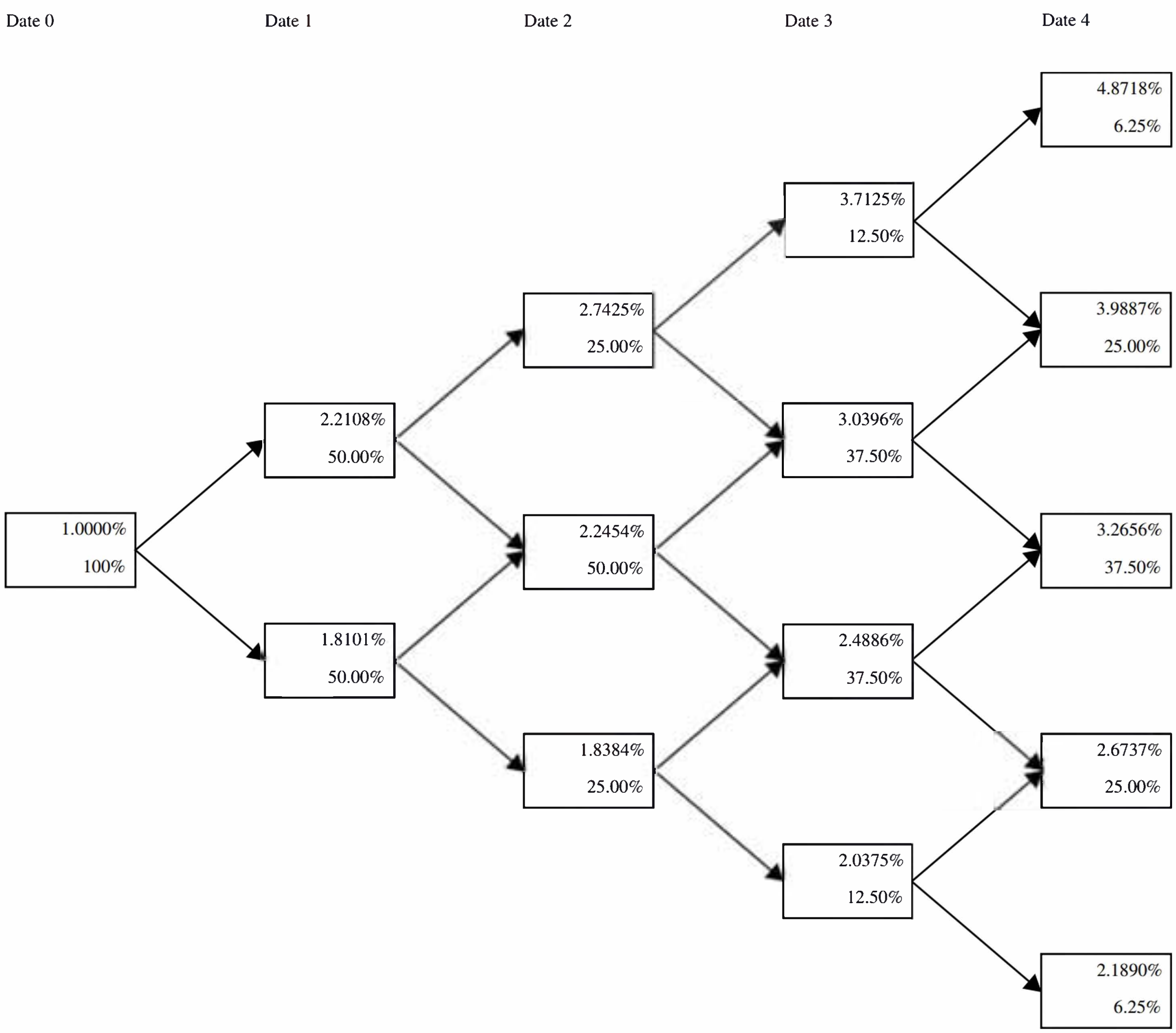
Table 4-4: Par Curve for Annual Payment Benchmark Government Bonds, Spot Rates, Discount Factors, and Forward Rates: Forward Rate

Maturity	Coupon Rate	Price	Discount Factor	Spot Rate	Forward Rate
1	1.00%	100	0.990099	1.0000%	
2	1.50%	100	0.970590	1.5038%	2.0101%
3	1.75%	100	0.949079	1.7574%	2.2665%
4	2.00%	100	0.923338	2.0140%	2.7878%
5	2.25%	100	0.893648	2.2743%	3.3223%

Using the methodology detailed in the "Arbitrage-Free Valuation Framework" reading, we can build a binomial interest rate tree based on the one-year forward rates produced above using an assumption for future interest rate volatility of 10%. The binomial tree based on these assumptions is shown in Figure 4-1.

The figure is structured so that each respective interest rate (either up or down) is shown above the probability of reaching that interest rate. So, for example, we can see that the one-year interest rate of 1.00% produced above has an equal chance of moving up to 2.2108% or down to 1.8101%. This process carries forward for the following years, out to year 5.

Figure 4-1: One-Year Binomial Interest Rate Tree for 10% Volatility



We can demonstrate that this is an arbitrage-free binomial interest rate tree by calculating the date 0 value of a 2.25% annual payment bond that is priced at par value per Tables 4-1–4-4. The results can be seen in Figure 4-2, where coupon and principal payments are shown to the right of each forward rate.

To prove that the binomial tree in Figure 4-1 is arbitrage-free, all we need to do is work backward from the right of Figure 4-2, discounting principal and interest payments to derive the bond's value.

For example, we solve date 4 bond values by discounting the principal and interest payments made at date 5 (102.25) by the associated interest rates in date 4.

102.25/1.048718 = 97.5

102.25/1.039887 = 98.3280

102.25/1.032656 = 99.0165

102.25/1.026737 = 99.5874

102.25/1.02189 = 100.0597

Then we discount the values of the bonds in date 4 (each value is equally weighted) that correspond to date 3 nodes by their respective interest rates. This process is repeated until we work our way back to date 0 to derive a bond value of 100.

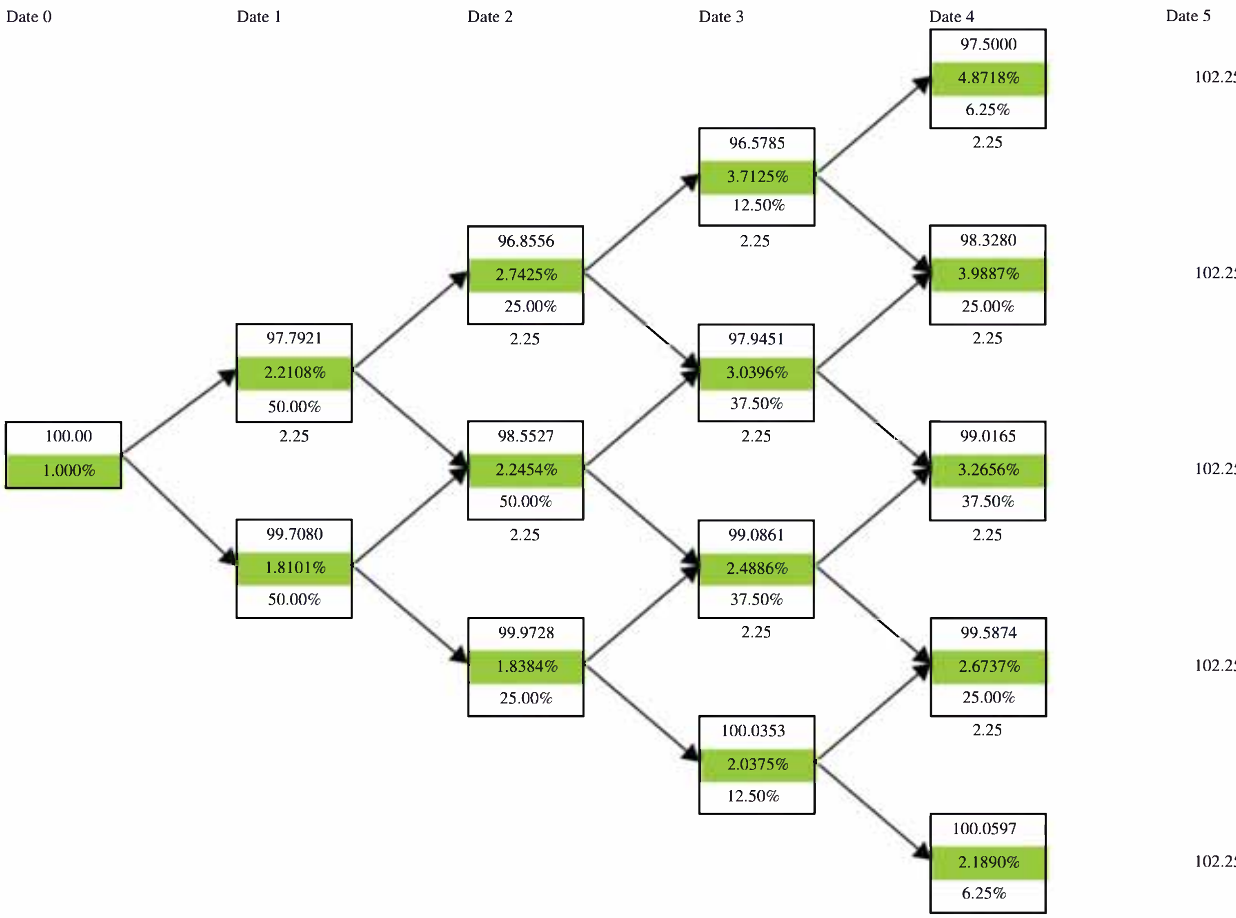
[(97.5 × 0.50) + (98.3280 × 0.50) + 2.25]/1.037125 = 96.5785

[(98.3280 × 0.50) + (99.0165 × 0.50) + 2.25]/1.030396 = 97.9451

[(99.0165 × 0.50) + (99.5874 × 0.50) + 2.25]/1.024886 = 99.0861

[(99.5784 × 0.50) + (100.0597 × 0.50) + 2.25]/1.020375 = 100.0353

Figure 4-2: Valuation of a 2.25% Annual Payment Government Bond

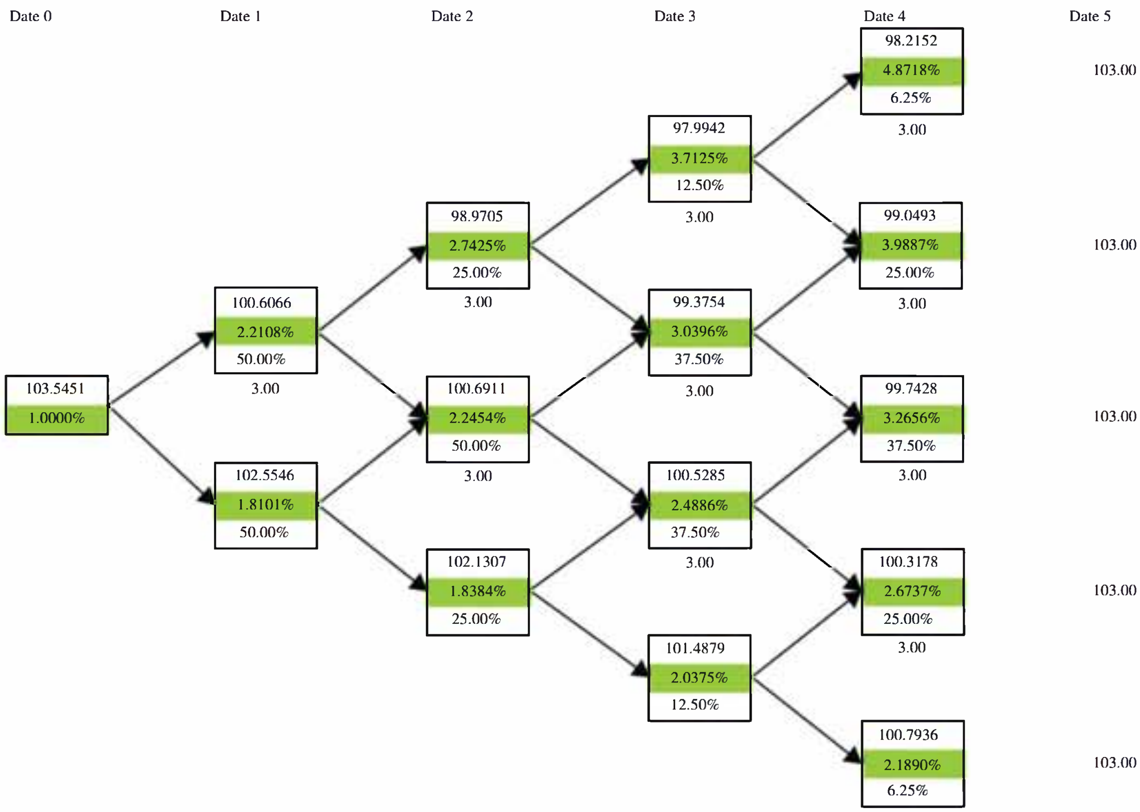


Now let's move on to working with corporate bonds that have expected default rates. In this example, we assume a five-year, 3.0% annual payment corporate bond with a hazard rate of 1.50%, a recovery rate of 30%, and the same interest rate assumptions used in Figure 4-2 to produce the bond's value, assuming no default (VND). The binomial tree for this bond is shown in Figure 4-3.

To assess the fair value of this bond, we need to take three steps:

1. Determine the value for the corporate bond assuming no default (VND).
2. Calculate the credit valuation adjustment (CVA).
3. Subtract the CVA from the fair value of the bond.

Figure 4-3: Value of a 3.0% Annual Payment Corporate Bond, Assuming No Default (VND)



As we can see from Figure 4-3, the value of the bond assuming no default is 103.5451. We could also have derived this value by discounting cash flows using the discount factors in Tables 4-1–4-4.

Cash Flow		Discount Factor		
3.0	×	0.9901	=	2.9703
3.0	×	0.9706	=	2.9118
3.0	×	0.9491	=	2.8472
3.0	×	0.9233	=	2.7700
103.0	×	0.8936	=	92.0457
				103.5451

Table 4-5 shows that the CVA for this bond is 4.9835, which is based on expected exposures that are derived from the binomial tree. For example, the expected exposure for date 1 is:

Date 1 exposure = 0.50 × 100.6066 + 0.50 × 102.5466 + 3.0 = 104.5806

This process can be repeated for dates 2–4 to derive expected exposures, with the expected exposure at year 5 being 103 (or par value plus interest). To derive the rest of the information to obtain the cumulative CVA, assume the discount factors from Tables 4-1–4-4 and follow the process discussed earlier to get LGD, POD, and CVA per year (Table 4-5).

Table 4-5: Credit Valuation Adjustment (CVA) for the 3.0% Annual Payment Corporate Bond

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	104.5806	73.2064	1.5000%	0.990099	1.0872
2	103.6208	72.5346	1.4775%	0.970590	1.0402
3	102.8992	72.0295	1.4553%	0.949079	0.9949
4	102.6834	71.8784	1.4335%	0.923338	0.9514
5	103.0000	72.1000	1.4120%	0.893648	0.9098
			7.2783%	CVA =	4.9835

Based on this CVA, the fair value of the bond is 103.5451 – 4.9835 = 98.5616. With this value in hand, we can then determine the bond's YTM using a net present value/IRR equation (Table 4-6).

Table 4-6. Net Present Value/IRR Equation

$$98.5616 = \frac{3.0}{(1+\text{YTM})} + \frac{3.0}{(1+\text{YTM})^2} + \frac{3.0}{(1+\text{YTM})^3} + \frac{3.0}{(1+\text{YTM})^4} + \frac{103}{(1+\text{YTM})^5}$$

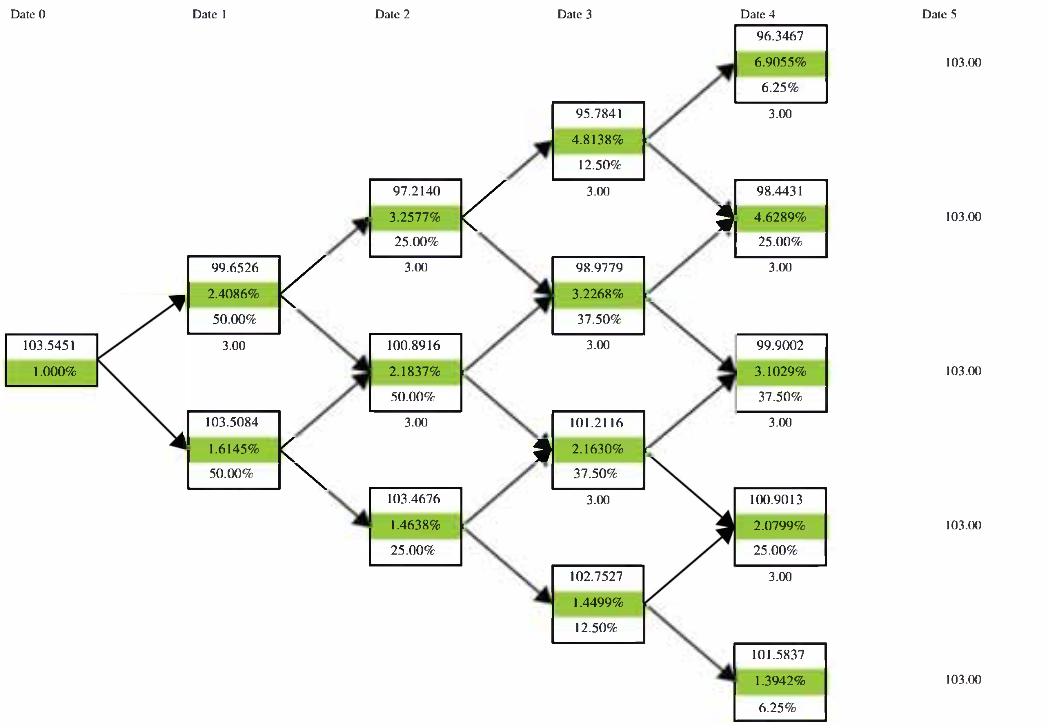
In this instance, the YTM is 3.3169%. Therefore, given that the five-year government bond yield is 2.25%, the credit spread for this bond is 3.3171% – 2.25% = 1.0669%.

The Effects of Higher Interest Rate Volatility

Changing the assumed level of interest volatility will have no effect on the fair value of a bond unless credit risk is involved or it has embedded put or call options. To demonstrate this, Figure 4-4 and Table 4-7 show a no-arbitrage binomial interest rate tree and CVA calculations for the same 3.0% annual pay corporate bond (without embedded options) used in the preceding calculations, but with 20% interest rate volatility.

Because interest volatility assumptions are different between Figures 4-3 and 4-4, the range of the forward rates is wider. For example, date 3 interest rates in Figure 4-3 range from 2.0375% to 3.7125% and in Figure 4-4, they range from 1.4499% to 4.8137%. However, as you can see from Figure 4-4, changing interest rate volatility assumptions has no effect on the value of the bond assuming no default (VND).

Figure 4-4: VND Calculation for the 3.0% Corporate Bond Assuming No Default and 20% Volatility



As mentioned earlier, the only time changing interest rate volatility assumptions will affect the price of a bond is if the bond has embedded options or when credit risk is involved. In Table 4-7 we demonstrate the effects of higher interest rate volatility on bonds with credit risk in that a slightly lower CVA is the result of changes in expected exposures to default loss due to differing interest rate assumptions. As a result of this new lower CVA, the value of the bond goes up slightly to 98.5619 (103.5451 – 4.9832) from 98.5616.

Table 4-7. CVA Calculation for the 3.0% Corporate Bond Assuming 20% Volatility

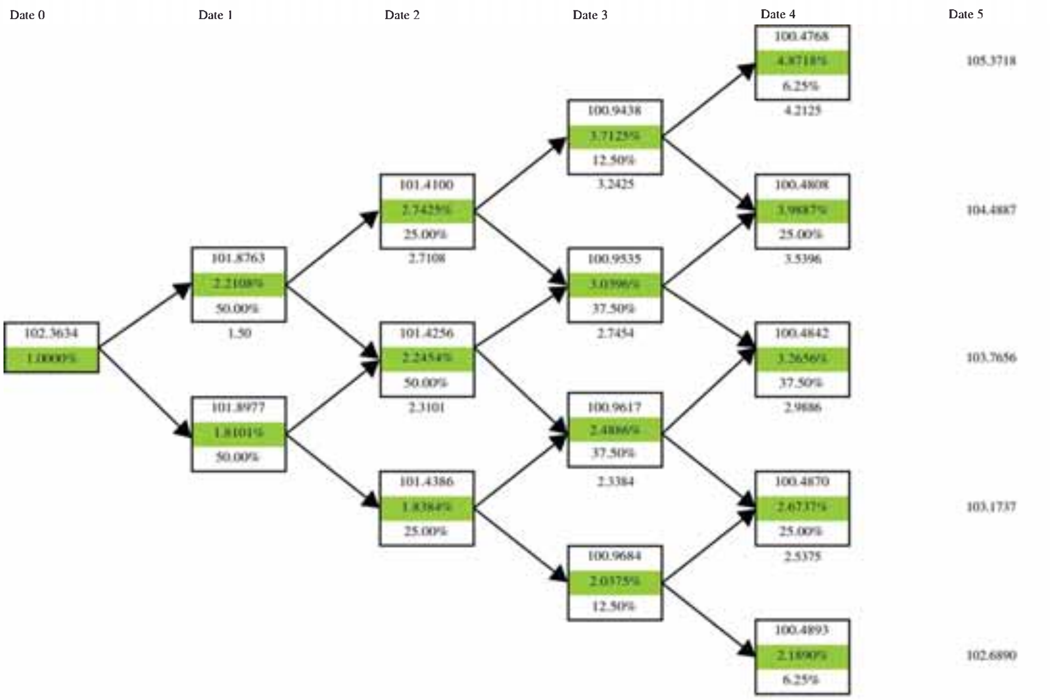
Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	104.5805	73.2064	1.5000%	0.990099	1.0872
2	103.6162	72.5313	1.4775%	0.970590	1.0401
3	102.8882	72.0217	1.4553%	0.949079	0.9948
4	102.6694	71.8686	1.4335%	0.923338	0.9513
5	103.0000	72.1000	1.4120%	0.893648	0.9098
			7.2783%	CVA=	4.9832

Floating Rate Notes

We can also apply the arbitrage-free valuation framework to a risky floating-rate note. To demonstrate, consider a five-year "floater" that pays 0.50% ("quoted margin") over the benchmark with an interest rate volatility assumption of 10%. Figure 4-5 shows that the

VND for the floater is 102.3634. When dealing with a floater, note that interest payments are made in arrears, which means that rates are set at the beginning of a period and paid at the end of a period. This is why interest payments are set to the right, depending on the realized rate in the binomial tree. For example, the interest payment in date 1 is 1.50% (or 1.00% + 0.50%) and the interest payment on date 5 is 103.7656 if the one-year rate on date 4 is 3.2656%.

Figure 4-5: Value of a Floating-Rate Note Paying the Benchmark Rate Plus 0.50% Assuming No Default and 10% Volatility



The corresponding calculations to derive the floating rate note's CVA are shown in Table 4-8. For this example of deriving the CVA, we demonstrate how to apply different hazard rates over various years should the credit risk of the issuer increase. For years 1–3, the hazard rate is 0.50% and for years 4–5 the hazard rate is 1.00%; the recovery rate is assumed to be 20% and 10% for those two periods.

Table 4-8. CVA Calculation for the Value of a Floating-Rate Note Paying the Benchmark Plus 0.50%

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	103.3870	82.7096	0.5000%	0.990099	0.4095
2	103.9354	83.1483	0.4975%	0.970590	0.4015
3	103.7251	82.9801	0.4950%	0.949079	0.3898
4	103.7757	93.3981	0.9851%	0.923338	0.8495
5	103.8315	93.4483	0.9752%	0.893648	0.8144
			3.4528%	CVA =	2.8647

When deriving the CVA for a floating-rate note, we need to take into account that the expected exposure recognizes that bond values on a given date follow the probabilities of reaching certain rates and that interest payments use probabilities from prior dates.

For example, the expected exposure for date 4 is 103.7757.

Discounted bond values at date 4:

$$= (0.0625 \times 100.4768) + (0.25 \times 100.4808) + (0.375 \times 100.4842) + (0.25 \times 100.4870) + (0.0625 \times 100.4893) = 100.4839$$

Discounted interest values at date 4:

$$= (0.125 \times 4.2125) + (0.375 \times 3.5396) + (0.375 \times 2.9886) + (0.125 \times 2.5375) = 3.2918$$

$$\text{Expected exposure} = 100.4839 + 3.2918 = 103.7757$$

We also need to take into account the differing hazard rates (POD) over the five-year period.

$$\text{POD date 2} = 0.50\% \times (100\% - 0.50\%) = 0.4975\%$$

$$\text{POD date 3} = 0.50\% \times (100\% - 0.50\%)^2 = 0.4950\%$$

This means the probability of survival into the fourth year = $(100\% - 0.50\%)^3 = 98.5075\%$

Therefore:

$$\text{POD date 4} = 98.5075\% \times 1.0\% = 0.9851\%$$

$$\text{POD date 5} = 1.0\% \times (100\% - 1.0\%) \times (100\% - 0.50\%)^3 = 0.9752\%$$

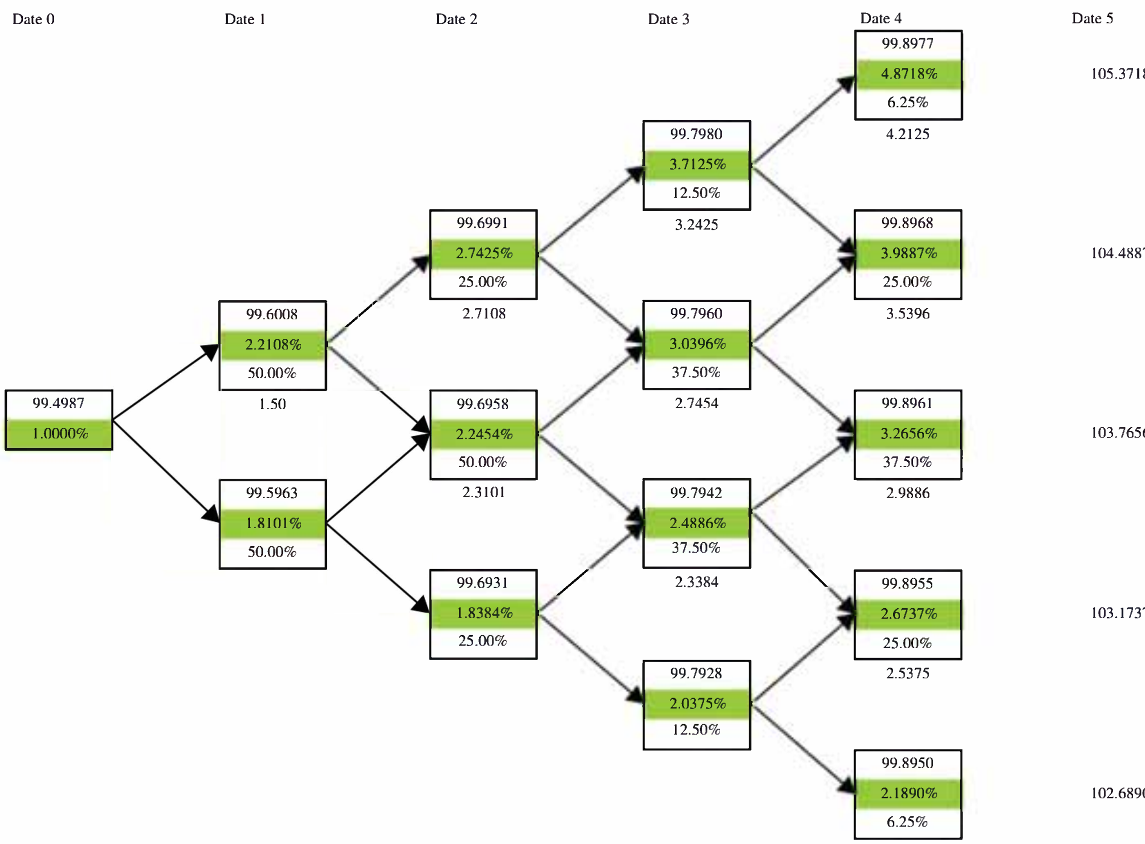
With a CVA for the floating rate note of 2.8647, the fair value is 99.4987 (102.3634 – 2.8647). The discount margin (DM) for a floating-rate bond is a yield measure, and given that the bond is priced below par value, the discount margin (DM) must be higher than the quoted floating rate margin of 0.50%.

In terms of determining the DM for this floating-rate note, we need to use a trial-and-error search, or use Excel Solver. To do this, we add a trial DM to the benchmark rates used to calculate values for each node in the binomial tree. Thereafter, the trial DM margin is adjusted until the date 0 value of the note equals the previously calculated price of 99.4987. This work is shown in Figure 4-6 and results in a DM of 0.6079%.

We can apply this DM to demonstrate that it results in the same value for the floating-rate note by using the VND and CVA models and adding the DM to the interest rates shown in the binomial tree at date 1:

$$[(0.50 \times 99.6008) + (0.50 \times 99.5963) + 1.50]/(1 + 1.00\% + 0.6079\%) = 99.4987$$

Figure 4-6: The Discount Margin for the Floating-Rate Note Paying the Benchmark Rate Plus 0.50% Assuming 10% Volatility

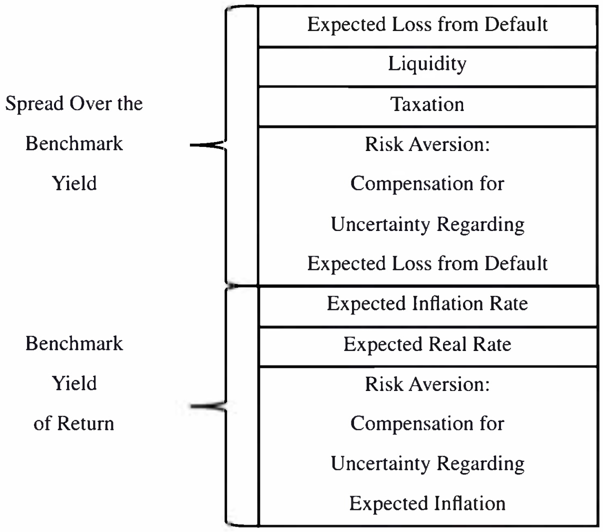


LESSON 5: INTERPRETING CHANGES IN CREDIT SPREADS

LOS 35f: Interpret changes in a credit spread. Vol 5, pp 234–238

Credit spreads between corporate and risk-free (benchmark) bonds change constantly and analysts must be able to explain why. Figure 5-1 shows the factors of corporate and benchmark bond yields for which investors require compensation. For example, risk-free bond investors need compensation for expected inflation, its uncertainty, and real interest rates. On top of benchmark yield concerns, corporate bond investors require compensation for expected losses (credit risk) and the uncertainty thereof, as well as liquidity and tax considerations that pertain to the specific issuer of the bond. The greater these concerns become for corporate bond investors, the greater the yield they will require to invest.

Figure 5-1: Components of a Corporate Bond Yield



Source: Smith 2017.

To further examine the connections between default probability, recovery rate, and credit spreads, we can go back to the results of the arbitrage-free framework for the five-year, 3.0% annual payment corporate bond that produced a value of 103.5451. We can also use the credit spread of 0.50% shown earlier in Lesson 2, Table 2-4 for a AAA-rated bond and an assumed recovery rate of 30%. Using a trial-and-error search and the credit spread of 0.50%, we find the annual probability of default is 0.71% (hazard rate). The results of these calculations are shown in Table 5-1, where the default probabilities and contributions to CVA differ from the previous example shown in Lesson 4, Table 4-5.

Table 5-1: CVA Calculations for the 3.00% Corporate Bond, Given a Default Probability of 0.71% and a Recovery Rate of 30%

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	104.5806	73.2064	0.7100%	0.990099	0.5146
2	103.6208	72.5346	0.7050%	0.970590	0.4963
3	102.8992	72.0295	0.7000%	0.949079	0.4785
4	102.6834	71.8784	0.6950%	0.923338	0.4612
5	103.0000	72.1000	0.6900%	0.893648	0.4446
			3.4999%	CVA =	2.3953

Based on this new CVA for the five-year, AAA-rated 3.0% corporate bond, we can determine its new fair value and credit spread.

$$\text{Fair value} = 103.5451 - 2.3953 = 101.1498$$

The yield to maturity (YTM) for the corporate bond is 2.7507% based on the use of an IRR/NPV equation.

$$101.1499 = \frac{3.0}{(1+\text{YTM})} + \frac{3.0}{(1+\text{YTM})^2} + \frac{3.0}{(1+\text{YTM})^3} + \frac{3.0}{(1+\text{YTM})^4} + \frac{103}{(1+\text{YTM})^5}$$

Therefore, the credit spread is the bond's YTM of 2.7507%, less the five-year par yield for the government bond, shown in Table 5-2, of 2.25%:

$$\text{Credit spread} = 2.7507\% - 2.25\% = 0.5007\%$$

In theory, we can conclude that the CVA and its resulting credit spread capture the credit risk for this corporate bond. However, the calculated credit spread is based entirely on credit risk, whereas in practice, the aforementioned risk factors in Figure 5-1 would all be incorporated into a bond's credit spread.

We can repeat this exercise for the remaining credit spreads from Lesson 2, Table 2-4 to determine the fair value and credit spread for other corporate bonds. We use a trial-and-error search to get the initial POD and cumulative POD for each assumed spread. These data are shown in Table 5-2.

Table 5-2: Default Probabilities Consistent with Given Credit Ratings, Spreads, and 30% Recovery Rate

Credit Rating	Credit Spread	Annual Default Probability	Cumulative Default Probability
AAA	0.50%	0.71%	3.50%
AA	0.70%	0.99%	4.85%
A	1.00%	1.40%	6.81%
BBB	1.75%	2.45%	11.66%
BB	2.75%	3.84%	17.78%
B	4.00%	5.58%	24.96%
CCC, CC, C	5.50%	7.68%	32.98%
D	7.00%	9.78%	40.22%

Using the credit risk model, we can examine the relationship between recovery rates and credit spreads. For example, we can see that the five-year, 3.0% A-rated annual payment corporate bond has a 1.0% credit spread and an initial POD (hazard rate) of 1.40% based on a 30% recovery rate. However, what would happen if we changed the recovery rate to 20%? Table 5-3 shows the credit risk table for this assumption.

Table 5-3: CVA Calculation for the 3.00% A-Rated Corporate Bond, Given a Default Probability of 1.40% and a Recovery Rate of 20%

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	104.5806	83.6645	1.4000%	0.990099	1.1597
2	103.6208	82.8967	1.3804%	0.970590	1.1107
3	102.8992	82.3194	1.3611%	0.949079	1.0634
4	102.6834	82.1467	1.3420%	0.923338	1.0179
5	103.0000	82.4000	1.3232%	0.893648	0.9744
			6.8067%	CVA =	5.3260

Reducing the recovery rate from 30% to 20% impacts the LGD and CVA for each year. Under this new assumption, the CVA is 5.3260, making the fair value of the bond 98.2191 (103.5451 – 5.3260).

The yield to maturity (YTM) for the corporate bond is 3.3932%, based on the use of an IRR/NPV equation:

$$98.2191 = \frac{3.0}{(1 + \text{YTM})} + \frac{3.0}{(1+\text{YTM})^2} + \frac{3.0}{(1+\text{YTM})^3} + \frac{3.0}{(1+\text{YTM})^4} + \frac{103}{(1+\text{YTM})^5}$$

Therefore, the credit spread is the bond's YTM of 3.3932% less the five-year par yield for the government bond, shown in Table 5-2, of 2.25%:

$$\text{Credit spread} = 3.3932\% - 2.25\% = 1.1432\%$$

This example of changing expected recovery rates explains how credit ratings agencies may notch credit ratings lower for certain debt issues within a company's capital stack. For example, an issuer may be A-rated based on a recovery rate of 30% for its senior unsecured debt. However, an analyst may review other bonds from an issuer such as its subordinate debt and determine that the recovery rate should only be 20%. This would lower the fair value of the subordinated debt relative to the senior secured debt, raise its credit spread, and warrant assigning a lower rating than A, such as A– or BBB+.

LESSON 6: THE TERM STRUCTURE OF CREDIT SPREADS

LOS 35g: Explain the determinants of the term structure of credit spreads and interpret a term structure of credit spreads. Vol 5, pp 240–247

Similar to a government bond yield curve, a credit-spread curve can be created for a single issuer of debt or a set of issuers across industries/sectors and credit ratings. This is referred to as the term structure of credit spreads and is a useful tool for issuers, underwriters, and investors in assessing the risk/return tradeoff associated with bond offerings. For example, issuers can use credit spreads when working with underwriters to determine terms and pricing for new debt or value outstanding debt to make tender offers. Also, portfolio managers can use credit-spread curves to make bids on new debt or value and trade existing debt.

There are several factors that determine the term structure of credit spreads:

1. **Credit quality:**
   a. Investment-grade bonds that have high ratings and low spreads can migrate in only one direction (downward) in terms of credit quality. This means that their term structure of credit spreads is usually flat in the short term and upward-sloping over the long term to account for greater probabilities of default over time that can occur due to conditions such as economic uncertainty, industry competition, and failing company fundamentals.
   b. High-yield bonds with low ratings are far more sensitive to economic and credit conditions. Generally speaking, poorly rated bonds have steep upward-sloping credit curves due to increasing chances of default over time. This can also occur during times of economic strength when credit spreads are low near term, but are expected to rise over longer periods as the economy cycles from expansion into recession. However, during times of economic weakness and high credit spreads, high-yield bonds can face an inverted credit curve, as the economy is expected to rebound in coming years. Additionally, when investors expect improving company fundamentals, high-yield bonds can face inverted credit curves, which happens when investors perceive improving company fundamentals (e.g., when a firm is bought out and investors expect the new owners to improve operations and profitability).
2. **Macroeconomic conditions:** The pace of economic growth heavily influences perceived credit risk. For example, as economic conditions strengthen, risk-free bond yields go up, while credit spreads go down (or tighten); the inverse is true during times of economic weakness.
3. **Market supply and demand:** Corporate bond markets are nowhere near as liquid as government debt markets, with most existing bonds not even trading on a regular basis. However, new bond issues represent the majority of trading volume and significantly influence the movement of credit spreads and the volatility thereof.
4. **Microeconomic conditions:** Industry-related issues such as peer-relative financial ratios, cash flows, leverage, and profitability can materially affect the structure of credit spreads for an issuer. However, company-specific fundamental analysis that uses forward-looking structural models that incorporate equity market valuations, equity volatility, and balance sheet information will also affect issuer credit spreads. The more volatility is present in these microeconomic factors, the steeper the credit curve becomes, and vice versa for declining volatility.

When analyzing a term structure of credit spreads, it is very important to choose an appropriate risk-free benchmark. Generally, the best choice is to use a frequently traded government bond that matches the maturity of the associated corporate bond. However, finding matching durations between corporate and the most liquid (on-the-run) government bonds can be difficult. In such instances, analysts can interpolate an appropriate yield based on two government bonds with the closest durations to the corporate bond. Analysts can also use swap curves because they often provide greater liquidity for less liquid (off-the-run) maturities.

Another issue that needs consideration is that term structure analysis should be based on bonds with similar credit characteristics, which usually includes senior, unsecured debt. This means that bonds with embedded options, lien provisions, or other unique structural issues should be excluded from the analysis.

We also need to be aware of how significantly default expectations can affect the shape of the term structure of credit spreads. We can do this by building on the data used in Lesson 1, Table 1-16 for a zero-coupon corporate bond. By keeping the recovery rate at 30% and changing the POD from 6.50% to 7.50%, we can see the credit spread goes up from 4.8422% to 5.5956% as shown in Table 6-1.

Table 6-1: Raising the Default Probability of the Four-Year, Zero-Coupon Corporate Bond

Year	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Discount Factor	PV of Expected Loss
1	86.3838	25.9151	60.4686	7.5000%	92.5000%	4.5351	0.95238	4.31919
2	90.7029	27.2109	63.4921	6.9375%	85.5625%	4.4048	0.90703	3.99525
3	95.2381	28.5714	66.6667	6.4172%	79.1453%	4.2781	0.86384	3.69561
4	100.0000	30.0000	70.0000	5.9359%	73.2094%	4.1551	0.82270	3.41843
				26.7906%			CVA =	15.4285

Recall that the price of the similar risk-free bond was 82.2702 and the new CVA is 15.4285. This means that the fair value of the corporate bond is now 66.8417 (82.2702 – 15.4285) and the yield to maturity (YTM) is calculated as follows:

$$100/(1 + \text{YTM})^4 = 66.8417$$

$$\text{YTM} = 10.5956\%$$

It is also important to note what can happen when investors seek greater compensation for rising default risk over longer periods of time. We illustrate this concept in Tables 6-2–6-4, where the credit spread goes up from 2.2203% to 2.8080%, and then to 3.3961%, when assuming increasing default probabilities.

- Years 1–4: 3.0%
- Years 5–7: 5.0%
- Years 8–10: 7.0%

Table 6-2: Increasing the Default Probability for Longer Times to Maturity

Year	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Discount Factor	PV of Expected Loss
1	86.3838	25.9151	60.4686	3.0000%	97.0000%	1.8141	0.9524	1.7277
2	90.7029	27.2109	63.4921	2.9100%	94.0900%	1.8476	0.9070	1.6758
3	95.2381	28.5714	66.6667	2.8227%	91.2673%	1.8818	0.8638	1.6256
4	100.0000	30.0000	70.0000	2.7380%	88.5293%	1.9166	0.8227	1.5768
				11.4707%			CVA =	6.6059

Fair value of the four-year risk-free bond = 100 × 0.822705 = 82.2705

CVA = 6.6059

Fair value of the corporate bond = 82.2705 – 6.6059 = 75.6644

$100/(1 + \text{YTM})^4 = 75.6644$

YTM = 7.2203%

Table 6-3: Increasing the Default Probability for Longer Times to Maturity

Year	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Discount Factor	PV of Expected Loss
1	74.6215	22.3865	52.2351	3.0000%	97.0000%	1.5671	0.95238	1.49243
2	78.3526	23.5058	54.8468	2.9100%	94.0900%	1.5960	0.90703	1.44766
3	82.2702	24.6811	57.5892	2.8227%	91.2673%	1.6256	8.86384	1.40423
4	86.3838	25.9151	60.4686	2.7380%	88.5293%	1.6556	0.82270	1.36210
5	90.7029	27.2109	63.4921	4.4265%	84.1028%	2.8105	0.78353	2.20206
6	95.2381	28.5714	66.6667	4.2051%	79.8977%	2.8034	0.74622	2.09196
7	100.0000	30.0000	70.0000	3.9949%	75.9028%	2.7964	0.71068	1.98736
				24.0972%			CVA =	11.9878

Credit spread = 7.2203% – 5.0% = 2.2203%

Fair value of the corporate bond = 71.0681 – 11.9878 = 59.0803

$100/(1 + \text{YTM})^7 = 59.0803$

YTM = 7.8080%

Credit spread = 7.8080% – 5.0% = 2.8080%

Table 6-4: Increasing the Default Probability for Longer Times to Maturity

Year	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Discount Factor	PV of Expected Loss
1	64.4609	19.3383	45.1226	3.0000%	97.0000%	1.3537	0.95238	1.28922
2	67.6839	20.3052	47.3788	2.9100%	94.0900%	1.3787	0.90703	1.25054
3	71.0681	21.3204	49.7477	2.8227%	91.2673%	1.4042	8.86384	1.21303
4	74.6215	22.3865	52.2351	2.7380%	88.5293%	1.4302	0.82270	1.17663
5	78.3526	23.5058	54.8468	4.4265%	84.1028%	2.4278	0.78353	1.90223
6	82.2702	24.6811	57.5892	4.2051%	79.8977%	2.4217	0.74622	1.80711
7	86.3838	25.9151	60.4686	3.9949%	75.9028%	2.4157	0.71068	1.71676
8	90.7029	27.2109	63.4921	5.3132%	70.5896%	3.3735	0.67684	2.28329
9	95.2381	28.5714	66.6667	4.9413%	65.6483%	3.2942	0.64461	2.12346
10	100.0000	30.0000	70.0000	4.5954%	61.0529%	3.2168	0.61391	1.97482
				38.9471%			CVA=	16.7371

Fair value of the 10-year risk-free bond = $100 \times 0.61391 = 61.3913$

CVA = 16.7371

Fair value of the corporate bond = $61.3913 - 16.7371 = 44.6542$

$100/(1 + \text{YTM})^{10} = 44.6542$

YTM = 8.3961%

Credit spread = 8.3961% – 5.0% = 3.3961%

Understanding credit spreads and their future direction is extremely important for investors who want to profit from constantly changing credit conditions. Many fixed-income managers produce the majority of their alpha by identifying times when the market has mispriced credit spreads. For example, if the market has priced a recession into credit spreads that a manager believes are unwarranted, they could profit by buying when spreads are wide and selling when they narrow if a recession fails to occur, and vice versa when markets expect too much economic strength. Additionally, managers can profit by understanding credit spreads as they relate to improving or declining company fundamentals. In these instances, managers buy bonds with high spreads from companies that are out of favor, and then profit from narrowing spreads as conditions improve, and vice versa when spreads are low and managers see worsening company fundamentals.

LESSON 7: CREDIT ANALYSIS FOR SECURITIZED DEBT

LOS 35h: Compare the credit analysis required for securitized debt to the credit analysis of corporate debt. Vol 5, pp 247–251

Securitized debt involves using a specific set of assets or receivables to collateralize debt, such as home mortgages, business loans, auto loans, and credit card receivables. This is separate from general debt obligations that are based on the balance sheet of a company. Lenders can materially benefit from the securitization of assets because they can package and sell debt into public markets, which allows them to reduce their balance sheet risk, free up capital to originate more loans and collect fees, and reduce the need to maintain regulatory capital requirements. Securitization of assets on a standalone basis also benefits companies because it tends to have lower financing costs relative to general debt obligations.

However, from an investor perspective, securitized debt is more complex than general debt obligations and requires fundamentally different credit analysis, as discussed next. This is why investors generally require a higher rate of return for securitized versus general debt obligations, even if they have similar credit ratings.

Securitized Debt Credit Analysis Considerations

Homogeneity refers to individual debt obligations within the asset pool of a securitized instrument being similar in nature (i.e., common underwriting criteria for borrowers), which allows credit analysts to draw generalized conclusions about the nature of loans. A good example of this is securitized auto loans, where people need minimum credit scores to get loans. Conversely, heterogeneity refers to underlying debt obligations not having common characteristics (i.e., differing underwriting criteria), thereby requiring scrutiny on a loan-by-loan basis. Examples of these types of securitized debt can include business project financing or real estate loans.

Granularity refers to the number of individual debt obligations within the asset pool of a securitized instrument. When an asset pool is highly granular, it may have hundreds of underlying creditors, which means credit analysts will likely base their analysis on summary statistics rather than looking at loans on an individual basis. Conversely, when a pool of assets is nongranular, credit analysts would need to look at individual loans to analyze a securitized instrument. What is important to understand is that the combination of homogeneity/heterogeneity and granularity/nongranularity drives the approach to credit analysis.

In general, when securitized instruments are comprised of homogeneous and highly granular loans, analysts look to summary statistics to conduct credit analysis. On the other hand, when loan pools are heterogeneous and nongranular, credit analysis needs to be performed on a loan-by-loan basis.

Origination and servicing of assets over the life of a loan pool is another major issue for credit analysis. Because investors are exposed to operational and counterparty risk, they must rely on the originator/servicer to establish and enforce eligibility criteria, maintain documentation, facilitate repayments, and manage delinquencies. For example, a servicer may be required to repossess an automobile should a borrower fail to make payments or enforce financial penalties and engage in collection efforts for credit card borrowers. Servicers may even be responsible for evicting and replacing tenants from commercial properties to maintain cash flow. This is why investors need to examine the creditworthiness of servicers and their track record in servicing loans over time during credit cycles.

The **structure** of a securitized instrument is another issue investors must consider. This includes analysis of originator and the obligor, which is usually a special-purpose entity (SPE) that acquires assets and issues securitized instruments, and any structural enhancements (i.e., overcollateralization and tiering the priority of claims or tranching). Investors also need to determine whether the originator can separate itself from the risk of bankruptcy in the securitized instrument through a "true sale" of assets to the SPE.

Credit enhancements are another major structural consideration when evaluating credit risk. These enhancements can include payout or performance triggers in the event of adverse credit conditions. For example, if borrowers fail to make payments to the pool of loans, this can trigger the amortization of the security. Another credit enhancement is the "excess spread," or the additional return that is built into the security to protect investors against expected or historic losses associated with the asset pool. Securitized debt can also include senior and subordinated tranches of debt, which means that investors that hold senior debt tranches have a return cushion in place, because subordinated tranches are designed to absorb initial losses.

READING 36: CREDIT DEFAULT SWAPS

LESSON 1: BASIC DEFINITIONS AND CONCEPTS

Introduction

A **credit derivative** is an instrument in which the underlying is a measure of a borrower's credit quality. Credit derivatives include total return swaps, credit spread options, credit-linked notes, and credit default swaps (CDS). CDS have emerged as the primary types of credit derivatives and are the focus of this reading.

LOS 36a: Describe credit default swaps (CDS), single-name and index CDS, and the parameters that define a given CDS product. Vol 5, pp 278–282

Credit Default Swaps: Basic Definitions and Concepts

A **credit default swap (CDS)** is a bilateral contract between two parties that transfers the credit risk embedded in a reference obligation from one party to another. It is essentially an insurance contract.

The **reference obligation** is the fixed-income security on which the protection is written (or whose credit risk is transferred). This is usually a bond, but can also be a loan in some cases.

The **protection buyer** makes a series of periodic payments (think of them as periodic insurance premium payments) to the protection seller during the term of the CDS. In return for this series of periodic premium payments (known as the **CDS fee** or **CDS spread**), the protection buyer obtains protection against default risk embedded in the reference obligation.

The **protection seller** earns the CDS spread over the term of the CDS in return for assuming the credit risk in the reference obligation. If a **credit event** (discussed later) occurs, the protection seller is obligated to compensate the protection buyer for **credit losses** by means of a specified **settlement procedure** (described later).

CDS are somewhat similar to put options. Put options give the option holder the right to sell (put) the underlying to the put writer if the underlying performs poorly relative to the strike price. The option holder is therefore compensated for poor performance. Similarly, CDS give the protection buyer compensation if the underlying (loan, bond, or bond portfolio) performs poorly. See Figure 1-1.

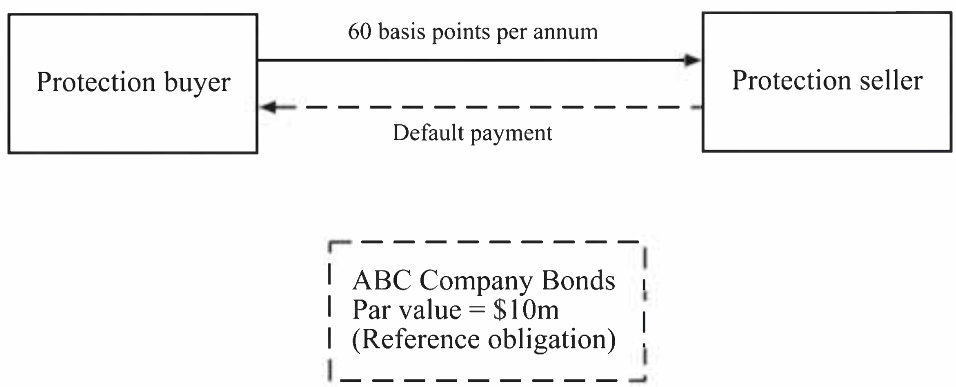
Simple Illustration of a CDS

To illustrate how CDSs work, assume that we purchase $10 million worth of 5-year bonds issued by ABC Company at par. In order to insulate our portfolio from ABC's credit risk, we enter a CDS on ABC Company as the protection buyer. This CDS has a notional amount of $10 million, a 5-year term, and a CDS premium of 60 bps (payable quarterly).

- If a credit event does not occur during the tenor of the swap, we will pay the swap counterparty a quarterly premium worth 0.006/4 × $10,000,000 = $15,000.
- In the case of a credit event, we would stop making premium payments and the CDS would be settled immediately.

- In a physical settlement, we (as the protection buyer) would:
  - Receive the notional amount ($10,000,000) from the swap counterparty (protection seller).
  - Deliver the ABC Company bonds that we hold (reference obligation) to the protection seller.
- In a cash settlement we would:
  - Receive a cash payment from the protection seller equal to the difference between the par value of the bonds and the post-default market value of the reference obligation.
  - For example, if ABC Company declared bankruptcy and the post-default value of the bonds were determined to be $4,500,000 in an auction (approximately 45% recovery rate of the par amount), we would receive a payment of 10,000,000 – 4,500,000 = $5,500,000.

Figure 1-1: Credit Default Swap



Types of CDS

> Note that the reference entity is not a counterparty in the swap contract, nor is its consent required for counterparties to enter into a CDS.

The **reference entity** is the entity issuing the **reference obligation** in a **single-name CDS** (where the CDS is on one borrower). The reference obligation is a particular instrument issued by the reference entity that serves as the designated instrument covered by the CDS. The designated instrument is usually a senior unsecured obligation (which can be referred to as **senior CDS**), but this instrument is not the only instrument covered by the CDS. Any debt issued by the reference entity that is **pari passu** (i.e., ranked equivalently in the priority of claims), or higher relative to the reference obligation is covered by the CDS. The payoff to the CDS is determined by the **cheapest-to-deliver** obligation, which is the instrument that can be purchased and delivered at the lowest cost, but has the same seniority as the reference obligation.

Example 1-1: Cheapest-to-Deliver Obligation

A company with several debt issues trading in the market undergoes a credit event. What is the cheapest-to-deliver obligation for a senior CDS contract?

A. An unsecured bond trading at 30% of par.
B. A 5-year senior unsecured bond trading at 40% of par.
C. A 2-year senior unsecured bond trading at 35% of par.

Solution: C

The cheapest-to-deliver, or lowest-priced, instrument is the 2-year senior unsecured bond as it is priced lower than the five-year senior unsecured bond. Bond A trades at a lower dollar price, but it is a **subordinated** unsecured bond, and therefore, does not qualify for delivery under a senior CDS.

Note that even if the protection buyer in the CDS actually held the 5-year bonds issued by the reference entity, her payoff on the CDS would still be based on the cheapest-to-deliver obligation (the 2-year bond on this case), not on the specific bonds that she holds.

Two other types of CDS are **index CDS** and **tranche CDS**.

An **index CDS** involves several reference entities (borrowers), which allows market participants to take positions on the credit of a portfolio of companies. The value of an index CDS is driven primarily by **credit correlations** (i.e., the extent to which the credit performance of companies covered by the index CDS are correlated). Generally speaking, the higher (lower) the correlation of default among index constituents, the higher (lower) the cost of obtaining credit protection on the index CDS.

A **tranche CDS** also covers a portfolio of borrowers, but only to a pre-specified amount of losses (similar to how tranches of asset-backed securities only cover a particular amount of losses).

Important Features of CDS Markets and Instruments

- The **International Swaps and Derivatives Association (ISDA)** publishes industry-supported conventions that facilitate the functioning of the global CDS market. Counterparties to a CDS contract typically agree that their contracts will follow ISDA specifications. Contract terms are specified in the **ISDA Master Agreement**, which both counterparties sign.
- Like any swap contract, a CDS also specifies a **notional principal**, which represents the size of the contract.
- The periodic premium paid by the protection buyer in a CDS pays to the protection seller is known as the **CDS spread**. The CDS spread is a return over LIBOR required to protect against credit risk and is sometimes also referred to as the **credit spread**.
- In recent years, **standard annual coupon rates** (basically standard credit spreads) have been established on CDS contracts. Previously, the rate was set at the credit spread so if a CDS required a credit spread (CDS spread) of 3% to compensate the seller for the credit risk inherent in the reference obligation, the protection buyer would make quarterly payments amounting to 3% annually. Now however, CDS rates are standardized (typically at 1% for CDS on an investment-grade company or index, and 5% for a CDS on a high-yield company or index).
    - Obviously, the standard rate may be too high or too low for a particular reference obligation, so the discrepancy is accounted for via an upfront payment known as the **upfront premium**.
    - The upfront premium is calculated by converting the difference between appropriate credit spread for the reference obligation and the standard rate on the CDS to a dollar present value amount.
        - If the standard rate is too low (to capture the credit risk), the protection buyer will pay the seller the upfront premium.

Note that the CDS coupon refers to the CDS spread, not to the coupon rate offered on the reference obligation. Do NOT confuse the two. We will revisit the upfront premium in more detail later in the reading.

    - If the standard rate is too high, the protection seller will pay the buyer an upfront premium.
- Regardless of which party makes the upfront payment, the value or price of the CDS can change over the contract term due to changes in the credit quality of the reference obligation. Consider a CDS with a 1% (standard) coupon rate that has an investment-grade company with a 1% credit spread as its reference entity (note that there would be no upfront premium in this case). The protection buyer agrees to pay 1% each year over the term of the CDS for protection against credit risk in the reference entity. Into the term of the CDS, if the reference entity experiences a decline in its credit quality, the protection buyer would benefit as it would still be paying 1% for coverage against a greater (than before) risk. Therefore, the value of the CDS to the protection buyer would increase. On the other hand, the protection seller would lose due to the decline in the reference entity's credit quality as it would still receiving 1% to cover a risk that now warrants a higher premium.

Note that the market value of a CDS captures all gains and losses from actual and perceived declines in the credit quality of the reference entity.

- When it comes to identifying counterparties to a CDS as the long and the short, things get confusing. Generally speaking, we tend to identify the buyer as the long and the seller as the short. However, in the CDS world, since the protection buyer promises to make a series of future payments, she is known as the short and the protection seller is known as the long.
    - The protection buyer, or short, benefits when things go badly (i.e., if the credit quality of the underlying instrument declines and CDS spreads increase).
    - The protection seller, or long, benefits when things go well (i.e., if the credit quality of the underlying instrument improves and CDS spreads decrease).

Note that the protection buyer can also be referred to as the party that is short on credit risk. She is also essentially short on the reference obligation.

LOS 36b: Describe credit events and settlement protocols with respect to CDS. Vol 5, pp 282–284

Credit and Succession Events

A **credit event** defines default by the reference entity. It is what triggers a payment from the protection seller on a CDS to the protection buyer. There are three types of credit events:

- **Bankruptcy**: A legal filing for temporary protection from creditors. In this period of respite, the company can either (1) talk to all types of creditors to come up with a **reorganization** plan to repay its loans or (2) go into **liquidation**, where the courts would determine payouts to various classes to debt.
- **Failure to pay**: When the borrower fails to make a scheduled principal or interest payment after a grace period, without a formal bankruptcy filing.
- **Restructuring**: This can refer to a number of possible events, including:
    - Reduction or deferral of principal or interest.
    - Change in seniority or priority of an obligation.
    - Change in the currency in which interest and principal repayments will be made.

  Note that in order to qualify as a credit event, the restructuring must be forced upon the company by its creditors (i.e., it must be involuntary). In the United States, restructuring is typically not a credit event as companies prefer to go down the bankruptcy route to reorganize.

Determination of whether there has been a credit event is made by a 15-member **Determinations Committee (DC)** within the ISDA. A supermajority vote of 12 members is required to declare a credit event.

The DC also plays a role in determining whether a **succession event** has occurred. A succession event occurs when there is a change in the corporate structure of the reference entity (e.g., divesture, spinoff, or merger) such that the ultimate obligor of the reference obligation becomes unclear. This question is obviously important for CDS counterparties, as the creditworthiness of the entity now (after the succession event) responsible for repaying the reference obligation becomes the primary driver of the value of the CDS. If a succession event is declared, the DC determines (based on contract law and the country's laws) who bears the ultimate responsibility for the debt in question.

Settlement Protocols

If the DC determines that a credit event has occurred, the counterparties in the CDS have the right, but not the obligation, to settle. Settlement typically occurs 30 days after declaration of a credit event, and can happen through a physical settlement or cash settlement.

- In a **physical settlement**, upon declaration of a credit event, the protection buyer delivers the underlying debt instrument to the protection seller. The par value of bonds delivered to the seller must equal the notional amount of the CDS. In return, the protection seller pays the buyer the par value of the debt (which is equal to the CDS notional amount).
- In a **cash settlement**, the idea is that the protection seller should pay the protection buyer an amount of cash equal to the difference between the notional amount and the current value of the bonds. Determining this amount is not an easy task as opinions can vary about how much money has actually been lost (in a default, a portion of the par value is generally recovered). See Example 1-2.
    - The percentage of the par value that is recovered is known as the **recovery rate**. This represents the percentage received by the protection buyer relative to the amount owed.
    - The **payout ratio** (which determines the amount that the protection seller must pay the protection buyer) is an estimate of the expected credit loss. The **payout amount** equals the payout ratio multiplied by the notional amount.

$$\text{Payout ratio} = 1 - \text{Recovery rate}$$

$$\text{Payout amount} = \text{Payout ratio} \times \text{Notional amount}$$

Note that actual recovery is a very long process, which can stretch way beyond the settlement date of the CDS. Therefore, to estimate an appropriate recovery rate, there is typically an auction for the defaulted bonds to determine their **post-default market value**. The protection seller then pays the protection buyer the difference between the par value (which is equal to the CDS notional amount) and the post-default market value of the bonds.

There can be a difference between the recovery rate determined through the auction and the actual recovery rate of the underlying bonds. This is an important consideration for the CDS protection buyer especially if it is long on the underlying bonds at the same time (as her position may not be perfectly hedged).

Example 1-2: Settlement Preference

XYZ Company files for bankruptcy, triggering payments on various CDS contracts written on its obligations. It has two series of senior unsecured bonds outstanding: Bond A trades at 20% of par, and Bond B trades at 30% of par.

Skyler owns $5 million of Bond A and owns $5 million of CDS protection.
Lydia owns $5 million of Bond B and owns $5 million of CDS protection.

1. Determine the recovery rate for both CDS contracts.
2. Explain whether Skyler would prefer to cash settle or physically settle her CDS contract or whether she is indifferent.
3. Explain whether Lydia would prefer to cash settle or physically settle her CDS contract or whether he is indifferent.

Solution

1. Bond A is the cheapest-to-deliver obligation (trading at 20% of par), so the recovery rate for both CDS contracts will be 20%.
2. Skyler would be indifferent. She can cash settle for a CDS payoff of €4 million [= (1 – 20%) × $5 million] and sell her bonds for $1 million (= 20% × $5 million), for a total payoff of $5 million. Alternatively, she can physically deliver all her bonds ($5 million face value) over to the protection seller in exchange for $5 million in cash.
3. Lydia would prefer a cash settlement because she holds Bond B, which holds more value (due to its higher recovery rate) than the cheapest-to-deliver obligation. She will receive a $4 million payout on her CDS contract (same as Skyler), but she can sell Bond B for $1.5 million (= 30% × $5 million), for a total payoff of $5.5 million. If she were to physically settle her contract, she would receive only $5 million, the face value of her bond holdings.

CDS Index Products

CDS indices allow participants to take a position on the credit risk of several companies simultaneously just like equity indices allow investors to gain exposure to stocks of several companies at the same time. CDS indices are all **equally-weighted**, so if there are 125 underlying reference entities, the settlement of one entity is based on $1/125^{th}$ of the notional.

A company called Markit has played a pivotal role in creating CDS indices. Markit updates the constituents of each CDS index every 6 months, but retains the old series as well. The series that is the latest one to be created is known as the **on-the-run series**, while older ones are known as **off-the-run series**. When an investor moves over from one series to the new one, the move is known as a **roll**. If an entity within an index defaults, it is removed from the index and settled as a single-name CDS (based on its pro-rata share in the index). The index continues to move forward with a smaller notional. See Example 1-3.

Example 1-3: Hedging and Exposure Using Index CDS

An investor sells $700 million of protection on the CDS IG index (which has 125 constituents). At the same time, she hedges her position on the credit risk of Company A (an index constituent) by purchasing $4 million of single-name CDS protection. Company A subsequently defaults.

1. What is the investor's net notional exposure to Company A?
2. What proportion of her exposure to Company A has she hedged?
3. What is the remaining notional on her index CDS trade?

Solution:

1. The investor has sold protection on \$5.6 million notional (= \$700 million/125) through the index CDS and has purchased protection on \$4 million notional through the single-name CDS. Her net notional exposure is therefore, \$1.6 million.
2. She has hedged 71.43% of her exposure (\$4 million out of \$5.6 million) via the single-name CDS.
3. After Company A's default, the remaining notional on the investor's index CDS position is \$694.4 million (= \$700m – \$5.6m).

A very important thing for you to understand at this stage is that a CDS does not provide protection against market-wide interest rate risk. It only provides protection from **credit risk.** For example, an increase in interest rates would lower the price of the reference bond, and you might be tempted to think that this would benefit the protection buyer, but note that this is not the case. Only in the case of a credit event is the settlement of a CDS triggered, and in the case of a credit event, the value of the reference bond is not based on the current interest rate environment relative to when the bonds were issued. The value of the reference bond depends on the recovery rate i.e., what percentage of the par value can be recovered from the issuing entity when it defaults.

LESSON 2: BASICS OF VALUATION AND PRICING

LOS 36c: Explain the principles underlying, and factors that influence, the market's pricing of CDS. Vol 5, pp 287–291

Basics of Valuation and Pricing

Basic Pricing Concepts

When we talk about pricing CDS, we are interested in determining the **CDS spread** or in the case of standardized CDS (with 1% or 5% coupon rates) determining the **upfront premium.** CDS pricing is a very complicated exercise, and you are not required to calculate the CDS spread or the upfront payment. What you do need to understand, however, are factors that influence CDS pricing.

The most important concept when it comes to CDS pricing is the **probability of default.** To illustrate some related concepts as they apply in CDS arena, let's work with an example of a 2-year annual-pay bond with a coupon rate of 10%, and a par value of \$1,000. In 1 year, a coupon payment of \$100 will be due and then a payment of \$1,100 (coupon + principal) will be due in 2 years. Each of these two payments carries the risk of default.

For a single-name CDS, when default occurs, the protection seller makes a payment to the protection buyer and the CDS is terminated. Therefore, in the context of valuing CDS, we will work with what is known as a **hazard rate** (a concept similar to conditional probability) that reflects the probability that an event (default) will occur given that it has not already occurred.

Continuing with our example, assume that the hazard rates for the two payments are 1% and 3%, respectively. We will also assume a 40% **recovery rate**, which is a common assumption for senior unsecured debt (on which CDS spreads are typically based). We assume that the following scenarios can occur:

We have also assumed that default can only occur at the end of the year (not during the course of the year).

- If default occurs on the $100 payment, the bondholder will receive $40 (= $100 × 40%) at the end of Year 1 and $440 (= $1,100 × 40%) at the end of Year 2. (We have assumed that if default occurs on the first payment, it will also occur on the second payment and that recovery rates on both will be the same).
    - The probability of this scenario is 0.01 or 1%.
- If default does not occur on the first payment, but does occur on the $1,100 payment, the bondholder receives $100 at the end of Year 1 and $440 at the end of Year 2.
    - The probability of this scenario is 0.99 × 0.03 = 0.0297 or 2.97%.
- If default does not occur at end of Year 1 or Year 2, the bondholder receives $100 at the end of Year 1 and $1,100 at the end of Year 2.
    - The probability of this scenario is 0.99 × 0.97 = 0.9603 or 96.03%.

We know that the probability of default is 1% on the first payment and 3% on the second, but now let's address the probability of default at any time during the loan. In order to compute this value, we must first compute the **probability of survival**, which is the complement of the probability of default at any time during the loan. Here, the probability of survival equals 0.99 × 0.97 = 96.03%. Therefore, the probability of default occurring at any point during the term of the loan is 100% − 96.03% = 3.97%. See Example 2-1.

The second important concept in credit analysis is the **loss given default**, the amount that will be lost if default occurs. In this example:

- The loss if the company defaults at the end of Year 1 would be $60 on the first payment and $660 on the second payment for a total loss of $720.
- The loss if the company defaults at the end of Year 2 would be $660.

After computing the probabilities of each scenario, and the loss given default in each scenario, we can compute the **expected loss**. The expected loss is calculated as the loss given default multiplied by the probability of default.

$$\text{Expected loss} = \text{Loss given default} \times \text{Probability of default}$$

In our example:

- There is a 1% chance of losing $720 (= 60 + 660).
- There is a 2.97% chance of losing $660.
- Therefore, the expected loss is (0.01 × $720) + (0.0297 × $660) = $26.80.

Example 2-1: Hazard Rate and the Probability of Survival

A company's hazard rate is a constant 2.5% per quarter. An investor sells 5-year CDS protection on the company with the premium paid quarterly over the next 5 years.

1. What is the probability of survival for the first quarter?
2. What is the conditional probability of survival for the second quarter?
3. What is the probability of survival through the second quarter?
4. What is the probability that the company will not default during the entire 5-year period?

Solution:

1. The probability of survival for the first quarter is 97.5% (100% minus the 2.5% quarterly hazard rate).
2. The conditional probability of survival for the second quarter is also 97.5% (since the hazard rate is constant at 2.5%). In other words, there is a 2.5% probability of default in the second quarter, given that the company has survived the first quarter.
3. The probability of survival through the first quarter is 97.5%, and the conditional probability of survival through the second quarter is also 97.5%. The probability of survival through the second quarter is therefore 97.5% × 97.5% = 95.06%. Note that the probability of default any time in the first two quarters equals 1 – 95.06% = 4.94%.
4. The probability that the company will not default through the entire 5-year period is calculated as $0.975^{20} = 0.6027$ or 60.27%. This means that the probability of default at any point during this period equals 1 – 0.6027 = 0.3973 or 39.73%. This somewhat simplified example illustrates how a low probability of default in any one period can actually turn into a relatively high probability of default over a longer period of time.

To understand CDS pricing further, we must recognize that there are two legs to the CDS (just like in any other swap):

- The **protection leg,** which refers to the contingent payment that the protection seller may have to make to the protection buyer.
- The **premium leg,** which refers to the series of periodic payments that the protection buyer promises to make to the protection seller.

In order to value the **protection leg,** we must compute the present value of the contingent obligation of the protection seller to the protection buyer. This is calculated as the difference between:

- the present value of the hypothetical value of the bond if it had no credit risk and
- the present value of the expected payoff on the (risky) bond, after accounting for the probability, amount and timing of each payment. Theoretically, this value reflects the price that the bond should be trading at in the market. In order to compute this value:
  - First we compute the expected payoff of each payment on the bond by multiplying the payment adjusted for the expected recovery rate by the probability of survival.
  - Next, we discount the expected payoff of each payment at the appropriate discount rate.
  - Then we sum all these amounts to come up with the expected payoff on the bond.

The difference between these two figures represents **credit exposure** (i.e., it reflects the present value of the contingent obligation of the protection seller to the protection buyer). The difference between the amount that an investor would pay for a bond that entails credit risk and what she would pay for an otherwise identical bond with no credit risk is what it would cost to eliminate the credit risk. Therefore, this difference represents the value of the protection leg.

Note that we could obtain the value of the bond and the (implicit) credit premium from the price it is trading at in the bond market, but then we would be assuming that the market is correctly pricing the credit risk, which is not always the case.

Value of protection leg = Credit risk = Value of risk-free bond – Expected payoff on risky bond

When it comes to valuing the **premium leg**, things may appear to be more straightforward (especially if there is a fixed standardized coupon rate). However, one factor that complicates this exercise is that once default occurs, regardless of exactly when it occurs during the term of the CDS, the protection buyer ceases to make payments to the protection seller. As a result, hazard rates must be applied to compute expected value of payments the protection buyer will make to the protection seller over the term of the instrument. See Example 2-2.

The difference in values of the protection leg and premium leg equals the amount of the **upfront payment**.

Upfront payment = Present value of protection leg – Present value of premium leg

If this difference is positive (negative), the protection buyer (seller) makes the upfront payment to the protection seller (buyer). In order to understand this, note that:

- PV of protection leg represents the value of the contingent payment that the protection seller will make to the buyer.
- PV of premium leg represents the value of expected premium payments that the protection buyer will make to the seller.
- If the value of payments that the seller will make over the term of the CDS is greater, the seller will receive a premium upfront so that overall, both the positions hold equal value at CDS inception.

Make sure you use the duration of the CDS, not the duration of the reference obligation in this formula.

Industry participants use the following specification to estimate the upfront payment (as a percentage) on CDS with standardized coupon rates:

Upfront premium % ≅ (Credit spread – Fixed coupon) × Duration of CDS

Make sure you know this: If the estimated upfront payment from this equation is positive, it means that the credit spread is actually greater than the fixed coupon on the CDS, so the protection buyer must make an upfront payment to the protection seller because she is making unjustifiably low premium payments on the CDS.

The following formula can help you determine who make the upfront payment:

Present value of credit spread = Upfront payment + Present value of fixed coupon

The upfront premium can also be converted into a price or dollar amount by subtracting the percentage premium from 100:

Price of CDS per 100 par = 100 – Upfront premium %

Example 2-2: Premiums and Credit Spreads

1. A high-yield company's 10-year credit spread is 550 bps, and the duration of the CDS is 6 years. Given that a (standardized) CDS on the company's debt carries a coupon of 5%, estimate the upfront premium required to buy 10-year CDS protection?
2. An investor sold 5-year protection on an investment-grade company and had to pay a 2% upfront premium to the buyer of protection. Given that the duration of the CDS was 3 years, compute the company's credit spread and the price of the CDS per 100 par?

Solution:

1. In this case, the investor wishes to buy protection on the CDS. Given that the credit spread on the issue is greater than the fixed coupon on the standardized CDS, the investor would have to make an upfront payment to buy protection (as she would be paying a lower credit spread on the CDS than required given the reference obligation's risk—5% versus 5.5%). In this case, the upfront premium would be 5.5% – 5% × 6 = 3% of the notional.
2. In this case, the investor has sold protection on the CDS and had to pay the upfront premium, which means that the coupon rate on the standardized CDS (1% on CDS on investment-grade bonds) is higher than the credit spread required given the company's level of risk. In our formula (reproduced below), this means that the upfront premium would be a negative number (as credit spread < fixed coupon).

$$\text{Upfront premium \%} \cong (\text{Credit spread} - \text{Fixed coupon}) \times \text{Duration of CDS}$$

$$-2\% = (\text{Credit spread} - 1\%) \times 3$$

$$\text{Credit spread} = 0.33\% \text{ or } 33 \text{ bps}$$

The price of the CDS per 100 par is calculated as:

$$\text{Price of CDS per 100 par} = 100 - \text{Upfront premium \%}$$

$$\text{Price of CDS per 100 par} = 100 - (-2) = 102$$

LESSON 3: APPLICATIONS OF CDS

LOS 36d: Describe the use of CDS to manage credit exposures and to express views regarding changes in shape and/or level of the credit curve.

Vol 5, pp 291–292

The Credit Curve

The **credit curve** presents credit spreads on a company's debt for a range of maturities. The **credit spread** refers to the spread on top of LIBOR required by investors to hold the debt instrument. With the evolution and high degree of efficiency in the CDS market, the credit curve for a borrower is essentially determined by CDS rates on its obligations. See Example 3-1.

One of the factors that influences the credit curve is the hazard rate:

- A constant hazard rate will result in a relatively flat credit curve.
- Upward (downward) sloping credit curves imply a greater (lower) likelihood of default in later years.

Example 3-1: Change in Credit Curve

A company's 5-year CDS trades at a credit spread of 200 bps, while its 10-year CDS trades at a credit spread of 400 bps.

1. Describe the implications of a change in the credit curve where the 5-year spread is unchanged, but the 10-year spread widens by 50 bps.

2. Describe the implications of a change in the credit curve where the 10-year spread is unchanged, but the 5-year spread widens by 400 bps.

Solution:

1. This change implies that the company's longer-term creditworthiness has deteriorated, but its short-term credit risk is unchanged. Perhaps the company has adequate liquidity for now, but may be expected to begin repaying debt or experience solvency issues after 5 years.
2. This change implies that the company's short-term credit risk has increased. In fact, since the short-term credit spread (600 bps) is now greater than the long-term credit spread (400 bps), one can conclude that the company's probability of default will decrease if it can survive the next 5 years.

Value Changes in CDS during Their Lives

Just like any other swap, the value of a CDS also fluctuates during its term. Some of the factors that cause changes in the value of a CDS are listed below:

- Changes in the credit quality or perceived credit quality of the reference entity.
- Changes in duration of the CDS (duration shortens through time).
- Changes in the probability of default, expected loss given default and in the shape of the credit curve.

Let's work with an example. Consider a 5-year CDS with the reference obligation being a bond issued by ABC Company. The CDS carries a standardized coupon rate of 5%, but the credit spread on ABC is actually 6%. This means that:

- The protection buyer is paying a CDS spread of 5% to obtain coverage on a reference obligation that actually demands a CDS spread of 6%.
- The present value of the protection leg (or credit risk) therefore exceeds the present value of the premium leg, so the upfront premium will be paid by the protection buyer to the protection seller.

We will continue with this example and talk about realizing this gain in the next section.

Now assume that during the term of this CDS, the credit spread on the reference obligation falls to 5.5% (i.e., its credit quality improves). If a new CDS (with the same remaining maturity and coupon as the original CDS) were created at this point in time, the upfront premium would still be paid by the protection buyer to the protection seller (since the actual credit spread is still greater than the coupon rate on the CDS—5.5% versus 5%), but it would be lower in amount than the upfront premium on the original CDS because there is now less risk in the reference obligation. However, you should note that the protection seller on the original CDS has gained while the protection buyer has lost out. The difference between the upfront premium on the original CDS and the upfront premium on the new CDS (which represents the current value of the original CDS) is the protection seller's gain and the protection buyer's loss. See Example 3-2.

The change in value of a CDS for a given change in the credit spread can be approximated as:

$$\text{Profit for protection buyer} \cong \text{Change in spread in bps} \times \text{Duration} \times \text{Notional}$$

The percentage price change in the CDS can be computed as:

$$\%\text{Change in CDS price} = \text{Change in spread in bps} \times \text{Duration}$$

Example 3-2: Profit and Loss from Change in Credit Spread

An investor purchased $5 million of 5-year CDS protection. The CDS contract has a duration of 4 years. The company's credit spread was originally 400 bps and widens to 600 bps.

1. Does the investor benefit or lose from the change in credit spread?
2. Estimate the CDS price change and the profit to the investor.

Solution:

1. The investor has purchased protection, so she benefits from an increase in the company's credit spread (as she is paying a lower premium than required by the company's current credit risk). In other words, she can now sell the protection for a higher premium.
2. The percentage price change is calculated as:

$$\text{Change in CDS price} = \text{Change in spread in bps} \times \text{Duration}$$
$$\text{Change in CDS price} = 200 \times 4 = 8\%$$

The profit to the investor is calculated as:

$$\text{Profit for protection buyer} \cong \text{Change in spread in bps} \times \text{Duration} \times \text{Notional}$$
$$\text{Profit for protection buyer} = 200 \times 4 \times \$5\text{m} = \$400{,}000$$

Monetizing Gains and Losses

Monetizing gains/losses refers to actually realizing the gains/losses on a CDS position. There are three ways to monetize the gain/loss on a CDS position.

The first method is to enter into a **new offsetting CDS**. Let's go back to the example in the previous section where we worked with a CDS on a bond issued by ABC Company. The original CDS carried a standardized coupon rate of 5%, but the credit spread on ABC was 6%, so an upfront premium was paid by the protection buyer to the protection seller. Later, during the term of the CDS, the credit spread on the reference obligation fell to 5.5%, which benefited the protection seller as the credit quality of the reference entity improved. The protection seller could lock in (or realize) her gain by entering into a new CDS with the same remaining term to maturity, reference obligation, and coupon rate as the original CDS, **but this time as the protection buyer**. Since the credit spread on the new CDS (5.5%) is still higher than the coupon rate on the new CDS (5%), the protection buyer on the new CDS would make an upfront premium payment to the protection seller on the new CDS, but the amount of this premium would be smaller than the premium that exchanged hands upon inception of the original CDS (as the difference between the credit spread and coupon rate has fallen from 6% – 5% = 1% to 5.5% – 5% = 0.5%).

Note that the offsetting transaction need not be with the same counterparty, but an offsetting transaction with the same counterparty does have its advantages.

Overall, from the perspective of the protection seller on the original CDS.

- She received an upfront premium on the original CDS and sold protection on the reference obligation.
- She paid an upfront payment on the new CDS and purchased protection on the reference obligation.
- Overall, she has eliminated all exposure to the credit risk of the reference obligation and monetized a gain equal to the difference between the higher premium received (as the protection seller) on the original CDS and the lower premium paid (as the protection buyer) on the new CDS.

A second way to monetize a gain/loss on a CDS position is to **exercise or settle the CDS upon default** via a cash or physical settlement (discussed earlier).

Finally, a third (least common) method occurs if there is no default. The parties simply **hold their positions until maturity**. By then, the seller has captured all the premium payments without having to make any payments to the protection buyer.

Applications of CDS

CDS are commonly used to:

- **manage credit exposures**, that is, take on or lay off credit risk in response to changing expectations, and/or
- take advantage of **valuation disparities**, that is, differences in the pricing of credit risk in the CDS market relative to another market (e.g., that of the underlying bonds).

Managing Credit Exposures

Generally speaking, purchasers of protection in the CDS market tend to be lenders who wish to reduce credit exposure to a borrower, while sellers of protection tend to primarily be dealers, who aim to make a profit from market making. Other than lenders and dealers, the CDS market has also become popular with speculators who aim to earn profits by taking positions on credit risk through CDS. If they expect credit spreads to increase (decrease), speculators would purchase (sell) protection on CDS to decrease (increase) exposure to credit risk.

A party takes a position on a **naked credit default swap** when it has no underlying exposure to the reference entity.

- In entering a naked CDS as the protection buyer, the investor is taking a position that the entity's credit quality will deteriorate (credit spreads will increase).
- In entering a naked CDS as the protection seller, the investor is taking a position that the entity's credit quality will improve (credit spreads will decline).

In a **long/short trade**, the party takes a long position in one CDS and a short position in another CDS, where the two swaps are based on **different** reference entities. Such a trade represents a bet that the credit quality of one entity will improve relative to that of the other. The investor would sell protection on the entity whose credit quality it expects to improve (go long on the CDS), and purchase protection on the entity whose credit quality it expects to deteriorate (go short on the CDS).

Similarly, an investor may undertake a long/short trade based on other factors, such as environmental, social, and governance (ESG) considerations. For example, an investor may take advantage of a company's poor ESG-related practices and policies relative to another company. She would short the CDS of a company with weak ESG practices and policies and go long the CDS of a company with strong ESG practices and policies.

A **curve trade** is a type of long/short trade. It involves buying a CDS of one maturity and selling a CDS with a different maturity, where both the CDS are on the **same** reference entity. Let's work with an example where we assume that the credit curve is currently upward sloping (long-term CDS rates are higher than short-term rates). Note that with an **upward-sloping** credit curve, a **steepening** (flattening) of the curve means that long-term credit risk has increased (decreased) relative to short-term credit risk. See Example 3-3.

- An investor who believes that long-term credit risk will increase relative to short-term credit risk (credit curve steepening) will purchase protection (or go short) on a long-term CDS and sell protection (or go long) on a short-term CDS.
    - A curve-steepening trade is bullish for the short run. It implies that the short-term outlook for the reference entity is better than the long-term outlook.
    - An investor who is bearish about a company's short-term creditworthiness will enter into a curve-flattening trade.

Note that the interpretation of the investor's position is the opposite if the credit curve is **currently downward sloping** (which is rarely the case and generally results from short-term stress in financial markets). In this case, a **flattening** (not steepening) of the curve suggests that long-term credit risk has increased relative to short-term credit risk.

Example 3-3: Curve Trading

An investor who owns some short-term bonds issued by a company has become increasingly worried about a default in the short term. However, she is not too concerned about a default in the long term. The company's 1-year duration CDS currently trades at 250 bps, and the 5-year duration CDS trades at 600 bps.

1. What kind of curve trade could the investor undertake to hedge the default risk?
2. Explain why an investor may prefer a curve trade to hedge against the company's default risk to a short position in one CDS.

Solution:

1. The investor expects the credit curve to flatten. She can hedge the default risk by buying protection in the 1-year CDS and selling protection on the 5-year CDS.
2. Purchasing protection in one CDS and selling protection in another CDS on the same reference entity reduces some of the risk because the investor is insulated from an unfavorable parallel shift in the credit curve. Further, the cost of one position will be subsidized (or even more than entirely offset) by the premium earned on the other.

Note that changes in the credit curve do no always mean a change in slope (steepening or flattening). It is possible for the general level of spreads across all maturities to shift by the same amount. In such a case, for a given change in credit spreads across all maturities, longer-term CDS values will be more affected than shorter-term CDS values (due to their higher durations).

LOS 36e: Describe the use of CDS to take advantage of valuation disparities among separate markets, such as bonds, loans, equities, and equity-linked instruments. Vol 5, pp 292–302

Note that the yield on a bond has many components (risk-free rate, funding spread, liquidity risk, etc.), so it is difficult to isolate the credit risk component of a bond's yield. Therefore, the success of a basis trade crucially depends on the accuracy of models used to compute the credit risk component of a bond's yield.

Valuation Differences and Basis Trading

A **basis trade** aims to profit from the difference in the credit spread implied by the price/yield on a bond, and the credit spread on a CDS on the same reference obligation with the same term to maturity. Theoretically speaking, the credit risk component of a bond's yield should be the same as the credit spread on a CDS as they both reflect compensation paid to the investor/protection seller for assuming the associated credit risk. Practically speaking, however, there may be a difference between the compensation for credit risk offered in the bond market and the CDS market due to differences of opinions, differences in models used by participants in the two markets, differences in liquidity in the two markets, and supply and demand conditions in the repo market. See Example 3-4.

Basis trades work on the assumption that any mispricing of credit risk across the bond and CDS markets is likely to be temporary and that credit spreads should converge once the disconnect has been recognized by participants. For example, assume that based on her analysis/calculations, a trader believes that for a particular bond/reference obligation the bond market implies a 4% credit risk premium, while the CDS market offers a 3% credit risk premium. The credit spread (CDS premium) is lower than the bond credit risk premium, which means that CDS market is pricing in too little credit risk, and/or the bond market is pricing in too much credit risk. In such a situation, the trader would:

- Buy protection from credit risk through the CDS at what may be an unjustifiably low rate.
- Buy the bond, thereby taking on credit risk and paying what may be an unjustifiably low price for the bond. (If the credit spread is too high, the bond's price would be too low).

Note that the investor does bear interest rate risk on the bond, but this risk can be hedged with a duration strategy or with interest rate derivatives. The basic idea is to eliminate all risks and to capitalize on the disparity between the price of credit risk in the CDS and bond markets.

Overall, she has no exposure to credit risk as she is protected from default risk on the bond through her short position on the CDS (protection buyer). If convergence occurs, the trader would capture the 1% differential in the two markets.

In order to determine the profit potential of a basis trade, we must decompose the bond yield into the risk-free rate plus the funding spread plus the credit spread. The risk-free rate plus the funding spread equals LIBOR, so the credit spread equals the excess of the yield on the bond over LIBOR. This credit spread must then be compared to the credit spread on the CDS. If the credit spread on the bond is higher (lower) than in the CDS market, it is said to be a negative (positive) basis.

Example 3-4: Bond versus CDS

A company's bond currently yields 5% and matures in 10 years. A 10-year CDS contract on the same bond has a credit spread of 2.75%. The investor can borrow in the market at a 2.0% interest rate.

1. Calculate the bond's credit spread.
2. Identify a basis trade that would exploit the current situation.

Solution:

1. The bond's credit spread equals its yield (5%) minus the investor's cost of funding (2.0%), which equals 3%.
2. The investor should buy protection in the CDS market at 2.75% and go long on the bond (which prices in a 3% credit risk premium). Overall, the investor will have no exposure to credit risk and will earn the 0.25% differential if and when the markets converge.

Example 3-5: Using CDS to Trade on a Leveraged Buyout

An investor believes that a company will soon undergo a leveraged buyout (LBO) transaction (where the company will raise large amounts of debt to repurchase equity from the market).

1. Why might the CDS spread change?
2. What positions would the investor take on the company's equity and on its credit in anticipation of the LBO?

Solution:

1. Issuing significant amounts of debt will increase the company's probability of default, translating into a widening of the CDS spread.
2. The investor might consider buying the stock and buying CDS protection. Both these positions will profit if the LBO occurs.
   - The stock price will rise from the repurchase.
   - The CDS spread will rise to reflect the higher probability of default.

Other types of trades that CDS are used in are:

- Trades that seek to take advantage of mispricing of credit risk between the CDS market and the credit risk reflected in the yields embedded in the company's unsecured debt instruments or capital leases. Note that such trades are very complicated as all claims are not paid off equally if default occurs (due to priority of claims).
- An arbitrage trade which aims to profit from the difference between the cost of an index CDS and the aggregate cost of index components.
- An arbitrage trade which aims to profit from the difference between the cost of a synthetic CDO and an otherwise identical actual (cash) CDO.

Synthetic CDOs and cash CDOs are described at length in the Fixed Income section.

Study Session 14:
Derivatives

Reading 37: Pricing and Valuation of Forward Commitments

LESSON 1: PRICING AND VALUATION OF FORWARD AND FUTURES CONTRACTS

LOS 37a: Describe and compare how equity, interest rate, fixed-income, and currency forward and futures contracts are priced and valued. Vol 5, pp 318–353

LOS 37b: Calculate and interpret the no-arbitrage value of equity, interest rate, fixed-income, and currency forward and futures contracts. Vol 5, pp 318–353

Introduction

A **forward commitment** is a derivative instrument in the form of a contract that provides the ability to lock in a price or rate at which one can buy or sell the underlying instrument at a specified future date, or exchange an agreed-upon amount of money at a specified series of dates.

Forward commitments include forwards, futures, and swaps. In Lesson 1 of this reading, we will cover forwards and futures, and in Lesson 2, we will cover swaps.

A **forward** is a contract between two parties, where one (the long position) has the obligation to buy, and the other (the short position) has an obligation to sell the underlying asset at a specified price (established at the inception of the contract) at a specified future date (also established at inception of the contract).

Principles of Arbitrage-Free Pricing and Valuation of Forward Commitments

At Level I, we were introduced to the concepts of arbitrage, replication, and pricing and valuation in the context of derivative instruments. At Level II we are going to dive into much more detail and work through some extensive calculations, especially when it comes to forward rate agreements (FRAs) and swaps. Let's begin with a bit of review from Level I.

Level I Recap

- The principle of **arbitrage** is based on the **law of one price**, which asserts that if two investments have the same or equivalent future cash flows regardless of what will happen in the future, then these two investments should have the same price today. Alternatively, if the law of one price is violated, someone could buy the cheaper asset/portfolio and sell the more expensive asset/portfolio, locking in a gain at no risk and with no commitment of capital (i.e., arbitrage profit).
- An arbitrageur relies on two fundamental rules: (1) do not use your own money, and (2) do not take any price risk. She aims to abide by both these rules and generate a positive cash flow today.
- **Price**, as it relates to forwards, futures, and swaps (note that options are different in this regard) refers to the fixed price (that is agreed upon at contract initiation) at which the underlying transaction will take place in the future.
    - Forward commitments do not require an outlay of cash at contract initiation, so there is no concept of a price being paid at the beginning.
    - As you will see in the illustrations that follow shortly, **forward commitment pricing** refers to determining the forward price that precludes arbitrage.

- The **value** of a forward commitment fluctuates in response to changes in the price of the underlying through the term of the contract.
    - **Forward commitment valuation** involves determining the value of the forward commitment at some point during the term of the contract.
- In determining forward commitment price and value, we will use the **carry arbitrage model** (also known as **cost-of-carry arbitrage model** or **cash-and-carry arbitrage model**).

Assumptions of Arbitrage

- Replicating instruments are identifiable and investable.
- Markets are frictionless.
- Short selling is allowed.
- Borrowing and lending are available at the risk-free rate.

Pricing and Valuation of Forward and Futures Contracts

In this section, we study pricing and valuation of forward *and* **futures contracts** based on the no-arbitrage approach. While there are material differences between forwards and futures (e.g., margin requirements, mark-to-market adjustments, and centralized clearing), we will simplify the discussion by focusing on cases where the carry arbitrage model can be used in both markets.

The **price** of a forward/futures contract is the fixed price or rate at which the underlying transaction will occur at contract expiration. The forward/futures price is agreed upon at initiation of the contract. Pricing a forward/futures contract means determining this **forward/futures price**.

The **value** of a forward/futures contract is the amount that a counterparty would need to pay, or would expect to receive, to get out of, or terminate, an (already-assumed) forward/futures position.

We will first work with a generic example to introduce you to the concepts and mechanics behind pricing and valuing forward/futures contracts. We will primarily work with forwards, but will highlight important distinctions between forwards and futures when the two types of contracts differ.

- The contract initiation date is denoted by $t = 0$.
- The contract expiration date is denoted by $t = T$.
- Any point in time between the contract initiation date and expiration date is denoted by $t = t$.
- Time is expressed as a fraction of years.

The **spot price (S)** of the underlying fluctuates during the term of the contract.

- S_0 refers to the spot price at initiation of the forward contract ($t = 0$).
- S_t refers to the spot price at a point in time during the term of the forward contract ($t = t$).
- S_T refers to the spot price at expiration of the forward contract ($t = T$).

The **forward price (F)** is determined at contract initiation. For a particular contract, it does not change over the term of the contract. The term $F_0(T)$ is used to refer to the forward price for a contract that was initiated at $t = 0$ and expires at $t = T$.

The **futures price (f)** is also determined at contract initiation. For a particular contract, it does not change over the term of the contract. The term $f_0(T)$ is used to refer to the futures price for a contract that was initiated at $t = 0$ and expires at $t = T$.

The **value of a forward contract (V)** changes over the term of the contract as the price of the underlying asset fluctuates.

- $V_0(T)$ refers to the value of a forward contract (expiring at $t = T$) at initiation ($t = 0$).
- $V_t(T)$ refers to the value of a forward contract (expiring at $t = T$) at a point in time during the term of the contract ($t = t$).
- $V_T(T)$ refers to the value of a forward contract (expiring at $t = T$) at expiration ($t = T$).

The **value of a futures contract (v)** also changes during the course of its term as the price of the underlying asset fluctuates.

- $v_0(T)$ refers to the value of a futures contract (expiring at $t = T$) at initiation ($t = 0$).
- $v_t(T)$ refers to the value of a futures contract (expiring at $t = T$) at a point in time during the term of the contract ($t = t$).
- $v_T(T)$ refers to the value of a futures contract (expiring at $t = T$) at expiration ($t = T$).

In the following section, we illustrate how forward contracts are valued at various points in time. Note that we will be taking the perspective of the long position on the contract when valuing a forward. Once the value of the long position has been determined, the value of the short position can be determined by simply changing the sign. Further, we will be assuming that the underlying asset entails no storage or carrying costs, and makes no payments to the owner of the asset over the term of the forward contract. In subsequent sections, we will relax those assumptions.

Valuing a Forward Contract at Expiration ($t = T$)

The long position on the forward has an obligation to buy the underlying asset for the agreed-upon (at contract initiation) price of $F_0(T)$ at contract expiration. The price of the underlying asset at expiration of the forward equals S_T. Therefore, the value of the long position on the forward contract at expiration equals the difference between:

- The current worth of the asset, S_T, which represents the current worth of the asset, or the price at which the underlying asset can be sold, and
- The price that the long position in the forward must pay to acquire the asset, $F_0(T)$.

$$\boxed{V_T(T) = S_T - F_0(T)}$$

If the value at expiration does not equal this amount, arbitrage profits can be made. For example, if the forward price established at contract initiation equals \$30 and the spot price at contract expiration equals \$35, then the value of the forward contract must equal \$35 – \$30 = \$5 at expiration.

- If the contract value at expiration were greater than \$5, it would mean that someone is willing to pay more than \$5 to obtain an obligation to buy something worth \$35 for \$30, which would not make sense.

- If the contract value were less than \$5, it would mean that someone is willing to accept less than \$5 to give up an obligation to buy something worth \$35 for \$30, which would not make sense either.

The value of the short position at contract expiration is given by:

$$\boxed{V_T(T) = F_0(T) - S_T}$$

Valuing a Forward Contract at Initiation (t = 0)

Now let's work with a forward contract that has a term of 1 year in two scenarios. The price (S_0) of the asset underlying the contract is currently \$100 and the risk-free rate (r) is 8%. Determine the value of the contract at initiation, $V_0(1)$, given that:

1. The forward price, $F_0(1)$, equals \$110.
2. The forward price, $F_0(1)$, equals \$106.

Note that no money changes hands at origination of the forward contract.

Scenario 1: $F_0(1)$ = \$110
In this scenario, arbitrage profits can be made through **carry arbitrage** by undertaking the following transactions simultaneously:

- Borrow \$100 at 8%.
- Purchase the underlying asset at the current spot price of \$100.
- Sell the underlying asset forward by taking the short position on the forward contract at a forward price of \$110.
- Borrow the known arbitrage profit in order to capture it today.

The following table illustrates the computation of arbitrage profits from this strategy:

t = 0 (At Contract Origination)		t = T (At Contract Expiration)	
Transaction	**Cash Flow**	**Transaction**	**Cash Flow**
Borrow \$100 @ 8%	\$100	Deliver the asset under the terms of the forward contract in return for $F_0(1)$	\$110
Buy asset at current price S_0	(\$100)	Repay loan plus interest	(\$108)
Take the short position in a forward contract on the asset with a forward price, $F_0(1)$ = \$110	\$0		
Net cash flow	**\$0**	**Expected arbitrage profit (net cash flow)**	**\$2**

Note the following:

- The series of steps listed are executed simultaneously in practice, not sequentially.
- For now, we are not concerned with compounding conventions, day count conventions, or the proxy for the risk-free interest rate. We simply work with annual compounding.

- By undertaking the steps described in the table, we adhere to the two fundamental rules of arbitrage:
    1. We do not use our own money to acquire the position, but borrow the funds.
    2. We do not take any price risk. Taking the short position on the forward removes price uncertainty. We do not consider other risks such as liquidity risk and counterparty credit risk.
- In order to capture the arbitrage profit today (net cash flow = \$2), the arbitrageur will borrow the present value (PV) of the known terminal profit (\$2) today. He will pocket this borrowed amount [= \$2 / (1 + 0.08) = \$1.85] and let the known terminal profit (\$2) take care of repaying this loan (including interest).

Scenario 2: F(0,1) = \$106

In this scenario, arbitrage profits can be made through **reverse carry arbitrage** by undertaking the following steps:

- Short the underlying asset at the current spot price of \$100.
- Invest or lend the proceeds at 8%.
- Buy the underlying asset forward by taking the long position on the forward contract at the forward price of \$106.
- Borrow the arbitrage profit in order to capture it today.

The following table illustrates the computation of arbitrage profits from this strategy:

t = 0 (At Contract Origination)		t = T (At Contract Expiration)	
Transaction	**Cash Flow**	**Transaction**	**Cash Flow**
Short the underlying asset at S_0	\$100	Take delivery of the asset under the terms of the forward contract by paying $F_0(1)$	(\$106)
Invest \$100 @ 8%	\$100		
Take the long position in a forward contract on the asset with a forward price, $F_0(1)$, of \$106	\$0	Receive investment amount plus interest	\$108
Net cash flow	**\$0**	**Expected terminal profit**	**\$2**

Get yourselves to think of the value to the long position as the current price of the underlying asset minus the present value of the obligation (forward price). This will save you from having to memorize some pretty intimidating formulas going forward.

Once again, as was the case with our first scenario (carry arbitrage), the arbitrageur will pre-capture the expected terminal profit by borrowing its present value (\$1.85) today.

Now let's understand the value of a forward contract. Recall that we are taking the perspective of the long position.

- The long position has the obligation to **pay** the forward price, $F_0(T)$, and take delivery of the underlying asset at contract expiration.
    - The value of this **obligation** at contract initiation equals $F_0(T) / (1 + r)^T$
- At expiration, the long position will **receive** the underlying asset, which will be worth S_T.
    - The value of this underlying **asset** at contract initiation equals S_0.
- Therefore, the value of the forward contract to the long position at contract initiation equals the current worth of the asset minus the present value of the obligation.
    - $V_0(T) = S_0 - [F_0(T) / (1 + r)^T]$

- The value to the short position at contract initiation is given as:
    - $V_0(T) = [F_0(T) / (1 + r)^T] - S_0$

The negative value to the long position in Scenario 1 implies that a positive value of \$1.85 accrues to the short position.

In Scenario 1, the value of the forward contract at initiation is calculated as:

$$V_0(0, T) = S_0 - \left[F_0(0, T)/(1 + r)^T\right]$$

$$V_0(0, 1) = 100 - \left[110/(1 + 0.08)^1\right] = -\$1.85$$

In Scenario 2, the value of the forward contract at initiation is calculated as:

$$V_0(0, T) = S_0 - \left[F_0(0, T)/(1 + r)^T\right]$$

$$V_0(0, 1) = 100 - \left[106/(1 + 0.08)^1\right] = \$1.85$$

It may help you to think of the forward price as the price that yields zero value to both the long and short positions at the inception of the forward contract. This is why we call it a **no-arbitrage forward price**.

These non-zero values at initiation would entice traders to engage in arbitrage until the price of the forward contract equals the **no-arbitrage forward price**.

- If $F_0(T) > FV(S_0)$, buy underlying and sell forward contract → The underlying's price will increase and forward price will fall until $F_0(T) = FV(S_0)$.
- If $F_0(T) < FV(S_0)$, buy forward contract and sell underlying → The forward price will increase and the underlying's price will fall until $F_0(T) = FV(S_0)$.

As a result of the no-arbitrage approach, forward and futures contracts are priced to have zero value to either counterparty at initiation; that is, $V_0(T) = v_0(T) = 0$. Such contracts are referred to as **at market**. No money changes hands at initiation.

Since $V_0(T) = v_0(T) = 0$, we can express the forward price in terms of the spot price of the asset:

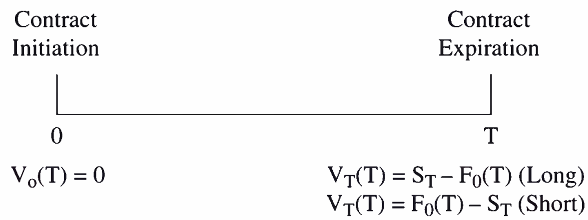
$$V_0(T) = S_0 - \left[F_0(T)/(1 + r)^T\right] = 0$$

$$F_0(T) = S_0(1 + r)^T \qquad \ldots \text{(Equation 1)}$$

From Equation 1, notice the following:

- An increase in the risk-free rate will lead to an increase in the forward price, and a decrease in the risk-free rate will lead to a decrease in the forward price. This relationship will generally hold as long as changes in interest rates do not influence the value of the underlying asset.
- The forward price is not influenced by the expected future spot price. The only factors that matter here are the risk-free rate and time to expiration. Any opinion that in the future the underlying will increase or decrease in value, even substantially, has no bearing on the forward price.

Figure 1-1: Value of a Forward Contract at Initiation and Expiration



Before moving on, note that at expiration, both forward and futures contracts are equivalent to a spot transaction in the underlying asset. This is known as **convergence**, implying that at contract expiration, the forward/futures price will be the same as the spot price; that is, $F_T(T) = f_T(T) = S_T$.

Valuing a Forward Contract during Its Life (t = t)

By now you should have digested the fact that the value of the forward contract to the long position equals the current price of the underlying minus the present value of the forward price. We now use this logic to derive the expression for the value of the forward contract at any point in time (t) during its life.

- The long position has an obligation to pay the forward price, $F_0(T)$, and take delivery of the underlying asset at contract expiration.
  - The value of this **obligation** at any point in time during the term of the contract equals $F_0(T) / (1 + r)^{T-t}$, or the forward price discounted over the remaining term of the forward contract.
- At expiration, the long position will receive the underlying asset, which will be worth S_T.
  - The value of this **asset** at any point in time during the term of the contract equals S_t.
- Therefore, the value of the forward contract to the long position at any point in time during the term of the contract equals the current value of the asset minus the present value of the obligation.
  - $V_t(T) = S_t - [F_0(T) / (1 + r)^{T-t}]$

Just one more thing that we need to re-emphasize before moving on: $F_0(T)$ represents the forward price that is agreed upon at the inception of the contract. Both spot and forward prices continue to fluctuate after inception of the contract, but for our purposes (to determine the value of a particular forward contract at any point in time) we compare the then-current spot price of the underlying asset to the present value of the initially agreed-upon (or fixed) forward price.

The value of a forward contract to the long position can also be calculated as the present value (over the remaining term of the contract) of the difference between (1) the current forward price, which is the forward price of an offsetting contract (one with the same underlying and term to expiration as the original contract), and (2) the forward price of the original contract. Due to changes in the price of the underlying, the prices of these contracts will most likely be different.

$V_t(T)$ = PV of differences in forward prices = $PV_{t,T}[F_t(T) - F_0(T)]$

where $PV_{t,T}(\ )$ means the present value at time t of an amount paid in T – t years (or at time T). . . . (Equation 2)

Note that once the offsetting forward is entered, the net position is not exposed to price risk.

Why We Would Need to Determine the Value of a Forward Contract

- To measure credit exposure.
- To mark to market for financial statement purposes or to adhere to the terms of the forward agreement.
- To determine how much money would need to be paid to terminate the contract.

Table 1-1 summarizes what we have learned so far.

Table 1-1: Value of a Forward Contract

Time	Long Position Value	Short Position Value
At initiation	Zero, as the contract is priced to prevent arbitrage	Zero, as the contract is priced to prevent arbitrage
During life of the contract	$S_t - \left[\dfrac{F_0(T)}{(1+r)^{T-t}}\right]$	$\left[\dfrac{F_0(T)}{(1+r)^{T-t}}\right] - S_t$
At expiration	$S_T - F_0(T)$	$F_0(T) - S_T$

Example 1-1: Calculating the Forward Price

Amanda holds an asset worth \$250, which she plans to sell in 6 months. To eliminate price risk, she decides to take a short position in a forward contract on the asset. Given an annual risk-free rate of 5%, calculate the no-arbitrage forward price of the contract.

Solution:

Forwards are priced to have zero value to either party at origination. Therefore, the forward price is calculated as:

$$F_0(T) = S_0(1+r)^T$$
$$F_0(6/12) = 250 \times (1+0.05)^{6/12} = \$256.17$$

Example 1-2: Calculating the Value of a Forward Contract during Its Life

In Example 1-1 we calculated the forward price as \$256.17. Suppose that 2 months into the term of the forward, the spot price of the underlying asset is \$262. Given an annual risk-free rate of 5%, calculate the values of the long and short positions in the forward contract.

Solution:

The value of the long position in the forward contract is calculated as:

$$V_t(T) = S_t - \left[F_0(T)/(1+r)^{T-t}\right]$$
$$V_{2/12}(6/12) = 262 - \left[256.17/(1+0.05)^{6/12-2/12}\right] = \$9.96 = \text{Value of long position}$$

The value of the short position is just the opposite of the value of the long position. Therefore, the value of the short position equals –\$9.96.

The value of the long position can also be computed as the present value of the difference between the two forward prices:

First, we need to determine the current forward price based on the current spot price.

$$F_t(T) = S_t(1 + r)^{T-t}$$
$$F_t(T) = \$262 \times (1 + 0.05)^{6/12-2/12}$$
$$F_t(T) = \$266.30$$

Now we can compute the value of the forward contract as:

$$V_t(T) = PV_{t,T}[F_t(T) - F_0(T)]$$
$$V_{2/12}(6/12) = (266.30 - 256.17)/(1 + 0.05)^{6/12-2/12}$$
$$V_{2/12}(6/12) = \$9.96$$

Notice that the current value of the forward contract (\$9.96) is greater than the difference (\$5.83) between the current underlying price (\$262) and the initial forward price (\$256.17). This is because of interest costs. Also, since the forward price has increased, the original contract holds positive value for the long position.

Example 1-3: Calculating the Value of a Forward Contract at Expiration

Continuing from Example 1-1, suppose that the spot price of the underlying asset at contract expiration is actually \$247. Given an annual risk-free rate of 5%, calculate the value of the long position.

Solution:

At expiration, the value of the long position in a forward contract is calculated as:

$$V_T(T) = S_T - F_0(T)$$
$$V_{6/12}(6/12) = 247 - 256.17 = -\$9.17$$

Before moving on, let's talk about futures contracts for a bit. As a result of the mark-to-market adjustment, the value of a futures contract any point in time during its term is simply the difference between the current futures price and the futures price that was applied to make the last mark-to-market adjustment.

- The market value of a long position in a futures contract value before marking to market is $v_t(T) = f_t(T) - f_{t-}(T)$.
- The market value of a short position in a futures contract value before marking to market is $v_t(T) = f_{t-}(T) - f_t(T)$.

Note: t– represents the point in time when the last mark-to-market adjustment was performed. Once futures contracts are marked to market, their value equals zero.

Carry Arbitrage Model When the Underlying Has Cash Flows

In this section, we incorporate various costs and benefits related to the underlying instrument into forward pricing. First, let's introduce some notation:

- γ = Carry benefits. These include dividends, foreign interest, and bond coupon payments that would arise from certain underlyings.
  - $\gamma_T = FV_{0,T}(\gamma_0)$ = Future value of underlying carry benefits.
  - $\gamma_0 = PV_{0,T}(\gamma_T)$ = Present value of underlying carry benefits.
- θ = Carry costs. These include financing or opportunity costs, waste, storage, and insurance.
  - $\theta_T = FV_{0,T}(\theta_0)$ = Future value of underlying carry costs.
  - $\theta_0 = PV_{0,T}(\theta_T)$ = Present value of underlying carry costs.

After accounting for carrying costs and benefits, the forward pricing equation can be expressed as:

$$\boxed{F_0(T) = (S_0 - \gamma_0 + \theta_0)\,(1 + r)^T} \qquad \text{. . . (Equation 3)}$$

or

$$\boxed{F_0(T) = S_0\,(1 + r)^T - (\gamma_0 - \theta_0)\,(1 + r)^T}$$

Note that only carry costs and benefits incurred over the term of the forward contract are accounted for when applying Equation 3. Putting this equation into words, the forward price is the future value of the underlying asset adjusted for carry cash flows.

- Carry costs, like the rate of interest, increase the burden of carrying the underlying instrument through time. Therefore, these costs are added in the forward pricing equation.
- Carry benefits, in contrast, decrease the burden of carrying the underlying instrument through time, so these benefits are subtracted in the forward pricing equation.

Valuing a Forward Contract When the Underlying Has Carry Benefits/Costs

Equation 2 can also be used to determine the value of a forward contract when the underlying has carry benefits or carry costs. Recall that Equation 2 computes the value of a forward contract during its term as the present value of the difference in forward prices:

$$\boxed{V_t(T) = \text{PV of differences in forward prices} = PV_{t,T}\,[F_t(T) - F_0(T)]} \qquad \text{. . . (Equation 2)}$$

Note that carry benefits and costs are accounted for in this equation in the computation of the two forward prices (using Equation 3): $F_t(T) = (S_t - \gamma_t + \theta_t)\,(1 + r)^{T-t}$ (see Example 1-6).

Just one more thing before moving on to some examples. The calculation of forward prices with discrete compounding is fairly straightforward. We have used discrete compounding in our earlier examples and to illustrate carry arbitrage. When the aim is to compute the forward price for a forward contract on an individual stock, we use discrete compounding.

However, when computing the forward price of a stock index, it is difficult to account individually for the numerous dividend payments that differ in amount and timing. In such a case, we work with continuous compounding (both the risk-free rate and the dividend yield are continuous rates). With continuous compounding, the forward price is calculated as:

$$\boxed{F_0(T) = S_0 e^{(r_c + \theta_c - \gamma_c)T}}$$

Example 1-4: Calculating the Price of a Forward Contract on an Index

An investor plans to enter into a 120-day forward contract on the S&P 500 index. The current value of the index is 1,241. Given a continuously compounded risk-free rate of 5.5% and a continuously compounded dividend yield of 2.5%, calculate the no-arbitrage price of the forward contract. Assume that there are 365 days in the year.

Solution:

$F_0(T) = S_0 \times e^{(rc - \gamma c)T}$

$F_0(120/365) = 1{,}241 \times e^{(0.055-0.025)\times 120/365} = 1{,}253.30$

Example 1-5: Calculating the Price of a Forward Contract on Dividend-Paying Stock

Sasha wants to purchase a stock of ABC Company in 150 days. The stock is currently priced at \$40 per share. It is expected to pay a dividend of \$0.60 in 30 days, \$0.80 in 120 days, and \$0.70 in 210 days. To hedge the interim price risk, Sasha enters the long position on a forward contract on the stock today. Given an annual risk-free rate of 5%, calculate the no-arbitrage forward price. Assume that there are 365 days in the year.

Solution:

We would expect the expiration of the forward contract to coincide with the point in time when Sasha wants to take delivery of the stock (T = 150/365). Therefore, we ignore the dividend expected to be paid in 210 days, as it will be paid **after** the expiration of the forward contract.

In order to compute the forward price, we first compute the present value (as of the contract initiation date) of dividends/benefits (γ_0) expected to be paid on the stock during the term of the forward contract:

$$\begin{aligned}\gamma_0 &= 0.60/1.05^{30/365} + 0.80/1.05^{120/365} \\ &= 0.5976 + 0.7873 = \$1.3849\end{aligned}$$

And then apply the formula for computing the forward price:

$$\begin{aligned}F_0(T) &= (S_0 - \gamma_0 + \theta_0)(1 + r)^T \\ &= (40 - 1.3849) \times (1 + 0.05)^{150/365} = \$39.3972\end{aligned}$$

Example 1-6: Calculating the Value of an Equity Forward Contract on a Stock during Its Life

In the previous example, we calculated the no-arbitrage price forward price as \$39.3972. Suppose that 70 days into this forward contract, the spot price of ABC stock is actually \$34. Given an annual risk-free rate of 5%, calculate the value of the long position on the contract.

The first thing that we need to do here is compute the current forward price for the offsetting contract:

$$F_t(T) = (S_t - \gamma_t + \theta_t)(1 + r)^{T-t}$$
$$F_{70/150}(150) = (S_{70/150} - \gamma_t)(1 + r)^{150/365-70/365}$$

To determine γ_t, bear in mind that the first dividend has already been paid, and that the second dividend (\$0.80) will be paid in another 50 days. We still do not need to account for the third dividend, as it will be paid after the forward contract expires. The present value of the second dividend as of t = 70 is computed as:

$$\gamma_t = \gamma_{70/150} = 0.80/(1 + 0.05)^{50/365} = \$0.7947$$

The forward price of a contract initiated at t = 70 for expiration at t = 150 is computed as:

$$F_{70/150}(150) = (34 - 0.7947)(1 + 0.05)^{150/365-70/365} = \$33.5623$$

Now we can compute the value of the forward contract as the present value of the differences in forward prices:

$$V_t(T) = PV_{t,T}[F_t(T) - F_0(T)]$$
$$= (\$33.5623 - \$39.3972)/(1 + 0.05)^{150/365-70/365} = -\$5.77$$

Important points:

- If a dividend payment is announced between the forward's valuation and expiration dates, assuming that the news announcement does not change the price of the underlying, the **value** of the original forward will fall.
  - This is because the original forward price is fixed, while the new forward price will be lower because of the dividend. The PV of the difference in forward prices (i.e., forward contract value) will therefore be lower.
- The mark-to-market adjustment in futures markets results in the value of a futures contract after settlement equaling zero. Therefore, the values of otherwise identical forward and futures contracts will likely be different.

Interest Rate Forward and Futures Contracts

Forward Rate Agreements

Introduction and Important Basic Concepts

A **forward rate agreement (FRA)** is a forward contract in which the underlying is an interest rate on a deposit. It involves two counterparties: (1) the fixed payer/floating receiver (long position) and (2) the floating payer/fixed receiver (short position).

- A long FRA position can be replicated by holding a longer-term Eurodollar time deposit and at the same time shorting (or owing) on a shorter-term Eurodollar time deposit.

This replication strategy is somewhat confusing for many candidates. Therefore, it would be helpful to just think about the long position in an FRA as the party that has committed to take a hypothetical loan, and the short as the party that has committed to give out a hypothetical loan, at the fixed FRA rate. However, no actual loan is made at FRA expiration, so there is no need to consider the creditworthiness of the parties in determining the FRA rate.

The payoffs on FRAs are determined by market interest rates (Libor) at **FRA expiration**.

- If Libor at FRA expiration is *greater* than the FRA rate, the long benefits. Effectively, the long has access to a loan at lower-than-market interest rates, while the short is obligated to give out a loan at lower-than-market interest rates.
  - Stated differently, the long benefits as the fixed payer and floating receiver.
- If Libor at FRA expiration is *lower* than the FRA rate, the short benefits. Effectively, the holder of the short position is able to invest her funds at higher-than-market interest rates, while the long is obligated to take a loan at higher-than-market interest rates.
  - Stated differently, the short benefits as the fixed receiver and floating payer.

Interest savings on the underlying hypothetical loan accrue at the end of the term of that loan, but the settlement payment on an FRA occurs **at the date of FRA expiration**. Therefore, we need to compute the present value (as of the date of FRA expiration) of the interest savings to compute the payoff on an FRA. This settlement mechanism is known as **advanced set, advanced settled**, referring to the fact that the interest rate effective on the underlying hypothetical loan is determined in advance (at the outset of the loan) and the contract is also settled in advance (at the outset of the loan). Contrast this with **settled in arrears**, which is the case with interest rate swaps and interest rate options (where the settlement payment is made at the end of the settlement period).

FRAs are quoted as "x × y," where "x" represents the number of months until the FRA expires and "y" equals the number of months until the hypothetical loan expires **starting from the date of inception of the FRA**. For example, a 1 × 4 FRA expires in 30 days. The hypothetical underlying loan has a term of 90 days (= y – x) and the loan matures in 120 days from today (FRA inception).

Libor is an add-on rate that is annualized on a 360-day basis. If Libor-90 (3-month Libor) is given as 4%, it implies that the actual unannualized rate for 90 days equals 4% × 90/360 = 1%.

Pricing a Forward Rate Agreement

The price of an FRA (forward price) represents the fixed interest rate at which the long position has the obligation to borrow (and the short position has the obligation to lend) funds for a specified period (term of the underlying hypothetical loan) starting at FRA expiration. Since there is no initial exchange of cash flows, in order to preclude arbitrage, the FRA price/rate is set such that the contract holds a value of zero at initiation. Essentially, pricing an FRA is a simple exercise of determining the forward rate consistent with two (given) spot rates, as illustrated in Example 1-7.

Example 1-7: Calculating the Price of an FRA

Thirty days from today Orix Inc. will need to borrow $1 million for 120 days. In order to hedge the interest rate risk associated with the anticipated borrowing, Orix takes the long position on a 1 × 5 FRA. Current 30-day Libor is 5% and 150-day Libor is 6%. Determine the price (forward rate) of this FRA.

Solution:

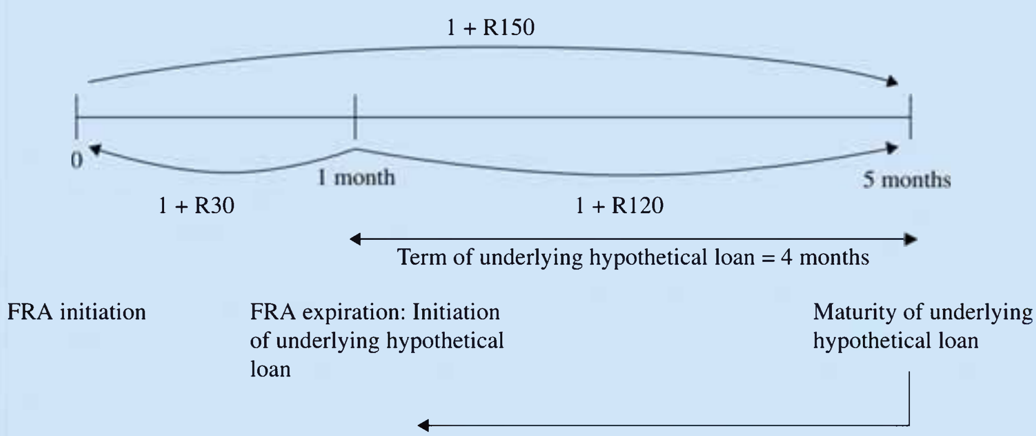
We are looking to establish the no-arbitrage forward rate (120-day Libor) that would make an investor indifferent between (1) borrowing for 150 days today at 150-day Libor, and (2) borrowing for 150 days by first borrowing for 30 days at 30-day Libor and then rolling over for 120 days at 120-day Libor (as of t = 30).

The first thing that we need to do is to unannualize the 30-day and 150-day Libor rates that are given.

$$\text{Unannualized rate on the 30-day loan} = R30 = 0.05 \times (30/360) = 0.004167$$
$$\text{Unannualized rate on the 150-day loan} = R150 = 0.06 \times (150/360) = 0.025$$

Based on the unannualized 30-day and 150-day rates, we can calculate the interest rate applicable on a 120-day loan that will be taken 30 days from today (which represents the price of the FRA) as follows:

$$\begin{aligned}
\text{Unannualized price of FRA} &= [(1 + R150)/(1 + R30)] - 1 \\
&= [(1 + 0.025)/(1 + 0.004167)] - 1 \\
&= 0.02075 \\
\text{Annualized price of FRA} &= 0.02075 \times 360/120 \\
&= 0.0622 \text{ or } 6.22\%
\end{aligned}$$



Takeaway: The FRA rate is really just a forward rate derived from the term structure of interest rates (or spot rates), even though the underlying is not an asset. Essentially, taking the long position on an FRA is equivalent to holding a longer-term Eurodollar time deposit and at the same time shorting (or owing) a shorter-term Eurodollar time deposit. In our example, the investor would take a long position on the 150-day Eurodollar time deposit and short the 30-day Eurodollar time deposit. This leaves the investor with no interest rate exposure over the initial 30-day period, and then, after 30 days, she actually has exposure to interest rate changes over the next 120 days.

If you'd rather just commit a formula to memory (LOL!!!), the forward price of an FRA can be calculated as follows:

$$FRA(0,h,m) = \left[\frac{1 + L_0(h+m)\left(\frac{h+m}{360}\right)}{1 + L_0(h)\left(\frac{h}{360}\right)} - 1\right]$$

where:

FRA(0,h,m) = The annualized rate on an FRA initiated at Day 0, expiring on Day h, and based on m-day Libor

h = Number of days until FRA expiration

m = Number of days in underlying hypothetical loan

h + m = Number of days from FRA initiation until end of term of underlying hypothetical loan

L_0 = (Unannualized) Libor rate today

Example 1-8: Calculating Interest on Libor Spot and FRA Payments

A company plans to deposit \$10 million in the bank in 30 days. It will keep the funds there for 90 days at applicable 90-day Libor. The company wants to protect itself from a decline in Libor rates over the next 30 days, so it decides to take a position on a 1 × 4 FRA today (t = 0). The appropriate discount rate for the FRA settlement cash flows is 0.45%. After 30 days, 90-day Libor is 0.75%.

1. Determine the interest actually paid on the bank deposit.
2. Determine the settlement payment if the FRA is initially priced at 0.70%.
3. Determine the settlement payment if the FRA is initially priced at 0.80%.

Solution:

Before answering these questions, we should determine what position the company will take on the FRA. The company wants to hedge itself against a decline in Libor-90 over the next 30 days, so it should expect the hedging instrument (the FRA) to result in cash inflows if Libor-90 falls. Therefore, the company will take a pay floating/receive fixed or short position on the FRA.

1. The interest paid at maturity of the deposit is calculated as:

$$\$10{,}000{,}000 \times (0.0075 \times 90/360) = \$18{,}750$$

2. The settlement payment is calculated as the present value of the interest savings on the underlying hypothetical loan.

$$\text{Interest savings} = 10{,}000{,}000 \times [(0.0070 - 0.0075)*(90/360)] = -\$1{,}250$$

The payoff on the FRA (PV of interest savings) is computed as:

$$\text{FRA payoff} = \$1{,}250/[1 + (0.0045 \times 90/360)] = -\$1{,}248.60$$

Since actual Libor-90 at FRA expiration (0.75%) is greater than the initial price of the FRA (0.70%), and the company has taken a short position on the FRA, the payoff to the company will be negative. It is obligated to give out a loan at 0.70% when the market rate is currently 0.75%. Stated differently, the short position receives the (lower) fixed rate and pays the (higher) floating rate.

3. The numbers that will be used for this computation are similar to part 2, except that Libor-90 at FRA expiration (0.75%) is now lower than the initial price of the FRA (0.80%). As the short position, the company benefits because it is obligated to give out a loan at a higher rate than the current market rate. Stated differently, the company (as the short position) will receive the higher fixed rate and pay the lower floating rate.

Note that this example provided you the rate that is used to determine the FRA settlement cash flow (0.45%). If this rate were not provided in the question, you would use current Libor-90 (as of FRA expiration), which is 0.75%, to discount the interest savings. Going forward, we will use the FRA rate at expiration as the discount rate to compute the payoff.

A generic formula used to compute the settlement payment of an FRA to the long position is:

$$\boxed{\text{FRA payoff} = \frac{\text{NP} \times [(\text{Market Libor} - \text{FRA rate}) \times \text{No.of days in the loan term}/360]}{= 1 + [\text{Market Libor} \times (\text{No.of days in the loan term}/360)]}}$$

Valuing an FRA Prior to Expiration

The value of an FRA during its term (before expiration) is computed as the present value of the difference in the interest savings based on (1) the new (current) FRA rate and (2) the old (original/initial) FRA rate. To value an FRA at any time prior to expiration, we need to follow these four steps:

1. Calculate the new implied forward rate based on current spot rates.
2. Calculate the interest savings based on this new or updated market forward rate.
3. Discount these interest savings for a number of days that equals the number of days remaining until FRA expiration plus the number of days in the term of the underlying hypothetical loan.
4. Make sure you use the appropriate discount rate as well.

Example 1-9: Calculating the Value of an FRA Prior to Expiration

Continuing from Example 1-7, suppose that 20 days into the term of the FRA, 10-day Libor is 5.50% and 130-day Libor is 7%. Given a notional principal of $1 million, calculate the value of Orix's long position on the FRA.

Solution:

We know that the FRA rate on the original contract was 6.22%. Our first task now is to compute the current FRA rate for an offsetting contract (same underlying and expiration date as the original FRA) based on the current term structure of Libor.

The first thing that we need to do is to unannualize the 10-day and 130-day Libor rates.

Unannualized rate on the 10-day loan = R10 = 0.055 × (10/360) = 0.001528
Unannualized rate on the 130-day loan = R130 = 0.07 × (130/360) = 0.025278

Based on these unannualized 10-day and 130-day rates, we calculate the current (as of t = 20) market forward rate applicable on a 120-day loan that will be taken 10 days from today as:

$$R120 = [(1 + R130)/(1 + R10)] - 1$$
$$R120 = [(1 + 0.025278)/(1 + 0.001528)] - 1 = 0.02371$$

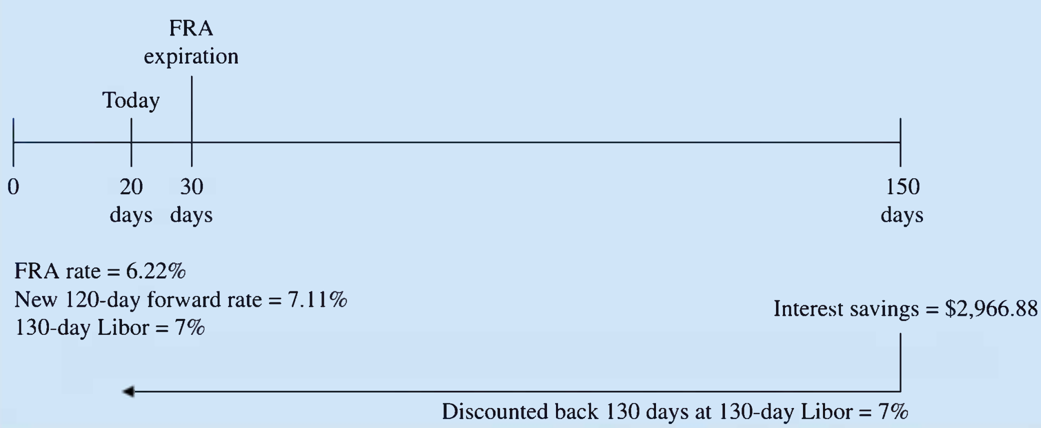
The annualized rate applicable on a 120-day loan 10 days from now equals 0.02371 × 360/120 = 0.0711 or 7.11%.

Next, we compute the interest savings based on the two (i.e., original and current) FRA rates.

$$\text{Interest savings} = [(0.0711 - 0.0622) \times 120/360)] \times 1{,}000{,}000 = \$2{,}966.88$$

The interest savings of $2,966.88 is expected to be realized at the end of the term of the underlying hypothetical loan (i.e., 130 days from today). Therefore (in order to determine the current value of the FRA), we must discount them at the current 130-day Libor for 130 days.

$$\text{PV(Interest savings)} = 2{,}966.88/[1 + (0.07 \times 130/360)] = \$2{,}893.74$$



Note that Orix took the long position on the original FRA. Over the subsequent 20 days, interest rates increased (new FRA rate = 7.11% compared to 6.22% initially), so the FRA holds positive value for Orix.

You can also use the following formula to determine the value of an FRA prior to expiration (super LOL):

$$\frac{\text{NP} \times [(\text{New forward rate} - \text{FRA rate}) \times \text{No.of days in the loan term}/360]}{1 + \{\text{New forward rate} \times [(\text{No.of days in the loan term} + \text{No.of days until contract expiration})/360]\}}$$

$$V_g(0,h,m) = \frac{1}{1 + L_g(h-g)\left(\frac{h-g}{360}\right)} - \frac{1 + \text{FRA}(0,h,m)\left(\frac{m}{360}\right)}{1 + L_g(h+m-g)\left(\frac{h+m-g}{360}\right)}$$

where g is the number of days since FRA initiation.

Fixed-Income Forward and Futures Contracts

When it comes to fixed-income forward and futures contracts, there are three unique factors that we need to account for when applying the carry arbitrage model:

1. In different markets, bond prices may be quoted as **clean prices** (without **accrued interest**) or **dirty/full prices** (that include accrued interest). While the quote convention for forward contracts, whether based on clean or dirty prices, usually corresponds to the quote convention in the respective bond market, it is necessary to understand the impact of quoting convention on derivative pricing. We will compute accrued interest using the following formula:

   Accrued interest = Accrual period × Periodic coupon amount

   $$\text{AI} = (\text{NAD}/\text{NTD}) \times (\text{C}/n)$$

   where:
   NAD = Number of accrued days since the last coupon payment
   NTD = Number of total days during the coupon payment period
   C = Stated annual coupon amount
   n = Number of coupon payments per year

2. We already know that futures contracts can be settled via physical delivery. When it comes to fixed-income futures, more than one bond can be delivered by the seller to settle the contract. Since those different bonds typically trade at different prices based on maturity and stated coupon, a **conversion factor adjustment** is applied to make all deliverable bonds approximately equal in value.
3. Since the conversion factor adjustment typically does not make all deliverable bonds exactly equal in value, a **cheapest-to-deliver bond** usually emerges. The short position on the contract will therefore choose to deliver this least expensive bond.

Markets Where Accrued Interest Is Included in the Bond Price Quote

In markets where bond prices and bond forward and futures quotes include accrued interest, the forward/futures price is computed using the same formula we applied earlier (Equation 3):

$F_0(T)$ = Future value of underlying adjusted for carry cash flows

$$F_0(T) = (S_0 - \gamma_0 + \theta_0)(1 + r)^T \quad \text{. . . (Equation 3)}$$

Markets Where Accrued Interest Is Not Included in the Bond Price Quote

For bonds quoted without accrued interest, we'll work with AI_0 = Accrued interest at $t = 0$.

- The spot price of the bond will be given as:
  - S_0 = Quoted bond price + Accrued interest = $B_0(T + Y) + AI_0$.
  - $B_0(T + Y)$ = Price of bond today.
    - T = Expiration of forward contract.
    - Y = Period from expiration of forward contract to maturity of bond.
- The present value of coupon income received during the term of the contract is γ_0.
- We know that there are no carry costs, so $\theta_0 = 0$.

The actual forward or futures price (the one that is used to compute the profit or loss on the position) is $F_0(T)$, but since there are multiple deliverable bonds in the market, there is a conversion factor adjustment, CF(T). Therefore, the price that is quoted is called the **quoted futures price**, denoted by QF_0.

$$F_0(T) = QF_0(T) \times CF(T) \text{ and } QF_0(T) = 1/CF(T) \times F_0(T)$$

where $F_0(T)$ is the future value of underlying adjusted for carry cash flows.

$$F_0(T) = (S_0 - PVCI_{0,T}) \times (1 + r)^T = [B_0(T + Y) + AI_0 - PVCI_{0,T}] \times (1 + r)^T$$

where $PVCI_{0,T}$ is the present value of coupon income received on the bond during the term of the forward contract.

So far, it is pretty straightforward. The price of the bond (S_0) has simply been broken down into the quoted bond price, $B_0(T + Y)$, which excludes accrued interest, and accrued interest (AI_0). We subtract the present value of coupon income received during the term of the contract ($PVCI_{0,T}$) from the price of the bond, and compound it over the term of the contract to attain the futures price (just like we subtracted the present value of dividends from the price of the stock to determine the forward price).

However, our formula is not complete yet. In a carry arbitrage for a stock, we would buy the stock now for S_0, and deliver it at contract expiration for S_T. Over here, we would buy the bond for $B_0(T + Y) + AI_0$ now and deliver it for $B_T(T + Y)$ **+ AI_T** at contract expiration.

The curriculum uses formulas that appear slightly different. Please rest assured that we are fundamentally doing the same thing. Our approach, in our humble opinion, is just a bit easier to follow.

Therefore, we need to account for the AI_T in our formula. Once we do this, our formula for the futures price of a bond quoted without accrued interest is given as:

$$F_0(T) = [B_0(T+Y) + AI_0 - PVCI_{0,T}] \times (1+r)^T - AI_T$$

And then finally, to obtain the expression for computing the quoted futures price, we apply the conversion factor adjustment:

$$QF_0(T) = 1/CF(T) \times [B_0(T+Y) + AI_0 - PVCI_{0,T}] \times (1+r)^T - AI_T$$

Example 1-10: Estimating a Bond Futures Price

U.S. Treasury futures have a contract value of $100,000. The underlying 1.5% Treasury bond is quoted at $110. One month has elapsed since the last coupon was paid, and the futures contract matures in another 2 months. There are no coupon payments due until after the futures contract expires, and the current risk-free rate is 1%. The conversion factor for this bond is 0.73581.

1. Compute accrued interest at initiation of the contract.
2. Compute accrued interest at expiration of the contract.
3. Compute the quoted futures price for the contract.

Solution:

1. The underlying is a Treasury bond. Treasury bonds make coupon payments semiannually, so the coupon paid every 6 months equals 1.5%/2 * 100 = $0.75.

   At t = 0 (the time of contract initiation), 1 month has elapsed in the coupon period (which is 6 months). Therefore, accrued interest is calculated as:

   $$AI_0 = \$0.75*1/6 = \$0.1252$$

2. At t = 2 (time of contract expiration), 3 months would have elapsed in the coupon period. Therefore:

   $$AI_T = \$0.75*3/6 = \$0.375$$

3. The quoted futures price (that accounts for the conversion factor) is calculated as:

$$QF_0(T) = 1/CF(T) * \{[B_0(T+Y) + AI_0 - PVCI_{0,T}] * (1+r)^T - AI_T\}$$

$$QF_0(2/12) = (1/0.73581) * \{[(110 + 0.125 - 0) * (1+0.01)^{2/12}] - 0.375\}$$

$$= \$149.40$$

Determining the Value of a Bond Forward Contract during Its Term

As was the case in earlier sections, the value of a bond forward contract during its term is simply the present value of the difference in **quoted forward/futures prices** between (1) the original bond forward/futures contract and (2) an offsetting contract. In the bond arena, we also need to apply the conversion factor adjustment to compute the **quoted** futures price.

Example 1-11: Estimating the Value of a Bond Forward Position

Suppose that 1 month ago, we purchased four Treasury bond forward contracts with 2 months to expiration and a contract notional of $100,000 each at a price of 149.40 (quoted as a percentage of par). The forward contracts now have 1 month to expiration. The underlying is a 1.5% Treasury bond quoted at 110 and has accrued interest of 0.125 (1 month since last coupon). At the contract expiration, the underlying bond will have accrued interest of 0.375, there are no coupon payments due until after the forward contract expires, and the current risk-free rate is 1%. If the current forward price is 150, compute the value of the forward position.

Solution:

The value of the bond forward position is simply the present value of the difference in the two quoted forward prices.

$$V_t(T) = \text{Present value of difference in forward prices} = PVt,_T[QF_t(T) - QF_0(T)]$$
$$= (150 - 149.40)/(1 + 0.01)^{1/12} = 0.5995$$

Note that this is 0.5995 per 100 of par because the forward price was quoted per 100 of par. Since we went long on four contracts with a notional of $100,000 each, the value of our long position is actually 0.5995/100 * 100,000 * 4 = $2,398.

Note that since interest rates are so low and the forward contract has a short maturity, the present value effect is nominal.

Currency Forward Contracts

Pricing a Currency Forward Contract

The determination of the forward price (forward exchange rate) of a currency forward is relatively straightforward. It relies on the concept of **interest rate parity**, which asserts that an investor would earn the same amount from the following two investment options that cover the same time horizon.

We will be working with exchange rate quotes of price currency per unit of base currency (PC/BC).

- Investing one unit of the price currency at the price currency risk-free rate.
- Converting one unit of price currency into base currency (at the current spot exchange rate, $S_{0,PC/BC}$), investing the base currency amount at the base currency risk-free rate (r_{BC}), and at the same time entering into a forward contract to convert the base currency proceeds back to the price currency at the forward rate $F_{0,PC/BC}$.

Interest rate parity: $F_{0,PC/BC} = \dfrac{S_{0,PC/BC} \times (1 + r_{PC})^T}{(1 + r_{BC})^T}$

where
r_{PC} = Price-currency risk-free rate
r_{BC} = Base-currency risk-free rate
T = Length of the contract in years

If you stick to our quoting convention, just remember that the currency in the numerator of the currency quote should have its risk-free rate in the numerator as well (r_{PC}).

Example 1-12: Calculating the Price of a Currency Forward Contract

Consider the following information:

Spot exchange rate = \$1.5/€
Annual risk-free rate in the United States = 4%
Annual risk-free rate in Germany = 7%

Calculate the forward exchange rate for a 1-year forward contract.

Solution:

$$F_{0,\$/€} = \frac{S_{0,\$/€} \times (1 + R_{\$})^{T}}{(1 + R_{€})^{T}}$$

$$F_{0,\$/€} = 1.5 \times (1.04/1.07) = \$1.4579/€$$

Bear in mind that:

- If the forward rate is higher than the spot rate, it means that the price currency risk-free rate is higher than the base currency risk-free rate.
- If the forward rate is lower than the spot rate, it means that the price currency risk-free rate is lower than the base currency risk-free rate.

If we assume continuous compounding, the forward exchange rate can be computed as:

$$\boxed{F_{0,PC/BC} = S_{0,PC/BC} \times e^{(rPC-rBC)\times T}}$$

Valuing a Currency Forward Contract

The value of the **long** position in a currency forward contract at any time prior to maturity can be calculated as the present value of the difference in foreign exchange forward prices. Note that the price currency risk-free rate is used in the denominator.

$$\boxed{V_t(T) = (F_{t,PC/BC} - F_{0,PC/BC})/(1 + r_{PC})^{T-t}}$$

Example 1-13: Calculating the Value of a Currency Forward Contract Prior to Expiration

A company has sold €20 million against the USD forward at a forward rate of $1.15/€ at t = 0. The current spot exchange rate is $1.10/€. The annual risk-free rates are 0.75% for the USD and 0.35% for the EUR. Three months remain until the forward contract expiration.

1. Determine the current forward exchange rate.
2. Determine the current value of the forward contract position held by the company.

Solution:

In answering these questions, the first thing you should do is ensure that the currency on which a position is taken, whether long or short, is treated as the base currency in all formulas. In this example, the company has sold EUR forward, so we will treat the EUR as the base currency.

1. The forward exchange rate is calculated using the interest rate parity formula, with the EUR as the base currency.

$$F_{0,PC/BC} = \frac{S_{0,PC/BC} \times (1 + r_{PC})^T}{(1 + r_{BC})^T}$$

$$F_{0,PC/BC} = 1.10 \times (1.0075/1.0035)^{3/12}$$

$$= \$1.10109/€$$

2. The value per EUR to the long on the forward contract is calculated as:

$$V_t(T) = (F_{t,PC/BC} - F_{0,PC/BC})/(1 + r_{PC})\ T^{-t}$$

$$= (1.10109 - 1.15)/(1 + 0.0075)^{0.25}$$

$$= \$0.048819/€$$

The contract holds a value of –$0.048819/€ to the long position. The company here is short on €20 million. The value of the position is therefore €20,000,000 * $0.048819/€ = $976,380. The forward price of the EUR falls over the period, so the short benefits. Notice that the value of the forward is attained in terms of the price currency (PC = USD).

LESSON 2: PRICING AND VALUATION OF SWAP CONTRACTS

LOS 37c: Describe and compare how interest rate, currency, and equity swaps are priced and valued. Vol 5, pp 353–372

LOS 37d: Calculate and interpret the no-arbitrage value of interest rate, currency, and equity swaps. Vol 5, pp 353–372

Introduction

The carry arbitrage model that we used to price and value forwards and futures can also be used to price and value swaps. Swaps can be valued using the (1) portfolio of underlying instruments approach or (2) the portfolio of forward contracts approach. We will be working with the former in this reading.

Plain-Vanilla Interest Rate Swaps (Level I Recap)

- A plain-vanilla interest rate swap involves the exchange of fixed interest payments for floating-rate payments.
- The party that wants to receive floating-rate payments and agrees to make fixed-rate payments is known as the **pay-fixed** side of the swap or the **fixed-rate payer/floating-rate receiver**.
- The party that wants to receive fixed payments and make floating-rate payments is the **pay-floating** side of the swap or the **floating-rate payer/fixed-rate receiver**.
- Note that there is no exchange of notional principal at initiation or expiration of the swap. The notional principal is simply used to determine the interest payment on each leg of the swap.
- Interest payments are not exchanged in full at each settlement date. Interest payments are netted, and the party that owes more in interest at a particular settlement date makes a payment equal to the difference to the other.
- As with forward contracts, there is an element of counterparty credit risk in swaps, as the party that owes the lower amount can default.
- The floating rate is usually quoted in terms of Libor plus a spread. The floating rate for any period is known at the beginning of the period, while the settlement payment is actually made at the end of each period. We will therefore assume that swaps work on an **advanced set, settled in arrears** basis.

The formula for the net payment made (received) by the fixed-rate payer is given by:

$$\text{Net fixed rate payment}_t = [\text{Swap fixed rate} - (\text{Libor}_{t-1} + \text{Spread})] * (\text{No.of days}/360) * \text{Notional principal}$$

Pricing versus Valuation of Swaps (Level I Recap)

The distinction between the price and value of a swap is comparable to the distinction that we made between the price and value of a forward contract earlier in the reading. The price of a forward is the forward rate/price that results in zero value to either party at initiation of the contract. Subsequently, over the term of contract, the value of the forward to the long/short position fluctuates as the price/rate of the underlying changes.
We will again work with a plain-vanilla interest rate swap to illustrate the difference between the price and the value of a swap.

- At the initiation of the swap, the **swap fixed rate** is set at a level at which the present value of the floating-rate payments (based on the current term structure of interest rates) equals the present value of fixed-rate payments so that there is zero value to either party. This swap fixed rate therefore represents the **price** of the swap.
- Over the term of the swap, as there are changes in the term structure of interest rates, the **value** of the swap will fluctuate.
    - If interest rates increase after swap initiation, the present value of floating-rate payments (based on the new term structure) will exceed the present value of fixed-rate payments (based on the swap fixed rate).
- The swap will have a positive value for the fixed-rate payer (floating-rate receiver).
- The swap will be an asset to the fixed-rate payer and a liability for the floating-rate payer.
    - If interest rates decrease after swap initiation, the present value of floating-rate payments will be lower than the present value of fixed-rate payments.
- The swap will have a positive value for the floating-rate payer (fixed-rate receiver).
- In this case, the swap will be an asset to the floating-rate payer and a liability for the fixed-rate payer.

The Value of a Floating-Rate Bond at Each Reset Date (Optional Material)

The first thing that you need to understand before getting into pricing and valuing swaps is that a floating-rate bond trades at par at issuance and at any reset date during its life. The reason for this is that at the reset date, the coupon rate on the bond is adjusted and brought into line with market interest rates, and when the coupon rate on a bond equals the market interest rate, it must trade at par.

For example, let's assume that today is the last reset date on a quarterly reset $1,000 floating-rate bond where the coupon rate equals Libor-90. If Libor-90 equals 6% today, it means that after 90 days, this bond:

- Will pay a coupon of 0.06 * 90/360 * 1,000 = $15, and
- Will return the principal amount (par value) of $1,000.

To determine the current value of this bond, we compute the present value of the $15 + $1,000 = $1,015 payment using the current market interest rate (Libor-90 = 6%) to discount it.

$$\boxed{1{,}015/[1 + (0.06*90/360)] = \$1{,}000}$$

Now let's go back in time to the second to last reset date. Suppose Libor-90 at this point in time is 4%. This means that after 90 days (at the last reset date) this bond:

- Will pay a coupon of 0.04 * 90/360 * 1,000 = $10, and
- Will be worth $1,000 (as determined previously).

To determine the current value of this bond, we compute the present value of $10 + $1,000 = $1,010 using the current market interest rate (Libor-90 = 4%) to discount it.

$$\boxed{1{,}010/[1 + (0.04*90/360)] = \$1{,}000}$$

This example shows that the price of a floating-rate bond will equal its par value at (1) issuance and at (2) any reset date. This takeaway will be important as we move into pricing and valuing swaps.

Note that when we talk about valuing swaps, we will discuss only scenarios where swaps are being valued at a settlement date. Valuing swaps in between settlement dates requires some adjustments, which are outside the scope of the Level II curriculum.

Plain-Vanilla Interest Rate Swaps as a Combination of Bonds

Taking a position as a fixed-rate payer (floating-rate receiver) on a plain-vanilla interest rate swap is equivalent to issuing a fixed-rate bond (on which fixed payments must be made) and using the proceeds to purchase a floating-rate bond (on which floating payments will be received). In other words, the fixed-rate payer can be viewed as being long on a floating-rate bond and short on a fixed-rate bond.

We assume that (1) both bonds are purchased at par, (2) the initial net cash flow is zero, (3) the payments at the end offset each other, (4) coupon dates on the bonds match settlement dates on the swap, and (5) the maturity of the bonds coincides with the expiration date of the swap.

- If interest rates increase, the fixed-rate payer benefits, as there is a positive difference between her (floating-rate) receipts and (fixed-rate) payments.
    - In terms of the positions on bonds, the value of the fixed-rate bond decreases as interest rates increase, but the value of the floating-rate bond remains at par. Since the fixed-rate payer is long on the floating-rate bond and short on the fixed-rate bond, she benefits from the increase in interest rates.
- If interest rates decrease, the fixed-rate payer loses out, as there is a negative difference between (floating-rate) receipts and (fixed-rate) payments.
    - In terms of the positions on bonds, the value of the fixed-rate bond increases as interest rates decrease, but the value of the floating-rate bond remains at par. Since the fixed-rate payer is long on the floating-rate bond and short on the fixed-rate bond, she loses out as a result of the decrease in interest rates.

Before moving on, we must mention that payments in an interest rate swap are a series of **net** interest payments. Since they are in the same currency, the coupon payments are not exchanged in their entire amounts by the counterparties, but are **netted**. Further, there is no exchange of (notional) principal at the initiation and final settlement of an interest rate swap. However, to be able to use the equivalence of interest rate swaps to bonds in order to price and value swaps, we introduce the entire coupon payments and notional principal as if they were actually exchanged. Since the overall amounts are the same, we have done no harm, but we are able to treat the counterparties in the swap as though they hold (opposite) positions in fixed-rate and floating-rate bonds.

Pricing a Swap: Determining the Swap Fixed Rate

To price an interest rate swap, we compute the swap fixed rate that results in the swap having zero value to either party at initiation.

We know that at issuance a floating-rate bond is priced at par, as the coupon rate in effect over the next settlement period equals the market interest rate. Our task is to compute the coupon rate on the fixed-rate bond (which represents the swap fixed rate) that would result in it having the same value as the floating-rate bond (i.e., par), and the swap having zero value at initiation.

Let's work with 1-year quarterly-pay bonds with a par value of $100 to illustrate swap pricing. The floating-rate bond will be worth $100 today. In order for the swap to have zero value to either party, the fixed-rate bond must also be worth $100 today. The value of the fixed-rate bond equals the present value of the four (fixed) coupon payments (C) and the principal payment. Given the term structure of interest rates as of today (swap initiation or, equivalently, bond issuance), we can use current Libor-90, Libor-180, Libor-270, and Libor-360 to calculate the coupon payment per $100 of principal (which represents the coupon rate on the fixed-rate bond, and the swap fixed rate) as:

$$100 = \frac{C}{1 + \left(\text{Libor-90} * \frac{90}{360}\right)} + \frac{C}{1 + \left(\text{Libor-180} * \frac{180}{360}\right)} + \frac{C}{1 + \left(\text{Libor-270} * \frac{270}{360}\right)} + \frac{C}{1 + \left(\text{Libor-360} * \frac{360}{360}\right)} + \frac{100}{1 + \left(\text{Libor-360} * \frac{360}{360}\right)}$$

We can represent the different Libor rates as of today (t = 0) as:

Libor-90 today = $L_0(90)$
Libor-180 today = $L_0(180)$
Libor-270 today = $L_0(270)$
Libor-360 today = $L_0(360)$

Based on the Libor rates, the present value factors (B) as of today (t = 0) can be represented as:

$$B_0(90) = \frac{1}{1 + [L_0(90) * (90/360)]}$$
$$B_0(180) = \frac{1}{1 + [L_0(180) * (180/360)]}$$
$$B_0(270) = \frac{1}{1 + [L_0(270) * (270/360)]}$$
$$B_0(360) = \frac{1}{1 + [L_0(360) * (360/360)]}$$

$B_0(90)$ refers to the current present value factor as of today (swap initiation, t = 0) for a payment that will be received on Day 90 of the swap.

Therefore, we can compute the (quarterly) swap fixed rate (by solving for C) as a percentage as:

$$100 = B_0(90) * C + B_0(180) * C + B_0(270) * C + B_0(360) * C + B_0(360) * 100$$
$$100 = C\left[B_0(90) + B_0(180) + B_0(270) + B_0(360)\right] + B_0(360) * 100$$

$$C = \frac{100 - B_0(360) * 100}{B_0(90) + B_0(180) + B_0(270) + B_0(360)}$$
$$C = \left[\frac{1 - B_0(360)}{B_0(90) + B_0(180) + B_0(270) + B_0(360)}\right] \times 100$$

Or more generally,

$$\text{Swap fixed rate} = \left[\frac{1 - B_0(N)}{B_0(1) + B_0(2) + B_0(3) + \ldots + B_0(N)}\right] \times 100$$

Example 2-1: Calculating the Fixed Rate on an Interest Rate Swap

Annualized Libor spot rates today are provided:

$L_0(90) = 0.0325$
$L_0(180) = 0.0375$
$L_0(270) = 0.0425$
$L_0(360) = 0.0460$

Consider a 1-year swap with quarterly payments and a notional principal of $20 million.

1. Determine the (annualized) fixed rate on the swap.
2. Calculate the quarterly fixed payments in dollars.

Solution:

1. In order to determine the fixed rate on the swap, we first need to calculate the present value factors based on the current term structure of Libor spot rates.

$$B_0(90) = \frac{1}{1 + [0.0325 * (90/360)]} = 0.9919$$

$$B_0(180) = \frac{1}{1 + [0.0375 * (180/360)]} = 0.9816$$

$$B_0(270) = \frac{1}{1 + [0.0425 * (270/360)]} = 0.9691$$

$$B_0(360) = \frac{1}{1 + [0.0460 * (360/360)]} = 0.9560$$

The quarterly fixed rate on the swap is then calculated as:

$$\text{Swap fixed rate} = \left[\frac{1 - B_0(N)}{B_0(1) + B_0(2) + B_0(3) + \ldots + B_0(N)}\right] \times 100$$

$$\text{Quarterly swap fixed rate} = \frac{1 - 0.9560 * 100}{0.9919 + 0.9816 + 0.9691 + 0.9560} = 1.128\%$$

Therefore, the (annual) swap fixed rate equals 0.01128 * (360/90) = 0.04512 = 4.512%.

2. Given a notional principal of $20 million, the quarterly payment on the pay-fixed side of the swap is calculated as:

Quarterly fixed payment = $20,000,000 * 0.01128 = $225,600

When working with Libor, always remember to unannualize the rates provided.

Valuing a Swap

To value a swap, we simply enter an offsetting swap. The floating-rate payments cancel each other out, leaving us with the difference in fixed rates. Therefore, the value of a plain-vanilla interest rate swap is the sum of the present values of the difference in swap fixed rates multiplied by the notional amount. (This statement should become clearer after Example 2-2.)

Example 2-2: Computing the Value of a Swap

Suppose that 6 months ago we entered into an 18-month swap as the fixed-rate receiver. The rate on that swap was 4.836%. Current market Libor rates are:

$L_{180}(90) = 0.0325$
$L_{180}(180) = 0.0375$
$L_{180}(270) = 0.0425$
$L_{180}(360) = 0.0460$

Notice that these are the same Libor rates as the previous example, so we know that the current swap fixed rate should equal 4.512%. Determine the value of the swap to the party receiving the fixed rate.

Solution:

The fixed rate on an offsetting swap (with the same underlying, settlement dates, day count convention, and expiration date as the original swap) entered into today (4.512%) is *lower* than the rate on the original swap (4.836%). The fixed-rate receiver on the original swap receives 4.836% and pays only 4.512% on the offsetting swap. The value of the swap to the fixed-rate payer is therefore positive, and is calculated as the difference in the present values of the two coupon streams. Note that since this a plain-vanilla interest rate swap, we do not have to consider an exchange of the notional amount at expiration. When we study currency swaps, we will have to consider cash flows at expiration as well.

Day (Note that we are at Day 180)	Cash Flows		Difference in Cash Flows	Discount Factor	Present Value of Difference in Cash Flows
	SFR = 4.836%	SFR = 4.512%			
270	241,800	225,600	16,200	90 days = 0.9919	16,068.78
360	241,800	225,600	16,200	180 days = 0.9816	15,901.92
450	241,800	225,600	16,200	270 days = 0.9691	15,699.42
540	241,800	225,600	16,200	360 days = 0.956	15,487.20
			Total	3.8986	**63,157.32**

The method in Example 2-2 should make intuitive sense. Through some algebra, we can obtain an equation for computing the value of a plain-vanilla interest rate swap. This formula is presented next. Note that PSFR stands for the **periodic** swap fixed rate. In our case, since this is a quarterly-pay swap, the swap fixed rates must be divided by 4 to compute the periodic swap fixed rates.

$V = NA * (PSFR_0 - PSFRt) * \text{Sum of PV factors of remaining coupon payments as of } t = t$

$V = \$20{,}000{,}000 * (0.04836/4 - 0.04512/4) * 3.8986 = \$63{,}157.32$

Note that if this value is positive, it means that the initial swap fixed rate is greater than the current swap fixed rate, which implies that the swap holds positive value for the **fixed-rate receiver** on the original swap.

Currency Swap Contracts

CFA Institute appears to have dropped the ball here. Currency swaps enjoyed detailed coverage at Level I prior to 2015, but are no longer covered at Level I. The material that follows is absolutely critical for you to understand currency swaps at Level II.

An Introduction to Currency Swaps

Currency Swaps

Suppose that a Chinese manufacturing firm, Chang Enterprises, wants to expand into the U.S. market by building a plant in the U.S. Chang needs a $10 million loan to finance this project. At the same time, a U.S. company, Jetex Industries, wants to expand into the Chinese market and build a manufacturing facility worth 69 million Chinese yuan. The ¥/$ exchange rate is ¥6.90/$, so effectively, both companies require the same amount of financing.

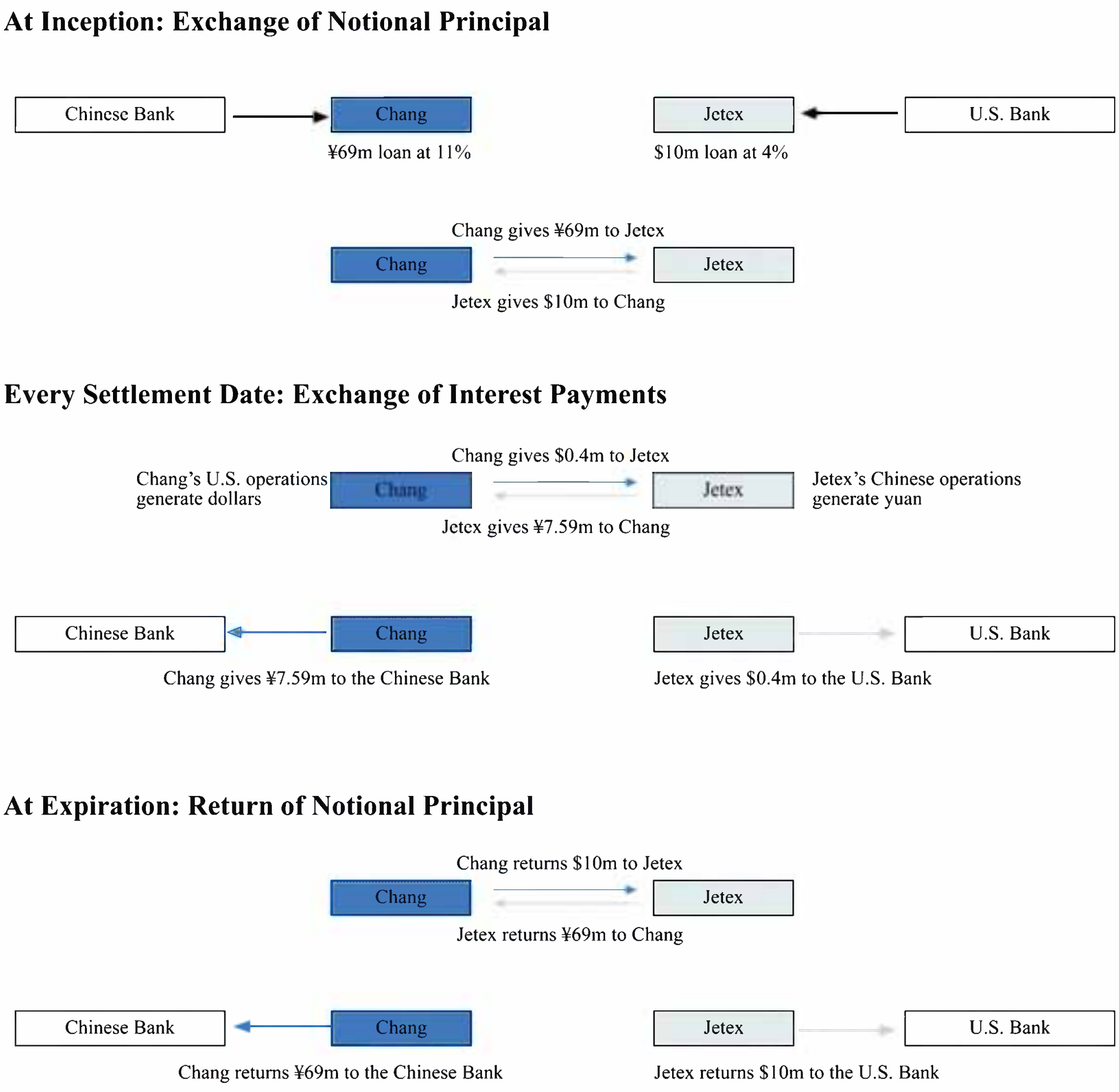
Jetex has a longstanding relationship with U.S. banks, but no credit history whatsoever in China. It can get a dollar loan in the U.S. at 4% and a yuan loan in China for 13%. Chang, on the other hand, has significant banking relationships in China, but none in the U.S. It can obtain a yuan loan in China at 11% and a dollar loan in the U.S. at 6%. Chang would like to take a dollar loan to fund its U.S. operations because its revenues (which will be used to service the loan) from the U.S. facility will also be in dollars. Similarly, Jetex plans to service the debt taken to build its Chinese facilities with revenues in China, which will be in yuan.

This example illustrates how currency swaps can be beneficial. Chang and Jetex will take loans in their domestic currencies, on which they will get relatively favorable terms, and then exchange the loans. In a simplistic scenario, ignoring commissions and other fees, Chang effectively could obtain a loan of $10 million at an interest rate of 4%, while Jetex could get a loan of ¥69 million at 11%. Notice that:

- Because these loans are taken to establish operations and fund the construction of manufacturing facilities, the notional principal is exchanged at swap initiation.
- Because interest payments must be made in full to the respective banks in different currencies, interest payments are not netted. Full interest payments are exchanged at every settlement date in different currencies.
- At the end of the swap the notional principal is exchanged again because the loan must be retired in different currencies. Remember that at the end of the swap ¥69 million will be exchanged for $10 million, irrespective of the prevailing exchange rate.

A currency swap is equivalent to issuing a fixed- or floating-rate bond in one currency, converting the proceeds into another currency, and using the proceeds to purchase a fixed- or floating-rate bond in the other currency. Therefore, currency swaps can be used to address both interest rate and currency exposures.

Figure 2-1: Fixed-for-Fixed Currency Swap.



Pricing Currency Swaps

The procedure for pricing and valuing currency swaps is similar to the one we illustrated for interest rate swaps. Let's begin by working with a fixed-for-fixed currency swap. We now have to deal with two term structures of interest rates (one for each country) and have to determine two swap fixed rates (one for each currency). In the analysis that follows, we look at things from the perspective of a U.S. investor, with the USD representing the domestic or price currency (DC), and the GBP representing the foreign or base currency (FC).

Let's start with the domestic currency leg of the swap (USD). The swap fixed rate on the USD leg is determined just as we illustrated earlier for a plain-vanilla interest rate swap (where both sides are denominated in USD). We know that the value of a floating-rate bond with a face value of $100 will be $100 at issuance. Therefore, the fixed rate on a plain-vanilla interest rate swap (in order to ensure that the swap has zero value at initiation) is the coupon rate on a fixed-rate USD bond that is trading at par ($100) at swap initiation.

The swap fixed rate on the GBP leg is determined just as we illustrated earlier for a plain-vanilla interest rate swap (where both sides would be denominated in GBP). In the United Kingdom, the value of a floating-rate bond with a face value of £100 will be £100 at issuance. Therefore, the fixed rate on a plain-vanilla interest rate swap in the United Kingdom will be the coupon rate that would make the present value of interest and principal payments on a fixed-rate bond equal to par (£100).

Notice that the present value of the fixed-rate payments on the USD bond (discounted at the U.S. swap fixed rate) does not equal the present value of the fixed-rate payments on the GBP bond (discounted at the U.K. swap fixed rate).

- The present value of the fixed-rate payments on the USD bond is **$**100.
- The present value of the fixed-rate payments on the GBP bond is **£**100.

Assuming a current (as of swap initiation) exchange rate of 1.50 USD/GBP, the value (in terms of USD) of the present value of the GBP fixed-rate payments is $1.50/£ * £100 = $150, which is greater than the present value of the USD fixed-rate payments ($100). To ensure that the currency swap has zero value at initiation (i.e., the present values of the payments on both sides of the currency swap are equal), we adjust the notional principal on the foreign currency leg of the swap. The notional principal on the foreign currency equals the notional principal on the domestic currency leg divided by the current spot rate (expressed as DC/FC), S_0. In our example, $100 / 1.50 = £66.67, which is the same as $100. Now the present values of payments on both sides of the swap are equal.

To summarize:

- The fixed rates on a currency swap are still the fixed rates on plain-vanilla interest rate swaps in the respective countries.
- All we need to do to ensure that there is zero value to either party at initiation of a currency swap is to adjust the notional principal in the foreign currency by dividing the notional principal in the domestic currency by the current spot rate (expressed as DC/FC).

Now let's move on to the other types of currency swaps. Once you've understood how a fixed-for-fixed currency swap is priced, the others are fairly straightforward to understand:

- For fixed-for-floating currency swaps:
    - The rate on the fixed side is just the fixed rate on a plain-vanilla interest rate swap in the given currency.
    - The payments on the floating side will have the same present value as the payments on the fixed side as long as the foreign notional principal is adjusted.
- For floating-for-floating currency swaps:
    - We do not need to determine the price (swap fixed rate), as both sides pay a floating rate.

- However, we do still need to adjust the foreign notional principal to ensure that the present values of payments on both sides of the swap are equal.

Example 2-3: Calculating the Fixed Rate on a Currency Swap

Consider a one-year currency swap with quarterly payments. The two currencies are the USD and EUR, and the current exchange rate is \$1.25/€. The current term structures of Libor (which is used in the United States) and Euribor (which is used in the Eurozone) are as follows:

Days	Libor (%)	Euribor (%)
90	3.25	3.80
180	3.75	4.65
270	4.25	5.90
360	4.60	6.75

1. Determine the fixed rate in euros on an annualized basis.
2. Given a notional principal of \$20 million, determine the quarterly cash flows on:
   i. A pay \$ fixed, receive € fixed currency swap.
   ii. A pay \$ fixed, receive € floating currency swap.

Solution:

1. In order to determine the fixed rate in euros, we first need to calculate the present value factors based on the term structure of Euribor spot rates.

$$B_0(90) = \frac{1}{1 + [0.0380 * (90/360)]} = 0.9906$$

$$B_0(180) = \frac{1}{1 + [0.0465 * (180/360)]} = 0.9773$$

$$B_0(270) = \frac{1}{1 + [0.0590 * (270/360)]} = 0.9576$$

$$B_0(360) = \frac{1}{1 + [0.0675 * (360/360)]} = 0.9368$$

The quarterly fixedrate on the swap is then calculated as:

$$\text{Swap fixed rate} = \left[\frac{1 - B_0(N)}{B_0(1)+B_0(2)+B_0(3) + \ldots +B_0(N)}\right] \times 100$$

$$\text{Quarterly fixed rate in EUR} = \frac{1 - 0.9368 * 100}{0.9906 + 0.9773 + 0.9576 + 0.9368} = 1.637\%$$

Annualized swap fixed rate in EUR = 1.637% * (360/90) = 6.549%

2. In order to compute the payments on the swap, we need to determine the notional principal in EUR by dividing the USD notional principal by the current spot rate (\$/€).

NP in EUR = \$20 million ÷ \$1.25/€ = €16 million

i. Pay $ fixed, receive € fixed currency swap:
   - At the initiation of the swap the "pay $ fixed" side would receive $20 million in return for €16 million.
   - Notice that the term structure of Libor is the same as in Example 2-1. Therefore, the quarterly fixed rate on the USD leg of the swap equals 1.128%. At the end of each quarter:
     - The "pay $ fixed" side would pay 1.128% on $20 million, which amounts to $225,600.
     - The "pay $ fixed" side would receive 1.637% on €16 million, which amounts to €261,948.
   - At the end of the year (at the end of the tenor of the swap), the "pay $ fixed" side would repay $20 million and get back €16 million.

ii. Pay $ fixed, receive € floating currency swap:
   - At the initiation of the swap the "pay $ fixed" side would receive $20 million in return for €16 million.
   - At the end of each quarter:
     - The "pay $ fixed" side would pay 1.128% on $20 million, which amounts to $225,600.
     - The "pay $ fixed" side would receive the relevant Euribor on €16 million. For example, 90-day Euribor at swap inception was 3.8%. Therefore, the first quarterly payment that the "pay $ fixed" side would receive would amount to €152,000 (= 0.038 * 90/360 * €16 million).
     - At the end of the year (at the end of the tenor of the swap), the "pay $ fixed" side would repay $20 million and get back €16 million.

Recall that for currency swaps, the notional amounts are actually exchanged at swap initiation and termination. Further, the periodic payments are not netted.

Don't be confused if you get slightly different numbers, as the difference may be attributable to rounding.

Valuing a Currency Swap

The value of a currency swap at any point in time during its tenor is determined in a similar manner as that of an interest rate swap, except that we need to account for fluctuations in exchange rates, which apply to (1) settlement payments during the tenor of the swap *and* (2) exchange of notional amounts at expiration of the swap.

Example 2-4: Calculating the Value of a Currency Swap after Initiation

Continuing from Example 2-3, suppose that 60 days into the tenor of the swap, the term structures for the USD and EUR are as follows:

Days	Libor (%)	Euribor (%)
30	3.10	3.50
120	3.25	4.15
210	3.80	5.05
300	4.35	6.10

Further, the exchange rate currently stands at $1.20/€. Given notional principals of $20 million and €16 million, calculate the value of a fixed-for-fixed currency swap.

Solution:

Note: From earlier examples, recall that the USD swap fixed rate at swap initiation was 4.512% and the EUR swap fixed rate was 6.549%. The notional amounts were $20 million and €16 million.

Now note that we mentioned earlier in the section that we will be valuing these swaps only as of a settlement date (because the curriculum explicitly states that valuation on a non-settlement date is "outside the scope of this reading"). In the blue-box example covering valuation of currency swaps, however, they value a swap on a non-settlement date, but they only work with a fixed-for-fixed currency swap. That is exactly what we will do here. Note that valuation on a non-settlement date only complicates valuation of a fixed-for-floating or floating-for-floating currency swap. Valuing a fixed-for-fixed currency swap on a non-reset date requires the same steps as valuing it on a reset date. This process is described next:

To determine the present values (as of t = 60) of the remaining fixed payments in USD and EUR, we need to recalculate the discount factors based on the current term structure of Libor and Euribor, respectively:

USD

$$B_{60}(90) = \frac{1}{1 + [0.0310 * (30/360)]} = 0.9974$$

$$B_{60}(180) = \frac{1}{1 + [0.0325 * (120/360)]} = 0.9893$$

$$B_{60}(270) = \frac{1}{1 + [0.0380 * (210/360)]} = 0.9783$$

$$B_{60}(360) = \frac{1}{1 + [0.0435 * (300/360)]} = 0.9650$$

The value of a fixed-rate leg of the swap is calculated as the present value of the remaining coupon payments and the notional principal (each discounted using the relevant discount factor based on the relevant Libor rate and time remaining until payment). For example, the discount factor (0.9974) for the next coupon payment (on Day 90 of the life of the swap) is based on current (as of Day 60 of the life of the swap) Libor-30 (3.10%) and 30 days until payment.

Day	Cash Flow	Discount Factor	Present Value
90	$225,600	30 days = 0.9974	$225,018.70
180	$225,600	120 days = 0.9893	$223,182.19
270	$225,600	210 days = 0.9783	$220,707.65
360	$20,225,600	300 days = 0.9650	$19,518,069.96
		Total = 3.93	**$20,186,978.51**

EUR

$$B_{60}(90) = \frac{1}{1 + [0.0350 * (30/360)]} = 0.9971$$

$$B_{60}(180) = \frac{1}{1 + [0.0415 * (120/360)]} = 0.9864$$

$$B_{60}(270) = \frac{1}{1 + [0.0505 * (210/360)]} = 0.9714$$

$$B_{60}(360) = \frac{1}{1 + [0.0610 * (300/360)]} = 0.9516$$

The value of the fixed-rate EUR leg is calculated as the present value of the expected EUR coupon payments and the notional principal. Based on the quarterly EUR fixed rate of 1.637%, the quarterly payments amount to €261,948.

Day	Cash Flow	Discount Factor	Present Value
90	€261,948	30 days = 0.9971	€261,185.71
180	€261,948	120 days = 0.9864	€258,373.34
270	€261,948	210 days = 0.9714	€254,451.78
360	€16,261,948	300 days = 0.9516	€15,475,287.08
		Total = 3.9065	**€16,249,297.9**

Next, we convert the present value of the EUR fixed rate payments into USD based on the current (as of swap valuation) exchange rate of $1.20/€.

PV (in USD) of fixed-rate EUR payments = €16,249,297.9 * $1.20/€ = $19,499,157.48

The value of the fixed-for-fixed currency swap can now be calculated as:

Pay $ fixed, receive € fixed = –$20,186,978.51 + $19,499,157.48 = –$687,821.03

Now that we've illustrated the intuitive way of valuing this swap, let's get to the curriculum's method. Some algebra will lead you to the following formula for valuing a currency swap:

$V = NA_{PC}$ * ($PSFR_{PC}$ * Sum of PV factors of remaining coupon payments$_t$ + PV factor for return of notional amount$_t$)

– $S_{t,PC/BC}$ * NA_{BC} * ($PSFR_{BC}$ * Sum of PV factors of remaining coupon payments$_t$ + PV factor for return of notional amount$_t$)

$$V = \$20m * (4.512\%/4 * 3.93 + .9650)$$
$$-(\$1.20/€)(€16m)(6.549\%/4 * 3.9065 + 0.9516)$$
$$= \$20{,}186{,}608 - 19{,}498{,}736 = \$687{,}872 \rightarrow \text{Difference in values is due to rounding.}$$

Note that this formula is different from the one for valuing an interest rate swap because we need to account for (1) the fluctuations in the exchange rate over the tenor of the swap and (2) the fact that coupon payments are not netted and notional amounts are also exchanged at swap expiration in a currency swap

The derivation of this formula is illustrated in our comprehensive video for this reading.

Equity Swap Contracts

Introduction

An **equity swap** is an over-the-counter (OTC) derivative contract in which two parties agree to exchange a series of cash flows. One party pays a variable series that will be determined by an equity security, and the other party pays either (1) a variable series determined by a different equity security or rate or (2) a fixed series. An equity swap is used to convert the returns from an equity investment into another series of returns that is derived from another equity series or fixed rate.

Equity Swaps

In an equity swap one party exchanges the return on a stock, portfolio, or an equity index for a fixed or floating interest payment.

Equity swaps distinguish themselves from other swaps in the following ways:

- One side of the swap can end up making the payment for both sides. For example, suppose an investor enters the pay-fixed side of an equity swap. She makes fixed payments in return for the return on an equity index over a period. If the return on the index is negative for a given settlement period (the index level falls), the pay-fixed investor will have to make the fixed-rate payment and the variable payment equal to the negative return on the equity index.
- The payment on an equity swap is not known till the end of the settlement period. In an interest rate or currency swap the next payment is known at the beginning of the settlement period.

Example 2-5: Equity Swaps

Jessica Aniston enters into a 3-year \$15 million quarterly swap as the fixed-rate payer and will receive the index return on the S&P 500. The swap fixed rate is 6% and the index is currently at 972. At the end of the next three quarters, the index level is 950, 1,004, and 995.

Calculate the net payment at each of the next three quarters and identify the party that makes the payment.

Solution:

The percentage change in the index over each quarter is calculated as:

Q1 = (950/972) – 1 = –2.26%
Q2 = (1,004/950) – 1 = 5.68%
Q3 = (995/1,004) – 1 = –0.90%

Jessica, as the fixed-rate payer, will pay 0.06/4 = 1.5% each quarter and receive the index return.

Jessica's net payments on the equity swap are as follows:

Q1: [1.5% – (–2.26%)] × \$15 million
= 3.76% × \$15 million
= \$564,000

Q2: (1.5% – 5.68%) × \$15 million
= –4.18% × \$15 million
= –\$627,000 (a negative payment means that Jessica receives this amount)

Q3: [1.5% – (–0.90%)] × \$15 million
= 2.4% × \$15 million
= \$360,000

Notice that Jessica makes the fixed and the floating payment for Quarters 1 and 3 because the return on the S&P 500 was negative over these periods.

In this section, we will price and value the following three types of equity swaps:

1. Pay a fixed rate and receive return on equity.
2. Pay a floating rate and receive return on equity.
3. Pay the return on one equity instrument and receive the return on another equity instrument.

Pricing Equity Swaps

- The price of a "pay a fixed rate and receive return on equity" swap refers to the fixed swap rate that will give the swap a zero value at inception. This fixed rate is determined in the same manner as the fixed rate in an interest rate swap or a currency swap:

$$\text{Swap fixed rate} = \left[\frac{1 - B_0(N)}{B_0(1)+B_0(2)+B_0(3) + \ldots + B_0(N)}\right] \times 100$$

- For a "pay a floating rate and receive return on equity" swap there is no fixed rate that we need to solve for. The market value of the swap at initiation equals zero.
- For a "pay the return on one equity instrument and receive the return on another equity instrument" swap there is no fixed rate that we need to solve for. The market value of the swap at initiation equals zero.

Valuing Equity Swaps

The value of a **pay-fixed, receive-return-on-equity** swap at any point in time is calculated as the difference between the value of the (hypothetical) equity portfolio and the present value of remaining fixed-rate payments:

> [(1 + Return on equity) * Notional amount] – PV of the remaining fixed-rate payments

The value of a **pay-floating, receive-return-on-equity** swap at any point in time is calculated as the difference between the value of the (hypothetical) equity portfolio and the par value of the bond (assuming that we are determining the value of the swap on a settlement date):

> [(1 + Return on equity) * Notional amount] – PV(Next coupon payment + Par value)

The value of a **pay-return-on-one-equity-instrument, receive-return-on-another-equity-instrument** swap is calculated as the difference between the values of the two (hypothetical) equity portfolios:

> [(1 + Return on Index2) * Notional amount]–[(1 + Return on Index 1) * Notional amount]

A very important thing for you to understand regarding equity swaps is that cash payments on the equity leg(s) are based on the percentage return on the underlying equity instrument over each settlement period, not on changes in price over each settlement period. For example, assume that the underlying equity portfolio is currently worth \$100. If at the next settlement date the portfolio is worth \$110 or \$91, the payment on the equity leg of the swap would be +10% or –10% (which is different from payments based on changes in price, which would be +\$10 or –\$9).

Further, the underlying equity portfolio is set to the notional amount at each payment date, and the dollar return on the equity leg of the swap is calculated based on the same notional principal for each settlement period (regardless of the past performance of the underlying equity instrument since swap initiation). The past performance of the underlying equity portfolio over previous settlement periods has no impact on the current value of the equity swap. **The current value of the hypothetical equity portfolio (and that of the swap) only depends on how the underlying equity instrument has performed since the last settlement date.**

Example 2-6: Pricing and Valuing Equity Swaps

Consider a 1-year equity swap that calls for quarterly payments. The investor will receive the return on the S&P 500 Index, which currently stands at 1,250.80, and will make fixed-rate payments. The current Libor term structure is as follows:

Days	Libor (%)
90	3.25
180	3.75
270	4.25
360	4.60

Sixty days into the tenor of the swap, the S&P 500 has moved up to 1,375.30 and the new term structure of Libor is as follows:

Days	Libor (%)
30	3.10
120	3.25
210	3.80
300	4.35

Given a notional principal of $20 million, determine:

1. The swap fixed rate.
2. The value of the equity swap to the fixed-rate payer after 60 days.
3. The value of the equity swap under the assumption that the swap calls for payments based on the return on the Dow Jones Industrial Average (DJIA), which stood at 10,062.74 at swap initiation and stands at 10,485.48 on Day 60.

Solution:

1. The Libor term structure is the same as in Example 2-1. Therefore, the annualized swap fixed rate equals 4.512%.
2. Note that the new (as of Day 60) term structure of Libor is the same as in Example 2-4. Further, the notional principal is also the same as in Example 2-4 ($20 million). Therefore, the present value of the remaining fixed-rate payments 60 days into the term of the swap equals $20,186,978.51. The computation of this figure was illustrated in Example 2-4.

   The S&P 500 Index has moved up from 1,250.80 to 1,375.30. Therefore, the value of $20 million invested in the index is calculated as:

   $$V_{60}(\text{S\&P 500}) = \$20{,}000{,}000 * (1{,}375.30/1{,}250.80) = \$21{,}990{,}725.94$$

   The value of the swap to the fixed-rate payer is calculated as the difference between the value of the hypothetical equity portfolio and the present value of the remaining fixed payments:

   $$V_{60}(\text{Fixed} - \text{rate payer}) = \$21{,}990{,}725.94 - \$20{,}186{,}978.51 = \$1{,}803{,}747.425$$

3. The DJIA has moved up from 10,062.74 to 10,485.48 on Day 60. Therefore, the value of $20 million invested in the index is calculated as:

   V_{60}(DJIA) = $20,000,000 * (10,485.48/10,062.74) = $20,840,208.53

   The value of the swap to an investor who pays the return on the DJIA and receives the return on the S&P 500 Index equals the difference between the two portfolios:

   V_{60} = $21,990,725.94 – $20,840,208.53 = $1,150,517.404

Example 2-7: Equity Swap Valuation

Six months ago, we entered a receive-fixed, pay-equity 5-year annual reset swap in which the fixed leg is based on a 30/360 day count and the notional amount was $10 million. The initial swap fixed rate was 2.5%, while the equity was trading at $127 per share. Today, all spot interest rates are 2.2% (the term structure is flat), while the equity is trading at $135.

1. Compute the value of the swap.
2. Compute the value of equity today that would take the current value of the swap to zero.

Solution:

1. First, we compute the present value of the payments on the implied fixed-rate bond by discounting its cash flows at current (new) spot rates.
   - The annual coupon payment equals 2.5% * $10,000,000 = $250,000.
   - The discount factors are computed using an annual rate of 2.2%. Based on the 30/360 convention, the discount rates are computed as follows:
     - For 0.5 years: 1 / [1 + (0.022 × 0.5)] = 0.98912
     - For 1.5 years: 1 / [1 + (0.022 × 1.5)] = 0.96805

Date (in years)	Cash Flow	Discount Factor	Present Value
0.5	$250,000	0.98912	$247,280.00
1.5	$250,000	0.96805	$242,012.50
2.5	$250,000	0.94787	$236,967.50
3.5	$250,000	0.92851	$232,127.50
4.5	$10,250,000	0.90992	$9,326,680.00
		Total	**$10,285,067.50**

   The value of the equity leg is 135 / 127 * 10,000,000 = $10,629,921.26.

   The value of the swap to the receive-fixed, pay-equity side is calculated as $10,285,067.50 – $10,629,921.26 = –$344,853.76.

2. In order for the swap to have a value of zero today, the equity return must match the return on the fixed leg. Due to the decline in interest rates, the value of the implied fixed-rate bond has gone to $10,285,067.50 (approximately risen by 2.85%). Therefore, the swap would have a value today of 0 if the value of equity today were 2.85% higher than on the last settlement date. This value can be calculated as (1 + 0.0285) * 127 = 130.62.

READING 38: VALUATION OF CONTINGENT CLAIMS

LESSON 1: INTRODUCTION AND THE ONE-PERIOD BINOMIAL MODEL

LOS 38a: Describe and interpret the binomial option valuation model and its component terms. Vol 5, pp 388–396

LOS 38c: Identify an arbitrage opportunity involving options and describe the related arbitrage. Vol 5, pp 391–392

LOS 38e: Describe how the value of a European option can be analyzed as the present value of the option's expected payoff at expiration. Vol 5, pp 394–395

A **contingent claim** is a derivative that gives the owner the right, but not the obligation, to a payoff based on an underlying asset, rate, or other derivative. Contingent claims include **options**, which are the focus of this reading.

Binomial Option Valuation Model

The binomial valuation approach is used extensively to value **path-dependent options**, which are options whose values are dependent not only on the value of the underlying at expiration, but also on the path taken by the underlying to get there. An American option is a path-dependent option because it can be exercised prior to expiration. In this section, we will use the binomial model to value both American and European options.

The **Black-Scholes-Merton (BSM) model**, which is described later in the reading, is used to value **path-independent options** only. A European option is path-independent because it depends only on the value of the underlying at expiration.

Notation

- $t = 0$ = Time of option initiation.
- $t = t$ = A point in time during the life of the option.
- $t = T$ = Option expiration.
- c_t and p_t = European call and put prices.
- C_t and P_t = American call and put prices.
- X = Exercise or strike price of option.
- S_t = Underlying stock price.
- PV = Present value.

Level I Recap

- Exercise/intrinsic values:
  - For calls: $c_T = \text{Max}\ [0, (S_T - X)]$
  - For puts: $p_T = \text{Max}\ [0, (X - S_T)]$
- An option's value is composed of its (1) exercise value and (2) time value.
  - Time value declines with the passage of time.
  - At option expiration, time value equals 0.
  - Time value is always non-negative due to the asymmetric nature of option payoffs.
- Put-call parity: $c + PV(X) = p + S$

The One-Period Binomial Model

Assumptions

- The underlying is the only uncertain factor that affects the price of the option.
- Time moves in discrete (not continuous) increments.
- Given the current price of the underlying asset, over the next period the price can move to one of two possible new prices.
- There are no costs or benefits from owning the underlying.

One-Period Binomial Model

In the one-period binomial model, the price of the underlying stock starts off at a given level, S, and can either:

- Move up by a factor of u to a new price, S^+ one period later, or
- Move down by a factor of d to a new price, S^- one period later.

Note that u and d are both total returns (i.e., one plus the rate of return). They are based on the volatility of the underlying. Generally speaking, the higher the volatility, the higher the value of S^+ and the lower the value of S^-.

Therefore:

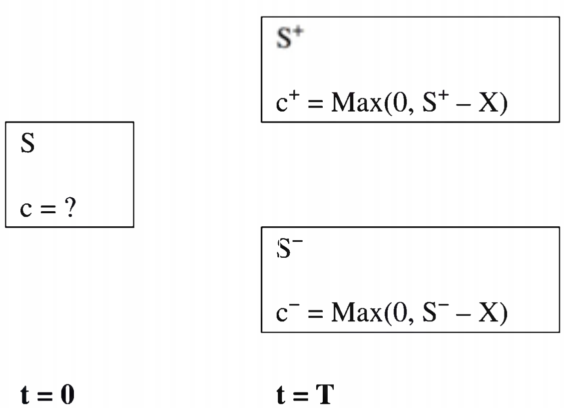
- $u = S^+/S$ and $d = S^-/S$, and
- $S^+ = Su$ and $S^- = Sd$.

Now let's bring in a European call option on the stock that expires in the next period and has an exercise price denoted by X.

- If the price of the stock in the next period equals S^+, the value of the call option (c^+) will equal $\text{Max}(0, S^+ – X)$, which represents its intrinsic value or the payoff at expiration.
- If the price of the stock in the next period equals S^-, the value of the call option (c^-) will equal $\text{Max}(0, S^- – X)$, which represents its intrinsic value or the payoff at expiration.

The aim of this exercise is to determine the price of the call option today (c).

Figure 1-1: One-Period Binomial Model



Now suppose that we are given the following information:

- $S = \$100$
- $u = 1.75$
- $d = 0.75$
- $X = 100$

We can calculate c^+ and c^- as follows:

$S^+ = 100(1.75) = \$175$
$c^+ = \text{Max}(0, 175 - 100) = \$75$
$S^- = 100(0.75) = \$75$
$c^- = \text{Max}(0, 75 - 100) = \$0$

The call can take a value of \$75 (if the stock price goes up) or \$0 (if the stock price goes down) in the next period. Therefore, the call is a risky investment. Now suppose that we want to **short** this call option today and go **long** on h number of stocks such that the portfolio is risk-free; that is, it maintains the same value regardless of the stock price in the next time period. Our portfolio (we'll call it Portfolio H) can be represented by $hS - c$.

$$\boxed{H = hS - c}$$

- If the stock price goes up (to \$175) in the next period, the value of our portfolio will be

$$\boxed{H^+ = hS^+ - c^+ = h(175) - 75}$$

  - The value of the stock holding will be h times the new stock price, \$175.
  - The call will be in-the-money, so our short position on the call will result in a payoff of –\$75.
- If the stock price goes down (to \$75) in the next period, the value of our portfolio will be

$$\boxed{H^- = hS^- - c^- = h(75) - 0}$$

  - The value of the stock holding will be h times the new stock price, \$75.
  - The call will be out-of-the-money, so our short position on the call will result in a payoff of 0.

In order for this portfolio to be risk-free (i.e., for the portfolio value to be the same in either scenario regardless of the stock price), $H^+ = h(175) - 75$ must equal $H^- = h(75) - 0$. We can therefore solve for the value of h that results in a risk-free portfolio:

$$\boxed{\begin{gathered} h(175) - 75 = h(75) - 0 \\ h = 0.75 \end{gathered}}$$

The following table proves that a portfolio consisting of a long position on 0.75 units of stock and a short position on 1 call option will result in the same value at the end of the next period regardless of whether the stock price increases or decreases.

Position	Value Today (H)	Value in Up State (H^+)	Value in Down State (H^-)
Long 0.75 stock	0.75 * 100 = $75	0.75 * 175 = 131.25	0.75 * 75 = 56.25
Short 1 call	–c	–(175 – 100) = –75	0
Overall	**75 – c**	**56.25**	**56.25**

From the example, you should be able to understand that for Portfolio H to be risk-free:

$$H^+ = H^-$$
$$hS^+ - c^+ = hS^- - c^-$$

Therefore:

$$h = \frac{c^+ - c^-}{S^+ - S^-}$$

This value, h, is known as the **hedge ratio**. Note that this hedge ratio is for calls. The hedge ratio for puts is addressed in the next section.

So now that we have proved to you that Portfolio H is a risk-free portfolio (it will have the same value in the next period regardless of which way the stock price moves), you should be able to understand that Portfolio H should grow at the risk-free rate. Therefore:

$$H^+ = H(1 + r)$$
$$H^- = H(1 + r)$$

We can work with the following three expressions that we have already derived to solve for the value of the call option today, c.

$$S^- = Sd \quad \text{… Expression 1}$$

$$h = \frac{c^+ - c^-}{S^+ - S^-} = \frac{c^+ - c^-}{(u - d)S} \quad \text{… Expression 2}$$

$$H^- = H(1 + r) \rightarrow H = H^- / (1 + r) \rightarrow hs - c = \frac{hs^- - c^-}{(1 + r)} \quad \text{… Expression 3}$$

Through some fairly extensive algebra, we come up with the following equation for the value of a call option:

$$c = \frac{\pi c^+ + (1 - \pi)c^-}{(1 + r)} \quad \text{… Equation 1}$$

where :

$$\pi = \frac{(1 + r - d)}{(u - d)}$$

Important:

- The price of the call option can be viewed as the **probability-weighted average** (or **expected value**) of the two possible next-period call prices (c^+ and c^-) discounted at the one-period **risk-free rate**.
- Do not confuse π and $1 - \pi$ with probabilities of up and down movements. One of the important takeaways here is that in this approach we do not need the probabilities of up and down movements that are expected by investors to determine the value of the call. The approach that we have used to value the call is known as risk-neutral valuation, where we assume that investors are risk-neutral and value assets by computing the expected future value (based on risk-neutral probabilities) and discounting it at the risk-free rate.
    - The probabilities computed in the models, π and $1 - \pi$, are therefore known as risk-neutral probabilities.
- The discount rate is not risk-adjusted. It is simply based on the estimated risk-free rate (r).

For put options, we obtain:

$$p = \frac{\pi p^+ + (1 - \pi)p^-}{(1 + r)} \quad \text{… Equation 2}$$

where :

$$\pi = \frac{(1 + r - d)}{(u - d)}$$

The No-Arbitrage Approach

This approach is very difficult to explain/derive in text form. I've done what I could here, but I will spend some time on this (and on the expectations approach) in our lecture video. As you go through the bullets that follow, keep referring to Table 1-1. At a bare minimum you should memorize the bullets that are italicized in the text that follows.

Once again, we work with call options. We will work with a portfolio consisting of h units of the underlying stock (worth hS) and write a call option on the stock. The current worth of our portfolio at $t = 0$ is $hS - c$.

- Given that h equals the hedge ratio, we know that our portfolio will be worth the same amount regardless of whether the stock rises or falls in the next period.
- In order to preclude arbitrage, if we borrow or lend our current portfolio (carry or reverse carry arbitrage), our net cash flow at T (in either scenario) from the trades should be zero.
- In order to have a net cash flow of zero at T (in either scenario), we would need a cash flow of $-hS^- + c^-$ if the stock falls or a cash flow of $-hS^+ + c^+$ if the stock rises.
    - Basically we are trying to offset the value of our long h units of stock and short call position at T (in either scenario).
    - Since h is the hedge ratio, $-hS^- + c^-$ and $-hS^+ + c^+$ will be worth the same amount at T.
        - We will work only with $-hS^- + c^-$ going forward.

Table 1-1: Writing One Call Hedged with h Units of Stock and Finance

Strategy	t = 0	Stock Falls T	Stock Rises T
Write one call (inflow at t = 0)	$+c$	$-c^-$	$-c^+$
Buy h units of stock (outflow at t = 0)	$-hS$	$+hS^-$	$+hS^+$
Borrow or lend	$-PV(hS^- + c^-)$ $= -PV(-hS^+ + c^+)$	$-hS^- + c^-$	$-hS^+ + c^+$
Net cash flow	$+c - hS$ $-PV(-hS^- + c^-)$	0	0

- In order to have a cash flow of $-hS^- + c^-$ (to offset the value of our **hS – c** portfolio at T), we will need to borrow/lend **PV $(-hS^- + c^-)$**) today.
  - We are not sure whether we will be borrowing or lending this amount, as its value depends on c, h, and S.
- So now we have a portfolio that is arbitrage-free (net cash flow at T equals 0) in either scenario. This portfolio (summing Column 2), $+c - hS - PV(-hS^- + c^-)$, should have a value of zero at t = 0 to preclude arbitrage.

The whole point of this exercise if to arrive at the expression for the value of a call option:

- $\boxed{+c - hS - PV(-hS^- + c^-) = 0 \rightarrow c = hS + PV(-hS^- + c^-)}$ ... Equation 3

- Also note that $c = hS + PV(-hS^- + c^-)$ can also be stated as $c = hS + PV(-hS^+ + c^+)$.
  - Recall that $-hS^- + c^-$ and $-hS^+ + c^+$ are the same.

Equation 3 is known as the single-period call valuation equation under the no-arbitrage approach.

It enables us to conclude that a long call option is equivalent to owning h shares of stock and financing the purchase by borrowing $PV(-hS^- + c^-)$. Essentially, a call option is equivalent to a leveraged position in the underlying. We will work with some numbers to illustrate this very important point in Example 2-2.

- *Note that h is (mathematically) positive for call options.*
- *It means that in order to hedge a short position on a call, we must go long on the underlying stock, and vice versa.*
- *If the stock price increases, the stock position will benefit but the short call position loses out, and vice versa.*

In the section on delta hedging (in the last lesson in this reading), you will learn that when creating a hedge, the position taken on the hedging instrument is actually the negative of the hedge ratio of the instrument. So, for example, if you have a long position on stock that you want to hedge with call options, you will have to take a negative or short position on the call options (the hedge ratio for calls is positive, so the negative of the hedge ratio will be negative). When the stock position increases in value, your short position on call options will lose money, thereby hedging your initial position.

For puts, the no-arbitrage approach leads to the following single-period put pricing equation:

$p = hS + PV(-hS^- + p^-)$ or equivalently $p = hS + PV(-hS^+ + p^+)$ … Equation 4

where:

$$h = \frac{p^+ - p^-}{S^+ - S^-} \le 0$$

- *Note that since p^+ is less than p^- (the exercise value of a put when the stock price rises will be lower than its exercise value when the stock price falls), the hedge ratio for puts is **negative**.*
- *This means that in order to hedge a long position on a put, we would actually go long on the underlying as well.*
    - *If the underlying falls, the put benefits, while the long underlying position loses out.*
- *Since h is negative for puts, a put is equivalent to a short position in the underlying (hS) and lending out the proceeds of the short sale. We will work with some numbers to illustrate this very important point in Example 2-2.*

*Finally, let us lay out the transactions for **writing** options*:

- *A written call is equivalent to selling the stock short and investing the proceeds.*
- *A written put is equivalent to buying the stock with borrowed funds.*

The Expectations Approach to Option Valuation

Notice that Equations 1 and 2 can be rewritten as:

$c = PV[\pi c^+ + (1-\pi)c^-]$ … Equation 1

$p = PV[\pi p^+ + (1-\pi)p^-]$ … Equation 2

where π [FV(1) – d] / (u – d).

Equations 1 and 2 express the value of the options as simply the present value of expected terminal option payoffs, where expected terminal option payoffs can be expressed as:

$E(c\ 1) = \pi c + + (1 - \pi)c^-$
$E(p\ 1) = \pi p + + (1 - \pi)p^-$

where c_1 and p_1 are the values of the options at Time 1.

The present values of these expected terminal option payoffs are based on the risk-free rate, r. Therefore, option values based on the **expectations approach** can be expressed as:

$$c = PV_r[E(c_1)]$$
$$p = PV_r[E(p_1)]$$

Before moving on, it is *very* important for you to understand that we just described the single-period expectations approach. The price of a call/put option, under this approach, is calculated directly from expected terminal option payoffs.

Before moving on to an example where we illustrate both approaches in a single-period setting, note that the expectations approach is often viewed as superior to the no-arbitrage approach because it replaces (1) the subjective future expectations and (2) the subjective risk-adjusted discount rate with (1) more objective risk-neutral probabilities and (2) the risk-free rate.

Example 1-1: One-Period Binomial Model

Pluto Inc.'s stock is currently trading at $40 per share. Calculate the value of an at-the-money call option on the stock given that the stock can either go up by 60% or go down by 37.5% over the next period. The risk-free rate equals 6%.

Solution:

We first compute the two possible values of the stock:

$$S^+ = Su = 40 * (1 + 0.6) = \$64$$
$$S^- = Sd = 40 * (1 - 0.375) = \$25$$

Then we calculate the intrinsic value of the call option at expiration in either scenario:

$$c^+ = \text{Max}(0, S^+ - X) = \text{Max}(0, 64 - 40) = \$24$$
$$c^- = \text{Max}(0, S^- - X) = \text{Max}(0, 25 - 40) = \$0$$

Next, we compute the risk-neutral probabilities:

$$\pi = \frac{1 + 0.06 - 0.625}{1.6 - 0.625} = 0.4462$$

$$1 - \pi = 1 - 0.4462 = 0.5538$$

Expectations Approach
We can calculate the value of the call option today based on expected terminal values:

$$c = \frac{(24 * 0.4462) + (0 * 0.5538)}{1.06} = \$10.10$$

No-Arbitrage Approach
First, we compute the optimal hedge ratio:

$$h = \frac{c^+ - c^-}{S^+ - S^-} = \frac{24 - 0}{64 - 25} = 0.61538$$

Then we apply the formula for the value of a call option (Equation 3):

$$c = hS + PV(-hS^- + c^-)$$
$$= (0.61538)(40) + [(-0.61538)(25) + 0] / (>1 + 0.06) = \$10.10$$

Binomial Tree

t = 0	t = T
	$S^+ = Su = \$64$ $c^+ = Max(0, S^+ - X)$ $c^+ = Max(0, 64 - 40)$
$S = \$40$ $c = \$10.10$	
	$S^- = Sd = \$25$ $c^- = Max(0, S^- - X)$ $c^- = Max(0, 25 - 40)$

One-Period Binomial Arbitrage Opportunity

If the actual price of the call option is different from the value computed from the binomial model, there is an arbitrage opportunity.

- If the price of the option is greater than the value computed from the model, the option is overpriced. To exploit this opportunity we would sell the option and buy h units of the underlying stock for each option sold.
- If the price of the option is lower than the value computed from the model, the option is underpriced. To exploit this opportunity we would buy the option and sell h units of the underlying stock for each option purchased.

$$h = \frac{c^+ - c^-}{S^+ - S^-}$$

Example 1-2: One-Period Binomial Arbitrage Opportunity

Continuing from Example 1-1, suppose that the call option is actually selling for \$11.50. Illustrate how an investor may exploit this arbitrage opportunity given that she trades 1,000 call options.

Solution:

Since the call option is selling for more than its actual value (\$11.50 versus \$10.10), it is overpriced. The investor should sell the option and buy h units of the underlying stock.

$$h = \frac{c^+ - c^-}{S^+ - S^-} = \frac{24 - 0}{64 - 25} = 0.6154$$

The investor should purchase 0.6154 units of the underlying stock for every option sold. Given that the investor trades 1,000 options, she should purchase approximately 615 shares (= 1,000 * 0.6154).

Recall from Example 1-1 that the company's share is currently selling for \$40. Further, since the investor is writing call options, her initial investment is reduced by \$11,500 (= 1,000 * \$11.50).

Her initial investment is calculated as:

$$\text{Initial investment} = (615 * \$40) - (1{,}000 * \$11.50) = \$13{,}100$$

At the end of the period, the possible values of her portfolio can be calculated as:

$$H^{+} = hS^{+} - c^{+} = (615 * \$64) - (1{,}000 * \$24) = \$15{,}360$$
$$H^{-} = hS^{-} - c^{-} = (615 * \$25) - (1{,}000 * \$0) = \$15{,}375$$

The values under the two scenarios should be the same. However, they differ slightly here because we rounded off the hedge ratio (h) to the nearest whole number. Going forward, we have used a portfolio value of \$15,360 in the next period.

Since the investor initially invested \$13,100 and ends up with \$15,360, her total return is calculated as:

$$(15{,}360 / 13{,}100) - 1 = 17.25\%$$

This return is risk-free and higher than the current risk-free rate of 6%. The pursuit of arbitrage profits will lead other investors to sell the call option on the stock as well, which will cause its price to decline until it reaches \$10.10. The specific steps that investors will take to exploit the arbitrage opportunity are as follows:

Today:

- Borrow \$13,100 for one year at 6%.
- Buy the hedged portfolio (purchase 615 shares of stock and write 1,000 calls) with an outlay of \$13,100.

One year later:

- Sell the hedged portfolio for \$15,360. The portfolio will be worth this amount regardless of the price of the underlying stock.
- Repay the borrowed funds along with interest. The total amount comes to \$13,886 (= 13,100 * 1.06).
- The arbitrage profit equals \$1,474 (= \$15,360 – \$13,886).

Binomial Put Option Pricing

To price a put option using the binomial model, we perform the same basic steps that we just described for a call, except that we use the put exercise rules instead of the call exercise rules to compute the intrinsic value of the option at expiration. The payoff of a put option equals:

$$\boxed{\text{Put payoff} = \text{Max}(0, X - S_T)}$$

Example 1-3: Valuing a Put Option

Atlas Inc.'s stock is currently trading at \$50. Calculate the value of a European put option on the stock with an exercise price of \$55 given that the stock can either go up by 40% or go down by 25% over the next period. The risk-free rate equals 6%.

Solution:

We first compute the two possible values of the stock:

$$S^+ = Su = 50 * (1 + 0.4) = \$70$$
$$S^- = Sd = 50 * (1 - 0.25) = \$37.50$$

Then we calculate the intrinsic value of the put option at expiration in either scenario:

$$p^+ = \text{Max}(0, X - S^+) = \text{Max}(0, 55 - 70) = \$0$$
$$p^- = \text{Max}(0, X - S^-) = \text{Max}(0, 55 - 37.50) = \$17.50$$

Next, we compute the risk-neutral probabilities:

$$\pi = \frac{1 + 0.06 - 0.75}{1.4 - 0.75} = 0.4769$$
$$1 - \pi = 1 - 0.4769 = 0.5231$$

Expectations Approach
We can calculate the value of the put option today based on expected terminal values:

$$p = \frac{(0 * 0.4769) + (17.50 * 0.5231)}{1.06} = \$8.64$$

No-Arbitrage Approach
First we compute the hedge ratio:

$$h = \frac{p^+ - p^-}{S^+ - S^-} = \frac{0 - 17.50}{70 - 37.50} = -0.53846$$

Then we apply the no-arbitrage put price formula (Equation 4):

$$p = hS + PV(-hS^- + p^-)$$
$$= (-0.53846)(50) + [(0.53846)(37.50) + 17.50] / (1 + 0.06) = \$8.64$$

Summary for this section: The no-arbitrage approach **and** the expectations approach can be used to value options in a **single-period setting**.

LESSON 2: THE TWO-PERIOD BINOMIAL MODEL

LOS 38a: Describe and interpret the binomial option valuation model and its component terms. Vol 5, pp 396–408

LOS 38b: Calculate the no-arbitrage values of European and American options using a two-period binomial model. Vol 5, pp 396–408

LOS 38f: Describe how the value of a European option can be analyzed as the present value of the option's expected payoff at expiration. Vol 5, pp 399–400

The Two-Period Binomial Model

The two-period binomial model requires a few more calculations, but the basic process is very similar to the one-period model. We make use of a "recombining" two-period binomial tree, which can effectively be viewed as three one-period binomial trees. (See Figure 2-1.)

Let's start by setting up the data for the binomial tree. We begin by computing the possible terminal values of the option at the end of Period 2.

At the end of the second period, the stock can take any of three possible values:

1. If the stock goes up in the first period and up again in the second, $S^{++} = Suu$.
2. If the stock goes up in the first period and down in the second, its value would be the same as if it were to go down in the first period and up in the second (hence the name "recombining" tree). Therefore, $S^{+-} = S^{-+} = Sud = Sdu$.
3. If the stock goes down in the first period and down again in the second, $S^{--} = Sdd$.

Since our horizon now stretches over two periods, we will work with a European option that expires at the end of the second period. Our next step is to compute the exercise value of the option at expiration (also known as the terminal value of the option) under each of the three possible price scenarios.

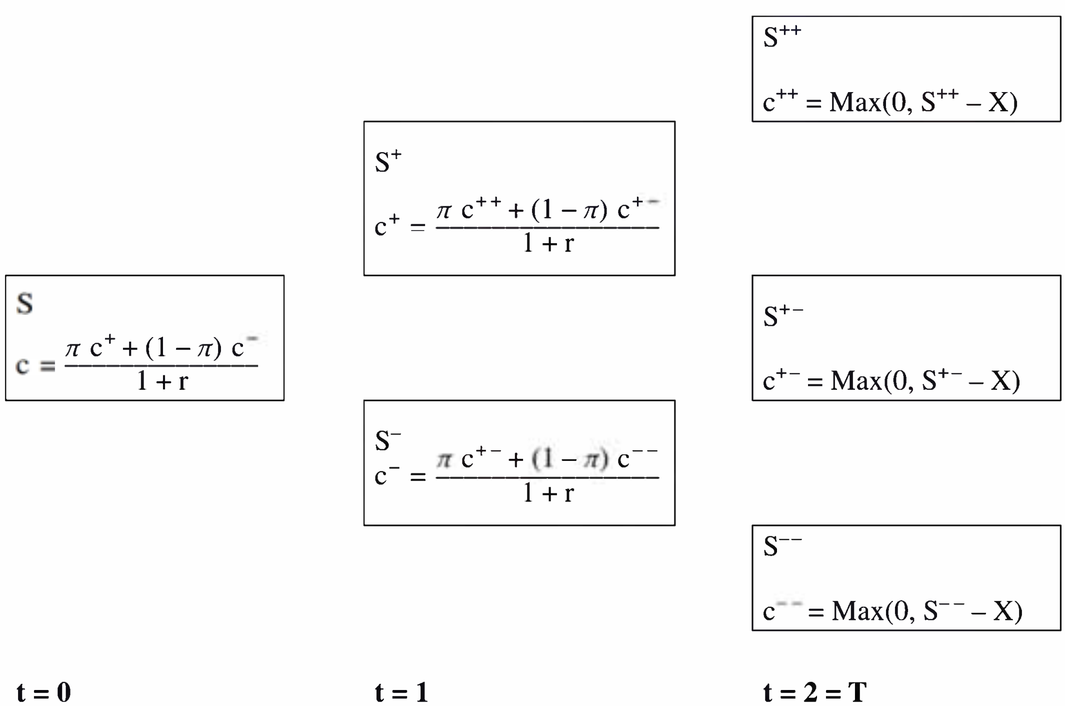
Important: We are working with a European call option that expires at the end of Period 2. Therefore, the value of the call at the nodes corresponding to Period 1 does not equal the option's intrinsic value (as it cannot be exercised at this point in time). Instead, it equals the probability-weighted present value of the terminal values.

1. If the stock price goes up to S^{++}, the value of the option, c^{++}, equals $\text{Max}(0, S^{++} - X)$.
2. If the stock price goes to S^{+-} (or S^{-+}), the value of the option, c^{+-}, equals $\text{Max}(0, S^{+-} - X)$.
3. If the stock price goes down to S^{--}, the value of the option, c^{--}, equals $\text{Max}(0, S^{--} - X)$.

Given S, u, d, and r, we calculate the risk-neutral probabilities of up and down moves, π and $1 - \pi$.

$$\boxed{\pi = \frac{1 + r - d}{u - d}}$$

Figure 2-1: One-Period Binomial Model



Using the possible terminal values of the option and risk-neutral probabilities, we compute the values of the option (c^+ and c^-) at each node at the end of the first period. Notice that we are working backward in time, starting from option expiration and ending at option initiation. This method is known as backward induction.

$$c^+ = \frac{\pi c^{++} + (1 - \pi)\, c^{+-}}{1 + r}$$

$$c^- = \frac{\pi c^{+-} + (1 - \pi)\, c^{--}}{1 + r}$$

Using these possible values of the option at the end of Period 1, we compute the value of the option today as:

$$c = \frac{\pi\, c^+ + (1 - \pi)\, c^-}{1 + r}$$

Finally, note that the hedge ratio, h (which is calculated as the difference between the two possible call prices for the next period divided by the difference between the two possible stock prices for the next period), is different at each node in the binomial tree.

$$h = \frac{c^+ - c^-}{S^+ - S^-}$$

$$h^+ = \frac{c^{++} - c^{+-}}{S^{++} - S^{+-}}$$

$$h^- = \frac{c^{+-} - c^{--}}{S^{+-} - S^{--}}$$

Example 2-1: Two-Period Binomial Model

Beta Inc.'s stock is currently trading at \$60. Calculate the value of an at-the-money European call option on the stock using a two-period binomial model given that the stock can either go up by 35% or go down by 20% each period. The risk-free rate equals 5% per period. Also compute the hedge ratio at each node in the tree.

Solution:

We first compute the possible values of the stock at each node in the binomial tree:

$t = 1$

$S^+ = Su = 60 * (1 + 0.35) = \$81$

$S^- = Sd = 60 * (1 - 0.20) = \$48$

$t = 2 = T$

$S^{++} = Suu = 60 * (1 + 0.35)^2 = \$109.35$

$S^{+-} = Sud = 60 * (1 + 0.35)\,(1 - 0.20) = \$64.80$

$S^{--} = Sdd = 60 * (1 - 0.20)^2 = \$38.40$

Then we calculate the intrinsic value of the call option at expiration (t = 2 = T) in each scenario:

$$c^{++} = \text{Max}(0, S^{++} - X) = \text{Max}(0, 109.35 - 60) = \$49.35$$

$$c^{+-} = \text{Max}(0, S^{+-} - X) = \text{Max}(0, 64.80 - 60) = \$4.80$$

$$c^{--} = \text{Max}(0, S^{--} - X) = \text{Max}(0, 38.40 - 60) = \$0$$

Next we compute the risk-neutral probabilities:

$$\pi = \frac{1 + 0.05 - 0.80}{1.35 - 0.80} = 0.4545$$

$$1 - \pi = 1 - 0.4545 = 0.5455$$

Then we compute the value of the call option at each node corresponding to t = 1:

$$c^{+} = \frac{(49.35 * 0.4545) + (4.80 * 0.5455)}{1.05} = \$23.86$$

$$c^{-} = \frac{(4.80 * 0.4545) + (0 * 0.5455)}{1.05} = \$2.08$$

Finally, we calculate the value of the call option today as:

$$c = \frac{(23.86 * 0.4545) + (2.08 * 0.5455)}{1.05} = \$11.41$$

Table 2-1 summarizes all our calculations. It also includes the values for the hedge ratio at the nodes corresponding to t = 0 and t = 1.

Table 2-1: Two-Period Binomial Example

		$S^{++} = Su^2 = \$109.35$ $c^{++} = \text{Max}(0, S^{++} - X)$ $c^{++} = \text{Max}(0, 109.35 - 60) = \$49.35$
	$S^{+} = Su = \$81$ $c^{+} = \$23.86$ $h^{+} = 1$	
$S = \$60$ $c = \$11.41$ $h = 0.66$		$S^{+-} = Sud = \$64.80$ $c^{+-} = \text{Max}(0, S^{+-} - X)$ $c^{+-} = \text{Max}(0, 64.80 - 60) = \$4.80$
	$S^{-} = Sd = \$48$ $c^{-} = \$2.08$ $h^{-} = 0.1818$	

$S^{--} = Sd^2 = \$38.40$

$c^{--} = \text{Max}(0, S^{--} - X)$

$c^{--} = \text{Max}(0, 38.40 - 60) = \$0$

t = 0 | **t = 1** | **t = T = 2**

$$h_c = \frac{23.86 - 2.08}{81 - 48} = 0.66$$

$$h_c^+ = \frac{49.35 - 4.80}{109.35 - 64.80} = 1$$

$$h_c^- = \frac{4.80 - 0}{64.80 - 38.40} = 0.1818$$

Notice that in the second hedge ratio calculation, the hedge ratio equals +1.0. This is because both the call options used in the calculation (c^{++} and c^{+-}) are in-the-money.

Example 2-2: Two-Period Binomial Model (No-Arbitrage Approach)

Beta Inc.'s stock is currently trading at \$60.

Calculate the value of at-the-money European call and put options on a stock given that the stock can either go up by 35% or go down by 20% each period. The risk-free rate equals 5% per period.

Solution:

Some of the computations required here have already been performed in Example 2-1:

$S^+ = \$81$	$c^{++} = \$49.35$	$h_c = 0.66$
$S^- = \$48$	$c^{+-} = \$4.80$	$h_c^+ = 1$
$S^{++} = \$109.35$	$c^{--} = \$0$	$h_c^- = 0.1818$
$S^{+-} = \$64.80$		
$S^{--} = \$38.40$		

In order to compute the hedge ratios for the put at various nodes, we need to first compute the terminal exercise values of the put:

$$p^{++} = \text{Max}(0, X - S^{++}) = \$0$$
$$p^{+-} = \text{Max}(0, X - S^{+-}) = \$0$$
$$p^{--} = \text{Max}(0, X - S^{--}) = \$21.60$$

Hedge ratios for the put option at each node at Year 1 are calculated as:

$$h_p^+ = \frac{0 - 0}{109.35 - 64.80} = 0$$

$$h_p^- = \frac{0 - 21.60}{64.80 - 38.40} = -0.8182$$

The real point of this example is to learn to interpret what a call really is. A call, as we have said before, is a leveraged position on a stock.

For c+, we are long 1 share of stock at a cost of \$81 (= 1 * 81), partially financed with a \$57.14 {= [(–1.0) (64.80) + 4.80)]/ (1.05)} loan.

Note that the loan amount can also be computed as the cost of the stock position minus the value of the option [\$57.14 = (1.0)(81) – 23.86].

From the no-arbitrage approach to option valuation, we know the following:

$$c = hS + PV(-hS^- + c^-) \text{ ... Equation 3}$$

$$p = hS + PV(-hS^- + p^-) \text{ ... Equation 4}$$

Based on Equation 3, we can come up with the following expressions for call values at the Year 1 nodes in the tree:

$$c^+ = h_c^+ S^+ + PV_{1,2}(-h_c^+ S^{+-} + c^{+-})$$

$$c^+ = (1.0)\ (81) + [(-1.0)\ (64.80) + 4.80)] / (1.05) = \$23.86$$

$$c^- = h_c^- S^- + PV_{1,2}(-h_c^- S^{--} + c^{--})$$

$$c^- = (0.1818)\ (48) + [(-0.1818)\ (38.40) + 0)] / (1.05) = \$2.08$$

Now that we have computed c^+ and c^-, we can compute c as:

$$c = hS + PV(-hS^- + c^-)$$

$$c = (0.66)\ (60) + [(-0.66)\ (48) + (2.08)] / (1.05) = \$11.41$$

A put, as we have said before, can be interpreted as lending that has been partially financed with a short position in shares.

For p^-, we are short 0.818 shares for total proceeds of (0.818) (48) = \$39.26 and have loaned out [(0.818)(64.80) + (0)] / (1.05) = \$50.48.

Note that the lending amount can also be computed as proceeds from the short sale plus the value of the option [\$50.48 = (0.818) (48) + 11.22].

Based on Equation 4, we can come up with the following expressions for put values at the Year 1 nodes in the tree:

$$p^+ = h_p^+ S^+ + PV_{1,2}(-h_p^+ S^{++} + p^{++})$$

$$p^+ = (0)\ (81) + [(-0)\ (109.35) + (0)] / (1.05) = \$0$$

Can also be calculated as

$$p^+ = h_p^+ S^+ + PV_{1,2}(-h_p^+ S^{+-} + p^{+-})$$

$$p^+ = (0)\ (81) + [(-0)\ (109.35) + (0)] / (1.05) = \$0$$

$$p^- = h_p^- S^- + PV_{1,2}(-h_p^- S^{-+} + p^{-+})$$

$$p^- = (-0.818)\ (48) + [(0.818)\ (64.80) + (0)] / (1.05) = \$11.22$$

Can also be calculated as

$$p^- = h_p^- S^- + PV_{1,2}(-h_p^- S^{-+} + p^{-+})$$

$$p^- = (-0.818)\ (48) + [(0.818)\ (64.80) + (0)] / (1.05) = \$11.22$$

Now that we have computed p^+ and p^-, we can compute h_p as:

$$h_p = \frac{0 - 11.22}{81 - 48} = -0.34$$

And now we can compute p as:

$$p = hS + PV(-hS^- + p^-)$$

$$p = (-0.34)\ (60) + [(0.34)\ (48) + (11.22)] / (1.05) = \$5.83$$

We are now going to work with the numbers calculated in Examples 2-1 and 2-2 (relating to the call option only) to illustrate two very important concepts related to the two-period binomial option valuation model: **dynamic replication** and **self-financing**.

Dynamic Replication

We know that $c = hS + PV(-hS^{-} + c^{-})$. Let's work with the numbers here to compute the value of the call at initiation.

$$c_0 = (0.66)\,(60) + [(-0.66)\,(48) + 2.08]\,/\,(1 + 0.05) = 39.60 - 28.19 = \$11.41$$

Now let's assume that the stock moves up and compute the value of our $c = hS + PV(-hS^{-} + c^{-})$ portfolio. The stock is now worth \$81 and the 28.19 borrowed must be repaid with interest

$$c = (0.66)\,(81) + -28.19(1 + 0.05) = \$23.86$$

Notice that this value is the same as the value of the call at Node S^{+}. The takeaway is that the portfolio of stock and borrowing dynamically replicates the value of the call through the binomial lattice.

Self-Financing

Comparing Nodes S and S^{+}, notice the following:

- The hedge ratio went up from 0.66 to 1.0, which means that additional shares were required to be purchased.
- At t = 1, the value (including interest) of the initial amount borrowed (at t = 0) was \$28.19 × 1.05 = \$29.60.
- The total amount borrowed as of t = 1 can be calculated as $PV(-hS^{+-} + c^{+-})$ = $[(-1.0)(64.80) + 4.80]/(1 + 0.05) = \$57.14$.

This shows that the strategy is self-financing. If additional funds are required to purchase more shares to maintain the hedge, the amount of borrowed funds will increase.

The Expectations Approach in a Two-Period Setting

Through some algebra (repeated substitutions), we can obtain the following expressions for the value of call and put options from a two-period binomial model.

$$c = PV\left[\pi^2 c^{++} + 2\pi(1-\pi)c^{+-}\left(1-\pi\right)^2 c^{--}\right] \quad \text{... Equation 5}$$

$$p = PV\left[\pi^2 p^{++} + 2\pi(1-\pi)p^{+-} + \left(1-\pi\right)^2 p^{--}\right] \quad \text{... Equation 6}$$

Notice that the two-period binomial model is again simply the present value of the expected terminal option payoffs based on the risk-neutral probabilities. The expected terminal option payoffs can be expressed as:

$$E(c_2) = \pi^2 c^{++} + 2\pi(1-\pi)\, c^{+-} + (1-\pi)^2 c^{--}$$
$$E(p_2) = \pi^2 c^{++} + 2\pi(1-\pi)\, p^{+-} + (1-\pi)^2 p^{--}$$

Therefore, the two-period binomial option values based on the expectations approach can be expressed as:

$$c = PV\,[\,E\pi(c_2)]$$
$$p = PV\,[\,E\pi(p_2)]$$

It is important to remember that the present value is computed over two periods in the two-period setting. Also keep in mind that, under this approach, the value of the option is computed based solely on possible terminal values (at $t = T = 2$). The possible values of the option at interim nodes ($t = 1$, in this example) play no role in option valuation (observe Equations 5 and 6).

Example 2-3: Expectations Approach in a Two-Period Setting

Continuing from Examples 2-1 and 2-2, compute the value of two-year call and put European options based on the expectations approach.

Solution:

Since the option expires in two years, we must compute the present value over two years.

We simply need to apply Equations 5 and 6 to the data computed in Examples 2-1 and 2-2:

$$c = PV\,[\,\pi^2 c^{++} + 2\pi(1-\pi)\, c^{+-} + (1-\pi)^2 c^{--}]$$
$$c = [(0.4545)^2(49.35) + 2(0.4545)(0.5455)(4.80) + (0.5455)^2(0)] \,/\, (1.05)^2 = \$11.41$$
$$p = PV\,[\,\pi^2 p^{++} + 2\pi(1-\pi)\, p^{+-} + (1-\pi)^2 c^{--}]$$
$$p = [(0.4545)^2(0) + 2(0.4545)(0.5455)(0) + (0.5455)^2(21.60)\,] \,/\, (1.05)^2 =$$ [redacted]

We can also use put-call parity to compute the value of the put once we have the value of the call:

$$c_0 = \frac{X}{(1+R_F)^T} = p_0 + S_0$$
$$p_0 = 11.41 + 60/1.05^2 - 60 = \$5.83$$

American-Style Options

An American-style call option (on non-dividend-paying stock) will never be exercised early, because the minimum value of the option will exceed its exercise value. Consider a situation where the underlying is a stock currently trading at \$50, and the call has an exercise price of \$5, making it deep in-the-money. The exercise value of the call is currently \$45, but the value of the option will be more than just \$45 because the option holds time value as well (assuming we are not at expiration). Even if you do not think the stock price will go any higher, it would make more sense to just sell the American call option (for more than \$45) than to exercise it (for a payoff worth only \$45).

Early exercise of an American put, however, may be warranted in certain cases. If the put is deep in-the-money, it may make more sense to exercise it and invest the sales proceeds at the risk-free rate to earn interest if it exceeds the time value embedded in the value of the put.

In Example 2-4, we examine how early exercise influences the value of an American put option. We will see that when early exercise has value, the no-arbitrage approach is the only way to value American options. The expectations approach, which is based only on possible terminal values, does not work since it ignores the possibility that early exercise may be optimal.

Example 2-4: Two-Period Binomial American-Style Put Option Valuation

Consider the following information:

- S_0 = \$72
- X = \$75
- u = 1.356
- d = 0.541
- T = 2
- r = 3%

Based on this information, we have computed the value of a European put option in Table 2-1. Now use the information provided to compute the value of an otherwise identical American put option.

Table 2-2: Valuing a European Put Option

t = 0	t = 1	t = T = 2
		S^{++} = \$132.389 p^{++} = 0
	S^{+} = \$97.632 p^{+} = \$8.61401 h^{+} = –0.27876	
S = \$72 p = \$18.16876 h = –0.43029		S^{+-} = \$52.81891 p^{+-} = \$22.18109
	S^{-} = \$38.952 p^{-} = [illegible] h^{-} = –1	
		S^{--} = \$21.07303 p^{--} = \$53.92697

Solution:

When working with an American put option, at each node, we need to check how the option value (the probability-weighted present value) compares with its current exercise value. If the exercise value is greater, then we must replace the probability-weighted present value with the exercise value in our backward induction process. Since an American option can be exercised at any point in time, its value at any node during its life must be the greater of its probability-weighted present value and intrinsic value.

So we work backward through the binomial tree and address whether early exercise is optimal at each step. Refer to Table 2-3 and the calculations that follow.

Table 2-3: Valuing an American Put Option

t = 0	t = 1	t = T = 2
		S^{++} = \$132.389 p^{++} = 0
	S^{+} = \$97.632 p^{+} = \$8.61401 h^{+} = –0.27876	
		S^{+-} = \$52.81891 p^{+-} = \$22.18109
S = \$72 p = \$19.0171 h = –0.46752		
	S^{-} = \$38.952 p^{-} = \$36.048 h^{-} = –1	
		S^{--} = \$21.07303 p^{--} = \$53.92697

At Node S^{+} the exercise value is 0, so we leave the p^{+} = \$8.61401 in there, but at Node S^{-} the exercise value is 75 – 38.952 = \$36.048, which is greater than \$33.86353. The early exercise premium at t = 1 if a down move occurs equals \$36.048 – \$33.86353 = \$2.18447.

We now use the updated values at t = 1 to compute the value of the put at t = 0.

$$p = PV\,[\pi p^{+} + (1 - \pi)\, p^{-}\,] = [(0.6)\ 8.61401 + (1 - 0.6)\ 36.048\ ] / 1.03 = \$19.0171$$

The exercise premium at t = 0 is computed as \$19.0171 – \$18.16876 = 0.8434.

As mentioned earlier, one key takeaway from this exercise is that we cannot simply use the PV of expected terminal option payoffs approach (Equation 6) to compute the value of an **American put option** in a **multi-period setting**. The PV of expected future option payoffs approach can only be used to value American puts in a single-period setting (Equation 2). When early exercise is a consideration for puts, we must address the possibility of early exercise at each node as we work backward through the tree.

Going back to **American calls** (on non-dividend-paying stock), there is no reason to exercise this option early. What you would find if you work backward through the binomial tree is that at each node the exercise values will be less than or equal to the call model values. Therefore, the American-style feature would have no effect on the hedge ratio or the option value at any node.

So far in this reading, we have assumed that the underlying does not make any dividend payments. If the underlying does make dividend payments, early exercise of an American call option **may be warranted**. When a stock pays a dividend, its value falls by the amount of the dividend on the ex-dividend date. If the ex-dividend date is very close to the option expiration date, and the dividend is so significant that it reduces the price of the stock below the exercise price (and takes the call option out-of-the-money), early exercise of the American call option would be warranted. The aforementioned is a very simplistic explanation, but all you need to understand is that while an American call on a non-dividend-paying stock will never be exercised early, there can be circumstances where an American call option on a dividend-paying stock would be exercised before expiration. As a result, the expectations approach (which only uses possible terminal payoffs) cannot be used to value an American call option on a stock that pays dividends.

Key Takeaways from This Lesson

- The two-period binomial lattice can be viewed as three one-period binomial models, one at Time 0 and two at Time 1.

In a multi-period setting:

- Generally speaking, European-style options **can** be valued based on the expectations approach in which the option value is determined as the present value of the expected future option payouts, where the discount rate is the risk-free rate and the expectation is taken based on the risk-neutral probability measure.
- Both American-style options and European-style options can be valued based on the no-arbitrage approach, which provides clear interpretations of the component terms. Option value is determined by working backward through the binomial tree to arrive at the correct current value.
- For American-style options, early exercise may influence the option values and hedge ratios as one works backward through the binomial tree. This is the case for American calls on dividend-paying stocks and American puts. Therefore, these options cannot be valued using the expectations approach. They can be valued only using the no-arbitrage approach.
    - American calls on non-dividend-paying stocks can be valued using the expectations approach because there is no reason to exercise them early.

LESSON 3: INTEREST RATE OPTIONS

LOS 38d: Calculate and interpret the values of an interest rate option using a two-period binomial model. **Vol 5, pp 409–411**

Interest Rate Options

In order to value interest rate options, we need to work with arbitrage-free interest rate trees. Derivation of the values in arbitrage-free interest rate trees is not required here, and it is covered in sufficient detail in the fixed income section. For our purposes here, the arbitrage-free interest rate tree is given, and the risk-neutral probability of an up/down move at each node is 50%.

In Table 3-1 we present a binomial lattice of one-year spot rates and corresponding one-year zero-coupon bond values covering two years. Note that the rates are expressed assuming annual compounding.

Table 3-1: Two-Year Interest Rate Lattice by Year

t = 0	t = 1	t = T = 2
		Maturity = 1 Value = 0.9617 Rate = 3.9804%
	Maturity = 1 Value = 0.9624 Rate = 3.9074%	
Maturity = 1 Value = 0.9704 Rate = 3.0502%		Maturity = 1 Value = 0.9681 Rate = 3.2942%
	Maturity = 1 Value = 0.9744 Rate = 2.6254%	
		Maturity = 1 Value = 0.9778 Rate = 2.2671%

The spot rates are fairly easy to compute from the bond value at each node. For example, the one-year spot rate at t = 0 is computed as:

$$1 / (1 + x)^1 = 0.9704 \rightarrow x = 0.030502 = 3.0502\%$$

The underlying instrument for an interest rate option is the one-year spot rate.

- A call option would be in-the-money when the current spot rate is above the exercise rate.
- A put option would be in-the-money when the current spot rate is below the exercise rate.

Interest rate option valuation is similar to the expectations approach described in the previous section. Example 3-1 illustrates interest rate option valuation.

Example 3-1: Valuation of Interest Rate Options

Using the interest rate lattice provided in Table 3-1, value two-year European-style interest rate call and put options with the one-year annually compounded spot rate as the underlying. Assume that the notional amount is $1 million and the exercise rate for both options is 3.3% of par.

Solution:

Under the expectations approach, the first step is to compute the exercise value of the options (per $1 of par) at option expiration (t = 2):

$$c^{++} = \text{Max}(0, S^{++} - X) = \text{Max}[0, 0.039804 - 0.033] = 0.006804$$
$$c^{+-} = \text{Max}(0, S^{+-} - X) = \text{Max}[0, 0.032942 - 0.033] = 0.0$$
$$c^{--} = \text{Max}(0, S^{--} - X) = \text{Max}[0, 0.022671 - 0.033] = 0.0$$
$$p^{++} = \text{Max}(0, X - S^{++}) = \text{Max}[0, 0.033 - 0.039804] = 0.0$$
$$p^{+-} = \text{Max}(0, X - S^{+-}) = \text{Max}[0, 0.033 - 0.032942] = 0.000058$$
$$p^{--} = \text{Max}(0, X - S^{--}) = \text{Max}[0, 0.033 - 0.022671] = 0.010329$$

Next, we compute the values of the call and put at each node at t = 1:

$$\begin{aligned}
c^{+} &= [\pi c^{++} + (1-\pi)sc^{+-}] / (1 + {}_1f_1) \\
&= [0.5(0.006804) + (1–0.5)\ 0.0] / (1 + 0.039074) \\
&= 0.003274 \\
c^{-} &= [\pi c^{+-} + (1-\pi)c^{--}] / (1 + {}_1f_1) \\
&= [0.5(0.00) + (1 - 0.5)\ 0.0] / (1 + 0.026254) \\
&= 0.0 \\
p^{+} &= [\pi p^{++} + (1-\pi)p^{+-}] / (1 + {}_1f_1) \\
&= [0.5(0.0) + (1 - 0.5)\ 0.000058] / (1 + 0.039074) \\
&= 0.000028 \\
p^{-} &= [\pi p^{+-} + (1-\pi)p^{--}] / (1 + {}_1f_1) \\
&= [0.5(0.000058) + (1 - 0.5)\ 0.010329] / (1 + 0.026254) \\
&= 0.005061
\end{aligned}$$

Notice that the one-year forward rates used to discount the expected payoffs are different (3.9074% vs. 2.6254%) in the two (up and down) scenarios. This is because interest rates are allowed to vary in this model (we are, after all, valuing an option on interest rates).

Finally, we can compute the current value of the call and put options as:

$$c = \left[\pi c^{+} + (1-\pi)\, c^{-}\right] / (1 + {}_{1}s_{0})$$
$$= [0.5(0.003274) + (1-0.5)\, 0.0\,] / (1 + 0.030502)$$
$$= 0.001589$$
$$p = \left[\pi p^{+} + (1-\pi)\, p^{-}\right] / (1 + {}_{1}s_{0})$$
$$= [0.5(0.000028) + (1-0.5)\, 0.005061\,] / (1 + 0.030502)$$
$$= 0.002469$$

Since the notional amount is $1 million, the call value equals 1,000,000 * 0.001589 = $1,589, while the put value equals 1,000,000 * 0.002469 = $2,469. Notice that final option results are obtained by discounting one period (t = 1) call and put values at the current spot rate provided in Table 3-1.

LESSON 4: THE BLACK-SCHOLES-MERTON MODEL

LOS 38f: Identify assumptions of the Black-Scholes-Merton option valuation model. Vol 5, pp 411–414

LOS 38g: Interpret the components of the Black-Scholes-Merton model as applied to call options in terms of a leveraged position in the underlying. Vol 5, pp 414–418

LOS 38h: Describe how the Black-Scholes-Merton model is used to value European options on equities and currencies. Vol 5, pp 422–424

The Black-Scholes-Merton (BSM) Option Valuation Model

Essentially, the BSM model is the no-arbitrage approach described in the previous section, but applied to a continuous time process.

Key Assumptions

- The underlying follows a statistical process called **geometric Brownian motion (GBM)**, which implies that returns follow the lognormal distribution. This basically means that the continuously compounded return is normally distributed.
- Prices are continuous, meaning that the price of the underlying instrument does not jump from one value to another, but moves smoothly from value to value.
- The underlying instrument is liquid, meaning that it can be easily bought and sold.
- Continuous trading is available, meaning that one can trade at every instant.
- Short selling of the underlying instrument with full use of the proceeds is permitted.
- Markets are frictionless, meaning that there are no transaction costs, regulatory constraints, or taxes.
- There are no arbitrage opportunities in the market.
- Options are European, so they cannot be exercised prior to expiration.
- The continuously compounded risk-free interest rate is known and constant. Further, borrowing and lending are allowed at the risk-free rate.
- The volatility of the return on the underlying is known and constant.
- If the underlying instrument pays a yield, it is expressed as a continuous, constant yield at an annualized rate.

BSM Model

In this section, we assume that the underlying is a non-dividend-paying stock. The BSM model for such stocks can be expressed as:

$$c = SN(d_1) - e^{-rT}XN(d_2)$$
$$p = e^{-rT}XN(-d_2) - SN(-d_1)$$

where:

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

σ, = The annualized standard deviation of the continuously compounded return on the stock

r = The continuously compounded risk-free rate of return

$N(d_1)$ = Cumulative normal probability of d_1. For example, if d_1 is 0.8252, then $N(d_1)$ will be 0.7954 (look up the normal distribution table).

Interpreting the BSM Model: Approach 1

The BSM model can be described as the present value of the expected option payoff at expiration.

- For calls, we can express the BSM model as $c = PV_r[E(c_T)]$ and for puts as $p = PV_r[E(p_T)]$, where $E(c_T) = Se^{rT}N(d_1) - XN(d_2)$ and $E(p_T) = XN(-d_2) - Se^{rT}N(-d_1)$.
- The discount factor in this context is simply e^{-rT}.

As was the case when we described Equations 1 and 2, the expectation here is based on risk-neutral probabilities (not actual probabilities) and the discount factor is based on the risk-free rate, not on the required rate of return (which includes a premium for risk).

Interpreting the BSM Model: Approach 2

The BSM model can be described as having two components: (1) a stock component and (2) a bond component.

- For call options, the stock component is $SN(d_1)$ and the bond component is $e^{-rT}XN(d_2)$.
  - The value of the call equals the stock component minus the bond component.
- For put options, the stock component is $SN(-d_1)$ and the bond component is $e^{-rT}XN(-d_2)$.
  - The value of the put equals the bond component minus the stock component.

Now think of the BSM model as a dynamically managed portfolio of the stock and zero-coupon bonds, and the idea is to replicate the payoffs on an option with this portfolio. Let's talk about call options first:

The cost of the replicating portfolio is expressed as $n_S S + n_B B$, where n_S denotes the number of shares of the underlying in the portfolio and n_B denotes the number of zero-coupon bonds in the portfolio. S is the current price of the stock and $e^{-rT}X$ is the current price (the present value) of the bond.

- Looking at the BSM equation for calls:
  - n_S equals $N(d_1)$, which is greater than 0. This means that we are buying the underlying.
  - n_B equals $-N(d_2)$, which is less than 0. This means that we are selling/shorting the bond, or borrowing.
  - Effectively, in taking a position on a call option, we are simply buying stock with borrowed money. Therefore, a call option can be viewed as a leveraged position in the stock.
- Looking at the BSM equation for puts:
  - n_S equals $-N(-d_1)$, which is less than 0. This means that we are shorting the stock.
  - n_B equals $N(-d_2)$, which is greater than 0. This means that we are buying the bond, or lending.
  - Effectively, in taking a position on a put option, we are simply buying bonds with the proceeds of shorting the stock.

Now consider the position of the **writer** of a put on a stock. For a put writer, n_S will be greater than 0 and n_B will be less than 0, which makes her position similar to that of the call option holder (who effectively takes a leveraged position on the stock). However, the put writer also receives the put premium today. This means that the put writer effectively takes such an extremely leveraged position on the stock that the amount borrowed actually exceeds the total cost of the underlying (hence the positive cash flow at position initiation).

Table 4-1 draws parallels between the no-arbitrage approach to the single-period binomial option valuation model and the BSM option valuation model.

Table 4-1: BSM and Binomial Option Valuation Model Comparison

Option Valuation Model Terms	Call Option		Put Option	
	Underlying	Financing	Underlying	Financing
Binomial model	hS	$PV(-hS^- + c^-)$	hS	$PV(-hS^- + p^-)$
BSM model	$N(d_1)S$	$-N(d_2)e^{-rT}X$	$-N(-d_1)S$	$N(-d_2)e^{-rT}X$

- The hedge ratio in the binomial model, h, is comparable to $N(d_1)$ in the BSM model.
- For call options, $-N(d_2)$ implies borrowing money or short selling $N(d_2)$ shares of a zero-coupon bond trading at $e^{-rT}X$.
- For put options, $N(-d_2)$ implies lending money or buying $N(-d_2)$ shares of a zero-coupon bond trading at $e^{-rT}X$.
- If the value of the underlying, S, increases, then the value of $N(d_1)$ also increases.
  - Therefore, the replicating strategy for calls requires continually buying shares in a rising market and selling shares in a falling market.
- Since market prices do not move continuously (prices can jump), hedges tend to be imperfect. Further, volatility cannot be known in advance (more on this later). As a result, options are typically more expensive than predicted by the BSM model.

Example 4-1: Illustration of BSM Model Component Interpretation

Consider the following information on call and put options on a stock: S = 50, X = 50, r = 5%, T = 1.0, and σ = 25%. Based on the BSM model, the following values are provided: PV(X) = 47.562, d_1 = 0.325, d_2 = 0.075, $N(d_1)$ = 0.627, $N(d_2)$ = 0.530, $N(-d_1)$ = 0.373, $N(-d_2)$ = 0.470, c = 6.17, and p = 3.73.

1. Describe the initial trading strategy required under the no-arbitrage approach to replicate the payoffs of the call option buyer.
2. Describe the initial trading strategy required under the no-arbitrage approach to replicate the payoffs of the put option buyer.

Solution:

1. The no-arbitrage approach to replicate the call option involves buying $N(d_1)$ = 0.627 shares of stock. This purchase is partially financed by selling $N(d_2)$ = 0.530 zero-coupon bonds, which are priced at PV(X) = 47.562 (which is the value for $B = e^{-rT}X$) per bond.

   Note that the cost of this replicating strategy is calculated as $n_S S + n_B B$ = 0.627(50) + (–0.530)47.562 = 6.14, which equals the price of the call (after accounting for rounding errors).
2. The no-arbitrage approach to replicate a put option involves buying $N(-d_2)$ = 0.470 units of the bond, and selling (short) $N(-d_1)$ = 0.373 of the stock.

   The cost of this replicating strategy is calculated as $n_S S + n_B B$ = –0.373(50) + (0.470)47.562 = 3.704, which equals the price of the put (after accounting for rounding errors).

Before moving on, note that $N(d_2)$ represents the probability that the call option expires in-the-money, while $1 - N(d_2) = N(-d_2)$ represents the probability that the put expires in-the-money. Remember that these statements are made regarding risk-neutral probabilities, not an investor's estimate of the probability of being in-the-money, nor the overall market's estimate.

BSM Model When the Underlying Offers Carry Benefits

Carry benefits include dividends for stock options, foreign interest rates for currency options, and coupon payments for bond options. For underlying instruments that entail carry costs (e.g., storage and insurance costs), those can simply be treated as negative carry benefits in the model. In order to apply the BSM model, the carry benefits must be modeled as a continuous yield, denoted by γ.

The carry-benefit-adjusted BSM model is as follows:

$$c = Se^{-\gamma T}N(d_1) - e^{-rT}XN(d_2)$$
$$p = e^{-rT}XN(-d_2) - Se^{-\gamma T}N(-d_1)$$

where:

$$d_1 = \frac{\ln(S/X) + \left(r - \gamma + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

From these formulas, note that dividends impact option values through their effects on (1) d_1, (2) d_2, and (3) the stock component of option value. The value of a put option can also be found based on carry-benefit-adjusted put-call parity:

$$p + Se^{-\gamma T} = c + e^{-rT}X$$

Once again, the BSM model (now with adjustments to accommodate carry benefits) can be interpreted in two ways.

Interpreting the BSM Model with Carry Benefits: Approach 1

The carry-benefit-adjusted BSM model can be described as the present value of the expected option payoff at expiration.

- For calls, $c = PV_r[E(c_T)]$, and for puts, $p = PV_r[E(p_T)]$, where $E(c_T) = Se^{(r-\gamma)T}N(d_1) - XN(d_2)$ and $E(p_T) = XN(-d_2) - Se^{(r-\gamma)T}N(-d_1)$.
- The discount factor remains e^{-rT}.
- Carry benefits basically lower the expected future value of the underlying.

Interpreting the BSM Model with Carry Benefits: Approach 2

The carry-benefit-adjusted BSM model can also be described as having two components: (1) a stock component and (2) a bond component.

- For call options, the stock component is $Se^{-\gamma T}N(d_1)$ and the bond component is $e^{-rT}XN(d_2)$.
  - The value of the call equals the stock component minus the bond component.
- For put options, the stock component is $Se^{-\gamma T}N(-d_1)$ and the bond component is $e^{-rT}XN(-d_2)$.
  - The value of the put equals the bond component minus the stock component.
- Note that d_1 and d_2 are both also reduced by carry benefits. The reduction in d_2, and hence $N(d_2)$, indicates a lower risk-neutral probability of the call being in-the-money.
- An increase in carry benefits will lower the value of a call option and increase the value of a put.

Now let's focus on options on stocks. For stock options, $\gamma = \delta$, which is the continuously compounded dividend yield. The dividend-yield-adjusted BSM model can be interpreted as a dynamically managed portfolio of the stock and zero coupon bonds.

- For call options, the number of shares of stock is given by $n_S = e^{-\gamma T}N(d_1)$, which is greater than 0, while the number of bonds remains $n_B = -N(d_2)$, which is less than 0.
- For put options, the number of shares is given by $n_S = -e^{-\gamma T}N(-d_1)$, which is less than 0, while the number of bonds remains $n_B = N(-d_2)$, which is greater than 0.
- As we just learned, dividends lower the values of d_1 and d_2, and therefore of $N(d_1)$ and $N(d_2)$.
    - This means that dividends lower the number of shares that must be bought in the call replicating portfolio, and reduce the number of shares that must be sold short in the put replicating portfolio.
        - Recall that these amounts are based on $N(d_1)$.
    - Further, higher dividends lower the number of bonds to sell short in the call replicating portfolio, and the number of bonds to purchase in the put replicating portfolio.
        - Recall that these amounts are based on $N(d_2)$.

Example 4-2: How the BSM Model Is Used to Value Stock Options

You want to apply the BSM model. The underlying is currently trading at \$5.31. You want to value an out-of-the-money call that expires in 3 months and has an exercise price of 5.60. The risk-free rate is 2%, the stock is yielding 0.25%, and the stock volatility is 35%. List the inputs of the BSM model and proceed to value the call option.

Solution:

We will use the following inputs to value the option: S = \$5.31, X = \$5.60, T = 0.25, r = 0.02, $\gamma = 0.0025$, and $\sigma = 0.35$.

$$c = Se^{-\gamma T}N(d_1) - e^{-rT}XN(d_2)$$

$$d_1 = \frac{\ln(S/X) + \left(r - \gamma + \sigma^2/2\right)T}{\sigma\sqrt{T}} = \frac{\ln(5.31/5.60) + \left(0.02 - 0.0025 + 0.35^2/2\right)0.25}{0.35\sqrt{0.25}}$$

$$= -0.19136$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.19136 - 0.35\sqrt{0.25} = -0.36636, \text{then}$$

$$c = 5.31(0.9994)(0.4241) - 0.9950(5.60)(0.3571) = \$0.26$$

Foreign Exchange Options

We are going to be working with foreign exchange quotes expressed as DC/FC. Please stick to this quoting convention to avoid unnecessary complications in valuing foreign currency options.

- For foreign exchange options, the carry benefit, γ, is the continuously compounded foreign risk-free interest rate, r^f. This is because the underlying, the foreign currency (currently worth the foreign exchange spot rate), can be invested at the foreign currency risk-free rate.

- The underlying and the exercise price must be quoted in the same currency unit (DC/FC).
- Volatility here refers to the volatility of the log return of the spot exchange rate.
- A currency option is for a certain quantity of foreign currency (the notional amount), just like an option on a stock is for a certain number of shares. For a stock option, the exercise price may be $1.50 per stock. Similarly, for a currency option, the exercise price may be $1.50 per GBP.

The BSM model for currencies also has two components: a foreign exchange component and a bond component.

- For call options, the foreign exchange component is $Se^{-r^fT}N(d_1)$ and the bond component is $e^{-rT}XN(d_2)$, where r is the domestic risk-free rate.
    - The value of a call option is simply the foreign exchange component minus the bond component.
- For put options, the foreign exchange component is $Se^{-r^fT}N(d_1)$ and the bond component is $e^{-rT}XN(-d_2)$.
    - The value of a put option is simply the bond component minus the foreign exchange component.

Example 4-3: BSM Model Applied to Value Options on Currency

An American exporter expects to receive a fixed EUR amount in 6 months' time. The current spot exchange rate is $1.15/€. The exporter is worried about a decline in the value of the EUR, because if the exchange rate falls, he will be able to buy fewer USD. As a result, he is considering buying an at-the-money spot put option on the EUR, which in essence is a call on the USD. The U.S. risk-free rate is 0.50% and the EUR risk-free rate is 1.00%. List the inputs for the BSM model in this case.

Solution:

In applying the BSM model here, we will work with $/€ exchange rate quotes. The exercise price equals the currency spot exchange rate, $1.15/€. The risk-free rate will be the U.S. rate of 0.50%, while the carry rate will be the EUR rate of 1.00%.

Key Takeaways from This Section

- A key assumption of the Black-Scholes-Merton option valuation model is that the return of the underlying instrument follows geometric Brownian motion, implying a lognormal distribution of the return.
- The BSM model can be interpreted as a dynamically managed portfolio of the underlying instrument and zero-coupon bonds.
- $N(d_1)$ can be interpreted as (1) the basis for the number of units of underlying instrument to replicate an option, (2) the primary determinant of delta, and (3) how much the option value will change for a small change in the underlying.
- $N(d_2)$ can be interpreted as (1) the number of zero-coupon bonds to acquire to replicate an option and (2) an estimate of the risk-neutral probability of an option expiring in-the-money.

LESSON 5: THE BLACK MODEL

LOS 38i: Describe how the Black model is used to value European options of futures. Vol 5, pp 422–424

LOS 38j: Describe how the Black model is used to value European interest rate options and European swaptions. Vol 5, pp 424–429

Black Option Valuation Model

The **Black model** is applicable to options on underlying instruments that are costless to carry, such as options on futures (e.g., equity index futures) and forward contracts (e.g., interest-rate-based options, such as caps, floors, and swaptions).

European Options on Futures

Assumptions:

- The futures price follows geometric Brownian motion (GBM).
- Margin requirements and marking to market are ignored.

Under the Black model, European-style options on futures are valued as:

$$c = e^{-rT}[F_0(T)N(d_1) - XN(d_2)]$$
$$p = e^{-rT}[XN(-d_2) - F_0(T)N(-d_1)]$$

where:

$$d_1 = \frac{\ln[F_0(T)/X] + (\sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

Note that:

- $F_0(T)$ denotes the futures price at Time 0 on a contract that expires at Time T.
- σ denotes the volatility of the futures price.
- The Black model is simply the BSM model in which the futures contract is assumed to reflect the carry arbitrage model.

Futures option put-call parity can be expressed as:

$$c = e^{-rT}[F_0(T) - X] + p$$

The Black model has two components, a futures component and a bond component.

- For call options, the futures component is $F_0(T)e^{-rT}N(d_1)$ and the bond component is again $e^{-rT}XN(d_2)$.
  - The call price is simply the futures component minus the bond component.
- For put options, the futures component is $F_0(T)e^{-rT}N(-d_1)$ and the bond component is again $e^{-rT}XN(-d_2)$.
  - The put price is simply the bond component minus the futures component.

Another way of looking at the Black model is that it is just computing the present value of the difference between the futures price and the exercise price, where the futures price and exercise price are adjusted by the N(d) functions.

- For call options, the futures price is adjusted by $N(d_1)$, and the exercise price is adjusted by $-N(d_2)$.
- For put options, the futures price is adjusted by $-N(-d_1)$, and the exercise price is adjusted by $+N(-d_2)$.

Example 5-1: European Options on Futures Index

Suppose the S&P 500 Index is currently trading at 2,130, and a futures contract on the index, which expires in 3 months, is trading at 2,150. Given specific assumptions regarding the continuously compounded risk-free rate, the dividend yield, and the volatility of the futures, we obtain the following results for options on a futures contract with an exercise price of 2,130.

Call	Put
$N(d_1) = 0.496$	$N(-d_1) = 0.504$
$N(d_2) = 0.468$	$N(-d_2) = 0.532$
c = \$43.45	p = \$45.73

Solution:

Based on this information, we can state that under the Black model:

- The value of the call option will equal the present value of the difference between the current futures price times $N(d_1) = 0.496$ and the exercise price times $N(d_2) = 0.468$.
- The value of the put option will equal the present value of the difference between the exercise price times $N(-d_2) = 0.532$ and the current futures price times $N(-d_1) = 0.504$.
- The underlying price and the exercise price used in the model will be 2,150 (the current futures price, not the current level of the index) and 2,130, respectively.

Interest Rate Options

The underlying for an interest rate option is a forward rate agreement (FRA) that expires on the option expiration date. In turn, the underlying of the FRA is a term deposit.

For example, consider an interest rate call option on 3-month Libor that expires in 6 months.

- The underlying on this option is an FRA on 3-month Libor that expires in 6 months.
- The underlying of the FRA is a 3-month Libor deposit that is made after 6 months and matures 9 months from option initiation (it is a 6 × 9 FRA).

For interest rate options:

- Interest rates are typically set in advance, but interest payments are made in arrears (referred to as advanced set, settled in arrears). This means that the interest rate is determined upon expiration of the option, but the payment is actually made at the end of the (hypothetical) deposit term. For example, if an option on 3-month Libor expires on May 15, the payoff on the option is determined based on the 3-month rate as of May 15, but the actual settlement payment is made on August 15.
    - Recall that FRAs typically work on an advanced set, advanced settled basis.
- Interest rates are quoted on an annualized basis, so if the underlying implied deposit is for less than a year, they must be unannualized. The accrual period on interest rate options is calculated on the actual number of days in the contract divided by the actual number of days in the year (ACT/ACT or ACT/365).
    - Recall that FRAs typically compute the accrual period based on 30/360.

The interest rate option valuation model described next is known as the **standard market model**. Before getting into the formulas, it is very important that you understand the timeline and symbols.

- t_{j-1} denotes the time to option expiration.
- t_{j-1} also denotes the time to FRA expiration.
- The term of the deposit underlying the FRA is denoted by t_m. This term starts at option/FRA expiration (t_{j-1}) and lasts until t_j (= $t_{j-1} + t_m$).
- Interest accrual on the underlying begins at the option expiration (t_{j-1}).
- FRA($0,t_{j-1},t_m$) denotes the fixed rate on a FRA at t = 0 that expires at t_{j-1}, where the underlying deposit matures at Time t_j (= $t_{j-1} + t_m$), with all times expressed on an annual basis.
- We assume the FRA is 30/360 day count.
    - For example, FRA(0,0.5,0.25) = 1.50% denotes the 1.5% fixed rate on a forward rate agreement that expires in 6 months with underlying deposit maturing 9 months from today. This means that the settlement amount on the FRA will be paid 9 months from today.
- R_X denotes the exercise rate on the option (expressed on an annual basis).
- σ denotes volatility of the interest rate. Specifically, it is the annualized standard deviation of the continuously compounded percentage change in the underlying FRA rate.

An interest rate call option gives the call buyer the right to a certain cash payment when the underlying interest rate exceeds the exercise rate. An interest rate put option gives the put buyer the right to a certain cash payment when the underlying interest rate is below the exercise rate.

Under the standard market model, the prices of interest rate call and put options can be expressed as:

$$c = (AP)\, e^{-r(t_{j-1} + t_m)}\left[FRA(0, t_{j-1}, t_m)\, N(d_1) - R_X N(d_2)\right]$$

$$p = (AP)\, e^{-r(t_{j-1} + t_m)}\left[R_X N(-d_2) - FRA(0, t_{j-1}, t_m)\, N(-d_1)\right]$$

where:

AP = Accrual period in years

$$d_1 = \frac{\ln\left[FRA(0, t_{j-1}, t_m) / R_X\right] + (\sigma^2/2)t_{j-1}}{\sigma\sqrt{t_{j-1}}}$$

$$d_2 = d_1 - \sigma\sqrt{t_{j-1}}$$

Note the following:

- The formulas here give the value of the option for a notional amount of 1. We would have to multiply this cost of the option by the notional amount to obtain the full cost of the option.
- The FRA rate, $FRA(0,t_{j-1},t_m)$, and the strike rate, R_X, are both stated on an annual basis. The option premium that we obtain after applying the formulas must be adjusted for the accrual period.
- Other (more subtle) differences between the standard market model and the Black model are:
  - The discount factor, $e^{-r(t_{j-1}+T_m)}$, does not apply to the option expiration, t_{j-1}. Instead, it is applied to the maturity of the deposit underlying the FRA. This is because settlement occurs in arrears.
  - The underlying is not a futures price, but a forward interest rate.
  - The exercise price is also an interest rate, not a price.
  - The time to the option expiration, t_{j-1}, is used in the calculation of d_1 and d_2.
  - Both the forward rate and the exercise rate should be expressed in decimal form and not as percent for the model to be accurate.

Analysis

The standard market model can be described as simply the present value of the expected option payoff at expiration.

- The standard market model for calls can be expressed as $c = PV[E(c_{tj})]$ where $E(c_{tj}) = (AP)[FRA(0,t_{j-1},t_m)N(d_1) - R_X N(d_2)]$.
- The standard market model for puts can be expressed as $p = PV[E(p_{tj})]$, where $E(p_{tj}) = (AP)[R_X N(-d_2) - FRA(0,t_{j-1},t_m)N(-d_1)]$.
- The present value term in this context is simply $e^{-rt_j} = e^{-r(t_{j-1}+t_m)}$. Again, note that we discount from Time t_j, the time when the deposit underlying the FRA matures.

Example 5-2: European Interest Rate Options

Assume that on June 15 you plan to borrow funds in 1 month's time to fund the purchase of an asset that you intend to hold for only 3 months (i.e., you plan to sell it on October 15). Current 3-month Libor is 0.75%, and the FRA rate for the period from July 15 to October 15 is 0.87%. You are concerned that borrowing costs will rise, so you want to purchase an interest rate call option with an exercise rate of 0.79%.

Solution:

In using the Black model to value this interest rate call option, the underlying rate would be 0.87%, which is the rate on the FRA for the borrowing period. Recall that an FRA rate serves as the underlying of interest rate options, not the current 90-day rate (0.75%), or the exercise rate of the option (0.79%). Note that this approach is unlike the BSM model, where the spot price is used as the underlying.

Further, the discount factor used in pricing this option will stretch from the period of option initiation (June 15) until the date that the option settlement payment is made (October 15, which is also the date when the deposit underlying the FRA matures).

Using Interest Options to Create Other Derivative Instruments

FRAs are the building blocks of interest rate swaps.

- If the exercise rate equals the current FRA rate, then a long position on an interest rate call option combined with a short position on an interest rate put option is equivalent to a receive-floating, pay-fixed FRA.
- If the exercise rate equals the current FRA rate, then a long position on an interest rate put option combined with a short position on an interest rate call option is equivalent to a receive-fixed, pay-floating FRA.

Floating-rate payments/receipts can be hedged with positions on interest rate call/put options.

- An interest rate cap is a series of interest rate call options that have expiration dates that correspond to the reset dates on a floating-rate loan. This portfolio of options (each option is known as a caplet) effectively places an upper limit on the floating rate applicable on the loan. Each of the caplets has an exercise rate equal to the desired cap rate.
- An interest rate floor is a series of interest rate put options that have expiration dates that correspond to the reset dates on a floating-rate bond. This portfolio of options (each option is known as a floorlet) effectively places a lower limit on the floating rate received on the bond. Each of the floorlets has an exercise rate equal to the desired floor rate.

Taking a long position on an interest rate cap and a short position on an interest rate floor with the same exercise rate is equal to a receive-floating, pay-fixed interest rate swap.

- When the market interest rate exceeds the exercise rate/swap fixed rate, the holder of the cap and receive-floating leg of the swap receives a net payment.
- When the market interest rate is lower than the exercise rate/swap fixed rate, the writer of the floor and receive-floating leg of the swap makes a net payment.

Taking a long position on an interest rate floor and a short position on an interest rate cap with the same exercise rate is equal to a receive-fixed, pay-floating interest rate swap.

- When the market interest rate exceeds the exercise rate/swap fixed rate, the writer of the cap and receive-fixed leg of the swap makes a net payment.
- When the market interest rate is lower than the exercise rate/swap fixed rate, the holder of the floor and receive-fixed leg of the swap receives a net payment.

Finally, if the exercise rate is set equal to the swap rate, then the value of the cap must be equal to the value of the floor. At initiation of an (at-market) interest rate swap, the swap fixed rate is set at a level that results in the swap having zero value to either counterparty. Therefore, if the exercise rate is set equal to the swap fixed rate, the value of the cap must equal the value of the floor, such that the initial cost of being long a cap and short the floor is also zero. This occurs when the cap and floor strikes are equal to the swap rate.

Swaptions

A **swaption** gives the holder the right, but not the obligation, to enter a swap at the predetermined swap rate (the exercise rate). It basically is an option to enter a swap. Interest rate swaps can be either receive fixed, pay floating or receive floating, pay fixed.

- A **payer swaption** is an option to enter a swap as the pay-fixed, receive-floating side.
- A **receiver swaption** is an option to enter a swap as the receive-fixed, pay-floating side.

The holder of a payer swaption hopes that the market swap fixed rate increases before expiration of the swaption.

- She would then exercise the swaption, and take the pay-fixed, receive-floating side of a swap at the predetermined (lower-than-market) exercise rate.
- At the same time, she would enter an offsetting at-market swap as the receive-fixed, pay-floating side. The swap fixed rate on this swap would be higher (in line with the market rate).
- The floating legs of both the swaps will offset, leaving the payer swaption holder with an annuity equal to the difference between the market (higher) fixed swap rate and the (lower) swaption exercise rate. Therefore, the value of the swaption will reflect an annuity. Going forward, we will use PVA to denote the present value of this annuity matching the forward swap payment.
    - $\text{PVA} = \sum_{j=1}^{n} PV_{0,t_j}(1)$
- This formula assumes a notional amount of 1.
    - It is also known as the **annuity discount factor**.

The Black model can be used to value swaptions. The formulas are as follows:

Payer swaption = (AP) PVA $[R_{FIX}N(d_1) - R_X N(d_2)]$

Receiver swaption = (AP) PVA $[R_X N(-d_2) - R_{FIX}N(-d_1)]$

where:

PVA = PV of annuity matching the forward swap payment based on a notional amount of 1

AP = Accrual period

$$d_1 = \frac{\ln(R_{FIX}/R_X) + (\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

R_{FIX} = Market swap fixed rate (annualized) at the time of swaption expiration (t = T)

R_X = The swaption exercise rate starting at Time T, again quoted on an annual basis. As before, we will assume a notional amount of 1.

σ = Volatility of the forward swap rate. Specifically, it represents the annualized standard deviation of continuously compounded percentage changes in the forward swap rate.

Note the following:

- Swap payments are advanced set, settled in arrears.
- As was the case with interest rate options, the actual premium would need to be scaled by the notional amount.
- Compared to the traditional Black model, the swaption model just described requires adjustments for (1) the accrual period (AP) and (2) the present value of an annuity (PVA).
- Other, more subtle differences between the Black model and the swaption model are as follows:
    - The payoff is not a single payment but a series of payments, and the PVA incorporates the option-related discount factor. There is no explicit discount factor in the formulas here.
    - Rather than the underlying being a futures price, the underlying is the fixed rate on a forward interest rate swap.
    - The exercise price is expressed as an interest rate.
    - Both the forward swap rate and the exercise rate should be expressed in decimal form and not as percent.

Analysis

The swaption model can be described as simply the present value of the expected option payoff at expiration.

- The payer swaption model value can be expressed as:
    - Payer swaption = PV[E($PAY_{SWN,T}$)] where E($PAY_{SWN,T}$) = $e^{rT}PAY_{SWN}$

- The receiver swaption model value can be expressed as:
    - Receiver swaption = PV[E($REC_{SWN,T}$)] where E($REC_{SWN,T}$) = $e^{rT}REC_{SWN}$
- The present value term in this context is simply e^{-rT}.

Alternatively, the swaption model can be described as having two components, a swap component and a bond component.

- For payer swaptions, the swap component is (AP)PVA(R_{FIX})N(d_1) and the bond component is (AP)PVA(R_X)N(d_2).
    - The value of a payer swaption is simply the swap component minus the bond component.
- For receiver swaptions, the swap component is (AP)PVA(R_{FIX})N($-d_1$) and the bond component is (AP)PVA(R_X)N($-d_2$).
    - The value of a receiver swaption is simply the bond component minus the swap component.

Equivalence of Swaps to Other Derivative Instruments

- Being long an interest rate cap and short an interest rate floor with the same exercise rate is comparable to taking a receive-floating, pay-fixed position on an interest rate swap.
    - In terms of swaptions, the position is equal to being long a payer swaption and short a receiver swaption with the same exercise rate.
- Being short an interest rate cap and long an interest rate floor with the same exercise rate is comparable to taking a pay-floating, receive-fixed position on an interest rate swap.
    - In terms of swaptions, the position is equal to being long a receiver swaption and short a payer swaption with the same exercise rate.
- Note that if the exercise rate is selected such that the receiver and payer swaptions have the same value, then the exercise rate is equal to the at-market forward swap rate.
- Being long a callable fixed-rate bond can be viewed as being long a straight fixed-rate bond and short a receiver swaption.
    - The holder of a callable bond effectively holds a straight bond and sells a call option on the bond to the issuer. The issuer, as the holder of this option, exercises this option when interest rates decline (when bond prices rise). Therefore, the holder of the callable bond loses out when interest rates decline, as the value of the callable bond is effectively capped.
    - Now let's equate the position of the holder of a callable bond to being long a straight bond and short a receiver swaption. The holder of the receiver swaption (the issuer) will exercise this option when swap rates decline. As the writer of the swaption, the callable bond holder will have to make settlement payments on the swap, which means that she loses out when interest rates decline.
- The **issuer** of a callable bond can effectively convert its position into a straight bond by selling a receiver swaption. The issuer benefits from an interest rate decline as the option embedded in the callable bond increases in value. However, the decrease in interest rates also means that it would need to make settlement payments on the swap (once the holder of the swaption exercises it). The issuer's positions on the embedded call and the receiver swaption effectively cancel each other out, leaving the issuer with just a straight bond.

Example 5-3: European Swaptions

A company with floating-rate debt outstanding is concerned about interest rates increasing over the next 6 months. The company wants to hedge its position by buying a payer swaption expiring in 6 months, which would offer it the choice to enter a 5-year swap locking in its borrowing costs. The current 6-month forward, 5-year swap rate is 2.85%, the current 5-year swap rate is 2.75%, and the current 6-month risk-free rate is 2.45%.

Solution:

Note the following:

- The underlying rate on this swaption will be 2.85%, which is the current 6-month forward, 5-year swap rate.
- The discount rate that will be used in the swaption model is 2.45%, not the current 5-year swap rate of 2.75%.
- The time to expiration of the swaption will be 6 months.
- The tenor of the swap underlying the swaption will be 5 years.

Key Takeaways from This Section

- The Black futures option model assumes the underlying is a futures or a forward contract.
- Interest rate options can be valued based on a modified Black futures option model in which the underlying is a forward rate agreement (FRA), there is an accrual period adjustment as well as an underlying notional amount, and care must be given to day-count conventions.
    - An interest rate cap is a portfolio of interest rate call options termed caplets, each with the same exercise rate and with sequential maturities.
    - An interest rate floor is a portfolio of interest rate put options termed floorlets, each with the same exercise rate and with sequential maturities.
- A swaption is an option on a swap.
    - A payer swaption is an option on a swap to pay fixed and receive floating.
    - A receiver swaption is an option on a swap to receive fixed and pay floating.
- Holding a fixed-rate callable bond can be viewed as being long on a straight fixed-rate bond and being short on a receiver swaption.

LESSON 6: OPTION GREEKS AND IMPLIED VOLATILITY

LOS 38k: Interpret each of the option Greeks. Vol 5, pp 430–439

LOS 38l: Describe how a delta hedge is executed. Vol 5, pp 430–431

LOS 38m: Describe the role of gamma risk in options trading. Vol 5, pp 431–436

LOS 38n: Define implied volatility and explain how it is used in options trading. Vol 5, pp 439–443

Option Greeks and Implied Volatility

In this section, we study option "Greeks" and their impact on option values. Each of the Greeks—delta, gamma, theta, rho, and vega—is an input in the BSM option valuation model. These measures are sometimes referred to as **static risk measures** in that they capture movements in option value for a movement in one of them, while holding all other factors constant. Our focus in this lesson will be on European stock options, where the underlying stock's dividend yield is denoted by δ.

Delta

Delta is defined as the change in the value of a particular instrument given a small change in the value of the stock, holding everything else constant. For example, the delta of a long position on one share of stock is +1.0, while the delta of a short position on one share of stock is –1.0.

Option delta measures the change in the value of an option given a small change in the value of the underlying stock, holding everything else constant. While it is a measure of the magnitude of change in the value of an option given a change in the underlying, delta does not measure the probability of the change in value. Option deltas are calculated as:

$$\text{Call option delta} = e^{-\delta T} N(d_1)$$

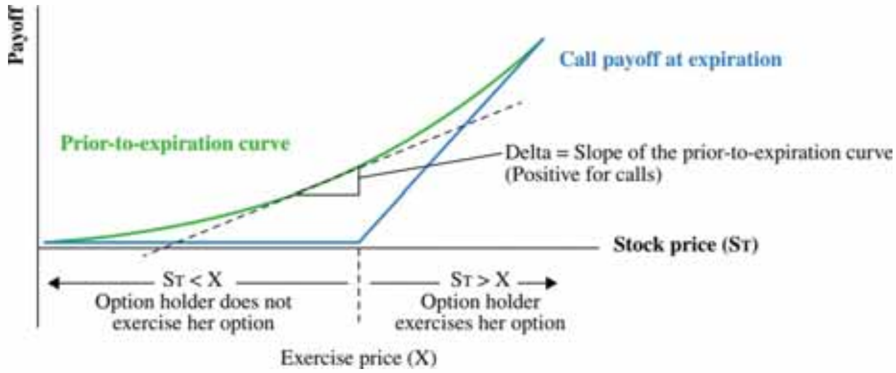
$$\text{Put option delta} = -\,e^{-\delta T} N(-\,d_1)$$

Recall that the value of an option equals the sum of its intrinsic value and time value. The straight lines in Figure 6-1 represent call and put option payoffs at option expiration. The curves illustrate the values of the options prior to expiration. We'll call these the "prior-to-expiration" curves. The vertical distances between the prior-to-expiration curves and the option payoff lines represent the time value of the options.

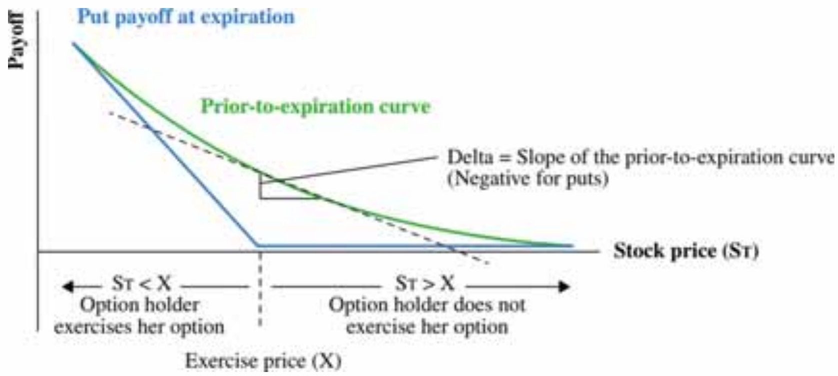
The slope of the prior-to-expiration curve equals the change in the value of the option for a one-unit change in the stock price, which basically equals the option's delta. Therefore, delta equals the slope of the prior-to-expiration curve.

Figure 6-1: European Option Payoff Diagrams and Delta

European Call Option:



European Put Option:



From the figure, note that:

- Call option values and underlying prices are positively related, so for a call option, delta is always positive. It will increase toward 1.0 as the underlying price moves up, and decrease toward 0 as the underlying price moves down.
    - When the call is deep in-the-money, as the stock price increases, the call option's value increases almost one-for-one with the increase in the stock price. The increase in intrinsic value of the option equals the increase in the stock price. Therefore, in this region, call option delta equals +1.0.
    - When the call is deep out-of-the-money, as the stock price decreases, the call option's value does not fall by much, as the intrinsic value of the option is already zero. Therefore, in this region, call option delta equals 0.
- Put option values and underlying prices are negatively related, so for a put option, delta is always negative. It will decrease toward –1.0 as the underlying price moves down, and increase toward 0 as the underlying price moves up.
    - When the put is deep in-the-money, as the stock price decreases, the put option's value increases almost one-for-one with the decrease in the stock price. The increase in intrinsic value of the option equals the decrease in the stock price. Therefore, in this region, put option delta equals –1.0.
    - When the put is deep out-of-the-money, as the stock price increases, the put option's value does not fall by much, as the intrinsic value of the option is already zero. Therefore, in this region, put option delta equals 0.

Option delta is also influenced by time to expiration. As the option approaches expiration, the time value of the option falls, which results in the prior-to-expiration curve edging closer and closer to the payoff curve. Therefore:

- If the underlying price remains unchanged, as a call option moves toward expiration:
  - Delta will move toward 1.0 if the call is in-the-money.
  - Delta will move toward 0 if the call is out-of-the-money.
- If the underlying price remains unchanged, as the put option moves toward expiration:
  - Delta will move toward –1.0 if the put is in-the-money.
  - Delta will move toward 0 if the put is out-of-the-money.

Delta hedging an option is the process of creating a portfolio that combines a position in options with a position on the underlying stock so that the combined position is insulated from changes in the price of the underlying stock. For a single-option delta hedge, we first compute option delta, and then buy or sell delta units of stock. Practically speaking, delta hedging is used to manipulate the delta of an entire portfolio, not just one option position.

A **delta neutral portfolio** refers to setting the portfolio delta all the way to zero. Theoretically, a delta neutral portfolio will not change in value for small changes in the underlying stock. For a delta neutral portfolio, it must be the case that:

Note that the hedging instrument could be the underlying stock, call options, or put options.

Preexisting portfolio delta + N_H $\text{Delta}_H = 0$

- N_H = Number of units of the hedging instrument.
- Delta_H = Delta of the hedging instrument.

Rearranging the previous expression, we obtain the following expression for N_H, the optimal number of hedging units:

$$\boxed{N_H = -\frac{\text{Portfolio delta}}{\text{Delta}_H}}$$

Note that if N_H is negative, then one must short the hedging instrument, and if N_H is positive, then one must go long the hedging instrument.

Consider the following examples:

- If the portfolio consists of 5,000 shares of stock at $25 per share, then portfolio delta is 5,000 (recall that long stock has a delta of +1.0). If the hedging instrument is also stock, then the optimal number of hedging units, N_H, is calculated as –5,000 / 1.0 = –5,000, or short 5,000 shares.
- If the portfolio delta is 3,000 and a particular call option with delta of 0.75 is used as the hedging instrument, then the optimal number of hedging units, N_H, is calculated as –3,000 / 0.75 = –4,000, or short 4,000 call options.
- If a portfolio of options has a delta of –2,500, and stock is used as the hedging instrument, then the optimal number of hedging units, N_H, is calculated as –(–2,500) / 1.0 = +2,500, or buy 2,500 shares of stock.

Delta can also be used to estimate the change in the price of an option in response to a change in the value of the underlying stock.

For calls: $\hat{c} - c \cong \text{Delta}_c(\hat{S} - S)$

For puts: $\hat{p} - p \cong \text{Delta}_p(\hat{S} - S)$

where $\hat{c}$, $\hat{p}$, and $\hat{S}$ denote some new value for the call, put, and stock, respectively.

Figure 6-2 illustrates estimates of call option value based on delta versus actual call values based on the BSM model.

Figure 6-2: Actual Call Values versus Delta-Estimated Call Values

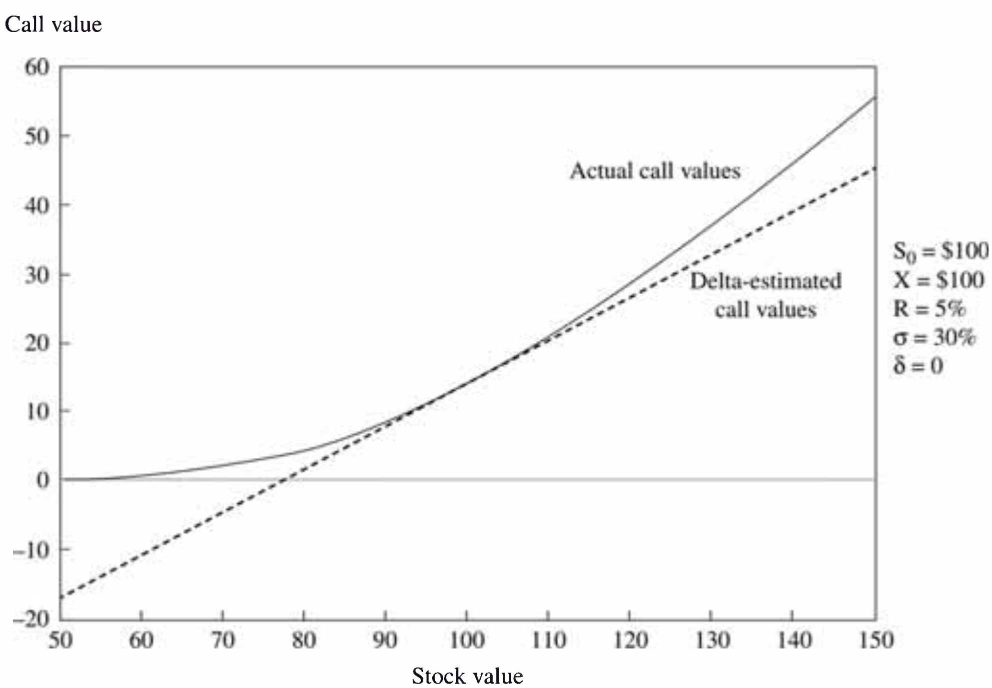


Exhibit 15, Volume 5, CFA Program Curriculum 2020

Notice the following:

- For very small changes in stock, the delta-based approximation is quite accurate. For example, when the stock goes from \$100 to \$99, the actual call line and delta-estimated call line are very close to each other.
- However, for large changes in stock, the actual call line is significantly above the delta-estimated call line. Importantly, this is the case for large increases **and** decreases in the price of the stock.
- The implication here is that delta-based estimation is not perfect, and it gets more and more inaccurate as the stock moves away from its initial value.
- The importance of the BSM assumption of continuous trading for avoiding hedging risk should become clear now. Hedging risk comes from the difference between these two lines, and the extent to which large changes in stock price occur.

Example 6-1: Delta Hedging

An investor has a short position in put options on 5,000 shares of stock. Call option delta is given as 0.532, while put option delta is given as –0.419. Assume that each option has one share of stock as the underlying.

1. Determine the delta hedge strategy for the preexisting portfolio given that the hedging instrument is stock.
2. Determine the delta hedge strategy for the preexisting portfolio given that the hedging instrument is call options.

Solution:

1. Put delta is given as –0.419. Since the investor has shorted puts, the delta of her position is +0.419. The delta of the preexisting portfolio of puts is calculated as 5,000 * 0.419 = 2,095.

   The delta of the hedging instrument is +1.0.

   $$N_H = -\frac{\text{Portfolio delta}}{\text{Delta}_H}$$

   Therefore, the number of hedging units is calculated as –2,095 / 1.0 = –2,095 or short sell 2,095 units of stock.
2. The delta of the existing portfolio is still 2,095. The delta of the hedging instrument (call options) is 0.532.

   Therefore, the number of hedging units is calculated as –2,095 / 0.532 = –3,938 or sell 3,938 call options.

Gamma

Option gamma is defined as the change in an option's delta given a small change in the value of the underlying stock, holding everything else constant. Option gamma measures the curvature in the option price–stock price relationship (more on this later).

- The gamma of a long or short position in one share of stock is zero because the delta of a share of stock never changes. Delta stays at +1.0 for a long position on a stock, and at –1.0 for a short position.
- The gammas for call and put options are the same and can be expressed as:

$$\text{Gamma}_c = \text{Gamma}_p = \frac{e^{-\delta T}}{S\sigma\sqrt{T}} n(d_1)\ .$$

where $n(d_1)$ is the standard normal probability density function.

- Gamma is always non-negative.
- As the stock price changes and as time to expiration changes, the gamma is also changing.

We learned from Figure 6-1 that delta is an accurate measure of the change in the value of an option for only small changes in the value of the underlying stock. When the price of the underlying stock changes by a more significant amount, the curvature of the prior-to-expiration curve comes into play and reduces the change in the option price that is explained by delta (which is only a linear approximation of the sensitivity of option values to changes in the underlying price). In this respect, option delta is similar to bond duration, which is only a linear measure of the sensitivity of bond prices to changes in yields.

Similar to the convexity adjustment effect (which is the second-order effect) that accounts for the curvature of the price-yield profile in the fixed income world, we have gamma in the options world. Gamma measures how sensitive delta is to changes in the price of the underlying stock. Stated differently, gamma measures the non-linearity risk or the risk that remains once the portfolio is delta neutral.

- When gamma is large, delta is very sensitive to changes in the value of the underlying stock and cannot provide a good approximation of how much the value of the option would change given a change in the price of the underlying stock.
- When gamma is small, delta is not as sensitive to changes in the value of the underlying stock and can provide a reasonably good approximation of how much the value of the option would change given a change in the price of the underlying stock.

Gamma is largest when there is great uncertainty regarding whether the option will expire in-the-money or out-of-the-money. This implies that gamma will tend to be large when an option is at-the-money and close to expiration. This also means that a delta hedge would work poorly when an option is at-the-money and close to expiration. On the other hand, when an option is deep in-the-money or deep out-of-the-money, gamma approaches zero, as changes in the price of the underlying do not have a significant impact on delta.

A **gamma neutral portfolio** implies the gamma is zero. If we want to alter the gamma and delta exposures of our portfolio, our first step would be to bring gamma to an acceptable level using options as the hedging instrument. Options have gamma, so they can be used to alter the portfolio's gamma exposure. Our second step would then be to alter the overall portfolio's (preexisting portfolio plus any options positions undertaken to alter gamma) delta by buying or selling stock. Stocks have positive delta but zero gamma, so adding positions on stock to our portfolio will change the overall portfolio's delta, while leaving gamma unchanged.

Gamma can be used to improve forecasts of changes in option prices. Changes in option prices that incorporate option gamma and delta can be computed as:

$$\text{For calls}: \hat{c} - c \approx \text{Delta}_c(\hat{S} - S) + \frac{\text{Gamma}_c}{2}(\hat{S} - S)^2$$

$$\text{For puts}: \hat{p} - p \approx \text{Delta}_p(\hat{S} - S) + \frac{\text{Gamma}_p}{2}(\hat{S} - S)^2$$

where $\hat{c}$, $\hat{p}$, and $\hat{S}$ denote new values for the call, put, and stock, respectively.

Figure 6-3 presents actual call values (based on the BSM model), delta-based estimates of call value, and delta-plus-gamma-based estimates of call value.

Figure 6-3: Call Values, Delta-Estimated Call Values, and Delta-Plus-Gamma-Estimated Call Values

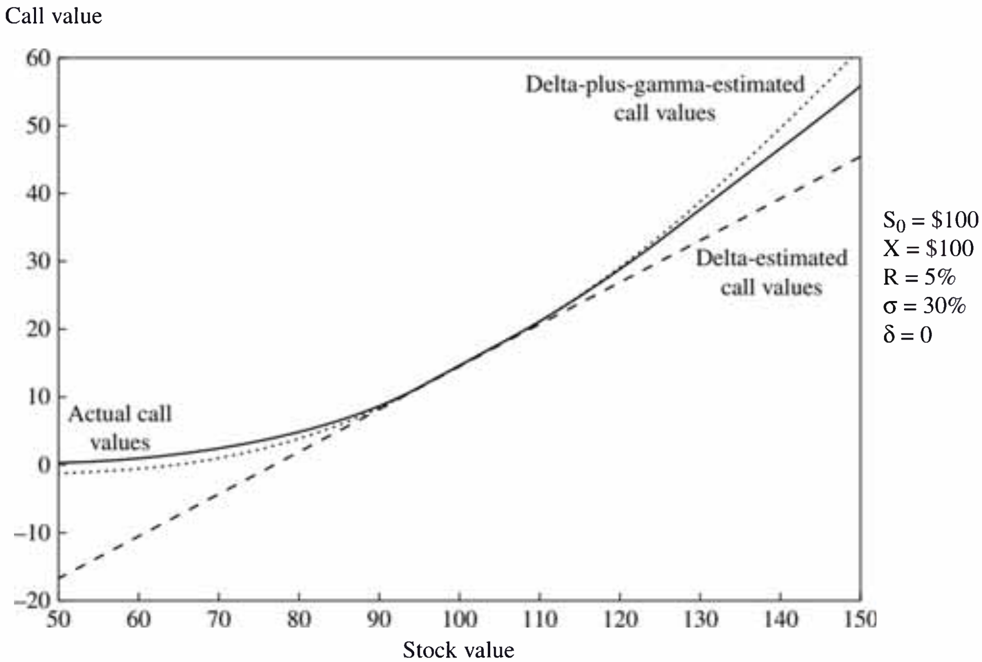


Exhibit 16, Volume 5, CFA Program Curriculum 2020

Notice the following:

- For very small changes in the stock, the delta approximation and the delta-plus-gamma approximations are fairly accurate.
- For large changes in the value of the stock, the delta-plus-gamma-based estimate is a more accurate approximation of the value of the call compared to the estimate based on delta alone. This is the case for large up **and** down moves in the stock.
- When the stock moves up by a large amount, the delta-plus-gamma-based estimate overestimates the price of the call. On the other hand, when the stock moves down by a large amount, the delta-plus-gamma-based estimate underestimates the price of the call.

Once again, note that if the BSM assumption of continuous trading holds, we would have no gamma risk, as stock prices would move continuously and smoothly. However, since stock prices often jump, it can be very difficult to create perfect hedges.

Theta

Option theta is defined as the change in the value of an option for a given small change in calendar time, holding everything else constant. We know that an option's value consists of its exercise value and its time value. Theta effectively measures the decline in an option's time value as it approaches expiration (also known as **time decay**). At expiration, of course, an option is worth only its exercise value (time value = 0).

Stocks do not have an expiration date, so stock theta is zero. Therefore, like gamma, portfolio theta cannot be adjusted by undertaking stock trades.

Theta is fundamentally different from delta and gamma in the sense that the passage of time does not involve any uncertainty. Time decay will always occur for options.

Theta is negative for both calls and puts. Further, the rate at which option values decrease accelerates as time to expiration decreases. (See Figure 6-4.)

Figure 6-4: Option Values and Time to Expiration

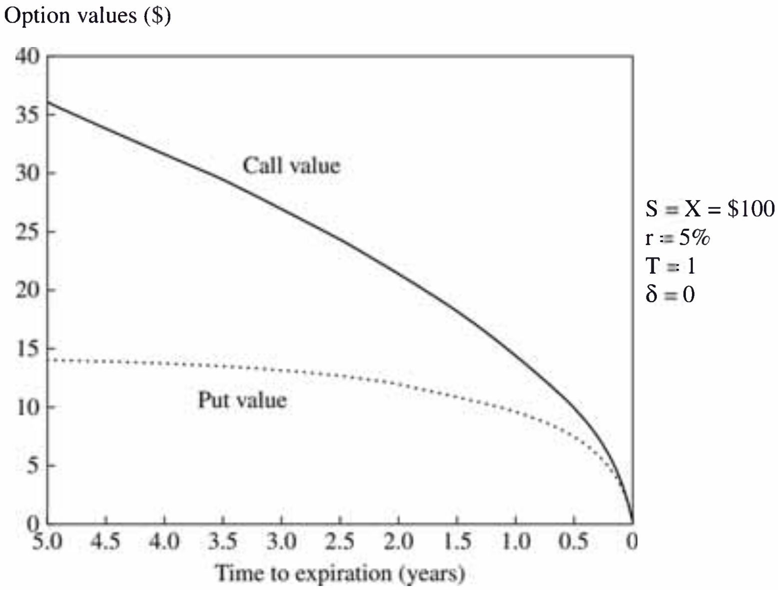


Exhibit 17, Volume 5, CFA Program Curriculum 2020

Vega

Vega measures the sensitivity of the value of an option to changes in volatility of the underlying stock. Volatility refers to the standard deviation of the continuously compounded return on the underlying stock.

- The vega of a call option is the same as the vega of an otherwise identical put option.
- Vega is positive for both calls and puts.
    - All other things remaining the same, an increase in volatility increases option values.
- Unlike other Greeks, vega is based on a parameter (future volatility) that is not observable in the market.
- Of all the Greeks, option values are most sensitive to volatility changes.
- Vega is highest when options are near- or at-the-money and close to expiration.
- Volatility is typically hedged using other options. However, note that volatility itself tends to be quite volatile. As a result, it is sometimes treated as a separate asset class or separate risk factor.
- If volatility approaches 0, option values approach their lower bounds.
    - The lower bound for a European call option is $\text{Max}[0, S_T - X/(1 + R_F)^T]$.
    - The lower bound for a European put option is $\text{Max}[0, X/(1 + R_F)^T - S_T]$.

Figure 6-5 presents the effect of volatility on option values.

Figure 6-5: Option Values and Volatility

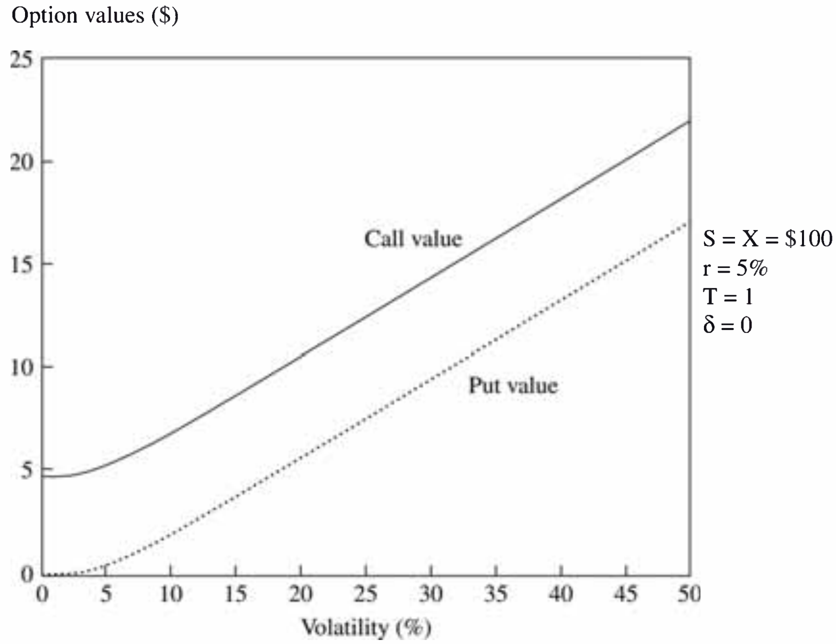


Exhibit 18, Volume 5, CFA Program Curriculum 2020

Rho

Rho measures the sensitivity of the price of an option to small changes in the risk-free rate, holding everything else constant.

Both call and put options on most assets are not very sensitive to changes in the risk-free rate. Generally speaking:

- The rho of call options is positive. Call option values increase in response to an increase in the risk-free rate.
    - Intuitively, buying a call options allows the investor to avoid the larger outlay required to purchase the stock. The investor can earn interest on that money, so the higher the risk-free rate, the higher the demand for (and value of) using call options.
- The rho of put options is negative. Put option values decrease in response to an increase in the risk-free rate.
    - Intuitively, buying a put option (to take a short position on a stock) rather than selling the stock itself prevents the investor from earning interest on the proceeds from the sale. Therefore, the higher the interest rate, the lower the demand for (and value of) put options.

Figure 6-6 shows the impact of changes in the risk-free rate on option values.

Figure 6-6: Option Values and Interest Rates

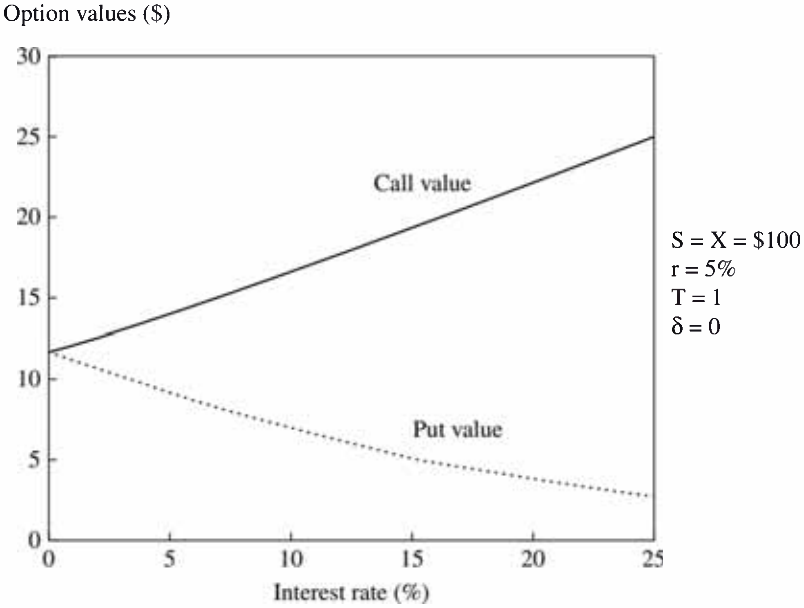


Exhibit 19, Volume 5, CFA Program Curriculum 2020

Notice that:

- When interest rates are zero, call and put option values are the same for at-the-money options.
- As interest rates rise, call options increase in value, while put option values decrease.

Implied Volatility

We have already mentioned that **future volatility** is a BSM model input variable that is not observable in the marketplace. While historical volatility can be estimated (based on historical data), it is dangerous to assume that the past can serve as a good guide for the future. Different investors will have different views of the future volatility.

Given option prices in the marketplace, and values for all other inputs in the model (the value of the underlying, the exercise price, the expiration date, the risk-free rate, and dividends paid by the underlying), the BSM model is often inverted and used to infer future volatility. This inferred volatility is known as **implied volatility** (the volatility built into, or implied by, current market prices of options).

> The process of inferring volatility is like using observed market prices to infer the yield to maturity on bonds.

Implied volatility enables us to understand the collective opinions of investors on the volatility of the underlying and the demand for options. If the demand for options increases, option prices would rise, and so would implied volatility.

Note that:

- While the BSM model assumes that volatility is constant, in practice it is quite common to observe different implied volatilities for otherwise identical calls and puts.
- Implied volatility also varies across time to expiration as well as across exercise prices.
    - The implied volatility with respect to time to expiration is known as the **term structure of volatility**.
    - The implied volatility with respect to the exercise price is known as the **volatility smile** or **volatility skew**.
    - A three-dimensional plot of the implied volatility with respect to both time to expiration and exercise prices is known as the **volatility surface**.
    - If BSM assumptions were true, we would expect the volatility surface to be flat.
- Implied volatility is also not constant through calendar time. An increase in implied volatility indicates an increased market price of risk.
    - For example, an increase in the implied volatility of a put indicates a higher cost of attaining downside protection, or that the market price of hedging is increasing.

      Volatility indexes measure the collective opinions of investors on the volatility of the broader market. Investors can now trade futures and options on various volatility indexes in an effort to manage their vega exposures. One of the best-known volatility indexes is the **Chicago Board Options Exchange S&P 500 Volatility Index**, known as the **VIX**.
- The VIX is quoted as a percent and represents the approximate implied volatility of the S&P 500 over the next 30 days.
- It is often referred to as the "fear index" because it is a gauge of market uncertainty.
    - An increase in the VIX index indicates greater investor uncertainty.
- Historically, the VIX has spiked up in times of market crisis, indicating great fear/uncertainty in the equity market. Since implied volatility reflects (1) beliefs regarding future volatility and (2) demand for risk-mitigating products like options, a higher reading of the VIX during a crisis reflects both higher expected future volatility and higher demand for buying rather than writing options.

In some markets, options are quoted in terms of volatility. The price is then computed by using the quoted volatility in an agreed-upon model. Volatility quotes allow investors to effectively trade volatility, which is the real (unknown) variable when it comes to option pricing. All other inputs—value of the underlying, exercise price, expiration, risk-free rate, and dividend yield—are observable. Quoted volatility allows traders to make effective comparisons across options with different exercise prices and expiration dates in a common unit of measure.

For example, consider two call options on the same stock. Option A has a longer term to expiration but a higher exercise price, whereas Option B has a shorter term to expiration but a lower exercise price. Based on this information alone, it is difficult to tell which option should be priced higher. But if Option A had a higher implied volatility, we would be able to conclude that after taking into account the effects of exercise price and term to expiration, Option A is the relatively more expensive one. Implied volatility quotes make it easier for investors to understand the current market price of various risk exposures.

Example 6-2: Implied Volatility in Option Trading across Markets

Suppose that a 6-month at-the-money call on the S&P 500 Index is available at 19% implied volatility, and a 3-month in-the-money put on Citigroup (C) is available at 24%. You believe that S&P 500 volatility should be around 23%, while C volatility should be around 18%. What trades would you undertake based on your views?

Solution:

You believe that S&P 500 call volatility is understated by the market, and that the C put volatility is overstated by the market, so you expect S&P 500 volatility to rise and C volatility to fall. Therefore, you would buy the S&P 500 call and sell the C put.

Key Takeaways from This Section

- Delta is defined as the change in the value of a portfolio for a given small change in the value of the underlying instrument, holding everything else constant.
- Delta hedging refers to managing the portfolio delta by entering additional positions to the portfolio.
- A delta neutral portfolio is one in which the portfolio delta is set and maintained at zero.
- A change in the option price can be estimated with a delta approximation.
- Because delta is used to make a linear approximation of the non-linear relationship that exists between the option price and the underlying price, there is an error that can be estimated by gamma.
- Gamma is defined as the change in a given portfolio delta for a given small change in the value of the underlying instrument, holding everything else constant.
- Gamma captures the non-linearity risk or the risk that remains once the portfolio is delta neutral.
- A gamma neutral portfolio is one in which the portfolio gamma is maintained at zero.
- The change in the option price can be better estimated by a delta-plus-gamma approximation compared to just a delta approximation.
- Theta is defined as the change in the value of an option given a small change in calendar time, holding everything else constant.
- Vega is defined as the change in a given portfolio for a given small change in volatility, holding everything else constant.
- Rho is defined as the change in a given portfolio for a given small change in the risk-free interest rate, holding everything else constant.
- Implied volatility is a measure of future volatility, whereas historical volatility is a measure of past volatility.
- Option prices reflect the beliefs of option market participant about the future volatility of the underlying, and the demand for purchasing relative to writing options.