ASSIGNMENT 3

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Exercise 3.1)

a)

Let C_1 and C_2 be two convex sets. Let $[P_1, P_2]$ be the line segment whose endpoints are P_1 and P_2.

$$[P_1, P_2] = \{t \times P_1 + (1-t) \times P_2 : 0 \leq t \leq 1\}.$$

If $C_1 \cap C_2$ contains 0 or 1 element, such intersection is convex by definitions. Otherwise, $C_1 \cap C_2$ is convex if $P_1, P_2 \in C_1 \cap C_2 \Rightarrow [P_1, P_2] \subseteq C_1 \cap C_2$:

$P_1, P_2 \in C_1 \cap\ C_2\ \Rightarrow P_1, P_2 \in C_1$ and also $P_1, P_2 \in C_2 \Rightarrow [P_1, P_2] \subseteq C_1$ and $[P_1, P_2] \subseteq C_2 \Rightarrow [P_1, P_2] \subseteq C_1 \cap\ C_2$

b)

Let $C_1 = \{1\}$ and $C_2 = \{2\}$ (both convex by definition)

$C_1 \cup C_2 = \{1, 2\}$

Applying the condition for convexity for $t = 0.5$:

$t \times 1 + (1-t) \times 2 \Rightarrow t + 2 - 2t \Rightarrow 2 - t \Rightarrow 1.5$

$\{1.5\} \notin C_1 \cup C_2 \Rightarrow$ The union of two convex sets is not necesseceraly convex.

c)

Let $C_1 = \{x \mid x \in \Re, 1 \leq x \leq 2\}$ and $C_2 = \{1.5\}$ (both convex sets)

$C_1 \setminus C_2 = \{x \mid x \in \Re, 1 \leq x \leq 2\} \setminus \{1.5\}$

Applying the condition for convexity for $t = 0.5$:

$t \times 1 + (1-t) \times 2 \Rightarrow t + 2 - 2t \Rightarrow 2 - t \Rightarrow 1.5$

$\{1.5\} \notin C_1 \setminus C_2 \Rightarrow$ The difference between two convex sets is not necesseraly convex.

d)

Let$'s$ pick two arbitrary point $x, y \in S$. We need to prove that $[x, y] \subseteq S$.

Considering an arbitrary $\lambda \in [0, 1]$:
As $x \in S$, $\langle c, x\rangle \leq b$ holds, so does $\langle c, \lambda x\rangle = \lambda\langle c, x\rangle \leq \lambda b$
As $y \in S$, $\langle c, y\rangle \leq b$ holds, so does $\langle c, (1-\lambda)y\rangle = (1-\lambda)\langle c, y\rangle \leq (1-\lambda)b$

Adding then the two equations together we get :
$\langle c, \lambda x\rangle + \langle c, (1-\lambda)y\rangle \leq \lambda b + (1-\lambda)b = b$

This proves that $\lambda x + (1-\lambda)y \in S \;\; \forall \lambda \in [0, 1] \Rightarrow$ The whole segment $[x, y] \in S$.

$[x, y] \subseteq S \Rightarrow S$ is convex.

e)

Let x, y be two vectors in the set, $x, y \in S$, and λ an arbitrary number, $\lambda \in [0, 1]$.

As $x \in S$, $\mathrm{Ax} \leq b$ holds, so does $A(\lambda x) = \lambda \cdot \mathrm{Ax} \leq \lambda\, b$
As $y \in S$, $\mathrm{Ay} \leq b$ holds, so does $A[(1-\lambda)y] = (1-\lambda) \cdot \mathrm{Ay} \leq (1-\lambda)\, b$

Adding the two equations :
$A(\lambda\, x) + A[(1-\lambda\,)y] \leq \lambda b + (1-\lambda)b = b$

Similarly to the point d), this proves that $\lambda x + (1-\lambda)y \in S \;\; \forall \lambda \in [0, 1] \Rightarrow$ The whole segment $[x, y] \in S$

The set S is a convex subset of $\mathfrak{R}^n$.

Exercise 3.2)

a)

Given $k \in \{1, \ldots, n\}$, $\theta \in [0, 1]$, and considering arbitrary $x, y \in \mathfrak{R}^n$. It is possible to apply the definition of convexity for a function:

$f(\theta x_k + (1-\theta)y_k) \leq \theta\, f(x_i) + (1-\theta)f(y_i)$

$\min_k[\theta x_k + (1-\theta)y_k] \leq \theta \cdot \; \min_i(x_i) + (1-\theta) \cdot \; \min_i(y_i)$

This statement does not hold for all k, it is possible to easily prove that by an example:

Let $x, y \in R^3$ with x = {25, 50, 5} and y = {10, 20, 12}

Applying the function for $\theta = 0.5$:

$\min_k[\theta x_k + (1-\theta)y_k] = \min[0.5 \cdot \; 5 + 0.5 \cdot 12] = 8.5$

$\theta \cdot \min_i(x_i) + (1-\theta) \cdot \min_i(y_i) = 0.5 \cdot 5 + 0.5 \cdot 10 = 7.5$

This proves that the function is not convex.

b)

Given $k \in \{1, \ldots, n\}$, $\theta \in [0,1]$, and considering arbitrary $x, y \in \mathfrak{R}^n$. It is possible to apply the definition of convexity for a function:

$f(\theta x_k + (1-\theta)y_k) \leq \theta\, f(x_i) + (1-\theta) f(y_i)$

(1) $\max_k[\theta x_k + (1-\theta)y_k] \leq \theta \cdot \max_i(x_i) + (1-\theta) \cdot \max_i(y_i)$

As $x_k \leq \max_i(x_i)$ and $y_k \leq \max_i(y_i) \quad \forall k, i$

it is possible to conclude that the equation (1) is true for all k, the function is thus convex.

c)

$\sqrt{x}$ is differentiable for $x > 0$, but not for $x = 0$, because the $\lim\limits_{x \to 0^-} \sqrt{x}$ does not exist as the function is not defined for $x < 0$.

$f'(x) = \frac{1}{2} x^{\left(\frac{1}{2}-1\right)} = \frac{1}{2\sqrt{x}}$, whose domain is $(0, +\infty)$, again, $f'(x)$ is not defined for $x < 0$, but in this case $x \neq 0$ must hold as well

$f''(x) = \frac{1}{2} x^{\left(\frac{1}{2}-1\right)} \times \left(-\frac{1}{2}\right) = -\frac{1}{4\sqrt{x^3}}$

The function is not convex as $f''(x) < 0\ \forall\, x \in \operatorname{dom}(f''(x))$.

d)

The function $f(x) = e^{\mathrm{ax}}$ is differentiable $\forall\, x \in \mathfrak{R}$.
$f'(x) = a \cdot e^{\mathrm{ax}}$
$f''(x) = a^2 \cdot e^{\mathrm{ax}}$
$f''(x) \geq 0\ \forall\, x \in \mathfrak{R} \Rightarrow f(x)$ is convex.

e)

$f(x)$ is differentiable $\forall\, x \in \mathfrak{R}$
$f'(x) = x^2 - 1 \Rightarrow f'(x) = 0 \Rightarrow x = \pm 1$
$f''(x) = 2x \Rightarrow$ positive only for $x \geq 0$
$f''(x) \geq 0\ \forall\, x \geq 0 \Rightarrow f(x)$ is convex only for $x \geq 0$.
$f(x)$is NOT a convex function.

Exercise 3.3)

Definitions:

- S is the set of scenarios
- C is the set of crops
- F is the set of fields
- $b_{f,c}$ is a binary variable indicating whether crop $c \in C$ is planted in the field $f \in F$.
- x_c indicates the amount of crop $c \in C$ planted.
- $z_{f,c}$ is an auxiliary matrix used to linearize the problem, it contains the number of acres of each crop $c \in C$ planted in a field $f \in F$
- c_c indicates the planting costs of crop $c \in C$.
- $F_{\max_f}$ indicates the maximum amount of acres available in field $f \in F$.
- $w^s{}_w$ and $w^s{}_{corn}$ indicate respectively the amount of wheat and of corn sold, $w, corn \in C$ in scenario $s \in S$
- $e^s{}_H, e^s{}_L$ indicate respectively the amount of sugar beet sold at high/low price in scenario $s \in S$
- $y^s{}_w$ and $y^s{}_{corn}$ indicate the amounts of wheat/corn purchased in scenario $s \in S$
- $p^s{}_c$ indicate the yields for each crop $c \in C$ in each scenario

$$\max_{x,b,y,w,e} \sum_{i=1}^{3} \left(\frac{\left(170w^i{}_W + 150w^i{}_c + 36e^i{}_H + 10e^i{}_L - 238y^i{}_w - 210y^i{}_c\right)}{3} \right) - \left(c^T \cdot \sum_{f \in F} z_{f,c} \right)$$

$$s.t.$$

$$\sum_{c \in C} b_{f,c} \leq 1 \quad \forall f \in F$$

$$F_i \leq F_{\max_i} \quad \forall i \in F$$

$$z_{i,j} \leq F_{\max_i} \cdot b_{i,j} \quad \forall i \in F, \forall j \in C$$

$$z_{i,j} \geq F_i - F_{\max_i} \cdot (1 - b_{i,j}) \quad \forall i \in F, \forall j \in C$$

$$p^s{}_w \cdot \sum_{i \in F} z_{i,w} + y^s{}_w - w^s{}_w = 200 \quad \forall s \in S$$

$$p^s{}_{\text{corn}} \sum_{i \in F} z_{i,\text{corn}} + y^s{}_{\text{corn}} - w^s{}_{\text{corn}} = 240 \quad \forall s \in S$$

$$p^s{}_{\text{sb}} \sum_{i \in F} z_{i,sb} - e^s{}_H - e^s{}_L = 0 \quad \forall s \in S$$

$$e^i{}_H \leq 6000 \quad \forall s \in S$$

$$w^i{}_c, y^i{}_c \geq 0 \quad \forall i \in S, \forall c \in C$$

$$e^i{}_L, e^i{}_H \geq 0 \quad \forall i \in S$$

$$x_c \geq 0 \quad \forall c \in C$$

$$F_f \geq 0 \quad \forall f \in F$$

$$z_{i,j} \geq 0 \quad \forall i \in F, \forall j \in C$$

Exercise 3.4)

a)

The optimal solution for the problem depends on the value that the random variable ω_1 assumes.

In particular, in order to minimize the sum of x_1 and x_2 while always satisfying the constraint, we define a stepwise function depending on ω_1. If ω_1 is between 0 and 1, it will be optimal to set $x_2 = 4$ and $x_1 = 0$, in order to satisfy the constraint by equaling it and at the same time minimising the sum of x_1 and x_2.

If $1 \leq \omega_1 \leq 3$, the optimal solution would be to divide 4 by the value of ω_1, so that the constraint is still satisfied, by equaling it, and the objective function assumes values that decrease with increasing ω_1.

The optimal solution is:

$$x_1^* = \begin{cases} 0 & \text{if } 0 \leq \omega_1 < 1 \\ \frac{4}{\omega_1} & \text{if } 1 \leq \omega_1 \leq 3 \end{cases}$$

$$x_2^* = \begin{cases} 4 & \text{if } 0 \leq \omega_1 < 1 \\ 0 & \text{if } 1 \leq \omega_1 \leq 3 \end{cases}$$

$$v^* = \begin{cases} 4 & \text{if } 0 \leq \omega_1 < 1 \\ \frac{4}{\omega_1} & \text{if } 1 \leq \omega_1 \leq 3 \end{cases}$$

b)

Calculate the CDF of v^*, F_{v^*}:

The range of the function v^* is $R_{v^*} = \left[\frac{4}{3}, 4\right]$

From R_{v^*} it is possible to say that:

$$F_{V^*}(v^*) = P(V^* \leq v^*) = 1 \quad \text{if } v^* \geq 4$$

$$F_{V^*}(v^*) = P(V^* \leq v^*) = 0 \quad \text{if } v^* < \frac{4}{3}$$

With values in the range of the function, the CDF can be derived as:

$$F_{V^*}(v^*) = P(V^* \leq v^*) \qquad \text{if } \frac{4}{3} \leq v^* < 4$$

$$= P\left(\frac{4}{\omega_1} \leq v^*\right) = P\left(\omega_1 \geq \frac{4}{v^*}\right) =$$

$$= 1 - P\left(\omega_1 \leq \frac{4}{v^*}\right)$$

$$= 1 - F_{\omega_1}\left(\frac{4}{v^*}\right)$$

$$= 1 - \int_0^{\frac{4}{v^*}} \frac{1}{3}\,\mathrm{dt} = 1 - \left[\frac{x}{3}\right]_0^{\frac{4}{v^*}} =$$

$$1 - \left(\frac{4}{3v^*} - 0\right) = 1 - \frac{4}{3v^*}$$

Then, F_{v^*} is equal to:

$$F_{v^*} = \begin{cases} 0 & if\, v^* < \frac{4}{3} \\ 1 - \frac{4}{3v^*} & if\, \frac{4}{3} \leq v^* < 4 \\ 1 & v^* \geq 4 \end{cases}$$

c)

The PDF of v^* is equal to:

$$f_{v^*}(v^*) = \frac{\mathrm{d}}{\mathrm{d}v^*} F_{v^*}(v^*) =$$

$$= \begin{cases} 0 & if\, v^* < \frac{4}{3} \\ \frac{4}{3v^{*2}} & if\, \frac{4}{3} \leq v^* < 4 \\ 0 & if\, v^* \geq 4 \end{cases}$$

Calculate the expected value $E(v^*)$:

$$E(v^*) = \int_{-\infty}^{\infty} v^* f_{v^*}(v^*)\, \mathrm{dv}^* =$$

$$\int_{\frac{4}{3}}^{4} v^* \frac{4}{3{v^*}^2}\, \mathrm{dv}^* =$$

$$\frac{4}{3}\int_{\frac{4}{3}}^{4} \frac{1}{v^*}\, \mathrm{dv}^* =$$

$$\frac{4}{3}\, \ln(v^*)]_{\frac{4}{3}}^{4} =$$

$$\frac{4}{3}\left(\ln(4) - \ln\left(\frac{4}{3}\right)\right) =$$

$$\frac{4}{3}\ln(3) = 1.46482$$