ΤΗΛ 415 - Στατιστική Επεξεργασία Σήματος για Τηλ/νίες
Εαρινό Εξάμηνο 2020

Σχολή Ηλεκτρολόγων Μηχανικών και Μηχανικών Υπολογιστών
Πολυτεχνείο Κρήτης

Εργασία 2
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Εργασία 2 - TEL415

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Άσκηση 2

α)

(i) Ν.Δ.ο $\nabla_X tr(X^T A X) = (A + A^T)X$

$$\nabla_X tr(X^T A X) = \nabla_X tr(A X X^T) = \nabla_X tr\left(A [\vec{x}_1 \dots \vec{x}_m] \begin{bmatrix} \vec{x}_1^T \\ \vdots \\ \vec{x}_m^T \end{bmatrix}\right) =$$

$$= \nabla_X tr\left(A \cdot (\vec{x}_1 \cdot \vec{x}_1^T + \dots + \vec{x}_m \vec{x}_m^T)\right) = \nabla_X \left(tr(A \vec{x}_1 \vec{x}_1^T) + \dots + tr(A \vec{x}_m \vec{x}_m)\right) =$$

$$= \nabla_X \left(tr(\vec{x}_1^T A \vec{x}_1) + \dots + tr(x_m^T A x_m)\right) = \nabla_X (\vec{x}_1^T A \vec{x}_1 + \dots + \vec{x}_m A \vec{x}_m) =$$

$$= \nabla_X (\vec{x}_1^T A \vec{x}_1) + \dots + \nabla_X (x_m^T A x_m) =$$

$$= \underbrace{[(A^T + A)\vec{x}_1, 0, \dots 0]}_{1 \times m} + \underbrace{[0, (A^T + A)\vec{x}_2, 0 \dots 0]}_{1 \times m} + \dots + [0, \dots 0, (A^T + A)\vec{x}_m] =$$

$$= \underbrace{[(A^T + A)\vec{x}_1, (A^T + A)x_2, \dots, (A^T + A)x_m]}_{1 \times m} = (A^T + A) \cdot [\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m] =$$

$$= (A^T + A) \cdot X$$

(ii) Ν.Δ.ο $\nabla_X tr(X^T B) = B$

$$\nabla_X tr(X^T B) = \nabla_X tr(B X^T) = \nabla_X tr\left([\vec{b}_1\ \vec{b}_2 \dots \vec{b}_m] \begin{bmatrix} \vec{x}_1^T \\ \vdots \\ \vec{x}_m^T \end{bmatrix}\right) =$$

$$= \nabla_X tr(\vec{b}_1 \vec{x}_1^T + \dots + \vec{b}_m \vec{x}_m^T) = \nabla_X tr(\vec{x}_1^T \vec{b}_1 + \dots + \vec{x}_m^T \vec{b}_m) =$$

$$= \nabla_X \left(tr(\vec{x}_1^T \vec{b}_1) + \dots + tr(\vec{x}_m^T b_m)\right) = \nabla_X (\vec{x}_1^T \vec{b}_1) + \dots + \nabla (\vec{x}_m^T \vec{b}_m) =$$

$$= [\vec{b}_1, 0, \dots, 0] + [0, \vec{b}_2, 0, \dots, 0] + \dots + [0, \dots, 0, \vec{b}_m] = [\vec{b}_1, \vec{b}_2, \dots, \vec{b}_m] = B$$

(iii) N.S.O $\nabla_X tr(B^T X) = B$

$$\nabla_X tr(B^T X) = \nabla_X tr(X B^T) = \nabla_X tr(X^T B) \overset{(ii)}{=} B$$

$\boxed{B}$)

(i) N.S.O $\nabla_w w^H A w = 2Aw$

$$\nabla_w \vec{w}^H A \vec{w} = \nabla_{\vec{x}} \vec{w}^H A w + j \cdot \nabla_{\vec{y}} \vec{w}^H A \vec{w} =$$

$$= \nabla_{\vec{x}} (\vec{x}^T - j\vec{y}^T) A(\vec{x} + j\vec{y}) + j \cdot \nabla_{\vec{y}} (\vec{x}^T - j\vec{y}^T) A(\vec{x} + j\vec{y}) =$$

$$= \nabla_x (\vec{x}^T A - j\vec{y}^T A)(\vec{x} + j\vec{y}) + j \nabla_y (\vec{x}^T A - j\vec{y}^T A)(\vec{x} + j\vec{y}) =$$

$$= \nabla_x \underbrace{(\vec{x}^T A\vec{x} + j\vec{x}^T A\vec{y} - j\vec{y}^T A\vec{x} - j\vec{y}^T A\vec{y})}_{\alpha} + j \nabla_y \underbrace{(\vec{x}^T A\vec{x} + j\vec{x}^T A\vec{y} - j\vec{y}^T A\vec{x} - j\vec{y}^T A\vec{y})}_{\beta} = (1)$$

$$\to \alpha = \nabla_x (x^T A x) + \nabla_x (j x^T A y) - \nabla_x (j y^T A x) - \nabla_x (j y^T A y) =$$

$$= \nabla_x (x^T A x) + \nabla_x (j x^T (Ay)) - \nabla_x (j (y A^T)^T x) - \nabla_x (j y^T A y) =$$

$$= (A^T + A)x + jAy - jyA^T + 0$$

$$\to \beta = \nabla_y (x^T A x) + \nabla_y (j x^T A y) - \nabla_y (j y^T A x) - \nabla_y (j y^T A y) =$$

$$= \nabla_y (x^T A x) + \nabla_y j (xA)^T y - \nabla_y (j y^T (Ax)) + \nabla_y (y^T A y) =$$

$$= 0 + jxA^T - jAx + (A^T + A)y$$

Άρα: (1) $\Rightarrow \alpha + j \cdot \beta = A^T x + Ax + jAy - jyA^T + 0 + j(xA^T - Ax + jA^T y + jAy) =$

$$= \cancel{xA^T} + Ax + jAy - \cancel{jyA^T} - \cancel{xA^T} + Ax + \cancel{jyA^T} + jAy =$$

$$= 2Ax + 2jAy = 2A(x + jy) = 2Aw$$

(ii) N.S.o $\nabla_w w^H b = 2\vec{b}$

$$\nabla_w w^H b = \nabla_x w^H b + j\nabla_y w^H b = \nabla_x (\vec{x}^T - j\vec{y}^T)\vec{b} + j\nabla_y (\vec{x}^T - j\vec{y}^T)\vec{b} =$$

$$= \nabla_x \vec{x}^T\vec{b} - \cancel{\nabla_x j\vec{y}^T\vec{b}} + j\left(\cancel{\nabla_y \vec{x}^T\vec{b}} - \nabla_y j\vec{y}^T\vec{b}\right) =$$

$$= \vec{b} + j(-j\vec{b}) = \vec{b} + \vec{b} = 2\vec{b}$$

(iii) N.S.o $\nabla_w b^H w = \vec{0}$

$$\nabla_w b^H w = \nabla_x b^H w + j\nabla_y b^H w = \nabla_x b^H(\vec{x} + j\vec{y}) + j\nabla_y b^H(\vec{x} + j\vec{y}) =$$

$$= \nabla_x b^H x + \cancel{\nabla_x j b^H y} + j\left(\cancel{\nabla_y b^H x} + \nabla_y j b^H y\right) =$$

$$= \nabla_x ((b^*)^T\vec{x}) + j\cdot\nabla_y (j(b^*)^T\vec{y}) = \vec{b}^* + j^2\cdot\vec{b}^* = \vec{b}^* - \vec{b}^* = \vec{0}$$

[δ]

(i) N.S.o $\nabla_w tr(W^H A W) = 2AW$

$$\nabla_w tr(W^H A W) = \nabla_w tr(A W W^H) = \nabla_w tr\left(A[\vec{w}_1 \dots \vec{w}_m]\begin{bmatrix} w_1^H \\ \vdots \\ w_m^H \end{bmatrix}\right) =$$

$$= \nabla_w tr\left(A\vec{w}_1\vec{w}_1^H + \dots + A\vec{w}_m\vec{w}_m^H\right) = \nabla_w\left(tr(Aw_1w_1^H) + \dots + tr(Aw_mw_m^H)\right) =$$

$$= \nabla_w\left(tr(w_1^H A w_1) + \dots + tr(w_m^H A w_m)\right) = \nabla_w w_1^H A w_1 + \dots + \nabla_w w_m^H A w_m =$$

$$= [2A\vec{w}_1, 0, \dots, 0] + [0, 2A\vec{w}_2, 0, \dots, 0] + \dots + [0, \dots, 0, 2A\vec{w}_m] =$$

$$= [2A\vec{w}_1, 2A\vec{w}_2, \dots, 2A\vec{w}_m] = 2A[\vec{w}_1, \dots, \vec{w}_m] = 2AW$$

(ii) N.S.o $\nabla_w tr(W^H B) = 2B$

$$\nabla_w tr(W^H B) = \nabla_w tr(BW^H) = \nabla_w tr\left([\vec{b}_1 \dots \vec{b}_m]\begin{bmatrix} \vec{w}_1^H \\ \vdots \\ \vec{w}_m^H \end{bmatrix}\right) =$$

$$= \nabla_w tr(\vec{b}_1 \cdot \vec{w}_1^H + \dots + \vec{b}_m \vec{w}_m^H) = \nabla_w tr(\vec{w}_1^H \vec{b}_1 + \dots + \vec{w}_m^H \vec{b}_m) =$$

$$= \nabla_w \left(tr(w_1^H b_1) + tr(w_2^H b_2) + \dots + tr(w_m^H b_2) \right) =$$

$$= \nabla_w (\vec{w}_1^H \vec{b}_1) + \dots + \nabla_w (\vec{w}_m^H \vec{b}_m) = [2\vec{b}_1, 0, \dots, 0] + [0, 2\vec{b}_2, 0, \dots, 0] +$$
$$+ \dots + [0, \dots, 0, 2\vec{b}_m] =$$

$$= [2\vec{b}_1, 2\vec{b}_2, \dots, 2\vec{b}_m] = 2B$$

(iii) Ν.δ.ο. $\nabla_w tr(B^H W) = \vec{0}$

$$\nabla_w tr(B^H W) = \nabla_w tr(W B^H) = \nabla_w tr\left([\vec{w}_1 \dots \vec{w}_m] \begin{bmatrix} \vec{b}_1^H \\ \vdots \\ \vec{b}_m^H \end{bmatrix} \right) =$$

$$= \nabla_w tr(\vec{w}_1 \vec{b}_1^H + \dots + \vec{w}_m \vec{b}_m^H) = \nabla_w \left(tr(\vec{w}_1 \vec{b}_1^H) + \dots + tr(\vec{w}_m \vec{b}_m^H) \right) =$$

$$= \nabla_w \left(tr(\vec{b}_1^H \vec{w}_1) + \dots + tr(\vec{b}_m^H \vec{w}_m) \right) = \nabla_w (\vec{b}_1^H \vec{w}_1) + \dots + \nabla_w (\vec{b}_m^H \vec{w}_m) =$$

$$= \underbrace{[0, \dots, 0]_{1\times m} + \dots + [0, \dots, 0]_{1\times m}}_{m} = [0, \dots, 0]_{1\times m} = \vec{0}$$

$\boxed{\text{Άσκηση 3}}$

$$F_W(w) = P(W \le w) = P(BX \le w) =$$
$$= P(B=1) \cdot P(BX \le w / B=1) + P(B=-1) \cdot P(BX \le w / B=-1) =$$
$$= \frac{1}{2} \cdot P(X \le w / B=1) + \frac{1}{2} P(-X \le w / B=-1) \overset{X, B}{\underset{\text{ανεξαρτ}}{=}} \frac{1}{2} P(X \le w) + \frac{1}{2} P(X \ge -w)$$
$$= \frac{1}{2} P(X \le w) + \frac{1}{2} (1 - P(X < w)) = \frac{1}{2} P\left(\frac{X-0}{1} \le \frac{w-0}{1} \right) + \frac{1}{2} (1 - P(X < w)) =$$

$$\overset{X \sim N(0,1)}{=} \frac{1}{2} \Phi_X(w) + \frac{1}{2} (1 - \Phi_X(-w)) = \frac{1}{2} \Phi_X(w) + \frac{1}{2} (1 - (1 - \Phi_X(w))) =$$

$$= \frac{1}{2} \Phi_X(w) + \frac{1}{2} \Phi_X(w) = \Phi_X(w) = F_X(w)$$

Παραγωγίζοντας:

$$f_W(w) = \left(F_W(w)\right)' = \left(F_X(w)\right)' = \left(\Phi_X(w)\right)' = f_X(w) \overset{X\sim N(0,1)}{=\!=}$$

$$= \frac{1}{\sqrt{2\pi \cdot 1}} \cdot e^{-\frac{(w-0)^2}{2\cdot 1}} = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{w^2}{2}}$$

Άρα $W \sim N(0,1)$

$\boxed{i)}$ Ασυσχέτιστα $R(Z), I(Z)$?

(α) $Z = X + j\cdot X$

$\sigma_{XX} = E\{(X-0)(X-0)\} = E\{X^2\} = \sigma_X^2 = 1$, άρα X,X <u>corralated!</u>

(β) $Z = X + j\cdot Y$

$$\sigma_{XY} = E\{(X-0)(Y-0)\} = E\{X\cdot Y\} \underset{\text{ανεξάρτ.}}{\overset{X,Y}{=\!=}} E\{X\}\cdot E\{Y\} =$$
$$= \mu_X \cdot \mu_Y = 0\cdot 0 = 0, \text{ άρα } \underline{\text{non-corralated!}}$$

(γ) $Z = X + j\cdot 2Y$

$\mu_{2Y} = E\{2Y\} = 2E\{Y\} = 2\cdot 0 = 0$

$$\sigma_{X2Y} = E\{(X-0)(2Y-0)\} = E\{X\cdot 2Y\} = 2E\{X\cdot Y\} \overset{(\beta)}{=} 0$$

άρα <u>non-corralated!</u>

(δ) $Z = X + jW = X + jBX$

$$\sigma_{XW} = E\{(X-0)(W-0)\} = E\{X\cdot B\cdot X\} =$$
$$= E\{B\cdot X^2\} \underset{B,X^2}{\overset{\text{ανεξαρτ.}}{=\!=}} E\{B\}\cdot E\{X^2\} \underset{①}{=} 0\cdot \sigma_X^2 = 0\cdot 1 = 0$$

① $E\{B\} = \frac{1}{2}\cdot 1 + \frac{1}{2}(-1) = \frac{1}{2} - \frac{1}{2} = 0$

άρα <u>non-corralated!</u>

$\boxed{ii}$ $R(Z), I(Z)$ από κοινού gaussian?

(α) $Z = X + jX$

Για να είναι από κοινού gaussian πρέπει αρχικά να ορίζεται ο $(C_{R(Z),I(Z)})^{-1}$.

$$C_{XX} = \begin{bmatrix} \sigma_X^2 & \sigma_{XX} \\ \sigma_{XX} & \sigma_X^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Για να ορίζεται ο C_{XX}^{-1} πρέπει:

$\det(C_{XX}) \neq 0 \Rightarrow 1\cdot 1 - 1\cdot 1 \neq 0 \Rightarrow 0 \neq 0$ ΑΤΟΠΟ

Τελικά: αφού δεν ορίζεται ο C_{XX}^{-1}, δεν ορίζεται η pdf της jointly gaussian των $R(Z), I(Z)$.

(β) $Z = X + j\cdot Y$

$$C_{XY} = \begin{bmatrix} \sigma_X^2 & \sigma_{YX} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad \det C_{XY} = 1$$

Άρα ορίζεται ο $(C_{XY})^{-1}$, οπότε ίσως η από κοινού pdf των X, Y να έχει τη μορφή gaussian, αφού X, Y gaussian.

Έστω X, Y jointly gaussian. Ξέρουμε ότι X, Y ανεξάρτητες.

Τότε: $$f_{X,Y}(X,Y) \triangleq \frac{1}{\sqrt{(2\pi)^2 \sigma_X^2 \sigma_Y^2}} \cdot e^{-\frac{1}{2}[X-\mu_X, Y-\mu_Y]\begin{bmatrix} 1/\sigma_X^2 & 0 \\ 0 & 1/\sigma_Y^2 \end{bmatrix}\begin{bmatrix} X-\mu_X \\ Y-\mu_Y \end{bmatrix}} =$$

$$= \left(\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(X-0)^2}{2}}\right) \cdot \left(\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(Y-0)^2}{2}}\right) = f_X(X) \cdot f_Y(Y)$$

Άρα όντως X, Y jointly gaussian!

{που σημαίνει X, Y ανεξάρτητες ΑΛΗΘΕΣ

(γ) $Z = X + j2Y$

$$C_{X,2Y} = \begin{bmatrix} \sigma_X^2 & \sigma_{2YX} \\ \sigma_{X2Y} & \sigma_{2Y}^2 \end{bmatrix} \overset{(2)}{=} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \ \det C_{X2Y} = 4, \text{ άρα υπάρχει ο } C_{X2Y}^{-1}$$

(2) $\sigma_{2Y}^2 = 2^2 \cdot \sigma_Y = 4 \cdot 1 = 4$

Ομοίως με το (β) και επειδή $X, 2Y$ ανεξάρτητες, οι $X, 2Y$ είναι jointly gaussian.

(δ) $Z = X + jW = X + jBX$

$$C_{X,W} = \begin{bmatrix} \sigma_X^2 & \sigma_{WX} \\ \sigma_{XW} & \sigma_W^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \det C_{X,W} = 1, \text{ άρα υπάρχει ο } C_{X,W}^{-1}$$

Ξέρουμε ότι X, W εξαρτημένες gaussian. Έστω ότι η από κοινού pdf τους είναι gaussian, άρα θα 'ναι της μορφής:

$$f_{X,W}(x,w) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}[x,w]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}} = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{w^2}{2}} =$$

$= f_X(x) \cdot f_W(w) \Rightarrow X, W$ εξαρτημένες <u>ΑΤΟΠΟ!</u>

Άρα X, W <u>δεν</u> είναι jointly gaussian.

$\boxed{iii}$ $Re(Z) + I(Z)$ είναι gaussian?

(α) $X + X = 2X$

$E\{2X\} = 2E\{X\} = 2 \cdot 0 = 0$

$var\{2X\} = 2^2 \cdot var\{X\} = 4 \cdot 1 = 4$

Επειδή $2X$ είναι γραμμικός συνδυασμός gaussian, και η $2X$ είναι gaussian, $N(0,4)$

β) $X+Y$

$$X+Y = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix}$$

Έστω $\vec{\rho} = \begin{bmatrix} X \\ Y \end{bmatrix}$. Τότε αν $\vec{\rho}$ gaussian με $N \sim (\vec{\mu}, C)$, τότε X, Y από κοινού gaussian και ξέρουμε ότι γραμμικός μετασχηματισμός του $\vec{\rho}$ θα 'ναι γκαουσιανή τυχαία μεταβλητή. Αρκεί ο A να έχει rank όσο και οι γραμμές του, που ισχύει εδώ αφού $\text{rank}(A) = 1$. Αρκεί, λοιπόν, ν.δ.ο X, Y από κοινού gaussian, κάτι που έδειξα στο (α). Άρα και $X+Y$ gaussian Τ.Μ. ως γραμμικός μετασχηματισμός $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = X+Y$, αφού A full row-rank

γ) $X+2Y$

$$X+2Y = \underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} X \\ Y \end{bmatrix}}_{\rho}$$

$\text{rank}(A) = 1$, full row-rank, και ομοίως με το β, επειδή $X, 2Y$ jointly gaussian, $X+Y$ gaussian

δ) $R = X + W = X + BX = (1+B)X$

$$f_R(r) = P(B=1) \cdot f_{R|B=1}(r) + P(B=-1) \cdot f_{R|B=-1}(r) =$$

$$= \frac{1}{2} \cdot \underbrace{f_{2X}(r)}_{N \sim (0,4)} + \frac{1}{2} \cdot \delta(0)$$, άρα R δεν είναι gaussian!

δεν έχει τυχαιότητα, $\delta(0) = +\infty$

$\boxed{iv)}$

(α) $Z = X + jX$, μιγαδική gaussian?

Αρκεί ν.δ.ο $Re(Z)$, $Im(Z)$ από κοινού gaussian, το οποίο ισχύει από $(ii)_\alpha$. Θα ελέγξω αν είναι proper.

$C_{X,X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Πρέπει $C_{11} = C_{22}$ και $C_{12} = C_{21} = 0$

ΙΣΧΥΟΥΝ, άρα Z proper CN

με pdf: $f_Z(z) = \frac{1}{\pi \sigma_Z^2} \cdot e^{-\frac{|z-\mu_Z|^2}{\sigma_Z^2}}$, $z \in \mathbb{C}$

$\Leftrightarrow f_Z(z) = \frac{1}{2\pi} \cdot e^{-\frac{|z|^2}{2}}$

$\mu_Z = \mu_X + j\mu_X = 0 + j \cdot 0 = 0$

$\sigma_Z^2 = \sigma_X^2 + \sigma_X^2 = 1 + 1 = 2$

~~β) Από $(ii)_\beta$ έχω δείξει ότι~~

(β) $Z = X + jY$ $C_{XY} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C_{11} = C_{22}$, $C_{12} = C_{21} = 0$ $\Big\}$ ΙΣΧΥΟΥΝ, άρα Z proper CN

Από $(ii)_\beta$ έχει δείξει ότι X, Y jointly gaussian, που αρκεί για να είναι η $Z = X + jY$ $CN \sim (0, 2)$

με pdf: $f_Z(z) = \frac{1}{2\pi} \cdot e^{-\frac{|z|^2}{2}}$, $z \in \mathbb{C}$.

(γ) $Z = X + j2Y$

Ομοίως, από $(ii)_\gamma$, $X, 2Y$ jointly gaussian, άρα $Z = X + j2Y$ μιγαδική gaussian. $CN \sim (0, 5)$

$C_{X,2Y} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, $C_{11} \neq C_{22}$, άρα Z δεν είναι proper CN

pdf από κοινού $X, 2Y$: $f_Z(z) \triangleq f_{X,Y}(x, y) \Rightarrow$

$f_Z(z) = \frac{1}{\sqrt{(2\pi)^2 1^2 5^2}} \cdot e^{-\frac{1}{2}[X-0, 2Y-0] \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} X \\ 2Y \end{bmatrix}} \Longrightarrow$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi 4}} \cdot e^{-\frac{(2y)^2}{8}} = \underset{\text{pdf } X}{f_X(x)} \cdot \underset{\text{pdf } 2Y}{f_{2Y}(2y)}$$

(δ) $Z = X + jW = X + jBX$

Επειδή από (ii)δ, X, W δεν είναι jointly gaussian, δεν είναι η $Z = X + jW$ gaussian.

Άσκηση 5

α) (i) Ν.δ.ο $\sigma_Z^2 = E\{|Z|^2\} - |\mu_Z|^2 = \sigma_X^2 + \sigma_Y^2$

$$\sigma_Z^2 = E\{|Z-\mu_Z|^2\} = E\{(Z-\mu_Z)(Z-\mu_Z)^*\} = E\{(Z-\mu_Z)(Z^* - \mu_Z^*)\} =$$

$$= E\{ZZ^* - Z\mu_Z^* - \mu_Z \cdot Z^* + \mu_Z \cdot \mu_Z^*\} =$$

$$= E\{Z \cdot Z^*\} - E\{Z \cdot \mu_Z^*\} - E\{\mu_Z \cdot Z^*\} + E\{\mu_Z \cdot \mu_Z^*\} =$$

* Επειδή $\mu_Z = \mu_X + j\mu_Y$, constant και $\mu_Z^* = \mu_X^* - j\mu_Y^*$ constant

$$= E\{|Z|^2\} + E\{|\mu_Z|^2\} - \mu_Z^* \cdot E\{Z\} - \mu_Z E\{Z^*\} =$$

* Επειδή $|\mu_Z|^2 = |\mu_X + j\mu_Y|^2 = \left(\sqrt{\mu_X^2 + \mu_Y^2}\right)^2 = \mu_X^2 + \mu_Y^2$ constant και $E\{Z^*\} = (E\{Z\})^*$

$$= E\{|Z|^2\} + |\mu_Z|^2 - \mu_Z^* \cdot \mu_Z - \mu_Z \cdot \mu_Z^* = E\{|Z|^2\} + |\mu_Z|^2 - 2|\mu_Z|^2 =$$

$$= \boxed{E\{|Z|^2\} - |\mu_Z|^2} =$$

$$= E\{|X+jY|^2\} - |\mu_X + j\mu_Y|^2 = E\{X^2 + Y^2\} - (\mu_X^2 + \mu_Y^2) =$$

$$= E(X^2) + E(Y^2) - \mu_X^2 - \mu_Y^2 = \boxed{\sigma_X^2 + \sigma_Y^2}$$

(ii) Ν.δ.ο $\tilde{\sigma}^2 = E\{Z^2\} - \mu_Z^2$

$$E\{Z^2\} = E\{(Z-\mu_Z)^2\} = E\{Z^2 - 2Z\mu_Z + \mu_Z^2\} =$$

$$= E\{Z^2\} - 2 \cdot \mu_Z \cdot E\{Z\} + E\{\mu_Z^2\} =$$

$$= E\{Z^2\} - 2 \cdot \mu_Z \cdot \mu_Z + \mu_Z^2 =$$

$$= E\{Z^2\} - \mu_Z^2$$

B/

(i) N.S.o $E\{Z^T\} = E\{Z\}^T$, Z complex matrix, A,B non random

* Έστω Z $m\times n$, άρα Z^T: $n\times m$ matrix.

$$E\{Z^T\} = E\left\{\begin{bmatrix} Z_{11} & \cdots & Z_{1m} \\ \vdots & & \vdots \\ Z_{n1} & \cdots & Z_{nm}\end{bmatrix}\right\} = \begin{bmatrix} E(Z_{11}) & \cdots & E(Z_{1m}) \\ \vdots & & \vdots \\ E(Z_{n1}) & \cdots & E(Z_{nm})\end{bmatrix}$$

$$(E\{Z\})^T = \begin{bmatrix} E(Z_{11}) & \cdots & E(Z_{1n}) \\ \vdots & & \vdots \\ E(Z_{m1}) & \cdots & E(Z_{mn})\end{bmatrix}^T = \begin{bmatrix} E(Z_{11}) & \cdots & E(Z_{1m}) \\ \vdots & & \\ E(Z_{n1}) & \cdots & E(Z_{nm})\end{bmatrix} = E\{Z^T\}$$

(ii) N.S.o $E\{Z^*\} = E\{Z\}^*$

$$E\{Z^*\} = E\left\{\begin{bmatrix} Z_{11}^* & \cdots & Z_{1n}^* \\ \vdots & & \vdots \\ Z_{m1}^* & \cdots & Z_{mn}^*\end{bmatrix}\right\} = \begin{bmatrix} E\{Z_{11}^*\} & \cdots & E\{Z_{1n}^*\} \\ \vdots & & \vdots \\ E\{Z_{m1}^*\} & \cdots & E\{Z_{mn}^*\}\end{bmatrix} =$$

$$= \begin{bmatrix} (E\{Z_{11}\})^* & \cdots & (E\{Z_{1n}\})^* \\ \vdots & & \vdots \\ (E\{Z_{m1}\})^* & \cdots & (E\{Z_{mn}\})^*\end{bmatrix} = \begin{bmatrix} E\{Z_{11}\} & \cdots & E\{Z_{1n}\} \\ \vdots & & \vdots \\ E\{Z_{m1}\} & \cdots & E\{Z_{mn}\}\end{bmatrix}^* =$$

$$= (E\{Z\})^*$$

(iii) N.S.o $E\{AZB\} = A\cdot E\{Z\}\cdot B$

$E\{AZB\} = E\{A(ZB)\} = E\{AZ + AB\} =$

$A_{x\cdot n}$
$Z_{n\cdot n}$
$B_{n\cdot y}$

$= E\{M + N\} = E\{M\} + E\{N\}$ ①

$M_{x \cdot n}$ με $M_{ij} = \sum_{k=1}^{n} a_{ik} Z_{kj}$ και $N_{x \cdot y}$ με $N_{ij} = \sum_{\lambda=1}^{n} a_{i\lambda} \beta_{\lambda j}$

και ①. $E\{M_{ij}\} = \sum_{k=1}^{n} a_{ik} E\{Z_{kj}\} = A \cdot E\{Z\}$

και $E\{N_{ij}\} = \sum_{\lambda=1}^{n} a_{i\lambda} \beta_{\lambda j} = A \cdot B$

Τελικά ① $\Rightarrow$ $A \cdot E\{Z\} + A \cdot B = A(E\{Z\} \cdot B) =$

$$= A \cdot E\{Z\} \cdot B$$

$\boxed{\gamma}$ (iv) N.δ.ο $C_{z_2, z_1} = C^H_{z_1, z_2}$ $\quad (E(Z))^H = E(Z^H)$

$$C^H_{z_1 z_2} = \left(E\{(Z_1 - \mu_{z_1})(Z_2 - \mu_{z_2})^H\}\right)^H =$$

$$= E\left\{\left((Z_1 - \mu_{z_1})(Z_2 - \mu_{z_2})^H\right)^H\right\} \overset{(AB)^H = B^H A^H}{=}$$

$$= E\{(Z_2 - \mu_{z_2})(Z_1 - \mu_{z_1})^H\} = C_{z_2, z_1}$$

(vi) N.δ.ο $\tilde{C}_{z_2 z_1} = \tilde{C}^T_{z_1, z_2}$

$$\tilde{C}^T_{z_1 z_2} = \left(E\{(Z_1 - \mu_{z_1})(Z_2 - \mu_{z_2})^T\}\right)^T = \quad (AB)^T = B^T A^T$$

$$= E\left\{\left((Z_1 - \mu_{z_1})(Z_2 - \mu_{z_2})^T\right)^T\right\} =$$

$$= E\{(Z_2 - \mu_{z_2})(Z_1 - \mu_{z_1})^T\} =$$

$$= \tilde{C}_{z_2 z_1}$$

Άσκηση 4

α) $R_y = E(\vec{y}\vec{y}^T) = E\{(x_1a_1 + x_2a_2 + N)(x_1a_1^T x_2 a_2^T + N^T)\} =$

$= E\{x_1^2 a_1 a_1^T + x_1 a_1 x_2 a_2^T + x_1 a_1 N^T + x_2 a_2 x_1 a_1^T + x_2^2 a_2 a_2 + x_2 a_2 N^T + N x_1 a_1 + N x_2 a_2^T + NN^T\} =$

$= a_1 a_1^T E(x_1^2) + \cancel{a_1 a_2^T \underbrace{E(x_1)E(x_2)}_{r_{x_1 x_2}=0}} + \cancel{a_1 E(x_1) \underset{0}{E(N^T)}} + \cancel{a_2 a_1^T \underbrace{E(x_1)E(x_2)}_{r_{x_1x_2}=0}} + a_2 a_2 E(x_2^2) +$

$+ \cancel{a_2 E(x_2) \underset{0}{E(N^T)}} + \cancel{a_1 E(x_1) \underset{0}{E(N)}} + \cancel{a_2^T E(x_2) \underset{0}{E(N)}} + E(NN^T) =$

$= a_1 a_1^T E(x_1^2) + a_2 a_2^T E(x_2^2) + E(\underset{n\times n}{NN^T}) =$
$= a_1 a_1^T \cdot \sigma_{x_1}^2 + a_2 a_2^T \cdot \sigma_{x_2}^2 + \sigma_N^2 \cdot I$

β) $x^T \cdot R_y \cdot x = x^T (a_1 a_1^T \cdot \sigma_{x_1}^2 + a_2 a_2^T \cdot \sigma_{x_2}^2 + \sigma_N^2 \cdot I) x =$

$= x^T a_1 a_1^T x \cdot \sigma_{x_1}^2 + x^T a_2 a_2^T x \cdot \sigma_{x_2}^2 + x^T I x \cdot \sigma_N^2 =$

$= (a_1^T x)^T (a_1^T \cdot x) \cdot \sigma_{x_1}^2 + (a_2^T \cdot x)^T \cdot (a_2^T x) \cdot \sigma_{x_2}^2 + \|\vec{x}\|^2 \cdot \sigma_N^2 =$

$= \underset{\geq 0}{\sigma_{x_1}^2} \cdot \underset{\substack{1\times1 \text{ real} \\ \text{και } \geq 0}}{(a_1^T \cdot x)^2} + \underset{\geq 0}{\sigma_{x_2}^2} \underset{\substack{1\times1 \text{ real} \\ \text{και } \geq 0}}{(a_2^T \cdot x)^2} + \underset{\geq 0}{\sigma_N^2} \cdot \underset{\geq 0}{\|\vec{x}\|^2} \geq \sigma_N^2 \cdot \|\vec{x}\|$ *(1)

γ) Η ιδιότητα λέει, ότι: Αν R_y symmetric $\Rightarrow x^T R_y x \geq \lambda_{min} \cdot \|\vec{x}\|^2$
Αρκεί ν.δ.ο R_y p.s.d, που από (β) δείξαμε ότι:

$x^T \cdot R_y \cdot x \geq \sigma_N^2 \cdot \|\vec{x}\| \geq 0 \quad \forall x \in \mathbb{R}^n \Rightarrow R_y$ είναι p.s.d,
άρα $\lambda_{min} \geq 0$
Επειδή R_y $n\times n$, δηλ. τετραγωνικός και

συμμετρικός, αφού $R_y^T = (a_1 a_1^T \sigma_{x_1}^2 + a_2 a_2^T \sigma_{x_2}^2 + \sigma_N^2 I)^T =$
$= (a_1 a_1^T)^T \cdot \sigma_{x_1}^2 + (a_2 a_2^T)^T \sigma_{x_2}^2 + \sigma_N^2 \cdot I^T =$
$= a_1 a_1^T \cdot \sigma_{x_1}^2 + a_2 a_2^T \cdot \sigma_{x_2}^2 + \sigma_N^2 \cdot I = R_y$
ισχύει η *(1) και επειδή δείξαμε στο (β)

όχι $\lambda_{min} = \sigma_N^2 > 0$

Άρα από τα παραπάνω, οι ιδιοτιμές του R_y είναι θετικές και real.

[ζ] $\hat{R}_y = \frac{1}{k} \cdot \sum_{i=1}^{k} \vec{y}_i \vec{y}_i^T = \frac{1}{k} \cdot \left(\vec{y}_1 \vec{y}_1^T + \vec{y}_2 \vec{y}_2^T + \dots + \vec{y}_k \vec{y}_k^T \right) =$

$$= \frac{1}{k} \cdot \underbrace{\begin{bmatrix} \vec{y}_1 & \vec{y}_2 & \dots & \vec{y}_k \end{bmatrix}}_{\bar{Y}} \underbrace{\begin{bmatrix} \vec{y}_1^T \\ \vec{y}_2^T \\ \vdots \\ \vec{y}_k^T \end{bmatrix}}_{\bar{Y}^T} = \frac{1}{k} \bar{Y} \cdot \bar{Y}^T$$

[η] $\hat{R}_y = \frac{1}{k} \cdot \bar{Y} \cdot \bar{Y}^T = \frac{1}{k} \cdot U \Sigma V^T \cdot (U \Sigma V^T)^T =$

$$= \frac{1}{k} \cdot U \Sigma \underbrace{\boxed{V^T \cdot V}}_{\Sigma \text{ diagonal}} \cdot \Sigma^T \cdot U^T = \frac{1}{k} U \Sigma \cdot I \cdot \Sigma^T U^T =$$

I, επειδή ορθογώνιος

$\Rightarrow \Sigma^T = \Sigma$

$$= \frac{1}{k} U \boxed{\Sigma \Sigma^T} U^T = \frac{1}{k} \cdot U \cdot \Sigma^2 \cdot U^T =$$

$$= \frac{1}{k} \cdot U \cdot \underbrace{\begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ & & \ddots \\ 0 & & \sigma_{min(n,k)}^2 \end{bmatrix}}_{n \times k} \cdot U^T =$$

$$= \begin{bmatrix} U_1 & U_2 & \dots & U_n \end{bmatrix} \begin{bmatrix} \frac{\sigma_1^2}{k} & & 0 \\ & \frac{\sigma_2^2}{k} & \\ & & \ddots \\ 0 & & \frac{\sigma_{min(n,k)}^2}{k} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}$$

Άρα ο $\hat{R}_y$ γράφεται στην μορφή $A = Q \Lambda Q^{-1}$ και έχει eigenvectors τις στήλες του U και eigenvalues $\lambda_i = \frac{\sigma_i^2}{k}$ (μη μηδενικά για $rank(\hat{R}_y) \geq n$)

18

θ) Αν στο (3) για R_y estimation χρησιμοποιήσουμε λιγότερα από n διανύσματα ($k<n$), ο R_y θα είναι γινόμενο 2 πινάκων $\bar{Y}=[\vec{Y}_1 \cdots \vec{Y}_n]$ και $\bar{Y}^T$, ο καθένας με rank $<n$, άρα $\mathrm{rank}(R_y) \leq n$, αφού $\mathrm{rank}(AB) \leq \min(\mathrm{rank}(A), \mathrm{rank}(B))$. Οπότε κάποια λ_i eigenvalue θα είναι μηδέν και έτσι:

Για $k<n$, $|\hat{R}_y| = \prod_i \lambda_i = \lambda_1 \cdot \lambda_2 \cdots \lambda_i \cdot 0 \cdots = 0$

Αν $\hat{R}_y$ full rank, τότε $k \geq n$, άρα η διακρίνουσα:

$$|\hat{R}_y| = \prod_i \lambda_i = \lambda_1 \cdot \lambda_2 \cdots \lambda_n = \frac{\sigma_1^2}{k} \cdot \frac{\sigma_2^2}{k} \cdots \frac{\sigma_n^2}{k} =$$

$$\Rightarrow |\hat{R}_y| = \frac{(\sigma_1 \sigma_2 \sigma_3 \cdots \sigma_n)^2}{k^n}$$

Τελικά:

$$|\hat{R}_y| = \begin{cases} 0 & k<n \\ \frac{(\sigma_1 \sigma_2 \cdots \sigma_n)^2}{k^n} & k \geq n \end{cases}$$