Acta Applicandae Mathematicae **56:** 121–138, 1999.

Symmetry Algebra of the Benjamin–Ono Equation

V. N. CHETVERIKOV*
Department of Applied Mathematics, Bauman Moscow State Technical University, 2-ja Baumanskaja ul. 5, 107005 Moscow, Russia, e-mail: chetveri@unix.ecfor.rssi.ru

(Received: 10 October 1998)

Abstract. The Lie algebra structure for symmetries of the Benjamin–Ono equation is completely described.

Mathematics Subject Classifications (1991): 45K05, 58A20, 58G20, 58G35.

Key words: integro-differential equations, Benjamin–Ono equation, boundary differential equations, symmetries.

1. Introduction

In this paper, we consider the Benjamin–Ono equation in the form

$$u_t = 2uu_x + Hu_{xx}, \tag{1}$$

where H is the Hilbert transform operator

$$Hu(x) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{u(z)}{x-z}\,\mathrm{d}z. \tag{2}$$

Here P denotes the principal value.

The following result was proved by Fokas and Fuchssteiner (see [4]). We first introduce the necessary notation. Let u be a function of x and t, $A(u)$, $B(u)$ be nonlinear operators from $C^\infty(\mathbb{R}^2)$ to $C^\infty(\mathbb{R}^2)$, and l_B be the derivative in the direction B, i.e.,

$$l_B(A) = \frac{\partial}{\partial\varepsilon} A(u+\varepsilon B(u))\Big|_{\varepsilon=0}. \tag{3}$$

Let $\{A, B\}$ denote the Lie bracket, $\{A, B\} = l_B(A) - l_A(B)$, ad_B denote the corresponding adjoint map, i.e.,

$$\mathrm{ad}_B(A) = \{A, B\}, \quad \tau(u) = -x(2uu_x + Hu_{xx}) - u^2 - \frac{3}{2}Hu_x.$$

Then

$$K_0 = u_x, \quad K_n = \mathrm{ad}_\tau^n(K_0),\ n = 1, 2, \ldots$$

* Partially supported by INTAS grant 96-0793.

are symmetries of Equation (1) and $\{K_n, K_m\} = 0$ for any $n, m \geqslant 0$.

Fuchssteiner [5] proved that the fields

$$G_{m,n} = \sum_{i=0}^{m} \frac{t^i}{i!} \operatorname{ad}_K^i \operatorname{ad}_x^m (K_n), \quad m = 1, 2, \ldots, n = 0, 1, 2, \ldots$$

are time-dependent symmetries for (1). Here $K = K_1 = 2uu_x + Hu_{xx}$.

Remark 1. The integral (2) converges, if $u = \phi(x)$ is a differentiable function on $\mathbb{R}$ satisfying the Hölder condition at infinity:

$$\phi(x) = \phi(\infty) + \mathrm{O}\left(\frac{1}{|x|^\mu}\right) \quad \text{as } x \to \infty,\ \phi(\infty) = \text{const},\ \mu > 0. \tag{4}$$

However, in order to calculate $G_{m,n}$, it is necessary to generalize the Hilbert transform operator in some cases when $\phi(x) \to \infty$ as $x \to \infty$. Namely, suppose

$$u = x^k \phi(x), \tag{5}$$

where k is a positive integer, $\phi(x)$ is a differentiable function satisfying (4), while $x\phi(x)$ does not satisfy condition (4). Then

$$Hu \overset{\text{def}}{=} x^k H\phi \tag{6}$$

(see [6], p. 52). The operator H is well defined in this way, since the presentation (5) is unambiguous.

In this paper we describe the Lie algebra structure on the set of symmetries found by Fokas and Fuchssteiner. Our main result is as follows.

THEOREM 1. *The Benjamin–Ono equation* (1) *possesses the following symmetry algebra $\mathcal{A}$: the generators of the linear space $\mathcal{A}$ are*

$$G_{m,n} = \sum_{i=0}^{m} \frac{t^i}{i!} \operatorname{ad}_K^i \operatorname{ad}_x^m \operatorname{ad}_\tau^n (K_0), \quad \text{for } n \geqslant 0, n+1 \geqslant m \geqslant 0.$$

The structure of Lie algebra on $\mathcal{A}$ is determined by relations

$$\{G_{m,n}, G_{l,k}\} = c(m,n,l,k) G_{m+l-1,n+k-1}, \tag{7}$$

where

$$\begin{aligned} &c(m,n,l,k) \\ &\quad = \frac{(n+1)!n!(k+1)!k!(n+k+1-m-l)!}{(n+1-m)!(k+1-l)!(n+k)!(n+k-1)!} \times \\ &\qquad \times (ln + l - km - m) \end{aligned} \tag{8}$$

for $n + k - 1 \geqslant 0, n + k + 1 - m - l \geqslant 0$ and $c(m,n,l,k) = 0$ otherwise.

Section 3 contains an outline of the geometry of boundary differential equations used here (see [1] for details). In Section 4, we calculate the classical symmetries of Equation (1).

2. Transformation the Benjamin–Ono Equation

We use the method developed in [1–3]. Let us first transform the Benjamin–Ono equation to a system of boundary differential equations.

It is clear [6] that

$$P\int_{-\infty}^{+\infty}\frac{f(z)}{x-z}\,\mathrm{d}z = P\int_{-\infty}^{+\infty}\frac{f(z)-f(x)}{x-z}\,\mathrm{d}z.$$

Since the right-hand integral has a singularity only at infinity, we have

$$\begin{aligned}&P\int_{-\infty}^{+\infty}\frac{f(z)-f(x)}{x-z}\,\mathrm{d}z\\ &\quad=\lim_{A\to+\infty}\left[\int_0^A\frac{f(z)-f(x)}{x-z}\,\mathrm{d}z+\int_{-A}^0\frac{f(z)-f(x)}{x-z}\,\mathrm{d}z\right].\end{aligned}$$

Substituting $-z$ for z in the last integral, we obtain

$$P\int_{-\infty}^{+\infty}\frac{f(z)}{x-z}\,\mathrm{d}z=\lim_{A\to+\infty}\int_0^A\left[\frac{f(z)-f(x)}{x-z}+\frac{f(-z)-f(x)}{x+z}\right]\mathrm{d}z.$$

Introduce a new function $v(x,z,t)$ in the following way. Suppose that

$$\begin{aligned}&v_2(x,z,t)=\frac{u_{11}(z,t)-u_{11}(x,t)}{x-z}+\frac{u_{11}(-z,t)-u_{11}(x,t)}{x+z},\\ &\lim_{z\to+\infty}v(x,z,t)=0,\end{aligned}$$

where $v_2=v_z$, $u_{11}=u_{xx}$, then

$$P\int_{-\infty}^{+\infty}\frac{u_{11}(z)}{x-z}\,\mathrm{d}z=-v(x,0,t).$$

Let f be a function of x,z,t, where $z\geqslant 0$. Denote by $f_{[x0]}$, $f_{[x\infty]}$, $f_{[z0]}$, $f_{[-z0]}$ the functions of x,z,t such that

$$\begin{aligned}&f_{[x0]}(x,z,t)=f(x,0,t), &&f_{[x\infty]}(x,z,t)=\lim_{A\to+\infty}f(x,A,t),\\ &f_{[z0]}(x,z,t)=f(z,0,t), &&f_{[-z0]}(x,z,t)=f(-z,0,t).\end{aligned}$$

Note that $f_{[x0]}=f$ if and only if the function f is independent of z.

We see that Equation (1) is equivalent to the system

$$u_3=2uu_1-v_{[x0]},\qquad u_{[x0]}=u,\qquad v_{[x\infty]}=0, \tag{9}$$

$$v_2=\frac{1}{\pi}\left[\frac{u_{11[z0]}-u_{11}}{x-z}+\frac{u_{11[-z0]}-u_{11}}{x+z}\right], \tag{10}$$

where $u_3 = u_t$. Equations (9)–(10) involve three independent variables x, z, t, two dependent variables u, v, and the restrictions of the dependent variables to the boundary sets $\{z = 0\}$ and $\{z \to +\infty\}$. We say that a differential equation with restrictions of dependent variables to boundary sets is a *boundary differential equation*.

3. Symmetries of Boundary Differential Equations

In this section we recall certain concepts of geometry of boundary differential equations [1].

Suppose $\pi: E \to M$ is a smooth vector bundle over a manifold M, $\Gamma(\pi)$ is the totality of sections of the bundle π, k is a nonnegative integer or infinity, $\pi_k: J^k(\pi) \to M$ is the corresponding bundle of k-jets, and $\mathcal{G}$ is a finite set of smooth mappings from M to M such that the identity map id_M belongs to $\mathcal{G}$; then by $\pi_k^{\mathcal{G}}$ we denote the Whitney product of the induced bundles: $\pi_k^{\mathcal{G}} = \bigoplus_{g\in\mathcal{G}} g^*(\pi_k)$. Let $J^k(\pi; \mathcal{G})$ denote the total space of the bundle $\pi_k^{\mathcal{G}}$. The set $J^k(\pi; \mathcal{G})$ is a finite-dimensional smooth manifold if k is a finite integer. This set is an infinite-dimensional manifold if k is infinity.

Suppose s is a section of the bundle π and y is a point of M; then by $[s]_y^k$ we denote the k-jet of s at y. Every point of $J^k(\pi; \mathcal{G})$ over $x \in M$ is a collection of k-jets $[s_g]^k_{g(x)} \in J^k(\pi)$, $g \in \mathcal{G}$, where $s_g \in \Gamma(\pi)$. The collection of the k-jets $\{[s_g]^k_{g(x)}\}_{g\in\mathcal{G}}$ is called the $(k, \mathcal{G})$*-jet of the collection of sections* $\{s_g\}$ *at the point* x. For any collection of sections $\{s_g\}_{g\in\mathcal{G}}$ denote by $j_k(\{s_g\})$ the following section of the bundle $\pi_k^{\mathcal{G}}$:

$$j_k(\{s_g\})(x) = \{[s_g]^k_{g(x)}\}_{g\in\mathcal{G}}. \tag{11}$$

If all sections s_g, $g \in \mathcal{G}$ coincide, then the corresponding collection of the k-jets is called $(k, \mathcal{G})$*-jet of the section* s *at the point* x and is denoted by $[s]_x^{(k,\mathcal{G})}$. The jet $[s]_x^{(k,\mathcal{G})}$ is interpreted as a class of sections tangent with order $\geqslant k$ to the section s at all points $g(x)$, $g \in \mathcal{G}$. Denote by $j_k(s)$ the section $x \mapsto [s]_x^{(k,\mathcal{G})}$ and by $J^k(\pi; \mathcal{G})_0$ the subset of $(k, \mathcal{G})$-jets of sections of the bundle π. We say that the manifold $J^k(\pi; \mathcal{G})$ is the *manifold* (or the *space*) *of* $(k, \mathcal{G})$*-jets* and the bundle $\pi_k^{\mathcal{G}}$ is the *bundle of* $(k, \mathcal{G})$*-jets*.

By a *system of boundary partial differential equations of order* $\leqslant k$ *imposed on sections of* π *with a set of maps* $\mathcal{G}$ (or simply by an *equation*), we mean a submanifold $\mathcal{E} \subset J^k(\pi; \mathcal{G})$. A section $s \in \Gamma(\pi)$ is a *solution of the equation* $\mathcal{E}$, if $j_k(s)(M) \subset \mathcal{E}$.

Remark 2. If $\mathcal{G} = \{\mathrm{id}_M\}$, then the manifold $J^k(\pi; \mathcal{G})$ coincides with the ordinary jet space $J^k(\pi)$. In this case, the concepts of the geometry of boundary differential equations coincide with the corresponding concepts of the geometry of partial differential equations [1, 7].

Suppose that $(x_1, \dots, x_n)$ are coordinates in a domain $U \subset M$, $(u^1, \dots, u^m)$ are coordinates in a fiber of the bundle $\pi|_U$, σ is a multi-index: $\sigma = (i_1, \dots, i_k)$, $1 \leqslant i_1 \leqslant n, \dots, 1 \leqslant i_k \leqslant n$, $|\sigma| = k$. Then (x_i, u^j_σ), $i = 1, \dots, n$, $j = 1, \dots, m$ are the corresponding coordinates in the manifold $J^\infty(\pi)$, $(u^j_{\sigma g})$ are the corresponding coordinates in a fiber of the bundle $g^*(\pi_\infty)$, $g \in \mathcal{G}$, and $u^j_{\sigma\, \mathrm{id}_M} = u^j_\sigma$. The collection

$$(x_i, u^j_{\sigma g}),\ i = 1, \dots, n,\ j = 1, \dots, m,\ g \in \mathcal{G},\ |\sigma| \geqslant 0 \tag{12}$$

is a *canonical coordinate system* in $J^\infty(\pi; \mathcal{G})$. Obviously, the coordinate $u^j_{\sigma g}$ at a point $\{[s_{\tilde g}]^\infty_{\tilde g(x)}\}_{\tilde g \in \mathcal{G}} \in J^\infty(\pi; \mathcal{G})$ is equal to $(\partial^{|\sigma|} s^j_g / \partial x^\sigma)(g(x))$, where $s^1_g, \dots, s^m_g$ are the components of the section s_g.

EXAMPLE 1. Consider system (9)–(10). The independent variables x, z, t are coordinates in the manifold $M_0 = \mathbb{R} \times [0, +\infty) \times \mathbb{R}$. Let us glue the points $(x, +\infty, t)$, $(-\infty, z, t)$, $(+\infty, z, t)$, $(-\infty, +\infty, t)$, $(+\infty, +\infty, t)$ to the manifold M_0. Introduce a manifold structure to the set $M = [-\infty, +\infty] \times [0, +\infty] \times \mathbb{R}$ such that the manifold M is diffeomorphic to the manifold $[-A, A] \times [0, A] \times \mathbb{R}$, where A is a positive number. Define the maps $[x0]$, $[x\infty]$, $[z0]$, $[-z0]$ from M to M by the rules

$$[x0]\colon (x, z, t) \mapsto (x, 0, t), \qquad [x\infty]\colon (x, z, t) \mapsto (x, +\infty, t),$$
$$[z0]\colon (x, z, t) \mapsto (z, 0, t), \qquad [-z0]\colon (x, z, t) \mapsto (-z, 0, t).$$

Take $\mathcal{G}_0 = \{\mathrm{id}_M, [x0], [x\infty], [z0], [-z0]\}$ and $\pi\colon \mathbb{R}^2 \times M \to M$, where the space $\mathbb{R}^2$ has the coordinates u, v. Equations (9)–(10) define the submanifold $\mathcal{E} \subset J^2(\pi; \mathcal{G}_0)$.

Define the bundle

$$\pi^{\mathcal{G}}_{k,l}\colon J^k(\pi; \mathcal{G}) \to J^l(\pi; \mathcal{G}) \text{ by } \pi^{\mathcal{G}}_{k,l}(\{\theta^g_k\}_{g \in \mathcal{G}}) = \{\theta^g_l\}_{g \in \mathcal{G}},$$

where $k > l$ or $k = \infty$. If $\mathcal{G}_1$ is a subset of $\mathcal{G}$, then the bundle $\pi^{\mathcal{G},\mathcal{G}_1}_k\colon J^k(\pi; \mathcal{G}) \to J^k(\pi; \mathcal{G}_1)$ is defined by $\pi^{\mathcal{G},\mathcal{G}_1}_k(\{\theta^g_k\}_{g \in \mathcal{G}}) = \{\theta^g_k\}_{g \in \mathcal{G}_1}$.

Evidently, we have

$$\pi^{\mathcal{G}_1}_k \circ \pi^{\mathcal{G},\mathcal{G}_1}_k = \pi^{\mathcal{G}}_k, \tag{13}$$
$$\pi^{\mathcal{G}_1}_{k,l} \circ \pi^{\mathcal{G},\mathcal{G}_1}_k = \pi^{\mathcal{G},\mathcal{G}_1}_l \circ \pi^{\mathcal{G}}_{k,l}, \tag{14}$$
$$\pi^{\mathcal{G}}_{k,l} \circ j_k(\{s_g\}) = j_l(\{s_g\}), \tag{15}$$
$$\pi^{\mathcal{G},\mathcal{G}_1}_k \circ j_k(\{s_g\}) = j_k(\{s_g\}_{g \in \mathcal{G}_1}), \tag{16}$$

where $\{s_g\}$ is an arbitrary set of sections of the bundle π with $g \in \mathcal{G}$ and $\{s_g\}_{g \in \mathcal{G}_1} = \{s_g \mid g \in \mathcal{G}_1\} \subset \{s_g\}$.

Let k be a positive integer, $\mathcal{F}(\pi; \mathcal{G})$ and $\mathcal{F}_k(\pi; \mathcal{G})$ denote the algebras of smooth functions on $J^\infty(\pi; \mathcal{G})$ and on $J^k(\pi; \mathcal{G})$, respectively, $\Lambda^*(\pi; \mathcal{G})$ denote the module of differential forms on $J^\infty(\pi;\mathcal{G})$. By definition, a vector field on the infinite-dimensional smooth manifold $J^\infty(\pi;\mathcal{G})$ is a derivation X of the algebra $\mathcal{F}(\pi;\mathcal{G})$ such that for any integer k, there exists an integer l such that $X(\mathcal{F}_k(\pi;\mathcal{G}))\subset\mathcal{F}_l(\pi;\mathcal{G})$. Denote by $D(\pi;\mathcal{G})$ the set of vector fields on $J^\infty(\pi;\mathcal{G})$. If π' is a bundle over M, $\mathcal{F}_k(\pi,\pi';\mathcal{G})$ denotes the set of sections of the bundle $(\pi_k^{\mathcal{G}})^*(\pi')$. An element $\varphi\in\mathcal{F}_k(\pi,\pi';\mathcal{G})$ can be interpreted as a nonlinear boundary differential operator from $\Gamma(\pi)$ to $\Gamma(\pi')$. For any k, the mapping $(\pi_{k+1,k}^{\mathcal{G}})^*$ embeds $\mathcal{F}_k(\pi,\pi';\mathcal{G})$ to $\mathcal{F}_{k+1}(\pi,\pi';\mathcal{G})$. Put

$$\mathcal{F}(\pi,\pi';\mathcal{G})=\bigcup_{k=0}^{\infty}\mathcal{F}_k(\pi,\pi';\mathcal{G}).$$

For any vector field X on M, we can assign a vector field $\widehat{X}$ on $J^\infty(\pi;\mathcal{G})$. Indeed, suppose that $\theta=\{[s_g]^\infty_{g(x)}\}$ is a point of the manifold $J^\infty(\pi;\mathcal{G})$, $\varphi\in\mathcal{F}(\pi;\mathcal{G})$; then, by definition, let us set

$$\widehat{X}(\varphi)(\theta)=X(j_k(\{s_g\})^*(\varphi))(x).$$

If a canonical coordinate system (12) in $J^\infty(\pi;\mathcal{G})$ is chosen, then

$$\widehat{\frac{\partial}{\partial x_i}}=\frac{\partial}{\partial x_i}+\sum_{\sigma,j,g,l}u^j_{\sigma l,g}\frac{\partial g^*(x_l)}{\partial x_i}\frac{\partial}{\partial u^j_{\sigma g}},$$

where $\sigma l=(i_1,\ldots,i_k,l)$ for $\sigma=(i_1,\ldots,i_k)$. The vector field $\widehat{\partial/\partial x_i}$ is denoted below by D_i and is called the *total derivative* with respect to x_i.

Further, if $\mathcal{G}$ is a semigroup of maps, then a smooth mapping $g_1\in\mathcal{G}$ can be lifted to the smooth mapping $\widehat{g_1}\colon J^k(\pi;\mathcal{G})\to J^k(\pi;\mathcal{G})$, where k is a positive integer or ∞. Namely, the mapping $\widehat{g_1}$ takes each $\theta_k=\{[s_g]^k_{g(x)}\}_{g\in\mathcal{G}}$ to $\tilde{\theta}_k=\{[s_{g\circ g_1}]^k_{g(x_1)}\}_{g\in\mathcal{G}}$, where $x_1=g_1(x)$. Then the following diagram

$$\begin{array}{ccc} J^k(\pi;\mathcal{G}) & \xrightarrow{\widehat{g_1}} & J^k(\pi;\mathcal{G}) \\ \downarrow{\scriptstyle\pi_k^{\mathcal{G}}} & & \downarrow{\scriptstyle\pi_k^{\mathcal{G}}} \\ M & \xrightarrow{g_1} & M \end{array} \tag{17}$$

is commutative and we have

$$\widehat{g_1\circ g_2}=\widehat{g_1}\circ\widehat{g_2},\qquad \widehat{g_1}\circ j_k(\{s_{\tilde g}\})=j_k(\{\tilde{s_g}\})\circ g_1,$$

where $g_2\in\mathcal{G}$ and $\tilde{s_g}=s_{g\circ g_1}$ for any $g\in\mathcal{G}$. The map $\widehat{g_1}$ induces the following action on coordinate functions: $\widehat{g_1}^*(x_i)=g_1^*(x_i)$, $\widehat{g_1}^*(u^j_{\sigma g})=u^j_{\sigma g_3}$, where $g_3=g\circ g_1$.

EXAMPLE 2. The maps $[x0]$, $[x\infty]$, $[z0]$, $[-z0]$ from Example 2 generate the semigroup

$$\begin{aligned}\mathcal{G}_1 = \{&\mathrm{id}_M,[x0],[x\infty],[z0],[-z0],[z\infty],[-z\infty],[00],[+\infty 0],[-\infty 0],\\ &[0\infty],[+\infty\infty],[-\infty\infty]\},\end{aligned}$$

where the map $[\bullet_1 \bullet_2]$ takes a point $(x, z, t) \in M$ to the point $(\bullet_1, \bullet_2, t) \in M$. Obviously, Equations (9)–(10) define a submanifold $\mathcal{E}$ of the manifold $J^2(\pi; \mathcal{G}_1)$ as well as of the manifold $J^2(\pi; \mathcal{G}_0)$ (see Example 1).

Suppose that $\mathcal{E} \subset J^k(\pi; \mathcal{G})$ is an equation, $\mathcal{G}$ is a semigroup, $\mathcal{E}^{(l)} \subset J^{k+l}(\pi; \mathcal{G})$ is the set of all points $\theta_{k+l} = \{[s_g]^{k+l}_{g(x)}\}_{g \in \mathcal{G}}$ such that any point $\widehat{g}(\theta_k)$, $g \in \mathcal{G}$, $\theta_k = \pi^{\mathcal{G}}_{k+l,k}(\theta_{k+l})$ belongs to $\mathcal{E}$ and the submanifold $j_k(\{s_g\})$ is tangent to $\mathcal{E}$ with order $\geqslant l$ at all points $\widehat{g}(\theta_k)$, $g \in \mathcal{G}$. Then the set $\mathcal{E}^{(l)}$ is called the lth *prolongation of the equation* $\mathcal{E}$.

If the equation $\mathcal{E}$ is given by the system

$$G_j\left(x, \dots, g^*\left(\frac{\partial^{|\sigma|} s^i}{\partial x^\sigma}\right), \dots\right) = 0, \quad j = 1, \dots, r, \tag{18}$$

where s^i, $i = 1, \dots, m$ are the components of a section $s \in \Gamma(\pi)$, $|\sigma| \leqslant k$, $g \in \mathcal{G}$, then its 0-prolongation $\mathcal{E}^{(0)}$ is given by the system

$$G_j\left({g_1}^*(x), \dots, (g \circ g_1)^*\left(\frac{\partial^{|\sigma|} s^i}{\partial x^\sigma}\right), \dots\right) = 0, \quad j = 1, \dots, r,\ g_1 \in \mathcal{G}$$

and the lth prolongation $\mathcal{E}^{(l)}$ is given by the system

$$\widehat{g_1}^*(D_\tau(G_j)) = 0, \quad j = 1, \dots, r,\ g_1 \in \mathcal{G},\ |\tau| \leqslant l,$$

where $\tau = (t_1, \dots, t_k)$, $D_\tau = D_{t_1} \circ \dots \circ D_{t_k}$, and D_t is the total derivative with respect to x_t.

For any l, the map $\pi^{\mathcal{G}}_{k+l+1,k+l}$ takes the set $\mathcal{E}^{(l+1)}$ to the set $\mathcal{E}^{(l)}$. We get the sequence of mappings $\mathcal{E}^{(l+1)} \to \mathcal{E}^l$, $l \geqslant 0$. The inverse limit of this sequence is called the *infinite prolongation* of the equation $\mathcal{E}$ and is denoted by $\mathcal{E}^\infty$. A point $\theta = [s]^{(\infty,\mathcal{G})}_x \in J^\infty(\pi; \mathcal{G})_0$ belongs to the set $\mathcal{E}^\infty$ if and only if the Taylor series of the section s at the points $g(x) \in M$, $g \in \mathcal{G}$ satisfies the equation $\mathcal{E}$. Therefore, we say that the points of the set $\mathcal{E}^\infty \cap J^\infty(\pi; \mathcal{G})_0$ are *formal solutions* of the equation $\mathcal{E}$.

Let us denote by $\mathcal{F}_l(\mathcal{E})$ the set of restrictions of smooth functions on $J^{k+l}(\pi; \mathcal{G})$ to $\mathcal{E}^{(l)}$. For any $l \geqslant 0$, we have the embedding $\mathcal{F}_l(\mathcal{E}) \subset \mathcal{F}_{l+1}(\mathcal{E})$. Elements of the set $\mathcal{F}(\mathcal{E}) = \bigcup_{l=0}^\infty \mathcal{F}_l(\mathcal{E})$ are called *smooth functions* on the infinite prolongation $\mathcal{E}^\infty$. In a similar way, we define differential i-forms on $\mathcal{E}^\infty$. The set $\mathcal{F}(\mathcal{E})$ is an algebra and the set $\Lambda^i(\mathcal{E})$ of differential i-forms on $\mathcal{E}^\infty$ is a module over $\mathcal{F}(\mathcal{E})$.

To define symmetries of boundary differential equation, we introduce some auxiliary concepts. A vector field X on $\mathcal{E}^\infty$ is called *vertical* if $(\pi_\infty)_*(X) = 0$, where $\pi_\infty \colon \mathcal{E}^\infty \to M$. Obviously, any vector field X on $\mathcal{E}^\infty$ locally has the form $Y + \sum_{i=1}^n a_i D_i$, where Y is a vertical field and D_i, $i = 1, \dots, n$, are the restrictions of the total derivatives to $\mathcal{E}^\infty$. Vector fields D_i are tangent to any submanifold $j_\infty(s)(M) \subset \mathcal{E}^\infty$. Therefore, fields D_i are trivial symmetries of any equation $\mathcal{E}^\infty$ and any nontrivial symmetry may be interpreted as a vertical field.

A differential 1-form ω on $J^k(\pi;\mathcal{G})$ is called a *Cartan form* on $J^k(\pi;\mathcal{G})$ if for any set of sections $\{s_g\}_{g\in\mathcal{G}}$ we have $[j_k(\{s_g\})]^*(\omega)=0$. Denote by $\mathcal{C}\Lambda^1(\mathcal{E}^l)$ the set of restrictions of Cartan forms on $J^{k+l}(\pi;\mathcal{G})$ to $\mathcal{E}^{(l)}$. From (16), it follows that $(\pi^{\mathcal{G}}_{k+l+1,k+l})^*(\mathcal{C}\Lambda^1(\mathcal{E}^l))\subset\mathcal{C}\Lambda^1(\mathcal{E}^{l+1})$ for any $l\geqslant 0$. Elements of the set $\mathcal{C}\Lambda^1(\mathcal{E}^\infty)=\bigcup_{l=0}^\infty\mathcal{C}\Lambda^1(\mathcal{E}^l)$ are called *Cartan forms* on $\mathcal{E}^\infty$.

THEOREM 2. *Suppose $\mathcal{E}$ is a boundary differential equation with a semigroup $\mathcal{G}$ and h is a section of the bundle $\pi_\infty\colon\mathcal{E}^\infty\to M$. Then there exists a solution $s\in\Gamma(\pi)$ of the equation $\mathcal{E}$ such that $h=j_\infty(s)$ if and only if for any $\omega\in\mathcal{C}\Lambda^1(\mathcal{E}^\infty)$ we have $h^*(\omega)=0$ and for any $g\in\mathcal{G}$ we have $\widehat{g}(h(M))\subset h(M)$.*

A vertical field X on $\mathcal{E}^\infty$ is called a *higher* (*infinitesimal*) *symmetry* of the equation $\mathcal{E}\subset J^k(\pi;\mathcal{G})$, if the conditions

$$X(\mathcal{C}\Lambda^1(\mathcal{E}^\infty))\subset\mathcal{C}\Lambda^1(\mathcal{E}^\infty)\quad\text{and}\quad\hat{g}^*\circ X=X\circ\hat{g}^*$$

hold for any $g\in\mathcal{G}$. The set of all higher symmetries of an equation $\mathcal{E}$ is a Lie algebra and is denoted by $\mathrm{sym}(\mathcal{E})$.

To make this definition work, we introduce some new concepts. Suppose that φ is a section of a bundle $(\pi_k^{\mathcal{G}})^*(\pi)$, $\mathcal{U}$ is a coordinate neighborhood in M and $\mathcal{U}^\infty=(\pi_\infty^{\mathcal{G}})^{-1}(\mathcal{U})$ is the corresponding coordinate neighborhood in $J^\infty(\pi;\mathcal{G})$. Then we define the vector field

$$\ni_{\varphi,\mathcal{U}}\overset{\text{def}}{=}\sum_{j,\sigma,g}\hat{g}^*(D_\sigma(\varphi^j))\frac{\partial}{\partial u^j_{\sigma g}}\tag{19}$$

on $\mathcal{U}^\infty$, where φ^j is the jth component of the restriction of φ to $\mathcal{U}^\infty$, D_σ is the σ-composition of the total derivatives. It can be proved that if $\mathcal{U},\mathcal{U}'\subset M$ are two coordinate neighborhoods in M, then the fields $\ni_{\varphi,\mathcal{U}}$ and $\ni_{\varphi,\mathcal{U}'}$ coincide on the neighborhood $(\pi_\infty^{\mathcal{G}})^{-1}(\mathcal{U}\cap\mathcal{U}')$. Thus, for any $\varphi\in\mathcal{F}(\pi,\pi;\mathcal{G})$ we have the field $\ni_\varphi$ on $J^\infty(\pi;\mathcal{G})$, which is called an *evolutionary derivation*.

Similar to the differential case [1, 7], a field $\ni_\varphi$, $\varphi\in\mathcal{F}(\pi,\pi;\mathcal{G})$ gives an evolution of sections of the bundle π. This evolution is given by the equations

$$\frac{\partial u^j}{\partial t}=\varphi^j\left(x,\dots,g^*\left(\frac{\partial^{|\sigma|}u^l}{\partial x^\sigma}\right),\dots\right),\quad j=1,\dots,m,$$

where $l=1,\dots,m$, $|\sigma|\leqslant k$, $g\in\mathcal{G}$.

Let $\mathcal{G}$ be a subsemigroup of a semigroup $\mathcal{G}'$ and $\varphi\in\mathcal{F}_k(\pi,\pi;\mathcal{G})$. It follows from (13) that the mapping

$$\left(\pi_k^{\mathcal{G}',\mathcal{G}}\right)^*\colon\mathcal{F}_k(\pi,\pi;\mathcal{G})\to\mathcal{F}_k(\pi,\pi;\mathcal{G}')$$

is an embedding. Therefore, we have the field $\ni_\varphi$ on $J^\infty(\pi;\mathcal{G})$ and the field $\ni_\varphi$ on $J^\infty(\pi;\mathcal{G}')$. By $\ni_\varphi^{\mathcal{G}}$, we denote the first field and by $\ni_\varphi^{\mathcal{G}'}$ the second one.

PROPOSITION 3. *If $\mathcal{G}$ is a subsemigroup of a semigroup $\mathcal{G}'$ and φ is an element of $\mathcal{F}_k(\pi,\pi;\mathcal{G})$, then the mapping $(\pi_k^{\mathcal{G}',\mathcal{G}})_*$ takes the field $\ni_\varphi^{\mathcal{G}'}$ to the field $\ni_\varphi^{\mathcal{G}}$.*

For two sections $\varphi, \psi \in \mathcal{F}(\pi, \pi; \mathcal{G})$ their *Jacobi bracket*, $\{\varphi, \psi\}$, is defined by

$$\ni_{\{\varphi,\psi\}} = [\ni_\varphi, \ni_\psi]. \tag{20}$$

It is easy to verify that $\{\varphi, \psi\} = \ni_\varphi(\psi) - \ni_\psi(\varphi)$. Therefore,

$$\{\varphi, \psi\}^j = \sum_{\alpha,\sigma,g} \left(\hat{g}^*(D_\sigma(\varphi^\alpha)) \frac{\partial \psi^j}{\partial u^\alpha_{\sigma g}} - \hat{g}^*(D_\sigma(\psi^\alpha)) \frac{\partial \varphi^j}{\partial u^\alpha_{\sigma g}} \right), \quad j = 1, \ldots, m. \tag{21}$$

For any function $\psi \in \mathcal{F}(\pi; \mathcal{G})$ define the *universal linearization operator* (or *derivative in the direction* ψ) $l_\psi : \mathcal{F}(\pi, \pi; \mathcal{G}) \to \mathcal{F}(\pi; \mathcal{G})$ by the rule

$$l_\psi(\varphi) = \ni_\varphi(\psi). \tag{22}$$

It is easy to prove that this definition of l_ψ is equivalent to definition (3).

Finally, let $\mathcal{E}$ be a boundary differential equation with a semigroup $\mathcal{G}$. For any $\varphi \in \mathcal{F}(\pi, \pi; \mathcal{G})$, denote by $\bar{\varphi}$ the restriction of φ to $\mathcal{E}^\infty$ and by $\mathcal{F}(\mathcal{E}, \pi; \mathcal{G})$ denote the set of these restrictions. By definition, put

$$l^{\mathcal{E}}_\psi = l_\psi\big|_{\mathcal{E}^\infty}, \qquad \{\bar{\varphi}, \bar{\theta}\}_{\mathcal{E}} = \{\varphi, \theta\}\big|_{\mathcal{E}^\infty}, \qquad \ni_{\bar{\varphi}} = \ni_\varphi\big|_{\mathcal{E}^\infty},$$

where $\theta \in \mathcal{F}(\pi, \pi; \mathcal{G})$.

THEOREM 4. *Suppose that an equation $\mathcal{E} \subset J^k(\pi; \mathcal{G})$ is given by system* (18) *and satisfies the condition*

$$\left(\pi_0^{\mathcal{G},\{\mathrm{id}_M\}} \circ \pi^{\mathcal{G}}_{\infty,0}\right)(\mathcal{E}^\infty) = J^0(\pi).$$

Then any higher symmetry of the equation $\mathcal{E}$ has the form $\ni_{\bar{\varphi}}$, $\bar{\varphi} \in \mathcal{F}(\mathcal{E}, \pi; \mathcal{G})$. The set of all solutions of the system

$$l^{\mathcal{E}}_{G_j}(\bar{\varphi}) = 0, \quad j = 1, \ldots, r, \ \bar{\varphi} \in \mathcal{F}(\mathcal{E}, \pi; \mathcal{G}) \tag{23}$$

forms a Lie algebra with respect to the bracket $\{\bullet, \bullet\}_{\mathcal{E}}$. The mapping $\bar{\varphi} \mapsto \ni_{\bar{\varphi}}$ is an isomorphism from this Lie algebra to $\mathrm{sym}(\mathcal{E})$.

In the differential case (see [1, 7]), a field X on the manifold $\mathcal{E} \subset J^k(\pi)$ is called a classical (infinitesimal) symmetry of the equation $\mathcal{E}$ if $X(\mathcal{C}\Lambda^1(\mathcal{E})) \subset \mathcal{C}\Lambda^1(\mathcal{E})$. This field can be lifted up to a field X^∞ on $\mathcal{E}^\infty$. Suppose that $\pi_{\infty,0}(\mathcal{E}^\infty) = J^0(\pi)$ and the vertical component of X^∞ coincides with the restriction of a certain evolutionary differentiation $\ni_\varphi$ to $\mathcal{E}^\infty$, where $\varphi \in \mathcal{F}(\pi, \pi; \mathcal{G})$. Then in a canonical coordinate system (x_i, u^j_σ) in $J^\infty(\pi)$ the components of the section φ are

$$\varphi^j = \sum_{i=1}^n a_i u^j_{(i)} + b_j, \quad j = 1, \ldots, m, \tag{24}$$

where $a_i, b_j, i = 1, \ldots, n, j = 1, \ldots, m$, are functions of the variables $x_1, \ldots, x_n$, $u^1, \ldots, u^m$. In this case, the mapping $(\pi_{k,0})_*$ takes the field X to the field

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} b_j \frac{\partial}{\partial u^j}.$$

Suppose now that $\ni_{\bar\varphi}$ is a higher symmetry of a boundary differential equation $\mathcal{E} \subset J^k(\pi; \mathcal{G})$, $\bar\varphi = \varphi|_{\mathcal{E}^\infty}$, $\varphi \in \mathcal{F}(\pi, \pi; \{\mathrm{id}_M\}) \subset \mathcal{F}(\pi, \pi; \mathcal{G})$, and the image of $\ni_\varphi$ under the projection $(\pi_\infty^{\mathcal{G}, \{\mathrm{id}_M\}})_*$ is the vertical component of a classical symmetry of the trivial equation $\widetilde{\mathcal{E}} = J^0(\pi)$. Then we say that the symmetry $\ni_{\bar\varphi}$ is *classical*. By Proposition (3), it follows that in a canonical coordinate system (12) in $J^\infty(\pi; \mathcal{G})$ a classical symmetry φ has the form (24).

4. Classical Symmetries of the Benjamin–Ono Equation

In this section, we calculate classical symmetries of system (9)–(10). It follows from (19), (22), and Example 1 that Equation (23) for system (9)–(10) has the form

$$D_3(U) = 2uD_1(U) + 2u_1U - V_{[x0]}, \tag{25}$$

$$D_2(V) = \frac{(D_1^2(U))_{[z0]} - D_1^2(U)}{\pi(x-z)} + \frac{(D_1^2(U))_{[-z0]} - D_1^2(U)}{\pi(x+z)}, \tag{26}$$

$$V_{[x\infty]} = 0, \tag{27}$$

$$U_{[x0]} = U, \tag{28}$$

where $V_{[x0]} = \widehat{[x0]}^*(V)$, $(D_1^2(U))_{[z0]} = \widehat{[z0]}^*(D_1(D_1(U)))$, etc.

Using (24) and the equation $u_2 = 0$, we get

$$U = \xi u_1 + \theta u_3 + \phi, \tag{29}$$

$$V = \xi v_1 + \eta v_2 + \theta v_3 + \psi, \tag{30}$$

where $\xi, \eta, \theta, \phi, \psi$ are functions of x, z, t, u, v. From (28), it follows that the functions ξ, θ, ϕ are independent of z, v.

Differentiating (26) with respect to $u_{111[z0]}$ and taking into account Equations (29), (30) and finally multiplying both sides by $\pi(x-z)$, we obtain

$$\eta + 2u_{[z0]}\theta = \xi_{[z0]} + 2u_{[z0]}\theta_{[z0]}. \tag{31}$$

In the same way, differentiating (26) with respect to $u_{111[-z0]}$ and multiplying by $\pi(x+z)$, we obtain

$$-\eta + 2u_{[-z0]}\theta = \xi_{[-z0]} + 2u_{[-z0]}\theta_{[-z0]}. \tag{32}$$

Summing up (31) and (32) and differentiating the result with respect to x or with respect to u, we get

$$2(u_{[z0]} + u_{[-z0]})\theta'_x = 0, \quad 2(u_{[z0]} + u_{[-z0]})\theta'_u = 0,$$

respectively. Therefore, the function θ does not depend on x, u. Hence, $\theta_{[z0]} = \theta = \theta_{[-z0]}$. It now follows from (31) and (32) that $\eta = \xi_{[z0]}$ and

$$\xi_{[z0]} + \xi_{[-z0]} = 0. \tag{33}$$

Since ξ is a function of x, t, u, it follows that the function $\xi_{[z0]}$ depends on $z, t, u_{[z0]}$ and the function $\xi_{[-z0]}$ depends on $z, t, u_{[-z0]}$. Therefore, from (33) it follows that ξ is independent of u.

Using (33), (29), (30), where $\xi = \xi(t, x)$, $\eta = \xi_{[z0]}$, $\theta = \theta(t)$, $\phi = \phi(t, x, u)$, $\psi = \psi(t, x, z, u, v)$, and differentiating (26) with respect to $u_{11[z0]}$ and with respect to $u_{11[-z0]}$, we obtain

$$-\frac{\xi - \xi_{[z0]}}{x - z} + \psi'_v = \phi'_{u[z0]} + \xi'_{x[z0]}, \tag{34}$$

$$-\frac{\xi + \xi_{[z0]}}{x + z} + \psi'_v = \phi'_{u[-z0]} + \xi'_{x[z0]}. \tag{35}$$

Subtracting (35) from (34), we get

$$\frac{-\xi + \xi_{[z0]}}{x - z} + \frac{\xi + \xi_{[z0]}}{x + z} = \Phi, \tag{36}$$

where $\Phi = \phi'_{u[z0]} - \phi'_{u[-z0]}$. The function Φ depends on $z, t, u_{[z0]}, u_{[-z0]}$. The left-hand side of (36) can depend on x, z, t only. It follows that Φ is a function of z, t.

Multiplying both sides of (36) by $x^2 - z^2$, we get

$$2x\xi_{[z0]} - 2z\xi = \Phi(x^2 - z^2). \tag{37}$$

Differentiating (37) with respect to x twice, we get $-2z\xi''_{xx} = 2\Phi$. Therefore ξ''_{xx} is a function of t. Finding the coefficient by z^3 in (37), we obtain $\xi''_{xx} = 0$. Hence, $\Phi = \phi'_{u[z0]} - \phi'_{u[-z0]} = 0$. As in the case of (33), from this equality it follows that ϕ'_u is independent of u.

Substituting $\xi_0 x + \xi_1$ for ξ and $\xi_0 z + \xi_1$ for $\xi_{[z0]}$ in (37), where ξ_0, ξ_1 are functions of t, we get $2(x - z)\xi_1 = 0$. Thus

$$\xi = \xi_0 x, \quad \eta = \xi_0 z, \quad \phi = \phi_0 u + \phi_1, \tag{38}$$

where ξ_0 is a function of t, ϕ_0, ϕ_1 are functions of x, t. Combining this with (34), we obtain $\psi'_v = \phi_{0[z0]} + 2\xi_0$. Therefore,

$$\psi = (\phi_{0[z0]} + 2\xi_0)v + \psi_0, \tag{39}$$

where ψ_0 is a function of x, z, t, u.

Taking into account (29), (30), (38), (39) and differentiating (25) with respect to $v_{[x0]}$, we get

$$\phi_0 + \theta'_t = \phi_{0[00]} + 2\xi_0. \tag{40}$$

It now follows that ϕ_0 is a function of t only, since other functions in (40) possess the same property. Hence, $\phi_{0[00]} = \phi_0$. Combining this with (40), we obtain

$$\xi_0 = \tfrac{1}{2}\theta'_t. \tag{41}$$

Differentiating (25) with respect to u_1, we get $\theta'_t u + \xi'_{0,t} x = 2\phi_0 u + 2\phi_1$. Hence,

$$\phi_0 = \tfrac{1}{2}\theta'_t, \qquad \phi_1 = \tfrac{1}{2}\xi'_{0,t} x. \tag{42}$$

Using (29), (30), (38), (39), (41), (42), we rewrite (26) as $\psi'_{0,z} = 0$. It follows that ψ_0 is a function of x, t, u. Similarly, we can rewrite (25) and (27) as $\frac{1}{2}\xi''_{0,tt} x = -\psi_{0[x0]}$ and $\xi_0 (zv_2)_{[x\infty]} + \psi_{0[x\infty]} = 0$. We have $\psi_{0[x0]} = \psi_0 = \psi_{0[x\infty]}$ and $(zv_2)_{[x\infty]} = u_{11[-\infty 0]} - u_{11[\infty 0]}$. This implies that if we consider only solutions satisfying the conditions

$$u_{11[-\infty 0]} = 0, \qquad u_{11[\infty 0]} = 0, \tag{43}$$

then $\psi_0 = 0$, $\xi''_{0,tt} = \frac{1}{2}\theta'''_{ttt} = 0$, and $\theta = at^2 + 2bt + c$, where a, b, c are constants. In this case,

$$U = a[t^2 u_3 + t(u + xu_1) + \tfrac{1}{2}x] + b[2tu_3 + u + xu_1] + cu_3, \tag{44}$$

$$\begin{aligned} V = {} & a[t^2 v_3 + t(3v + xv_1 + zv_2)] + \\ & + b[2tv_3 + 3v + xv_1 + zv_2] + cv_3. \end{aligned} \tag{45}$$

Clearly, the symmetries (44)–(45) preserve the condition (43). In other words, $(D_1^2(U))_{[-\infty 0]} = 0$ and $(D_1^2(U))_{[\infty 0]} = 0$ whenever conditions (43) hold.

If we consider solutions vanishing at infinity, then

$$u_{[-\infty 0]} = 0, \quad u_{[\infty 0]} = 0. \tag{46}$$

In this case, a symmetry (U, V) must satisfy the conditions $U_{[-\infty 0]} = 0$ and $U_{[\infty 0]} = 0$. Obviously, the symmetries (44)–(45) satisfy these conditions iff $a = 0$.

In the notation of the Introduction, the symmetry (44)–(45) with $a = 0$, $b = 0$, $c = 1$ corresponds to $G_{0,1}$. The symmetry (44)–(45) with $a = 0$, $b = -2$, $c = 0$ corresponds to $G_{1,1}$. The symmetry (44)–(45) with $a = 8$, $b = 0$, $c = 0$ corresponds to $G_{2,1}$. Thus we proved the following theorem.

THEOREM 5. (a) *The linear span of the fields $G_{0,1}$, $G_{1,1}$, and $G_{2,1}$ is the classical symmetry algebra for system* (9), (10), (43).

(b) *The linear span of the fields $G_{0,1}$ and $G_{1,1}$ is the classical symmetry algebra for system* (9), (10), (46).

Remark 3. The symmetries $G_{0,0} = u_1$ and $G_{1,0} = -2tu_1 - 1$ are higher symmetries in our sense. For example, if $U = G_{1,0}$, then

$$V = -2tv_1 - \frac{2t}{\pi}\left[\frac{u_{11[z0]} - u_{11}}{x - z} - \frac{u_{11[-z0]} - u_{11}}{x + z}\right].$$

It can be proved that the symmetries $G_{m,n}$ are nonlocal ones for $n \geqslant 2$.

5. Lie Algebra Structure on the Set of Symmetries of the Benjamin–Ono Equation

The aim of this section is to prove Theorem 1. To do this, we need the following lemma.

LEMMA 6. *Suppose that the right-hand side K of a scalar evolution equation $\mathcal{E}$: $u_t = K$ is independent of t. Then φ is a symmetry of $\mathcal{E}$ polynomial in t if and only if*

$$\varphi = \sum_{i=0}^{m} \frac{t^i}{i!} \operatorname{ad}_K^i(\varphi_0),$$

where the function φ_0 is independent of t and $\operatorname{ad}_K^{m+1}(\varphi_0) = 0$.

Proof. In our case, Equation (23) on a symmetry φ has the form

$$\frac{\partial \varphi}{\partial t} = \{\varphi, K\}. \tag{47}$$

Indeed, it follows from (22), (19), and (21) that

$$\begin{aligned} l^{\mathcal{E}}_{u_t - K}(\varphi) &= l_{u_t}(\varphi)|_{\mathcal{E}^\infty} - l_K(\varphi) \\ &= \frac{\partial \varphi}{\partial t} + \ni_K(\varphi) - l_K(\varphi) = \frac{\partial \varphi}{\partial t} - \{\varphi, K\} = 0. \end{aligned}$$

Suppose that $\varphi = \sum_{i=0}^m t^i \varphi_i$, where functions φ_i are independent of t; then from (47) we get

$$\sum_{i=1}^{m} i t^{i-1} \varphi_i = \sum_{i=0}^{m} t^i \{\varphi_i, K\}.$$

Hence, for $i = 1, \ldots, m$ we have

$$\varphi_i = \frac{1}{i}\{\varphi_{i-1}, K\} = \frac{1}{i} \operatorname{ad}_K(\varphi_{i-1}) = \frac{1}{i!} \operatorname{ad}_K^i(\varphi_0) \quad \text{and} \quad \{\varphi_m, K\} = 0.$$

This completes the proof. □

Before proving the theorem, we make some preliminary observations. While calculating commutators we shall use the known properties of the Hilbert transform operator:

$$H(fHg) + H(gHf) = -fg + (Hf)(Hg), \tag{48}$$

$$H(Hf) = -f, \tag{49}$$

$$D \circ H = H \circ D, \tag{50}$$

$$H(x^k) = 0, \tag{51}$$

$$H(xf) = xHf \tag{52}$$

and the well-known properties of the linearization operator:

$$l_{H \circ \psi} = H \circ l_\psi, \qquad l_{D(\psi)} = D \circ l_\psi, \qquad l_{\phi\psi} = \phi l_\psi + \psi l_\phi, \tag{53}$$

where f, g are functions of the form (5) with $\phi(\infty) = 0$, D is the total derivative with respect to x, ϕ and ψ are nonlinear operators. For the functions f and g of the class $\mathrm{O}(1/|x|^{\mu})$ as $x \to \infty$, $\mu > 0$, the proof of properties (48), (49), (50), and (51) for $k = 0$ can be found in the book [6]. In a general case, these properties and (52) easily follow from definition (6). Properties (53) are deduced from definition (3).

Denote $\tau^{m,n} = \mathrm{ad}_x^m \mathrm{ad}_\tau^n(K_0)$ for $m \geqslant 0, n \geqslant 0$ (for definitions of τ and K_0, see the Introduction).

Proof. The proof of Theorem 1 is in 11 steps.

Step 1: By direct calculations, we prove that

$$\tau^{0,0} = u_1, \tag{54}$$

$$\tau^{0,1} = K, \tag{55}$$

$$\tau^{1,1} = -2D(xu), \tag{56}$$

$$\tau^{2,1} = 4x, \tag{57}$$

$$\tau^{1,2} = 6\tau, \tag{58}$$

$$\{\tau, \tau^{1,1}\} = 2\tau, \tag{59}$$

$$\{x, \tau^{1,1}\} = -4x, \tag{60}$$

$$\{x, \tau^{0,1}\} = -\tau^{1,1}. \tag{61}$$

Step 2: We prove

$$\mathrm{ad}_{\tau^{1,1}}(\tau^{m,n}) = (2 + 2n - 4m)\tau^{m,n} \tag{62}$$

by induction on m and n. The case $m = 0, n = 0$ is obtained by direct calculations. Using (61), the Jacobi identity, and the induction hypothesis, we obtain

$$\begin{aligned} \mathrm{ad}_{\tau^{1,1}}(\tau^{0,n}) &= \mathrm{ad}_{\tau^{1,1}} \mathrm{ad}_\tau(\tau^{0,n-1}) \\ &= \mathrm{ad}_\tau \mathrm{ad}_{\tau^{1,1}}(\tau^{0,n-1}) + \mathrm{ad}_{\{\tau,\tau^{1,1}\}}(\tau^{0,n-1}) \\ &= 2n \,\mathrm{ad}_\tau(\tau^{0,n-1}) + 2\,\mathrm{ad}_\tau(\tau^{0,n-1}) = (2 + 2n)\tau^{0,n} \end{aligned}$$

for $n > 0$. Similarly, for $m > 0$ we get

$$\begin{aligned} \mathrm{ad}_{\tau^{1,1}}(\tau^{m,n}) &= \mathrm{ad}_x \mathrm{ad}_{\tau^{1,1}}(\tau^{m-1,n}) + \mathrm{ad}_{\{x,\tau^{1,1}\}}(\tau^{m-1,n}) \\ &= (6 + 2n - 4m)\,\mathrm{ad}_x(\tau^{m-1,n}) - 4\,\mathrm{ad}_x(\tau^{m-1,n}) \\ &= (2 + 2n - 4m)\tau^{m,n}. \end{aligned}$$

Step 3: By induction on m we prove that

$$\mathrm{ad}_{\tau^{0,1}}(\tau^{m,n}) = 2m(m - 2 - n)\tau^{m-1,n}. \tag{63}$$

The case $m = 0$ follows from the results by Fokas and Fuchssteiner (see Introduction). As above, for $m > 0$ we obtain

$$\begin{aligned} \mathrm{ad}_{\tau^{0,1}}(\tau^{m,n}) &= \mathrm{ad}_x \mathrm{ad}_{\tau^{0,1}}(\tau^{m-1,n}) + \mathrm{ad}_{\{x,\tau^{0,1}\}}(\tau^{m-1,n}) \\ &= 2(m-1)(m-3-n)\,\mathrm{ad}_x(\tau^{m-2,n}) - \mathrm{ad}_{\tau^{1,1}}(\tau^{m-1,n}) \\ &= 2m(m-2-n)\tau^{m-1,n}. \end{aligned}$$

Step 4: The relations

$$\tau^{n+2,n} = 0 \tag{64}$$

and

$$\tau^{n+1,n} = (-1)^{n+1}[(n+1)!]^2 x^n \tag{65}$$

are proved by induction on n. The case $n = 0$ is obtained by direct calculations. It follows from the results by Fokas and Fuchssteiner that

$$\{\tau^{0,n-1}, \tau^{0,2}\} = 0. \tag{66}$$

Using the Jacobi identity and the induction hypothesis, we get

$$\mathrm{ad}_x^{n+3}\{\tau^{0,n-1}, \tau^{0,2}\} = \binom{n+3}{3}\{\tau^{n,n-1}, \tau^{3,2}\} = 0, \tag{67}$$

where $\binom{n}{k}$ denotes the binomial coefficient. By (58) and by the definition of $\tau^{0,n}$, we have $\{\tau^{0,n-1}, \tau^{1,2}\} = 6\tau^{0,n}$. Like in the case of (67), we obtain

$$\mathrm{ad}_x^{n+2}\{\tau^{0,n-1}, \tau^{1,2}\} = \binom{n+2}{2}\{\tau^{n,n-1}, \tau^{3,2}\} = 6\tau^{n+2,n}.$$

Combining this with (67), we obtain (64).

As above,

$$\begin{aligned} &\mathrm{ad}_x^{n+2}\{\tau^{0,n-1}, \tau^{0,2}\} \\ &\quad = \binom{n+2}{3}\{\tau^{n-1,n-1}, \tau^{3,2}\} + \binom{n+2}{3}\{\tau^{n,n-1}, \tau^{2,2}\} = 0, && (68) \\ &\mathrm{ad}_x^{n+1}\{\tau^{0,n-1}, \tau^{1,2}\} \\ &\quad = \binom{n+1}{2}\{\tau^{n-1,n-1}, \tau^{3,2}\} + \binom{n+1}{1}\{\tau^{n,n-1}, \tau^{2,2}\} = 6\tau^{n+1,n}. && (69) \end{aligned}$$

Solving system (68)–(69), we get

$$\{\tau^{n,n-1}, \tau^{2,2}\} = -\frac{12}{n+1}\tau^{n+1,n}. \tag{70}$$

By direct calculations, we obtain $\tau^{2,2} = 12D(x^2u)$. Combining this with (70) and with the induction hypothesis, we obtain (65).

Step 5: Using (63) and (65), we get

$$\begin{aligned} \tau^{n,n} &= (-1)^n (n!)^2 (n+1) D(x^n u), \\ \tau^{n-1,n} &= \tfrac{1}{2}(-1)^{n-1}(n!)^2(n+1)D\left[x^{n-1}(u^2 + Hu_1) + \frac{n-1}{2}x^{n-2}Hu\right]. \end{aligned}$$

Hence

$$\{\tau^{n+1,n}, \tau\} = -\frac{2}{n+2}\tau^{n+1,n+1},$$
$$\{\tau^{n,n}, \tau\} = -\frac{2(2n-1)}{(n+1)(n+2)}\tau^{n,n+1}. \tag{71}$$

Step 6: Using induction on m, we obtain

$$\{\tau^{m,n}, \tau^{0,2}\} = c(m, n, 0, 2)\tau^{m-1,n+1}, \tag{72}$$

where $c(m, n, 0, 2)$ satisfies (8). Namely, it follows from the definition of $\tau^{0,2}$, the Jacobi identity, (63), and (71) that

$$\begin{aligned}\{\tau^{n+1,n}, \tau^{0,2}\} &= \{\tau^{n+1,n}, \{\tau^{0,1}, \tau\}\} \\ &= \{\{\tau^{n+1,n}, \tau^{0,1}\}, \tau\} + \{\tau^{0,1}, \{\tau^{n+1,n}, \tau\}\} \\ &= -2(n+1)\{\tau^{n,n}, \tau\} - \frac{2}{n+2}\{\tau^{0,1}, \tau^{n+1,n+1}\} \\ &= -\frac{12}{n+2}\tau^{n,n+1}.\end{aligned}$$

Suppose that $n+1 > m > 0$; then $2(m+1)(m-1-n) \neq 0$. Using (63), (66), the Jacobi identity, and the induction hypothesis for $\{\tau^{m+1,n}, \tau^{0,2}\}$, we get

$$\begin{aligned}\{\tau^{m,n}, \tau^{0,2}\} &= \frac{1}{2(m+1)(m-1-n)}\{\{\tau^{m+1,n}, \tau^{0,1}\}, \tau^{0,2}\} \\ &= \frac{1}{2(m+1)(m-1-n)}\{\{\tau^{m+1,n}, \tau^{0,2}\}, \tau^{0,1}\} \\ &= \frac{c(m+1, n, 0, 2)}{2(m+1)(m-1-n)}\{\tau^{m,n+1}, \tau^{0,1}\} \\ &= c(m, n, 0, 2)\tau^{m-1,n+1}.\end{aligned}$$

Step 7: Using (72), we obtain

$$\begin{aligned}\{\tau^{m,n}, \tau^{1,2}\} &= \{\tau^{m,n}, \{\tau^{0,2}, x\}\} \\ &= \{\{\tau^{m,n}, \tau^{0,2}\}, x\} + \{\tau^{0,2}, \{\tau^{m,n}, x\}\} \\ &= c(m, n, 0, 2)\{\tau^{m-1,n+1}, x\} + \{\tau^{0,2}, \tau^{m+1,n}\} \\ &= c(m, n, 1, 2)\tau^{m,n+1}.\end{aligned} \tag{73}$$

Step 8: Combining (58), (72), (73), and the Jacobi identity, we get

$$\begin{aligned}\{\tau^{m,n}, \tau^{0,3}\} &= \tfrac{1}{6}\{\tau^{m,n}, \{\tau^{0,2}, \tau^{1,2}\}\} \\ &= \tfrac{1}{6}\{\{\tau^{m,n}, \tau^{0,2}\}, \tau^{1,2}\} + \tfrac{1}{6}\{\tau^{0,2}, \{\tau^{m,n}, \tau^{1,2}\}\} \\ &= \tfrac{1}{6}c(m, n, 0, 2)\{\tau^{m-1,n+1}, \tau^{1,2}\} + \tfrac{1}{6}c(m, n, 1, 2)\{\tau^{0,2}, \tau^{m,n+1}\} \\ &= c(m, n, 0, 3)\tau^{m-1,n+2}.\end{aligned} \tag{74}$$

Step 9: For $m+n \geqslant l+k \geqslant 3$ we prove by induction on $l+k$ that

$$\{\tau^{m,n}, \tau^{l,k}\} = c(m,n,l,k)\tau^{m+l-1,n+k-1}, \tag{75}$$

where $c(m,n,l,k)$ satisfies (8). The case $l+k=3$ follows from (73), (74), and (57). Combining (58), (73), the Jacobi identity, and the induction hypothesis, for $l=0$, $k>3$ we obtain

$$\begin{aligned}
\{\tau^{m,n}, \tau^{0,k}\} &= \tfrac{1}{6}\{\tau^{m,n}, \{\tau^{0,k-1}, \tau^{1,2}\}\} \\
&= \tfrac{1}{6}\{\{\tau^{m,n}, \tau^{0,k-1}\}, \tau^{1,2}\} + \tfrac{1}{6}\{\tau^{0,k-1}, \{\tau^{m,n}, \tau^{1,2}\}\} \\
&= \tfrac{1}{6}c(m,n,0,k-1)\{\tau^{m-1,n+k-2}, \tau^{1,2}\} \\
&\quad + \tfrac{1}{6}c(m,n,1,2)\{\tau^{0,k-1}, \tau^{m,n+1}\} \\
&= c(m,n,0,k)\tau^{m-1,n+k-1}.
\end{aligned}$$

Similarly, for $l>0$, $k+l>3$ we have

$$\begin{aligned}
\{\tau^{m,n}, \tau^{l,k}\} &= \{\tau^{m,n}, \{\tau^{l-1,k}, x\}\} \\
&= \{\{\tau^{m,n}, \tau^{l-1,k}\}, x\} + \{\tau^{l-1,k}, \{\tau^{m,n}, x\}\} \\
&= c(m,n,l-1,k)\{\tau^{m+l-2,n+k-1}, x\} + \{\tau^{l-1,k}, \tau^{m+1,n}\} \\
&= c(m,n,l,k)\tau^{m+l-1,n+k-1}.
\end{aligned}$$

Step 10: We prove (75) for $k=0$ by induction on n and m. The relations

$$\tau^{1,0} = -1, \tag{76}$$

$$\{x, \tau^{1,0}\} = 0, \tag{77}$$

$$\{\tau, \tau^{1,0}\} = \tau^{1,1}, \tag{78}$$

$$\{\tau^{0,0}, \tau^{1,0}\} = 0 \tag{79}$$

are proved by direct calculations. Using the definition of $\tau^{0,n}$, the Jacobi identity, the induction hypothesis, (78), and (62), we obtain

$$\begin{aligned}
\{\tau^{0,n}, \tau^{1,0}\} &= \mathrm{ad}_{\tau^{1,0}}\,\mathrm{ad}_{\tau}(\tau^{0,n-1}) \\
&= \mathrm{ad}_{\tau}\,\mathrm{ad}_{\tau^{1,0}}(\tau^{0,n-1}) + \mathrm{ad}_{\{\tau,\tau^{1,0}\}}(\tau^{0,n-1}) \\
&= n(n-1)\,\mathrm{ad}_{\tau}(\tau^{0,n-2}) + \mathrm{ad}_{\tau^{1,1}}(\tau^{0,n-1}) \\
&= n(n-1)\tau^{0,n-1} + 2n\tau^{0,n-1} \\
&= c(0,n,1,0)\tau^{0,n-1}
\end{aligned}$$

for $n>0$. Similarly, using (77), for $m>0$ we get

$$\begin{aligned}
\{\tau^{m,n}, \tau^{1,0}\} &= \mathrm{ad}_{\tau^{1,0}}\,\mathrm{ad}_{x}(\tau^{m-1,n}) \\
&= \mathrm{ad}_{x}\,\mathrm{ad}_{\tau^{1,0}}(\tau^{m-1,n}) \\
&= n(n+1)\,\mathrm{ad}_{x}(\tau^{m-1,n-1}) = c(m,n,1,0)\tau^{m,n-1}.
\end{aligned}$$

Finally, we have $\{\tau^{0,n}, \tau^{0,0}\} = 0$. This result was in fact obtained by Fokas and Fuchssteiner (they used different notation). As above, for $m > 0$ we obtain

$$\begin{aligned}\{\tau^{m,n}, \tau^{0,0}\} &= \mathrm{ad}_{\tau^{0,0}}\, \mathrm{ad}_x(\tau^{m-1,n})\\ &= \mathrm{ad}_x\, \mathrm{ad}_{\tau^{0,0}}(\tau^{m-1,n}) - \mathrm{ad}_{\{\tau^{0,0},x\}}(\tau^{m-1,n})\\ &= -(m-1)n(n+1)\,\mathrm{ad}_x(\tau^{m-2,n-1}) - \mathrm{ad}_{\tau^{1,0}}(\tau^{m-1,n})\\ &= c(m,n,0,0)\tau^{m-1,n-1}.\end{aligned}$$

Step 11: Thus from steps 2, 3, 6, 9, 10 it follows that formula (75) holds for any m, n, l, k. Combining this with (55), we get $\mathrm{ad}_K^{m+1}(\tau^{m,n}) = 0$ for any $m \geqslant 0, n \geqslant 0$. By Lemma 6, $G_{m,n} = \sum_{i=0}^{m} t^i/i!\,\mathrm{ad}_K^i(\tau^{m,n})$ is a symmetry of Equation (1). Moreover, the commutator of symmetries is a symmetry. From (75) it follows that the constant term (that is, the coefficient at t^0, cf. Lemma 6) of the symmetry

$$\varphi = \{G_{m,n}, G_{l,k}\} - c(m,n,l,k)G_{m+l-1,n+k-1}$$

vanishes. If we combine this with Lemma 6, we obtain $\varphi = 0$. This concludes the proof. □

Acknowledgements

The author is grateful to P. Olver for calling his attention to the Benjamin–Ono equation and to I. S. Krasil'shchik for interest in this work.

References

1. Krasil'shchik, I. S. and Vinogradov, A. M. (eds): *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*, Trans. Math. Monogr. 182, Amer. Math. Soc., Providence, RI, 1999.
2. Chetverikov, V. N. and Kudryavtsev, A. G.: A method for computing symmetries and conservation laws of integro-differential equations, *Acta Appl. Math.* **41**(1–3) (1995), 45–56.
3. Chetverikov, V. N. and Kudryavtsev, A. G.: Modelling integro-differential equations and a method for computing their symmetries and conservation laws, *Amer. Math. Soc. Transl. (2)* **167** (1995), 1–22.
4. Fokas, A. S. and Fuchssteiner, B.: The hierarchy of the Benjamin–Ono equation, *Phys. Lett. A* **86**(6) (1981), 341–345.
5. Fuchssteiner, B.: Mastersymmetries, higher order time-dependent symmetries and conserved densities of nonlinear evolution equations, *Progr. Theor. Phys.* **70**(6) (1983), 1508.
6. Gakhov, F. D.: *Boundary-Value Problems*, Nauka, Moscow, 1977 (in Russian).
7. Krasil'shchik, I. S., Lychagin, V. V., and Vinogradov, A. M.: *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*, Gordon and Breach, New York, 1986.