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Internship report

Novel realizations of warp drive spacetimes as solutions of general relativity

Author :
Antony Frackowiak
antony.frackowiak@etu.univ-lyon1.fr

Under the direction of :
Thomas BUCHERT
thomas.buchert@ens-lyon.fr

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Abstract

In this study, we explore novel realizations of warp drive spacetimes as exact solutions of general relativity. Building upon the standard 3+1 decomposition of spacetime, we analyze how specific geometric and kinematic constraints lead to a classification of warp field models beyond the conventional Alcubierre framework. Our approach is based on a class of metrics that characterize motion in terms of a one-component shift vector field with a constant lapse function.

In this report, we investigate the general consequences of motion within this restricted class of metrics by analyzing the Einstein equations and their projections onto spatial hypersurfaces, together with the energy and momentum conservation laws. We formulate the results in the form of a lemma for the subcase of irrotational motion, and as a theorem for the general case. Analytical and numerical methods are employed to construct specific examples of these exact solutions to general relativity.

This study extends classical warp drive models into a broader relativistic framework, providing a deeper understanding of the geometric and physical concepts beyond the preliminary ideas presented in the literature on the possibility of hyper-fast motion in general relativity.

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Notations used

- The comma denotes derivative with respect to x^j, $\partial/\partial x^j$ (in Eulerian formalism).
- The $\boldsymbol{\partial}$ denotes the nabla operator.
- The semicolon ; represents the covariant derivative in 4-dimensional spacetime.
- A double vertical slash || indicates a covariant derivative on a 3-dimensional hypersurface.
- A vertical slash | for the derivative with respect to X^k, $\partial/\partial X^k$ (in Lagrangian formalism).
- We define here the scalar invariants associated with any tensor of rank-2 with spatial indices, T_{ij}:

  First invariant:

  $$\mathrm{I}(\boldsymbol{T}) = \mathrm{Tr}(\boldsymbol{T}) = T^i{}_i \, .$$

  Second invariant:

  $$\mathrm{II}(\boldsymbol{T}) = \frac{1}{2}[(\mathrm{Tr}(\boldsymbol{T}))^2 - \mathrm{Tr}(\boldsymbol{T}^2)] \, ,$$

  where $\mathrm{Tr}(T^2) = T_{ij}T^{ij}$.
- Einstein's summation convention of summing over double indices, with Latin indices $i, j, k, \ldots = 1, 2, 3$ and Greek indices $\alpha, \beta, \gamma, \ldots = 0 \ldots, 3$.
- We will set the speed of light $c = 1$.

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Introduction — the warp drive context

The theoretical possibility of achieving effective superluminal travel within the framework of general relativity has attracted substantial interest since the introduction of the so-called "warp drive" metric by Miguel Alcubierre in 1994 [1]. This metric constructs a spacetime geometry, formulated in the 3+1 decomposition of spacetime, wherein a localized "warp bubble" moves through space by contracting it in front and expanding it behind, thereby allowing faster-than-light travel from the perspective of distant observers, without violating local causality. However, the Alcubierre model faces substantial physical and structural challenges. Notably, the shape and dynamics of the warp bubble or warp field are imposed in an *ad hoc* manner, rather than being derived from self-consistent solutions of the Einstein field equations. The velocity profile, typically encoded via a shift vector, is externally prescribed. In addition, it requires negative energy densities that violate the classical energy conditions. These limitations raise fundamental concerns about the physical realism and predictive power of the model. In contrast, the present work aims to derive warp field configurations directly from the geometric and kinematical properties of spacetime, ensuring that all aspects of the warp field — its shape, evolution, and energy-momentum content — are governed by exact solutions of Einstein's equations.

Among these developments, our previous work [2] explored the Alcubierre spacetime from a kinematical perspective, emphasizing the transition between Eulerian and Lagrangian frames in an inertial setting. This study highlighted the central role of expansion, shear, and vorticity in understanding the internal structure of the warp field and established a foundation for reinterpreting warp geometries within a Lagrangian framework. More recently, the theoretical landscape has evolved further with the work of Hamed Barzegar and Thomas Buchert [3], who critically assessed the standard formulation of warp metrics — referred to as "Restricted Warp Drives" (**R**-Warp) — and proposed a more general covariant formulation called the "Tilted Warp Drive" (**T**-Warp). Their analysis reveals that most existing models impose strong and often unphysical constraints, such as flow-orthogonality (**R1**), a prescribed one-component coordinate velocity field (**R2**) and flat spatial sections (**R3**). Although these constraints effectively suppress covariant motion, covariant vorticity, and spatial curvature, which we consider essential for realistic descriptions of motion through curved spacetime, the present work retains but fully exploits these restrictions.

This report extends these lines of inquiry by pursuing a fully geometric, metric-based approach (see Synge's G-method [4] and [5]), to Einstein's equations, as opposed to the alternative matter-based strategy where one specifies the energy-momentum content *a priori*. In the metric-based approach, one begins with a chosen spacetime geometry — defined through the lapse, shift, and spatial metric — and subsequently derives the implied energy-momentum tensor. While both methods are legitimate within general relativity, the geometric approach is particularly well-suited to the exploration of exact solutions and constraints on physically admissible configurations, especially in cases involving spacetime engineering such as warp drives.

Here, we consider a class of 3+1 decomposed spacetimes with constant lapse and a shift vector possessing only one non-zero component. This setting encompasses the Alcubierre model as a limiting case, while also allowing for more general configurations. Our objective is to construct new exact solutions that respect the full structure of the Einstein equations, free from some of the physically limiting assumptions typically imposed.

Moreover, it should be noted that we are making notable changes and corrections to the series of the three articles [6], [7] and [8], by Osvaldo L. Santos-Pereira, Everton M.C. Abreu and Marcelo B. Ribeiro: we correct the writing of the components of the Einstein tensor, and we provide an exact decomposition of the general energy-momentum tensor including anisotropic stresses and momentum flux with the correct component expressions. Finally, in these works, only an application to the Alcubierre model as an a priori hypothesis is made that does not allow for dynamical changes of the warp field.

This report is divided into three main sections. In the first section we present the theoretical framework of the 3+1 decomposition of general relativity, detailing the kinematical decomposition of motion and its implications for energy-momentum structure. Section 2 focuses on Einstein's equations in this formalism, with particular attention to constraint and evolution equations relevant to warp configurations. Finally, in the third section, we apply this framework to construct and analyze generalized warp drive spacetimes, constructing some examples and discussing the resulting physical implications on the Alcubierre model in particular.

1 General and restricted frameworks: kinematics

1.1 General 3+1 metric and kinematics

The description of the warp drive by Alcubierre [1] and later by Natário [9], adopted in the majority of the literature on this subject has many limitations. For this reason we compare in what follows a *classic* representation of warp drive spacetimes with a new more general one. We review the kinematics arising in the different descriptions following [3]. We first present the general properties of motion in the covariant framework of general relativity.

The description of a warp drive spacetime is made using the standard 3+1 decomposition of spacetime, i.e. we have the four-dimensional line-element:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \ = -(N^2 - N_k N^k)\mathrm{d}t^2 + 2N_i \mathrm{d}x^i \mathrm{d}t + h_{ij}\mathrm{d}x^i \mathrm{d}x^j \,, \tag{1}$$

with N the lapse function, $\boldsymbol{N} = N^i \boldsymbol{\partial}_i$ the shift vector, and the Riemannian spatial metric h_{ij}. We so decompose spacetime into spatial hypersurfaces labelled by the coordinate time t, Σ_t. We have the unit normal $\boldsymbol{n}$ to these hypersurfaces:

$$\boldsymbol{n} = \frac{1}{N}(1, -\boldsymbol{N}) \,, \quad \boldsymbol{n}^\flat = -N(1, \boldsymbol{0}) \,, \tag{2}$$

where the $\flat$ indices represent the metric dual 1-form of a given vector field tangent to the given manifold. Furthermore, using $\boldsymbol{v} = v^i \boldsymbol{\partial}_i$, the covariant spatial velocity field, the fluid 4-velocity $\boldsymbol{u}$ can be expressed as:

$$\boldsymbol{u} = \gamma(\boldsymbol{v})(\boldsymbol{n} + \boldsymbol{v}) \,, \tag{3}$$

with $n^\mu v_\mu = 0$ and $\gamma(\boldsymbol{v}) := -n^\mu u_\mu = (1 - v^\mu v_\mu)^{-1/2}$.
Here $\boldsymbol{v}$ is a measure of tilt between the normal frames and the fluid frames, also measured through the covariant Lorentz-factor $\gamma(\boldsymbol{v})$. So we can find the following expressions for $\boldsymbol{v}$ and $\boldsymbol{u}$:

$$\boldsymbol{v} = \frac{1}{N}(\boldsymbol{N} + \boldsymbol{V}) \,, \quad \boldsymbol{u} = \frac{\gamma(\boldsymbol{v})}{N}(N\boldsymbol{n} + \boldsymbol{N} + \boldsymbol{V}) \tag{4}$$

with

$$\frac{\gamma(\boldsymbol{v})}{N} = [N^2 - (N^i + V^i)(N_i + V_i)]^{-1/2} \,,$$

where

$$\boldsymbol{V} := \frac{d\boldsymbol{x}}{dt} \,,$$

is the fluid's *coordinate velocity* [3]. The components of $\boldsymbol{u}$ and $\boldsymbol{v}$ are:

$$\boldsymbol{u} = \frac{\gamma(\boldsymbol{v})}{N}(1, \boldsymbol{V}) \,, \quad \boldsymbol{u}^\flat = \frac{\gamma(v)}{N}(-N^2 + N^i(N_i + V_i), \boldsymbol{N}^\flat + \boldsymbol{V}^\flat). \tag{5}$$

We define also the vector field $\boldsymbol{\partial}_t$, called *time vector*, tangent to the congruence of coordinate observers' worldlines which, *a priori*, can be null, space- or time-like, and characterizes the lapse function and the shift vector field introduced above:

$$\boldsymbol{\partial}_t = N\boldsymbol{n} + \boldsymbol{N} \,, \quad N = -n_\mu(\boldsymbol{\partial}_t)^\mu \,, \quad N_i = h_{ij}(\boldsymbol{\partial}_t)^j \,. \tag{6}$$

1.2 Restricted 3+1 metric and kinematics

Now, we can describe the *classic* case like a "restricted warp drive", called **R**-Warp. In this case, we have three main restrictions. First, named **R1**, flow-orthogonality, the 4-velocity $\boldsymbol{u}$ is assumed to follow the normal congruence defined by $\boldsymbol{n}$. So $\boldsymbol{u} = \boldsymbol{n}$, we have no tilt. Second, named **R2**, lapse function and shift vector are fixed. The final restriction is the restriction to flat spatial hypersurfaces, **R3**. Current warp drives are based on the spatial coordinate velocity, denoted by $V = V_S(t, \boldsymbol{x})$, where $\boldsymbol{x}_S$ locates the spaceship S moving along the normal congruence $\boldsymbol{n}$ measured by the coordinate observer moving along $\hat{\boldsymbol{\partial}}_t$ at each hypersurface, where $\hat{\boldsymbol{\partial}}_t$ is a normalized time-like 4-velocity tangent to the observer's worldline, what we call *coordinate observer*:

$$\hat{\boldsymbol{\partial}}_t := |N^2 - N^k N_K|^{-1/2} \boldsymbol{\partial}_t \,. \tag{7}$$

We require in particular $N = 1$ and $\boldsymbol{N} = -\boldsymbol{V}_S$. Third is to consider only a reduced class of solutions on flat spatial hypersurfaces. In conclusion, we have $\boldsymbol{u}_S = (1, \boldsymbol{V}_S)$ and $\boldsymbol{u}^\flat_S = (-1, \boldsymbol{0})$ for the **R**-Warp model. The four-dimensional line-element (1) becomes:

$$\mathrm{d}s^2 = -(1 - V_k V^k)\mathrm{d}t^2 - 2V_i \mathrm{d}x^i \mathrm{d}t + h_{ij}\mathrm{d}x^i \mathrm{d}x^j \,, \tag{8}$$

To go beyond these limitations we have to create a new representation of motion, tilted warp drives, called **T**-Warp. In this representation $\boldsymbol{u}$ is not aligned with the normal $\boldsymbol{n}$ of the foliation. Moreover, we choose a Lagrangian coordinate system in order to compare with **R**-Warp more easily. Here the 4-velocity of the spaceship, the time vector and the 4-velocity of the coordinate observer are the same, $\boldsymbol{u}_S = \hat{\boldsymbol{\partial}}_t = \boldsymbol{\partial}_t$. We consider the same assumptions taken in [3]. In conclusion, we have $\boldsymbol{u}_S = (1, \boldsymbol{0})$ and $\boldsymbol{u}_S = (-1, \gamma(\boldsymbol{v}_S)\boldsymbol{v}^\flat_S)$. We can compare these two warp representations (see figure 1 in [3]). In this report we confine ourselves to the **R**-Warp concept.

2 General framework: dynamics

2.1 3+1 foliation and Einstein equations

From the line element (1), with the lapse function $N = 1$ and the shift vector $\boldsymbol{N}$, which will be later assumed to have a single component $N^x = -V$. We also take an Euclidean metric $h_{ij} = \delta_{ij}$. From this we obtain the components of the metric $g_{\mu\nu}$. Let us also introduce the operator $\boldsymbol{b} = b_{\alpha\beta}\boldsymbol{d}x^\alpha \otimes \boldsymbol{d}x^\beta$ that projects tensors onto the local rest frames of the fluid (orthogonal to $\boldsymbol{u}$):

$$b_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu \,, \quad b_{\alpha\mu}u^\alpha = 0 \,, \quad b^\mu{}_\alpha b^\alpha{}_\nu = b^\mu{}_\nu \,, \quad b^{\alpha\beta}b_{\alpha\beta} = 3 \,. \tag{9}$$

Note that the projector $\boldsymbol{b}$ reduces here to the projector $\boldsymbol{h} = h_{\alpha\beta}\boldsymbol{d}x^\alpha \otimes \boldsymbol{d}x^\beta$, where above $\boldsymbol{u}$ is replaced by the normal $\boldsymbol{n}$. They usually differ because of the tilt between $\boldsymbol{u}$ and $\boldsymbol{n}$, here we are in the special case where these two projectors are identical.

We place ourselves in a framework of general relativity, in order to analyze gravitational dynamics, we introduce several fundamental objects. We define Christoffel symbols that are used to express the covariant derivative:

$$\Gamma^\alpha{}_{\beta\mu} = \frac{1}{2} g^{\alpha\nu}\left(\partial_\beta g_{\mu\nu} + \partial_\mu g_{\beta\nu} - \partial_\nu g_{\beta\mu}\right) \,. \tag{10}$$

From this connection, we define the Riemann tensor, which encodes the curvature of spacetime:

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\nu\beta} - \partial_\nu \Gamma^\alpha{}_{\mu\beta} + \Gamma^\alpha{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\beta} - \Gamma^\alpha{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\beta} \,. \tag{11}$$

By contracting this tensor, we obtain the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R by contracting again:

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} \,, \quad R = g^{\mu\nu}R_{\mu\nu} \,. \tag{12}$$

From the Ricci tensor, we define the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \,. \tag{13}$$

This tensor directly links the geometry of spacetime to its energy-momentum content through Einstein's equations, including the cosmological constant Λ:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \,, \tag{14}$$

where $T_{\mu\nu}$ is the energy-moment tensor (EMT). This describes the distribution of energy, pressure and material flow in spacetime. To better understand its properties, we use its standard decomposition for a general fluid [4]:

$$T_{\mu\nu} = \epsilon u_\mu u_\nu + 2q_{(\mu}u_{\nu)} + p_{\mu\nu} \,, \tag{15}$$

with

$$\epsilon := u^\alpha u^\beta T_{\alpha\beta} \,, \quad q_\mu := -b^\alpha{}_\mu u^\beta T_{\alpha\beta} \,, \quad p_{\mu\nu} = p b_{\mu\nu} + \pi_{\mu\nu} := b^\alpha{}_\mu b^\beta{}_\nu T_{\alpha\beta} \,, \tag{16}$$

$$u^\mu q_\mu = 0\,, \quad u^\mu \pi_{\mu\nu} = 0\,, \quad b^{\mu\nu}\pi_{\mu\nu} = 0\,.$$

ϵ denotes the energy density of the fluid in its rest frames, q_μ the spatial momentum flux vector, p the isotropic pressure, and $\pi_{\mu\nu}$ the spatial and traceless anisotropic stress.

Now we introduce some kinematic properties of a fluid with the decomposition of the 4-covariant velocity u into the 4-covariant acceleration and the kinematic parts [10]:

$$u_{\nu;\mu} = -u_\mu a_\nu + \frac{1}{3}\Theta b_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}\,, \tag{17}$$

where:

$$\Theta = u^\mu{}_{;\mu}\,, \quad a_\mu := u^\mu u_{\mu;\nu}\,, \quad \sigma_{\mu\nu} = b^\alpha{}_\mu b^\beta{}_\nu u_{(\beta;\alpha)} - \frac{1}{3}\Theta b_{\mu\nu}\,, \quad \omega_{\mu\nu} = b^\alpha{}_\mu b^\beta{}_\nu u_{[\beta;\alpha]}\,, \tag{18}$$

with Θ the expansion rate, $\boldsymbol{a}$ the covariant acceleration of the fluid, $\boldsymbol{\sigma}$ the shear tensor and $\boldsymbol{\omega}$ the vorticity tensor.

From the property $T^{\mu\nu}{}_{;\nu} = 0$ [10] we deduce the energy conservation law,

$$u_\mu T^{\mu\nu}{}_{;\nu} = 0 \quad \Longleftrightarrow \quad \dot{\epsilon} + \Theta(\epsilon + p) = -a_\mu q^\mu - q^\mu{}_{;\mu} - \pi^{\mu\nu}\sigma_{\mu\nu}\,. \tag{19}$$

and the momentum conservation law:

$$b_{\alpha\mu} T^{\mu\nu}{}_{;\nu} = 0 \quad \Longleftrightarrow \quad -(\epsilon + p)a_\mu = b^\alpha{}_\mu p_{;\alpha} + b_{\mu\alpha}\dot{q}^\alpha + \frac{4}{3}\Theta q_\mu + q^\alpha(\sigma_{\alpha\mu} + \omega_{\alpha\mu}) + b_{\mu\alpha}\pi^{\alpha\beta}{}_{;\beta}\,. \tag{20}$$

To better understand how the energy-momentum tensor is related to the curvature, we can use the Gauss-Codazzi equations, which provide a connection between the intrinsic geometry of the hypersurfaces and the ambient spacetime geometry. The Gauss equation relates the curvature of the hypersurface to the curvature of the ambient spacetime via the Ricci tensors $R_{\mu\nu}$ and $\mathcal{R}_{ij}$ [11]:

$$R_{\mu\nu} b^\mu{}_i b^\nu{}_j = \mathcal{R}_{ij} - 2K_{ik}K^k{}_j + KK_{ij}\,. \tag{21}$$

We must examine the extrinsic curvature of the hypersurface Σ_t. The extrinsic curvature K_{ij} is a measure of how the hypersurface is curved within the ambient spacetime and plays a crucial role in understanding the flow of energy and momentum across the boundary of the hypersurface. It reflects the rate of change of the normal vector $\boldsymbol{n}$ along the hypersurface.

The extrinsic curvature of a spatial hypersurface Σ_t (the second fundamental form) embedded into the spacetime is given by the evolution of the metric (the first fundamental form):

$$K_{ij} = -\frac{1}{2}\mathcal{L}_n h_{ij}\,, \tag{22}$$

where h_{ij} is the metric induced on the hypersurface Σ_t, in other words, the restriction of the metric $g_{\mu\nu}$ to the hypersurface. $\mathcal{L}_n h_{ij}$ is the Lie derivative of the metric tensor, here along the normal vector field. The Lie derivative of a tensor h_{ij} along a vector field, here n^μ, is given by:

$$\mathcal{L}_n h_{ij} = n^\mu \partial_\mu h_{ij} + h_{i\mu}\partial_j n^\mu + h_{\mu j}\partial_i n^\mu\,. \tag{23}$$

Here we are in the restricted Euclidean case, $\partial_\mu h_{ij} = \partial_\mu \delta_{ij} = 0$, so that the extrinsic curvature reduces to:

$$K_{ij} = -\frac{1}{2}(\partial_j n_i + \partial_i n_j)\,. \tag{24}$$

2.2 Evolution and constraint equations

Einstein's equations provide the evolution equation for K_{ij} [11]:

$$\frac{\partial K_{ij}}{\partial t} = \mathcal{R}_{ij} + KK_{ij} - 2K_{ik}K^k{}_j + 4\pi[(p - \epsilon)h_{ij} - 2p_{ij}] - \Lambda h_{ij}\,, \tag{25}$$

with the pressure p and note the trace of $p_{ij} = p h_{ij} + \pi_{ij}$ is $3p$.
We can obtain the energy constraint (or Hamiltonian constraint) [12],

$$\mathcal{R} - K^i{}_j K^j{}_i + K^2 = 16\pi G\epsilon + 2\Lambda\,, \tag{26}$$

which can also be expressed as [3],

$$\mathcal{R} + 2\mathrm{II}(\boldsymbol{\Theta}) = 16\pi G\epsilon + 2\Lambda\,, \tag{27}$$

where $\mathrm{II}(\boldsymbol{\Theta})$ is the second principal invariant of the expansion tensor as defined by $\Theta_{ij} := -K_{ij}$. And the momentum constraints:

$$K^i{}_{j\,||i} - K_{||j} = 8\pi G q_j\,, \tag{28}$$

These constraints play a central role in 3+1 formalism (see lecture notes of Eric Gourgoulhon for more details [11]). These four constraint equations govern the evolution and consistency of spacetime slices over time, and the total number of independent components is thus 10, respectively 6, 1 and 3 for (25), (28) and (27), i.e. the same as the original Einstein equations.

3 Several restricted cases of R-Warp

3.1 Coordinate acceleration and coordinate vorticity

Since we assume that hypersurfaces are flat, we choose the coordinate representation in terms of globally inertial, nonrotating coordinates, as in most of the literature on warp drive spacetimes. Let us define the coordinate acceleration:

$$A^i(t, x^j) := \frac{\mathrm{d}}{\mathrm{d}t} V^i\,, \tag{29}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} := \partial_t + V^j \partial_j\,, \quad A^i(t, x^j) = \partial_t V^i + V^j \partial_j V^i\,. \tag{30}$$

The acceleration gradient reads:

$$A_{i,j} = \frac{\mathrm{d}}{\mathrm{d}t} V_{i,j} + V_{i,k} V^k{}_{,j}\;. \tag{31}$$

We have the following kinematic decomposition of the velocity gradient: [1]

$$V_{i,j} := V_{(i,j)} + V_{[i,j]} =: \Theta_{ij} + \Omega_{ij} = \frac{1}{3}\Theta\delta_{ij} + \Sigma_{ij} + \Omega_{ij}\,, \tag{33}$$

with Ω_{ij} the coordinate vorticity tensor and Σ_{ij} the shear tensor. From here we can replace the expression for the velocity gradient in (31):

$$A_{i,j} = \frac{\mathrm{d}}{\mathrm{d}t}\Theta_{ij} + \frac{\mathrm{d}}{\mathrm{d}t}\Omega_{ij} + (\Theta_{ik} + \Omega_{ik})(\Theta^k{}_j + \Omega^k{}_j)\,. \tag{34}$$

We will later refer to the trace part of the coordinate acceleration gradient,

$$\begin{aligned} A^j{}_{,j} &= \frac{\mathrm{d}}{\mathrm{d}t}\Theta^j_j + (V^j{}_{,k})(V^k{}_{,j})\,, \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{I}(\boldsymbol{\Theta}) + \mathrm{I}(\boldsymbol{\Theta})^2 - 2\mathrm{II}(\boldsymbol{\Theta}) + \Omega^2\,, \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{I}(\boldsymbol{\partial V}) + \mathrm{I}(\boldsymbol{\partial V})^2 - 2\mathrm{II}(\boldsymbol{\partial V})\,, \end{aligned} \tag{35}$$

where Ω^2 and Σ^2 refer to the scalar values defined below. By definition, vorticity is the rotational component of velocity:

$$\boldsymbol{\Omega} := \boldsymbol{\partial} \times \boldsymbol{V}\,. \tag{36}$$

We defined above the coordinate vorticity tensor Ω_{ij}; the coordinate vorticity vector Ω^i and the coordinate vorticity scalar Ω^2 are defined as follows:

$$\Omega_{ij} = V_{[i,j]}\,, \quad \Omega^i = -\frac{1}{2}\epsilon^{ijk}\Omega_{jk}\,, \quad \boldsymbol{\Omega}^2 = \Omega^i\Omega_i = \frac{1}{2}\Omega_{ij}\Omega^{ij} =: \Omega^2\,. \tag{37}$$

[1] We denote symmetrization and anti-symmetrization as follows:

$$V_{(i,j)} = \frac{1}{2}(V_{i,j} + V_{j,i})\,, \quad V_{[]i,j]} = \frac{1}{2}(V_{i,j} - V_{j,i})\,. \tag{32}$$

The definition of the shear tensor Σ_{ij} and the shear scalar $\boldsymbol{\Sigma}^2$ are:

$$\Sigma_{ij} = V_{(i,j)} - \frac{1}{3}\Theta\delta_{ij}\,, \quad \boldsymbol{\Sigma}^2 = \frac{1}{2}\Sigma_{ij}\Sigma^{ij} =: \Sigma^2\,. \tag{38}$$

Moreover, here I and II are the first and second principal scalar invariants of the velocity gradient denoted by $\boldsymbol{\partial V}$:

$$\begin{aligned} \mathrm{I}(\Theta) = \mathrm{I}(\boldsymbol{\partial V}) = \boldsymbol{\Theta}\,, \quad \mathrm{II}(\boldsymbol{\partial V}) = \frac{1}{3}\boldsymbol{\Theta}^2 - \boldsymbol{\Sigma}^2 + \boldsymbol{\Omega}^2\,, \\ 2\mathrm{II}(\boldsymbol{\partial V}) = 2\mathrm{II}(\Theta) + 2\Omega^2 = \boldsymbol{\partial}\cdot[\boldsymbol{V}(\boldsymbol{\partial}\cdot\boldsymbol{V}) - (\boldsymbol{V}\cdot\boldsymbol{\partial})\boldsymbol{V}]\,, \end{aligned} \tag{39}$$

and we refer to the anti-symmetric part of the coordinate acceleration gradient [2]:

$$A_{[i,j]} = \frac{\mathrm{d}}{\mathrm{d}t}\Omega_{ij} + 2\Theta_{k[i}\Omega^k_{\ j]}\,, \tag{40}$$

which separates the effects of expansion rate and vorticity in the dynamics. Note that the time-evolution of Θ_{ij} is governed by the Einstein equations (25) for $K_{ij} = -\Theta_{ij}$.

Now that we understand the link between coordinate acceleration and coordinate vorticity, we are going to take a closer look. From (40), with coordinate vorticity vector $\boldsymbol{\Omega}$, we recover the transport equation for the vorticity:

$$\boldsymbol{\partial}\times\boldsymbol{A} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Omega} + \boldsymbol{\Omega}(\boldsymbol{\partial}\cdot\boldsymbol{V}) - (\boldsymbol{\Omega}\cdot\boldsymbol{\partial})\boldsymbol{V}\,. \tag{41}$$

3.2 Subcase: one-component coordinate velocity

With this background established, we are going to focus on a subcase that we will deal with in detail. In this case, we consider a single-component velocity field, $\boldsymbol{V}(t,x,y,z) = V(t,x,y,z)\boldsymbol{e}_x$. This implies:

$$\boldsymbol{V} = (V,0,0)\,, \quad \boldsymbol{A} = (A,0,0)\,. \tag{42}$$

We distinguish two cases: one without coordinate vorticity and a second, complete case, with it. If $\boldsymbol{V}(t,x)$ has only one spatial dependence we do not have coordinate vorticity, on the contrary if $\boldsymbol{V}(t,x,y,z)$ we can have it.

We therefore obtain the following components for coordinate vorticity:

$$\boldsymbol{\Omega} = \Big(0, \quad \partial_z V, \quad -\partial_y V\Big)\,, \tag{43}$$

$$(\Omega_{ij}) = \begin{pmatrix} 0 & \frac{1}{2}\partial_y V & \frac{1}{2}\partial_z V \\ -\frac{1}{2}\partial_y V & 0 & 0 \\ -\frac{1}{2}\partial_z V & 0 & 0 \end{pmatrix}\,, \quad \Omega^2 = \frac{1}{4}\Big[(\partial_y V)^2 + (\partial_z V)^2\Big]\,. \tag{44}$$

The curl of $\boldsymbol{\Omega}$ is:

$$\boldsymbol{\partial}\times\boldsymbol{\Omega} = (-\partial_y^2 V - \partial_z^2 V, \quad \partial_x(\partial_y V), \quad \partial_x(\partial_z V))\,. \tag{45}$$

We notice immediately that without coordinate vorticity all these terms are zero. We calculate also the components of the transport equation for coordinate vorticity (41):

$$\boldsymbol{\partial}\times\boldsymbol{A} = \begin{pmatrix} 0 \\ \partial_z A \\ -\partial_y A \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_t\Omega^y + V\partial_x\Omega^y + \Omega^y\partial_x V \\ \partial_t\Omega^z + V\partial_x\Omega^z + \Omega^z\partial_x V \end{pmatrix}\,. \tag{46}$$

These expressions and combinations of them will enable us to greatly simplify the calculations that follow.

Moreover, in the particular case of one component, the trace of the extrinsic curvature (24), which in this context equals to $K := -\Theta$, becomes:

$$\Theta = \partial_x V\,, \tag{47}$$

[2] For simpler notation: $\Theta_{k[i}\Omega^k_{\ j]} = \frac{1}{2}\left(\Theta_{ki}\Omega^k_{\ j} - \Theta_{kj}\Omega^k_{\ i}\right)$

and the expansion and the shear tensor coefficients (33), can be expressed in components [2]:

$$(\Theta_{ij}) = \begin{pmatrix} \partial_x V & \frac{1}{2}\partial_y V & \frac{1}{2}\partial_z V \\ \frac{1}{2}\partial_y V & 0 & 0 \\ \frac{1}{2}\partial_z V & 0 & 0 \end{pmatrix}, \quad (\Sigma_{ij}) = \begin{pmatrix} \frac{2}{3}\partial_x V & \frac{1}{2}\partial_y V & \frac{1}{2}\partial_z V \\ \frac{1}{2}\partial_y V & -\frac{1}{3}\partial_x V & 0 \\ \frac{1}{2}\partial_z V & 0 & -\frac{1}{3}\partial_x V \end{pmatrix}, \tag{48}$$

for $V(t,x,y,z)$.

To describe the main properties of the chosen models, we need to detail the components of Einstein's equations (14). To do this, we use *SageMath* [13] and its SageManifolds tool. We start with the shape of the metric and after computation we obtain the 10 Einstein's equations (see appendix A for more details).

By obtaining all its components, we can identify the results obtained with *SageManifolds*, and decompose the velocity-dependent components as a function of kinematical and dynamical variables. We can therefore obtain analytical solutions to Einstein's equations.

Here we look in particular the following quantities projected onto the hypersurface orthogonal to the vector n^μ. In this particular point of view, the EMT and the Einstein tensor in mixed indices notation becomes (compare to the EMT $T_{\mu\nu}$ and $G_{\mu\nu}$ in appendix A):

$$(T^\mu{}_\nu) = \begin{pmatrix} -\epsilon + q_0 & q^x & q^y & q^z \\ -\epsilon V - q^x + V q_0 - V p + \pi_{10} & p + \pi^x{}_x + q^x V & \pi^x{}_y + q^y V & \pi^x{}_z + q^z V \\ -q^y + \pi_{20} & \pi^y{}_x & p + \pi^y{}_y & \pi^y{}_z \\ -q^z + \pi_{30} & \pi^z{}_x & \pi^z{}_y & p + \pi^z{}_z \end{pmatrix}, \tag{49}$$

$$(G^\mu{}_\nu) = \begin{pmatrix} -G_{00} - VG_{10} & -G_{01} - VG_{11} & -G_{02} - VG_{12} & -G_{03} - VG_{13} \\ -VG_{00} + (-V^2+1)G_{10} & -VG_{01} + (-V^2+1)G_{11} & -VG_{02} + (-V^2+1)G_{12} & -VG_{03} + (-V^2+1)G_{13} \\ G_{20} & G_{21} & G_{22} & G_{23} \\ G_{30} & G_{31} & G_{32} & G_{33} \end{pmatrix}.$$

In addition, the mixed stress tensor $p^\mu{}_\nu$ and the momentum flux vector q^μ become purely spatial, they can be expressed as follows (see appendix A):

$$(p^\mu{}_\nu) = h^{\mu\alpha} p_{\alpha\nu} = g^{\mu k} p_{k\nu} + n^\alpha n^\mu p_{\alpha\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -Vp + \pi_{10} & p + \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{20} & \pi_{21} & p + \pi_{22} & \pi_{23} \\ \pi_{30} & \pi_{31} & \pi_{32} & p + \pi_{33} \end{pmatrix}, \quad q^\mu = h^{\mu\alpha} h^\sigma{}_\alpha n^\nu T_{\sigma\nu}\,. \tag{51}$$

$p_{\mu\nu}$ is said to be purely spatial if it satisfies two essential conditions which we verify in our case:

$$\begin{aligned} &1. \quad p_{\mu\nu} h^\mu{}_\alpha h^\nu{}_\beta = p_{\alpha\beta}\,, \\ &2. \quad p_{\mu\nu} n^\mu n^\nu = 0 \quad \Rightarrow \quad p_{00} + 2Vp_{01} + V^2 p_{11} = 0\,. \end{aligned} \tag{52}$$

In the same way we have $\pi^\mu{}_\nu = h^{\mu\alpha}\pi_{\alpha\nu}$ which can be read in the stress tensor expression. And the mixed shear tensor $\sigma^\mu{}_\nu = h^{\mu\alpha}\sigma_{\alpha\nu}$ (18):

$$\sigma^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -V\frac{2}{3}\partial_x V & \frac{2}{3}\partial_x V & \frac{1}{2}\partial_y V & \frac{1}{2}\partial_z V \\ -V\frac{1}{2}\partial_y V & \frac{1}{2}\partial_y V & -\frac{1}{3}\partial_x V & 0 \\ -V\frac{1}{2}\partial_z V & \frac{1}{2}\partial_z V & 0 & -\frac{1}{3}\partial_x V \end{pmatrix}. \tag{53}$$

It is important to note that a spatial tensor can have temporal components if it respects the orthogonality relations (see appendix B). [3]

In the mixed representation it is easier to have an intuitive meaning of the physical quantities we study. Firstly, the tensor traces can be calculated immediately, and secondly we can read the momentum flux vector instantly.

To see what this implies for other representations see appendix B.

3.3 Case 1: without coordinate vorticity (Lemma)

We are going to prove the following Lemma.

Lemma 1. *Any solution to the Einstein equations in a 3+1 formalism of general relativity, with lapse $N = 1$, shift vector $\boldsymbol{N} = (-V(t,x), 0, 0)$, where the single coordinate velocity component $V(t,x)$ only depends on one spatial and a temporal coordinate, and for an Euclidean spatial metric $h_{ij} = diag(1,1,1)$ corresponds to a spacetime with the following properties:*

The metric can be written in global inertial (nonrotating) coordinates:

$$\begin{aligned} \mathrm{d}s^2 &= -(1 - V^2(t,x))\mathrm{d}t^2 + 2V(t,x)\mathrm{d}t\mathrm{d}x + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= -\mathrm{d}t^2 + (\mathrm{d}x + V(t,x)\mathrm{d}t)(\mathrm{d}x + V(t,x)\mathrm{d}t) + \mathrm{d}y^2 + \mathrm{d}z^2 \,, \end{aligned} \tag{54}$$

where $V(t,x)$ obeys $\partial_t V + V\partial_x V = A(t,x)$, and $p + \pi^x{}_x = -\epsilon = \frac{\Lambda}{8\pi G}$, $\boldsymbol{q} = \boldsymbol{0}$, and where the stress tensor takes the following form:

$$(p^\mu{}_\nu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -Vp + \pi^x{}_0 & p + \pi^x{}_x & 0 & 0 \\ 0 & 0 & p - \frac{1}{2}\pi^x{}_x & \\ 0 & 0 & 0 & p - \frac{1}{2}\pi^x{}_x \end{pmatrix} .$$

The Einstein equations reduce to the form:

$$\begin{aligned} &G^\mu{}_\nu + \Lambda\delta^\mu{}_\nu - 8\pi G T^\mu{}_\nu = 0 \quad \Leftrightarrow \\ &diag(0, \quad 0, \quad -\partial_x A, \quad -\partial_x A) + diag(\Lambda, \Lambda, \Lambda, \Lambda) - 8\pi G\, diag\left(\frac{\Lambda}{8\pi G}, \quad \frac{\Lambda}{8\pi G}, \quad \frac{3}{2}\pi^x{}_x, \quad \frac{3}{2}\pi^x{}_x\right) = 0 \,, \end{aligned}$$

so that the only nontrivial equations constrain the coordinate acceleration:

$$\boldsymbol{\partial} \cdot \boldsymbol{A} = \partial_x A = \Lambda - 4\pi G(\epsilon + 3p) = \frac{3}{2}(\Lambda - 8\pi G p) = 12\pi G \pi^x{}_x \;;\; \boldsymbol{\partial} \times \boldsymbol{A} = \boldsymbol{0} \,.$$

Furthermore, no additional conditions arise from the conservation equations.

For a perfect fluid source, $\pi^x{}_x = 0$, the motion is without loss of generality inertial, $A = 0$, and $p = -\epsilon$.

Proof.

- We start with the metric (54), where the coordinate velocity has the following form: $V(t,x)$. Here we deal with mixed components like it is expressed in (51), we just obtain three diagonal components for $p^i{}_j$ and $q^i = h^{ij}q_j = q_i$.

  After calculating the elements of the EMT tensor decomposition for a fluid (16), we obtain by comparing with Einstein's equations (125):

$$\epsilon = -\frac{\Lambda}{8\pi G} \,, \quad q^i = q_i = (0,0,0) \,. \tag{55}$$

- For the stress tensor in (49), We first notice that $p^y{}_y = p^z{}_z$, so we also have $\pi^y{}_y = \pi^z{}_z$. We can express all terms as functions of $\pi^x{}_x$, because of the trace-free condition for $\pi^i{}_j$:

$$\pi^x{}_x + 2\pi^y{}_y = 0 \quad \Rightarrow \quad \pi^y{}_y = -\frac{1}{2}\pi^x{}_x \,. \tag{56}$$

[3] It is interesting to note that a single projection with the inverse metric $g^{\mu\nu}$ alone is not enough to conclude that they are spatial, for which we need to add the orthogonality conditions. Unlike a projection with the inverse metric $h^{\mu\nu}$, which implicitly includes orthogonality.

The Einstein equations give a relation between the stress tensor and the coordinate velocity:

$$(p^{\mu}{}_{\nu}) = \frac{1}{8\pi G}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -V\Lambda & \Lambda & 0 & 0 \\ 0 & 0 & -(\partial_x V)^2 - V\partial_x^2 V - \partial_t(\partial_x V) + \Lambda & 0 \\ 0 & 0 & 0 & -(\partial_x V)^2 - V\partial_x^2 V - \partial_t(\partial_x V) + \Lambda \end{pmatrix}. \tag{57}$$

- We recognize the coordinate acceleration (31) in $p^y{}_y$ and $p^z{}_z$. Now we calculate the trace $p^{\mu}{}_{\mu}$:

$$p^{\mu}{}_{\mu} = p^i{}_i = p^x{}_x + p^y{}_y + p^z{}_z = \frac{1}{8\pi G}(3\Lambda - 2\partial_x A) = 3p\,, \tag{58}$$

$$\Rightarrow \quad p = \frac{1}{8\pi G}\left(\Lambda - \frac{2}{3}\partial_x A\right)\,. \tag{59}$$

We deduce the form the the acceleration divergence:

$$\partial_x A = \frac{3}{2}(\Lambda - 8\pi G p) = 12\pi G \pi^x{}_x \quad (i)\,,$$

where in the second expression we used (49). Therefore we have:

$$p + \pi^x{}_x = \frac{\Lambda}{8\pi G} \quad (ii)\,.$$

- Note that the only non-diagonal component $p^x{}_0$ gives us the same information if we use the expression of $T^x{}_0$ from (49). Because of (123), $\pi_{x0} = -VT_{xx} + pV$ and $T_{xx} = p + \pi_{xx}$ so we deduce again:

$$-\epsilon V - q^x + Vq_0 - Vp + \pi^x{}_0 = 0 \quad \Rightarrow \quad -\epsilon V - VT_{xx} = 0 \quad \Rightarrow \quad p + \pi^x{}_x = \frac{\Lambda}{8\pi G}\,.$$

- We now consider the conservation laws (19) and (20).

  - The energy conservation equation (19) gives us:

$$\dot\epsilon + \Theta(\epsilon + p) = -\pi^i{}_j\sigma^j{}_i \quad \Rightarrow \quad \partial_x V(\epsilon + p + \pi^x{}_x) = 0 \quad (iii)\,,$$

where $\dot\epsilon = 0$ because the energy density (55) is constant here, and we used the shear tensor components from (48) for the last term:[4]

$$\pi^i{}_j\sigma^j{}_i = \frac{2}{3}\pi^x{}_x\partial_x V - \frac{2}{3}\pi^y{}_y\partial_x V = \frac{2}{3}\pi^x{}_x\partial_x V + \frac{1}{3}\pi^x{}_x\partial_x V = \pi^x{}_x\partial_x V\,.$$

Using (55), we see that the relation (iii) is equivalent to (ii) for nonvanishing velocity divergence.

  - The momentum conservation law (20), with $q_i = (0,0,0)$ and $q^i = h^{ij}q_j = (0,0,0)$, leaves us with:

$$a^{\mu} = (\partial_x p + \partial_x\pi^x{}_x, \quad \partial_y p + \partial_y \pi^y{}_y, \quad \partial_z p + \partial_z\pi^y{}_y) \quad (iv - a)\,.$$

The covariant acceleration a^{μ} is zero because the lapse is constant, from [11]:

$$a^{\mu} = h^{\mu\nu}[\ln(N)]_{||\nu}\,, \text{so} \quad a^{\mu} = 0 \quad (iv - b)\,, \tag{60}$$

so that, together with (56), we obtain:

$$\partial_x(p + \pi^x{}_x) = 0\ ,\ \partial_y(p - (1/2)\pi^x{}_x) = 0\ ,\ \partial_z(p - (1/2)\pi^x{}_x) = 0\ . \quad (v)$$

We conclude from (iv) and (v) that $p + \pi^x{}_x = f(t)$, and then from (iii) that $f(t) = \frac{\Lambda}{8\pi G}$. From (i) and (iii) we then have:

$$\partial_x A = 12\pi G\pi^x{}_x \quad (vi)\,,$$

which is equivalent to (i). Notice that in (v) the y- and z-derivatives vanish, since $A(t,x)$, so that p and $\pi^x{}_x$ are also, via (i), only functions of (t,x) in general.

- Relation (v) shows that for a perfect fluid source the divergence of $A(t,x)$ is zero, so that with (ii) and (55) we have $p = -\epsilon$ and $A = A(t)$. This time-dependence can be removed by a global translation on flat space so that, without loss of generality, we have $A = 0$ in this case.

□

[4] Note that $\pi^x{}_0 \neq 0$ and $\sigma^x{}_0 \neq 0$, but $\pi^0{}_x = 0$ and $\sigma^0{}_x = 0$ so the sum on this particular indices $\pi^x{}_0\sigma^0{}_x + \pi^0{}_x\sigma^x{}_0 = 0$. And that brings us back to the sum of spatial indices.

3.4 Discussion of Case 1

Having obtained the general form of the equations and the constraints they must satisfy, we are going to look more specifically at the general solution as well as the admissible coordinate velocity profiles and how we can find them. To do this, we will use the results obtained during my Master 1 internship [2]. We are going to switch from an Eulerian to a Lagrangian study.

According to **Lemma 1**, if we assume a perfect fluid source, ${\pi^x}_x = 0$, then $p = -\epsilon$, and we obtain without loss of generality the system of inertial motion for $V(x,t)$, and we have Euler's equation governing inertial motion according to **Lemma 1** for $A = 0$. The system for inertial motion has been studied in [2]:

$$\frac{\partial}{\partial t}\boldsymbol{V} + (\boldsymbol{V}\cdot\boldsymbol{\partial})\boldsymbol{V} = \boldsymbol{0}\ , \quad \frac{\partial}{\partial t}V^i + V^j {V^i}_{,j} = 0\ . \tag{61}$$

We will show examples of this case. In the Lagrange point of view, we follow the trajectories $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{X}, t)$ of a fluid parcel, where $\boldsymbol{X}$ labels trajectories of fluid elements. This transformation is possible since spatial sections are flat. The solution of the above coupled nonlinear system of differential equations (61) can be solved in the Lagrangian coordinate system:

$$\frac{\mathrm{d}}{\mathrm{d}t}V^i = \frac{\partial}{\partial t}\Big|_{\boldsymbol{X}} V^i = 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}t} := \frac{\partial}{\partial t}\Big|_{\boldsymbol{x}} + (\boldsymbol{v}\cdot\boldsymbol{\partial}) = \frac{\partial}{\partial t}\Big|_{\boldsymbol{X}}\ . \tag{62}$$

The solution of (61) is a constant velocity field along trajectories:

$$V^i(t, X^k) = V_0^i(X^k)\ . \tag{63}$$

We integrate the previous equation and assume that at $t = t_0$, the position at that moment is $f^i(t_0, X^k) = X^i$, so we obtain the mapping from Lagrangian to Eulerian coordinates as follows:

$$x^i = f^i(t, X^j) = X^i + V_0^i(X^j)(t - t_0)\,, \tag{64}$$

with x^i the Eulerian position coordinate. We deduce from this that $X^k = h^k(t, x^j)$, where $\boldsymbol{h} := \boldsymbol{f}^{-1}$. With the help of the inverse mapping $\boldsymbol{h}$ we are then able to express the admissible functions of the velocity in the Eulerian frame. So we express X^k from (64):

$$V^i(t, x^j) = V_0^i(X^k = h^k(x^j, t))\,, \tag{65}$$

with $V_0^i(X^k)$ the initial velocity field components.

We remind here that the Jacobian is:

$$J(t, X^k) = \det\left(\frac{\partial f^i(t, X^k)}{\partial X^j}\right) = \det\left(\delta_{ij} + \frac{\partial V_0^i}{\partial X^j}(t - t_0)\right)\ , \tag{66}$$

which describes how a change in the coordinates of a reference system X affects the variables of the system x, specifically how the position x changes with respect to X in the given flow.

In the present case of a one-component velocity, we now study concrete examples.

3.4.1 Example 1

Using a given initial condition,

$$V_0(X^k) = a\sin(bX)\,, \quad a = const.\,, \quad b = const.\,, \tag{67}$$

we obtain from (65) the Eulerian positions and determine $V(t, x)$:

$$x = f(t, X) = X + a\sin(bX)(t - t_0) \Rightarrow V(t, x) = a\sin(bh(t, x))\ . \tag{68}$$

For this example we solve the velocity expression in Eulerian coordinates numerically, starting from the initial conditions V_0. For that, we must invert the map $x = f(t, X)$ to find $X = h(t, x)$, which we do for this example representing all cases that are not analytically solvable. We use numerical root-finding methods to determine X, and thereafter we can compute $V(t, x)$.

We choose arbitrary values for a and b to represent the velocity curves obtained, figure 1:

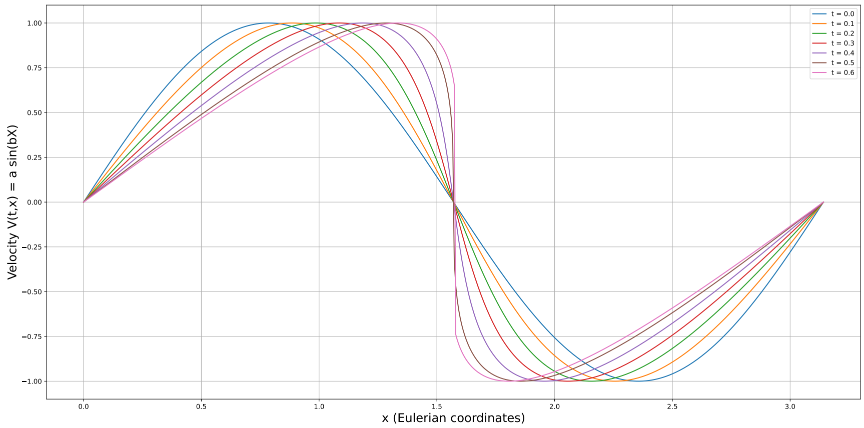


Figure 1: Example of the velocity in Eulerian space, $V(t,x) = a\sin(bh(t,x))$, over the range of times $t = [0, 0.6]$ versus x. Here we take $t_0 = 0$, $a = 1$ and $b = 2$.

As expected from the above-mentioned work [2], a caustic is formed. This happens around $t \approx 0.5$, we can see the velocity is multi-valued after this time. This form of velocity profile looks complex but the motion is still inertial $A = 0$.

We can verify that the trace of V is simply the divergence of the velocity field also known as rate of expansion:

$$\mathrm{I} = \partial_x V = \frac{\dot{J}}{J} = \frac{ab\cos(bX)}{1 + ab\cos(bX)(t - t_0)}\,, \tag{69}$$

where J denotes the Jacobian (66). The expansion rate as well as other variables can be treated according to the inversion above.

3.4.2 Example 2

Using as initial condition,

$$V_0(X^k) = 1 - \frac{1}{2}X^2\,, \tag{70}$$

we obtain from (65) $V(t,x)$ explicitly because it admits an explicit expression for the inverse mapping:

$$x = f(t,X) = X + \left(1 - \frac{1}{2}X^2\right)(t - t_0)\,, \tag{71}$$

$$\Rightarrow V(t,x) = 1 - \frac{1}{2}X^2 = 1 - \frac{1}{2(t-t_0)^2}\left(1 \pm \sqrt{1 - 2(t-t_0)(x - (t - t_0))}\right)^2 . \tag{72}$$

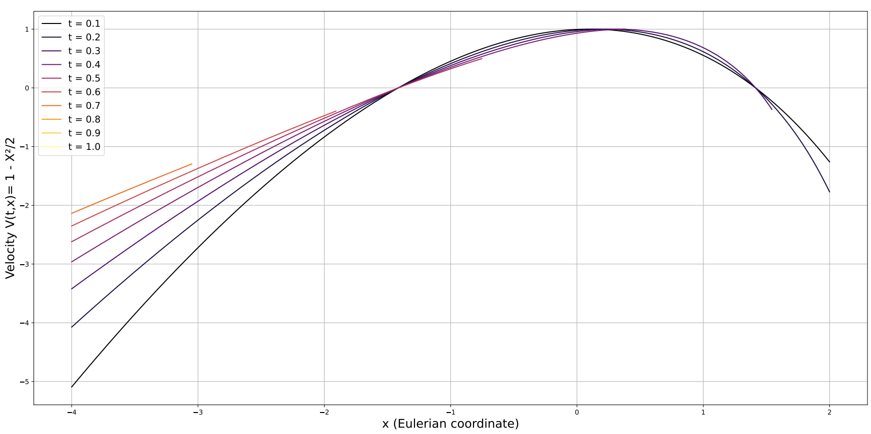


Figure 2: Example of the velocity profile in Eulerian space, $V(t,x)$, see (71), over the range of times $t = [0.1, 1.0]$ versus x. Here we take $t_0 = 0$.

It should be noted that the regime remains laminar as long as the square root has a real value $1 - 2(t - t_0)(x - (t - t_0)) \geq 0$, i.e. over an interval: $x \leq \frac{1+2(t-t_0)^2}{2(t-t_0)}$. And this is under the condition $t \neq t_0$. We obtain the following curves, figure 2.

We can see the formation of a caustic as in the previous example, but here the trajectories were cut off when the boundary of x is reached. Finally, the motion is still inertial $A = 0$, for the same reasons as before, and we can explicitly calculate relevant variables, e.g. the rate of expansion:

$$\mathrm{I} = \partial_x V = \frac{\dot{J}}{J} = \frac{-X}{1 - X(t - t_0)} = \frac{-1 \pm \sqrt{1 - 2(t - t_0)(x - (t - t_0))}}{(t - t_0)\sqrt{1 - 2(t - t_0)(x - (t - t_0))}} \,. \tag{73}$$

We now move to the more interesting general one-component case.

3.5 Case 2: with coordinate vorticity (Theorem)

We are going to prove the following theorem for the general one-component case.

Theorem 1. *Any solution to the Einstein equation in a 3+1 formalism of general relativity, with lapse $N = 1$, shift vector $\boldsymbol{N} = (-V(t, x, y, z), 0, 0)$, where the single coordinate velocity component $V(t, x, y, z)$ depends on three spatial coordinates and a temporal coordinate, and for an Euclidean spatial metric $h_{ij} = diag(1, 1, 1)$, corresponds to a spacetime with the following properties:*

The metric can be written in global inertial (nonrotating) coordinates:

$$\mathrm{d}s^2 = -(1 - V^2(t, x, y, z))\mathrm{d}t^2 + 2V(t, x, y, z)\mathrm{d}t\mathrm{d}x + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,, \tag{74}$$

where $V(t, x, y, z)$ obeys $\partial_t V + V\partial_x V = A(t, x, y, z)$, the energy density and the momentum flux density attain the following forms:

$$\epsilon = -\frac{1}{8\pi G}\left(\Omega^2 + \Lambda\right) \,, \quad \boldsymbol{q} = \frac{1}{8\pi G}(\boldsymbol{\partial} \times \boldsymbol{\Omega}) \,. \tag{75}$$

The mixed stress tensor takes the following form:

$$(p^{\mu}{}_{\nu}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -Vp + \pi_{10} & p + \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{20} & \pi_{21} & p + \pi_{22} & \pi_{23} \\ \pi_{30} & \pi_{31} & \pi_{32} & p + \pi_{33} \end{pmatrix} .$$

The Einstein equations reduce to the form:

$$G^{\mu}{}_{\nu} + \Lambda\delta^{\mu}{}_{\nu} - 8\pi G T^{\mu}{}_{\nu} = 0 \quad \Leftrightarrow$$
$$\begin{pmatrix} -\Omega^2 + \frac{2}{3}\partial_x A & \partial_y\Omega^z - \partial_z\Omega^y & -\partial_x\Omega^z & \partial_x\Omega^y \\ -(V^2+1)(\partial_y\Omega^z - \partial_z\Omega^y) + 4\Omega^2 & -3\Omega^2 + V(\partial_y\Omega^z - \partial_z\Omega^y) & \frac{1}{2}(\partial_y A - \Omega^z\partial_x V) - \partial_x\Omega^z & \frac{1}{2}(\partial_z A + \Omega^y\partial_x V) + \partial_x\Omega^y \\ -8\pi G q^y + \frac{1}{2}(-V\partial_y A + \partial_x V\Omega^z) & \frac{1}{2}(\partial_y A - \Omega^z\partial_x V) & -\partial_x A + \Omega^2 - \frac{1}{2}(\Omega^y)^2 & -\frac{1}{2}\Omega^y\Omega^z \\ -8\pi G q^z + \frac{1}{2}(-V\partial_z A - \partial_x V\Omega^y) & \frac{1}{2}(\partial_z A + \Omega^y\partial_x V) & -\frac{1}{2}\Omega^y\Omega^z & -\partial_x A + \Omega^2 - \frac{1}{2}(-\Omega^z)^2 \end{pmatrix}$$
$$+ \begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ 0 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & \Lambda \end{pmatrix} - 8\pi G \begin{pmatrix} -\epsilon - Vq^x & q^x & q^y & q^z \\ -\epsilon V - q^x - V^2 q^x - Vp + \pi^x{}_0 & q^x V + p\pi^x{}_x & q^y V + \pi^x{}_y & q^z V + \pi^x{}_z \\ -q^y + \pi^y{}_0 & \pi^y{}_x & \pi^y{}_y + p & \pi^z{}_x \\ -q^z + \pi^z{}_0 & \pi^z{}_x & \pi^z{}_y & \pi^z{}_z + p \end{pmatrix} = 0 \,,$$

so that the only nontrivial equations constrain the coordinate acceleration:

$$\boldsymbol{\partial} \cdot \boldsymbol{A} = \partial_x A = \frac{3}{2}\left(\Lambda - \Omega^2 - 8\pi G p\right) = \Lambda - 4\pi G(\epsilon + 3p) - 2\Omega^2 = 12\pi G \pi^x{}_x + 3\Omega^2 \,. \tag{78}$$

$$\boldsymbol{\partial} \times \boldsymbol{A} = \begin{pmatrix} 0 \\ \partial_z A \\ -\partial_y A \end{pmatrix} = \begin{pmatrix} 0 \\ 16\pi G \pi^x{}_z - \Omega^y \partial_x V \\ -16\pi G \pi^x{}_y - \Omega^z \partial_x V \,. \end{pmatrix} \tag{79}$$

Furthermore, an evolution equation for the momentum flux density arises from the momentum conservation law:

$$-\boldsymbol{\partial} \cdot \boldsymbol{p} = \dot{q}^i + \frac{4}{3}\Theta q^i + q^j \sigma_j{}^i \,.$$

For a perfect fluid source, $\pi^x{}_x = 0$ and $\boldsymbol{q} = 0$, the motion reduces to the case of ***Lemma 1****.*

Proof.

- We start with the metric (74), where the coordinate velocity has the following form: $V(t,x,y,z)$. After calculating the elements of the EMT tensor decomposition for a fluid (16), we obtain by comparing with (125):

$$\epsilon = -\frac{1}{32\pi G}\left[(\partial_y V)^2 + (\partial_z V)^2 + 4\Lambda\right] \,, \tag{80}$$

$$q^i = \frac{1}{16\pi G}\left(\quad -\partial_y^2 V - \partial_z^2 V \,, \quad \partial_x(\partial_y V) \,, \quad \partial_x(\partial_z V) \right) \,, \tag{81}$$

$$\begin{aligned}
p^x{}_0 &= \frac{1}{32\pi G} V\left[3(\partial_y V)^2 + 3(\partial_z V)^2 - 4\Lambda\right] + \frac{1}{16\pi G}(\partial_y^2 V - \partial_z^2 V) \,, \\
p^y{}_0 &= -\frac{1}{16\pi G}[(V^2+1)\partial_x(\partial_y V) + 2V\partial_x V \partial_y V + V\partial_t(\partial_y V)] \,, \\
p^z{}_0 &= -\frac{1}{16\pi G}[(V^2+1)\partial_x(\partial_z V) + 2V\partial_x V \partial_z V + V\partial_t(\partial_z V)] \,, \\
p^x{}_x &= -\frac{1}{32\pi G}\left[3(\partial_y V)^2 + 3(\partial_z V)^2 - 4\Lambda\right] \,, \\
p^y{}_y &= -\frac{1}{32\pi G}\left\{4\left[(\partial_x V)^2 + V\partial_x^2 V - \Lambda + \partial_t(\partial_x V)\right] - (\partial_y V)^2 + (\partial_z V)^2\right\} \,, \\
p^z{}_z &= -\frac{1}{32\pi G}\left\{4\left[(\partial_x V)^2 + V\partial_x^2 V - \Lambda + \partial_t(\partial_x V)\right] + (\partial_y V)^2 - (\partial_z V)^2\right\} \,, \\
p^x{}_y &= p^y{}_x = \frac{1}{16\pi G}[V\partial_x(\partial_y V) + 2\partial_x V \partial_y V + \partial_t(\partial_y V)] \,, \\
p^x{}_z &= p^z{}_x = \frac{1}{16\pi G}[V\partial_x(\partial_z V) + 2\partial_x V \partial_z V + \partial_t(\partial_z V)] \,, \\
p^y{}_z &= p^z{}_y = \frac{1}{16\pi G}\partial_y V \partial_z V \,.
\end{aligned} \tag{82}$$

- We can note two important remarks:
  First, the only non-diagonal component $p^i{}_0$ gives us the same information of what follows, if we use the expression of T^i_0 (49). We have: $p^x{}_0 = -VT_{xx}$, $p^y{}_0 = -VT_{yx}$ and $p^z{}_0 = -VT_{zx}$.
  Second, we replace easily the 0-components in the EMT expressions by using the orthogonality relation, from (131) we obtain: $q^0 = -Vq^x$. Similarly, for the anisotropic stress tensor we can express all 0-components through spatial terms.

- From the vorticity equations (44), (45), the expressions of the coordinate acceleration and the components of the vorticity conservation law (46), we can rewrite and simplify ϵ and q^i, and obtain the above equations as written in the **Theorem 1**. Notice the spatial momentum flux vector q^i is equal to the curl of the vorticity vector, which is the content of the momentum constraints.

  From the components of $p^i{}_j$ we can read off the components of the anisotropic stress tensor; its trace and its non-diagonal elements are written as follows:

$$p^i{}_j = p\delta^i_j + \pi^i{}_j \quad \Rightarrow \quad p^k{}_k = 3p \,, \quad \pi^k{}_k = 0 \,, \tag{83}$$

$$
\begin{aligned}
p + {\pi^x}_x &= \frac{1}{8\pi G}(-3\Omega^2 + \Lambda) \quad (i)\,,\\
p + {\pi^y}_y &= \frac{1}{8\pi G}\left[-\partial_x A + \Lambda + \Omega^2 - \frac{1}{2}(\Omega^y)^2\right] \quad (ii)\,,\\
p + {\pi^z}_z &= \frac{1}{8\pi G}\left[-\partial_x A + \Lambda + \Omega^2 - \frac{1}{2}(-\Omega^z)^2\right] \quad (iii)\,,\\
{\pi^x}_y = {\pi^y}_x &= \frac{1}{16\pi G}\left(\partial_y A - \Omega^z \partial_x V\right)\,,\\
{\pi^x}_z = {\pi^z}_x &= \frac{1}{16\pi G}\left(\partial_z A + \Omega^y \partial_x V\right)\,,\\
{\pi^y}_z = {\pi^z}_y &= -\frac{1}{16\pi G}\Omega^y\Omega^z\,.
\end{aligned}
\tag{84}
$$

- From the trace of ${p^i}_j$, we have with (i), (ii) and (iii):

$$
\begin{aligned}
3p = \frac{1}{8\pi G}(-3\Omega^2 + 3\Lambda - 2\partial_x A) \quad &\Rightarrow \quad p = \frac{1}{8\pi G}\left(-\Omega^2 + \Lambda - \frac{2}{3}\partial_x A\right)\,,\\
4\pi G(3p + \epsilon) &= (-2\Omega^2 + \Lambda - \partial_x A) \quad (iv)\,.
\end{aligned}
\tag{85}
$$

We replace p in the ${p^i}_j$ components and obtain the tensor ${p^i}_j$ in terms of the coordinate acceleration:

- From (iv) and expression of ϵ we have:

$$
\boldsymbol{\partial} \cdot \boldsymbol{A} = \partial_x A = \Lambda - 4\pi G(\epsilon + 3p) - 2\Omega^2 \quad (v)\,. \tag{86}
$$

- From (46) we have:

$$
\boldsymbol{\partial} \times \boldsymbol{A} = \begin{pmatrix} 0 \\ \partial_z A \\ -\partial_y A \end{pmatrix} = \begin{pmatrix} 0 \\ 16\pi G {\pi^x}_z - \Omega^y \partial_x V \\ -16\pi G {\pi^x}_y - \Omega^z \partial_x V \end{pmatrix}\,. \tag{87}
$$

- The energy conservation equation (19), with expression (48) ${\sigma^i}_j = h^{ik}\sigma_{kj}$, gives us:

$$
\begin{aligned}
&\dot\epsilon + \Theta(\epsilon + p) + {\pi^i}_j {\sigma^i}_j = 0\\
\Leftrightarrow \quad &\dot\epsilon + \Theta(\epsilon + p) = \left(-\frac{2}{3}{\pi^x}_x + \frac{1}{3}{\pi^y}_y + \frac{1}{3}{\pi^z}_z\right)\partial_x V - \frac{1}{2}({\pi^x}_y + {\pi^y}_x)\partial_y V - \frac{1}{2}({\pi^x}_z + {\pi^z}_x)\partial_z V\\
\Leftrightarrow \quad &\dot\epsilon + \partial_x V\left(\epsilon + p + \frac{2}{3}{\pi^x}_x - \frac{1}{3}{\pi^y}_y - \frac{1}{3}{\pi^z}_z\right) + {\pi^x}_y \partial_y V + {\pi^x}_z \partial_z V = 0 \quad (vi)\,,
\end{aligned}
$$

because ${q^\mu}_{;\mu} = 0$. The total time derivative of the energy density is:

$$
\dot\epsilon = \frac{\mathrm{d}}{\mathrm{d}t}\epsilon = \frac{\partial}{\partial t}\epsilon + V\partial_x \epsilon = -\frac{1}{8\pi G}\left(\frac{\partial}{\partial t}\Omega^2 + V\partial_x \Omega^2\right) = -\frac{1}{8\pi G}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Omega^2\right)\,.
$$

Using the vorticity transport equation (41), respectively (46), we obtain for the evolution of the rotational energy:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\Omega^2\right) = \partial_y A\,\partial_y V + \partial_z A\,\partial_z V - \Omega^2 \partial_x V\,,
$$

which yields:

$$
\dot\epsilon = -\frac{1}{4\pi G}\left[\partial_y A\,\partial_y V + \partial_z A\,\partial_z V - \partial_x V \Omega^2\right] \quad (vii)\,.
$$

We calculate separately the elements of (vi) with the components expression of $p+\pi^i{}_i$: (i), (ii) and (iii). We start with (vii). Later we join the terms together:

$$(\alpha) \quad \dot{\epsilon} = -\pi^x{}_y\partial_y V - \pi^x{}_z\partial_z V + \frac{1}{8\pi G}\partial_x V(4\Omega^2)\,,$$

$$(\beta) \quad \partial_x V(\epsilon+p) = \partial_x V\left[-\frac{1}{8\pi G}(\Lambda+\Omega^2)+p\right] = \partial_x V\left[\frac{1}{8\pi G}\left(-2\Omega^2-\frac{2}{3}\partial_x A\right)\right]$$

using in the second step $\quad p = \dfrac{1}{8\pi G}\left(\Lambda-\Omega^2-\dfrac{2}{3}\partial_x A\right)$.

$$(\gamma) \quad \pi^i{}_j\sigma^i{}_j = \partial_x V\left(\frac{2}{3}\pi^x{}_x - \frac{1}{3}\pi^y{}_y - \frac{1}{3}\pi^z{}_z\right) + \partial_y V\pi^x{}_y + \partial_z V\pi^x{}_z = \partial_x V\pi^x{}_x + \partial_y V\pi^x{}_y + \partial_z V\pi^x{}_z\,,$$

using $\pi^x{}_y = \pi^y{}_x$ and $\pi^x{}_z = \pi^z{}_x$, and the trace-free condition for $\pi^i{}_j$. Overall we have:

$$(\alpha)+(\beta)+(\gamma)=0 \quad \Leftrightarrow \quad \frac{1}{8\pi G}\partial_x V\left(2\Omega^2-\frac{2}{3}\partial_x A+8\pi G\pi^x{}_x\right)=0 \quad \Leftrightarrow \quad \pi^x{}_x = \frac{1}{8\pi G}\left(\frac{2}{3}\partial_x A-2\Omega^2\right).$$

We see that we cannot separate the diagonal $\pi^i{}_j$ terms from the pressure p and energy density ϵ, which are present here in the term $\partial_x A$. In addition, we note that this relation is equivalent to (i) if we introduce the p expression from (iv), so the energy conservation equation does not provide any additional restriction.

- The momentum conservation law (20), gives us, with vanishing covariant acceleration (60) and vanishing covariant vorticity:

$$0 = h^{\alpha\mu}p_{;\alpha} + h^\mu{}_\alpha\dot{q}^\alpha + \frac{4}{3}\Theta q^\mu + q^\alpha\sigma_\alpha{}^\mu + h^\mu{}_\alpha\pi^{\alpha\beta}{}_{;\beta} \quad (x)\,,$$

where the projector onto the spatial hypersurfaces is like in the case 1 (119), and the shear tensor (48). This gives us three components, which we will detail below using (x):

$$\begin{aligned}
h^{\alpha\mu}p_{;\alpha} &= (\partial_x p, \quad \partial_y p, \quad \partial_z p)\,,\\
h^\mu{}_\alpha\dot{q}^\alpha &= h^\mu{}_\alpha\left(\frac{\partial}{\partial t}q^\alpha + V^\mu q^\alpha{}_{,\mu}\right),\\
&= \frac{1}{8\pi G}\big(\ \partial_t(\partial\times\Omega)^x + (V\partial_x)(\partial\times\Omega)^x, \quad \partial_t(\partial\times\Omega)^y + (V\partial_x)(\partial\times\Omega)^y,\\
&\quad \partial_t(\partial\times\Omega)^z + (V\partial_x)(\partial\times\Omega)^z\big)\,,\\
&= (\dot{q}^x, \quad \dot{q}^y, \quad \dot{q}^z)\,,\\
\frac{4}{3}\Theta q^\mu &= \frac{1}{6\pi G}\partial_x V((\partial\times\Omega)^x, \quad (\partial\times\Omega)^y, \quad (\partial\times\Omega)^z) = \frac{4}{3}\partial_x V(q^x, \quad q^y, \quad q^z)\,,\\
q^\alpha\sigma_\alpha{}^\mu &= \frac{1}{8\pi G}\Big((\partial\times\Omega)^x\frac{2}{3}\partial_x V + (\partial\times\Omega)^y\frac{1}{2}\partial_y V + (\partial\times\Omega)^z\frac{1}{2}\partial_z V, \quad -(\partial\times\Omega)^y\frac{1}{3}\partial_x V + (\partial\times\Omega)^x\frac{1}{2}\partial_y V,\\
&\quad -(\partial\times\Omega)^z\frac{1}{3}\partial_x V + (\partial\times\Omega)^x\frac{1}{2}\partial_z V\Big)\,,\\
&= \left(q^x\frac{2}{3}\partial_x V + q^y\frac{1}{2}\partial_y V + q^z\frac{1}{2}\partial_z V, \quad -q^y\frac{1}{3}\partial_x V + q^x\frac{1}{2}\partial_y V, \quad -q^z\frac{1}{3}\partial_x V + q^x\frac{1}{2}\partial_z V\right),\\
h^\mu{}_\alpha\pi^{\alpha\beta}{}_{;\beta} &= \left(\pi^{xx}{}_{,x} + \pi^{xy}{}_{,y} + \pi^{xz}{}_{,z}, \quad \pi^{yx}{}_{,x} + \pi^{yy}{}_{,y} + \pi^{yz}{}_{,z}, \quad \pi^{zx}{}_{,x} + \pi^{zy}{}_{,y} + \pi^{zz}{}_{,z}\right).
\end{aligned}$$

We can rewrite this equation solely in terms of spatial components, given that $q^0=0$ and $h^{00}=0$:

$$-\boldsymbol{\partial}\cdot\boldsymbol{p} = \dot{q}^i + \frac{4}{3}\Theta q^i + q^j\sigma_j{}^i\,,$$

Finally, (x) becomes:

$$\Leftrightarrow -\begin{pmatrix} \partial_x p + \pi^{xx}{}_{,x} + \pi^{xy}{}_{,y} + \pi^{xz}{}_{,z} \\ \partial_y p + \pi^{yx}{}_{,x} + \pi^{yy}{}_{,y} + \pi^{yz}{}_{,z} \\ \partial_z p + \pi^{zx}{}_{,x} + \pi^{zy}{}_{,y} + \pi^{zz}{}_{,z} \end{pmatrix} = \begin{pmatrix} \dot{q}^x + \frac{4}{3}\partial_x V q^x + q^x \frac{2}{3}\partial_x V + q^y \frac{1}{2}\partial_y V + q^z \frac{1}{2}\partial_z V \\ \dot{q}^y + \frac{4}{3}\partial_x V q^y - q^y \frac{1}{3}\partial_x V + q^x \frac{1}{2}\partial_y V \\ \dot{q}^z + \frac{4}{3}\partial_x V q^z - q^z \frac{1}{3}\partial_x V + q^x \frac{1}{2}\partial_z V \end{pmatrix} \quad (xi)\,.$$

We express the above using coordinate acceleration by calculating the curl of the vorticity transport equation (46):

$$\boldsymbol{\partial} \times (\boldsymbol{\partial} \times \boldsymbol{A}) = \boldsymbol{\partial} \times \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Omega} + \boldsymbol{\Omega}(\boldsymbol{\partial} \cdot \boldsymbol{V}) - (\boldsymbol{\Omega} \cdot \boldsymbol{\partial})\boldsymbol{V} \right),$$

$$\Leftrightarrow \frac{1}{16\pi G} \begin{pmatrix} -\partial_y^2 A - \partial_z^2 A \\ \partial_x(\partial_y A) \\ \partial_x(\partial_z A) \end{pmatrix} = \begin{pmatrix} \dot{q}^x - 2q^y \partial_y V - 2q^z \partial_z V + q^x \partial_x V \\ \dot{q}^y + 2q^y \partial_x V + \frac{1}{16\pi G}\partial_y V \partial_x^2 V \\ \dot{q}^z + 2q^z \partial_x V + \frac{1}{16\pi G}\partial_z V \partial_x^2 V \end{pmatrix}$$

So (xi) simplifies and reads:

$$-\boldsymbol{\partial} \cdot \boldsymbol{p} = \begin{pmatrix} \frac{-\partial_y^2 A - \partial_z^2 A}{16\pi G} + \frac{5}{2} q^z \partial_z V + \frac{5}{2} q^y \partial_y V + q^x \partial_x V \\ \frac{\partial_x(\partial_y A)}{16\pi G} - q^y \partial_x V + q^x \frac{1}{2}\partial_y V - \frac{1}{16\pi G}\partial_y V \partial_x^2 V \\ \frac{\partial_x(\partial_z A)}{16\pi G} - q^z \partial_x V + q^x \frac{1}{2}\partial_z V - \frac{1}{16\pi G}\partial_z V \partial_x^2 V \end{pmatrix} \quad (xii)\,.$$

From there, we can verify the consistency of (xii), by calculating the divergence of the $p^i{}_j$ terms (84). After some calculations, we obtain the same components.

It is immediately apparent that the various components are entirely dependent on the vorticity component. The directional dependence of the pressure terms is therefore simply an effect of vorticity.

- We note that this more general case can be easily reduced to **Lemma 1** if we put the coordinate vorticity to zero. Furthermore, for a perfect fluid we have $\boldsymbol{q} = \boldsymbol{0}$ and vanishing anisotropic stresses. From the former we conclude $\boldsymbol{\partial} \times \boldsymbol{\Omega} = \boldsymbol{0}$ and, together with $\boldsymbol{\partial} \cdot \boldsymbol{\Omega} = \boldsymbol{0}$, we have $\Delta\boldsymbol{\Omega} = \boldsymbol{0}$, i.e. $\boldsymbol{\Omega}$ is harmonic and for suitable boundary or fall-off conditions we go again back to the case of **Lemma 1**.

□

3.6 Discussion of Case 2

First, we discuss the general case, if we do not make the assumption of inertial motion, we have:

$$8\pi G\epsilon = -(\Lambda + \Omega^2)\,, \quad \partial_x A = \frac{3}{2}(\Lambda - \Omega^2) - 12\pi G p = 12\pi G \pi^x{}_x + 3\Omega^2\,, \tag{88}$$

and

$$\partial_z A = 16\pi G \pi^x{}_z - \Omega^z \partial_x V\,, \quad \partial_y A = 16\pi G \pi^x{}_y + \Omega^y \partial_x V\,. \tag{89}$$

From a physical point of view, we would expect to have equations of state that link the pressure sources to the energy density. However, for $\Lambda = 0$, we need a negative energy density in regions where Ω^2 is non-zero, which is generally considered non-physical within standard energy conditions in general relativity. In particular, the dominant energy condition and the weak energy condition are violated. Also, setting $\Lambda = 0$, we conclude that the acceleration divergence is negative for positive pressure, i.e. also a negative pressure is needed to drive acceleration of the warp field.

If we would assume an equation of state of the form $p = F(\epsilon)$, we may look at the simplest case of a stiff equation of state $p = \epsilon$. In that case we find the acceleration sourced by Λ only, $\partial_x A = 3\Lambda$, which implies a homogeneous acceleration $A = 3\Lambda x$. We could absorbe this acceleration into a background field as is commonly done in cosmology. More generally, for other equations of state $p = F(\epsilon)$, one could compute the effective energy and pressure distributions directly from the vorticity profile $\Omega^2(t, x)$, enabling a wider class of models. Each such function would correspond to a different acceleration profile, with possible implications for the shape and evolution of the warp field.

If we assume now that the anisotropic pressure components are null, ${\pi^i}_j = 0$, we have an explicit form for the gradient of A. From (88) and (89) we can write:

$$\partial_x A = 3\Omega^2\,, \quad \partial_y A = \Omega^y \partial_x V\,, \quad \partial_z A = -\Omega^z \partial_x V\,. \tag{90}$$

If we would, on the other hand, assume a Dark Energy equation of state, $p = -\epsilon$, we have from (88), $\partial_x A = -3\Omega^2 = 12{\pi^x}_x + 3\Omega^2$. Thus, a fluid source with no anisotropic pressure, ${\pi^i}_j = 0$, would imply $\Omega = 0$ and we fall back to the result of **Lemma 1**. Note that, for perfect fluids, the momentum flux also has to vanish, as noted in the proof, and we also obtain the result of **Lemma 1**. In the extreme case of a fluid without pressure sources, the acceleration is sourced by the rotational energy, $\partial_x A = (3/2)\Lambda - (9/2)\Omega^2$, which again implies a deceleration for negative or null Λ.

Now, we are going to discuss the example analyzed in [2], which can now be seen as a subclass of exact solutions of Einstein's equations. We see from **Theorem 1** that the case of inertial motion arises as a solution, if the following conditions are met. From (86) and (87) we have in the case $A = 0$:

$$\Lambda - 4\pi G(\epsilon + 3p) = 2\Omega^2\,, \quad 16\pi G {\pi^x}_z = \Omega^y \partial_x V\,, \quad 16\pi G {\pi^x}_y = -\Omega^z \partial_x V\,. \tag{91}$$

Since $8\pi G\epsilon = -(\Lambda + \Omega^2)$, we obtain from the first expression relations between the coordinate vorticity scalar and the pressure sources (again in the case $A = 0$):

$$\Lambda - 8\pi G p = \Omega^2 = -4\pi G {\pi^x}_x\,, \tag{92}$$

containing the solution of **Lemma 1** for the subcase $\Omega = 0$. Here, for $A = 0$, we obtain the equations of state:

$$p = \epsilon + \frac{\Lambda}{4\pi G}\,, \quad {\pi^x}_x = 2\epsilon + \frac{\Lambda}{4\pi G}\,, \quad p - {\pi^x}_x = -\epsilon\,. \tag{93}$$

In the other expressions we assume that the vorticity components are generated by anisotropic stress components and we calculate:

$$\Omega = \frac{1}{2}\sqrt{(\Omega^y)^2 + (-\Omega^z)^2} = 8\pi G \frac{\sqrt{({\pi^x}_y)^2 + ({\pi^x}_z)^2}}{\partial_x V}\,, \quad \partial_x V \neq 0\,. \tag{94}$$

We can link all ${\pi^i}_j$ components to the vorticity scalar, e.g.:

$${\pi^x}_x = -\frac{(4\pi G)^2[({\pi^x}_y)^2 + ({\pi^x}_z)^2]}{(\partial_x V)^2}\,. \tag{95}$$

A note on the literature

Finally, we are making modifications and some corrections to the following articles [6], [7] and [8]. First of all, the components of the Einstein tensor G_{22} and G_{33} are corrected in Appendix A. Then, we correctly include in this work the anisotropic pressure tensor as well as the momentum flux, contrary to the realization made in [7] with erroneous components and wrong equations, e.g. on page 34, the trace of $\pi_{\alpha\beta}$ is incorrect. Moreover, in these articles, only the specific Alcubierre profile is analyzed, making the strong assumption of determining the form and time-evolution of the velocity profile *a priori*. While we have one degree of freedom left to allow for this, any further specific assumption on the sources would lead to contradictions with the result from the Einstein equations. As we will demonstrate below, this specific velocity profile just represents a translation of initial data without a dynamical change of the warp field.

3.6.1 Illustration for Alcubierre initial conditions

We take up the work done in my internship report [2], and reuse the results in order to apply them to the case of the function given by Alcubierre [1]. We emphasize that we can only use the function of Alcubierre as initial condition, the admissible velocity profiles follow from the solution and cannot be given *a priori*, since we removed the freedom of one function by adding one assumption. We take the initial velocity of the form:

$$V_S(t, X^k) = V_S(t_0, X^k) =: V_0(X^k) = v_S(t_0) W(r_S(t_0, X^k))\,, \tag{96}$$

where the (now Lagrangian) radial distance from the trajectory is:

$$r_S(t_0, X^k) = [(X - (X_S + v_S(t_0)W(r_S(t_0, X))))^2 + Y^2 + Z^2)]^{1/2}\,, \tag{97}$$

and the shape of the initial warp field window function is:

$$W(r_S) = \frac{\tanh(\sigma(r_S + R)) - \tanh(\sigma(r_S - R))}{2\tanh(\sigma R)}\,, \tag{98}$$

with R a fixed initial (Lagrangian) radius, and σ a constant that determines the inverse thickness of the wall of the "warp bubble".

Here X, Y, Z are the Cartesian components of $\boldsymbol{X}$. X_S is the fixed position of the centre of the warp bubble in Lagrangian space, and x_S is its position in the Eulerian space; it can be obtained as follows:

$$x_S(t, X^k) = f_S(t, X^k) = X + V_0(X^k)(t - t_0) = X + v_S(t_0)W(r_S(t_0, X^k))(t - t_0)\,. \tag{99}$$

We express the Lagrangian evolution equations of the kinematic properties, notably with the help of the Jacobian (66):

$$J = 1 + (t - t_0)\Theta(t_0, X^k)\,, \tag{100}$$

and we obtain, with the evolution equations (for more details see [8]), the three expressions for the expansion rate Θ, the shear scalar Σ^2 and the vorticity scalar Ω^2 as follows. For the expansion rate we have:

$$\Theta(t, X^k) = \frac{\Theta(t_0, X^k)}{1 + (t - t_0)\Theta(t_0, X^k)}\,. \tag{101}$$

For the shear scalar we use the fact demonstrated in [8], that the second scalar invariant vanishes for one-component velocity fields, $\mathrm{II} = 0 = \frac{1}{3}\Theta^2 - \Sigma^2 + \Omega^2$, so that we have:

$$\Sigma^2(t, X^k) = \frac{1}{3}\Theta^2(t, X^k) + \Omega^2(t, X^k)\,. \tag{102}$$

To calculate $\Omega(t, X^k)^2$ we know the integral of the Kelvin-Helmholtz vorticity transport in terms of the vorticity vector (41). Since we demand $\boldsymbol{\partial} \times \boldsymbol{A} = \boldsymbol{0}$, we have Cauchy's integral [14] in three dimensions:

$$\boldsymbol{\Omega} = \frac{(\boldsymbol{\Omega}(t_0) \cdot \boldsymbol{\partial_X})\boldsymbol{f}}{J}\,, \tag{103}$$

where $\boldsymbol{\partial_X}$ denotes the nabla operator with respect to Lagrangian coordinates, and where J is given by the Jacobian determinant for inertial motion (100).

In the present case, vorticity is in the (y, z)-plane, displacement only occurs along the x-direction, and therefore the Lagrangian convective term reduces to the initial vorticity and Cauchy's integral reduces to:

$$\Omega^y(t, X^k) = \frac{\Omega^y(t_0)}{J} = \frac{\Omega^y(t_0)}{1 + (t - t_0)\Theta(t_0, X^k)}\,, \quad \Omega^z(t, X^k) = \frac{\Omega^z(t_0)}{J} = \frac{\Omega^z(t_0)}{1 + (t - t_0)\Theta(t_0, X^k)}\,,$$

$$\Rightarrow \quad \boldsymbol{\Omega}(t, X^k) = \frac{1}{J}\begin{pmatrix} 0 \\ \Omega^y(t_0) \\ \Omega^z(t_0) \end{pmatrix}\,. \tag{104}$$

From the definition $\Omega^2 = \frac{1}{2}\Omega_{ij}\Omega^{ij} = \frac{1}{2}\boldsymbol{\Omega}^2$ we obtain:

$$\Omega^2(t, X^k) = \frac{\Omega^2(t_0, X^k)}{J^2} = \frac{\Omega^2(t_0, X^k)}{(1 + (t - t_0)\Theta(t_0, X^k))^2}\,. \tag{105}$$

We can deduce the exact form of ${\pi^x}_x$ (88) in the present case with (105), for null-coordinate acceleration:

$${\pi^x}_x(t, X^k) = -\frac{\Omega^2(t, X^k)}{4\pi G}\,. \tag{106}$$

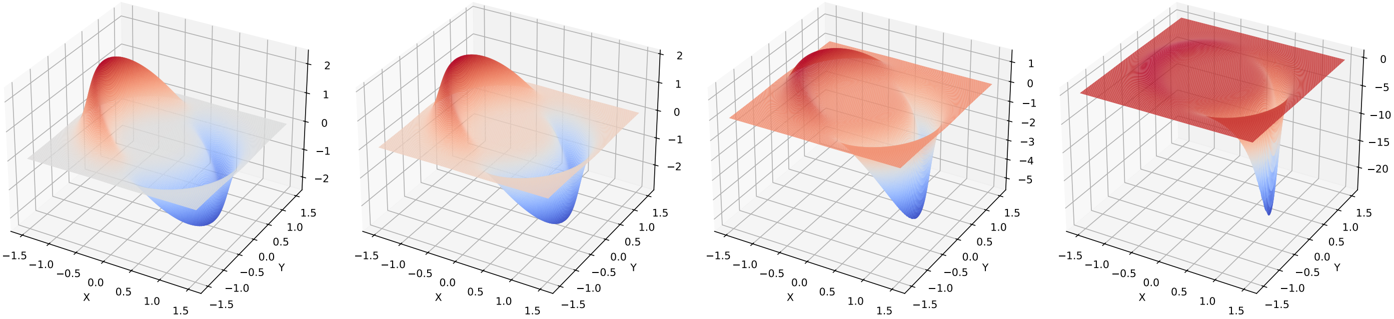
The energy density and the pressure attain this form:

$$\epsilon(t, X^k) = -\frac{1}{8\pi G}[\Lambda + \Omega(t, X^k)^2]\,, \quad p(t, X^k) = \frac{1}{8\pi G}[\Lambda - \Omega(t, X^k)^2]\,. \tag{107}$$

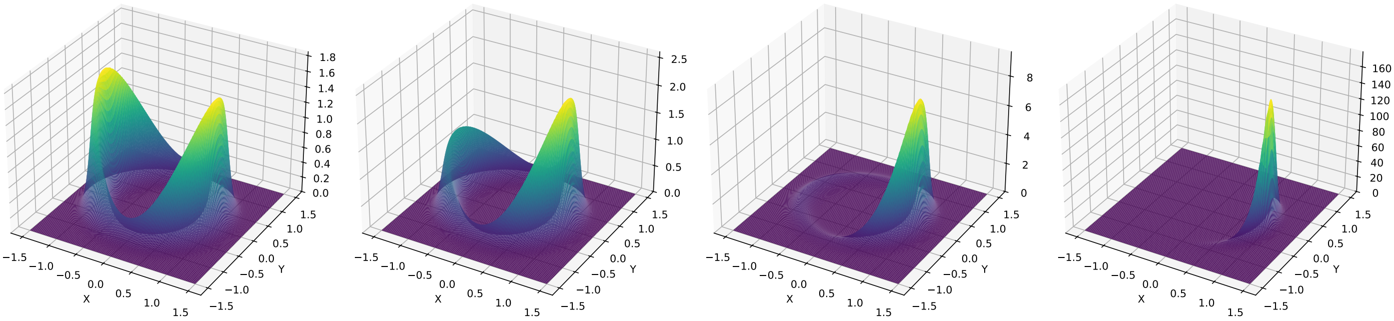
We compute in the same way the anisotropic stress scalar in inertial motion, for details see appendix C. In this case, from the **Theorem 1**, with the assumption of inertial motion we obtain:

$$\Pi^2(t, X^k) = \frac{1}{(8\pi G)^2}[4\Omega^2(t, X^k) + \Omega^2(t, X^k)\Theta^2(t, X^k)] \,. \tag{108}$$

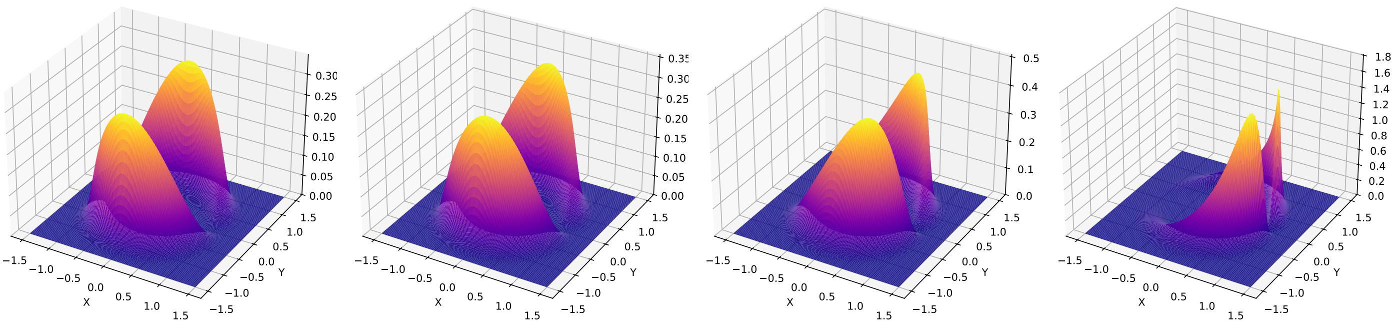
In figure (3) we show the Lagrangian evolution of these fields at different times:



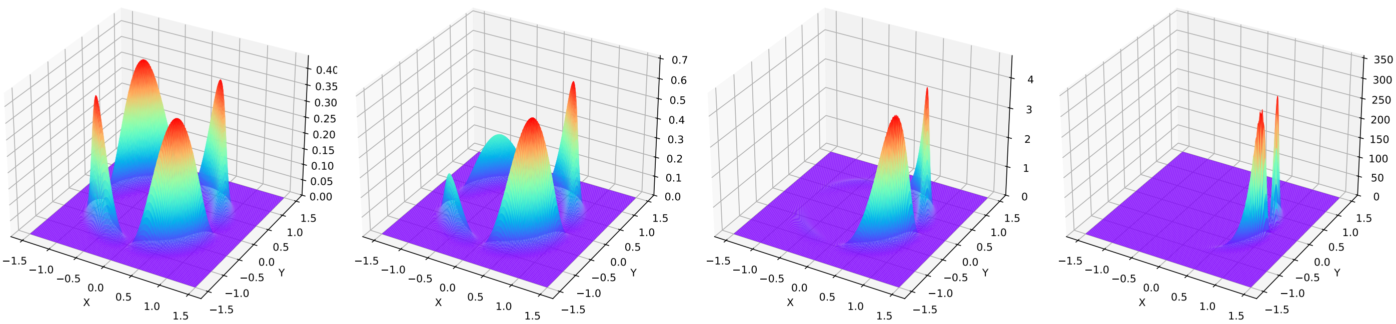
(a)



(b)



(c)



(d)

Figure 3: Evolution of the Θ (a), Σ^2 (b), Ω^2 (c) and Π^2 (d) fields at different times t. We have chosen different colors to distinguish the fields. At initial time, here $t = 0$, left figure, Eulerian and Lagrangian coordinates coincide and we see the initial data. At $t = 0.10$ and $t = 0.25$ we can observe a marked deformation of the fields. The vorticity scalar and shear scalar fields move significantly with the rate of expansion field. At a time close to the computable limit $t = 0.4$, right figure, the deformation of Θ at the front will become infinite. We can see the same phenomenon on all the other scalar fields (Here we set the gravitational constant $G = 1$ and the others variables: $v_S = 0,9$, $\sigma = 5$, $R = 1$, $t_0 = 0$).

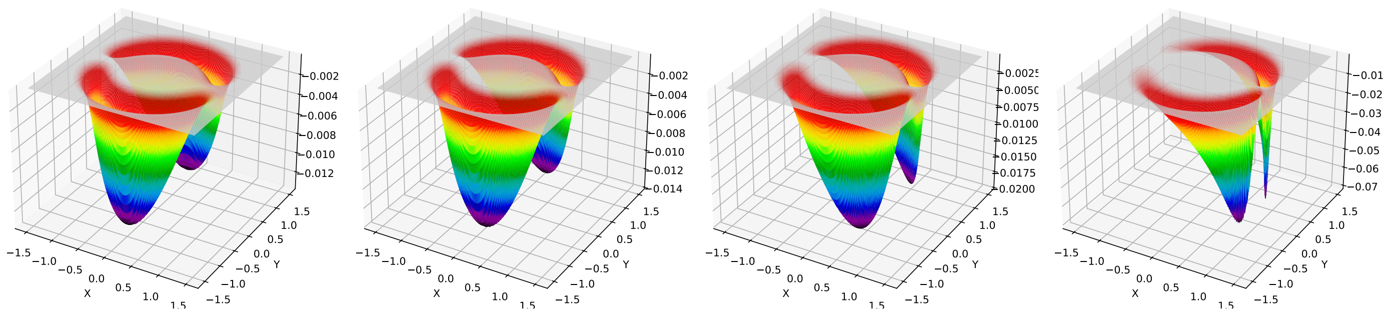
Moreover, if the cosmological constant is set to 0, we see that $\epsilon = p$, and:

$$\epsilon = p = -\frac{\Omega^2}{8\pi G}\,. \tag{109}$$

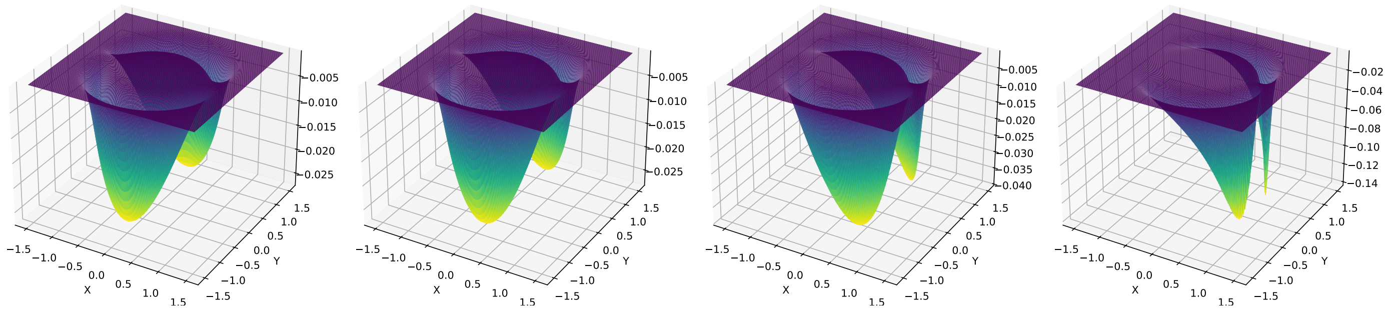
We then explicitly have a simple relation between energy density, pressure and $\pi^x{}_x$ anisotropic term; they only depend on the coordinate vorticity:

$$\epsilon = p = 2\pi^x{}_x\,. \tag{110}$$

In figure (4), we represent these Lagrangian fields with their evolution in time:



(a)



(b)

Figure 4: Evolution of the ϵ or p (a) and $\pi^x{}_x$ (b) fields at different times t. It should be noted that the colors are different in order to more clearly distinguish the fields. We note the shape inverse resemblance to the vorticity scalar of figure (3). Furthermore, the anisotropic term evolves two times faster than energy density and pressure. Here we take $t_0 = 0$, $\Lambda = 0$ and $G = 1$.

3.6.2 Remarks on the Alcubierre model as a solution

Above we considered the assumption of inertial motion along the solution to close the system of Einstein equations. However, a priori, according to the Synge-G method, the model given by Alcubierre is also a solution, although physically less interesting, since the warp field is assumed to stay spherical in the coarse of time. We are now in the position to derive the necessary sources to support the Alcubierre solution.

In the following, we study the general representation of Alcubierre's warp field and calculate its coordinate acceleration field. The externally given proper speed of the bubble $v_S(t) = \frac{\partial x_S(t)}{\partial t}$ and, more importantly, the window function $W(r_S)$ (98) can induce coordinate acceleration.

In this general case, $A(t, x^k) \neq 0$, it is possible to obtain the exact form of $\pi^x{}_x$ (88) and $\pi^x{}_z$, $\pi^x{}_z$ (89) that support Alcubierre's warp field. From Einstein's equations we have the relations:

$$\begin{aligned}
\pi^x{}_x &= \frac{1}{12\pi G}\left[\partial_x A_S(t,x^k) - 3\Omega^2\right] = \frac{1}{12\pi G}\left[\partial_x\left(\frac{\mathrm{d}V_S(t,x^k)}{\mathrm{d}t}\right) - 3\Omega^2\right],\\
\pi^x{}_y &= \frac{1}{16\pi G}[\partial_y A_S(t,x^k) - \Omega^y\partial_x V] = \frac{1}{16\pi G}\left[\partial_y\left(\frac{\mathrm{d}V_S(t,x^k)}{\mathrm{d}t}\right) - \Omega^y\partial_x V\right],\\
\pi^x{}_z &= \frac{1}{16\pi G}[\partial_z A_S(t,x^k) + \Omega^z\partial_x V] = \frac{1}{16\pi G}\left[\partial_z\left(\frac{\mathrm{d}V_S(t,x^k)}{\mathrm{d}t}\right) + \Omega^z\partial_x V\right].
\end{aligned} \tag{111}$$

The gradient of $A(t,x^k)$ enters these equations, so we first calculate the coordinate acceleration directly in the Eulerian frame [5]:

$$
\begin{aligned}
A_S(t,x^k) &= \frac{\mathrm{d}V_S(t,x^k)}{\mathrm{d}t} = \frac{\partial v_s(t)}{\partial t}W(r_S) + v_S(t)\frac{\mathrm{d}W(r_S)}{\mathrm{d}t} =: a_S(t)W(r_S) + v_S(t)\frac{\partial W}{\partial r_S}\frac{\mathrm{d}r_S}{\mathrm{d}t}\,,\\
&= a_S(t)W(r_S) + v_S^2(t)\frac{\sigma(x-x_S(t))}{2r_S\tanh(\sigma R)}[\mathrm{sech}^2(\sigma(r_S+R)) - \mathrm{sech}^2(\sigma(r_S-R))][-1+W(r_S)]\,,\\
&= a_S(t)W(r_S) + v_S^2(t)g(r_S)\frac{(x-x_S)}{r_S}[W(r_S)-1]\,,\\
\text{where}\quad g(r_S) &= \frac{\partial W}{\partial r_S} = \frac{\sigma}{2\tanh(\sigma R)}[\mathrm{sech}^2(\sigma(r_S+R)) - \mathrm{sech}^2(\sigma(r_S-R))]\,,
\end{aligned}
$$

with a_S the acceleration of the warp bubble and we deduce for the gradient of A:

$$
\begin{aligned}
\partial_x A_S(t,x^k) &= a_S(t)\frac{(x-x_S)}{r_S}g(r_S) + v_S^2(t)\left[h(r_S)\frac{(x-x_S)^2}{r_S^2}(W(r_S)-1) + g(r_S)\left(\frac{r_S^2-(x-x_S)^2}{r_S^3}(W(r_S)-1) + \frac{(x-x_S)^2}{r_S^2}g(r_S)\right)\right]\\
&= a_S(t)\frac{(x-x_S)}{r_S}g(r_S) + v_S^2(t)\left[h(r_S)\frac{(x-x_S)^2}{r_S^2}(W(r_S)-1) + g(r_S)\left(\frac{r_S^2-(x-x_S)^2}{r_S^3}(W(r_S)-1)\right)\right] + \Theta^2\,,\\
\partial_y A_S(t,x^k) &= a_S(t)\frac{y}{r_S}g(r_S) + v_S^2(t)\left[h(r_S)\frac{(x-x_S)y}{r_S^2}(W(r_S)-1) + g(r_S)\left(-\frac{(x-x_S)y}{r_S^3}(W(r_S)-1) + \frac{(x-x_S)y}{r_S^2}g(r_S)\right)\right]\\
&= a_S(t)\frac{y}{r_S}g(r_S) + v_S^2(t)\left[h(r_S)\frac{(x-x_S)y}{r_S^2}(W(r_S)-1) + g(r_S)\left(-\frac{(x-x_S)y}{r_S^3}(W(r_S)-1)\right)\right] + \frac{\Theta^2 y}{x-x_S}\,,\\
\partial_z A_S(t,x^k) &= a_S(t)\frac{z}{r_S}g(r_S) + v_S^2(t)\left[h(r_S)\frac{(x-x_S)z}{r_S^2}(W(r_S)-1) + g(r_S)\left(-\frac{(x-x_S)z}{r_S^3}(W(r_S)-1) + \frac{(x-x_S)z}{r_S^2}g(r_S)\right)\right]\\
&= a_S(t)\frac{z}{r_S}g(r_S) + v_S^2(t)\left[h(r_S)\frac{(x-x_S)z}{r_S^2}(W(r_S)-1) + g(r_S)\left(-\frac{(x-x_S)z}{r_S^3}(W(r_S)-1)\right)\right] + \frac{\Theta^2 z}{(x-x_S)}\,,\\
\text{with}\quad h(r_S) &= \frac{\partial^2 W}{\partial r_S^2} = \frac{\sigma^2}{\tanh(\sigma R)}[\mathrm{sech}^2(\sigma(r_S+R))\tanh(\sigma(r_S+R)) - \mathrm{sech}^2(\sigma(r_S-R))\tanh(\sigma(r_S-R))]\,.
\end{aligned}
$$

To understand how the acceleration field changes when we move the warp, we also calculate the following derivative:

$$
\partial_{r_S}A_S(t,x^k) = \frac{x-x_S}{r_S}\partial_x A_S(t,x^k) + \frac{y}{r_S}\partial_y A_S(t,x^k) + \frac{z}{r_S}\partial_z A_S(t,x^k)\,. \tag{113}
$$

We can see in figure (5) the following fields: the velocity $V_S(t,x^k)$, its derivative with respect to r_S and the derivative with respect to r_S of the coordinate acceleration A_S. We can observe that the shape of the warp is derived from the derivative of the window function and that the coordinate acceleration will take place in the x direction, with a positive amplitude at the front against a negative one at the back of the bubble wall. Furthermore, we have observed that this acceleration encompasses the shape of the bubble, which is situated between two accelerating fronts that tend to offset each other. This is more visible in a 2D view of the amplitude (right figure). This helps us understand the dynamics of such a rigid warp field.

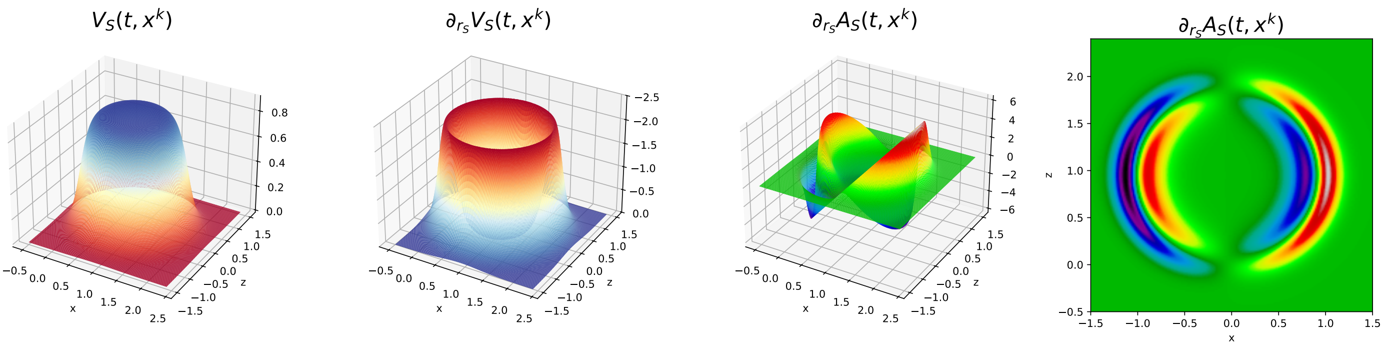


Figure 5: Representation of the Alcubierre velocity field $V_S(t,x^k)$ and its derivative with respect to r_S, as well as the derivative of the coordinate acceleration field with respect to r_S. Here we take $v_S = 0,9$, $\sigma = 5$, $R = 1$, $t_0 = 0$ (See appendix (D) for the norm of $\partial_{r_S}A_S(t,x^k)$).

[5] Note we use the following derivation laws: $(\tanh u)' = u'\,\mathrm{sech}^2(u)\,,\quad (\mathrm{sech}^2(u))' = -2u'\,\mathrm{sech}^2(u)\tanh(u)$

We now use the above calculations to give some further illustrations for the action of the gradient of the coordinate acceleration. For this we calculate the vector:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \partial_x V_S(t,x^k) \\ \partial_y V_S(t,x^k) \\ \partial_z V_S(t,x^k) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial_t}\partial_x V_S(t,x^k) + V_S(t,x^k)\partial_x^2(V_S(t,x^k)) \\ \frac{\partial}{\partial_t}\partial_y V_S(t,x^k) + V_S(t,x^k)\partial_x(\partial_y V_S(t,x^k)) \\ \frac{\partial}{\partial_t}\partial_y V_S(t,x^k) + V_S(t,x^k)\partial_x(\partial_y V_S(t,x^k)) \end{pmatrix},
$$

$$
= \begin{pmatrix} a_S(t)g(r_S)\frac{x-x_S}{r_S} + v_S^2(t)\left[h(r_S)\frac{(x-x_S)^2}{r_S^2}(W(r_S)-1) + g(r_S)\left(\frac{r_S^2-(x-x_S)^2}{r_S^3}(W(r_S)-1)\right)\right] \\ a_S(t)g(r_S)\frac{y}{r_S} + v_S^2(t)\left[h(r_S)\frac{(x-x_S)z}{r_S^2}(W(r_S)-1) + g(r_S)\left(-\frac{(x-x_S)z}{r_S^3}(W(r_S)-1)\right)\right] \\ a_S(t)g(r_S)\frac{z}{r_S} + v_S^2(t)\left[h(r_S)\frac{(x-x_S)z}{r_S^2}(W(r_S)-1) + g(r_S)\left(-\frac{(x-x_S)z}{r_S^3}(W(r_S)-1)\right)\right] \end{pmatrix}. \tag{114}
$$

From (112) and (114) we can infer the following relation:

$$
\partial_x A_S(t,x^k) = \frac{\mathrm{d}}{\mathrm{d}t}(\partial_i V_S(t,x^k)) - \Theta^2\,,
$$

$$
\partial_y A_S(t,x^k) = \frac{\mathrm{d}}{\mathrm{d}t}(\partial_i V_S(t,x^k)) - \Theta^2\frac{y}{x-x_S}\,,
$$

$$
\partial_z A_S(t,x^k) = \frac{\mathrm{d}}{\mathrm{d}t}(\partial_i V_S(t,x^k)) - \Theta^2\frac{z}{x-x_S}\,.
$$

To visualize the shape of these terms, here for example for the x-components, we show in figure (6) the x-component of the coordinate acceleration gradient, $\partial_x A_S(t,x^k)$, compared with $\frac{\mathrm{d}}{\mathrm{d}t}\partial_x V_S(t,x^k)$. Subtracting these two fields equals to Θ^2 (which also illustrates the Raychaudhuri equation (35)).

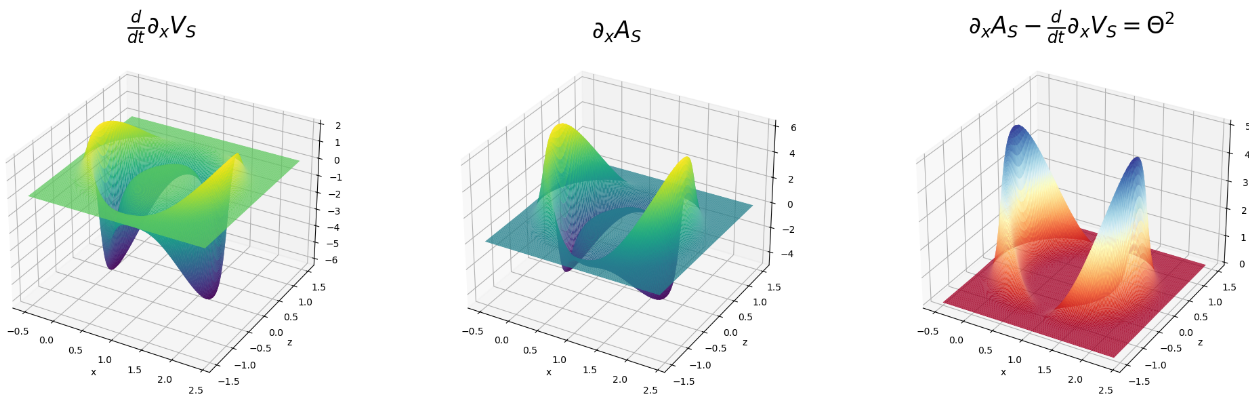


Figure 6: Representation of the Alcubierre x-components of the total derivative with respect to time of the gradient of the velocity and the gradient of the coordinate acceleration. The field on the right shows the difference of these two which equals Θ^2. (Here we take $v_S = 0.9$, $\sigma = 5$, $R = 1$, $t_0 = 0$).

We now add a proof that the form of the Alcubierre warp field is sustained in time. For this we perform a change of coordinates into the comoving (Eulerian) system of the warp field:

$$
\text{We put} \quad q_x = x - x_S(t),\ q_y = y,\ q_z = z, \quad \Rightarrow \quad r_S(\boldsymbol{q}) = [q_x^2 + q_y^2 + q_z^2]^{-1/2}\,. \tag{115}
$$

We then calculate the total time-derivative of $V_S = v_S(t)W(r_S)$:

$$
\frac{\mathrm{d}V_S}{\mathrm{d}t} = a_S(t)W(r_S) + v_S(t)\frac{\mathrm{d}W(r_S)}{\mathrm{d}t} = a_S(t)W(r_S) + v_S(t)\frac{\partial W(r_S)}{\partial r_S}\frac{\mathrm{d}r_S}{\mathrm{d}t} = v_S(t)\frac{\partial W(r_S)}{\partial q_x}\frac{\mathrm{d}q_x}{\mathrm{d}t} = 0\,. \tag{116}
$$

Since the new coordinates are comoving, $\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \boldsymbol{0}$, we obtain a rigid transport of the warp field for the case $a_S = 0$. We deduce that the form $W(r_S)$ and therefore all properties derived from it, is simply translated in space at the externally given speed $v_S = const.$ in this case. The external acceleration $a_S(t)$ changes the amplitude, not the form of the warp field.

We finally calculate some average properties of the warp field. First, we calculate the average of $\langle\Theta\rangle = \langle I(\Theta)\rangle$:

$$\text{We put}\quad \begin{cases} \zeta = x - x_S = r_S \sin\theta\cos\phi \\ y = r_S\sin\theta\sin\phi \\ z = r_S\cos\Theta \end{cases} \quad \Rightarrow \Theta = v_S(t)\frac{\partial W(r_S)}{\partial r_S}\frac{x-x_S}{r_S} = v_S(t)\frac{\partial W(r_S)}{\partial r_S}\sin\theta\cos\phi\,,$$

$$\begin{aligned}\langle\Theta\rangle &= v_S(t)\int_0^R\int_0^{2\pi}\int_0^{\pi}\frac{\partial W(r_S)}{\partial r_S}\sin\theta\cos\phi\,\mathrm{d}\phi\mathrm{d}r_S\mathrm{d}\theta \\ &= v_S(t)\int_0^R\frac{\partial W(r_S)}{\partial r_S}r_S^2\mathrm{d}r_S\int_0^{2\pi}\cos\phi\,\mathrm{d}\phi\int_0^{\pi}(\sin\theta)^2\mathrm{d}\theta = 0 \quad\text{with}\quad \int_0^{2\pi}\cos\phi\,\mathrm{d}\phi = 0\,.\end{aligned}$$

We deduce that the shape of the average rate of expansion remains zero in time. Furthermore, the form of it is a sphere. We deduce to conservation of the comoving volume $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}(\mathbb{S}) = 0$ as follows. The volume of the warp field is:

$$\mathrm{Vol}(\mathbb{S}) = \int_0^R\int_0^{2\pi}\int_0^{\pi} r_S^2\sin\theta\,\mathrm{d}r_S\mathrm{d}\theta\mathrm{d}\phi = \frac{4}{3}\pi R^3\,. \tag{117}$$

In summary, our more general study of the warp drive introduced by Alcubierre, which seems very simple at first glance, opens up possibilities that have never been investigated before and leads to numerous further paths of investigation.

Conclusion

In this report, we have explored novel configurations of warp drive spacetimes by employing a purely geometric approach within the 3+1 formalism of general relativity. By focusing on a restricted class of spacetimes — namely those with constant lapse and a one-component shift vector — we have constructed and analyzed exact solutions to the Einstein field equations, both with and without coordinate vorticity. This approach offers a consistent framework that can potentially avoid many of the unphysical assumptions inherent in earlier warp drive models, such as the *ad hoc* prescription of velocity profiles or the neglect of dynamic constraints imposed by general relativity.

We have shown that under specific kinematic and geometric conditions, particularly in the irrotational (vorticity-free) case, the dynamical degrees of freedom are reduced substantially, with energy-momentum content satisfying simple relations. In the more general, vorticity case, new constraints emerge that couple the rotational structure of the shift vector to the energy density, pressure, and anisotropic stresses. We proposed a dynamical generalization of the Alcubierre warp field model and provided a more solid theoretical foundation for investigating warp drive.

Future investigations could extend this framework by relaxing some of the restrictive assumptions imposed in this work, such as the flatness of spatial hypersurfaces or the one-component nature of the shift vector. For example, a more general case is a three-component velocity field like in Natário class metrics [9]. A particularly promising direction lies in exploring tilted warp drive spacetimes, **T**-Warp [3], beyond the **R**-Warp setting, where the fluid 4-velocity is no longer aligned with the foliation normal. Finally, more general analyses may provide further understanding of the global causal structure and stability of such warped spacetimes, and assess their viability through employing results from realistic cosmological models.

Note: My M1 internship report, as well as all the *Python* and *SageManifolds* codes for this study can be consulted if required at this GitHub address: `https://github.com/AntonyFrackowiak/Warp_drive.git`**.**

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Special thanks to Prof. Thomas Buchert for giving me the opportunity to work with him on this subject during my two master's internships. I would also like to thank Dr. Hamed Barzegar for his help in coding with *SageManifolds*.

Appendix

A Decomposition of Einstein's equations

In the following we present the kinematic component decomposition of EMT in the **R**-warp framework described; in this case the components of the metric read:

$$g_{\mu\nu} = \begin{pmatrix} V^2-1 & -V & 0 & 0 \\ -V & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1 & -V & 0 & 0 \\ -V & -V^2+1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{118}$$

and where $b_{\mu\nu} = h_{\mu\nu} = diag(0,1,1,1)$, with $n^\mu = (1,V,0,0)$ and $n_\mu = (-1,0,0,0)$. We have the following result:

$$(b^\mu{}_\nu) = (h^\mu{}_\nu) = \delta^\mu{}_\nu + n^\mu n_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -V & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{119}$$

$$(b_{\mu\nu}) = (h_{\mu\nu}) = g_{\mu\nu} + n_\mu n_\nu = \begin{pmatrix} V^2 & -V & 0 & 0 \\ -V & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{120}$$

$$(b^{\mu\nu}) = (h^{\mu\nu}) = g^{\mu\nu} + n^\mu n^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{121}$$

Using (15) and (16) we deduce in this particular case the following components:

$$\epsilon := n^\alpha n^\beta T_{\alpha\beta}\,, \quad q_\mu := -h^\alpha{}_\mu n^\beta T_{\alpha\beta}\,, \quad p h_{\mu\nu} + \pi_{\mu\nu} := h^\alpha{}_\mu h^\beta{}_\nu T_{\alpha\beta}\,, \quad h^{\mu\nu}\pi_{\mu\nu} = 0\,. \tag{122}$$

where the non-zero components are as follows:

$$\begin{aligned}
\epsilon = n^0 n^0 T_{00} + n^0 n^1 T_{01} + n^1 n^0 T_{10} + n^1 n^1 T_{11} \quad &\Rightarrow \quad \epsilon = T_{00} + V T_{01} + V T_{10} + V^2 T_{11}\,, \\
q_0 = -h^1{}_0 n^0 T_{10} - h^1{}_0 n^1 T_{11} \quad &\Rightarrow \quad q_0 = V T_{10} + V^2 T_{11}\,, \\
q_1 = -h^1{}_1 n^0 T_{10} - h^1{}_1 n^1 T_{11} \quad &\Rightarrow \quad q_1 = -T_{10} - V T_{11}\,, \\
q_2 = -h^2{}_2 n^0 T_{20} - h^2{}_2 n^1 T_{21} \quad &\Rightarrow \quad q_2 = -T_{20} - V T_{21}\,, \\
q_3 = -h^3{}_3 n^0 T_{30} - h^3{}_3 n^1 T_{31} \quad &\Rightarrow \quad q_3 = -T_{30} - V T_{31}\,, \\
q^0 = h^{00} q_0 + h^{01} q_1 \quad &\Rightarrow \quad q^0 = 0\,, \\
q^1 = h^{10} q_0 + h^{11} q_1 \quad &\Rightarrow \quad q^1 = q_1\,, \\
q^2 = h^{22} q_2 \quad &\Rightarrow \quad q^2 = q_2\,, \\
q^3 = h^{33} q_3 \quad &\Rightarrow \quad q^3 = q_3 \\
p h_{00} + \pi_{00} = h^1{}_0 h^1{}_0 T_{11} \quad &\Rightarrow \quad p V^2 + \pi_{00} = V^2 T_{11}\,, \\
p h_{11} + \pi_{11} = h^1{}_1 h^1{}_1 T_{11} \quad &\Rightarrow \quad p + \pi_{11} = T_{11}\,, \\
p h_{22} + \pi_{22} = h^2{}_2 h^2{}_2 T_{22} \quad &\Rightarrow \quad p + \pi_{22} = T_{22}\,, \\
p h_{33} + \pi_{33} = h^3{}_3 h^3{}_3 T_{33} \quad &\Rightarrow \quad p + \pi_{33} = T_{33}\,,
\end{aligned} \tag{123}$$

$$
\begin{aligned}
ph_{01}+\pi_{01}=ph_{10}+\pi_{10}={h^1}_0{h^1}_1T_{11} \quad &\Rightarrow \quad -pV+\pi_{01}=-pV+\pi_{10}=-VT_{11}\,,\\
\pi_{02}=\pi_{20}={h^1}_0{h^2}_2T_{12} \quad &\Rightarrow \quad \pi_{02}=\pi_{20}=-VT_{12}\,,\\
\pi_{03}=\pi_{30}={h^1}_0{h^3}_3T_{13} \quad &\Rightarrow \quad \pi_{03}=\pi_{30}=-VT_{13}\,,\\
\pi_{12}=\pi_{21}={h^1}_1{h^2}_2T_{12} \quad &\Rightarrow \quad \pi_{12}=\pi_{21}=T_{12}\,,\\
\pi_{13}=\pi_{31}={h^1}_1{h^3}_3T_{13} \quad &\Rightarrow \quad \pi_{13}=\pi_{31}=T_{13}\,,\\
\pi_{23}=\pi_{32}={h^2}_2{h^3}_3T_{23} \quad &\Rightarrow \quad \pi_{23}=\pi_{32}=T_{23}\,,
\end{aligned}
$$

So we obtain the 10 EMT components as follows:

$$
(T_{\mu\nu})=\begin{pmatrix}
\epsilon-2q_0+\pi_{00}+V^2p & -q_1+\pi_{01}-Vp & -q_2+\pi_{02} & -q_3+\pi_{03}\\
-q_1+\pi_{01}-Vp & p+\pi_{11} & \pi_{12} & \pi_{13}\\
-q_2+\pi_{02} & \pi_{21} & p+\pi_{22} & \pi_{23}\\
-q_3+\pi_{03} & \pi_{31} & \pi_{32} & p+\pi_{33}
\end{pmatrix}. \tag{124}
$$

And the 10 Einstein tensor components:

$$
\begin{aligned}
&G_{00}=8\pi GT_{00}-\Lambda g_{00} \Leftrightarrow -\frac{1}{4}(3V^2+1)\left[(\partial_yV)^2+(\partial_zV)^2\right]-V\left(\partial_y^2V+\partial_z^2V\right)=8\pi G(\epsilon-2q_0+\pi_{00}+V^2p)-\Lambda(V^2-1)\,,\\
&G_{11}=8\pi GT_{11}-\Lambda g_{11} \Leftrightarrow -\frac{3}{4}\left[(\partial_yV)^2+(\partial_zV)^2\right]=8\pi G(p+\pi_{11})-\Lambda\,,\\
&G_{22}=8\pi GT_{22}-\Lambda g_{22} \Leftrightarrow -(\partial_xV)^2-V\partial_x^2V-\partial_t(\partial_xV)+\frac{1}{4}\partial_y^2V-\frac{1}{4}\partial_z^2V=8\pi G(p+\pi_{22})-\Lambda\,,\\
&G_{33}=8\pi GT_{33}-\Lambda g_{33} \Leftrightarrow -(\partial_xV)^2-V\partial_x^2V-\partial_t(\partial_xV)-\frac{1}{4}\partial_y^2V+\frac{1}{4}\partial_z^2V=8\pi G(p+\pi_{33})-\Lambda\,,\\
&G_{01}=G_{10}=8\pi GT_{01}-\Lambda g_{01} \Leftrightarrow \frac{3}{4}V\left[(\partial_yV)^2+(\partial_zV)^2\right]+\frac{1}{2}(\partial_y^2V+\partial_z^2V)=8\pi G(-q_1+\pi_{01}-Vp)+\Lambda V\,,\\
&G_{02}=G_{20}=8\pi GT_{02}-\Lambda g_{02} \Leftrightarrow -V\partial_xV\partial_yV-\frac{1}{2}V\partial_t(\partial_yV)-\frac{1}{2}(V^2+1)\partial_x(\partial_yV)=8\pi G(-q_2+\pi_{02})\,,\\
&G_{03}=G_{30}=8\pi GT_{03}-\Lambda g_{03} \Leftrightarrow -V\partial_xV\partial_zV-\frac{1}{2}V\partial_t(\partial_zV)-\frac{1}{2}(V^2+1)\partial_x(\partial_zV)=8\pi G(-q_3+\pi_{03})\,,\\
&G_{12}=G_{21}=8\pi GT_{12}-\Lambda g_{12} \Leftrightarrow \frac{1}{2}V\partial_x(\partial_yV)+\frac{1}{2}\partial_t(\partial_yV)+\partial_xV\partial_yV=8\pi G\pi_{12}\,,\\
&G_{13}=G_{31}=8\pi GT_{13}-\Lambda g_{13} \Leftrightarrow \frac{1}{2}V\partial_x(\partial_zV)+\frac{1}{2}\partial_t(\partial_zV)+\partial_xV\partial_zV=8\pi G\pi_{13}\,,\\
&G_{23}=G_{32}=8\pi GT_{23}-\Lambda g_{23} \Leftrightarrow \frac{1}{2}\partial_yV\partial_zV=8\pi G\pi_{23}\,.
\end{aligned} \tag{125}
$$

B Covariant notation and orthogonality relations

It is interesting to decompose all the EMT components (16) in a covariant notation. Here we use the complete spacetime metric $g_{\mu\nu}$. We do that for the two cases of section 3.2. We use also results of Appendix (A).

- Case 1: We have the same expressions for the energy density and the heat vector (55), but the decomposition of $p_{\mu\nu}$ and the anisotropic stress tensor read:

$$(p_{\mu\nu}) = \frac{1}{8\pi G}\begin{pmatrix} V^2\Lambda & -V\Lambda & 0 & 0 \\ -V\Lambda & \Lambda & 0 & 0 \\ 0 & 0 & \Lambda - \partial_x A & 0 \\ 0 & 0 & 0 & \Lambda - \partial_x A \end{pmatrix} = \begin{pmatrix} V^2 p + \pi_{00} & -Vp + \pi_{0x} & 0 & 0 \\ -Vp + \pi_{x0} & p + \pi_{xx} & 0 & 0 \\ 0 & 0 & p + \pi_{yy} & 0 \\ 0 & 0 & 0 & p + \pi_{zz} \end{pmatrix},$$

$$(\pi_{\mu\nu}) = \begin{pmatrix} \pi_{00} & \pi_{0x} & \pi_{0y} & \pi_{0z} \\ \pi_{x0} & \pi_{xx} & \pi_{xy} & \pi_{xz} \\ \pi_{y0} & \pi_{yx} & \pi_{yy} & \pi_{yz} \\ \pi_{z0} & \pi_{zx} & \pi_{zy} & \pi_{zz} \end{pmatrix} = \begin{pmatrix} T_{00} - \epsilon - 2q_0 - V^2 p & T_{10} - q_1 + Vp & T_{20} - q_2 & T_{30} - q_3 \\ T_{01} - q_1 + Vp & \pi_{xx} & 0 & 0 \\ T_{02} - q_2 & 0 & \pi_{yy} & 0 \\ T_{30} - q_3 & 0 & 0 & \pi_{zz} \end{pmatrix},$$

$$(\pi_{\mu\nu}) = \frac{1}{8\pi G}\begin{pmatrix} \frac{2}{3}V^2\partial_x A & -\frac{2}{3}V\partial_x A & 0 & 0 \\ -\frac{2}{3}V\partial_x A & \frac{2}{3}\partial_x A & 0 & 0 \\ 0 & 0 & -\frac{1}{3}\partial_x A & 0 \\ 0 & 0 & 0 & -\frac{1}{3}\partial_x A \end{pmatrix}.$$

Now we write the components of the anisotropic stress tensor, its trace and its non-diagonal elements:

$$\pi_{\mu\nu} = p_{\mu\nu} - ph_{\mu\nu} \quad \text{and} \quad h^{\mu\nu}\pi_{\mu\nu} = 0\,,$$

$$\Rightarrow \pi_{00} = p_{00} - ph_{00} = V^2\left(\frac{\Lambda}{8\pi G} - p\right)\,,$$

$$\Rightarrow \pi_{0x} = \pi_{x0} = p_{0x} - ph_{0x} = V\left(\frac{-\Lambda}{8\pi G} + p\right)\,,$$

$$\Rightarrow \pi_{xx} = p_{xx} - ph_{xx} = \frac{\Lambda}{8\pi G} - p\,,$$

$$\Rightarrow \pi_{yy} = p_{yy} - ph_{yy} = \frac{\Lambda}{8\pi G} - \frac{1}{8\pi G}\partial_x A - p\,,$$

$$\Rightarrow \pi_{zz} = p_{zz} - ph_{zz} = \frac{\Lambda}{8\pi G} - \frac{1}{8\pi G}\partial_x A - p\,.$$

So we obtain:

$$h^{xx}\pi_{xx} + h^{yy}\pi_{yy} + h^{zz}\pi_{zz} = 0 \quad \Rightarrow 3\frac{\Lambda}{8\pi G} - 2\frac{1}{8\pi G}\partial_x A - 3p = 0\,, \quad \text{if} \quad p = \frac{1}{8\pi G}\left(\Lambda - \frac{2}{3}\partial_x A\right)\,. \tag{127}$$

- Orthogonality of heat flux $n_\mu q^\mu = 0$:

$$n_0 q^0 + n_1 q^1 + n_2 q^2 + n_3 q^3 = 0 \quad \text{is verified.} \tag{128}$$

- Orthogonality of the tracefree part of the stress tensor $n^\mu \pi_{\mu\nu} = 0$:

$$\nu = 0 \quad \Rightarrow \quad n^0\pi_{00} + n^1\pi_{10} = 0 \quad \text{is verified.} \tag{129}$$

$$\nu = 1 \quad \Rightarrow \quad n^0\pi_{01} + n^1\pi_{11} = 0 \quad \text{is verified.} \tag{130}$$

- Case 2: We have the same expression of energy density and the heat vector (75). But the special decomposition for $p_{\mu\nu}$ and the anisotropic stress tensor are:

$$(p_{\mu\nu}) = \frac{1}{8\pi G}\begin{pmatrix} V^2\left[\left(\frac{2}{3}\partial_x A - 2\Omega^2\right)+p\right] & -V\left[\left(\frac{2}{3}\partial_x A - 2\Omega^2\right)+p\right] & -V\frac{1}{2}\left(\partial_y A - \Omega^z\partial_x V\right) & -V\frac{1}{2}\left(\partial_z A + \Omega^y\partial_x V\right) \\ -V\left[\left(\frac{2}{3}\partial_x A - 2\Omega^2\right)+p\right] & \left(\frac{2}{3}\partial_x A - 2\Omega^2\right)+p & \frac{1}{2}\left(\partial_y A - \Omega^z\partial_x V\right) & \frac{1}{2}\left(\partial_z A + \Omega^y\partial_x V\right) \\ -V\frac{1}{2}\left(\partial_y A - \Omega^z\partial_x V\right) & \frac{1}{2}\left(\partial_y A - \Omega^z\partial_x V\right) & \left[-\partial_x A + \Lambda + \Omega^2 - \frac{1}{2}(\partial_z V)^2\right] & -\frac{1}{2}\Omega^y\Omega^z \\ -V\frac{1}{2}\left(\partial_z A + \Omega^y\partial_x V\right) & \frac{1}{2}\left(\partial_z A + \Omega^y\partial_x V\right) & -\frac{1}{2}\Omega^y\Omega^z & \left[-\partial_x A + \Lambda + \Omega^2 - \frac{1}{2}(\partial_y V)^2\right] \end{pmatrix}$$

$$= \begin{pmatrix} V^2 p + \pi_{00} & -Vp + \pi_{0x} & \pi_{0y} & \pi_{0z} \\ -Vp + \pi_{x0} & p + \pi_{xx} & \pi_{xy} & \pi_{xz} \\ \pi_{y0} & \pi_{yx} & p + \pi_{yy} & \pi_{yz} \\ \pi_{z0} & \pi_{zx} & \pi_{zy} & p + \pi_{zz} \end{pmatrix},$$

$$(\pi_{\mu\nu}) = \frac{1}{8\pi G}\begin{pmatrix} V^2\left(\frac{2}{3}\partial_x A - 2\Omega^2\right) & -V\left(\frac{2}{3}\partial_x A - 2\Omega^2\right) & -V\frac{1}{2}\left(\partial_y A - \Omega^z\partial_x V\right) & -V\frac{1}{2}\left(\partial_z A + \Omega^y\partial_x V\right) \\ -V\left(\frac{2}{3}\partial_x A - 2\Omega^2\right) & \frac{2}{3}\partial_x A - 2\Omega^2 & \frac{1}{2}\left(\partial_y A - \Omega^z\partial_x V\right) & \frac{1}{2}\left(\partial_z A + \Omega^y\partial_x V\right) \\ -V\frac{1}{2}\left(\partial_y A - \Omega^z\partial_x V\right) & \frac{1}{2}\left(\partial_y A - \Omega^z\partial_x V\right) & -\frac{1}{3}\partial_x A + 2\Omega^2 - \frac{1}{2}(\partial_z V)^2 & -\frac{1}{2}\Omega^y\Omega^z \\ -V\frac{1}{2}\left(\partial_z A + \Omega^y\partial_x V\right) & \frac{1}{2}\left(\partial_z A + \Omega^y\partial_x V\right) & -\frac{1}{2}\Omega^y\Omega^z & -\frac{1}{3}\partial_x A + 2\Omega^2 - \frac{1}{2}(\partial_y V)^2 \end{pmatrix}.$$

- Orthogonality of heat flux $n_\mu q^\mu = 0$ and $q_\mu n^\mu = 0$:

$$\begin{aligned} n_0 q^0 + n_1 q^1 + n_2 q^2 + n_3 q^3 = 0 \quad &\text{is verified.} \\ q_0 n^0 + q_1 n^1 + q_2 n^2 + q_3 n^3 = 0 \quad &\text{is verified.} \end{aligned} \tag{131}$$

- Orthogonality of the tracefree part of the stress tensor $n^\mu \pi_{\mu\nu} = 0$:

$$\begin{aligned} \nu = 0 \quad &\Rightarrow \quad n^0\pi_{00} + n^1\pi_{10} = 0 \quad \text{is verified.} \\ \nu = 1 \quad &\Rightarrow \quad n^0\pi_{01} + n^1\pi_{11} = 0 \quad \text{is verified.} \\ \nu = 2 \quad &\Rightarrow \quad n^0\pi_{02} + n^1\pi_{12} = 0 \quad \text{is verified.} \\ \nu = 3 \quad &\Rightarrow \quad n^0\pi_{03} + n^1\pi_{13} = 0 \quad \text{is verified.} \end{aligned} \tag{132}$$

C Stress anisotropic scalar

We take the ${p^i}_j$ and p expression respectively from (84) and from (85) components of **Theorem 1** and deduce the anisotropic stress components for the Alcubierre's case for inertial motion (*idem* $\partial_i A = 0$). With this we can calculate the stress anisotropic scalar Π^2. We start to replace the pressure $p = \frac{1}{8\pi G}(-\Omega^2 + \Lambda)$:

$$\begin{aligned} {\pi^x}_x &= \frac{1}{8\pi G}(-3\Omega^2 + \Lambda) - p \quad \Rightarrow \quad {\pi^x}_x = \frac{1}{8\pi G}(-2\Omega^2)\,, \\ {\pi^y}_y &= \frac{1}{8\pi G}\left[\Lambda + \Omega^2 - \frac{1}{2}(\Omega^z)^2\right] - p \quad \Rightarrow \quad {\pi^y}_y = \frac{1}{8\pi G}\left[2\Omega^2 - \frac{1}{2}(\Omega^z)^2\right], \\ {\pi^z}_z &= \frac{1}{8\pi G}\left[\Lambda + \Omega^2 - \frac{1}{2}(-\Omega^y)^2\right] - p \quad \Rightarrow \quad {\pi^z}_z = \frac{1}{8\pi G}\left[2\Omega^2 - \frac{1}{2}(\Omega^y)^2\right]. \end{aligned} \tag{133}$$

We have $\pi^{\mu\nu} = h^{\mu\alpha}h^{\nu\beta}\pi_{\alpha\beta}$ and we see immediately that the 0-components are null. Furthermore, ${\pi^\mu}_\nu = h^{\mu\alpha}\pi_{\alpha\nu}$. We deduce that we can use only spatial components and $\pi^{ij} = {\pi^i}_j = \pi_{ij}$. So we can simplify the scalar expression as follows:

$$\Pi^2 = \frac{1}{2}\pi^{ij}\pi_{ij} = \frac{1}{2}(\pi_{xx}^2 + \pi_{yy}^2 + \pi_{zz}^2 + 2\pi_{xy}^2 + 2\pi_{xz}^2 + 2\pi_{yz}^2)\,. \tag{134}$$

Now we replace and simplify:

$$\pi_{xx}^2 = \frac{1}{(8\pi G)^2}(4\Omega^4)\,,$$

$$\pi_{yy}^2 = \frac{1}{(8\pi G)^2}\left[4\Omega^4 - 2\Omega^2(\Omega^z)^2 + \frac{1}{4}(\Omega^z)^4\right],$$

$$\pi_{zz}^2 = \frac{1}{(8\pi G)^2}\left[4\Omega^4 - 2\Omega^2(\Omega^y)^2 + \frac{1}{4}(\Omega^y)^4\right],$$

$$\Rightarrow \quad \pi_{xx}^2 + \pi_{yy}^2 + \pi_{zz}^2 = \frac{1}{(8\pi G)^2}\left[12\Omega^2 - 8\Omega^2 + 4\Omega^4 - \frac{1}{4}2(\Omega^z)^2(\Omega^y)^2\right] = \frac{1}{(8\pi G)^2}\left[8\Omega^2 - \frac{1}{4}2(\Omega^z)^2(\Omega^y)^2\right],$$

$$\pi_{xy}^2 = \frac{1}{(8\pi G)^2}\frac{1}{4}(\Omega^z)^2(\partial_x V)^2\,,$$

$$\pi_{xz}^2 = \frac{1}{(8\pi G)^2}\frac{1}{4}(\Omega^y)^2(\partial_x V)^2\,,$$

$$\pi_{yz}^2 = \frac{1}{(8\pi G)^2}\frac{1}{4}(\Omega^y)^2(\Omega^z)^2\,,$$

$$\Rightarrow \quad 2\pi_{xy}^2 + 2\pi_{xz}^2 + 2\pi_{yz}^2 = \frac{1}{(8\pi G)^2}\left[2\Omega^2\Theta^2 + \frac{1}{4}2(\Omega^z)^2(\Omega^y)^2\right],$$

$$\Rightarrow \quad \pi_{xx}^2 + \pi_{yy}^2 + \pi_{zz}^2 + 2\pi_{xy}^2 + 2\pi_{xz}^2 + 2\pi_{yz}^2 = \frac{1}{(8\pi G)^2}\left(8\Omega^2 + 2\Omega^2\Theta^2\right)\,. \tag{135}$$

So we obtain the stress anisotropic scalar:

$$\Pi^2 = \frac{1}{(8\pi G)^2}(4\Omega^2 + \Omega^2\Theta^2)\,. \tag{136}$$

D Figure Appendix

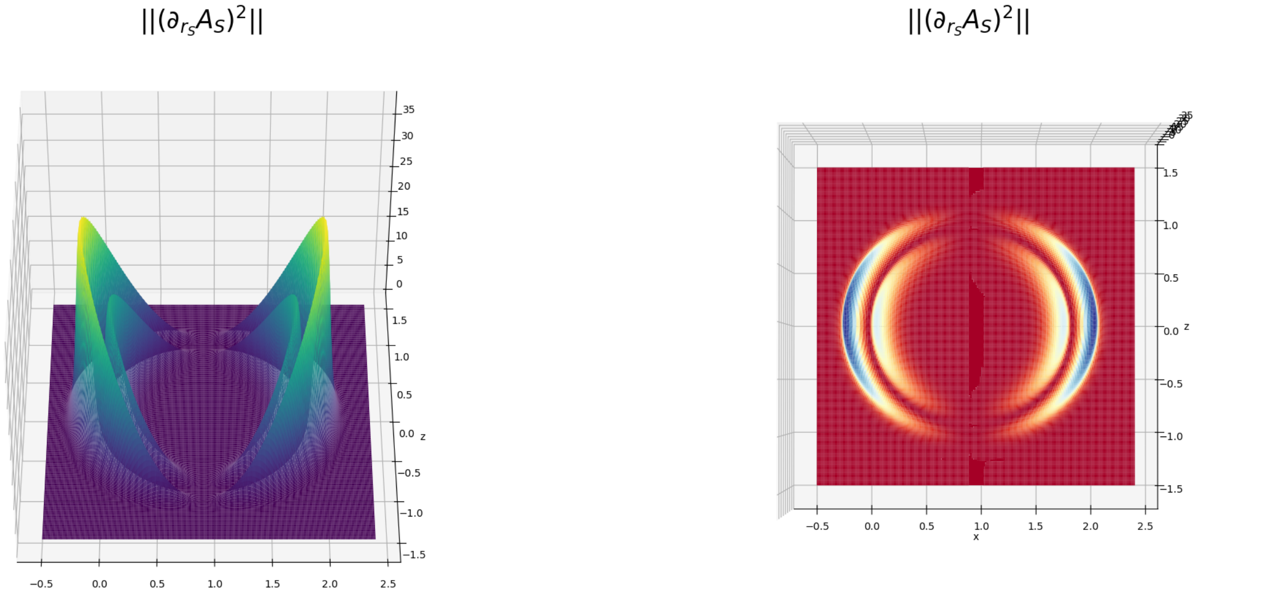

Figure 7: Normalization of the derivative of the coordinate acceleration field. Two views are proposed, one facing us with the direction of movement along x from left to right and another from above.

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