R_Warp_design_without_vorticity

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1 R-Warp design with one velocity component (case without vorticity)

[1]:
```
%display latex
```

2 1. Manifold

[2]:
```
M = Manifold(4, 'M', structure="Lorentzian")
N = Manifold(3, 'N', ambient=M, structure="Riemannian")
```

3 2. Chart

[3]:
```
C.<t,x,y,z> = M.chart(r't:(-oo,+oo) x:(-oo,+oo) y:(-oo,+oo) z:(-oo,+oo)')
```

[4]:
```
C0.<x0,y0,z0> = N.chart(r'x0:(-oo,+oo) y0:(-oo,+oo) z0:(-oo,+oo)')
```

4 3. Metric

[5]:
```
g=M.metric(name='g')
```

5 3.1. Functions

[6]:
```
V = M.scalar_field(function('V')(t,x), name='V')
```

[7]:
```
A = M.scalar_field(function('A')(t,x), name='A')
```

6 3.2. Components of the metric

[8]:
```
g[0,0]=-1 + V**2
g[1,1]=1
g[0,1]=-V
g[2,2]=1
```

```
g[3,3]=1
```

[9]:
```
g.display()
```

[9]:
$$g = \left(V\left(t, x\right)^2 - 1\right) \mathrm{d}t \otimes \mathrm{d}t - V\left(t, x\right) \mathrm{d}t \otimes \mathrm{d}x - V\left(t, x\right) \mathrm{d}x \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z$$

[10]:
```
g[:]
```

[10]:
$$\left(\begin{array}{rrrr} V\left(t, x\right)^2 - 1 & -V\left(t, x\right) & 0 & 0 \\ -V\left(t, x\right) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

7 3.3. Inverse metrical

[11]:
```
ginv = g.inverse()
ginv[:]
```

[11]:
$$\left(\begin{array}{rrrr} -1 & -V\left(t, x\right) & 0 & 0 \\ -V\left(t, x\right) & -V\left(t, x\right)^2 + 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

8 Replacement $A = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x}$

[12]:
```
dV = V.differential()
```

[13]:
```
ptV = dV[C.frame(),0]
```

[14]:
```
pxV = dV[C.frame(),1]
```

[15]:
```
pxV.expr()
```

[15]:
$$\frac{\partial}{\partial x}V\left(t, x\right)$$

[17]:
```
repptV = A - V*pxV
remplacement = { ptV.expr(): repptV.expr() }

expr_modifiee = ptV.expr().subs(remplacement)
show(expr_modifiee)
```

$$-V\left(t, x\right) \frac{\partial}{\partial x}V\left(t, x\right) + A\left(t, x\right)$$

[18]:
```
ptV.expr().subs(remplacement)
```

[18]:
$$-V\left(t, x\right) \frac{\partial}{\partial x}V\left(t, x\right) + A\left(t, x\right)$$

9 4. Connection

[19]:
```
nab = g.connection()
```

10 5. Christoffel symbols

[20]:
```
nab.display(only_nonredundant=True)
```

[20]:

$$\begin{array}{lcl} \Gamma^{t}{}_{t\,t} & = & V\left(t, x\right)^{2} \frac{\partial\,V}{\partial x} \\ \Gamma^{t}{}_{t\,x} & = & -V\left(t, x\right) \frac{\partial\,V}{\partial x} \\ \Gamma^{t}{}_{x\,x} & = & \frac{\partial\,V}{\partial x} \\ \Gamma^{x}{}_{t\,t} & = & \left(V\left(t, x\right)^{3} - V\left(t, x\right)\right) \frac{\partial\,V}{\partial x} - \frac{\partial\,V}{\partial t} \\ \Gamma^{x}{}_{t\,x} & = & -V\left(t, x\right)^{2} \frac{\partial\,V}{\partial x} \\ \Gamma^{x}{}_{x\,x} & = & V\left(t, x\right) \frac{\partial\,V}{\partial x} \end{array}$$

11 5.1. Curvature

[21]:
```
Ric=g.ricci()
Scal=Ric['_{ij}']*ginv['^{ij}']
Ein = Ric-(Scal/2)*g
Riem = g.riemann()
```

12 5.2. Riemann and Ricci Tensors

[22]:
```
Riem.display()
```

[22]:

$$\begin{aligned} \mathrm{Riem}\left(g\right) = & \left(-V\left(t, x\right) \left(\frac{\partial\,V}{\partial x}\right)^{2} - V\left(t, x\right)^{2} \frac{\partial^2\,V}{\partial x^2} - V\left(t, x\right) \frac{\partial^2\,V}{\partial t\partial x} \right) \frac{\partial}{\partial t} \otimes \mathrm{d} t \otimes \mathrm{d} t \otimes \mathrm{d} x + \left(V\left(t, x\right) \left(\frac{\partial\,V}{\partial x}\right)^{2} + V\left(t, x\right)^{2} \frac{\partial^2\,V}{\partial x^2} + V\left(t, x\right) \frac{\partial^2\,V}{\partial t\partial x} \right) \frac{\partial}{\partial t} \otimes \mathrm{d} t \otimes \mathrm{d} x \otimes \mathrm{d} t \\ & + \left(\left(\frac{\partial\,V}{\partial x}\right)^{2} + V\left(t, x\right) \frac{\partial^2\,V}{\partial x^2} + \frac{\partial^2\,V}{\partial t\partial x} \right) \frac{\partial}{\partial t} \otimes \mathrm{d} x \otimes \mathrm{d} t \otimes \mathrm{d} x + \left(-\left(\frac{\partial\,V}{\partial x}\right)^{2} - V\left(t, x\right) \frac{\partial^2\,V}{\partial x^2} - \frac{\partial^2\,V}{\partial t\partial x} \right) \frac{\partial}{\partial t} \otimes \mathrm{d} x \otimes \mathrm{d} x \otimes \mathrm{d} t \\ & + \left(-\left(V\left(t, x\right)^{2} - 1\right) \left(\frac{\partial\,V}{\partial x}\right)^{2} - \left(V\left(t, x\right)^{2} - 1\right) \frac{\partial^2\,V}{\partial t\partial x} - \left(V\left(t, x\right)^{3} - V\left(t, x\right)\right) \frac{\partial^2\,V}{\partial x^2} \right) \frac{\partial}{\partial x} \otimes \mathrm{d} t \otimes \mathrm{d} t \otimes \mathrm{d} x \\ & + \left(\left(V\left(t, x\right)^{2} - 1\right) \left(\frac{\partial\,V}{\partial x}\right)^{2} + \left(V\left(t, x\right)^{2} - 1\right) \frac{\partial^2\,V}{\partial t\partial x} + \left(V\left(t, x\right)^{3} - V\left(t, x\right)\right) \frac{\partial^2\,V}{\partial x^2} \right) \frac{\partial}{\partial x} \otimes \mathrm{d} t \otimes \mathrm{d} x \otimes \mathrm{d} t \\ & + \left(V\left(t, x\right) \left(\frac{\partial\,V}{\partial x}\right)^{2} + V\left(t, x\right)^{2} \frac{\partial^2\,V}{\partial x^2} + V\left(t, x\right) \frac{\partial^2\,V}{\partial t\partial x} \right) \frac{\partial}{\partial x} \otimes \mathrm{d} x \otimes \mathrm{d} t \otimes \mathrm{d} x + \left(-V\left(t, x\right) \left(\frac{\partial\,V}{\partial x}\right)^{2} - V\left(t, x\right)^{2} \frac{\partial^2\,V}{\partial x^2} - V\left(t, x\right) \frac{\partial^2\,V}{\partial t\partial x} \right) \frac{\partial}{\partial x} \otimes \mathrm{d} x \otimes \mathrm{d} x \otimes \mathrm{d} t \end{aligned}$$

[23]:
```
Ric.display()
```

[23]:

$$\mathrm{Ric}\left(g\right) = \left(\left(V\left(t,x\right)^{2}-1\right)\left(\frac{\partial\,V}{\partial x}\right)^{2}+\left(V\left(t,x\right)^{2}-1\right)\frac{\partial^{2}\,V}{\partial t\partial x}+\left(V\left(t,x\right)^{3}-V\left(t,x\right)\right)\frac{\partial^{2}\,V}{\partial x^{2}}\right)\mathrm{d}t\otimes\mathrm{d}t+\left(-V\left(t,x\right)\left(\frac{\partial\,V}{\partial x}\right)^{2}-V\left(t,x\right)^{2}\frac{\partial^{2}\,V}{\partial x^{2}}-V\left(t,x\right)\frac{\partial^{2}\,V}{\partial t\partial x}\right)\mathrm{d}t\otimes\mathrm{d}x+\left(-V\left(t,x\right)\left(\frac{\partial\,V}{\partial x}\right)^{2}-V\left(t,x\right)^{2}\frac{\partial^{2}\,V}{\partial x^{2}}-V\left(t,x\right)\frac{\partial^{2}\,V}{\partial t\partial x}\right)\mathrm{d}x\otimes\mathrm{d}t+\left(\left(\frac{\partial\,V}{\partial x}\right)^{2}+V\left(t,x\right)\frac{\partial^{2}\,V}{\partial x^{2}}+\frac{\partial^{2}\,V}{\partial t\partial x}\right)\mathrm{d}x\otimes\mathrm{d}x$$

13 5.3. Components of Ricci

[24]:
```
Ric[0,0]
```

[24]:

$$\left(V\left(t,x\right)^{2}-1\right)\left(\frac{\partial\,V}{\partial x}\right)^{2}+\left(V\left(t,x\right)^{2}-1\right)\frac{\partial^{2}\,V}{\partial t\partial x}+\left(V\left(t,x\right)^{3}-V\left(t,x\right)\right)\frac{\partial^{2}\,V}{\partial x^{2}}$$

[25]:
```
Ric[1,1]
```

[25]:

$$\left(\frac{\partial\,V}{\partial x}\right)^{2}+V\left(t,x\right)\frac{\partial^{2}\,V}{\partial x^{2}}+\frac{\partial^{2}\,V}{\partial t\partial x}$$

[26]:
```
Ric[1,1].expr().subs(remplacement)
```

[26]:

$$\frac{\partial}{\partial x}V\left(t,x\right)^{2}+V\left(t,x\right)\frac{\partial^{2}}{(\partial x)^{2}}V\left(t,x\right)+\frac{\partial^{2}}{\partial t\partial x}V\left(t,x\right)$$

[27]:
```
Ric[2,2]
```

[27]: 0

[28]:
```
Ric[3,3]
```

[28]: 0

[29]:
```
Ric[0,1]
```

[29]:

$$-V\left(t,x\right)\left(\frac{\partial\,V}{\partial x}\right)^{2}-V\left(t,x\right)^{2}\frac{\partial^{2}\,V}{\partial x^{2}}-V\left(t,x\right)\frac{\partial^{2}\,V}{\partial t\partial x}$$

[30]:
```
Ric[0,2]
```

[30]: 0

[31]:
```
Ric[0,3]
```

[31]: 0

[32]:
```
Ric[1,2]
```

[32]:

0

[33]:
```
Ric[1,3]
```

[33]:

0

[34]:
```
Ric[2,3]
```

[34]:

0

14 5.4. Ricci scalar

[35]:
```
Scal.expr()
```

[35]:

$$2\frac{\partial}{\partial x}V(t,x)^2 + 2V(t,x)\frac{\partial^2}{(\partial x)^2}V(t,x) + 2\frac{\partial^2}{\partial t\partial x}V(t,x)$$

15 5.5. Components of Einstein tensor

[36]:
```
Ein[:]
Ein.display()
```

[36]:

$$\mathrm{Ric}(g) - \text{unnamed metric} = \left(-\left(\frac{\partial V}{\partial x}\right)^2 - V(t,x)\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial t\partial x}\right)\mathrm{d}y\otimes\mathrm{d}y + \left(-\left(\frac{\partial V}{\partial x}\right)^2 - V(t,x)\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial t\partial x}\right)\mathrm{d}z\otimes\mathrm{d}z$$

16 6. Energy Momentum Tensor (EMT)

[37]:
```
var('Lambda_')
var('G')
T = 1/(8*pi*G) * ( Ein + g*Lambda_ )
```

[38]:
```
T[:]
```

[38]:

$$\begin{pmatrix} \frac{\Lambda V(t,x)^2-\Lambda}{8\pi G} & -\frac{\Lambda V(t,x)}{8\pi G} & 0 & 0 \\ -\frac{\Lambda V(t,x)}{8\pi G} & \frac{\Lambda}{8\pi G} & 0 & 0 \\ 0 & 0 & -\frac{\left(\frac{\partial V}{\partial x}\right)^2+V(t,x)\frac{\partial^2 V}{\partial x^2}-\Lambda+\frac{\partial^2 V}{\partial t\partial x}}{8\pi G} & 0 \\ 0 & 0 & 0 & -\frac{\left(\frac{\partial V}{\partial x}\right)^2+V(t,x)\frac{\partial^2 V}{\partial x^2}-\Lambda+\frac{\partial^2 V}{\partial t\partial x}}{8\pi G} \end{pmatrix}$$

17 6.1. Components of EMT

18 6.1.1. Lapse function $N_0 = \sqrt(|g^{00}|)$

[39]:
```
N_0 = sqrt(abs(ginv[0,0]))
N_0.display()
```

[39]:

$$(t, x, y, z) \mapsto 1$$

19 6.1.2. Compute shift vector N^i

[40]:
```
N = [ginv[0, i] for i in range(1, 4)]
N[0],N[1], N[2]
```

[40]:

$$(-V(t, x), 0, 0)$$

20 6.1.3. Define the fluid 4-velocity u^μ

[41]:
```
u = M.vector_field(name="u")
u[0] = 1 / N_0 # u_MU = 1
for i in range(1, 4):
    u[i] = - N[i - 1]/ N_0  # Spatial components : u_MU[1]=-V(t,x), u_MU[2]=0,␣
↪u_MU[3]=0
u.display()
```

[41]:

$$u = \frac{\partial}{\partial t} + V(t, x)\frac{\partial}{\partial x}$$

[42]:
```
u_down = u.down(g)
u_down[:]
```

[42]:

$$[-1, 0, 0, 0]$$

21 6.1.5. Energy density ϵ

[43]:
```
Ein[:]
```

[43]:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\left(\frac{\partial V}{\partial x}\right)^2 - V(t, x)\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial t \partial x} & 0 \\ 0 & 0 & 0 & -\left(\frac{\partial V}{\partial x}\right)^2 - V(t, x)\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial t \partial x} \end{pmatrix}$$

[44]:
```
uu= u['^i'] * u['^j']
uu[:]
```

[44]:

$$\begin{pmatrix} 1 & V(t,x) & 0 & 0 \\ V(t,x) & V(t,x)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[45]:
```
epsilon=uu['^{ij}'] * T['_{ij}']
epsilon.display()
```

[45]:

$$\begin{array}{llcl} & M & \longrightarrow & \mathbb{R} \\ & (t,x,y,z) & \longmapsto & -\frac{\Lambda}{8\pi G} \end{array}$$

22 6.1.6. Pressure tensor $p_{ij} = ph_{ij} + \pi_{ij}$

[46]:
```
T[1,1], T[1,2], T[1,3]
```

[46]:

$$\left(\frac{\Lambda}{8\pi G}, 0, 0\right)$$

[47]:
```
T[2,1], T[2,2], T[2,3]
```

[47]:

$$\left(0, -\frac{\left(\frac{\partial V}{\partial x}\right)^2 + V(t,x)\frac{\partial^2 V}{\partial x^2} - \Lambda + \frac{\partial^2 V}{\partial t \partial x}}{8\pi G}, 0\right)$$

[48]:
```
T[3,1], T[3,2], T[3,3]
```

[48]:

$$\left(0, 0, -\frac{\left(\frac{\partial V}{\partial x}\right)^2 + V(t,x)\frac{\partial^2 V}{\partial x^2} - \Lambda + \frac{\partial^2 V}{\partial t \partial x}}{8\pi G}\right)$$

23 6.1.7. Define the spatial projector $b^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu = h^{\mu\nu}$

[49]:
```
b = u['^i'] * u['^j'] + ginv
b[:]
```

[49]:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

24 6.1.8. Spatial heat vector $q_k = -b^{ik}u^jT_{ij}$

[50]:
```
bu = -b['^{ik}']*u['^j']
q2 = bu['^{ikj}']*T['_{ij}']
q2.display()
```

[50]: 0

[51]:
```
d =  g+u_down['_i'] * u_down['_j']
d[:]
```

[51]:

$$\begin{pmatrix} V(t,x)^2 & -V(t,x) & 0 & 0 \\ -V(t,x) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[52]:
```
r = u['^i'] * u_down['_j']
r[:]
```

[52]:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -V(t,x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[53]:
```
# f^i_j = u^i * u_j
f = M.tensor_field(2, 0, name='f')
for i in range(4):
    for j in range(4):
        f[i, j] = u[i] * u_down[j]  # u^i * u_j

# Ajout de delta^i_j
for i in range(4):
    f[i, i] += 1
f[:]
```

[53]:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -V(t,x) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[54]:
```
T[:]
```

[54]:

$$\begin{pmatrix} \frac{\Lambda V(t,x)^2-\Lambda}{8\pi G} & -\frac{\Lambda V(t,x)}{8\pi G} & 0 & 0 \\ -\frac{\Lambda V(t,x)}{8\pi G} & \frac{\Lambda}{8\pi G} & 0 & 0 \\ 0 & 0 & -\frac{\left(\frac{\partial V}{\partial x}\right)^2+V(t,x)\frac{\partial^2 V}{\partial x^2}-\Lambda+\frac{\partial^2 V}{\partial t\partial x}}{8\pi G} & 0 \\ 0 & 0 & 0 & -\frac{\left(\frac{\partial V}{\partial x}\right)^2+V(t,x)\frac{\partial^2 V}{\partial x^2}-\Lambda+\frac{\partial^2 V}{\partial t\partial x}}{8\pi G} \end{pmatrix}$$

[55]:
```
p_mixed = ginv['^{ik}'] * T['_{kj}']
p_mixed[:]
```

[55]:

$$\begin{pmatrix} \frac{\Lambda}{8\pi G} & 0 & 0 & 0 \\ 0 & \frac{\Lambda}{8\pi G} & 0 & 0 \\ 0 & 0 & -\frac{\left(\frac{\partial V}{\partial x}\right)^2+V(t,x)\frac{\partial^2 V}{\partial x^2}-\Lambda+\frac{\partial^2 V}{\partial t\partial x}}{8\pi G} & 0 \\ 0 & 0 & 0 & -\frac{\left(\frac{\partial V}{\partial x}\right)^2+V(t,x)\frac{\partial^2 V}{\partial x^2}-\Lambda+\frac{\partial^2 V}{\partial t\partial x}}{8\pi G} \end{pmatrix}$$

[56]:
```
T11 = ginv[1,1] * T[1,1] +ginv[1,0] * T[0,1]
T11
```

[56]:

$$\frac{\Lambda}{8\pi G}$$

[57]:
```
T22 = ginv[2,2] * T[2,2]
T22
```

[57]:

$$-\frac{\left(\frac{\partial V}{\partial x}\right)^2 + V(t,x)\frac{\partial^2 V}{\partial x^2} - \Lambda + \frac{\partial^2 V}{\partial t\partial x}}{8\pi G}$$

[58]:
```
Ein_mixed = ginv['^{ik}'] * Ein['_{kj}']
Ein_mixed[:]
```

[58]:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\left(\frac{\partial V}{\partial x}\right)^2 - V(t,x)\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial t\partial x} & 0 \\ 0 & 0 & 0 & -\left(\frac{\partial V}{\partial x}\right)^2 - V(t,x)\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial t\partial x} \end{pmatrix}$$

[59]:
```
p_mixed = b['^{ik}'] * T['_{kj}']
p_mixed[:]
```

[59]:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{\Lambda V(t,x)}{8\pi G} & \frac{\Lambda}{8\pi G} & 0 & 0 \\ 0 & 0 & -\frac{\left(\frac{\partial V}{\partial x}\right)^2 + V(t,x)\frac{\partial^2 V}{\partial x^2} - \Lambda + \frac{\partial^2 V}{\partial t\partial x}}{8\pi G} & 0 \\ 0 & 0 & 0 & -\frac{\left(\frac{\partial V}{\partial x}\right)^2 + V(t,x)\frac{\partial^2 V}{\partial x^2} - \Lambda + \frac{\partial^2 V}{\partial t\partial x}}{8\pi G} \end{pmatrix}$$

[60]:
```
T[:]
```

[60]:

$$\begin{pmatrix} \frac{\Lambda V(t,x)^2 - \Lambda}{8\pi G} & -\frac{\Lambda V(t,x)}{8\pi G} & 0 & 0 \\ -\frac{\Lambda V(t,x)}{8\pi G} & \frac{\Lambda}{8\pi G} & 0 & 0 \\ 0 & 0 & -\frac{\left(\frac{\partial V}{\partial x}\right)^2 + V(t,x)\frac{\partial^2 V}{\partial x^2} - \Lambda + \frac{\partial^2 V}{\partial t\partial x}}{8\pi G} & 0 \\ 0 & 0 & 0 & -\frac{\left(\frac{\partial V}{\partial x}\right)^2 + V(t,x)\frac{\partial^2 V}{\partial x^2} - \Lambda + \frac{\partial^2 V}{\partial t\partial x}}{8\pi G} \end{pmatrix}$$

[61]:
```
g[:]
```

[61]:

$$\begin{pmatrix} V(t,x)^2 - 1 & -V(t,x) & 0 & 0 \\ -V(t,x) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[62]:
```
ginv[:]
```

[62]:

$$\begin{pmatrix} -1 & -V(t,x) & 0 & 0 \\ -V(t,x) & -V(t,x)^2+1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$