Université Claude Bernard Lyon 1

Master 1 Physique fondamentale et applications

Internship report

Characterization and modeling of the properties of the Alcubierre metric in an inertial approach.

Author :
Antony Frackowiak
antony.frackowiak@etu.univ-lyon1.fr

Under the direction of :
Thomas BUCHERT
thomas.buchert@ens-lyon.fr

Duration : 6 weeks

Abstract

In this study, we investigate the kinematic properties of the Alcubierre warp drive metric using both Eulerian and Lagrangian coordinate representations. We begin by establishing the theoretical framework for the transition between these two coordinate systems in a system moving under inertia, with the aim of describing its exact solution. The Alcubierre metric, known for its potential to allow apparent superluminal travel, is examined in detail. We derive the metric in a 3+1 formalism and explore the assumptions made by Alcubierre. The kinematic decomposition of the velocity gradient, including expansion, shear, and vorticity tensors, is analyzed to understand the deformation and rotation of space around the warp bubble. This work aims to generalize Alcubierre's approach within a Lagrangian framework and provides new insights into the behavior of warp fields under inertial motion. This study contributes to the broader context of inertial systems in cosmology, where similar principles are applied to models of large-scale structures in the Universe.

May-June 2024

Introduction

In 1994 Miguel Alcubierre published an article [1] in which he introduced the concept of warp drive. It is claimed that this metric form would allow to travel at an apparent superluminal speed with respect to an external observer. After this publication many others have followed with the aim of improving or studying the properties of this metric in more detail and in particular to respond to the problems of negative energy such as the work carried out by José Natário [2] and Shaun D.B. Fell and Lavinia Heisenberg [3]. This is the context of our study. We are particularly interested in the kinematic properties of an Alcubierre warp bubble in Eulerian and Lagrangian coordinate representations, and we propose a generalization in the framework of Lagrangian inertial motion.
This report is divided into three sections. In the first section we introduce the theory of the transition between an inertial system in Eulerian and Lagrangian coordinates. In the second section, we begin with a preamble about the Alcubierre metric, and we continue with investigating kinematical properties of its warp field. Finally, in the third section, we investigate a proposal of how to generalize Alcubierre's model and we explain the new kinematic properties based on a Lagrangian approach for the initial conditions from the previous section. In addition, we present a numerical simulation of the new proposal and explain its limitations.

This work is part of a larger framework, as the inertial system also plays a key role in cosmology, where the Zel'dovich approximation as a first-order Lagrangian perturbation model forms a well-known analytical approach [4] and [5]. It successfully describes the formation of the large-scale structure of the Universe compared to full N-body simulations [6]. And finally, this approximation can be transformed to the inertial system by rescaling the time and space coordinates [7].

In everything that follows, we will place ourselves in a Galilean frame of reference. That is to say, we will consider a body that respects the principle of inertia, if no external force is exerted on this body in motion, it continues its course following a geodesic of space-time. In other words, the acceleration is set to zero and we will have a constant velocity field $\boldsymbol{v}$. We will see that the assumption of inertial motion imposed in the Eulerian or in the Lagrangian representation are different.

Notations used

- The comma denotes derivative with respect to x^j, $\partial/\partial x^j$ (in Eulerian formalism).
- A vertical slash $|$ for the derivative with respect to X^k, $\partial/\partial X^k$ (in Lagrangian formalism).
- Einstein's summation convention of summing over double indices, with Latin indices $i, j, k, \ldots = 1, 2, 3$ and Greek indices $\alpha, \beta, \ldots = 0 \ldots, 3$.
- We will set the speed of light $c = 1$.

1 Inertial system in Eulerian and Lagrangian coordinates

We will be interested in the transition from an Eulerian (inertial) coordinate system $\boldsymbol{x}$ to a Lagrangian (non-inertial) coordinate system $\boldsymbol{X}$. In the Eulerian approach the dependent variables are $\varrho(t, \boldsymbol{x})$ and $\boldsymbol{v}(t, \boldsymbol{x})$, whereas in the Lagrangian approach $\boldsymbol{f}(t, \boldsymbol{X})$ is the only dependent field variable.

1.1 Inertial system

We start from the equation of conservation of momentum, which follows from the fundamental principle of dynamics. Using the conditions set out earlier in the introduction, we have:

$$\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{0} \ , \quad \frac{\partial}{\partial t} v^i + v^j {v^i}_{,j} = 0 \ , \tag{1}$$

supplemented by the continuity equation that describes the conservation of mass for a collection of fluid elements as they move with the flow. This equation can be expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho + \varrho \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \ . \tag{2}$$

Equation (1) is called Euler's equation. It is a set of coupled nonlinear partial differential equations. We cannot easily know the explicit solutions. This is why we will first go through a system of Lagrangian coordinates in which the solution can be found. In a second step, we will return to the Eulerian space.

1.2 Transition from Eulerian to Lagrangian formalism

In the Lagrange point of view, we follow the trajectories $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{X}, t)$ of the body studied, where $\boldsymbol{X}$ labels trajectories of fluid elements. We have for (1):

$$\frac{\mathrm{d}}{\mathrm{d}t} v^i = \frac{\partial}{\partial t} v^i = 0 \ , \quad \frac{\mathrm{d}}{\mathrm{d}t} := \frac{\partial}{\partial t}\Big|_{\boldsymbol{x}} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) = \frac{\partial}{\partial t}\Big|_{\boldsymbol{X}} \ . \tag{3}$$

The solution of (3) is then indeed simple:

$$v^i(t, X^k) = v_0^i(X^k) \ . \tag{4}$$

We integrate the previous equation and assume that at $t = t_0$, the position at that moment is $f^i(t_0, X^k) = X^i$, so we obtain the mapping from Lagrangian to Eulerian coordinates as follows:

$$x^i = f^i(t, X^j) = X^i + v_0^i(X^j)(t - t_0) \ , \tag{5}$$

with x^i the Eulerian position coordinate. We deduce from this that $X^k = h^k(t, x^j)$, where $\boldsymbol{h} := \boldsymbol{f}^{-1}$. With the help of the inverse mapping $\boldsymbol{h}$ we are then able to express the velocity and the density in the Eulerian frame. So we express X^k from (5):

$$v^i(t, x^j) = v_0^i(X^k = h^k(x^j, t)) \ , \quad \varrho(t, x^j) = \varrho(t, X^k = h^k(x^j, t)) \ , \tag{6}$$

with $v_0^i(X^k)$ the initial velocity field components.

Equation (2) admits the following exact integral (using (6)):

$$\varrho(t, x^j) = \frac{\varrho_0(X^k = h^k(t, x^j))}{J(t, X^k = h^k(x^j, t))} \ . \tag{7}$$

To show this result we start from the conservation of the total rest mass M_{D_t} in a spatial domain D_t:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} M = \frac{\mathrm{d}}{\mathrm{d}t} \int_{D_t} \varrho(t, x^j) d^3x \ . \tag{8}$$

We change for a Lagrangian domain D_{t_0}:

$$\int_{D_{t_0}} \frac{\mathrm{d}}{\mathrm{d}t} (\varrho(t, X^k) J(t, X^k)) d^3X = 0 \ . \tag{9}$$

Then we obtain the solution:

$$\frac{\mathrm{d}}{\mathrm{d}t} (\varrho J) = 0 \Rightarrow \varrho J = C(X^k) \ , \tag{10}$$

with the Jacobian:

$$J(t, X^k) = \det\Big(\frac{\partial f^i(t, X^k)}{\partial X^j}\Big) = \det\Big(\delta_{ij} + \frac{\partial v_0^i}{\partial X^j}(t - t_0)\Big) \ . \tag{11}$$

At $t = t_0$, we have the initial density $\varrho(t_0, X^k) = \varrho_0(X^k)$ and we find (7).

Now we can numerically evaluate the expressions (6) in Eulerian space for given initial conditions.

In addition, we will look at the expression of the Jacobian in terms of velocity of the object of study. This will enable us to calculate the derivative of the new coordinates with respect to the old ones, in order to deduce the temporal evolution of the warp field's properties. The Jacobian (11) of the quadratic matrix can be explicitly written in terms of principal scalar invariants:

$$J(t, X^k) = 1 + (t - t_0)\mathrm{I} + (t - t_0)^2\mathrm{II} + (t - t_0)^3\mathrm{III} \ , \tag{12}$$

with I the trace, II the dispersion of diagonal components and III the determinant of the matrix:

$$\mathrm{I} = \mathrm{Tr}\Big(\frac{\partial v_0^i}{\partial X^k}\Big) \ , \tag{13}$$

$$\mathrm{II} = \frac{1}{2}\Big(\mathrm{I}^2 - \frac{\partial v_0^i}{\partial X^k}\frac{\partial v_0^k}{\partial X^i}\Big) \ , \tag{14}$$

$$\mathrm{III} = \frac{1}{3}\Big(\frac{\partial v_0^i}{\partial X^k}\frac{\partial v_0^k}{\partial X^j}\frac{\partial v_0^j}{\partial X^i} - \mathrm{I}^3\Big) + \mathrm{I} \cdot \mathrm{II} \ . \tag{15}$$

2 Application to the Alcubierre metric

We will now study the Alcubierre warp drive metric. After a brief introduction on it we will explain how he formulated his concept in an Eulerian approach and then try to generalize his model in a Lagrangian approach. The study of kinematic properties of the Alcubierre warp field will be carried out beyond the analysis of the expansion rate that is found in the literature.

2.1 Metric in 3+1 form and Alcubierre's assumptions

Let us briefly present Alcubierre's metric [1], in the 3+1 formalism of general relativity,

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \; = -(N^2 - N_i N^i)\mathrm{d}t^2 + 2N_i \mathrm{d}x^i \mathrm{d}t + \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j \; , \tag{16}$$

where N is the lapse function and N^i the shift vector field. The metric components in space are γ_{ij}.

Alcubierre uses a flat spatial metric, $\gamma_{ij} = \delta_{ij}$, hence considers the x^i as global inertial coordinates, and considers one-dimensional motion. He specifies the coordinate setting by defining lapse and shift in the above metric as follows:

$$N = 1 \; , \quad N^x = -v_S(t) W(r_S(t)) \; , \quad N^y = N^z = 0 \; , \tag{17}$$

where the speed along the trajectory defined for the spaceship S, $x_S(t) = f_S(t, X_S)$, for any constant position X_s in a Lagrangian coordinate system attached to the center of the warp bubble, is given by:

$$v_S(t) = \frac{\mathrm{d}x_S(t)}{\mathrm{d}t} \; , \tag{18}$$

and the radial distance from the trajectory is:

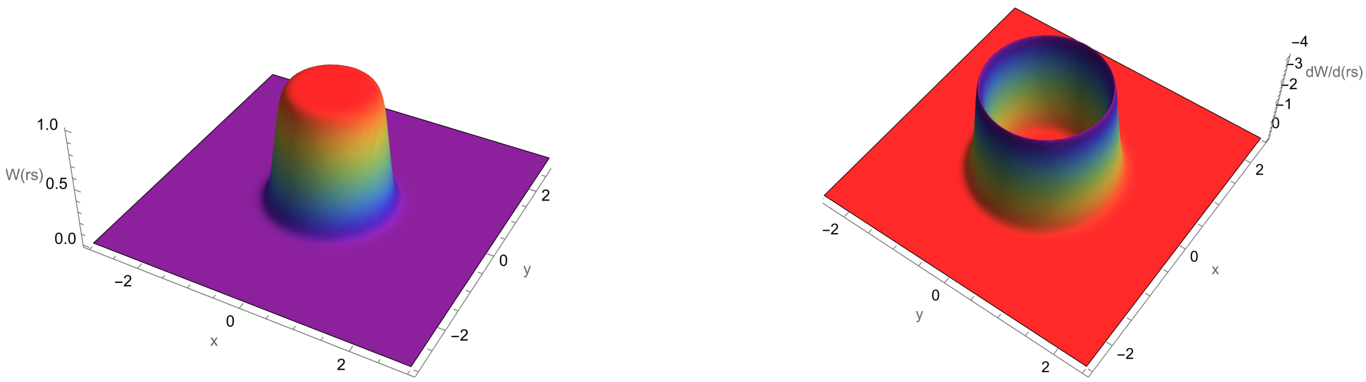
$$r_S(t, \boldsymbol{x}) = [(x - x_S(t))^2 + y^2 + z^2)]^{1/2} \; . \tag{19}$$

Furthermore, the function $W(r_S(t, \boldsymbol{x}))$ determines the shape of the warp field:

$$W(r_S(t, \boldsymbol{x})) = \frac{\tanh\left(\sigma(r_S + R)\right) - \tanh\left(\sigma(r_S - R)\right)}{2\tanh\left(\sigma R\right)} \; , \tag{20}$$

with R a fixed Eulerian radius, and σ a constant that determines the inverse thickness of the wall of the 'warp bubble'. In figure 1 we can see the characteristic form of this function (a), together with its derivative with respect to r_S (b) that we will need later:

$$\frac{\partial W(r_S(t, \boldsymbol{x}))}{\partial r_S} = \frac{1}{2}\coth(R\sigma)\left[-\sigma\,\mathrm{sech}^2\left[\sigma(r_S - R)\right] + \sigma\,\mathrm{sech}^2\left[\sigma(r_S + R)\right]\right] \; . \tag{21}$$



(a) (b)

Figure 1: Representation of the Alcubierre function W in 3D (a) and its derivative with respect to r_S (b), with $\rho = \sqrt{y^2 + z^2}$, $\sigma = 8$, $R = 1$ and $x_s = 0$.

Alcubierre assumes $\boldsymbol{N} = -\boldsymbol{V}_S(t, x^i) = -\boldsymbol{v}_S(t)W(r_S(t, \boldsymbol{x}))$, with $\boldsymbol{V}_S(t, x^i)$ the coordinate velocity. We know the extrinsic curvature tensor K_{ij} from [1], and the expansion rate of the volume elements as seen by an Eulerian observer Θ (figure 2):

$$K_{ij} = \frac{1}{2}(\partial_i N_j + \partial_j N_i) \ , \tag{22}$$

$$\Theta = -\mathrm{Tr}(K_{ij}) = v_S(t)\frac{\partial W(r_S(t, \boldsymbol{x}))}{\partial r_S(t, \boldsymbol{x})}\left(\frac{x - x_S(t)}{r_S(t, \boldsymbol{x})}\right) \ . \tag{23}$$

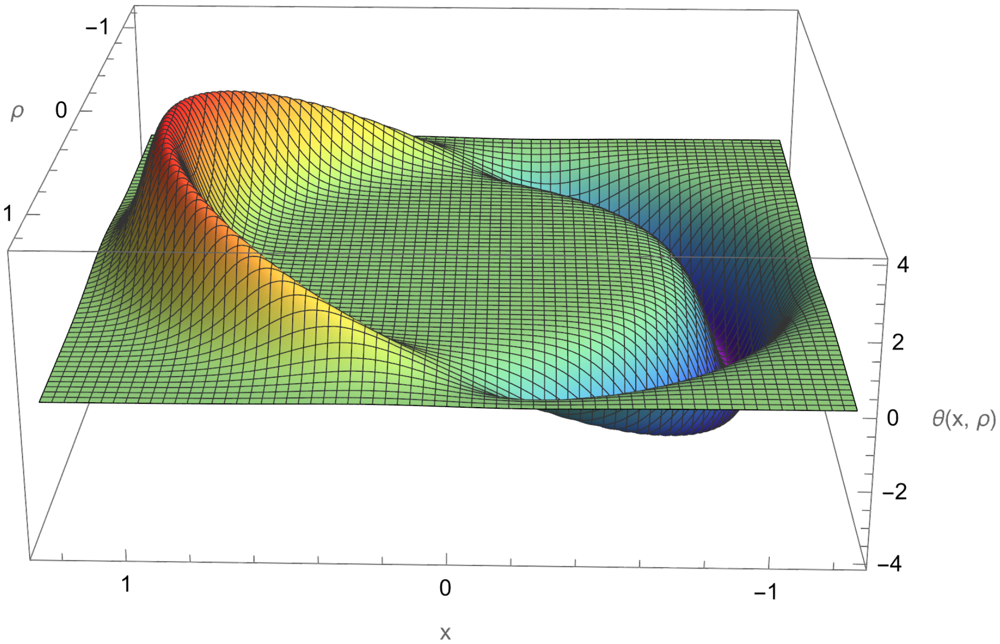


Figure 2: Expansion of the normal volume elements, with $\rho = \sqrt{y^2 + z^2}$, $\sigma = 8$, $R = 1$ and $x_s = 0$.

2.2 Alcubierre's kinematic description of a warp field

The previous definition of the Alcubierre metric is realized in an Eulerian description. From (5) we can interpret the change in coordinates. We will first go into details on the kinematical properties, and we then move to a Lagrangian description.

We have the following kinematic decomposition of the velocity gradient:[1]

$$v_{i,j} = v_{(i,j)} + v_{[i,j]} = \Theta_{ij} + \Omega_{ij} = \frac{1}{3}\Theta\delta_{ij} + \Sigma_{ij} + \Omega_{ij} \ , \tag{24}$$

where Θ_{ij} and Θ are the expansion tensor and the rate of expansion, respectively, Σ_{ij} are the coefficients of the shear tensor, and Ω_{ij} those of the vorticity tensor. The shear is defined by

$$\Sigma_{ij} = v_{(i,j)} - \frac{1}{3}\Theta\delta_{ij} \ . \tag{25}$$

It represents the deformation of the space without volume change. As for the rate of vorticity,

$$\Omega_{ij} = v_{[i,j]} \ , \tag{26}$$

it represents the rotation of the velocity field. We also define the shear scalar Σ and the vorticity scalar Ω:

$$\Sigma^2 = \frac{1}{2}\Sigma_{ij}\Sigma^{ij} \ , \quad \Omega^2 = \frac{1}{2}\Omega_{ij}\Omega^{ij} \ . \tag{27}$$

[1] We denote symmetrization and anti-symmetrization as follows:

$$v_{(i,j)} = \frac{1}{2}(v_{i,j} + v_{j,i}) \ , \ v_{[i,j]} = \frac{1}{2}(v_{i,j} - v_{j,i}).$$

They characterize respectively the deformation of space in the warp bubble and the rotation of the volume elements in the bubble, transported with the moving spaceship.

Now in the case of Alcubierre in a Cartesian coordinate system, we assume a motion in x direction only, and we have one component of the velocity $\boldsymbol{V}_S(t,x) = V_S(t,x)\delta_1^i$. We find for (23), (25), (26) and (27):

$$(\Sigma_{ij}) = \begin{pmatrix} \frac{2}{3}\frac{\partial V_S}{\partial x} & \frac{1}{2}\frac{\partial V_S}{\partial y} & \frac{1}{2}\frac{\partial V_S}{\partial z} \\ \frac{1}{2}\frac{\partial V_S}{\partial y} & -\frac{1}{3}\frac{\partial V_S}{\partial x} & 0 \\ \frac{1}{2}\frac{\partial V_S}{\partial z} & 0 & -\frac{1}{3}\frac{\partial V_S}{\partial x} \end{pmatrix} = v_S \frac{\partial W(r_S)}{\partial r_S} \begin{pmatrix} \frac{2}{3}\frac{\partial r_S}{\partial x} & \frac{1}{2}\frac{\partial r_S}{\partial y} & \frac{1}{2}\frac{\partial r_S}{\partial z} \\ \frac{1}{2}\frac{\partial r_S}{\partial y} & -\frac{1}{3}\frac{\partial r_S}{\partial x} & 0 \\ \frac{1}{2}\frac{\partial r_S}{\partial z} & 0 & -\frac{1}{3}\frac{\partial r_S}{\partial x} \end{pmatrix}$$

$$= \frac{v_S}{r_s}\frac{\partial W(r_S)}{\partial r_S} \begin{pmatrix} \frac{2}{3}(x-x_s) & \frac{1}{2}y & \frac{1}{2}z \\ \frac{1}{2}y & -\frac{1}{3}(x-x_s) & 0 \\ \frac{1}{2}z & 0 & -\frac{1}{3}(x-x_s) \end{pmatrix} \tag{28}$$

$$(\Omega_{ij}) = \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial V_S}{\partial y} & \frac{1}{2}\frac{\partial V_S}{\partial z} \\ -\frac{1}{2}\frac{\partial V_S}{\partial y} & 0 & 0 \\ -\frac{1}{2}\frac{\partial V_S}{\partial z} & 0 & 0 \end{pmatrix} = v_S \frac{\partial W(r_S)}{\partial r_S} \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial r_S}{\partial y} & \frac{1}{2}\frac{\partial r_S}{\partial z} \\ -\frac{1}{2}\frac{\partial r_S}{\partial y} & 0 & 0 \\ -\frac{1}{2}\frac{\partial r_S}{\partial z} & 0 & 0 \end{pmatrix}$$

$$= \frac{v_S}{r_S}\frac{\partial W(r_S)}{\partial r_S} \begin{pmatrix} 0 & \frac{1}{2}y & \frac{1}{2}z \\ -\frac{1}{2}y & 0 & 0 \\ -\frac{1}{2}z & 0 & 0 \end{pmatrix} \tag{29}$$

$$\Theta = v_S \frac{\partial W(r_S)}{\partial r_S}\frac{\partial r_S}{\partial x} = v_S \frac{\partial W(r_S)}{\partial r_S}\frac{x - x_s}{r_S}\ , \tag{30}$$

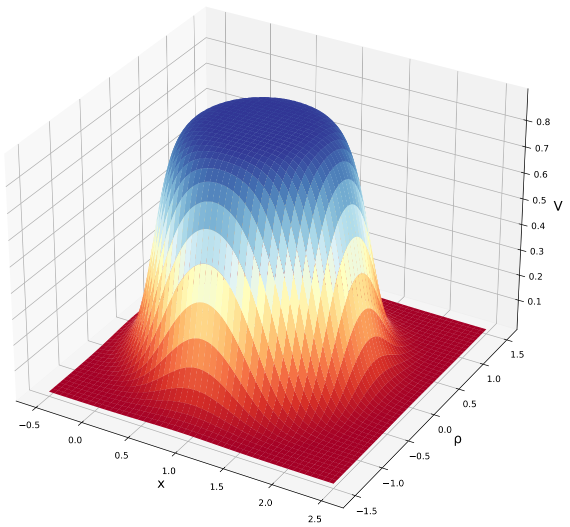
$$\Sigma^2 = v_S^2 \Big(\frac{\partial W(r_S)}{\partial r_S}\Big)^2 \left[\frac{1}{3}\Big(\frac{\partial r_S}{\partial x}\Big)^2 + \frac{1}{4}\Big(\frac{\partial r_S}{\partial y}\Big)^2 + \frac{1}{4}\Big(\frac{\partial r_S}{\partial z}\Big)^2\right]$$

$$= \frac{v_S^2}{r_S^2}\Big(\frac{\partial W(r_S)}{\partial r_S}\Big)^2 \Big(\frac{1}{3}(x - x_S)^2 + \frac{1}{4}y^2 + \frac{1}{4}z^2\Big) \tag{31}$$

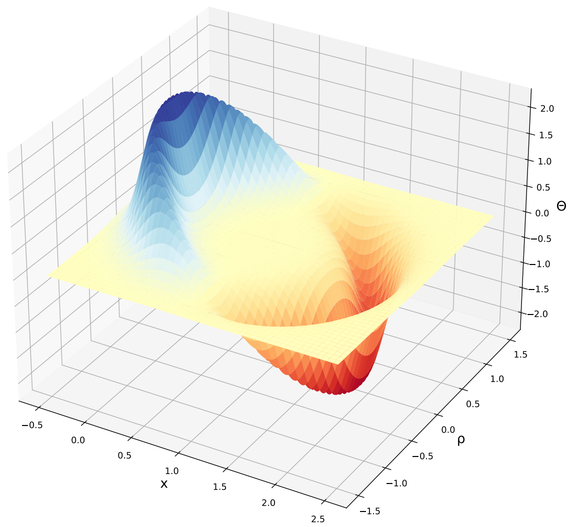
$$\Omega^2 = v_S^2 \Big(\frac{\partial W(r_S)}{\partial r_S}\Big)^2 \left[\frac{1}{4}\Big(\frac{\partial r_S}{\partial y}\Big)^2 + \frac{1}{4}\Big(\frac{\partial r_S}{\partial z}\Big)^2\right]$$

$$= \frac{v_S^2}{r_S^2}\Big(\frac{\partial W(r_S)}{\partial r_S}\Big)^2 \Big(\frac{1}{4}y^2 + \frac{1}{4}z^2\Big)\ . \tag{32}$$

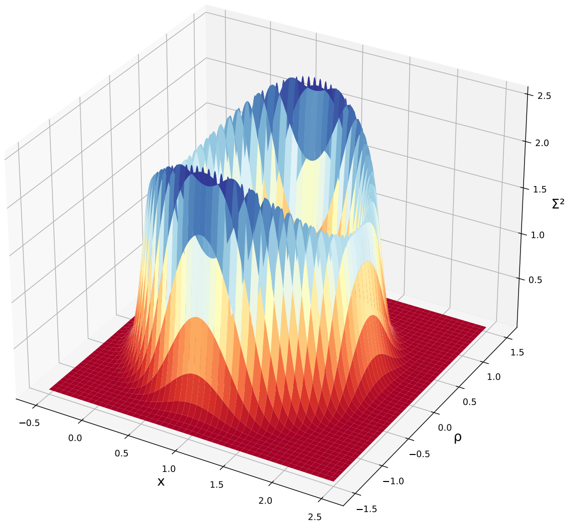
After numerical computation with *Python* we obtain the plot of these properties for a choice of arbitrary parameters $\sigma = 5$, $R = 1$ and for different times t to look at the time evolution. We have chosen to plot in three dimensions (3D), the projection of the kinematical properties in relation to the plane (x, ρ), where $\rho = \sqrt{y^2 + z^2}$. The colors are used to enhance 3D visualization. At time $t = 1$, figure 3, shows the characteristic shape of a warp bubble with radius $R = 1$. As might be expected, when time evolves, we have a displacement in x-direction proportional to the constant velocity v_s with no change in bubble shape. Hence, in this first Eulerian approach, the kinematic variables remain constant.



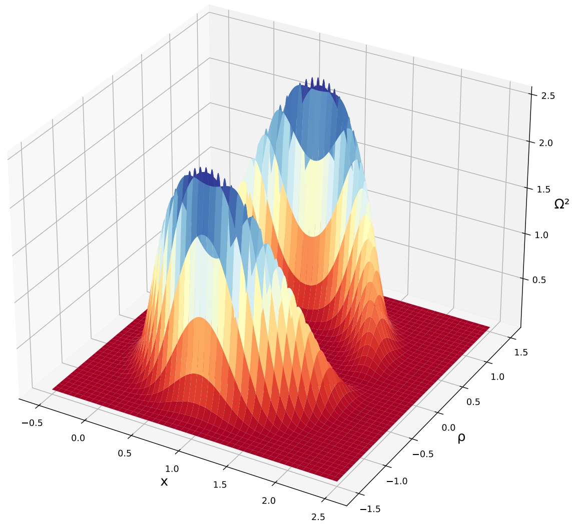
(a) The velocity profile V_S



(b) The rate of expansion Θ



(c) The shear scalar Σ^2



(d) The vorticity scalar Ω^2

Figure 3: 3D representation of kinematical quantities at $t = 1$.

3 New Lagrangian model using Alcubierre initial conditions

We will study in this part a generalization of Alcubierre's warp field in a Lagrangian system of coordinates that moves only in the x-direction. First, we show that the second and third principal scalar invariants (13) are null in this approach. Secondly, we will express the Jacobian and derive the new velocity, density, expansion, shear and vorticity fields. Finally, we plot the warp bubble under these new conditions.

3.1 Lagrangian kinematic fields

Based on the equations of the Jacobian (12) we can express in terms of kinematical variables the components of (13) in the general case:

$$\mathrm{I} = \Theta \ , \tag{33}$$

$$\mathrm{II} = \frac{1}{3}\Theta^2 - \Sigma^2 + \Omega^2 \ , \tag{34}$$

$$\mathrm{III} = \frac{1}{27}\Theta^3 + \frac{1}{3}\Sigma_{ij}\Sigma_{jk}\Sigma_{ki} + \frac{1}{3}\Theta(\Omega^2 - \Sigma^2) + \Omega_i\Omega_j\Sigma_{ij} \ , \tag{35}$$

with the coordinate vorticity vector $\Omega_i := \frac{1}{2}(\boldsymbol{\nabla} \times \boldsymbol{V})^i = -\frac{1}{2}\epsilon_{ijk}\Omega_{jk}$ [11].

In the Alcubierre's case, if we calculate II, with (30), (31) and (32), we have zero:

$$\mathrm{II} = \frac{1}{3}\left(v_S \frac{\partial W(r_S)}{\partial r_S}\frac{x - x_s}{r_S}\right)^2 - \frac{v_S^2}{r_S^2}\left(\frac{\partial W(r_S)}{\partial r_S}\right)^2\left(\frac{1}{3}(x - x_S)^2 + \frac{1}{4}y^2 + \frac{1}{4}z^2\right) + \frac{v_S^2}{r_S^2}\left(\frac{\partial W(r_S)}{\partial r_S}\right)^2\left(\frac{1}{4}y^2 + \frac{1}{4}z^2\right) = 0 \ . \tag{36}$$

In the same way, we can calculate III with two different tools, as before or with this formula, in special case of the Alcubierre model with one component velocity $\boldsymbol{V}_S(t,x) = V_S(t,x)\delta_1^i$, the principal invariants of the velocity gradient can be represented in terms of divergences of vector fields [11]:

$$\mathrm{II} = \mathrm{II}(\boldsymbol{\nabla}\boldsymbol{V}_S) = \frac{1}{2}\boldsymbol{\nabla}\cdot\left[\boldsymbol{V}_S(\boldsymbol{\nabla}\cdot\boldsymbol{V}_S) - (\boldsymbol{V}_S\cdot\boldsymbol{\nabla})\boldsymbol{V}_S\right] \ , \tag{37}$$

$$\mathrm{III} = \mathrm{III}(\boldsymbol{\nabla}\boldsymbol{V}_S) = \frac{1}{3}\boldsymbol{\nabla}\cdot\left(\frac{1}{2}\boldsymbol{\nabla}\cdot\left[\boldsymbol{V}_S(\boldsymbol{\nabla}\cdot\boldsymbol{V}_S) - (\boldsymbol{V}_S\cdot\boldsymbol{\nabla})\boldsymbol{V}_S\right]\boldsymbol{V}_S - \left[\boldsymbol{V}_S(\boldsymbol{\nabla}\cdot\boldsymbol{V}_S) - (\boldsymbol{V}_S\cdot\boldsymbol{\nabla})\boldsymbol{V}_S)\right]\cdot\boldsymbol{\nabla}\boldsymbol{V}_S\right) \ . \tag{38}$$

It is simple to proof that III annd III in the Alcubierre case are identically zero. After numerical computation on *Python*, with the same condition of the section 2.2, we can verify this result for II. In addition, we can do the same for III. So, in this case, we have this simple expression for the Jacobian:

$$J = 1 + (t - t_0)\Theta(t_0, \boldsymbol{X}) \ . \tag{39}$$

From (6) and (7) we obtain with (39) for the density:

$$\varrho(t, \boldsymbol{X}) = \frac{\varrho(t, \boldsymbol{X})}{J(t, \boldsymbol{X})} = \frac{\varrho_0(t_0, \boldsymbol{X})}{1 + (t - t_0)\Theta(t_0, \boldsymbol{X})} \ , \tag{40}$$

and for the velocity, with the expression of the coordinate velocity:

$$V_S(t, \boldsymbol{X}) = v_S W(r_S(t, \boldsymbol{X})) \ , \tag{41}$$

where,

$$r_S(t, \boldsymbol{X}) = [(X - (X_S + v_S W(r_S(t, X))))^2 + Y^2 + Z^2)]^{1/2} \ . \tag{42}$$

Here X, Y, Z are the Cartesian components of $\boldsymbol{X}$. We have X_S is the fixed position of the centre of the warp bubble in Lagrangian space and x_S in the Eulerian space, and its value can be obtained as follows:

$$x_S(t, \boldsymbol{X}) = f_S(t, \boldsymbol{X}) = X + V_0(\boldsymbol{X})(t - t_0) = X + \int_{t_0}^{t} V_S(t, \boldsymbol{X})dt = X + \int_{t_0}^{t} v_S W(r_S(t, \boldsymbol{X}))dt \ . \tag{43}$$

Now, we want to express the Lagrangian evolution equations of the kinematical properties, in the particular case of Alcubierre.

First, for the evolution of field density, from (40) we demonstrate [6]:

$$\dot{\varrho}(t,\boldsymbol{X}) = -\frac{\varrho_0(t_0,\boldsymbol{X})}{J^2(t,\boldsymbol{X})}\dot{J}(t,\boldsymbol{X}) \ , \tag{44}$$

where the overdot stands for the total time-derivative $\frac{\mathrm{d}}{\mathrm{d}t}$. Using the continuity equation $\dot{\varrho} + \varrho(\boldsymbol{\nabla}\cdot\boldsymbol{V}) = 0$ and $\Theta = \boldsymbol{\nabla}\cdot\boldsymbol{V}$, we find:

$$\dot{J} = \Theta(t,\boldsymbol{X})J \Rightarrow \Theta(t,\boldsymbol{X}) = \frac{\dot{J}}{J} \ . \tag{45}$$

Then, for the evolution of the kinematical variables we start with (1) and make the derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_S)^i{}_{,j} = -(V_S)^i{}_{,k}(V_S)^k{}_{,j} \ . \tag{46}$$

We insert (46) into (24) to obtain the evolution equations in the general case [10]:

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 + 2(\Omega^2 - \Sigma^2) \ , \tag{47}$$

$$\dot{\Sigma}_{ij} = -\frac{2}{3}\Theta\Sigma_{ik} - \Sigma_{ik}\Sigma_{kj} - \Omega_{ik}\Omega_{kj} + \frac{2}{3}\left(\Sigma^2 - \Omega^2\right)\delta_{ij} \ , \tag{48}$$

$$\dot{\Omega}_{ij} = -\frac{2}{3}\Theta\Omega_{ij} - \Sigma_{ik}\Omega_{kj} - \Omega_{ik}\Sigma_{kj} \ . \tag{49}$$

For the Alcubierre's case we can have easily the evolution in time of $\Theta(t,\boldsymbol{X})$ with (45) and (39):

$$\Theta(t,\boldsymbol{X}) = \frac{\Theta(t_0,\boldsymbol{X})}{1+(t-t_0)\Theta(t_0,\boldsymbol{X})} \ , \tag{50}$$

where $\Theta(t_0,\boldsymbol{X})$ is the initial field calculated in section 2.2.

For $\Omega(t,\boldsymbol{X})$ we know the integral of the Kelvin–Helmholtz vorticity transport in terms of the vorticity vector $\boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\nabla}_0 \times \boldsymbol{V}_S$ [5]. For an inertial general case, we obtain:

$$\begin{aligned}\boldsymbol{\Omega}(t,\boldsymbol{X}) &= \frac{1}{J}(\boldsymbol{\Omega}(t_0,\boldsymbol{X})\cdot\boldsymbol{\nabla}_0)\boldsymbol{f}(t,\boldsymbol{X}) \\ &= \frac{(\boldsymbol{\Omega}(t_0,\boldsymbol{X})\cdot\boldsymbol{\nabla}_0)(\boldsymbol{X}+V_0(\boldsymbol{X})(t-t_0))}{J} \\ &= \frac{\boldsymbol{\Omega}(t_0,\boldsymbol{X})}{J} + \frac{(\boldsymbol{\Omega}(t_0,\boldsymbol{X})\cdot\boldsymbol{\nabla}_0)V_0(\boldsymbol{X})(t-t_0)}{J} \ .\end{aligned} \tag{51}$$

In the case of Alcubierre, with $\Omega_X(t_0) = 0$, $\Omega_Y(t_0) = \frac{1}{2}(V_S)_{|Z}$, $\Omega_Z(t_0) = -\frac{1}{2}(V_S)_{|Y}$, this reduces to:

$$\boldsymbol{\Omega}(t,\boldsymbol{X}) = \frac{1}{1+(t-t_0)\Theta(t_0,\boldsymbol{X})}\begin{pmatrix} 0 \\ \frac{1}{2}(V_S)_{|Z} \\ -\frac{1}{2}(V_S)_{|Y}\end{pmatrix} \ , \tag{52}$$

since the directional derivative term for the deviations in time vanishes.

From the above we obtain $\boldsymbol{\Omega}^2 = \Omega^2 = \frac{1}{2}\Omega_{ij}\Omega^{ij}$:

$$\Omega^2(t,\boldsymbol{X}) = \frac{(V_S)^2_{|Y} + (V_S)^2_{|Z}}{4(1+(t-t_0)\Theta(t_0,\boldsymbol{X}))^2} \ . \tag{53}$$

Finally, for $\Sigma^2(t,\boldsymbol{X})$, with the help of (33) and (36), we have:

$$\Sigma^2(t,\boldsymbol{X}) = \frac{1}{3}\Theta^2(t,\boldsymbol{X}) + \Omega^2(t,\boldsymbol{X}) \ . \tag{54}$$

Now that we have these expressions, we can plot the time evolution of the kinematic fields in Lagrangian space and, by transforming to Eulerian coordinates, also in Eulerian space. This latter point we had to postpone to future work.

3.2 Outlook

In our case we focused on the $\Theta(t, \boldsymbol{X})$ plot at multiple times. For that we use (50), with the initial $\Theta(t_0, X)$ value. We took the same conditions for the free parameters, $\sigma = 5$, $R = 1$, $v_S = 0.9$. It should be noted that we encounter an instability during the motion when $1+\Theta(t_0, X)(t-t_0) \leq 0$, i.e. a singularity occurs because the trajectories cross each other and form a caustic line [6]. We have multi-valued V for a given X after this critical time t_c; we show this phenomenon in figure 4. We can determine analytically and numerically the value of t_c for which we have a Θ_{min} minimum. For that, we make the derivative of Θ with respect to time and find the value for which this function has a minimum. We obtain this condition for $\Theta_{min} \approx -2.249$ and $t_c = \frac{-1}{\Theta_{min}} + t_0 \approx 0.445$.

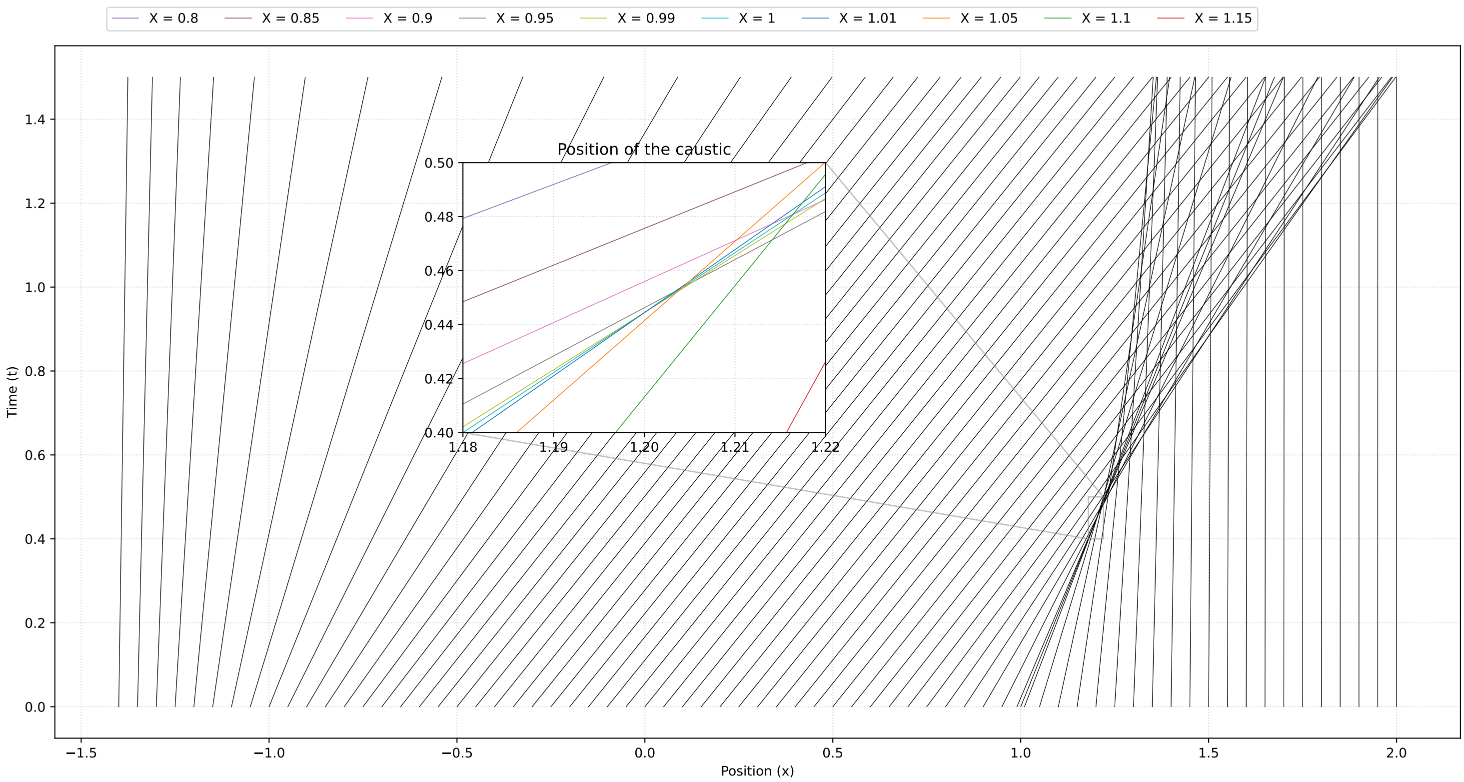
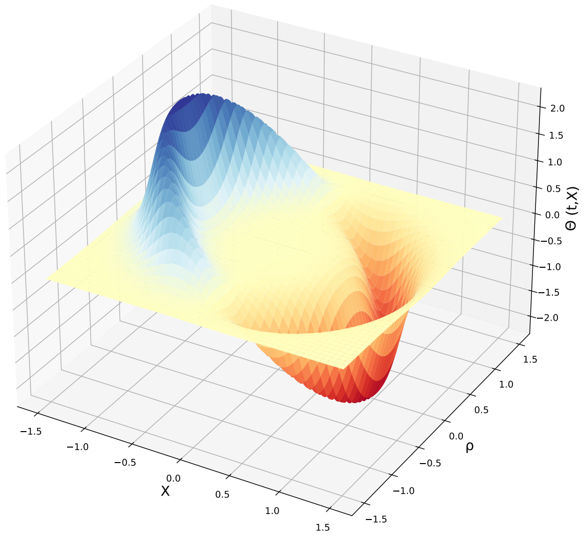


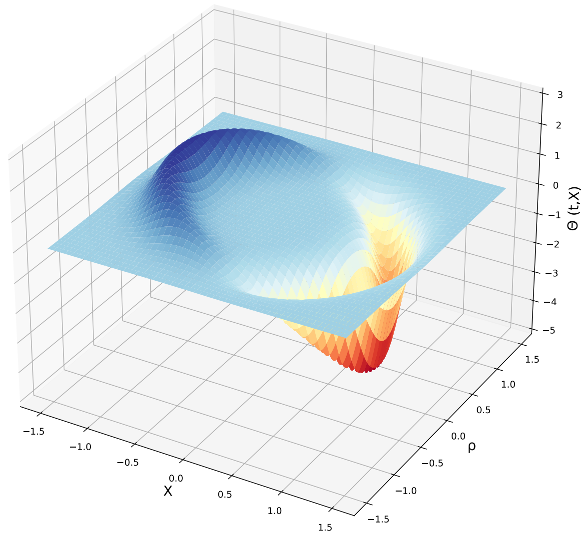
Figure 4: Family of trajectories in Eulerian space, $x = f_S = X + v_S W(r_S(t_0, \boldsymbol{X}))(t-t_0)$, with $v_S = 0.9$, over the range of $X = [-1.4, 2.0]$ versus time. We see a caustic line for a critical time, approximately $t \approx 0.445$, here two infinitesimally close trajectories cross each other for the first time in the Eulerian coordinate system. (Here we take $t_0 = 0$.)

In figure 5 we show the new Lagrangian $\Theta(t, \boldsymbol{X})$ field at different times.

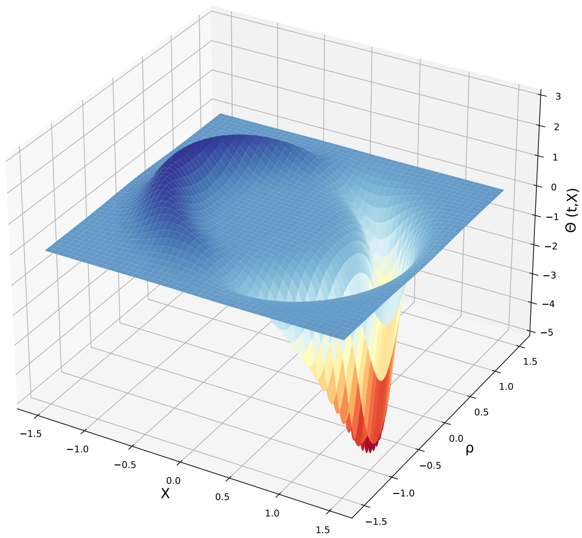
The next step is to transform the function $\Theta(t, \boldsymbol{X})$ in Lagrangian space to the Eulerian space. For that we have to replace the coordinate X by x for the same values of Θ and plot the kinematic fields for the new model in the Eulerian approach. We were able to observe in time the instability of the warp field along its trajectories, and with the transformation to Eulerian space we will then see a split of the singularity and so-called multistream systems [7].



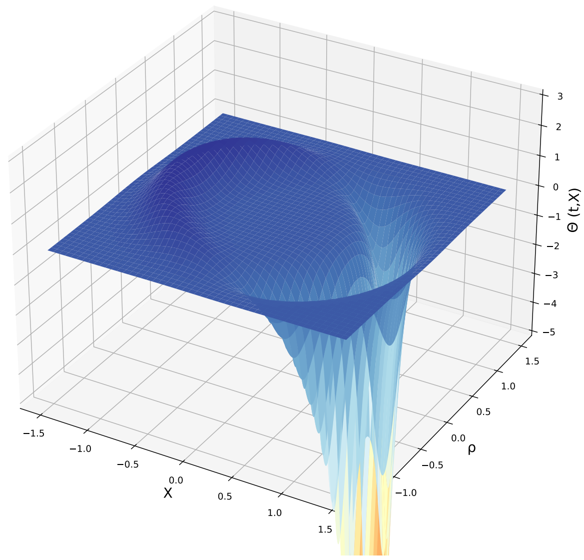
(a) At $t = 0.0$



(b) At $t = 0.2$



(c) At $t = 0.3$



(d) At $t = 0.4$

Figure 5: Evolution of the Θ field at different times t. At initial time, here $t = 0$, (a) Eulerian and Lagrangian coordinates coincide and we see the initial data. At $t = 0.2$ (b) and $t = 0.3$ (c) we can observe a marked deformation of the field. At the front, an increasingly deep trough is created, while at the rear, the deformation diminishes. At a time close to the computable limit $t = 0.4$ (d), the deformation at the front will become infinite and that at the rear zero. (Here we take $t_0 = 0$.)

We can clearly conclude that this new point of view from an observer outside the warp bubble is radically different from that of section 2.2, which did not change as the trajectory progressed. Now the temporal evolution of the field influences its shape along the trajectory, as seen by an outside observer. In particular, the expansion, shear and vorticity experience a singularity at the time of caustic formation.

Conclusion

In this work, we have shown that the transition from an Eulerian to a Lagrangian coordinate system allows us to establish exact solutions of the kinematic fields of an Alcubierre warp bubble in an inertial case. In addition, the time-dependence highlighted by this new approach enables us to challenge the classical view of the shape of this object from an external Eulerian point of view. This study opens the door to a wide range of new research questions. For example, a more detailed study of each new Lagrangian kinematic property would be interesting. Then, research into the behavior of matter by adding a density parameter would be interesting. After that, a study of this object in an accelerated field, beyond the inertial case, would open up other fields of applications including a link to cosmological applications, if in the present investigation we include an expanding background field.

Note: All the *Python* and *Mathematica* codes for this study can be consulted if required.

Acknowledgements

Special thanks to Prof. Thomas Buchert for giving me the opportunity to work with him on this subject. I would also like to thank Dr. Hamed Barzegar for his help in coding during writing of this report.

References

[1] Miguel Alcubierre, "The warp drive: hyper-fast travel within general relativity", *Classical and Quantum Gravity*, L73-L77 [arXiv:gr-qc/0009013] (1994)

[2] José Natário, "Warp drive with zero expansion", *Classical and Quantum Gravity*, pp.1157-1165 [arXiv:gr-qc/0110086] (2002)

[3] Shaun D.B. Fell and Lavinia Heisenberg, "Positive Energy Warp Drive from Hidden Geometric Structures", *Classical and Quantum Gravity*, 155020 [arXiv:2104.06488] (2021)

[4] Sergei F. Shandarin and Yakov B. Zeldovich "The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium", *Reviews of Modern Physics*, pp. 185-220 (1989)

[5] Thomas Buchert, "Lagrangian theory of gravitational instability of Friedmann–Lemaître cosmologies and the 'Zel'dovich approximation'.", *M.N.R.A.S.*, **254**, 729 (1992)

[6] Thomas Buchert, Lecture notes 2, *Introduction à la Relativité Générale (M1)*, Newtonian Gravitation, Intrinsic point of view, ENS (2024)

[7] Thomas Buchert and Alvaro Domínguez, "Modeling multi-stream flow in collisionless matter: approximations for large-scale structure beyond shell-crossing.", *Astron. Astrophys.*, pp.395-402 [arXiv:astro-ph/9702139] (1998)

[8] Thomas Buchert, Lecture notes 1, *Introduction à la Relativité Générale (M1)*, Newtonian Gravitation, Extrinsic point of view, ENS (2023)

[9] Thomas Buchert, Lecture notes 3, *Cosmologie et Systèmes Gravitationnels (M2)*, Gravitational Instability, structure formation, Eulerian and Lagrangian perturbation approach, ENS (2016)

[10] Thomas Buchert, "Lagrangian perturbation approach to the formation of large-scale structure." in *Proc. International School Enrico Fermi* Course CXXXII *Dark Matter in the Universe*, Varenna 1995, ed. S. Bonometto,J.R. Primack and A. Provenzale, IOS Press Amsterdam, pp.543-64. [arXiv:astro-ph/9509005] (1996)

[11] Jürgen Ehlers and Thomas Buchert, "Newtonian Cosmology in Lagrangian Formulation: Foundations and Perturbation Theory", *General Relativity and Gravitation* pp.733-764 [arXiv:astro-ph/9609036] (1997)