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Implied Volatility Functions: Empirical Tests

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ABSTRACT

Derman and Kani (1994), Dupire (1994), and Rubinstein (1994) hypothesize that asset return volatility is a deterministic function of asset price and time, and develop a deterministic volatility function (DVF) option valuation model that has the potential of fitting the observed cross section of option prices exactly. Using S&P 500 options from June 1988 through December 1993, we examine the predictive and hedging performance of the DVF option valuation model and find it is no better than an ad hoc procedure that merely smooths Black–Scholes (1973) implied volatilities across exercise prices and times to expiration.

EXPECTED FUTURE VOLATILITY PLAYS a central role in finance theory. Consequently, accurately estimating this parameter is crucial to meaningful financial decision making. Finance researchers generally rely on the past behavior of asset prices to develop expectations about volatility, documenting movements in volatility as they relate to prior volatility and/or variables

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in the investors' information set. As useful as such investigations have been, they are by nature backward looking, using past behavior to project forward. An alternative approach, albeit less explored in the literature, is to use reported option prices to infer volatility expectations.[1] Because option value depends critically on expected future volatility, the volatility expectation of market participants can be recovered by inverting the option valuation formula.

The volatility expectation derived from reported option prices depends on the assumptions underlying the option valuation formula. The Black–Scholes (1973) model, for example, assumes the asset price follows geometric Brownian motion with constant volatility. Consequently, all options on the same asset should provide the same implied volatility. In practice, however, Black–Scholes implied volatilities tend to differ across exercise prices and times to expiration.[2] S&P 500 option-implied volatilities, for example, form a "smile" pattern prior to the October 1987 market crash. Options that are deep in the money or out of the money have higher implied volatilities than at-the-money options. After the crash, a "sneer"[3] appears—the implied volatilities decrease monotonically as the exercise price rises relative to the index level, with the rate of decrease increasing for options with shorter time to expiration.

The failure of the Black–Scholes model to describe the structure of reported option prices is thought to arise from its constant volatility assumption.[4] It has been observed that when stock prices go up volatility goes down, and vice versa. Accounting for nonconstant volatility within an option valuation framework, however, is no easy task. With stochastic volatility, option valuation generally requires a market price of risk parameter, which, among other things, is difficult to estimate. An exception occurs when volatility is a deterministic function of asset price and/or time. In this case, option valuation based on the Black–Scholes partial differential equation remains possible, although not by means of the Black–Scholes formula itself. We refer to this special case as the "deterministic volatility function" (DVF) hypothesis.

Derman and Kani (1994a,b), Dupire (1994), and Rubinstein (1994) develop variations of the DVF approach. Their methods attempt to decipher the cross section of option prices and deduce the future behavior of volatility as anticipated by market participants. Rather than positing a structural form for the volatility function, they search for a binomial or trinomial lattice that achieves an *exact* cross-sectional fit of reported option prices. Rubinstein, for

[1] See Breeden and Litzenberger (1978), Bick (1988), and Bates (1996a, 1996b).

[2] Rubinstein (1994) examines the S&P 500 index option market. Similar investigations have also been performed for the Philadelphia Exchange foreign currency option market (e.g., Taylor and Xu (1993)), and for stock options traded at the London International Financial Futures Exchange (e.g., Duque and Paxson (1993)) and the European Options Exchange (e.g., Heynen (1993)).

[3] Webster (1994, p. 1100) defines a sneer as "a scornful facial expression marked by a slight raising of one corner of the upper lip."

[4] Putting it succinctly, Black (1976, p. 177) says that "if the volatility of a stock changes over time, the option formulas that assume a constant volatility are wrong."

example, uses an "implied binomial tree" whose branches at each node are designed (either by choice of up-and-down increment sizes or probabilities) to reflect the time variation of volatility.

The goal of this paper is to assess the time-series validity of assuming volatility is a deterministic function of asset price and time. We do this by answering the question: Is the asset price behavior revealed by these methods validated by the actual, subsequent behavior of asset prices? We do not perform statistical analysis on the asset prices themselves, however, as this would require years of observations. Instead, we consider the future behavior of option prices. This approach represents a powerful statistical procedure that more rapidly yields a verdict on the validity of the DVF approach.

To implement this approach, we simply move out-of-sample to assess whether the volatility function implied today is the same one embedded in option prices tomorrow. If the estimated volatility function is stable through time, this finding supports the DVF approach as an important new way to identify the underlying process of financial market prices and for setting hedge ratios and valuing exotic options. On the other hand, if the estimated function is not stable, we must conclude that valuation and risk management using the DVF approach is unreliable and that other explanations for the Black–Scholes implied volatility patterns must be sought.

The paper is organized as follows. In Section I, we document the historical patterns of the Black–Scholes implied volatilities. In Section II, we provide a brief overview of the implied tree approach. Section III outlines our empirical procedure. We show how it is related to the implied tree specification, we review our computational procedure for option valuation under deterministic volatility, and we describe the data. In Section IV, we estimate the implied volatility functions using the DVF model on S&P 500 index option prices, and we describe the model's goodness-of-fit and the time-series behavior of its implied parameter values. In Sections V and VI, we assess the time-series validity of the implied volatility functions. Section V examines how well the implied functions predict option prices one week later, and Section VI assesses whether the DVF approach improves hedging performance. In Section VII, we examine several variations of the model and the procedure to ascertain the robustness of our approach. Section VIII concludes with a summary of the main results and some suggestions for future research.

I. Black–Scholes Implied Volatility Patterns

The motivation for considering deterministic volatility functions in option valuation arises from apparent deficiencies of the Black–Scholes model. These deficiencies are most commonly expressed in cross section as the relation between the Black–Scholes implied volatility and option exercise price. In this section, we illustrate this relation for S&P 500 index options and describe its implications for option valuation.

S&P 500 index options are used in our illustration because, as Rubinstein (1994) argues, this option market provides a context where the Black–Scholes conditions seem most reasonably satisfied. We use only one cross section of option prices, from April 1, 1992, but the pattern on this day is typical of those since the October 1987 stock market crash. The data for the example include all bid and ask price quotes for call options during the half-hour interval of 2:45 to 3:15 p.m. (CST). To compute the implied volatilities, we use the Black–Scholes call option formula,

$$c = (S - PVD)N(d_1) - Xe^{-rT}N(d_2), \tag{1}$$

where $S - PVD$ is the index level net of the present value of expected dividends paid over the option's life, X is the option's exercise price, T is the time to expiration, r is the risk-free interest rate, σ is the volatility rate,

$$d_1 = \frac{\ln[(S - PVD)/X] + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}, \tag{2}$$

$$d_2 = d_1 - \sigma\sqrt{T}, \tag{3}$$

and $N(d)$ is the cumulative unit normal density function with upper integral limit d. To proxy for the risk-free rate, the rate on a T-bill of comparable maturity is used. The actual cash dividends paid during the option's life are used to proxy for expected dividends. For each option price, the implied volatility is computed by solving for the volatility rate (σ) that equates the model price with the observed bid or ask quote.[5]

Figure 1 illustrates the typical pattern in the S&P 500 implied volatilities. Strikingly, the volatilities do not all lie on a horizontal line. This pattern is often called the volatility "smile" and constitutes evidence against the Black–Scholes model. In the figure, the "smile" actually appears to be more of a "sneer." The smile label arose prior to the 1987 crash when, in general, the volatilities were symmetric around zero moneyness, with in-the-money and out-of-the-money options having higher implied volatilities than at-the-money options. The sneer pattern displayed in Figure 1, however, is more indicative of the pattern since the crash, with call (put) option implied volatilities decreasing monotonically as the call (put) goes deeper out of the money (in the money).

Figure 1 also illustrates that the sneer is influenced by the time to expiration of the underlying options.[6] The implied volatilities of seventeen-day

[5] We use the reported index level for this exercise. Since this index is stale, the implied volatilities of call options will be biased downward or upward depending on whether the index is above or below its true level. With puts, the bias is opposite. By using only call options, the bias for each option is in the same direction. Longstaff (1995) has shown that using the wrong index level will create a smile, but a much fainter one than observed.

[6] It is important to recognize that the moneyness variable in Figure 1 is adjusted by the square root of time. Without the adjustment, the slope of the sneer steepens as the option's life grows shorter. This is consistent with Taylor and Xu (1993), who demonstrate that more complex valuation models (such as jump diffusion) can generate time-dependence in the sneer even when volatility is constant over time.

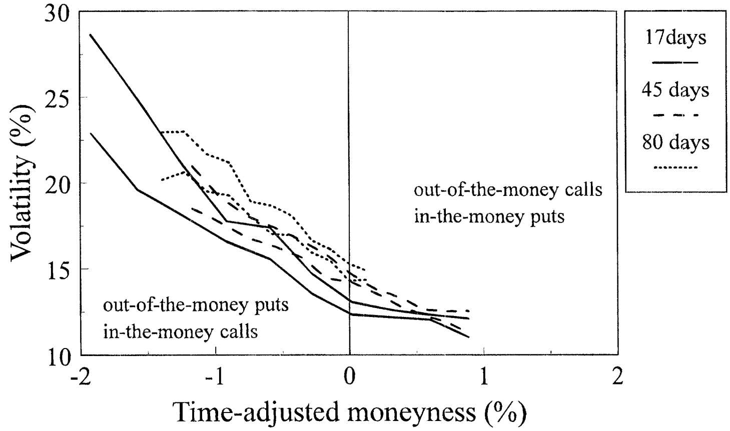

Figure 1. Black–Scholes implied volatilities on April 1, 1992. Implied volatilities are computed from S&P 500 index call option prices for the April, May and June 1992 option expirations. The lower line of each pair is based on the option's bid price, and the upper line is based on the ask. Time-adjusted moneyness is defined as $[X/(S - PVD) - 1]/\sqrt{T}$, where S is the index level, PVD is the present value of the dividends paid during the option's life, X is the option's exercise price, and T is its number of days to expiration.

options are generally lower than the forty-five-day options, which, in turn, are lower than the eighty-day options. This pattern suggests that the local volatility rate modeled within the DVF framework is a function of time.

The differences in implied volatilities across exercise prices shown in Figure 1 appear to be economically significant. The bid-implied volatility for the short-term, in-the-money call, for example, exceeds the ask-implied volatility for the short-term, at-the-money call,[7] implying the possibility of an arbitrage profit. A strategy of selling in-the-money calls and buying at-the-money calls to capture the "arbitrage profits" is more complex that merely spreading the options, however, and requires dynamic rebalancing through time. The differences among the implied volatilities, however, are too large to be accounted for by the costs of dynamic rebalancing, as can be shown using the Constantinides (1997) bounds.

[7] The variation in the difference between bid and ask volatilities depends on two factors. First, although bid/ask spreads are competitively determined, they tend to vary systematically with option moneyness. In part, this may be caused by the CBOE's rules governing the maximum spreads for options with different premia. The rules state that the maximum bid/ask spread is (a) 1/4 for options whose bid price is less than \$2, (b) 3/8 for bid prices between \$2 and \$5, (c) 1/2 for bid prices between \$5 and \$10, (d) 3/4 for bid prices between \$10 and \$20, and (e) 1 for bid prices above \$20. See the Chicago Board Options Exchange (1995, pp. 2123–2124). Second, the sensitivity of option price to the volatility parameter is highest for at-the-money options, with in-the-money and out-of-the-money having much lower sensitivities. As a result, for a given spread between the bid and ask price quotes, the range of Black/Scholes implied volatilities will be lowest for at-the-money options and will become larger as the options move deeper in or out of the money.

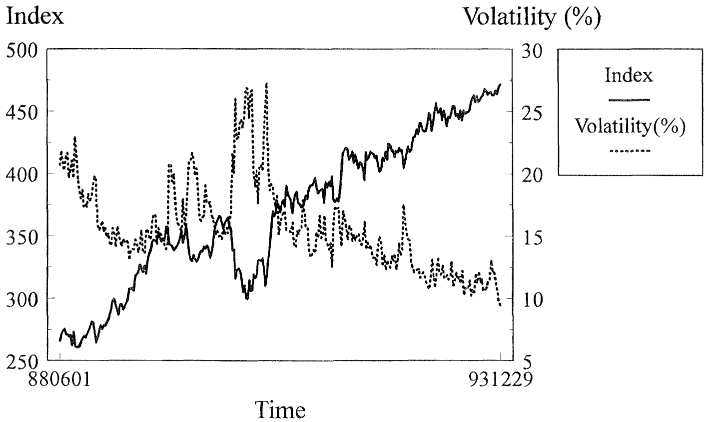

Figure 2. S&P 500 index level and Black–Scholes implied volatility each Wednesday during the period of June 1988 through December 1993.

The differences raise a question concerning the source of the Black–Scholes model's apparent deficiency. One possibility is that the constant volatility assumption is violated, or that the distribution of asset prices at expiration is not lognormal. In this context, the emergence of the volatility sneer after the crash might be explained by an increase in investors' probability assessment of downward moves in the index level. The nonlognormality of prices is also consistent with what has become known as the "Fischer Black effect." Black (1976) writes

> I have believed for a long time that stock returns are related to volatility changes. When stocks go up, volatilities seem to go down; and when stocks go down, volatilities seem to go up. (p. 177)

This inverse time-series relation between stock returns and volatility changes has been documented in a number of empirical studies. Most of the studies use stock returns to measure volatility, but the effect is also apparent when volatility is measured using option prices. Figure 2 shows the level of Black–Scholes implied volatility during the sample period of our study, June 1, 1988 through December 31, 1993. As the S&P 500 index level trends up, the level of implied volatility trends down. The correlation in the first differences of these series is -0.570.

In addition to the DVF approach considered in this paper, a number of option valuation models are capable of explaining the behavior documented in Figures 1 and 2. The stochastic volatility models of Heston (1993) and Hull and White (1987), for example, can explain them when the asset price and volatility are negatively correlated. The negative correlation is what produces the sneer, not the stochastic feature itself. Similarly, the jump model of Bates (1996a) can generate these patterns when the mean jump is negative. Deterministic volatility models, however, are the simplest because they preserve the arbitrage argument that underlies the Black–Scholes model. Unlike stochastic volatility and jump models, they do not require additional assumptions about investor preferences for risk or additional securities that can be used to hedge volatility or jump risk. Therefore, only the parameters that govern the volatility process need be estimated.

II. The Implied Tree Approach

The implied tree approach developed by Derman and Kani (1994a,b), Dupire (1994), and Rubinstein (1994) assumes the local volatility rate is a flexible but deterministic function of the asset price and time. The aim of the approach is to develop an asset price lattice that is consistent with a cross section of option prices. The general procedure for doing this involves: (a) estimating the risk-neutral probability distribution of asset prices at the end of the lattice, and (b) determining the up and down step sizes and probabilities throughout the lattice that are consistent with the implied probability distribution.

The implied probability distributions obtained from step (a) typically seem consistent with the apparent deficiencies of the Black–Scholes model. In particular, for S&P 500 options, the distributions tend to exhibit negative skewness and excess kurtosis relative to the lognormal distribution. The excess kurtosis is a well-known feature of historical stock returns and the skewness is consistent with the notion that after the crash investors increased their assessment of the probability of stock market declines. Indeed, as Rubinstein reports, the implied probability assessment of decreases is so severe that it is "quite common" to observe a bimodal distribution. In other words, for index levels far enough below the mean, the implied probability actually increases.

Step (b) in the approach involves constructing the asset price tree. At any node in the tree, we can deduce the move volatility, which in the limit converges to the local volatility rate, $\sigma(S,t)$. The structure of these volatilities is typically consistent with the empirical evidence regarding stock volatility. Specifically, an inverse relation exists between the index level and volatility—as the index falls, volatility increases (Black (1976)). Moreover, the relation is asymmetric—the increase in volatility for decreases in the index tends to be larger than the decrease in volatility for higher index levels (Schwert (1989, 1990)).

The reasonableness of these implied dynamics provides indirect support for the implied tree approach. But, given the many potential applications stemming from this approach, more comprehensive tests seem necessary. The approach yields an estimate of how the asset price evolves over time, and this estimate could be used to value other derivatives on the same asset (e.g., American and exotic options) or as the basis for more exact hedge ratios. Moreover, to the extent that the asset is a stock market index, the estimates could be used in more general asset pricing and volatility estimation contexts. The reliability of the approach in these settings depends critically on how well we can estimate the dynamics of the underlying asset price from a cross section of option prices. This assessment is the purpose of this paper.

III. Empirical Methodology

In this section, we begin by describing the intuition for our valuation method vis-à-vis the implied tree approach. As in the implied tree approach, our method assumes the local volatility rate is a deterministic function of asset price and time. Next we provide a formal description of the deterministic volatility function valuation framework, and we specify the structure of the volatility functions that we test in our analysis. The final subsection describes the data.

A. Intuition

The implied tree approach uses a cross section of option prices to imply the tree (and, hence, to implicitly estimate the volatility function) that achieves an exact fit of observed option prices. An exact fit is possible because there are as many degrees of freedom in defining the tree as there are observed option prices. With so much freedom in parameter selection, however, the possibility exists that the approach overfits the data.

We examine this possibility by evaluating the time-series reliability of the implied parameter estimates. The logic of our test is straightforward. First, we use today's option prices to estimate the parameters of the underlying process, that is, the implied tree. Then, we step forward in time. If the original tree was correct, then the subtree stemming out of the node realized today must again be correct. Equivalently, option values from this subtree (using the new asset price) should be correct. If, however, the volatility function is not stable through time, then the out-of-sample option values are inaccurate. This finding suggests that the cross-sectional fit has not identified the true volatility function or the true stochastic process for the underlying asset.

Using a tree-based approach to implement this test suffers from a practical limitation. Suppose we estimate an implied tree today, and then step forward in time to use the remainder of the tree. The likelihood that the realized asset price falls exactly on a node of our original tree is remote. Indeed, the realized price is virtually certain to fall between nodes or entirely outside the span of the tree. Consequently, using the tree for out-of-sample option valuation would require interpolation or extrapolation techniques.

To avoid this complication, we specify from the start an interpolative functional form for the volatility process. We consider a number of alternatives based on a Taylor series approximation in S and t. Once we specify the function, we can estimate its parameters by obtaining the best fit of the option values under deterministic volatility with the observed option prices.

This deterministic volatility function (DVF) approach to fitting the data is slightly different from the implied tree approach, but the spirit of the two approaches is the same. Both fundamentally concern obtaining estimates of the deterministic volatility function. What we will show is that, even with fairly parsimonious models of the volatility process, we achieve an "almost exact" fit of observed option prices. The crucial question, then, concerns the stability of these estimates over time. Using more elaborate models such as those embedded in the lattice-based approaches presents an even greater danger of overfitting reported prices and deteriorating the quality of prediction.

The fact that we allow for pricing errors in our approach may seem inconsistent with the implied tree approach. Rubinstein requires that all option values computed using the implied tree fall within their respective bid and ask prices observed in the market—that is, that no arbitrage opportunities exist. More recent research, however, relaxes this requirement. Jackwerth and Rubinstein (1996), for example, advocate using bid/ask midpoint prices, as we do, rather than the bid/ask band due to the tendency to "(overfit) the data by following all the small wiggles" when the no-arbitrage constraint is imposed. As a result, they allow for "small" deviations from market prices, and use the sum of squared dollar errors (as we do also) in their objective function in fitting the implied tree.

B. Option Valuation under Deterministic Volatility

Option valuation when the local volatility rate is a deterministic function of asset price and time is straightforward. In this case, the partial differential equation describing the option price dynamics is the familiar Black–Scholes (1973) equation,

$$-\frac{1}{2}\sigma^2(F,t)F^2\frac{\partial^2 c}{\partial F^2} = \frac{\partial c}{\partial t}, \tag{4}$$

where F is the forward asset price for delivery on the expiration date of the option, c is the forward option price, $\sigma(F,t)$ is the *local* volatility of the price F, and t is current time.[8] We use forward prices, rather than spot prices, for both the option and the underlying asset to avoid the issue of randomly fluctuating interest rates.

Equation (4) is called the *backward* equation of the Black–Scholes model (expressed in terms of forward prices). The call option value is a function of F and t for a fixed exercise price X and date of expiration T. At time t when F is known, however, the cross section of option prices (with different exercise prices and expiration dates) can also be considered to be functionally

[8] Bergman, Grundy, and Wiener (1996) examine the implications of specifying volatility as a function of the underlying spot or forward asset price. They also illustrate a number of reasons for which volatility may be a (possibly nonmonotonic) function of the asset price.

related to X and T. For European-style options, Breeden and Litzenberger (1978) and Dupire (1994) show that the forward option value, $c(X,T)$, must be a solution of the *forward* partial differential equation,[9]

$$\frac{1}{2}\sigma^2(X,T)X^2\frac{\partial^2 c}{\partial X^2} = \frac{\partial c}{\partial T}, \tag{5}$$

with the associated initial condition, $c(X,0) = \max(F - X,0)$. The volatility function in equation (5) is the same one as in equation (4), but the arguments, F and t, are replaced by X and T. Equation (4) requires the local volatility that prevails at the *present time* when the date is t and the index level is F; equation (5) uses the *future* local volatility that will prevail on the expiration date, T, when the underlying index is then at level X.

The advantage of using the forward equation to value European-style options (such as those on the S&P 500 index) is that all option series with a common time to expiration can be valued simultaneously—a considerable computational cost saving when using numerical procedures.[10] To infer volatility functions from American-style option prices, however, requires solving the backward equation (4) for each option series.

C. Specifying the Volatility Function

We estimate the volatility function, $\sigma(X,T)$, by fitting the DVF option valuation model to reported option prices at time t. Because $\sigma(X,T)$ is an arbitrary function, we posit a number of different structural forms including:

$$\text{Model 0:} \quad \sigma = \max(0.01, a_0); \tag{6}$$

$$\text{Model 1:} \quad \sigma = \max(0.01, a_0 + a_1X + a_2X^2); \tag{7}$$

$$\text{Model 2:} \quad \sigma = \max(0.01, a_0 + a_1X + a_2X^2 + a_3T + a_5XT); \text{ and} \tag{8}$$

$$\text{Model 3:} \quad \sigma = \max(0.01, a_0 + a_1X + a_2X^2 + a_3T + a_4T^2 + a_5XT). \tag{9}$$

Model 0 is the volatility function of the Black–Scholes constant volatility model. Model 1 attempts to capture variation in volatility attributable to asset price, and Models 2 and 3 capture additional variation attributable to time. A minimum value of the local volatility rate is imposed to prevent negative values.

[9] Because the option price, c, and the underlying asset price, F, are expressed as forward prices (forward to the maturity date of the option), equations (4) and (5) ignore interest and dividends. We account for these factors in our definition of forward prices. See Section III.D below.

[10] We solve equation (5) using the Crank–Nicholson finite-difference method.

We choose quadratic forms for the volatility function, in part because the Black–Scholes implied volatilities for S&P 500 options tend to have a parabolic shape. The volatility function could also be estimated using more flexible nonparametric methods such as kernel regressions[11] or splines. As noted above, however, we want to avoid overparameterization. In Section VII below, we verify that the quadratic form of the DVF, despite the parabolic branches, leads to robust empirical results.

D. Data Selection

Our sample contains reported prices of S&P 500 index options traded on the Chicago Board Options Exchange (CBOE) over the period June 1988 through December 1993.[12] S&P 500 options are European-style and expire on the third Friday of the contract month. Originally, these options expired only at the market close and were denoted by the ticker symbol SPX. In June 1987, when the Chicago Mercantile Exchange (CME) changed its S&P 500 futures expiration from the close to the open, the CBOE introduced a second set of options with the ticker symbol NSX that expired at the open. Over time, the trading volume of this "open-expiry" series grew to surpass that of the "close-expiry" series, and on August 24, 1992, the CBOE reversed the ticker symbols of the two series. Our sample contains SPX options throughout: close-expiry until August 24, 1992, and open-expiry thereafter. During the first subperiod, the option's time to expiration is measured as the number of calendar days between the trade date and the expiration date; during the second, we use the number of calendar days remaining less one.

As we noted earlier, we estimate each of the volatility functions once each week during the sample. We use Wednesdays for these estimations because fewer holidays fall on a Wednesday than on any other trading day. When a particular Wednesday is a holiday, we use the immediately preceding trading day.

To estimate the volatility functions, we express both the index level and option price as forward prices. Constructing the forward index level requires the term structure of default-free interest rates and the daily cash dividends on the index portfolio. We proxy for the riskless interest rate by using the T-bill rates implied by the average of the bid and ask discounts reported in the *Wall Street Journal*. The t_i-period interest rate is obtained by interpolating the rates for the two T-bills whose maturities straddle t_i. The daily cash dividends for the S&P 500 index portfolio are collected from the *S&P 500 Information Bulletin*. To compute the present value of the dividends paid

[11] See Ait-Sahalia and Lo (1998).

[12] The sample begins in June 1988 because it was the first month for which Standard and Poors began reporting daily cash dividends for the S&P 500 index portfolio. See Harvey and Whaley (1992) regarding the importance of incorporating discrete daily cash dividends in index option valuation.

during the option's life, PVD, the daily dividends are discounted at the rates corresponding to the ex-dividend dates and summed over the life of the option; that is,

$$PVD = \sum_{i=1}^{n} D_i e^{-r_i t_i}, \tag{10}$$

where D_i is the ith cash dividend payment, t_i is the time to ex-dividend from the current date, r_i *is the* t_i-period riskless interest rate, and n is the number of dividend payments during the option's life.[13] The implied forward price of the S&P 500 index is therefore

$$F = (S - PVD)e^{rT}, \tag{11}$$

where S is the reported index level and T is the time to expiration of the option. To create a forward option price, we multiply the average of the option's bid and ask price quotes[14] by the interest accumulation factor appropriate to the option's expiration, e^{rT}.

Three exclusionary criteria are applied to the data. First, we eliminate options with fewer than six or more than one hundred days to expiration. The shorter-term options have relatively small time premiums, hence the estimation of volatility is extremely sensitive to nonsynchronous option prices and other possible measurement errors. The longer-term options, on the other hand, are unnecessary because our objective is only to determine whether the volatility function remains valid over a span of one week. Including these options would simply deteriorate the cross-sectional fit.

Second, we eliminate options whose absolute "moneyness," $|X/F - 1|$, is greater than 10 percent. Like extremely short-term options, deep in- and out-of-the-money options have small time premiums and hence contain little information about the volatility function. Moreover, these options are not actively traded, and price quotes are generally not supported by actual trades.

Finally, we only use those options with bid/ask price quotes during the last half hour of trading (2:45 to 3:15 p.m. (CST)). Fearing imperfect synchronization with the option market,[15] we do not use the reported S&P 500

[13] The convention introduces an inconsistency, with small consequences, between option prices of different maturities. The inconsistency takes two forms. First, our forward index level assumes that the dividends to be paid during the option's life are certain, so the index cannot fall below the promised amount of dividends during this period. This barrier is different for each maturity date. This is an inconsistency in the specification of the process for the index. Second, the volatility function that we are estimating is actually a volatility of the forward price to the maturity date of the option. To be completely rigorous, we should model the forward price process for each maturity, with the appropriate cross-maturity constraints on price imposed, and estimate a separate volatility function for each.

[14] Using bid/ask midpoints rather than trade prices reduces noise in the cross-sectional estimation of the volatility function.

[15] See Fleming, Ostdiek, and Whaley (1996).

index level or the S&P 500 futures price[16] in our estimation. Instead, we infer the current index level simultaneously,[17] together with the parameters of the volatility function, from the cross section of option prices. In this way, our empirical procedure relies only on observations from a single market, with no auxiliary assumption of market integration.[18] The procedure, however, requires that the option prices are reasonably synchronous—hence the need for a tight time window. The cost of this criterion is a reduction in the number of option quotes, but the cost is not too onerous because we have quotes for an average of 44 (and a range from 14 to 87) option series during the last half-hour each Wednesday.[19] Seventeen of the 292 Wednesday cross sections had only one contract expiration available; 141 had two; 129 had three; and 5 had four.

IV. Estimation Results

Using the S&P 500 index option data described in the previous section, we now estimate the four volatility functions specified in equations (6) through (9). As noted earlier, Model 0 is the Black–Scholes constant volatility model. Model 1 allows the volatility rate to vary with the index level but not with time. Models 2 and 3 attempt to capture additional variation due to time. A fifth volatility function, denoted Model S, is also estimated. Model S switches among the volatility functions given by Models 1, 2, and 3, depending on whether the number of different option expiration dates in a given cross section is one, two, or three, respectively. Model S is introduced because some cross sections have fewer expiration dates available, undermining our ability to estimate precisely the relation between the local volatility rate and time.

This section focuses on identifying the "best" volatility function given the structure of S&P 500 index option prices. First, each local volatility function is estimated by minimizing the sum of squared dollar errors between the reported option prices and their DVF model values. Summary statistics on the goodness-of-fit and on coefficient stability are provided. Next, we illustrate the shape of the implied probability functions for options of different times to expiration.

[16] For a detailed description of the problems of using a reported index level in computing implied volatility, see Whaley (1993, Appendix).

[17] In doing so, we impose the "cross-futures constraint" that the futures prices for different maturities should reflect the same underlying cash index level.

[18] This is not quite true since we use Treasury bill rates in computing forward prices.

[19] To assess the reasonableness of using the 2:45 to 3:15 p.m. window for estimation, we compute the mean absolute return and the standard deviation of return of the nearby S&P 500 futures (with at least six days to expiration) by fifteen-minute intervals throughout the trading day across the days of the sample period. The results indicate that the lowest mean absolute return and standard deviation of return occur just before noon. The end-of-day window is only slightly higher, but the beginning-of-day window is nearly double. We choose to stay with the end-of-day window for ease in interpreting the results.

A. Goodness-of-Fit

To assess the quality of the fitted models, five measurements are made each week. These are defined as follows:

(i) The root mean squared valuation error (RMSVE) is the square root of the average squared deviations of the reported option prices from the model's theoretical values.
(ii) The mean outside error (MOE) is the average valuation error outside the bid/ask spread. If the theoretical value is below (exceeds) the option's bid (ask) price, the error is defined as the difference between the theoretical value and the bid (ask) price, and, if the theoretical value is within the spread, the error is set equal to zero. A positive value of MOE, therefore, means that the model value is too high on average, and a negative value means the model value is too low. This measure is used primarily to detect biases in specific option categories.
(iii) The average absolute error (MAE) is the average absolute valuation error outside the bid/ask spread. This measure illustrates the exactness with which each model fits within the quoted bid and ask prices over all option categories.
(iv) The frequency (FREQ) indicates the proportion of observations where the specified model has a lower RMSVE than Model S.
(v) Finally, the Akaike (1973) Information Criterion (AIC) is calculated to appraise the potential degree of overfitting. The AIC penalizes the goodness-of-fit as more degrees of freedom are added to the model in a manner similar to an adjusted R^2. The lowest value of the AIC identifies the "best" model based on in-sample performance. Of course, overfitting is best detected by going out of sample (see Section V).

Table I contains the average RMSVEs, MOEs, and MAEs across the 292 days (one day each week) during the sample period of June 1988 through December 1993. The average RMSVE results show a strong relationship between the local volatility rate and the asset price. Where the volatility rate is a quadratic function of asset price (Model 1), the average RMSVE of the DVF model is less than one-half that of the Black–Scholes constant volatility model (Model 0), 30.1 cents versus 65.0 cents, for all options in the sample. Time variation also appears important. In moving from Model 1 to Model 2, the average RMSVE in the full sample is reduced even more (from 30.1 cents to 23.0 cents), albeit not so dramatically. The addition of the time variable to the volatility function appears to be important, although most of the incremental explanatory power appears to come from the cross-product term, XT.[20] Adding a quadratic time to expiration term (Model 3) reduces the average RMSVE to its lowest level of the assumed specifications, 22.6 cents. Model S's RMSVE is virtually the same. The average MOE and MAE measurement

[20] Model 2 is also estimated without the time variable with little difference in explanatory power.

criteria lead to the same conclusions for the overall sample. The MAE shows that with Model 3 an essentially exact fit, within the bid-ask spread, has been achieved because the average absolute error outside the spread is a mere 5 cents.

Once goodness-of-fit is adjusted to account for the number of parameters in the volatility function, more parsimonious volatility functions appear to work best. The AIC results are reported in Table I as the proportion of days during the sample period that a particular model is judged the best specification. The results indicate that Model 2 provides the best fit of the cross section of S&P 500 index option prices, having the lowest AIC in 67.1 percent of the 292 days in the sample. The next best performer is Model 1, which is even more parsimonious than Model 2 and does not have time variation in the local volatility rate function, outperforming the other models in 25.7 percent of the days in the sample. The more elaborate Model 3, which had the lowest RMSVE, does not perform well once the penalty for the additional variables is imposed—performing best in less that 7 percent of the days in the sample. All in all, the results indicate that the deterministic volatility function need not be very elaborate to describe the observed structure of index option prices accurately.

The MOE values reported for the Black–Scholes model (Model 0) show that the theoretical value exceeds the ask price on average for call options, 16.6 cents, and is below the bid price for put options, -23.9 cents. This behavior arises from the character of our sample (i.e., the number of calls versus the number of puts, and the number of in-the-money options versus the number of out-of-the-money options). When the options are stratified by option type and moneyness, the Black–Scholes model value appears to be too low (relative to the bid price) for in-the-money calls and for out-of-the-money puts. This is consistent with the implied volatility sneer shown in Figure 1. With all options forced to have the same volatility in the estimation of Model 0, the variation in implied volatility translates into valuation errors. Options with Black–Scholes implied volatilities higher (lower) than average are valued too low (high).

Figure 3 shows the dollar valuation errors (i.e., the model values less the bid/ask midpoints) of Model 0 for the subsample of call options with 40 to 70 days to expiration. Also shown are normalized bid/ask spreads (i.e., the bid/ask prices less the bid/ask midpoint). Note first that the bid/ask spreads are as high as one dollar for deep in-the-money calls on the left of the figure. As we move right along the horizontal axis, the maximum bid/ask spread stays at a dollar until the moneyness variable is about -2.5 percent, and then the maximum spread begins to decrease as the calls move further out-of-the-money. This spread behavior is consistent with the CBOE's maximum spread rules described earlier. The average bid/ask spread across all option series used in our estimation is approximately 47 cents.

Figure 4 shows the valuation errors of Model 3 for calls with 40 to 70 days to expiration. The DVF model improves the cross-sectional fit. Where the valuation errors are outside the bid/ask spread, they appear randomly, with

Table I

Average S&P 500 Index Option Dollar Valuation Errors Using the Deterministic Volatility Function (DVF) Model

RMSVE is the root mean squared dollar valuation error averaged across all days in the sample period from June 1988 through December 1993. MOE is the average of the mean valuation error outside the observed bid/ask quotes across all days in the sample (a positive value indicates the theoretical value exceeds the ask price on average; a negative value indicates the theoretical value is below the bid price). MAE is the average of the mean absolute valuation error outside the observed bid/ask quotes across all days in the sample. AIC is the proportion of days during the sample period that the model is judged best by the Akaike Information Criterion. FREQ is the frequency of days, expressed as a ratio of the total number of days, on which a particular model has a lower daily RMSVE than Model S. Model 0 is the Black/Scholes constant volatility model. Models 1, 2, and 3 specify that the local volatility rate is linear in (a) X and X^2, (b) X, X^2, T, and XT, and (c) X, X^2, T, T^2, and XT, respectively, where X is the option's exercise price and T is its time to expiration. Model S switches between Models 1, 2, and 3 depending upon whether the number of option expirations in the cross-section is one, two or three, respectively. Moneyness is defined as $X/F - 1$, where F is the forward index level.

Panel A: Aggregate Results

DVF	All Options				Call Options			Put Options		
model	RMSVE	MOE	MAE	AIC	RMSVE	MOE	MAE	RMSVE	MOE	MAE
0	0.650	−0.034	0.348	0.003	0.651	0.166	0.360	0.643	−0.239	0.338
1	0.301	0.022	0.095	0.257	0.300	0.036	0.095	0.296	0.009	0.096
2	0.230	−0.009	0.052	0.671	0.222	0.000	0.047	0.233	−0.020	0.058
3	0.226	−0.011	0.050	0.062	0.218	−0.002	0.044	0.230	−0.020	0.057
S	0.227	−0.010	0.050	0.007	0.218	−0.002	0.050	0.230	−0.020	0.057

			Days to Expiration								
Moneyness (%)		DVF	Less than 40			40 to 70			More than 70		
Lower	Upper	model	RMSVE	MOE	FREQ	RMSVE	MOE	FREQ	RMSVE	MOE	FREQ
Panel B: Call Options Only											
−10	−5	0	0.313	0.000	0.429	0.644	−0.290	0.042	1.051	−0.623	0.016
		1	0.236	−0.012	0.593	0.246	−0.021	0.491	0.342	−0.035	0.210
		2	0.257	−0.026	0.271	0.224	−0.014	0.241	0.234	−0.002	0.306
		3	0.259	−0.027	0.214	0.221	−0.013	0.264	0.220	0.003	0.073
		S	0.259	−0.027		0.223	−0.014		0.220	0.003	

−5	0	0	0.401	0.143	0.133	0.403	−0.075	0.124	0.583	−0.227	0.091
		1	0.352	0.136	0.179	0.228	0.023	0.434	0.255	−0.029	0.462
		2	0.205	0.020	0.218	0.192	0.014	0.239	0.210	0.022	0.394
		3	0.197	0.014	0.267	0.187	0.012	0.248	0.205	0.026	0.053
		S	0.198	0.014		0.187	0.012		0.205	0.026	
0	5	0	0.721	0.607	0.014	0.836	0.661	0.000	0.850	0.627	0.016
		1	0.384	0.251	0.087	0.234	0.037	0.347	0.260	−0.099	0.315
		2	0.177	0.058	0.225	0.180	0.014	0.216	0.198	−0.027	0.386
		3	0.166	0.047	0.293	0.171	0.007	0.266	0.189	−0.021	0.087
		S	0.167	0.048		0.173	0.008		0.189	−0.021	
5	10	0	0.441	0.367	0.228	0.905	0.804	0.028	1.096	0.975	0.014
		1	0.154	−0.020	0.574	0.203	−0.094	0.491	0.317	−0.205	0.108
		2	0.150	−0.081	0.277	0.204	−0.102	0.245	0.241	−0.130	0.297
		3	0.151	−0.082	0.287	0.205	−0.108	0.236	0.245	−0.135	0.068
		S	0.151	−0.082		0.207	−0.107		0.246	−0.135	
Panel C: Put Options Only											
−10	−5	0	0.608	−0.524	0.000	1.193	−1.073	0.000	1.660	−1.521	0.000
		1	0.188	−0.114	0.792	0.216	−0.108	0.473	0.308	−0.176	0.177
		2	0.237	−0.163	0.242	0.201	−0.096	0.234	0.161	−0.049	0.365
		3	0.237	−0.164	0.273	0.194	−0.088	0.228	0.152	−0.035	0.073
		S	0.238	−0.165		0.196	−0.091		0.152	−0.035	
−5	0	0	0.446	−0.287	0.054	0.753	−0.546	0.010	1.018	−0.769	0.024
		1	0.291	0.123	0.200	0.197	0.006	0.406	0.218	−0.047	0.435
		2	0.166	−0.020	0.275	0.154	−0.003	0.238	0.182	0.026	0.347
		3	0.162	−0.026	0.279	0.152	−0.001	0.218	0.181	0.037	0.081
		S	0.162	−0.026		0.151	−0.002		0.181	0.037	
0	5	0	0.299	0.046	0.268	0.335	0.080	0.231	0.435	0.103	0.190
		1	0.353	0.129	0.137	0.235	0.016	0.330	0.284	−0.057	0.254
		2	0.206	0.012	0.243	0.192	0.002	0.245	0.224	−0.003	0.397
		3	0.203	0.010	0.289	0.190	0.000	0.208	0.225	−0.001	0.071
		S	0.203	0.010		0.190	0.001		0.225	−0.001	
5	10	0	0.409	−0.063	0.224	0.369	0.076	0.405	0.610	0.233	0.230
		1	0.247	0.021	0.513	0.294	−0.035	0.386	0.367	−0.058	0.190
		2	0.244	0.016	0.194	0.291	−0.036	0.203	0.314	−0.038	0.330
		3	0.243	0.016	0.259	0.291	−0.037	0.229	0.309	−0.037	0.060
		S	0.243	0.016		0.291	−0.036		0.310	−0.037	

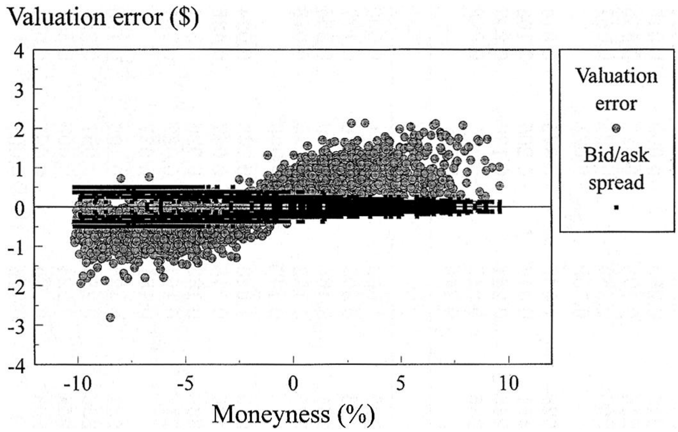

Figure 3. Dollar valuation estimation errors of Deterministic Volatility Function Model 0 (i.e., the Black–Scholes model) for S&P 500 call options with 40 to 70 days to expiration. The solid squares correspond to normalized bid/ask price quotes (i.e., the bid and ask prices less the average of the bid and ask prices). The circles correspond to valuation errors (i.e., the theoretical option value less the bid/ask midpoint). Moneyness is defined as $X/F - 1$, where F is the forward index level and X is the option's exercise price.

a slight tendency for the DVF model to undervalue deep in-the-money and deep out-of-the-money calls and to overvalue at-the-money calls. Overall, however, Model 3's fit appears quite good. The MOE across all calls in this category is just -2.6 cents, in contrast to an MOE of more than 25 cents for the Black–Scholes model.

Model 3 also appears to eliminate the relation between valuation error and the option's days to expiration. For the Black–Scholes model, the valuation errors generally increase with days to expiration. For deep in-the-money calls with fewer than 40 days to expiration, for example, the RMSVE is 31.3 cents; it is 64.4 cents for calls between 40 and 70 days to expiration; and 105.1 cents for calls with more than 70 days to expiration. For the same call options, the RMSVEs for Model 3 are 25.9, 22.1, and 22.0 cents, respectively.

The results in Table I support the notion that a relatively parsimonious model can accurately describe the observed structure of S&P 500 index option prices.[21] The implied tree approach can achieve an exact fit of option prices by permitting as many degrees of freedom as there are option prices. Our results suggest that such a complete parameterization may be undesirable. Based on the AIC, Model 2 does "best," with the local volatility rate being a function of X, X^2, T, and XT. Moreover, where Model 2 does not

[21] To test if the estimation results are driven by the presence of outliers, we examine the valuation errors of the various models. We identify unusually large errors for three days during the sample period. When we eliminate these days from the summary results, the magnitudes of the average errors reported in Table I are reduced by only small amounts. Consequently, we report the results for the full sample.

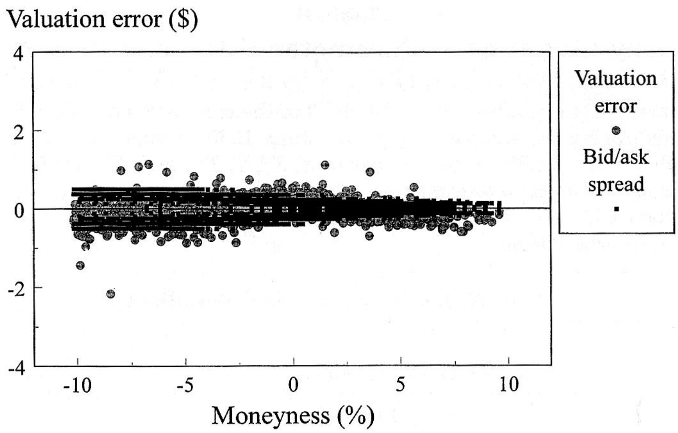

Figure 4. Dollar valuation estimation errors of Deterministic Volatility Function Model 3 for S&P 500 call options with 40 to 70 days to expiration. The solid squares correspond to normalized bid/ask price quotes (i.e., the bid and ask prices less the average of the bid and ask prices). The circles correspond to valuation errors (i.e., the theoretical option value less the bid/ask midpoint). Moneyness is defined as $X/F - 1$, where F is the forward index level and X is the option's exercise price.

perform best, the more parsimonious Model 1, with local volatility being only a function of X and X^2, usually does best. Together, these two simple models outperform the others in 92.8 percent of the 292 cross sections of index option prices examined. Based on these in-sample results, parsimony in the specification of the volatility function appears to be warranted.

B. Average Parameter Estimates and Parameter Stability over Time

The average parameters estimated for each of the volatility functions are also informative. Model 0 is, of course, the constant volatility model of Black–Scholes. When this model is fitted each week during our 292-week sample period, the mean estimated coefficient, $\bar{a}_0$, is 15.72 percent. Recall that Figure 2 shows the level of the Black–Scholes implied volatility on a week-by-week basis. Over the sample period, implied market volatility fell from more than 20 percent to less than 10 percent. Volatility reached a maximum of 27.16 percent on January 16, 1991, the height of the Gulf War. The minimum implied volatility, 9.43 percent, occurred on December 29, 1993, the last date of the sample period.

Model 3 has six parameters, and the averages (standard deviations) of the model's six parameter estimates across the 292 cross sections are reported in Panel A of Table II. The standard deviation of the parameter estimates indicates that there is considerable variation in the coefficient estimates from week to week, implying perhaps that the volatility function is not stable through time. If the parameter estimates are highly correlated, however, the errors affecting them may cancel out when option prices are looked at.

Table II

Summary Statistics of Parameter Estimates Obtained for Deterministic Volatility Function (DVF) Model 3

Below are summary statistics from fitting Model 3 to the cross-section of S&P 500 index option prices each week during the sample period from June 1988 through December 1993. Model 3 specifies that the local volatility rate is linear in X, X^2, T, T^2, and XT, where X is the option's exercise price and T is its time to expiration. The parameter estimates, a_1, a_2, a_3, a_4, and a_5, are the estimated coefficients of each of these terms, respectively. The parameter estimate, a_0, is the estimated intercept term.

Panel A: Means and Standard Deviations

Coefficient Estimate	Mean	Standard Deviation
a_0	131.8	69.5
a_1	−0.3529	0.447
a_2	0.00008611	0.00768
a_3	−0.2260	1.94
a_4	−0.0001666	0.00237
a_5	0.05275	0.0593

Panel B: Correlations

Coefficient Estimate	a_2	a_3	a_4	a_5
a_1	−0.969	0.596	−0.811	−0.291
a_2		−0.589	0.853	0.182
a_3			−0.114	−0.232
a_4				0.093

To check this possibility, we compute the correlation among the parameter estimates across the 292 weeks in the sample period and report them in Panel B of Table II. As the values show, the correlations are generally quite large. The correlation between the linear and quadratic terms in Model 3, for example, is −0.969, indicating that in weeks where a_1 is high, a_2 is low and vice versa.

In order to examine explicitly the issue of coefficient stability, Figure 5 has four panels containing plots of the time-series estimates of the Black–Scholes implied volatility (i.e., $\hat{a}_0$ in Model 0) as well as of the time-series estimates of the three main coefficients of Model 3 (i.e., $\hat{a}_0$, $\hat{a}_1$, and $\hat{a}_2$). The figures in Panels A and B indicate that the intercept coefficient of the DVF function fluctuates largely in unison with the Black–Scholes implied volatility. Week after week, the coefficient appears to simply record the movements in the level of the volatility. It is to be suspected that, were the coefficients kept constant from one week to the next, very little of the volatility movement would be captured by the movement in the level of the index itself. The plots in Figure 5 themselves are is not entirely meaningful, however, since the movements in the individual coefficients are highly cor-

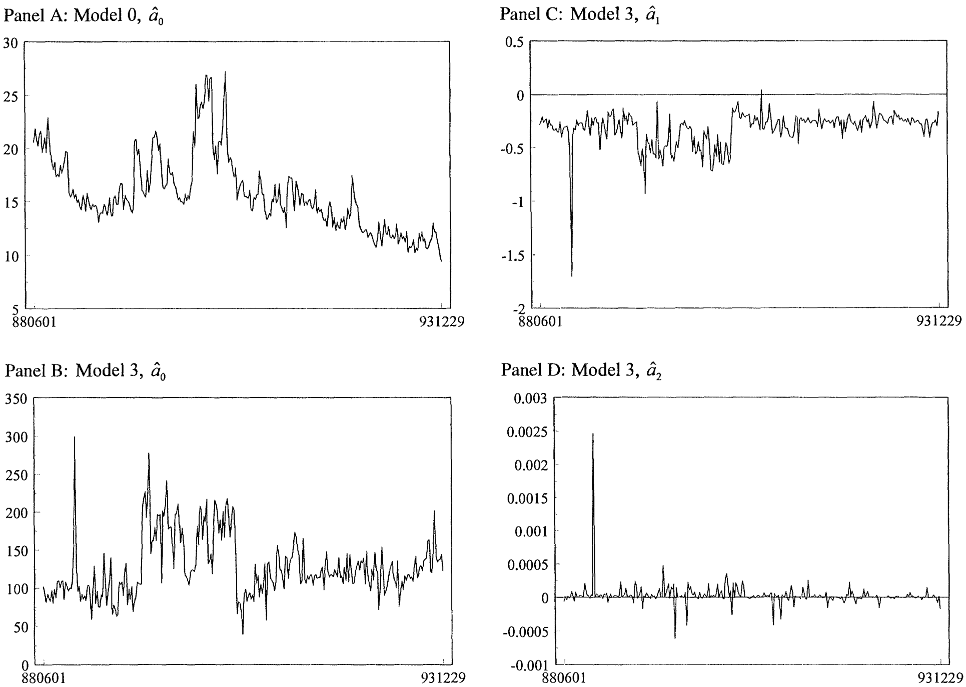

Figure 5. Parameter estimates for a_0 of Model 0 (Panel A) and a_0, a_1, and a_2 of Model 3 (Panels B, C, and D, respectively) each Wednesday during the sample period of June 1988 through December 1993. The parameter estimates are obtained by fitting Model 0 and Model 3 to the cross section of S&P 500 index option prices each week.

related, and they may, to some extent, offset each other when combined to generate fitted volatility levels. We are ultimately interested in the movements of the fitted volatility in the neighborhood of the money.

To examine this further, Figure 6 shows the time series of the explained at-the-money volatility of each week (with contemporaneous coefficients) minus the explained at-the-money volatility of the same week calculated on the basis of the *previous week's* coefficients. The figure, therefore, portrays the week-to-week changes in the level of the DVF function at the money which result from changes in the coefficients, and which remain "unexplained" by the DVF function and index level changes. It is apparent that these unexplained weekly changes in (annualized) volatility are very large and routinely reach several percentage points.

This evidence indicates that the in-sample estimates for the DVF model seem to be unstable. This inference implies that changes in the coefficient estimates may not be entirely due to economic factors, but may be the result of overfitting. Therefore, it seems critical that we should measure the economic significance of the DVF model in terms of valuation prediction errors. This is exactly the procedure applied in Section V.

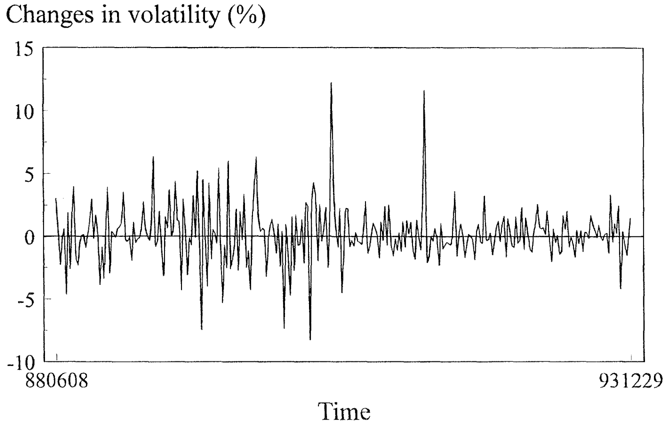

Figure 6. Week-to-week change in the local volatility rate of Model 3 arising from changes in the coefficient estimates, for the period June 1988 through December 1993. The change is defined as the difference between the estimated local volatility rate at the current index level using contemporaneous coefficient estimates of Model 3 less the estimated local volatility rate using the previous week's coefficient estimates.

C. Implied Probability Distribution

The estimated coefficients of the volatility functions can also be used to deduce the shape of the risk-neutral probability distribution at the option expiration dates.[22] To illustrate, we first use the estimated coefficients of Model 3 on April 1, 1992. On April 1, 1992, the S&P 500 options had three different expiration months, April, May, and June 1992, with 17, 45, and 80 days to expiration, respectively. Based on these expirations, the estimated volatility function implies the three probability distributions shown in Figure 7. All distributions are skewed to the left, exactly the opposite of the right-skewness implied by the Black–Scholes assumption of lognormally distributed asset prices. The wider variances for the May and then June expirations merely reflect the greater probability of large price moves over a longer period of time. Our implied distributions do not exhibit the bimodality that was present in Rubinstein (1994). This likely results from the fact that our volatility functions are more parsimonious than those implicitly used within his binomial lattice framework.

V. Prediction Results

The estimation results reported in the last section indicate that the volatility function embedded in index option prices is not particularly elaborate. The AIC indicates that only linear and quadratic terms in asset price

[22] The identification of state price densities from option prices has been the goal of much of David Bates' work. See Bates (1996a, 1996b). See also Ait-Sahalia and Lo (1998).

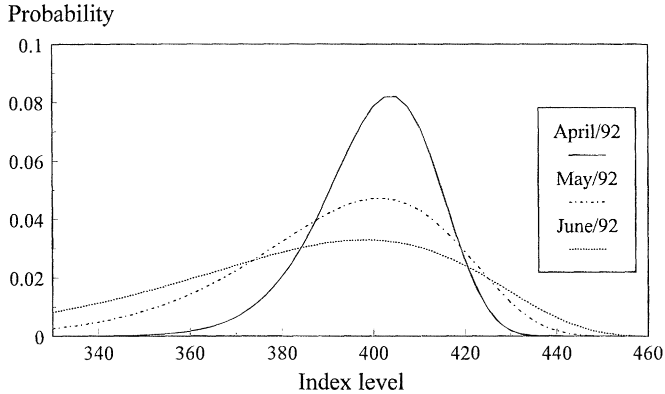

Figure 7. Risk-neutral probability density functions for April, May, and June 1992 S&P 500 option expirations on April 1, 1992. The probability distributions are based on the parameter estimates of Deterministic Volatility Function Model 3.

are necessary and only linear terms in time. A critical assumption of the model, however, is that the volatility function is stable through time, an assumption we already have reason to doubt. In this section, we evaluate how well each week's estimated volatility function values the same options one week later.

A. Goodness-of-Fit

Table III provides the summary statistics for the prediction errors. The RMSVE, MOE, and MAE values in the table are computed in the same manner as in the previous section. The prediction errors are generally quite large, at least relative to the estimation errors reported in Table I. The average RMSVE reported in Panel A is about 56 cents out-of-sample across all days for all DVF models except Model 0, and the in-sample error for these models is about 23 cents. The MAE statistics tell essentially the same story but more dramatically: The almost exact fit achieved for Model 3 (i.e., a 5-cent MAE in sample) has deteriorated to nearly 30 cents within a week. New market information induces a shift in the level of overall market volatility from week to week.

The prediction errors for calls and puts reported in Panel A are about the same size. As in the case of the estimation errors, the average MOE for Model 0 is positive for calls and negative for puts, depending on the character of the sample. When the options are stratified by option type and moneyness, we see that the Black–Scholes model value is too low (relative to the bid price) for in-the-money calls and out-of-the-money puts and is too high (relative to the ask price) for out-of-the-money calls and in-the-money puts. This pattern is particularly clear in Figure 8, which is the analogue of Figure 3, but for the prediction stage.

Table III
Average S&P 500 Index Option Dollar Prediction Errors Using the Deterministic Volatility Function (DVF) Model

RMSVE is the root mean squared dollar valuation error averaged across all days in the sample period from June 1988 through December 1993. ΔRMSVE is the incremental RMSVE when comparing successive models. The t-ratio indicates the significance of that increment. MOE is the average of the mean valuation error outside the observed bid/ask quotes across all days in the sample (a positive value indicates the theoretical value exceeds the ask price on average; a negative value indicates the theoretical value is below the bid price). MAE is the average of the mean absolute valuation error outside the observed bid/ask quotes across all days in the sample. FREQ is the frequency of days, expressed as a ratio of the total number of days, on which a particular model has a lower daily RMSVE than the Ad Hoc Model. Model 0 is the Black/Scholes constant volatility model. Models 1, 2, and 3 specify that the local volatility rate is linear in (a) X and X^2, (b) X, X^2, T, and XT, and (c) X, X^2, T, T^2, and XT, respectively, where X is the option's exercise price and T is its time to expiration. Model S switches between Models 1, 2, and 3 depending upon whether the number of option expirations in the cross-section is one, two or three, respectively. The Ad Hoc (AH) Model specifies that Black/Scholes implied volatility is linear in (a) X and X^2, (b) X, X^2, T, and XT, and (c) X, X^2, T, T^2, and XT, respectively, depending upon whether the number of option expirations in the cross-section is one, two or three. Moneyness is defined as $X/F - 1$, where F is the forward index level.

Panel A: Aggregate Results

DVF	All Options					Call Options					Put options				
Model	RMSVE	ΔRMSVE	t-ratio	MOE	MAE	RMSVE	ΔRMSVE	t-ratio	MOE	MAE	RMSVE	ΔRMSVE	t-ratio	MOE	MAE
0	0.784			−0.017	0.449	0.790			0.180	0.458	0.762			−0.219	0.440
1	0.557	−0.227	−7.26	−0.043	0.285	0.556	−0.234	−7.85	−0.022	0.284	0.551	−0.211	−6.62	−0.064	0.289
2	0.559	0.002	0.15	−0.067	0.294	0.551	−0.005	−0.28	−0.045	0.287	0.562	0.011	0.78	−0.091	0.305
3	0.556	−0.003	−1.19	−0.065	0.291	0.549	−0.002	−0.60	−0.044	0.285	0.557	−0.005	−1.82	−0.088	0.300
S	0.555	−0.001	−0.61	−0.066	0.292	0.548	−0.001	−1.05	−0.045	0.286	0.557	0.000	−0.09	−0.090	0.301
AH	0.498	−0.057	−2.46	−0.054	0.238	0.491	−0.057	−2.19	−0.002	0.235	0.493	−0.064	−2.85	−0.107	0.243

Moneyness (%)		DVF	Days to Expiration								
			Less than 40			40 to 70			More than 70		
Lower	Upper	Model	RMSVE	MOE	FREQ	RMSVE	MOE	FREQ	RMSVE	MOE	FREQ
Panel B: Call Options Only											
−10	−5	0	0.348	−0.004	0.484	0.640	−0.290	0.332	1.106	−0.688	0.073
		1	0.310	−0.024	0.609	0.436	−0.082	0.469	0.602	−0.180	0.419
		2	0.324	−0.045	0.534	0.458	−0.096	0.417	0.607	−0.057	0.476
		3	0.330	−0.045	0.523	0.461	−0.098	0.417	0.604	−0.049	0.444
		S	0.328	−0.047	0.516	0.459	−0.098	0.427	0.603	−0.051	0.452
		AH	0.344	−0.016		0.443	−0.079		0.553	−0.107	
−5	0	0	0.524	0.179	0.377	0.625	−0.063	0.347	0.794	−0.325	0.212
		1	0.472	0.136	0.444	0.547	−0.014	0.400	0.691	−0.252	0.364
		2	0.425	0.015	0.489	0.577	−0.067	0.396	0.713	−0.057	0.386
		3	0.428	0.010	0.500	0.583	−0.069	0.382	0.706	−0.036	0.379
		S	0.426	0.009	0.500	0.581	−0.069	0.382	0.707	−0.040	0.379
		AH	0.428	0.084		0.469	−0.023		0.545	−0.119	
0	5	0	0.799	0.656	0.142	0.989	0.743	0.149	0.930	0.599	0.213
		1	0.504	0.234	0.396	0.571	−0.025	0.385	0.664	−0.313	0.409
		2	0.428	0.062	0.458	0.589	−0.108	0.353	0.710	−0.100	0.370
		3	0.424	0.054	0.476	0.592	−0.114	0.353	0.686	−0.080	0.378
		S	0.425	0.054	0.476	0.593	−0.114	0.353	0.692	−0.082	0.370
		AH	0.408	0.163		0.443	0.008		0.507	−0.172	
5	10	0	0.475	0.397	0.257	0.981	0.860	0.113	1.042	0.921	0.149
		1	0.220	−0.010	0.525	0.364	−0.105	0.472	0.461	−0.252	0.459
		2	0.222	−0.066	0.436	0.389	−0.156	0.425	0.461	−0.088	0.554
		3	0.221	−0.068	0.436	0.394	−0.162	0.406	0.469	−0.096	0.527
		S	0.222	−0.067	0.436	0.394	−0.162	0.406	0.466	−0.095	0.527
		AH	0.209	−0.024		0.344	−0.101		0.437	−0.237	

Table III—*Continued*

Moneyness (%)		DVF	Days to Expiration								
			Less than 40			40 to 70			More than 70		
Lower	Upper	Model	RMSVE	MOE	FREQ	RMSVE	MOE	FREQ	RMSVE	MOE	FREQ
Panel C: Put Options Only											
−10	−5	0	0.591	−0.502	0.039	1.176	−1.051	0.024	1.742	−1.602	0.010
		1	0.276	−0.127	0.615	0.440	−0.200	0.506	0.623	−0.362	0.313
		2	0.309	−0.176	0.459	0.479	−0.232	0.434	0.532	−0.215	0.396
		3	0.313	−0.173	0.468	0.475	−0.225	0.434	0.520	−0.189	0.438
		S	0.310	−0.177	0.468	0.477	−0.227	0.434	0.522	−0.189	0.438
		AH	0.308	−0.196		0.415	−0.244		0.497	−0.283	
−5	0	0	0.540	−0.281	0.265	0.849	−0.551	0.144	1.135	−0.867	0.113
		1	0.446	0.109	0.416	0.540	−0.061	0.383	0.654	−0.232	0.371
		2	0.430	−0.040	0.419	0.581	−0.122	0.363	0.711	−0.018	0.355
		3	0.431	−0.042	0.409	0.580	−0.121	0.368	0.698	0.016	0.371
		S	0.429	−0.044	0.416	0.581	−0.122	0.368	0.699	0.010	0.363
		AH	0.369	−0.050		0.434	−0.115		0.511	−0.130	
0	5	0	0.443	0.081	0.435	0.611	0.156	0.393	0.691	0.077	0.357
		1	0.484	0.117	0.406	0.579	−0.066	0.422	0.714	−0.293	0.413
		2	0.446	−0.002	0.435	0.615	−0.132	0.398	0.775	−0.135	0.405
		3	0.443	−0.003	0.435	0.615	−0.133	0.403	0.751	−0.112	0.421
		S	0.444	−0.004	0.442	0.616	−0.133	0.403	0.754	−0.116	0.421
		AH	0.409	−0.001		0.486	−0.090		0.572	−0.185	
5	10	0	0.444	−0.059	0.372	0.493	0.116	0.490	0.634	0.249	0.400
		1	0.277	0.025	0.567	0.400	−0.071	0.569	0.520	−0.143	0.500
		2	0.273	0.015	0.563	0.427	−0.094	0.497	0.562	−0.055	0.480
		3	0.273	0.015	0.532	0.424	−0.092	0.497	0.551	−0.057	0.470
		S	0.273	0.015	0.550	0.425	−0.094	0.503	0.551	−0.056	0.480
		AH	0.347	−0.037		0.459	−0.103		0.508	−0.125	

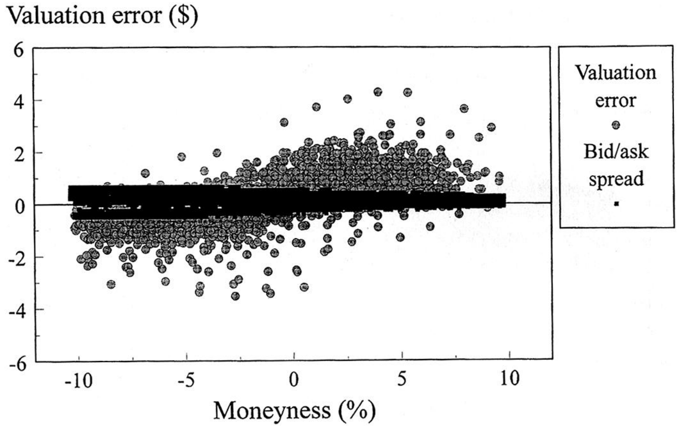

Figure 8. Dollar valuation prediction errors of Deterministic Volatility Function Model 0 (i.e., the Black–Scholes model) for S&P 500 call options with 40 to 70 days to expiration. The theoretical values are based on the implied volatility function from the previous week. The solid squares correspond to normalized bid/ask price quotes (i.e., the bid and ask prices less the average of the bid and ask prices). The circles correspond to valuation errors (i.e., the theoretical option value less the bid/ask midpoint). Moneyness is defined as $X/F - 1$, where F is the forward index level and X is the option's exercise price.

Interestingly, the average MOE is smaller for Model 1 than for Models 2, 3, and S. This suggests that the time variable in the more elaborate volatility functions is unnecessary. Apparently, the time variable serves only to overfit the data at the estimation stage. The fact that the valuation prediction errors for the models that include the time variable are more negative than those of Model 1 indicates that the implied volatility functions predict a larger decrease in volatility over the week than actually transpires.

At-the-money options have the largest valuation prediction errors for all times to expiration. This arises because at-the-money options are the most sensitive to volatility (where time premium is the highest). For a given error in the estimated volatility rate, the dollar valuation error is larger for at-the-money options than for either in-the-money or out-of-the-money options. Figure 9, which is the analogue of Figure 4, illustrates that the prediction errors of Model 4 do not display the characteristic patterns across the spectrum of moneyness that we identify above for Model 0.

B. An "Ad Hoc" Strawman

A troubling aspect of the analysis thus far is that, although the RMSVEs seem large for all practical purposes, we have not yet indicated what size of prediction error should be considered "large." One way to gauge the prediction errors is to measure them against a benchmark. To account for the sneer patterns in Black–Scholes implied volatilities, many marketmakers simply smooth the implied volatility relation across exercise prices (and days

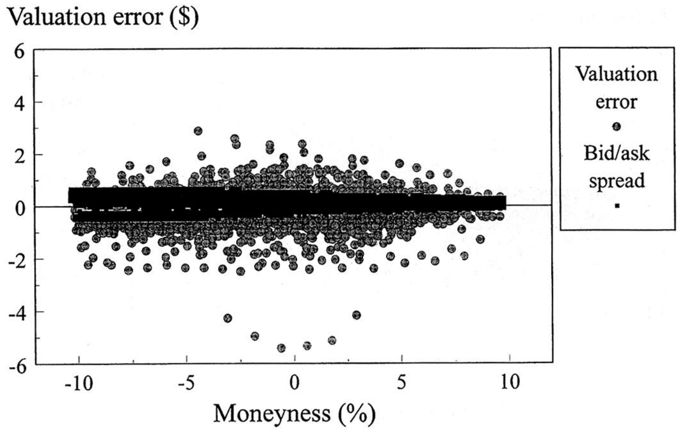

Figure 9. Dollar valuation prediction errors of Deterministic Volatility Function Model 3 for S&P 500 call options with 40 and 70 days to expiration. The theoretical values are based on the implied volatility function from the previous week. The solid squares correspond to normalized bid/ask price quotes (i.e., the bid and ask prices less the average of the bid and ask prices). The circles correspond to valuation errors (i.e., the theoretical option value less the bid/ask midpoint). Moneyness is defined as $X/F - 1$, where F is the forward index level and X is the option's exercise price.

to expiration) and then value options using the smoothed relation. To operationalize this practice, we fit the Black–Scholes model to the reported structure of option prices each week using Model S to describe the Black–Scholes implied volatility. Obviously, applying the Black–Scholes formula in this context is internally inconsistent because the Black–Scholes formula is based on an assumption of constant volatility. Nonetheless, the procedure is a variation of what is applied in practice as a means of predicting option prices.[23] The DVF option valuation model, which is based on an internally consistent specification, should dominate this "ad hoc" approach.

To create our strawman, we use a two-step procedure similar to the one we used for the DVF models. On day t, we fit Model S to the Black–Scholes implied volatilities, and then, on day $t + 7$, we apply the Black–Scholes formula using the volatility levels from estimated regression. The valuation prediction errors computed in this fashion are also included in Table III. As the table shows, the errors using the ad hoc model (AH) are almost uniformly smaller than those of the DVF approach. The average RMSVE across the entire sample period is 49.8 cents for the ad hoc Black–Scholes procedure, whereas it is more than 55 cents for the best DVF option valuation model. The average MAE is 23 cents for the ad hoc Black–Scholes procedure and 28.5 cents for Model 1. In viewing the various option categories, the greatest pricing improvement appears to be for at-the-money options, whose

[23] The Black/Scholes procedure cannot serve to predict American or exotic option prices from European option prices, which is the major benefit claimed for the implied volatility tree approach.

average RMSVEs are reduced by 10 cents or more. Put simply, the deterministic volatility approach does not appear to be an improvement over the, albeit theoretically inconsistent, ad hoc procedure used in practice. One possible interpretation of this evidence is that there is little economic meaning to the deterministic volatility function implied by option prices.

The reason the ad hoc strawman performs marginally better than the DVF model can be seen by examination of Figure 10, which is identical in its format to Figure 6; that is, we show the time series of the explained at-the-money-implied volatility of each week (with contemporaneous coefficients) minus the explained at-the-money implied volatility of the same week calculated on the basis of the *previous week's* coefficients. A comparison of Figure 10 with Figure 6 shows that the coefficients of the ad hoc model are somewhat more stable than those of the DVF Model 3.

C. A t-Test of Equivalence between the Various Models[24]

Panel A of Table III also reports the results of statistical tests of the equivalence between models. The tests are based on West (1996), and, for ease of reference, we use his notation. Let f_t be the (6×1) vector of root mean squared prediction errors at time t corresponding to the six models 0, 1, 2, 3, S, and AH. Let Ef be the population value, and let f be the sample average. Then, if we know the population values for all the parameters, $f - Ef$ is asymptotically normal with the variance-covariance matrix:

$$S_{ff} = \sum_{j=-\infty}^{\infty} E(f_t - Ef)(f_{t-j} - Ef)'. \tag{12}$$

On the basis of this observation and a generalized method of moments reasoning, we determine $\bar{f}$, a 6×1 constant vector, such that

$$\min_{\bar{f}} \sum_t (f_t - \bar{f})'\Omega(f_t - \bar{f}), \tag{13}$$

where Ω is the Newey-West heteroskedasticity-consistent, 6×6 variance-covariance matrix with fifteen weekly lags which accounts for the possibility of correlation across the models and serially correlated errors. In this way, we obtain asymptotic t-ratios for the root mean square prediction errors of our six models. Panel A of Table III reports both the incremental root mean squared prediction errors of each model compared to the previous one in the list as well as their corresponding t-ratios.

The test results reported in Table III indicate that DVF Model 1 is a significant improvement over the straight Black–Scholes Model 0. The incremental average root mean squared error is -22.7 cents and its t-ratio is -7.26. The incremental improvements in going from DVF Models 1 to 2,

[24] We are extremely grateful to Ken West for his correspondence outlining the steps of the procedure below.

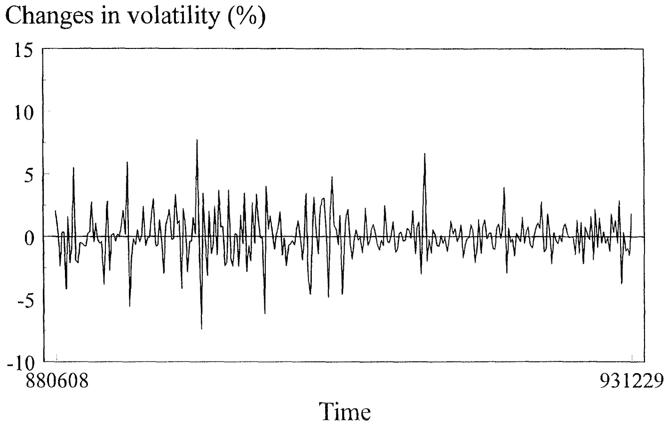

Figure 10. Week-to-week change in the volatility rate of the ad hoc Black–Scholes model arising from changes in the coefficient estimates, for the period June 1988 through December 1993. The change is defined as the difference between the estimated volatility rate at the current index level using contemporaneous coefficient estimates of the ad hoc Black–Scholes model less the estimated volatility rate using the previous week's coefficient estimates.

2 to 3, and 3 to S, however, are insignificant. Finally, the ad hoc model is an improvement over the Model S. The incremental average root mean squared error is −5.7 cents and its *t*-ratio is −2.46.[25]

VI. Hedging Results

A key motivation for developing the DVF option valuation model is to provide better risk management. If volatility is a deterministic function of asset price and time, then setting hedge ratios based on the DVF option valuation model should present an improvement over the constant volatility model. In this section, we evaluate the performance of a hedge portfolio formed on day t and unwound one week later. Galai (1983) shows that the return on such a discretely adjusted option hedge portfolio has three components: (a) the riskless return on investment, (b) the return from the discrete adjustment of the hedge, and (c) the return from the change in the deviation of the actual option price from the change in the theoretical value.

[25] Admittedly, the assumption that we know the population values for all the parameters is incorrect. Each time we move down one week, we calculate the errors of the new week on the basis of the parameters of the old week. The t-statistics do not take into account the standard errors of the parameter estimation performed in the old week. As shown by West (1996), the needed correction would depend on the expected value of the derivative of the prediction errors taken with respect to the parameter values. This would require repeated numerical calculation of the time series of errors, varying one parameter each time. There is no theoretical reason why this derivative would be equal to zero in our application, whereas it would have been if our estimators had been designed to optimize the prediction.

Since all option prices used in our analysis are forward prices, the riskless return component of the hedge portfolio is zero. Furthermore, because our focus is on model performance and not on the issues raised by discrete-time readjustment, we assume that the hedge portfolio is continuously rebalanced through time. Consequently, the hedge portfolio error is defined as:

$$\epsilon_t = \Delta c_{\text{actual},t} - \Delta c_{\text{model},t}, \tag{14}$$

where $\Delta c_{\text{actual},t}$ is the change in the reported option price from day t until day $t + 7$ and $\Delta c_{\text{model},t}$ is the change in the model's theoretical value.

The proof of equation (14) is straightforward. The hedging error that results from the continuous rebalancing using the hedge ratio, h, is

$$\Delta c_{\text{actual},t} - \int_t^{t+7} h(S_u, u)\, dS_u. \tag{15}$$

If we had the correct model to determine h, the two quantities in equation (15) would be equal to one another, not as a real number equality but with probability one or at the very least in the sense that their difference would have an expected value of zero and zero variance. Therefore, it must be the case that the integral term equals $\Delta c_{\text{model},t}$. In other words, when the hedge is continuously rebalanced, the hedging error is simply equal to the time increment in the valuation error.

Table IV contains a summary of the hedging error results. Across the overall sample period, Model 0—the Black–Scholes constant-volatility model—performs best of all the deterministic volatility function specifications. Its average root mean squared hedging error (RMSHE) is 45.5 cents, compared with 48.9, 50.5, 50.6, and 50.5 cents for Models 1 through 3 and Model S, respectively. The intuition for this result is that, although the model's option values are systematically incorrect, its errors are stable (or, at least, strongly serially dependent as suits a specification error), unlike the less parsimonious models. Within the class of DVF models considered, the results again indicate that, the more parsimonious volatility functions provide better hedging performance.

The ad hoc Black–Scholes procedure described in the preceding section also performs well from a hedging standpoint. The average RMSHE is only 46.7 cents. Consistent with the prediction results reported in Table III, the DVF option valuation model does not appear to be an improvement.

To further distinguish between the hedging errors of the different models, we run t-tests similar to the ones we performed for the prediction errors. The results are shown in Panel A of Table IV. With a t-ratio of 1.99, Model 1 represents a significant worsening relative to the plain Black–Scholes model.

Table IV

Average S&P 500 Index Option Dollar Hedging Errors Using the Deterministic Volatility Function (DVF) Model

RMSHE is the root mean squared dollar hedging error averaged across all days in the sample period from June 1988 through December 1993. ΔRMSHE is the incremental RMSHE when comparing successive models. The t-ratio indicates the significance of that increment. FREQ is the frequency of days, expressed as a ratio of the total number of days, on which a particular model has a lower daily RMSHE than the Ad Hoc Model. Models 1, 2, and 3 specify that the local volatility rate is linear in (a) X and X^2, (b) X, X^2, T, and XT, and (c) X, X^2, T, T^2, and XT, respectively, where X is the option's exercise price and T is its time to expiration. Model S switches between Models 1, 2, and 3 depending upon whether the number of option expirations in the cross-section is one, two or three, respectively. The Ad Hoc (AH) Model specifies that Black/Scholes implied volatility is linear in (a) X and X^2, (b) X, X^2, T, and XT, and (c) X, X^2, T, T^2, and XT, respectively, depending upon whether the number of option expirations in the cross-section is one, two or three. Moneyness is defined as $X/F - 1$, where F is the forward index level.

Panel A: Aggregate Results

DVF	All Options			Call Options			Put Options		
Model	RMSHE	ΔRMSHE	t-ratio	RMSHE	ΔRMSHE	t-ratio	RMSHE	ΔRMSHE	t-ratio
0	0.455			0.445			0.443		
1	0.489	0.034	1.99	0.491	0.046	2.56	0.472	0.029	1.55
2	0.505	0.016	3.04	0.500	0.009	1.44	0.496	0.024	4.50
3	0.506	0.001	0.85	0.505	0.005	4.14	0.492	−0.004	−1.89
S	0.505	−0.001	−1.90	0.503	−0.002	−2.51	0.492	0.000	0.41
AH	0.467	−0.038	−2.13	0.454	−0.049	−2.39	0.450	−0.042	−2.42

Moneyness (%)		DVF	Days to Expiration					
			Less than 40		40 to 70		More than 70	
Lower	Upper	Model	RMSHE	FREQ	RMSHE	FREQ	RMSHE	FREQ
			Panel B: Call Options Only					
−10	−5	0	0.367	0.519	0.349	0.541	0.454	0.455
		1	0.334	0.515	0.404	0.453	0.518	0.409
		2	0.315	0.535	0.433	0.394	0.552	0.409
		3	0.319	0.550	0.441	0.382	0.568	0.398
		S	0.316	0.554	0.439	0.382	0.568	0.398
		AH	0.370		0.388		0.482	
−5	0	0	0.406	0.534	0.394	0.591	0.489	0.473
		1	0.466	0.455	0.524	0.403	0.642	0.333
		2	0.452	0.487	0.560	0.333	0.673	0.344
		3	0.457	0.484	0.569	0.328	0.686	0.333
		S	0.454	0.487	0.567	0.328	0.686	0.333
		AH	0.432		0.444		0.510	
0	5	0	0.388	0.367	0.367	0.534	0.422	0.488
		1	0.359	0.480	0.459	0.391	0.536	0.417
		2	0.364	0.476	0.483	0.385	0.552	0.429
		3	0.364	0.476	0.486	0.379	0.555	0.417
		S	0.363	0.480	0.485	0.385	0.555	0.417
		AH	0.335		0.384		0.443	
5	10	0	0.431	0.245	0.381	0.250	0.363	0.303
		1	0.203	0.510	0.256	0.462	0.304	0.485
		2	0.169	0.551	0.238	0.500	0.301	0.394
		3	0.168	0.571	0.238	0.500	0.306	0.394
		S	0.169	0.571	0.237	0.500	0.307	0.394
		AH	0.228		0.238		0.285	

Table IV—*Continued*

			Days to Expiration					
Moneyness (%)		DVF	Less than 40		40 to 70		More than 70	
Lower	Upper	Model	RMSHE	FREQ	RMSHE	FREQ	RMSHE	FREQ
Panel C: Put Options Only								
−10	−5	0	0.277	0.353	0.292	0.370	0.421	0.510
		1	0.252	0.419	0.352	0.380	0.473	0.429
		2	0.241	0.443	0.395	0.360	0.539	0.388
		3	0.240	0.455	0.395	0.360	0.540	0.367
		S	0.239	0.461	0.396	0.370	0.541	0.367
		AH	0.211		0.251		0.412	
−5	0	0	0.352	0.510	0.407	0.466	0.443	0.553
		1	0.437	0.361	0.521	0.338	0.565	0.395
		2	0.434	0.384	0.580	0.345	0.644	0.303
		3	0.433	0.384	0.579	0.351	0.642	0.316
		S	0.432	0.384	0.579	0.351	0.643	0.316
		AH	0.348		0.402		0.464	
0	5	0	0.420	0.506	0.443	0.509	0.479	0.607
		1	0.388	0.557	0.515	0.509	0.544	0.506
		2	0.408	0.502	0.551	0.441	0.596	0.461
		3	0.404	0.524	0.547	0.460	0.588	0.449
		S	0.405	0.531	0.547	0.460	0.588	0.449
		AH	0.425		0.467		0.522	
5	10	0	0.358	0.473	0.442	0.491	0.453	0.410
		1	0.272	0.597	0.370	0.545	0.384	0.492
		2	0.283	0.587	0.377	0.500	0.401	0.492
		3	0.279	0.587	0.368	0.527	0.389	0.492
		S	0.280	0.602	0.369	0.527	0.389	0.492
		AH	0.359		0.429		0.397	

VII. Robustness

The results reported in the last three sections offer evidence that the volatility functions implied by index option prices are not stable through time. In developing our test procedures, however, we make a number of methodological decisions, some of which could be questioned in the sense that the final results may have been different had other methodologies been adopted. In this section, we investigate the robustness of our results by checking three issues of this kind. The first issue pertains to the choice of the quadratic functional forms given in equations (6) through (9). In particular, allowing volatility to grow quadratically with state variables violates the slow-growth assumptions necessary for the existence of a solution to the stochastic differential equation. The second issue pertains to the trade-off between the cross-sectional and the time-series goodness-of-fit. Derman and Kani (1994a,b), Dupire (1994), and Rubinstein (1994) recommend using a single cross section of option prices to parameterize the DVF model. In this way, the arbitrage-free spirit of the model is maintained. But are the model's predictions improved by using multiple cross sections simultaneously? The third issue concerns the uniformity of the results over various subsamples. Does the DVF model work better during specific subperiods? We study these three issues below.

A. Functional Form

The quadratic functional forms given in equations (6) through (9) may seem questionable for two related reasons. First, the use of the parabolic branches, for which there is no basis in fact and which are purely extrapolative in nature, may influence our results. Of course, the probability weights received by values of the underlying asset far from the current value become extremely small very quickly (at an exponential rate), so they probably play a negligible role in the analysis. Nonetheless, this conjecture is worth checking. Second, it is questionable, mathematically speaking, to let the volatility grow quadratically with the state variable because such a volatility function violates the assumptions for existence of the solution of a stochastic differential equation (so-called "slow-growth" and "Lipschitz" conditions).

In order to allay these two fears simultaneously, we perform a simple experiment. In place of estimating Model 3 in an unconstrained manner, we truncate the local volatility rate at a maximum level of 50 percent annually, that is,

$$\text{Model 3t:} \quad \sigma = \max(0.01, \min(a_0 + a_1 X + a_2 X^2 + a_3 T + a_4 T^2 + a_5 XT, 0.50)), \tag{16}$$

and then redo all of the steps of the analysis. The results appear in Table V. For convenience, the earlier results of Tables I, III, and IV for Model 3 are also presented. Examining the various entries of the table, we find that the

Table V

Average S&P 500 Index Option Dollar Valuation, Prediction, and Hedging Errors Using the Deterministic Volatility Function (DVF) Model 3 without and with Maximum Truncation

RMSVE is the root mean squared dollar valuation error averaged across all days in the sample period from June 1988 through December 1993. MOE is the average of the mean valuation error outside the observed bid/ask quotes across all days in the sample (a positive value indicates the theoretical value exceeds the ask price on average; a negative value indicates the theoretical value is below the bid price). MAE is the average of the mean absolute valuation error outside the observed bid/ask quotes across all days in the sample. "Estimation" refers to in-sample valuation errors, "prediction" refers to out-of-sample valuation errors, and "hedge" refers to hedging errors. Model 3 specifies that the local volatility rate is linear in X, X^2, T, T^2, and XT, where X is the option's exercise price and T is its time to expiration. Model 3t has the same structural form as Model 3 except that the local volatility rate is truncated at a maximum of 50 percent. Moneyness is defined as $X/F - 1$, where F is the forward index level.

Panel A: Aggregate Results

Test	DVF Model	All Options			Call Options			Put Options		
		RMSVE	MOE	MAE	RMSVE	MOE	MAE	RMSVE	MOE	MAE
Estimation	3	0.226	−0.011	0.050	0.218	−0.002	0.044	0.230	−0.020	0.057
Estimation	3t	0.228	−0.010	0.051	0.220	−0.002	0.045	0.232	−0.019	0.057
Prediction	3	0.556	−0.065	0.291	0.549	−0.044	0.285	0.557	−0.088	0.300
Prediction	3t	0.556	−0.067	0.292	0.549	−0.046	0.286	0.557	−0.089	0.301
Hedge	3	0.506			0.505			0.492		
Hedge	3t	0.506			0.504			0.492		

Panel B: Call Options Only

Moneyness (%)				Days to Expiration					
				Less than 40		40 to 70		More than 70	
Lower	Upper	Test	DVF Model	RMSVE	MOE	RMSVE	MOE	RMSVE	MOE
−10	−5	estimation	3	0.259	−0.027	0.221	−0.013	0.220	0.003
		estimation	3t	0.258	−0.027	0.222	−0.014	0.227	−0.001
		prediction	3	0.330	−0.045	0.461	−0.098	0.604	−0.049
		prediction	3t	0.327	−0.046	0.459	−0.096	0.605	−0.070
		hedge	3	0.319		0.441		0.568	
		hedge	3t	0.318		0.438		0.563	
−5	0	estimation	3	0.197	0.014	0.187	0.012	0.205	0.026
		estimation	3t	0.201	0.018	0.190	0.014	0.209	0.023
		prediction	3	0.428	0.010	0.583	−0.069	0.706	−0.036
		prediction	3t	0.425	0.014	0.579	−0.065	0.715	−0.056
		hedge	3	0.457		0.569		0.686	
		hedge	3t	0.456		0.568		0.681	
0	5	estimation	3	0.166	0.047	0.171	0.007	0.189	−0.021
		estimation	3t	0.173	0.053	0.176	0.009	0.194	−0.023
		prediction	3	0.424	0.054	0.592	−0.114	0.686	−0.080
		prediction	3t	0.424	0.058	0.588	−0.109	0.697	−0.098
		hedge	3	0.364		0.486		0.555	
		hedge	3t	0.364		0.487		0.552	
5	10	estimation	3	0.151	−0.082	0.205	−0.108	0.245	−0.135
		estimation	3t	0.151	−0.081	0.204	−0.107	0.248	−0.137
		prediction	3	0.221	−0.068	0.394	−0.162	0.469	−0.096
		prediction	3t	0.222	−0.068	0.390	−0.160	0.464	−0.099
		hedge	3	0.168		0.238		0.306	
		hedge	3t	0.168		0.241		0.301	

Table V—*Continued*

Panel C: Put Options Only

Moneyness (%)				Days to Expiration					
				Less than 40		40 to 70		More than 70	
Lower	Upper	Test	DVF Model	RMSVE	MOE	RMSVE	MOE	RMSVE	MOE
−10	−5	estimation	3	0.237	−0.164	0.194	−0.088	0.152	−0.035
		estimation	3t	0.236	−0.162	0.194	−0.089	0.156	−0.041
		prediction	3	0.313	−0.173	0.475	−0.225	0.520	−0.189
		prediction	3t	0.310	−0.174	0.473	−0.226	0.542	−0.202
		hedge	3	0.240		0.395		0.540	
		hedge	3t	0.238		0.395		0.539	
−5	0	estimation	3	0.162	−0.026	0.152	−0.001	0.181	0.037
		estimation	3t	0.164	−0.021	0.150	−0.001	0.181	0.032
		prediction	3	0.431	−0.042	0.580	−0.121	0.698	0.016
		prediction	3t	0.428	−0.038	0.578	−0.118	0.712	−0.009
		hedge	3	0.433		0.579		0.642	
		hedge	3t	0.433		0.579		0.642	
0	5	estimation	3	0.203	0.010	0.190	0.000	0.225	−0.001
		estimation	3t	0.206	0.012	0.192	0.002	0.226	−0.003
		prediction	3	0.443	−0.003	0.615	−0.133	0.751	−0.112
		prediction	3t	0.442	0.000	0.612	−0.129	0.759	−0.126
		hedge	3	0.404		0.547		0.588	
		hedge	3t	0.404		0.548		0.588	
5	10	estimation	3	0.243	0.016	0.291	−0.037	0.309	−0.037
		estimation	3t	0.244	0.016	0.292	−0.038	0.312	−0.038
		prediction	3	0.273	0.015	0.424	−0.092	0.551	−0.057
		prediction	3t	0.272	0.015	0.424	−0.094	0.550	−0.061
		hedge	3	0.279		0.368		0.389	
		hedge	3t	0.280		0.371		0.389	

truncation makes no difference. The in-sample valuation errors, the out-of-sample prediction errors, and the hedging errors are virtually identical to those of the unconstrained version of the model for both the overall sample and the various option categorizations. In other words, the parabolic branches of the quadratic DVF models have not, in any way, obfuscated the analysis.

B. Two-Week Estimation

The second robustness test addresses the issue of the deterministic volatility function's stationarity. If we had truly believed in the permanency of the DVF model, we would have attempted to fit it to the entire five-year data sample with the same values of the coefficients throughout. It should be apparent by now, however, that no meaningful fit would have been obtained. To give the model the benefit of the doubt, we adopt the procedure advocated by the model's developers and fit it to the cross section of options available on one day only, and then determine whether the model could survive at least one week. By using comparatively little information, however, we may have introduced sampling variation. This sampling variation, as opposed to true parameter instability, may be responsible for the poor fit one week later. In order to address this possibility, we now redo the entire analysis, using the cross sections of two successive weeks for the in-sample estimation. We then investigate the quality of the fit out of sample by moving ahead by one more week, so that a total of three weeks are involved in the test.

The results for Model 3 are shown in Table VI. Not surprisingly, the in-sample fit (estimation mode) deteriorates slightly when going from one-week to two-week estimation. Forcing the same coefficient structure on two cross sections of option prices necessarily reduces in-sample performance. For the "All Options" category, for example, the RMSVE increases from 22.6 cents to 30.8 cents. For calls, the increase is from 21.8 cents to 29.7 cents, and, for puts, 23.0 cents to 31.4 cents.[26]

The prediction results are also reported in Table VI. Overall the improvement in out-of-sample performance is small. For the whole sample, the RMSVE is reduced from 55.5 cents to 54.4 cents. The subcategory results are mixed. For short-term options, the prediction performance is reduced, but, for intermediate and long-term options, the performance is improved. All in all, the results indicate that the additional variation in the time to expiration brought about by using two cross sections of option prices captures slightly better the relation between the local volatility rate and time.

The hedging performance results are noticeably improved as a result of the two-week estimation. For the full sample, Table VI shows that the RMSHE is reduced from 50.5 cents to 48.2 cents. Reductions in the RMSHE are also observed for the call and put option categories as well as for most of the

[26] The slight differences between the one-week results reported in Table VI and those reported in Table V arise because one cross section of option prices is lost in the two-week estimation procedure.

Table VI

Average S&P 500 Index Option Dollar Valuation, Prediction, and Hedging Errors Using the Deterministic Volatility Function (DVF) Model 3

RMSVE is the root mean squared valuation error averaged across all days in the sample period from June 1988 through December 1993. MOE is the average of the mean valuation error outside the observed bid/ask quotes across all days in the sample (a positive value indicates the theoretical value exceeds the ask price on average; a negative value indicates the theoretical value is below the bid price). MAE is the average of the mean absolute valuation error outside the observed bid/ask quotes across all days in the sample. "Estimation" refers to in-sample valuation errors, "prediction" refers to out-of-sample valuation errors, and "hedge" refers to hedging errors. Model "3-1week" specifies that the local volatility rate is linear in X, X^2, T, T^2, and XT and is fitted to a single week's cross-section, where X is the option exercise price, and T is the time to expiration. Model "3-2 weeks" specifies that the local volatility rate is linear in X, X^2, T, T^2, and XT and is fitted to two weeks' cross sections. Moneyness is defined as $X/F - 1$, where F is the forward index level.

Panel A: Aggregate Results

	DVF	All Options			Call Options			Put Options		
Test	Model	RMSVE	MOE	MAE	RMSVE	MOE	MAE	RMSVE	MOE	MAE
Estimation	3-1 week	0.226	−0.011	0.050	0.218	−0.002	0.044	0.230	−0.020	0.057
Estimation	3-2 weeks	0.308	−0.018	0.098	0.297	0.009	0.087	0.314	−0.047	0.112
Prediction	3-1 week	0.555	−0.063	0.292	0.548	−0.043	0.286	0.556	−0.086	0.301
Prediction	3-2 weeks	0.544	−0.047	0.283	0.537	−0.008	0.274	0.548	−0.089	0.298
Hedge	3-1 week	0.505		0.503		0.491				
Hedge	3-2 weeks	0.482		0.476		0.473				

Panel B: Call Options Only

Moneyness (%)				Days to Expiration					
				Less than 40		40 to 70		More than 70	
Lower	Upper	Test	DVF Model	RMSVE	MOE	RMSVE	MOE	RMSVE	MOE
−10	−5	estimation	3-1 week	0.260	−0.027	0.221	−0.013	0.220	0.003
		estimation	3-2 weeks	0.265	−0.032	0.311	−0.050	0.325	−0.057
		prediction	3-1 week	0.330	−0.045	0.461	−0.098	0.604	−0.049
		prediction	3-2 weeks	0.331	−0.039	0.474	−0.104	0.585	−0.105
		hedge	3-1 week	0.316		0.438		0.563	
		hedge	3-2 weeks	0.309		0.395		0.523	
−5	0	estimation	3-1 week	0.197	0.014	0.186	0.012	0.205	0.026
		estimation	3-2 weeks	0.253	0.027	0.271	0.003	0.297	−0.007
		prediction	3-1 week	0.427	0.012	0.583	−0.069	0.706	−0.036
		prediction	3-2 weeks	0.420	0.039	0.579	−0.038	0.665	−0.066
		hedge	3-1 week	0.454		0.568		0.681	
		hedge	3-2 weeks	0.429		0.529		0.652	
0	5	estimation	3-1 week	0.166	0.047	0.172	0.007	0.189	−0.021
		estimation	3-2 weeks	0.239	0.103	0.260	0.044	0.249	0.021
		prediction	3-1 week	0.423	0.056	0.592	−0.114	0.686	−0.080
		prediction	3-2 weeks	0.427	0.124	0.560	0.014	0.621	−0.009
		hedge	3-1 week	0.364		0.487		0.552	
		hedge	3-2 weeks	0.346		0.453		0.534	
5	10	estimation	3-1 week	0.151	−0.082	0.205	−0.108	0.245	−0.135
		estimation	3-2 weeks	0.151	−0.035	0.217	−0.058	0.258	−0.085
		prediction	3-1 week	0.212	−0.058	0.394	−0.162	0.469	−0.096
		prediction	3-2 weeks	0.214	−0.003	0.372	−0.031	0.406	−0.017
		hedge	3-1 week	0.168		0.241		0.301	
		hedge	3-2 weeks	0.188		0.227		0.289	

Table VI—*Continued*

Panel C: Put Options Only

Moneyness (%)				Days to Expiration					
				Less than 40		40 to 70		More than 70	
Lower	Upper	Test	DVF Model	RMSVE	MOE	RMSVE	MOE	RMSVE	MOE
−10	−5	estimation	3-1 week	0.237	−0.164	0.194	−0.088	0.152	−0.035
		estimation	3-2 weeks	0.280	−0.204	0.323	−0.208	0.308	−0.166
		prediction	3-1 week	0.309	−0.170	0.475	−0.225	0.520	−0.189
		prediction	3-2 weeks	0.339	−0.209	0.513	−0.281	0.544	−0.287
		hedge	3-1 week	0.235		0.395		0.539	
		hedge	3-2 weeks	0.230		0.352		0.490	
−5	0	estimation	3-1 week	0.162	−0.026	0.151	−0.001	0.181	0.037
		estimation	3-2 weeks	0.243	−0.053	0.268	−0.070	0.271	−0.002
		prediction	3-1 week	0.429	−0.039	0.580	−0.121	0.698	0.016
		prediction	3-2 weeks	0.444	−0.064	0.586	−0.138	0.665	−0.046
		hedge	3-1 week	0.431		0.579		0.642	
		hedge	3-2 weeks	0.413		0.546		0.610	
0	5	estimation	3-1 week	0.203	0.010	0.191	0.000	0.225	−0.001
		estimation	3-2 weeks	0.264	0.017	0.276	−0.003	0.299	0.004
		prediction	3-1 week	0.441	−0.001	0.615	−0.133	0.751	−0.112
		prediction	3-2 weeks	0.450	0.008	0.593	−0.066	0.670	−0.062
		hedge	3-1 week	0.405		0.548		0.588	
		hedge	3-2 weeks	0.407		0.523		0.557	
5	10	estimation	3-1 week	0.243	0.016	0.291	−0.037	0.309	−0.037
		estimation	3-2 weeks	0.261	0.013	0.303	−0.038	0.326	−0.026
		prediction	3-1 week	0.271	0.017	0.424	−0.092	0.551	−0.057
		prediction	3-2 weeks	0.293	0.012	0.393	−0.062	0.479	−0.029
		hedge	3-1 week	0.280		0.371		0.389	
		hedge	3-2 weeks	0.286		0.355		0.365	

option subcategories. Apparently, the two-week estimation has removed some of the sampling variation and has identified a coefficient structure that is more stable through time.

The overall performance of Model 3 estimated using two weeks of index option prices, however, still does not match the overall performance of the ad hoc Black–Scholes model fitted to a *single* cross section of prices (recall Table IV where the RMSHE is reported as 46.7 cents). Indeed, if one estimates the ad hoc Black–Scholes model using two cross sections of option prices, its sampling variation is reduced by even more than it is for Model 3. Though not shown in the table, its RMSHE falls to 40.8 cents. For calls and puts separately, the RMSHEs of the ad hoc Black–Scholes model estimated using the two-week estimation are 39.3 and 41.0 cents compared with 47.6 and 47.3 cents for Model 3. In other words, while increasing the amount of information used in estimation has identified coefficients that are more stable through time, the hedging performance of the ad hoc Black–Scholes model estimated using two cross sections of option prices shows even greater dominance over the DVF model than it does when only one cross section is used.

C. Analysis of Subsamples

The final issue has to do with performance through time. Does the DVF model perform better in some periods but not in others? To answer this question, we summarize the estimation, prediction, and hedging errors by calendar year. The results are reported in Table VII.

First, with respect to in-sample performance, the results are qualitatively robust across the sample. Using the AIC, Model 2 most frequently does best at describing the cross section of option prices in all subperiods. This is followed by the performance of Model 1. Again, parsimony in the volatility structure appears warranted. With respect to prediction, the ad hoc Black–Scholes model does best in every year except 1988, when its RMSVE is only slightly higher than Model 2's. Indeed, the outperformance is quite extraordinary in 1990, when its RMSVE is 54.2 cents versus 73.3 cents for Model 2. Finally, with respect to hedging performance, the ad hoc Black–Scholes (and the constant volatility Black–Scholes) model again dominates. All in all, the results of Table VII indicate that the poor performance of the DVF model is not driven by a particular subperiod of the sample. The DVF model performs poorly relative to an ad hoc procedure.

VIII. Summary and Conclusions

Claims that the Black and Scholes (1973) valuation formula no longer holds in financial markets are appearing with increasing frequency. When the Black–Scholes formula is used to imply volatilities from reported option prices, the volatility estimates vary systematically across exercise prices and times to expiration. Derman and Kani (1994a,b), Dupire (1994), and Rubinstein (1994) argue that this systematic behavior is driven by changes in the

Table VII

Average S&P 500 Index Option Dollar Valuation, Prediction, and Hedging Errors By Year Using the Deterministic Volatility Function (DVF) Model

RMSVE (RMSHE) is the root mean squared valuation (hedging) error averaged across all days in the sample period from June 1988 through December 1993. MAE is the average of the mean absolute valuation error outside the observed bid/ask quotes across all days in the sample. AIC is the proportion of days during the sample period that the model is judged best by the Akaike Information Criterion. Model 0 is the Black/Scholes constant volatility model. Models 1, 2, and 3 specify that the local volatility rate is linear in (a) X and X^2, (b) X, X^2, T, and XT, and (c) X, X^2, T, T^2, and XT, respectively, where X is the option's exercise price and T is its time to expiration. Model S switches between Models 1, 2, and 3 depending upon whether the number of option expirations in the cross-section is one, two or three, respectively. The Ad Hoc (AH) Model specifies that Black/Scholes implied volatility is linear in (a) X and X^2, (b) X, X^2, T, and XT, and (c) X, X^2, T, T^2, and XT, respectively, depending upon whether the number of option expirations in the cross-section is one, two or three. Moneyness is defined as $X/F - 1$, where F is the forward index level.

Panel A: Estimation

Year	DVF Model	Overall RMSVE	Overall MAE	Overall AIC	Calls RMSVE	Calls MAE	Puts RMSVE	Puts MAE
1988	0	0.435	0.208	0.000	0.433	0.215	0.435	0.200
	1	0.229	0.063	0.258	0.205	0.050	0.248	0.075
	2	0.178	0.032	0.677	0.178	0.032	0.177	0.033
	3	0.176	0.031	0.065	0.176	0.030	0.174	0.031
	S	0.178	0.032	0.000	0.178	0.032	0.177	0.032
1989	0	0.485	0.244	0.000	0.483	0.251	0.484	0.237
	1	0.256	0.085	0.192	0.266	0.090	0.240	0.079
	2	0.164	0.028	0.808	0.159	0.025	0.167	0.032
	3	0.164	0.028	0.000	0.159	0.025	0.167	0.032
	S	0.164	0.028	0.000	0.159	0.025	0.166	0.032
1990	0	0.838	0.475	0.000	0.861	0.522	0.814	0.433
	1	0.339	0.117	0.250	0.348	0.126	0.323	0.108
	2	0.223	0.048	0.712	0.230	0.053	0.212	0.043
	3	0.216	0.043	0.000	0.220	0.046	0.206	0.040
	S	0.216	0.043	0.038	0.220	0.046	0.207	0.040
1991	0	0.621	0.332	0.000	0.613	0.326	0.622	0.335
	1	0.281	0.085	0.302	0.294	0.087	0.265	0.082
	2	0.209	0.042	0.679	0.206	0.036	0.209	0.048
	3	0.209	0.042	0.019	0.205	0.036	0.209	0.048
	S	0.209	0.042	0.000	0.206	0.036	0.209	0.048
1992	0	0.708	0.390	0.000	0.712	0.401	0.698	0.381
	1	0.284	0.082	0.135	0.288	0.081	0.278	0.086
	2	0.226	0.048	0.731	0.227	0.041	0.225	0.057
	3	0.222	0.045	0.135	0.222	0.038	0.222	0.055
	S	0.223	0.046	0.000	0.223	0.038	0.222	0.055
1993	0	0.725	0.385	0.019	0.718	0.385	0.719	0.386
	1	0.388	0.124	0.404	0.358	0.115	0.405	0.139
	2	0.359	0.104	0.423	0.316	0.088	0.386	0.125
	3	0.352	0.100	0.154	0.306	0.083	0.382	0.123
	S	0.353	0.100	0.000	0.308	0.084	0.381	0.123

Table VII—*Continued*

		Panel B: Prediction						Panel C: Hedging		
	DVF	Overall		Calls		Puts		Overall	Calls	Puts
Year	Model	RMSVE	MAE	RMSVE	MAE	RMSVE	MAE	RMSVE	RMSVE	RMSVE
1988	0	0.546	0.290	0.554	0.308	0.524	0.273	0.308	0.299	0.305
	1	0.412	0.201	0.400	0.200	0.420	0.201	0.349	0.358	0.331
	2	0.394	0.187	0.388	0.190	0.397	0.185	0.365	0.366	0.356
	3	0.398	0.193	0.392	0.196	0.402	0.190	0.365	0.368	0.356
	S	0.394	0.188	0.388	0.191	0.397	0.185	0.365	0.367	0.355
	AH	0.397	0.185	0.373	0.174	0.414	0.194	0.329	0.300	0.344
1989	0	0.612	0.346	0.614	0.350	0.605	0.345	0.386	0.376	0.394
	1	0.494	0.258	0.497	0.258	0.483	0.261	0.411	0.389	0.429
	2	0.475	0.256	0.468	0.245	0.477	0.270	0.441	0.417	0.460
	3	0.475	0.256	0.469	0.246	0.476	0.269	0.441	0.417	0.459
	S	0.474	0.255	0.468	0.245	0.476	0.269	0.440	0.417	0.459
	AH	0.462	0.240	0.465	0.247	0.446	0.236	0.404	0.381	0.413
1990	0	0.996	0.589	1.016	0.624	0.968	0.560	0.529	0.544	0.490
	1	0.671	0.364	0.677	0.375	0.661	0.353	0.608	0.634	0.570
	2	0.733	0.424	0.746	0.443	0.718	0.407	0.649	0.675	0.613
	3	0.731	0.424	0.744	0.445	0.714	0.405	0.651	0.678	0.613
	S	0.731	0.423	0.744	0.445	0.713	0.405	0.649	0.678	0.611
	AH	0.542	0.260	0.538	0.261	0.539	0.257	0.511	0.505	0.494
1991	0	0.819	0.484	0.811	0.474	0.807	0.487	0.501	0.508	0.473
	1	0.649	0.370	0.658	0.367	0.636	0.371	0.559	0.551	0.541
	2	0.640	0.372	0.638	0.362	0.639	0.384	0.571	0.555	0.564
	3	0.642	0.374	0.641	0.364	0.640	0.385	0.572	0.557	0.564
	S	0.640	0.372	0.638	0.362	0.638	0.383	0.571	0.555	0.564
	AH	0.594	0.323	0.601	0.328	0.576	0.322	0.550	0.545	0.516
1992	0	0.820	0.474	0.829	0.479	0.784	0.459	0.435	0.410	0.432
	1	0.473	0.216	0.482	0.212	0.455	0.220	0.424	0.439	0.402
	2	0.465	0.216	0.464	0.204	0.461	0.230	0.423	0.425	0.408
	3	0.451	0.207	0.449	0.195	0.449	0.220	0.422	0.427	0.404
	S	0.451	0.206	0.449	0.195	0.448	0.220	0.423	0.429	0.403
	AH	0.403	0.164	0.412	0.162	0.386	0.167	0.372	0.371	0.352
1993	0	0.809	0.446	0.815	0.447	0.781	0.445	0.506	0.468	0.502
	1	0.581	0.266	0.553	0.253	0.596	0.288	0.523	0.518	0.495
	2	0.577	0.263	0.532	0.234	0.608	0.302	0.523	0.503	0.513
	3	0.569	0.259	0.530	0.234	0.595	0.292	0.521	0.516	0.496
	S	0.573	0.261	0.532	0.236	0.600	0.295	0.522	0.512	0.501
	AH	0.507	0.208	0.473	0.187	0.524	0.235	0.477	0.469	0.437

volatility rate of asset returns. They hypothesize that volatility is a deterministic function of asset price and time, and they provide appropriate binomial or trinomial option valuation procedures to account for this.

In this paper, we apply the deterministic volatility option valuation approach to S&P 500 index option prices during the period June 1988 through December 1993. We reach the following conclusions. First, although there is unlimited flexibility in specifying the volatility function and it is always possible to describe exactly the reported structure of option prices, our results indicate that a parsimonious model works best in sample according to the Akaike Information Criterion. Second, when the fitted volatility function is used to value options one week later, the DVF model's prediction errors grow larger as the volatility function specification becomes less parsimonious. In particular, specifications that include a time parameter do worst of all, indicating that the time variable is an important cause of overfitting at the estimation stage. Third, hedge ratios determined by the Black–Scholes model appear more reliable than those obtained from the DVF option valuation model. In sum, "simpler is better."

Overall, our results suggest at least two possible avenues for future investigation. First, the deterministic volatility framework could be generalized. The volatility surface, for example, may be related to past changes in the index level. Such a generalized volatility surface is probably the last candidate model that can be considered before resorting to fully stochastic volatility processes—processes that are difficult to estimate and that do not permit option valuation by the absence of arbitrage.[27]

Second, thought should be given to appropriate statistical test designs for competing volatility structures. The "null hypothesis" being investigated is that volatility is an exact function of asset price and time, so that options can be valued exactly by the no-arbitrage condition. Any deviation from such a strict theory, no matter how small, should cause a test statistic to reject it.[28] If a source of error had been introduced, some restriction on the sampling distribution of the error could be deduced and could provide a basis for a testing procedure.[29]

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[27] For an empirical test of stochastic volatility models that uses out-of-sample performance, as we recommend, see Bakshi, Cao, and Chen (1997).

[28] The same difficulty arises in any empirical verification of an exact theory. See MacBeth and Merville (1980), Whaley (1982), and Rubinstein (1985).

[29] Jacquier and Jarrow (1995) introduce two kinds of errors in the Black/Scholes model: model error and market error, which they distinguish by assuming that market errors occur rarely. Other approaches to the problem include Lo (1986) who introduces parameter uncertainty, Clément, Gouriéroux, and Monfort (1993) who randomize the martingale pricing measure to account for an incomplete market, and Bossaerts and Hillion (1997) whose error is due to discreteness in hedging.

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