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Arbitrage-free conditions for implied volatility surface by Delta[☆]

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ABSTRACT

Implied volatility surface provided by Deltas and maturities (IVS-DM) is widely used in financial fields, especially in foreign exchange options market, since it can effectively describe the characteristics of the volatilities. The main purpose of this paper is to develop arbitrage-free conditions for the IVS-DM. We propose sufficient conditions for the IVS-DM to be static arbitrage-free, which are proved to be necessary under a mild assumption. To test the arbitrage opportunities in the market data, we build a practical tool, which is more accurate than existing ones such as early warning indicator tests. For illustration, arbitrage tests are conducted on a simulated IVS-DM to show the effectiveness of the conditions. The results are consistent with the tests for the implied volatility surface provided by strikes and maturities. Furthermore, empirical examinations are implemented on EURUSD and USDJPY currency options to ensure the feasibility of the proposed conditions.

1. Introduction

Implied volatility surface (IVS), obtained from the classical Black-Scholes (BS) formula (Black et al., 1973), is a key input parameter for market practitioners in financial applications such as the pricing of options and volatility derivatives (Bianconi, Maclachlan, & Sammon, 2015; Harry & Mijatović, 2011; Sousa, Cruzeiro, & Guerra, 2017), the construction of local volatility models (Derman et al., 1994; Labuschagne & Boetticher, 2016), and the extraction of market information from option data (Kim & Kim, 2003; Wang, 2007).

Most asset pricing and risk management analysis of financial products are under the arbitrage-free assumption. However, it is inevitable to obtain the IVS with arbitrage opportunities since some of the options are illiquid, exotic or non-listed (Hentschel, 2003). Furthermore, the interpolation or extrapolation of the quoted IVS will lead to arbitrage points, even when the original set is arbitrage-free (Kahalé, 2004; Laurini, 2015). For example, arbitrage violations of the IVS may result in negative transition probabilities and local volatilities, which can lead to divergence in solving the generalized BS partial differential equation of the underlying asset (Matthias, 2009). In addition, if there exist some arbitrage opportunities in the IVS, arbitrageurs may construct investment portfolios to make profits such as bull spread, bear spread, butterfly spread (Hull, 2009). Therefore, it is crucial to test whether the IVS quoted in the market is arbitrage-free or not.

There are two quotation styles for the IVS in the market. One is provided by different strikes and maturities (IVS-SM), and the

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other is provided by different Deltas and maturities (IVS-DM). Generally, the equity, commodity and interest rate option markets commonly quote volatility using IVS-SM, while the foreign exchange (FX) options market uses IVS-DM (see Wystup, 2007; Clark et al., 2011). The choice of Delta as the parameter describing the phenomenon of volatility smile is natural and sensible, since it allows better coverage of the volatility smile for many maturities. Otherwise if a strike is employed, the volatility of an out-of-money option which is close to expire would be similar to the volatility of at-the-money option which is far from expire. Moreover, traders often use Delta to hedge risk. Hence it is more convenient to quote prices in Delta.

Most existing literatures study static arbitrage-free, which means that there is no buy and hold arbitrage opportunities trading in the surface (D'Aspremont, 2004; Glaser & Heider, 2012). More precisely, a call option price surface (COPS) is free of static arbitrage if and only if there exists a non-negative martingale on a filtered probability space such that European call option prices can be written as the expectation of their final payoffs under the risk neutral measure (Cox et al., 2005; Gatheral & Jacquier, 2014).

Static arbitrage-free conditions for the IVS-SM are considered in many literatures since they can be derived from the conditions of COPS directly. For the IVS-SM, Lee (2005) proposed necessary conditions for it to be arbitrage-free. Roper (2009) presented sufficient and close to necessary conditions. Moreover, the implied volatilities are interpolated to construct a complete surface with parameterized or non-parameterized methods. For the parameterized method, varies models have been discussed, such as the polynomial model, Gatheral's SVI model (Gatheral, 2004; Gatheral & Jacquier, 2014). Dumas, Fleming, and Whaley (1998) assumed that the implied volatility is a quadratic deterministic volatility function of asset price and time. While they did not consider the arbitrage-free conditions. In Kotzé, Labuschagne, Nair, and Padayachi (2013), they built the implied volatility surface with the quadratic deterministic function under no spread and calendar spread conditions. Then, they used these conditions in the options on single stock futures data to show how all the no-arbitrage conditions were implemented and tested practically.

For the non-parameterized method, many interpolation and smoothing methods are used to construct the IVS (Roper & Bulirsch, 2013; Fengler, 2006), such as the standard bi-linear or bi-cubic interpolation/spline methods and thin plate spline methods (Kamp, 2009). Kahalé (2004) presented an interpolation method to obtain arbitrage-free implied volatility for each single-maturity with three steps. While, the input data is required to be arbitrage-free to ensure the algorithm works. Matthias (2009) proposed an approach for smoothing the implied volatility smile in an arbitrage-free way based on natural spline method under suitable shape constraints. This approach is simple to implement. It works when input data is scarce and not arbitrage free. However, there are few results on the construction of the arbitrage-free IVS-DM, thus it is difficult to test arbitrage points in the market data.

One objective of this paper is to establish static arbitrage-free conditions for the IVS-DM. This is a complicated problem since the arbitrage-free conditions depend on many variables, such as the implied volatilities, strikes, maturities, and current spot values, etc. In addition, to construct the arbitrage-free IVS, one needs to estimate under highly nonlinear constrains (Matthias, 2009). On the one hand, the relations among Delta, strike, and implied volatility are nonlinear. On the other hand, there exist no globally parametric restrictions on the IVS. Hence the arbitrage-free conditions can not be transformed from the IVS-SM to the IVS-DM directly.

We propose seven items for the IVS-DM arbitrage-free conditions such as conditions on maturities, partial derivatives on Deltas and bound conditions. These conditions can ensure the monotonicity and convexity of COPS, which means that there are no calendar spread, call spread, butterfly spread arbitrage opportunities in the whole surface. Furthermore, in our work, the domestic interest rate and the foreign interest rate (dividend rate) are assumed to be non-zero. It is more general and realistic since in the market, these rates are important factors for asset pricing and risk management. The proposed conditions for the IVS-DM to be free of arbitrage are sufficient. They are meanwhile necessary under a mild condition.

Another objective of this paper is to examine the arbitrage opportunities in the market IVS-DM with the proposed conditions. The proposed arbitrage-free conditions have solid theoretical foundations for monotonicity, convexity, boundness and other properties. However, the empirical early warning indicator (EWI) method is simply an indication of the volatility convexity.

To show the effectiveness of the proposed arbitrage-free conditions, comprehensive empirical studies on the performance of the IVS-DM arbitrage tests for the FX options market are carried out. Firstly, an IVS-DM is simulated for the arbitrage test. Simultaneously, the IVS-SM can be derived form the corresponding COPS. Arbitrage test results for these two kinds of surfaces are consistent under both arbitrage and arbitrage-free situations. Secondly, the proposed arbitrage conditions are implemented on the market EURUSD and USDJPY option IVS-DM on December 6, 2016. Compared with EWI test, which is usually used by financial institutions due to its simple implementation, the proposed conditions are more accurate to test arbitrage points. In addition, the calendar spread arbitrage can be tested with the proposed conditions while not by the EWI method. Therefore, both simulated and empirical data show the feasibility of the arbitrage-free condition for the IVS-DM.

The rest of this paper is organized as follows. Section 2 describes the problem formulation. Section 3 analyzes the arbitrage-free conditions for the COPS and proposes the sufficient conditions for the IVS-DM, which are proved to be necessary under a mild assumption. Section 4 carries out several simulation results on simulated and empirical data. The conclusion and future work are presented in Section 5.

For simplicity, Table 1 list the notations and abbreviations used in this paper.

2. Problem formulation

2.1. Definitions and basic facts

To study the IVS-DM, we focus on the FX options market in this paper. Assuming that the domestic and foreign risk free interest rates are constant, the BS pricing formula for the FX European call option can be written as follows (Black et al., 1973; Garman &

Table 1
Notations and Abbreviations.

Notations and Abbreviations	
$\mathbb{R}$	Real numbers
K	Strike price
t	Current time
T	Maturity date
τ	Time to maturity date (years), equal to $T-t$
S	Current spot price of the underlying asset (spot exchange rate)
r_d	Domestic interest rate (continuously compounded)
r_f	Foreign interest rate (continuously compounded)
Δ	Sensitivity of option price to its underlying price
Δ_s	Spot Delta
Δ_f	Forward Delta
COPS	Call option price surface
σ	Constant implied volatility
$\sigma(K, \tau)$	Implied volatility varying with strikes and maturities
$\sigma(\Delta, \tau)$	Implied volatility varying in Deltas and maturities
IVS-SM	Implied volatility surface provided by strikes and maturities
IVS-DM	Implied volatility surface provided by Deltas and maturities
$C^{BS}(K, \tau, \sigma, S, r_d, r_f)$	Call option price with strike price K and time to maturity τ years

Kohlhagen, 1983),[1]

$$C^{BS}(K, \tau, \sigma, S, r_d, r_f) = Se^{-r_f\tau}N(d_+(K, \tau, \sigma)) - Ke^{-r_d\tau}N(d_-(K, \tau, \sigma)) \tag{1}$$

with the final conditions and boundary conditions given by

$$C^{BS}(K, 0, \sigma, S, r_d, r_f) = (S-K)^+,$$
$$C^{BS}(0, \tau, \sigma, S, r_d, r_f) = Se^{-r_f\tau},$$
$$\lim_{K\to\infty} C^{BS}(K, \tau, \sigma, S, r_d, r_f) = 0,$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy,$$
$$d_+(K, \tau, \sigma) = \frac{1}{\sigma\sqrt{\tau}}\left[ln\frac{S}{K} + \left(r_d - r_f + \frac{\sigma^2}{2}\right)\tau\right],$$
$$d_-(K, \tau, \sigma) = \frac{1}{\sigma\sqrt{\tau}}\left[ln\frac{S}{K} + \left(r_d - r_f - \frac{\sigma^2}{2}\right)\tau\right] = d_+(K, \tau, \sigma) - \sigma\sqrt{\tau},$$
$$(S-K)^+ = max(S-K, 0),$$

and σ is a constant volatility. We will drop some of the variables of C^{BS} depending on the context. As shown in Eq. (1), the BS model assumes a constant volatility. However, traded options in the market imply different volatilities for different time to maturities τ and strike prices K, this phenomenon is called the volatility "smile" or "skew" (Dupire, 1997; Romo, 2014). The element of the IVS-SM is $\sigma(K, \tau)$, which is defined as follows.

Definition 1. Supposing the market price of an European option is C_{market}, the implied volatility varying with K and τ is the unique parameter $\sigma(K, \tau) > 0$ satisfying

$$C^{BS}(K, \tau, \sigma(K, \tau), S, r_d, r_f) = C_{market}.$$

2.2. Greeks

In the options market, the Greeks are important risk management and hedging indicators. The Greeks are the partial derivatives of the BS equation and also used in sensitivity analysis. Traders can acquire important information from Greeks. We list some Greeks in Table 2. More details about Greeks and the relations among Greeks are presented in Hakala and Wystup (2002).

As one of the Greeks, Delta represents the instantaneous derivative of option price with respect to changes of the underlying price. Delta of an FX option is the percentage of the foreign notional one must buy when selling the option to a hedged position. According to the quotation styles, there are four different kinds of Deltas. Reiswich and Wystup (2010) and Reiswich and Wystup (2012) have introduced the quoting conventions in detail.

[1] According to the put-call parity, the implied volatility for a European call option is the same as that for an European put option with the same expire date and strike price (Daglish, Hull, & Suo, 2007). In this paper, we consider call options.

Spot Delta (Δ_s)

Spot Delta (Δ_s) is the ratio of the change in present value of the option to the change in spot price. It is used to hedge in the spot market and given by

$$\Delta_s(K, \sigma, \tau, \phi) = e^{-r_f \tau}\phi N(\phi d_+(K, \tau, \sigma)), \tag{2}$$

where $\phi = 1$ for a call option, $\phi = -1$ for a put option.

Put-call Delta parity:

$$\Delta_s(K, \sigma, \tau, +1) - \Delta_s(K, \sigma, \tau, -1) = e^{-r_f \tau}.$$

In FX markets, one would buy Δ_s times the foreign notional N to hedge a short option position. With the help of put-call parity, one can calculate the corresponding Delta of a put (call) option when given the Delta of a call (put) option.

Forward Delta (Δ_f)

Since financial crisis in 2008, it became unfeasible for banks to agree on spot Deltas (see Clark et al., 2011). In market practice, people use forward Delta Δ_f to construct FX smiles since it does not include any discounting. Δ_f of the option is the ratio of the change in present value of the option to the change in forward price. It is used to hedge in the forward market and is given by

$$\Delta_f(K, \sigma, \tau, \phi) = \phi N(\phi d_+(K, \tau, \sigma)). \tag{3}$$

Put-call Delta parity:

$$\Delta_s(K, \sigma, \tau, +1) - \Delta_s(K, \sigma, \tau - 1) = 100\%.$$

Note that the Delta of a call and the Delta of the corresponding put add to 100%, it is often used in FX option.

Premium-Adjusted Spot Delta ($\Delta_{s,pa}$)

The actual hedge must be changed if the premium is paid in foreign currency, this type of Delta is called premium-adjusted Delta ($\Delta_{s,pa}$). It can be calculated as follows,

$$\begin{aligned} &\Delta_{s,pa}(K, \sigma, \tau, \phi) = \phi e^{-r_f \tau}\tfrac{K}{f} N(\phi d_-(K, \tau, \sigma)), \\ &f = Se^{(r_d - r_f)}\tau. \end{aligned} \tag{4}$$

Put-call Delta parity:

$$\Delta_{s,pa}(K, \sigma, \tau, +1) - \Delta_{s,pa}(K, \sigma, \tau - 1) = e^{-r_f \tau}\frac{K}{f}.$$

Premium-Adjusted Forward Delta ($\Delta_{f,pa}$)

Similar to a spot Delta, a premium payment in foreign currency leads to an adjustment of forward Delta. The premium-adjusted forward Delta is given as

$$\begin{aligned} &\Delta_{f,pa}(K, \sigma, \tau, \phi) = \phi \tfrac{K}{f} N(\phi d_-(K, \tau, \sigma)), \\ &f = Se^{(r_d - r_f)\tau} \end{aligned} \tag{5}$$

Put-call Delta parity:

$$\Delta_{s,pa}(K, \sigma, \tau, +1) - \Delta_{s,pa}(K, \sigma, \tau - 1) = \frac{K}{f}.$$

Since there are four kinds of Deltas quotations in FX markets, one may wonder when to use

2.3. BS formula by Delta

In FX markets the moneyness of options is always expressed in terms of Deltas. Hence it is natural to calculated the moneyness and options prices in terms of Delta (Clark et al., 2011; Wystup, 2007). The BS formula (1) can be rewritten with the known Δ as follows.[2] For $\tau \leqslant 1$,

$$C^{BS}(K(\Delta,\tau),\tau,\sigma,S,r_d,r_f) = S\Delta - e^{-r_d\tau}K(\Delta,\tau)N(d_-(K(\Delta,\tau),\tau,\sigma)),$$

where

$$K(\Delta,\tau) = Se^{(r_d-r_f)\tau-\sigma\sqrt{\tau}N^{-1}(e^{r_f\tau}\Delta)+\frac{\sigma^2\tau}{2}}, \tag{6}$$

$$d_+(K(\Delta,\tau),\tau,\sigma) = N^{-1}(e^{r_f\tau}\Delta),$$
$$d_-(K(\Delta,\tau),\tau,\sigma) = N^{-1}(e^{r_f\tau}\Delta) - \sigma\sqrt{\tau}.$$

For $\tau \geqslant 1$,

$$C^{BS}(K(\Delta,\tau),\tau,\sigma,S,r_d,r_f) = e^{-r_f\tau}S\Delta - e^{-r_d\tau}K(\Delta,\tau)N(d_-(K(\Delta,\tau),\tau,\sigma)),$$

where

$$K(\Delta,\tau) = Se^{(r_d-r_f)\tau-\sigma\sqrt{\tau}N^{-1}(\Delta)+\frac{\sigma^2\tau}{2}}, \tag{7}$$

$$d_+(K(\Delta,\tau),\tau,\sigma) = N^{-1}(\Delta),$$
$$d_-(K(\Delta,\tau),\tau,\sigma) = N^{-1}(\Delta) - \sigma\sqrt{\tau}.$$

Hence one can calculate strike price K with known Δ and τ by Eq. (6) or (7). Similarly, the element of the IVS-DM is $\sigma(\Delta,\tau)$, which can be defined as follows.

Definition 2. The implied volatility varying with Δ and τ of the price C_{market} is the unique parameter $\sigma(\Delta,\tau) > 0$ satisfying

$$C^{BS}(K(\Delta,\tau),\tau,\sigma(\Delta,\tau),S,r_d,r_f) = C_{market}.$$

This paper considers the conditions for the implied volatility surface provided by Deltas and maturities, i.e., IVS-DM or $\sigma(\Delta,\tau)$ to be arbitrage-free.

3. Arbitrage-free conditions

3.1. Review of static arbitrage-free conditions for COPS

D'Aspremont (2004) and Roper (2009) gave conditions for the COPS to be free of static arbitrage under zero interest and dividend yield. Here we consider the case with a constant domestic and foreign risk free interest rate r_d and r_f for FX option.

Lemma 3.1. *Let $S > 0$, r_d and r_f be constant. If C: $(0,\infty)\times[0,\infty)\to\mathbb{R}$ satisfies the following conditions,*

(A1) *(Monotonicity in τ)*
$C(K,\cdot)$ *is non-decreasing, for* $K > 0$.
(A2) *(Monotonicity in K)*
$C(\cdot,\tau)$ is non-increasing, for $\tau \geqslant 0$.
(A3) *(Convexity in K)*
$C(\cdot,\tau)$ is a convex function, for $\tau \geqslant 0$.
(A4) *(Large strike limit)*
$\lim_{K\to\infty} C(K,\tau) = 0$, for $\tau \geqslant 0$.
(A5) *(Bounds)*
$\max(e^{-r_f\tau}S - e^{-r_d\tau}K, 0) \leqslant C(K,\tau) \leqslant e^{-r_f\tau}S$, for $K > 0$ and $\tau \geqslant 0$.
(A6) *(Expiry Value)*
$C(K,0) = (S-K)^+$, for $K > 0$. *Then for the function*

$$\widehat{C}(K,\tau) = \begin{cases} Se^{-r_f\tau}, & \text{if} \quad K = 0,\\ C(K,\tau) & \text{if} \quad K > 0,\end{cases}$$

[2] Since in market practice, spot Delta is used for the time to maturities less than one year and forward Delta is used for longer dates, we consider two Deltas in the following context.

(i) *there exists a non-negative Markov martingale* (Kellerer, 1972) *X with the property that*

$$\hat{C}(K, \tau) = E((X_\tau - K)^+ | X_0 = S)$$

for all $K \geqslant 0$ *and* $\tau \geqslant 0$. *Thus the call price surface is free of static arbitrage.*

(ii) *All of the conditions* $(A1) \sim (A6)$ *are necessary conditions for* $\hat{C}$ *to be the conditional expectation of the call option under the assumption that X is a (non-negative) martingale.*

Proof. See Proposition 4.3.4 and Theorem 6.2.1 in Roper (2009) for the proof of necessity and sufficiency, respectively. □

Intuitively, each condition in Lemma 3.1 corresponds to a practical meaning of being arbitrage-free in the options market. Monotonicity in τ of condition $(A1)$ ensures that there is no calendar arbitrage in the option price, i.e., the prices of call options with long maturities should be more expensive than the one with short maturities. Monotonicity in K of condition $(A2)$ prevents that out of money option price worth more than in the money option, and makes sure that there is no call spread arbitrage. Convexity in K of condition $(A3)$ implies that there is no butterfly spread arbitrage in the option price (see Rebonato, 2005). The bound conditions of $(A5)$ mean that the option price should satisfy the lower and upper bounds in case that the chance of risk free arbitrage happens (see Hull, 2009).

3.2. Static Arbitrage-free conditions for IVS-DM

In this section, we derive sufficient and close to necessary conditions for the IVS-DM to be free of static arbitrage. We define static arbitrage-free of an IVS as follows.

Definition 3. An IVS is free of static arbitrage if and only if the corresponding COPS

$$C(K, \tau) = C^{BS}(K, \tau, \sigma(K, \tau), S, r_d, r_f)$$

or

$$C(K(\Delta, \tau), \tau) = C^{BS}(K(\Delta, \tau), \tau, \sigma(\Delta, \tau), S, r_d, r_f)$$

is free of static arbitrage.

Some other definitions are provided in the following for the convenience of analysis.

Definition 4. Let

$$B(x, \theta) = \begin{cases} (e^{-r_f\tau} - e^{-r_d\tau + x})^+, & if \quad \theta = 0, \\ e^{-r_f\tau}N(d_1(x, \theta)) - e^{-r_d\tau + x}N(d_2(x, \theta)), & if \quad \theta \in (0, \infty), \\ e^{-r_f\tau}, & if \quad \theta = \infty, \end{cases}$$

where $x = ln(\frac{K}{S})$ and $x \in \mathbb{R}$, $d_1(x, \theta) = \frac{-x + (r_d - r_f)\tau}{\theta} + \frac{\theta}{2}$, $d_2(x, \theta) = \frac{-x + (r_d - r_f)\tau}{\theta} - \frac{\theta}{2} = d_1(x, \theta) - \theta$. Thus the BS pricing formula (1) can be written as follows,

$$C^{BS}(K, \tau, \sigma) = SB(ln(K/S), \sigma\sqrt{\tau}).$$

Particularly, according to Eqs. (6) and (7), K can be calculated if Δ and $\sigma(\Delta, T)$ are given. We now introduce a time-scale implied volatility in Delta form as follows.

Definition 5. Time scaled implied volatility (in Delta form) is a function defined by (Clark et al., 2011; Daglish et al., 2007)

$$\Xi(\Delta, \tau) = \sigma(\Delta, \tau)\sqrt{\tau},$$

where $\Delta \in [0, 1]$ and $\tau \geqslant 0$. Ξ is the implied volatility scaled by the square root of time.

Remark 1. When the $IVS_{\Delta - T}$ is given, the strike prices can be calculated with $K(\Delta, \tau)$. Thus the Black–Scholes equation can be written as

$$C^{BS}(K(\Delta, \tau), \tau, \sigma(\Delta, \tau)) = SB(ln(K(\Delta, \tau)/S), \Xi(\Delta, \tau)),$$

where $K(\Delta, \tau)$ is given in Eqs. (6) or (7).

Theorem 3.2. *Let the time-scale implied volatility* $\Xi(\Delta, \tau)$ *as a function of* Δ *and* τ *satisfy the following conditions,*[3]

(B1) *(Smoothness)*

[3] *In this theorem, we will sometimes omit the arguments of the functions, e.g.,* Ξ *for* $\Xi(\Delta, \tau)$, d_- *for* $d_-(\Delta, \tau)$, *and K for* $K(\Delta, \tau)$, *which is calculated with Eqs.* (6) and (7) *for* $\tau \leqslant 1$ *and* $\tau > 1$, *respectively. In addition,* $d_2(x(\Delta, \tau), \Xi(\Delta, \tau)) = d_-(K(\Delta, \tau), \tau, \sigma(\Delta, \tau))$. *See details for other notations of this theorem in* Appendix A.

$\Xi(\Delta,\cdot)$ is first differentiable for $\Delta \in [0,1]$; $\Xi(\cdot,\tau)$ is twice differentiable for $\tau \geqslant 0$.
(B2) *(Positivity)*

$$\Xi(\Delta,\tau) > 0$$

for $\Delta \in [0,1]$ and $\tau \geqslant 0$.
(B3) *(Conditions on τ)*

$$[r_f N(d_-) + \Xi\frac{\partial N^{-1}(e^{r_f\tau}\Delta)}{\partial\tau}N(d_-) + n(N^{-1}(e^{r_f\tau}\Delta)-\Xi)\frac{\partial N^{-1}(e^{r_f\tau}\Delta)}{\partial\tau}] + [N^{-1}(e^{r_f\tau}\Delta)N(d_-)-\Xi N(d_-)-n(N^{-1}(e^{r_f\tau}\Delta)-\Xi)]\frac{\partial\Xi}{\partial\tau} \geqslant 0, \quad \text{for } \tau \leqslant 1,$$

and

$$\left[r_f N(d_-)-r_f\frac{S}{K}e^{(r_d-r_f)\tau}\Delta\right] + [N^{-1}(\Delta)N(d_-)-\Xi N(d_-)-n(N^{-1}(\Delta)-\Xi)]\frac{\partial\Xi}{\partial\tau} \geqslant 0, \quad \text{for } \tau > 1,$$

where d_- is short for $d_-(K(\Delta,\tau),\tau) = N^{-1}(e^{r_f\tau}\Delta)-\Xi$.
(B4) *(First Order conditions on Δ)*

$$\frac{A_i-B_i\frac{\partial\Xi}{\partial\Delta}}{C_i\frac{\partial\Xi}{\partial\Delta}-D_i} \leqslant 0, \tag{8}$$

where for $\tau \leqslant 1$ and $i = 1$, we define,

$$\begin{aligned}
&A_1=e^{r_d\tau}\tfrac{S}{K} + \Xi\tfrac{\partial N^{-1}(e^{r_f t}\Delta)}{\partial\Delta}N(d_-)-n(N^{-1}(e^{r_f\tau}\Delta)-\Xi),\\
&B_1=\Xi N(d_-)-N^{-1}(e^{r_f t}\Delta)N(d_-)-n(N^{-1}(e^{r_f\tau}\Delta)-\Xi),\\
&C_1=\Xi-N^{-1}(e^{r_f t}\Delta),\\
&D_1=\Xi\tfrac{\partial N^{-1}(e^{r_f t}\Delta)}{\partial\Delta}.
\end{aligned}$$

for $\tau > 1$ and $i = 2$, we define,

$$\begin{aligned}
&A_2=e^{(r_d-r_f)\tau}\tfrac{S}{K} + \Xi\tfrac{\partial N^{-1}(\Delta)}{\partial\Delta}N(d_-)-n(N^{-1}(\Delta)-\Xi)\tfrac{\partial N^{-1}(\Delta)}{\partial\Delta},\\
&B_2=\Xi N(d_-)-N^{-1}(\Delta)N(d_-)-n(N^{-1}(\Delta)-\Xi),\\
&C_2=\Xi-N^{-1}(\Delta),\\
&D_2=\Xi\tfrac{\partial N^{-1}(\Delta)}{\partial\Delta}.
\end{aligned}$$

(B5) *(Second Order conditions on Δ)*

$$\frac{\partial M_i(\Delta)}{\partial\Delta}-M_i(\Delta)\frac{\frac{\partial^2 K}{\partial\Delta^2}}{\frac{\partial K}{\partial\Delta}} \geqslant 0, \tag{9}$$

where for $\tau \leqslant 1$ and $i = 1$,

$$\begin{aligned}
&M_1(\Delta) = S-\tfrac{\partial K}{\partial\Delta}e^{-r\tau}N(d_-)-Ke^{-r\tau}\tfrac{\partial N(d_-)}{\partial\Delta},\\
&\tfrac{\partial M_1(\Delta)}{\partial\Delta} = -e^{-r\tau}\left[\tfrac{\partial^2 K}{\partial\Delta^2}N(d_-) + 2\tfrac{\partial K}{\partial\Delta}\tfrac{\partial N(d_-)}{\partial\Delta} + K\tfrac{\partial^2 N(d_-)}{\partial\Delta^2}\right],\\
&\tfrac{\partial K}{\partial\Delta} = K\left[\Xi\tfrac{\partial\Xi}{\partial\Delta}-\tfrac{\partial\Xi}{\partial\Delta}N^{-1}(e^{r_f t}\Delta)-\Xi\tfrac{\partial N^{-1}(e^{r_f t}\Delta)}{\partial\Delta}\right],\\
&\tfrac{\partial^2 K}{\partial\Delta^2} = \tfrac{\partial K}{\partial\Delta}\left[\Xi\tfrac{\partial\Xi}{\partial\Delta}-\tfrac{\partial\Xi}{\partial\Delta}N^{-1}(e^{r_f t}\Delta)-\Xi\tfrac{\partial N^{-1}(e^{r_f t}\Delta)}{\partial\Delta}\right]\\
&+ K\left[\left(\tfrac{\partial\Xi}{\partial\Delta}\right)^2 + \Xi\tfrac{\partial^2\Xi}{\partial\Delta^2}-\tfrac{\partial^2\Xi}{\partial\Delta^2}N^{-1}(e^{r_f t}\Delta)-2\tfrac{\partial\Xi}{\partial\Delta}\tfrac{\partial N^{-1}(e^{r_f t}\Delta)}{\partial\Delta}\right.\\
&\left.-\ \Xi\tfrac{\partial^2 N^{-1}(e^{r_f t}\Delta)}{\partial\Delta^2}\right];
\end{aligned}$$

for $\tau > 1$ and $i = 2$,

$$
\begin{aligned}
M_2(\Delta) &= Se^{-r_f\tau} - \frac{\partial K}{\partial \Delta} e^{-r\tau} N(d_-) - Ke^{-r\tau} \frac{\partial N(d_-)}{\partial \Delta}, \\
\frac{\partial M_2(\Delta)}{\partial \Delta} &= -e^{-r\tau}\left[\frac{\partial^2 K}{\partial \Delta^2} N(d_-) + 2\frac{\partial K}{\partial \Delta}\frac{\partial N(d_-)}{\partial \Delta} - K\frac{\partial^2 N(d_-)}{\partial \Delta^2}\right], \\
\frac{\partial K}{\partial \Delta} &= K\left[\Xi\frac{\partial \Xi}{\partial \Delta} - \frac{\partial \Xi}{\partial \Delta} N^{-1}(\Delta) - \Xi\frac{\partial N^{-1}(\Delta)}{\partial \Delta}\right], \\
\frac{\partial^2 K}{\partial \Delta^2} &= \frac{\partial K}{\partial \Delta}\left[\Xi\frac{\partial \Xi}{\partial \Delta} - \frac{\partial \Xi}{\partial \Delta} N^{-1}(\Delta) - \Xi\frac{\partial N^{-1}(\Delta)}{\partial \Delta}\right] \\
&\quad + K\left[\left(\frac{\partial \Xi}{\partial \Delta}\right)^2 + \Xi\frac{\partial^2 \Xi}{\partial \Delta^2} - \frac{\partial^2 \Xi}{\partial \Delta^2} N^{-1}(\Delta) - 2\frac{\partial \Xi}{\partial \Delta}\frac{\partial N^{-1}(\Delta)}{\partial \Delta} \right. \\
&\quad \left. - \Xi\frac{\partial^2 N^{-1}(\Delta)}{\partial \Delta^2}\right].
\end{aligned}
\tag{10}
$$

(B6) *(Limitation Behavior)*

$$\lim_{\Delta \to 0} K(\Delta, \tau) = +\infty$$

for $\tau \geqslant 0$.

(B7) *(Value at maturity)*

$$\Xi(\Delta, 0) = 0$$

for $\Delta \in [0, 1]$.

Then

$$C(K(\Delta, \tau), \tau, \sigma(\Delta, \tau)) = SB(ln(K(\Delta, \tau)/S), \sigma(\Delta, \tau)\sqrt{\tau})$$

is a call price surface which is free of static arbitrage. In particular, there exists a non-negative Markov martingale X with the property that $C(\Delta, \tau) = E((X_\tau - K)^+ | X_0 = S)$ *for all* $\Delta \in [0, 1]$ *and* $\tau \geqslant 0$.

Proof. See Appendix A for the details of the proof. □

Remark 2. The arbitrage-free conditions $(B3) \sim (B7)$ of Theorem 3.2 can be reversed except for condition $(B6)$, since the large limit condition $\lim_{\Delta \to 0} K(\Delta, \tau) = +\infty$ can not always hold. However, it is regarded as a sufficient and close to necessary condition for the IVS-DM to be arbitrage-free.

Remark 3. For applications, the practitioner can test whether the IVS-DM is static arbitrage-free or not by simply considering the conditions of $(B3) \sim (B5)$. These conditions imply that there are no calendar spread arbitrage, call spread arbitrage and butterfly spread arbitrage, respectively.

4. Simulated and empirical tests

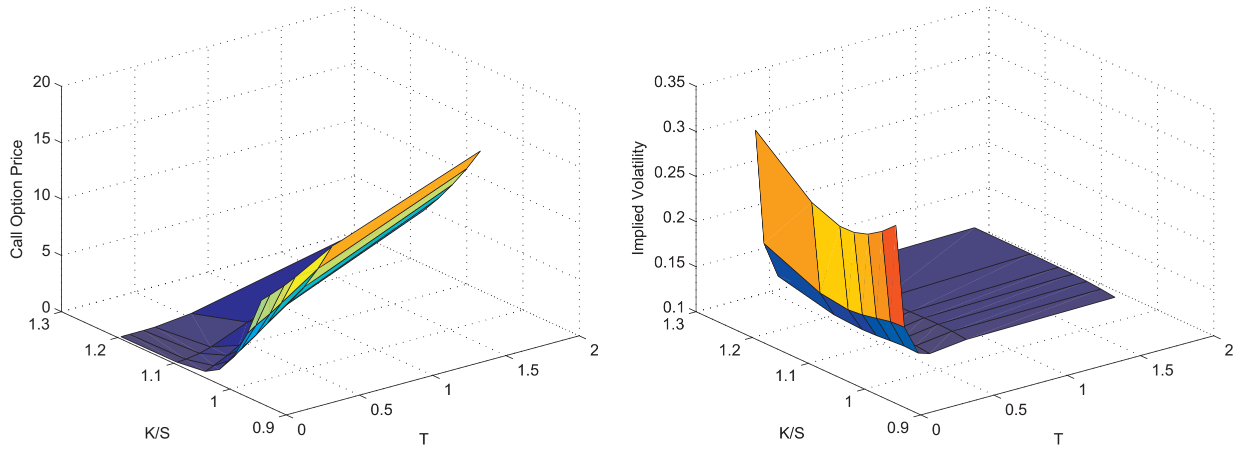
In this section we conduct some arbitrage tests on a simulated arbitrage-free IVS-DM and an arbitraged one, respectively. Afterwards, the empirical tests are implemented on the EURUSD and USDJPY option IVS-DM with the conditions of $(B3) \sim (B5)$ of Theorem 3.2. For the simulated data, the COPS is constructed firstly and then transferred into the IVS-SM and IVS-DM, respectively. Thus it is feasible to compare the test results of the surfaces. For the empirical data, the test results of the proposed arbitrage-free conditions are compared with the EWI method, which is widely used in financial institutions. Furthermore, the calendar and butterfly spread portfolios are constructed to testify the results of the proposed conditions and EWI method. All the test results show that the conditions of $(B3) \sim (B5)$ in Theorem 3.2 are accurate and efficient to examine arbitrage opportunities in the IVS-DM.

4.1. Arbitrage test on simulated IVS-DM

In this section, the proposed conditions are implemented for an arbitrage test on the simulated IVS-DM. An arbitrage-free COPS consisting of 6 maturities and 8 various strikes for an asset is constructed. The current spot price S of the underlying is 170.76.

Fig. 1a and b show the COPS and IVS-SM, respectively.[4] We note that for each maturity, the option price is monotonically decreasing and convex with respect of strikes. These properties ensure that there are no arbitrage opportunities in strikes according to Lemma 3.1. Fig. 2 shows the IVS-DM calculated with Eqs. (2) and (3) for each maturity. Conditions $(B4)$ and $(B5)$ of Theorem 3.2 are carried out on IVS-DM to test the arbitrage opportunities in Delta. To verify whether the arbitrage examine results are consistent with

[4] The time to maturities are $\tau = 8, 29, 64, 92, 183, 554$ (days), respectively. The strikes are $K = 136.608, 153.684, 162.222, 166.491, 170.760, 175.029, 179.298, 187.836, 204.912$, respectively. Figures are presented by moneyness K/S and annualized time to maturities $T/365$.



(a) Arbitrage-free COPS. (b) Arbitrage-free IVS-SM.

Fig. 1. Arbitrage-free Surfaces.

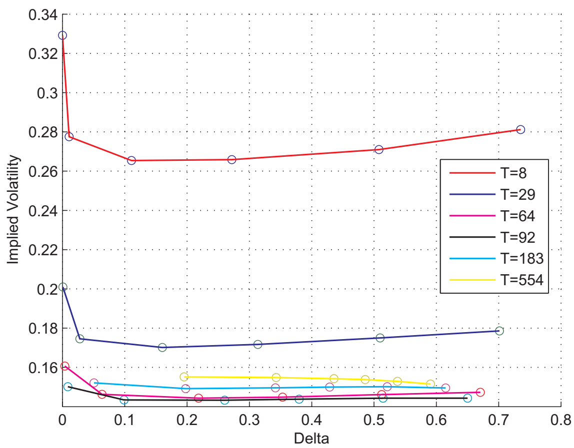


Fig. 2. Arbitrage-free IVS-DM.

the IVS-SM, the Durrleman's condition (condition ($IV3$)) and slope bound (condition ($IV4$)) of Theorem 6.2.11 in Roper (2009) are also implemented on IVS-SM.[5] Due to the space limitation, the results of $\tau = 92$ days on the IVS-DM and IVS-SM are presented in panel A of Table 3.[6] From panel A, we note that the values of the first order condition ($B4$) and second order condition on Delta ($B4$) of Theorem 3.2 indicate no arbitrage in Delta, which is consistent with the test result on the IVS-SM.

Furthermore, the arbitrage-free COPS is adjusted to an arbitrage one. The option prices of the last four strikes for each maturity are set to be monotonically increasing, which violates the arbitrage-free property of the COPS. The adjusted arbitrage COPS, IVS-SM and IVS-DM are shown in Fig. 3a, Fig. 3b and Fig. 4, respectively. Panel B of Table 3 shows the examine results for $\tau = 92$ days. The values in red imply that there exist arbitrage opportunities in these two surfaces. Note that the conditions on the IVS-DM are more sensible than the IVS-SM since the values of the conditions change signs earlier.

All the above test results show that the proposed conditions for the IVS-DM are consistent with the conditions of the IVS-SM and COPS. Furthermore, the proposed conditions are more sensible to the arbitrage point than the IVS-SM arbitrage-free conditions.

4.2. Arbitrage test on empirical data

In this part, the proposed conditions ($B3$) $\sim$ ($B5$) of Theorem 3.2 are implemented on EURUSD and USDJPY currency option IVS-DM. The test results are compared with the EWI method, which is designed on the convexity of the smile, while failing the test does not mean there is definitive arbitrage, it only means that the volatility is no longer convex. Testing the surface may need further

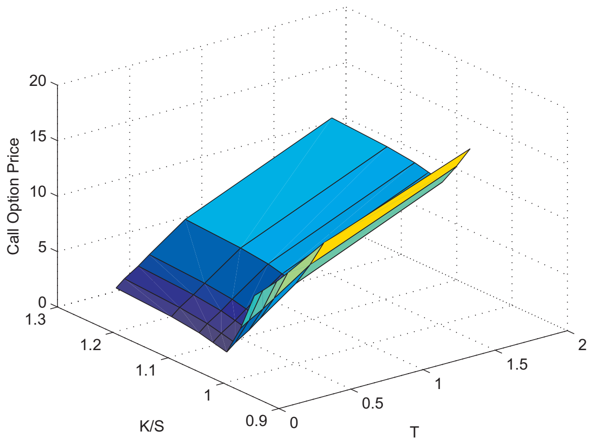
[5] Risk free rates are set to be zero in this simulation for the comparability of the conditions.

[6] In this paper, backward difference method is used to approximate the derivatives in the proposed arbitrage-free conditions

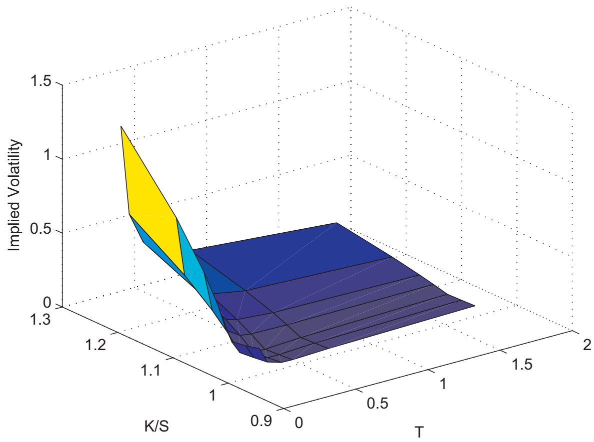
Table 3
Arbitrage test for IVSs.

Panel A. Arbitrage-free Surfaces					
			IVS-DM		
$(B4)$	−0.610	−0.486	−0.357	−0.241	−0.087
$(B5)$	33.827	31.032	32.450	38.059	70.498
			IVS-SM		
$(IV3)$	0.931	0.969	1.007	1.041	1.035
$(IV4)$	−1.640	−1.218	−0.950	−0.768	−0.549
Panel B. Arbitrage Surfaces					
			IVS-DM		
$(B4)$	−0.666	−0.536	−0.024	0.016	−0.006
$(B5)$	33.154	34.945	−143.613	−262.712	−389.341
			IVS-SM		
$(IV3)$	1.014	5.423	0.062	−0.334	−0.131
$(IV4)$	−1.554	−1.213	−1.019	−0.923	−0.817

This table presents the arbitrage results of the call spread and butterfly spread arbitrage for original surfaces and adjusted surfaces. In panel A, for the arbitrage test on the IVS-DM, the negative values of the first order conditions $(B4)$ and the positive values of the second conditions $(B5)$ imply that there is no arbitrage opportunity on *Delta*. For the arbitrage test on the IVS-SM, the positive values of the Durrleman's condition $(IV3)$ and negative values of slope bound condition $(IV4)$ imply that there is no arbitrage opportunity on K. In panel B, the opposite values in red color imply there exist arbitrage opportunities.



(a) Arbitrage COPS.



(b) Arbitrage IVS-SM.

Fig. 3. Arbitrage Surfaces.

examinations.

The conditions tested by EWI method are as follows.

$$Vol(10C) - Vol(25C) \geqslant 0.6 * [Vol(ATM) - Vol(25C)],$$
$$Vol(10P) - Vol(25P) \geqslant 0.6 * [Vol(ATM) - Vol(25P)],$$

where "x P (C)" means $\Delta = x\%$ for the put (call) option. ATM is approximated with $\Delta = 50\%$. $Vol(\cdot)$ is the implied volatility. A general form of the above equations is given as

$$EWI_{\Delta_1,\Delta_2,\Delta_3} = \frac{Vol(\Delta_1) - Vol(\Delta_2)}{|\Delta_1 - \Delta_2|} - \frac{Vol(\Delta_2) - Vol(\Delta_3)}{|\Delta_2 - \Delta_3|}, \tag{11}$$

where Δ_i, $i = 1, 2, 3$, are on the right or left side of ATM with decreasing distance to ATM. For example, if $\Delta_1 = 10C$, $\Delta_3 = 25C$, $\Delta_3 = ATM$, respectively, then

$$EWI_{\Delta_1,\Delta_2,\Delta_3} = \frac{Vol(10C) - Vol(25C)}{|10\% - 25\%|} - \frac{Vol(25C) - Vol(ATM)}{|25\% - 50\%|}. \tag{12}$$

If $EWI_{\Delta_1,\Delta_2,\Delta_3} \geqslant 0$, this volatility slice is arbitrage-free. Fig. 5 shows the good and bad situations for the EWI test. Good situation

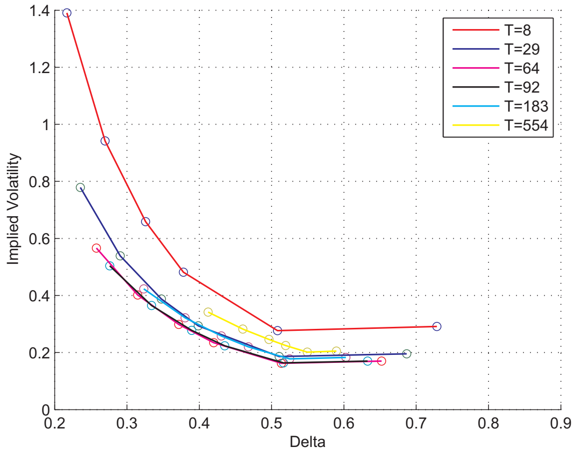


Fig. 4. Arbitrage IVS-DM.

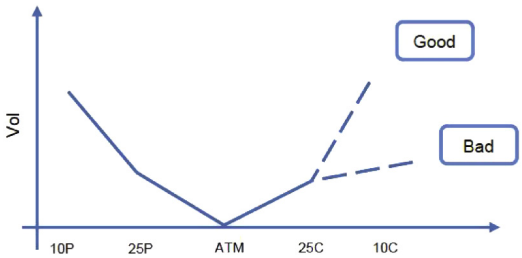


Fig. 5. Good and Bad Situation of EWI.

Table 4
EURUSD and USDJPY IVS-DM.

Tenor	Delta(%) 5 P	10 P	25 P	ATM	25 C	10 C	5 C
			EURUSD IVS-DM				
2016-12-7	–	15.825	14.038	13.600	13.775	15.375	–
2016-12-13	–	16.425	15.625	15.006	14.837	15.013	–
2020-12-4	–	13.032	11.757	11.070	11.220	11.895	–
2022-12-6	–	13.107	12.113	11.519	11.751	12.357	–
			USDJPY IVS-DM				
2016-12-7	20.880	20.450	18.575	17.700	17.400	18.200	18.658
2016-12-13	18.073	18.063	16.725	15.825	15.775	16.188	16.018
2020-12-4	17.716	15.892	13.535	12.185	12.385	13.654	15.061
2022-12-6	18.785	16.875	14.106	12.594	12.756	14.238	15.595

"x P (C)" means $\Delta = x\%$ for the put (call) option, ATM is approximated with $\Delta = 50\%$.

Source of the data: Reuters database.

means there are no arbitrage opportunities in the implied volatility for a specific maturity τ. However, EWI method only examines the convexity of Delta, which is not a necessary and sufficient condition for the arbitrage test. It can not provide any information whether there exist calendar arbitrage opportunities.

The EURUSD and USDJPY IVS-DM on December 6, 2016 are presented in Table 4.[7] Table 5 shows the EWI test results of these two currencies. We note there is no arbitrage point in EURUSD IVS-DM, while there may exist arbitrage opportunities in USDJPY according to EWI method. It needs further research to test whether there exist arbitrage opportunities.

For the proposed method, condition ($B3$) of Theorem 3.2 is implemented to test calendar arbitrage opportunities, condition ($B4$)

[7] There are nineteen maturities on the IVS-DM of December 6, 2016 in total. Only two maturities within one year and two maturities longer than one year are presented due to the space limitation.

Table 5
EWI Test for EURUSD and USDJPY IVS-DM.

	EWI Test for EURUSD		EWI Test for USDJPY	
Tenor	$EWI_{10P,25P,50P}$	$EWI_{10C,25C,50C}$	$EWI_{5P,10P,15P}$	$EWI_{5C,10C,15C}$
2016-12-7	0.100	0.100	−0.040	0.040
2016-12-13	0.030	0.020	−0.090	−0.060
2020-12-4	0.060	0.040	0.210	0.200
2022-12-6	0.040	0.030	0.200	0.170

The negative values imply that there may exist arbitrage opportunities on Delta by EWI method. It needs further research to test whether there exist arbitrage opportunities.

Table 6
Conditions on τ for USDJPY Option.

	Delta						
Tenor	5 P	10 P	25 P	ATM	25 C	10 C	5 C
2016-12-13	0.148	0.065	0.019	0.005	0.001	0.000	0.000
2017-1-5	0.150	0.066	0.018	0.005	0.001	0.000	0.000
2018-12-6	−0.007	−0.006	−0.006	−0.008	−0.009	−0.010	−0.009
2020-12-4	−0.014	−0.011	−0.009	−0.011	−0.011	−0.011	−0.010
2022-12-6	−0.022	−0.016	−0.012	−0.013	−0.013	−0.011	−0.010

The negative values in this table imply that there exist calendar spread arbitrage opportunity on τ.

Table 7
Conditions on Delta for USDJPY Option.

Tenor	10 P	25 P	ATM	25 C	10 C
			First Order conditions on Δ		
2016-12-7	-0.910	−0.807	−0.528	−0.254	−0.090
2016-12-13	-0.896	−0.792	−0.528	−0.244	−0.091
2020-12-4	-0.893	−0.713	−0.374	−0.109	−0.026
2022-12-6	−0.865	−0.677	−0.318	−0.076	−0.014
			Second Order conditions on Δ		
2016-12-7	6.289	2.648	2.612	3.328	5.774
2016-12-13	16.055	6.815	6.148	7.903	14.226
2020-12-4	115.851	66.451	68.220	76.871	103.185
2022-12-6	194.609	86.946	89.456	94.883	117.653

The negative values of the first order and second conditions on Δ imply that there is no arbitrage opportunity on *Delta*.

and (*B*5) are utilized to examine call spread and butterfly spread arbitrage opportunities. The results show that there exist calendar arbitrage opportunities for both EURUSD and USDJPY option, while there is no call spread and butterfly spread arbitrage opportunity. The test results for EURUSD are consistent with the proposed conditions and EWI method. Here we only present the results for the USDJYP option due to space limitation. Negative values in Table 6 show that there exist calendar spread arbitrage points in the USDJPY options IVS. While calendar arbitrage can not be testes with EWI method. For call spread and butterfly arbitrage testing, since condition (*B*4) and (*B*5) in Table 7 are negative and positive, respectively, there are no arbitrage opportunities in the dimension of Δ, which is contrary to the results of EWI.

To prove the reliability of our conclusion, a calendar spread and a butterfly spread portfolio are constructed.[8] The calendar spread consists of a short call option with maturity date of 2020-12-4 and a long call with maturity date of 2022-12-6. These two options will be written with the same strike price $K = 99.776$. on the valuation date of 2016-12-6, the option prices for the short and call are 11.821 (JPY) and 10.732 (JPY), respectively. The spread for the two options is $10.732 - 11.821 = -1.089$. The negative spread implies that there exist calendar arbitrage points in the time scale. For the butterfly spread, we use the volatilities on the date of 2016-12-13 to see whether there exist any arbitrage opportunities on that day. Fig. 6 represents prices of the butterfly spread across Deltas.[9] From Fig. 6, we note that, there exists no butterfly arbitrage opportunity at the maturity date of 2016-12-3 since the butterfly spreads are positive, which is in agreement with our tests on Delta. The EWI arbitrage test results show that there exist arbitrage opportunities,

[8] It is noted that a test for butterfly spread arbitrage is the more "powerful" than call spread arbitrage. We therefore focus on butterfly spread arbitrage tests. If this arbitrage does not exist, we deem a test for call spread arbitrage.

[9] All the other trade data such as the domestic risk free rate of USD and foreign risk free rate (USD) are needed on the valuation date of 2016-12-6 to calculate the option prices. Both the calendar and butterfly tests are conducted on unit notional.

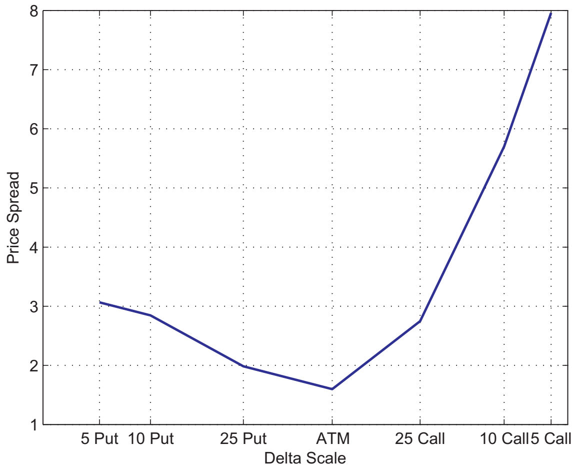


Fig. 6. Butterfly Spread.

which is proved to be incorrect according to the butterfly spread portfolio. Hence the EWI test may sometimes fail to examine arbitrage in Delta.

5. Conclusion

In this paper, we studied the sufficient conditions for the IVS-DM to be arbitrage-free. These conditions are proved to be necessary under a mild assumption. Although the forms of the conditions seem "complicated", they can be used to examine the arbitrage phenomenon with approximate method. These conditions are useful tools for financial practitioners to examine the possible arbitrage chances in the market. For the arbitrageurs, they can catch the opportunities which will last for a short period to gain profits. For the risk managers, it is important to find the arbitrage points in advance since they will influence the pricing of the financial products. Compared with the EWI method, the proposed arbitrage-free conditions for the IVS-DM are proved to be practical and reliable to test arbitrage opportunities.

Appendix A. The Proof of Theorem 3.2

Proof. It is sufficient to check that if the conditions $(B1) \sim (B7)$ hold, then the conditions in Lemma 3.1 are satisfied. Without losing any generality, here we only prove the situation of $\tau \leqslant 1$. For $0 < \sigma(\Delta, \tau)\sqrt{\tau} < \infty$, we have

$$\begin{aligned} C(K(\Delta, \tau), \tau, \sigma(\Delta, \tau)) &= SB(ln(K(\Delta, \tau)/S), \sigma(\Delta, \tau)\sqrt{\tau}) \\ &= S\Delta - e^{-r\tau}K(\Delta, \tau)N(d_-(\Delta, \tau), \tau, \sigma), \end{aligned} \tag{1}$$

where

$$x(\Delta, \tau) = ln(S/K(\Delta, \tau)).$$

(A1) (Monotonicity in τ)

Taking first partial derivatives of τ on both sides of the option price (1) yields

$$\frac{\partial C}{\partial \tau} = -\frac{\partial K}{\partial \tau}e^{-r\tau}N(d_-) + rKe^{-r\tau}N(d_-) - Ke^{-r\tau}\frac{\partial N(d_-)}{\partial \tau},$$

where

$$\begin{aligned} &\tfrac{\partial K}{\partial \tau} = K[(r-r_f) + \Xi\tfrac{\partial \Xi}{\partial \tau} - \tfrac{\partial \Xi}{\partial \tau}N^{-1}(e^{r_f\tau}\Delta) - \Xi\tfrac{\partial N^{-1}(e^{r_f\tau}\Delta)}{\partial \tau}], \\ &\tfrac{\partial N^{-1}(e^{r_f\tau}\Delta)}{\partial \tau} = \tfrac{1}{n(N^{-1}(e^{r_f\tau}\Delta))}r_f e^{r_f\tau}\Delta, \\ &N(d_-) = N(N^{-1}(e^{r_f\tau}\Delta) - \Xi), \\ &\tfrac{\partial N(d_-)}{\partial \tau} = n(N^{-1}(e^{r_f\tau}\Delta) - \Xi)\left(\tfrac{\partial N^{-1}(e^{r_f\tau}\Delta)}{\partial \tau} - \tfrac{\partial \Xi}{\partial \tau}\right). \end{aligned}$$

Thus

$$\frac{\partial C}{\partial \tau} = Ke^{-r\tau}\left[r_f N(d_-) + \Xi\frac{\partial N^{-1}(e^{r_f\tau}\Delta)}{\partial \tau}N(d_-) + n(N^{-1}(e^{r_f\tau}\Delta)-\Xi)\frac{\partial N^{-1}(e^{r_f\tau}\Delta)}{\partial \tau}\right] + Ke^{-r\tau}[N^{-1}(e^{r_f\tau}\Delta)N(d_-)-\Xi N(d_-)$$
$$-n(N^{-1}(e^{r_f\tau}\Delta)-\Xi)]\frac{\partial \Xi}{\partial \tau}.$$

According to Condition $(B3)$, we have

$$\frac{\partial C(\tau, K)}{\partial \tau} \geqslant 0.$$

(A2) (Monotonicity in K)

According the chain rule, taking the first partial derivatives of K on both sides of the option price Eq. (1) yields,

$$\frac{\partial C}{\partial K} = \frac{\frac{\partial C}{\partial \Delta}}{\frac{\partial K}{\partial \Delta}}$$
$$= \frac{S - \frac{\partial K}{\partial \Delta}e^{-r\tau}N(d_-) - Ke^{-r\tau}\frac{\partial N(d_-)}{\partial \Delta}}{\frac{\partial K}{\partial \Delta}},$$

where

$$\frac{\partial K}{\partial \Delta} = K\left[\Xi\frac{\partial \Xi}{\partial \Delta} - \frac{\partial \Xi}{\partial \Delta}N^{-1}(e^{r_f t}\Delta) - \Xi\frac{\partial N^{-1}(e^{r_f t}\Delta)}{\partial \Delta}\right],$$
$$\frac{\partial N(d_-)}{\partial \Delta} = n(N^{-1}(e^{r_f\tau}\Delta)-\Xi)\left(\frac{\partial N^{-1}(e^{r_f\tau}\Delta)}{\partial \Delta} - \frac{\partial \Xi}{\partial \Delta}\right),$$
$$\frac{\partial N^{-1}(e^{r_f t}\Delta)}{\partial \Delta} = \frac{1}{n(N^{-1}(e^{r_f t}\Delta))}e^{r_f t}.$$

Denote

$$A_1 = e^{r\tau}\frac{S}{K} + \Xi\frac{\partial N^{-1}(e^{r_f t}\Delta)}{\partial \Delta}N(d_-) - n(N^{-1}(e^{r_f\tau}\Delta)-\Xi),$$

$$B_1 = \Xi N(d_-) - N^{-1}(e^{r_f t}\Delta)N(d_-) - n(N^{-1}(e^{r_f\tau}\Delta)-\Xi),$$
$$C_1 = \Xi - N^{-1}(e^{r_f t}\Delta),$$
$$D_1 = \Xi\frac{\partial N^{-1}(e^{r_f t}\Delta)}{\partial \Delta},$$

then, we have

$$\frac{\partial C}{\partial K} = \frac{A_1 - B_1\frac{\partial \Xi}{\partial \Delta}}{K(C_1\frac{\partial \Xi}{\partial \Delta} - D_1)}.$$

According to Condition $(B4)$, we have

$$\frac{\partial C}{\partial K} \leqslant 0.$$

(A3) (Convexity in K)

Similarly, taking the second partial derivatives of K on both sides of the option price Eq. (1) yields,

$$\frac{\partial^2 C}{\partial K^2} = \frac{\frac{\partial\left(\frac{\partial C}{\partial K}\right)}{\partial \Delta}}{\frac{\partial K}{\partial \Delta}},$$

where

$$\frac{\partial\left(\frac{\partial C}{\partial K}\right)}{\partial \Delta} = \partial\left(\frac{S - e^{-r\tau}\frac{\partial K}{\partial \Delta}N(d_-) - Ke^{-r\tau}\frac{\partial N(d_-)}{\partial \Delta}}{\frac{\partial K}{\partial \Delta}}\right)/\partial \Delta.$$

Denoting that $M_1(\Delta) = S - \frac{\partial K}{\partial \Delta}e^{-r\tau}N(d_-) - Ke^{-r\tau}\frac{\partial N(d_-)}{\partial \Delta}$, we have

$$\frac{\partial(\frac{\partial C}{\partial K})}{\partial \Delta} = \frac{\frac{\partial M_1(\Delta)}{\partial \Delta}\frac{\partial K}{\partial \Delta} - M(\Delta)\frac{\partial^2 K}{\partial \Delta^2}}{\left(\frac{\partial K}{\partial \Delta}\right)^2},$$

where

$$\frac{\partial M_1(\Delta)}{\partial \Delta} = - e^{-r\tau}\left[\frac{\partial^2 K}{\partial \Delta^2}N(d_-) + 2\frac{\partial K}{\partial \Delta}\frac{\partial N(d_-)}{\partial \Delta} + K\frac{\partial^2 N(d_-)}{\partial \Delta^2}\right],$$

$$\frac{\partial^2 K}{\partial \Delta^2} = \frac{\partial K}{\partial \Delta}\left[\Xi\frac{\partial \Xi}{\partial \Delta} - \frac{\partial \Xi}{\partial \Delta}N^{-1}(e^{r_f t}\Delta) - \Xi\frac{\partial N^{-1}(e^{r_f t}\Delta)}{\partial \Delta}\right]$$
$$+ K\left[\left(\frac{\partial \Xi}{\partial \Delta}\right)^2 + \Xi\frac{\partial^2 \Xi}{\partial \Delta^2} - \frac{\partial^2 \Xi}{\partial \Delta^2}N^{-1}(e^{r_f t}\Delta) - 2\frac{\partial \Xi}{\partial \Delta}\frac{\partial N^{-1}(e^{r_f t}\Delta)}{\partial \Delta}\right.$$
$$\left. - \Xi\frac{\partial^2 N^{-1}(e^{r_f t}\Delta)}{\partial \Delta^2}\right],$$

$$\frac{\partial N^2(d_-)}{\partial \Delta^2} = - n(N^{-1}(e^{r_f \tau}\Delta) - \Xi)(N^{-1}(e^{r_f \tau}\Delta) - \Xi)\left(\frac{\partial N^{-1}(e^{r_f \tau}\Delta)}{\partial \Delta} - \frac{\partial \Xi}{\partial \Delta}\right)^2$$
$$+ n(N^{-1}(e^{r_f \tau}\Delta) - \Xi)\left(\frac{\partial^2 N^{-1}(e^{r_f \tau}\Delta)}{\partial^2 \Delta} - \frac{\partial^2 \Xi}{\partial \Delta^2}\right)$$

$$\frac{\partial^2 N^{-1}(e^{r_f t}\Delta)}{\partial \Delta^2} = \frac{e^{2 r_f t}}{n(N^{-1}(e^{r_f t}\Delta))}(N^{-1}(e^{r_f \tau}\Delta))\frac{\partial N^{-1}(e^{r_f t}\Delta)}{\partial \Delta}.$$

Since

$$\frac{\partial^2 C}{\partial K^2} = \frac{\frac{\partial M_1(\Delta)}{\partial \Delta}\frac{\partial K}{\partial \Delta} - M_1(\Delta)\frac{\partial^2 K}{\partial \Delta^2}}{(\frac{\partial K}{\partial \Delta})^3}.$$

According to condition ($B5$), we have

$$\frac{\partial^2 C}{\partial K^2} \geqslant 0.$$

(A4) (Large strike limit)

For the BS formula by Delta

$$C(K(\Delta, \tau), \tau, \sigma(\Delta, \tau)) = S\Delta - e^{-r\tau}K(\Delta, \tau)N(d_-),$$

we have

$$\lim_{\Delta \to 0} S\Delta = 0.$$

To prove the large strike limit condition for the COPS, it remains to shows that

$$\lim_{\Delta \to 0} e^{x(\Delta,\tau)}N(d_-) = 0.$$

Notice that

$$d_-(K(\Delta, \tau), \tau, \sigma(\Delta, \tau)) = d_2(x(\Delta, \tau), \Xi(\Delta, \tau)),$$

according to the Arithmetic-Geometric mean inequality and the condition ($B2$),

$$d_2(x(\Delta, \tau), \Xi(\Delta, \tau)) = \frac{(r-r_f)\tau}{\Xi(\Delta, \tau)} - \frac{x(\Delta, \tau)}{\Xi(\Delta, \tau)} - \frac{\Xi(\Delta, \tau)}{2} \leqslant \frac{(r-r_f)\tau}{\Xi(\Delta, \tau)} - \sqrt{2x(\Delta, \tau)},$$

thus

$$0 \leqslant e^{x(\Delta,\tau)}N(d_2(\Delta, \tau)) \leqslant e^{x(\Delta,\tau)}N\left(\frac{(r-r_f)\tau}{\Xi(\Delta, \tau)} - \sqrt{2x(\Delta, \tau)}\right).$$

Since

$$\lim_{\Delta \to 0} K(\Delta, \tau) = +\infty,$$

and $x(\Delta, \tau) \to +\infty$ as $K(\Delta, \tau) \to +\infty$, by L'Hopital's Rule we have $e^{x(\Delta,\tau)}N(-\sqrt{2x(\Delta, \tau)}) \to 0$as $x(\Delta, \tau) \to +\infty$. Thus

$$\lim_{\Delta \to 0} C(K(\Delta, \tau), \tau, \sigma(\Delta, \tau)) = 0.$$

(A5) (Bounds)

It is clear that

$$\left(e^{-r_f\tau} - e^{-r\tau + x(\Delta,\tau)}\right)^+ \leqslant B(x(\Delta, \tau), \Xi(\Delta, \tau)) \leqslant e^{-r_f\tau},$$

multiplying this equation by S, we have the bounds condition of the call price holds.

(A6) (Expiry Value)

We have that $\Xi(\Delta, \tau) = 0$ for $\tau = 0$, thus

$$C(K(\Delta, \tau), \tau, \sigma(\Delta, \tau)) = SB(x(\Delta, \tau), 0) = (S - K(\Delta, 0))^{+}.$$

□

Appendix B. Supplementary data

Supplementary data associated with this article can be found, in the online version, athttps://doi.org/10.1016/j.najef.2018.08.011.

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