Geometry 2 - Circles

TSS Math Club

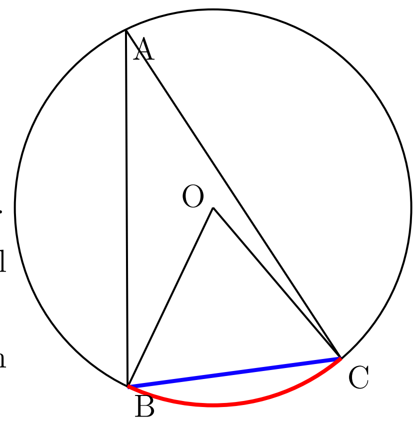
Oct 2022

1 Basic property of Circles

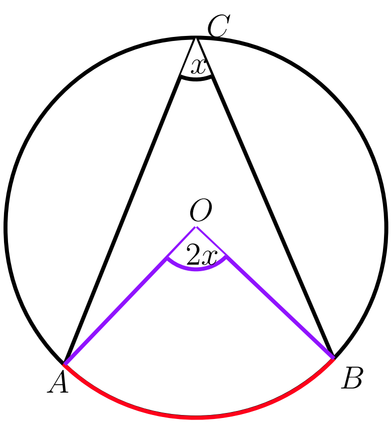
1.1 Definition of Circles

1.2 Terms to describe geometric object related to circles

- Center: Point O.
- Radius: Length from center to perimeter.
- Arc: A curved line on the circumference of a circle.
- Chord: A straight line between two points on a circle.
- Central angle: $\angle BOC$ would be an example of a central angle.
- Inscribed angle: $\angle BAC$ would be an example of an inscribed angle.

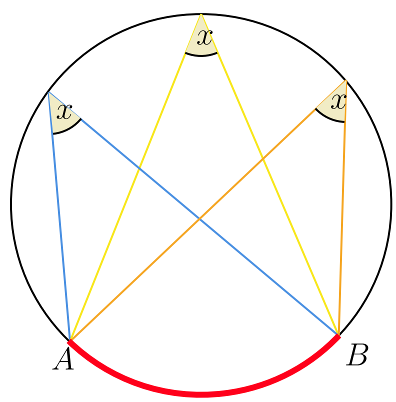


1.3 Central angle is twice any inscribed angle



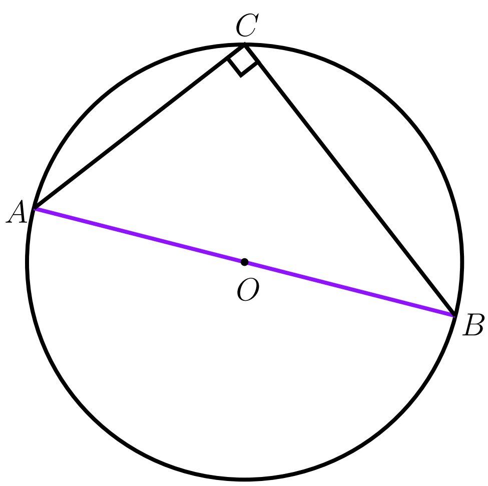
Create a line from point C that goes through point O and hits the circumference of the circle. Name this point P, and label $\angle ACO$ as y.
Since triangle $\triangle COA$ is an isosceles ($CO = AO$), $\angle ACO$ and $\angle CAO$ are the same, and thus $\angle AOP$ is equal to $2y$.
Label $\angle BCO$ as z, and since $\triangle COB$ is also an isosceles triangle, $\angle BOC$ is equivalent to $2z$.
Therefore, $\angle AOB = 2y + 2z = 2y + z = 2x$.

1.4 Inscribed angles subtended by the same arc are equal



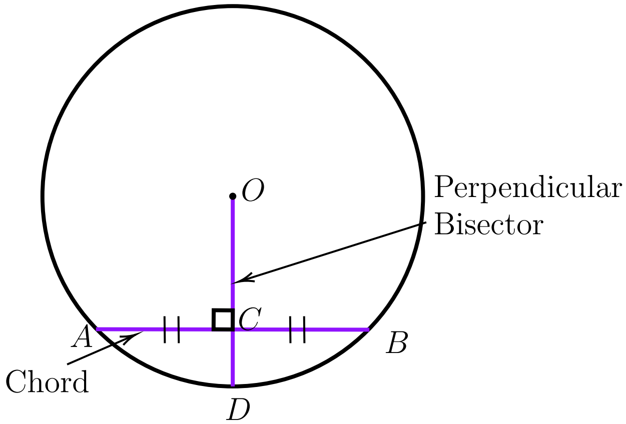
Since an inscribed angle is half of its center angle with the same arc (ref 1.3), and all three angles share the same arc, the inscribed angles are all equal.

1.5 Angle subtended by a diameter is $90°$



Both angles share the same arc, thus $\angle ACB$ is half of $\angle AOB$ $90°$ is half of $180°$.

1.6 Perpendicular chord theorem

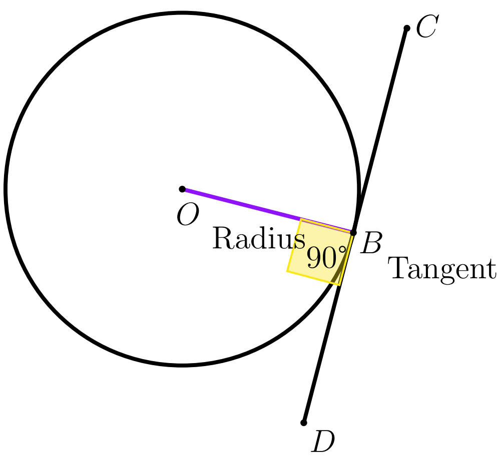


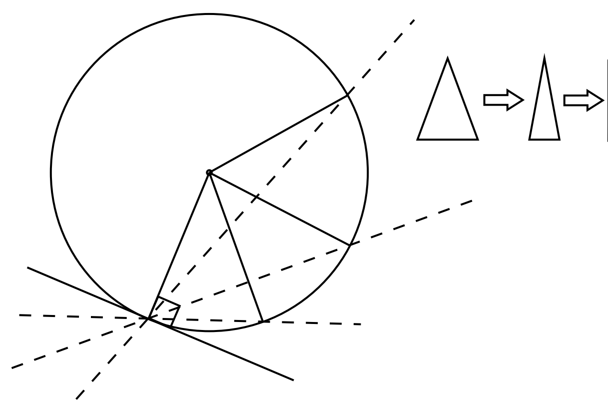
Connect point A to O, and point O to B. Note: AO and BO are both the radii of the circle. As a result, $\triangle AOC$ and $\triangle BOC$ are congruent triangles, with $AB = CB$. Therefore, the radius bisects AB.

1.7 Tangent to a circle

1.7.1 Definition: A line that intersects a circle at only one point.

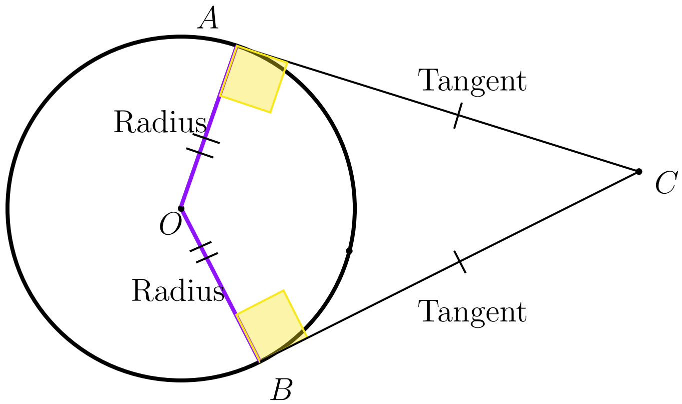
1.7.2 The radius from the center of the circle to the point of tangency is perpendicular to the tangent line





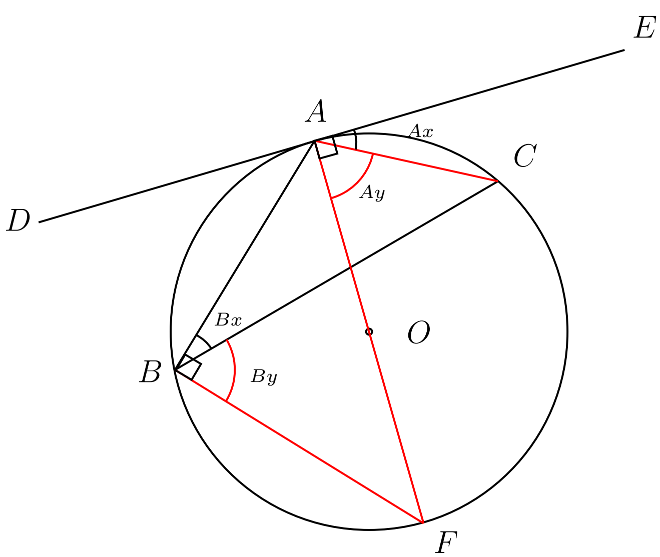
By having a line intersect a circle at random, a triangle is made when connecting the points of intersection to the center of the circle. The interior angle that is made can then be measured to be, presumably, less than 90° (when there are two points of intersection made).
By bringing one of the points of intersection closer to the other, the triangle created becomes thinner and thinner, until it becomes a line that has an interior angle of 90°, and thus becomes tangent to the circle (intersects the circle at 1 point, and at a 90° angle).

1.7.3 The length of tangents from a point to a circle are equal



Since $AO = BO$, OC is shared, and $\angle OAC = \angle OBC$ (SSA), $AC = BC$ (Hypotenuse Length Theorem).

1.7.4 Tangent-Chord Theorem: the angle formed between a chord and a tangent line to a circle is equal to the inscribed angle on the other side of the chord



To prove that $\angle EAC$ is eqivalent to $\angle ABC$, draw a line from point A that passes through the center and intersects with the circumference of the circle. Since DE is tangent to the circle, $\angle EAF$ is 90°. $\angle CBF$ is also 90°, since $\angle CBF$ shares the same arc as $\angle EAF$ (ref. 1.4).

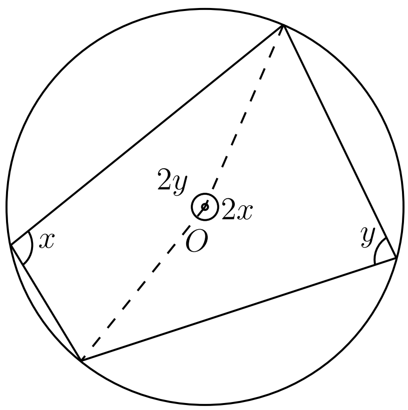
$\angle EAF$ can be considered as the sum of Ax and Ay. We know that By and Ay are the same angle because the share the same arc. Thus, we know $Ax = Bx$, since $90° - Ay = 90° - By = Ax = Bx$.

2 Cyclic Quadrilateral (Four points cyclic)

2.1 Definition

A quadrilateral which has all its four vertices lying on the perimeter of a circle.

2.2 Opposite angles are added up to $180°$

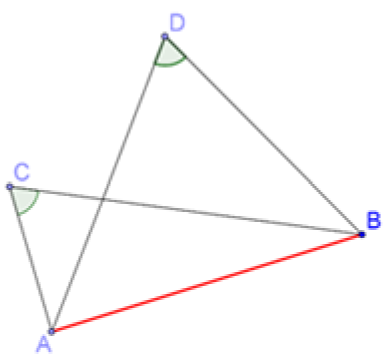


$$2x + 2y = 360°$$
$$(x + y)/2 = 360°/2$$
$$x + y = 180°$$

2.3 How to prove four points cyclic

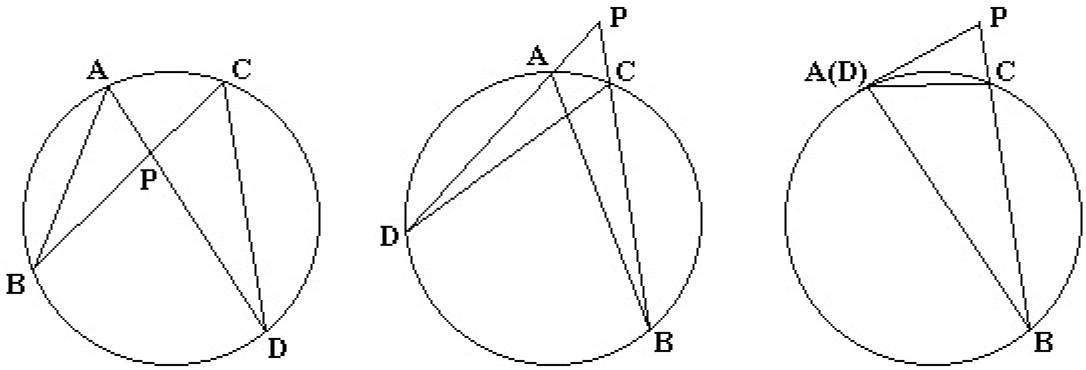
2.3.1 Prove these four points lies equally distance to another point — the center of the circle

2.3.2 Two equal angles subtend a segment (chord in the circle)



3 Similar triangles involving a circle

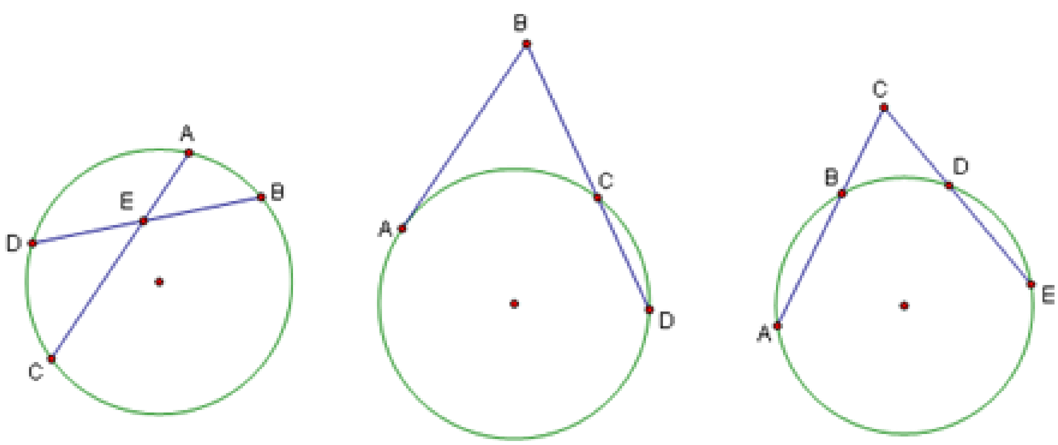
3.1 Identify as many similar triangles as possible



3.2 Power of a point

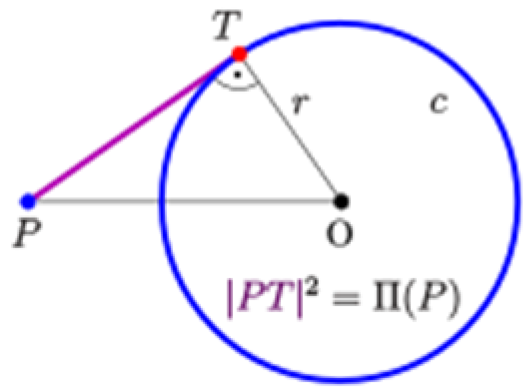
3.2.1 Definition:

3.2.2 Power of point is fixed regardless the choice of chord



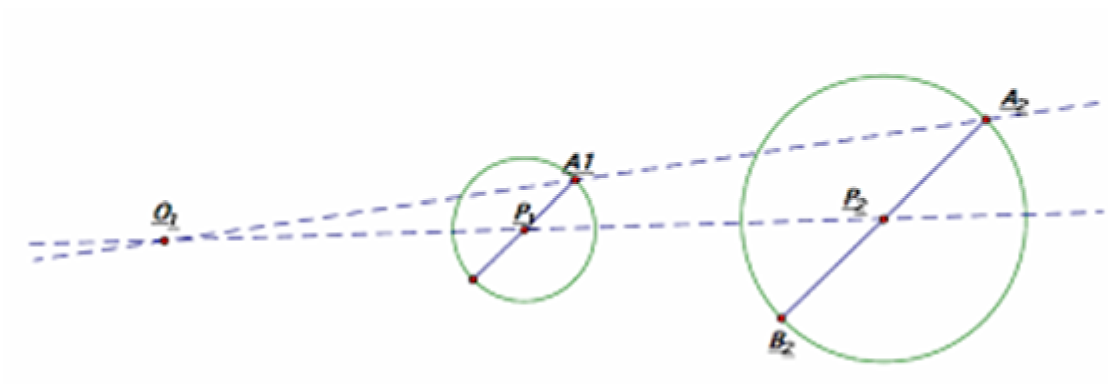
3.2.3 Power of a point formula

$$PoP = PO^2 + r^2$$



3.3 Homothety involving circles

3.3.1 Homothety of a circle is a circle

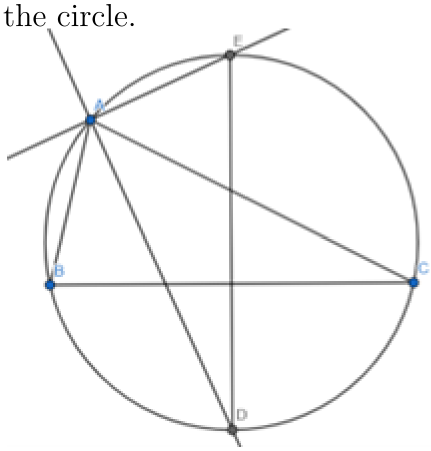


3.3.2 Ratios in the homothety

4 Problems

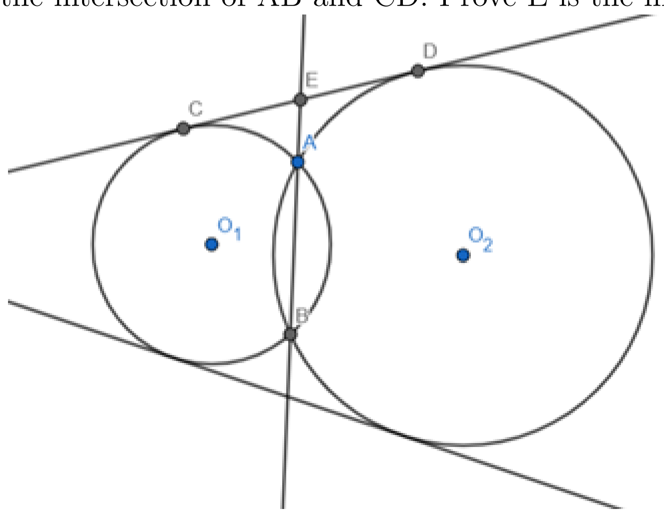
4.1 Problem

Given AD AE are the internal, external angle bisector of angle A, such that D,E are the intersection of the angle bisectors with the circumcircle. Prove DE is a diameter of the circle.



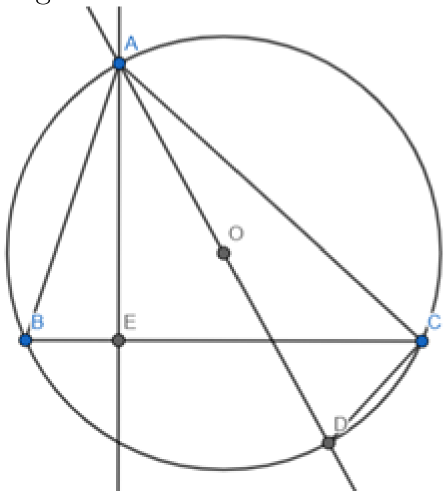
4.2 Problem

Given Circle C1, C2 intersect at A, B, CD is the common tangent to both circles, E is the intersection of AB and CD. Prove E is the midpoint of CD.



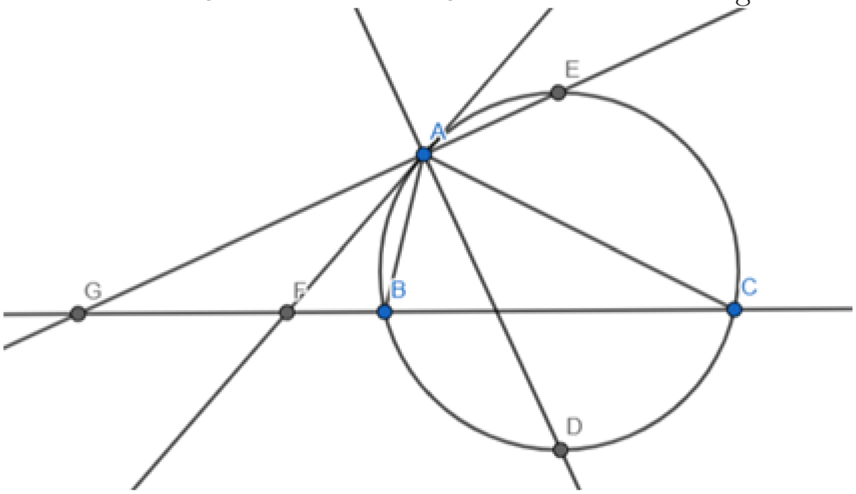
4.3 Theorem

In a triangle abc=4RS



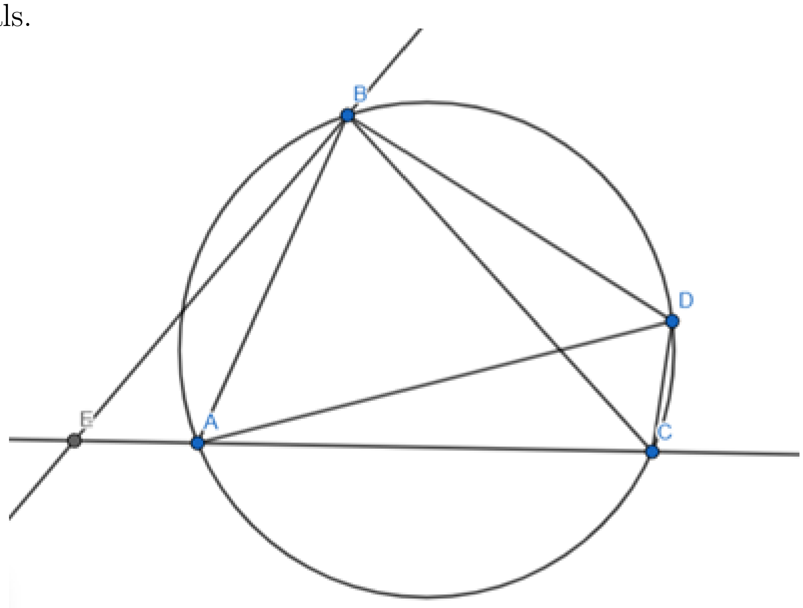
4.4 Problem

Given AE is the external angle bisector of angle A, AE intersects BC at G, the tangent at A intersects BC at F. Prove AFG is an isosceles triangle.



4.5 Ptolemy's theorem

If a quadrilateral is inscribed in a circle then the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides. Or ab+cd=xy where a,b,c,d are the sides of the quadrilateral and x,y are the diagonals.



4.6 Problem

In $\triangle ABC$, point D is inside of ABC such that $\angle DAC = \angle DCA = 30°$ and $\angle DBA = 60°$. E is the midpoint on BC and F is a trisect point on AC such that $CF = \frac{CA}{3}$. Prove $DE \perp EF$.