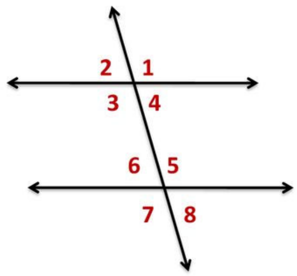
Geometry 1

1. Parallelism and basic geometry

1.1.Parallel lines:

- Parallel postulate:
    - If a line segment intersects two straight lines forming two interior angles on the same side that are less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.
- Definition:
  Two lines in two-dimensional Euclidean space are said to be parallel if they do not intersect.
- Parallel Lines and Transversal

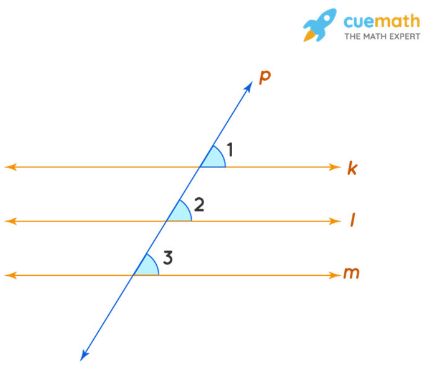
Properties of parallel lines cut by a transversal



Pair of angles are equal
Each pair of **corresponding angles** are **equal**
Each pair of **interior alternate angles** are **equal**
Each pair of **exterior alternate angles** are **equal**
Each pair of **vertically opposite angles** are **equal**

Sum of pair of angles are 180^0
Each pair of **interior angles** on the **same side of the transversal** are **supplementary** (i.e 180^0)
Each pair of **exterior angles** on the **same side of the transversal** are **supplementary** (i.e 180^0)
Sum of pair of **adjacent supplementary angles** are **supplementary** (i.e 180^0)

1.2.Parallel is transitive (i.e. l1//l2,l2//l3 -> l1//l3)



$\because k//l$

$\therefore \angle 1 = \angle 2$

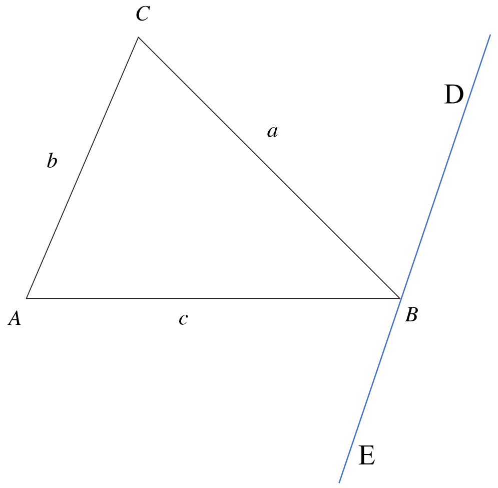
$\because l//m$

$\therefore \angle 2 = \angle 3$

$\therefore \angle 1 = \angle 3$

$\therefore k//m$

1.3.Sum of interior angles of a triangle is 180°



Draw line DE through B such that DE//AC

$\because AC//DE$

$\therefore \angle A = \angle ABE$

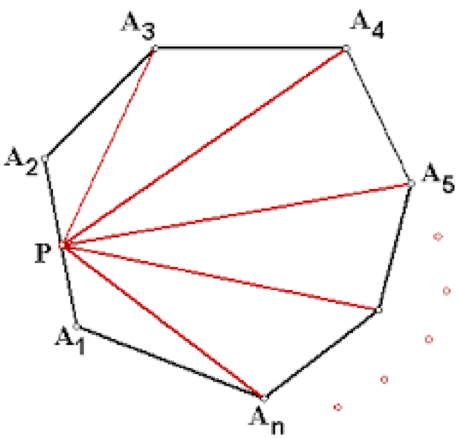
$\angle B = \angle CBD$

$\therefore \angle A + \angle B + \angle C$

$= \angle CBD + \angle CBA + \angle ABE$

$= 180°$

1.4.Sum of interior angles of a n-gon is (n-2)x180°

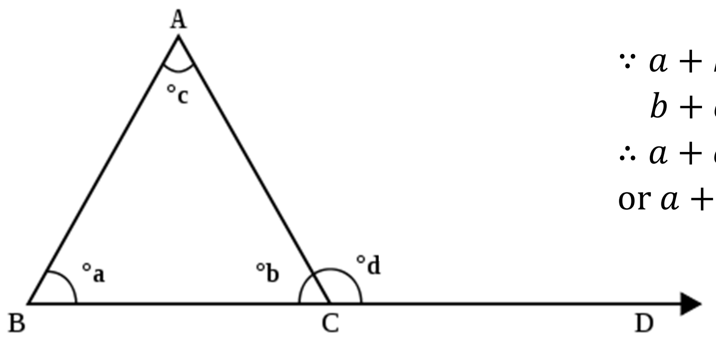


It can be partitioned into n-2 triangles.

Since the sum of interior angles of a n-gon is the sum of all the interior angles of the triangles, by the theorem above, the interior angles = number of triangles x 180° or

(n-2)x180°.

1.5.Exterior angle theorem



$\because a + b + c = 180°$

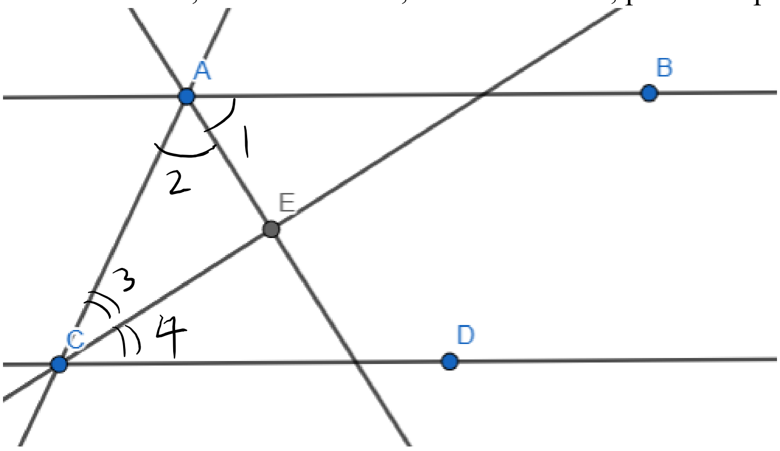
$b + d = 180°$

$\therefore a + c - d = 0$

or $a + c = d$

1.6.Another problem

Given AB//CD, AE bisect BAC, CE bisect ACD, prove AE perpendicular to CE.



$\because AE, CE\ are\ bisectors$

$\therefore \angle 2 = \dfrac{\angle BAC}{2}$

$and \angle 3 = \dfrac{\angle ACD}{2}$

$\therefore \angle 2 + \angle 3 = \dfrac{\angle BAC + \angle ACD}{2}$

$\because AB//CD$

$\therefore \angle BAC + \angle ACD = 180°$

$\therefore \angle 2 + \angle 3 = \dfrac{\angle BAC + \angle ACD}{2} = \dfrac{180°}{2} = 90°$

$\therefore \angle AEC = 180° - (\angle 2 + \angle 3) = 90°$

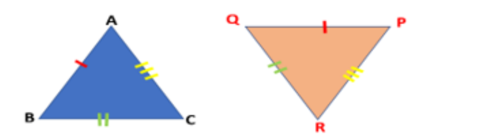
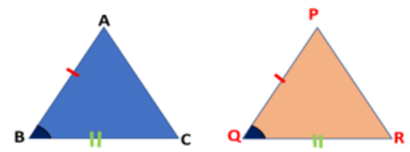
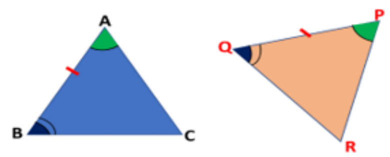
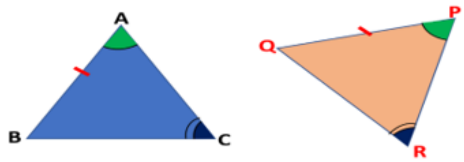
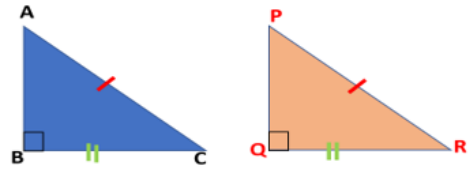
$\therefore$ AE is perpendicular to CE

2. Congruence

2.1.Definition: The same shape and size.

2.2.Method to prove congruency

Congruent Triangles

SSS (Side – Side – Side) Congruency		*When the three sides of a triangles are equal to the other three sides of another triangle.* $\Delta ABC \cong \Delta PQR$
SAS (Side – Angle – Side) Congruency		*When two sides and the included angle on one triangle is equal to the two sides and included angle of another triangle.* $\Delta ABC \cong \Delta PQR$
ASA (Angle – Side – Angle) Congruency		*When two angles and the included side of one triangle are equal to two angles and the included side of another triangle.* $\Delta ABC \cong \Delta PQR$
AAS (Angle – Angle – Side) Congruency		*When two angles and the non-included side of one triangle are equal to two angles and the non-included side of another triangle.* $\Delta ABC \cong \Delta PQR$
RHS (Right Angle – Hypotenuse – Side) Congruency		*When hypotenuse and one side of a right triangle is equal to the hypotenuse and other side of another right triangle.* $\Delta ABC \cong \Delta PQR$

How to write in a contest:

$$\text{In the } \Delta ABC \text{ and } \Delta A'B'C'$$
$$\begin{cases} Condition\ 1 \\ Condition\ 2 \\ Condition\ 3 \end{cases}$$
$$\therefore \Delta ABC \cong \Delta A'B'C'$$

2.2.1. Side-Side-Angle (SSA), the ambiguous case.

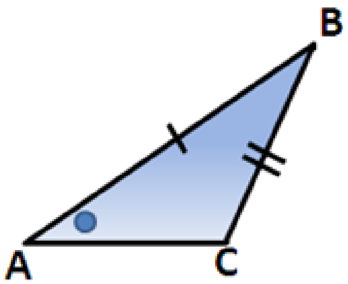


Figure 3-A

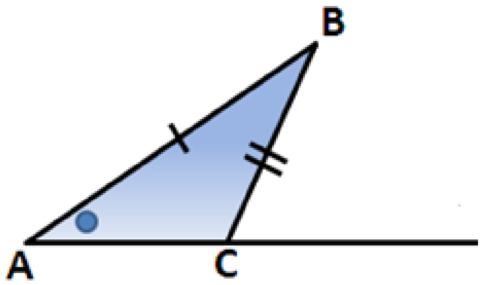


Figure 3-B

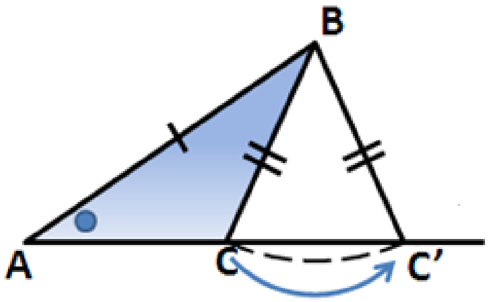


Figure 3-C

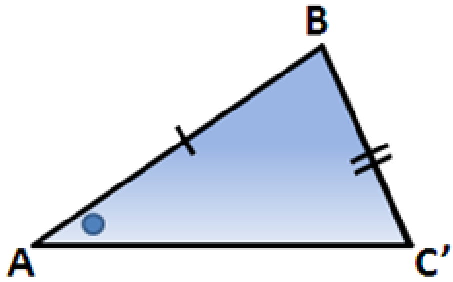
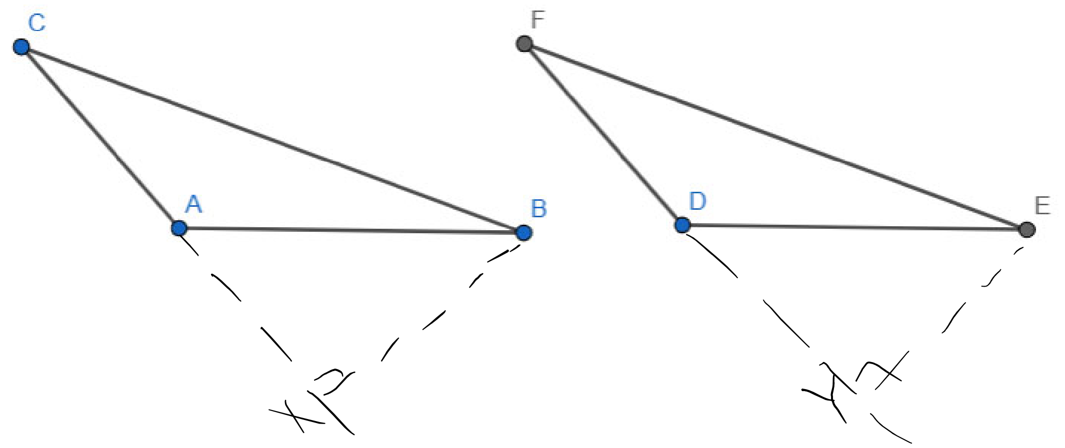


Figure 3-D

2.2.2. Proof related to the ambiguous case

Given AB=DE, BC=EF, angle A =angle D>90°, prove triangle ABC congruent to triangle DEF.



Extend CA and draw BX perpendicular to CA at X

Extend FD and draw EY perpendicular to FD at Y

$\because$ *the perpendicular*

$\therefore \angle X = \angle Y = 90°$

$\because \angle BAC = \angle EDF$

$\therefore \angle BAX = \angle EDY$

InΔABX andΔDEY

$$\begin{cases} \angle X = \angle Y \\ \angle \mathrm{BAX} = \angle \mathrm{EDY} \\ AB = DE \end{cases}$$

$\therefore \Delta \mathrm{ABX} \cong \Delta \mathrm{DEY}(\mathrm{AAS})$

$\therefore BX = EY$

In $Rt\Delta$CBX and RtΔFEY

$$\begin{cases} \mathrm{BX} = \mathrm{EY} \\ \mathrm{BC} = \mathrm{EF} \end{cases}$$

$\therefore Rt\Delta\mathrm{CBX} \cong \mathrm{Rt}\Delta\mathrm{FEY}(\mathrm{HL})$

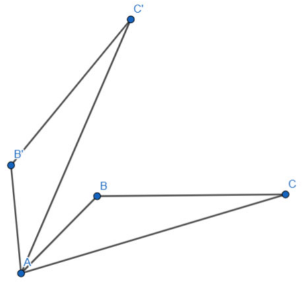
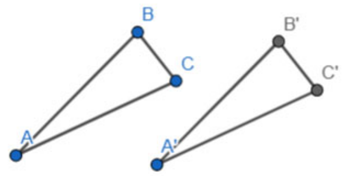
$\therefore \angle C = \angle F$

In ΔABC andΔDEF

$$\begin{cases} \angle \mathrm{C} = \angle \mathrm{F} \\ \angle \mathrm{BAC} = \angle \mathrm{EDF} \\ AB = DE \end{cases}$$

$\therefore \Delta \mathrm{ABC} \cong \Delta \mathrm{DEF}(\mathrm{AAS})$

Useful congruencies

Rotation	Translation
	

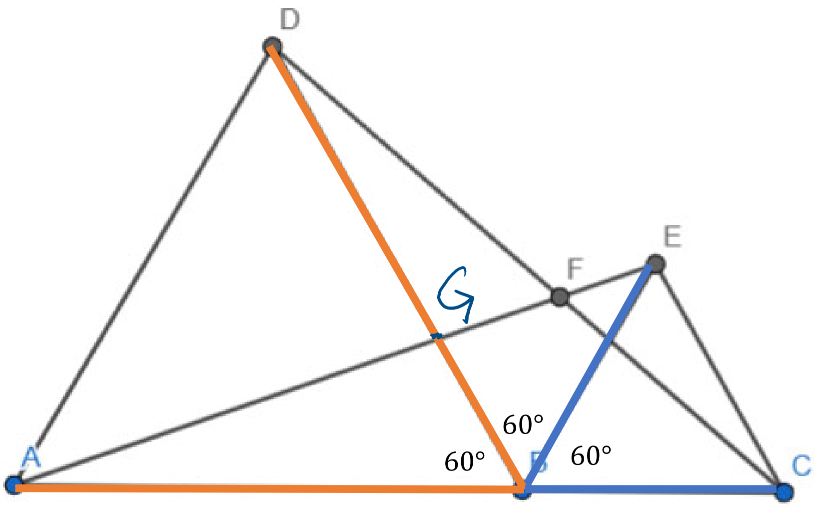
2.3. Properties of congruency

2.3.1. Equal side length

2.3.2. Equal angles

2.4. Problems

2.4.1. Assume ABD and BCE are equilateral triangles, find angle DFA



$\because ABD\ and\ BCE\ are\ equilateral\ triangles$

$\therefore AB = BD, BC = BE$

$\angle ABD = \angle CBE = 60°$

$\therefore \angle DBE = 180° - 60° - 60° = 60°$

$\therefore \angle ABE = \angle \mathrm{DBC} = 120°$

InΔABE and ΔDBC

$$\begin{cases} \mathrm{AB} = \mathrm{DB} \\ \angle ABE = \angle \mathrm{DBC} \\ \mathrm{BE} = \mathrm{BC} \end{cases}$$

$\therefore \Delta\mathrm{ABE} \cong \Delta\mathrm{DBC}(\mathrm{SAS})$

$\therefore \angle EAB = \angle CDB = x$

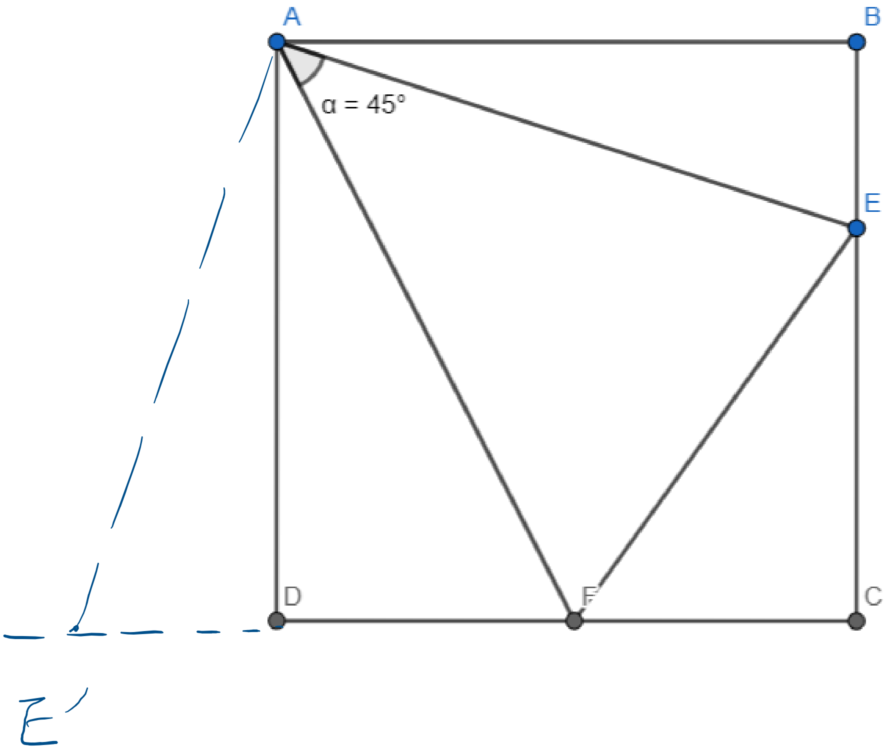
$\therefore \angle DGA = \angle EAB + \angle DBA$

$= \angle CDB + \angle DFA = 60° + x$

$\therefore \angle \mathrm{DFA} = 60°$

2.4.2.

Given ABCD is a square, and angle EAF = 45°, prove AE bisect BEF.



Extend CD to E' such that E'D=BE

Connect E'A

$\because ABCD \text{ is a square}$

$\therefore AD = AB$

$\angle B = \angle ADE' = 90°$

$In \triangle ADE' and \triangle ABE$

$$\begin{cases} \mathrm{AD} = \mathrm{AB} \\ \angle B = \angle ADE' \\ \mathrm{DE'} = \mathrm{BE} \end{cases}$$

$\therefore \triangle ADE' \cong \triangle ABE(SAS)$

$\therefore AE' = AE$

$\angle \mathrm{AEB} = \angle \mathrm{E'}$

$\angle E'AD = \angle EAB$

$\therefore \angle E'AF = \angle E'AD + \angle DAF$

$= \angle EAB + \angle DAF = 90° - \angle EAF = 45°$

$In \triangle AFE' and \triangle AFE$

$$\begin{cases} \mathrm{AF} = \mathrm{AF} \\ \angle EAF = \angle E'AF = 45° \\ \mathrm{AE'} = \mathrm{AE} \end{cases}$$

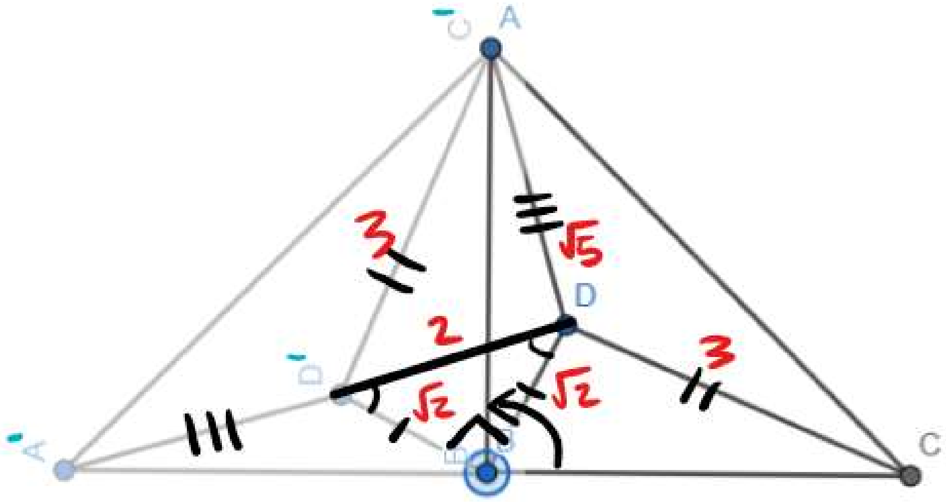
$\therefore \triangle AFE' \cong \triangle AFE(\mathrm{SAS})$

$\therefore \angle AEF = \angle E' = \angle AEB$

$\therefore$ AE bisect BEF

2.4.3.

Given AB=BC and angle ABC = 90°, AD=$\sqrt{5}$, BD=$\sqrt{2}$, DC=3, find angle ADB.



Rotate ABC 90° counterclockwise on point B, so that AB and BC overlap.

∠ABD + ∠DBC = 90°, therefore ∠DBD' = ∠ABD + ∠D'BA = 90°

Since BDD' is an isosceles with ∠DBD' = 90°,

∠BD'D = ∠BDD' = $\frac{180° - 90°}{2}$ = 45°

BD = BD' = $\sqrt{2}$, so DD' = $\sqrt{a^2 + b^2}$ = c => $\sqrt{\sqrt{2}^2 + \sqrt{2}^2}$ = 2

Applying the pythagorean theorem can determine whether ∠ADD' is 90° or not:

$a^2 + b^2 = c^2 \Rightarrow DD'^2 + DA^2 = D'A^2 \Rightarrow 2^2 + \sqrt{5}^2 = 3^2 \Rightarrow 9 = 9$

Therefore, AD'D is a right angle triangle, with ∠ADD' being 90°.

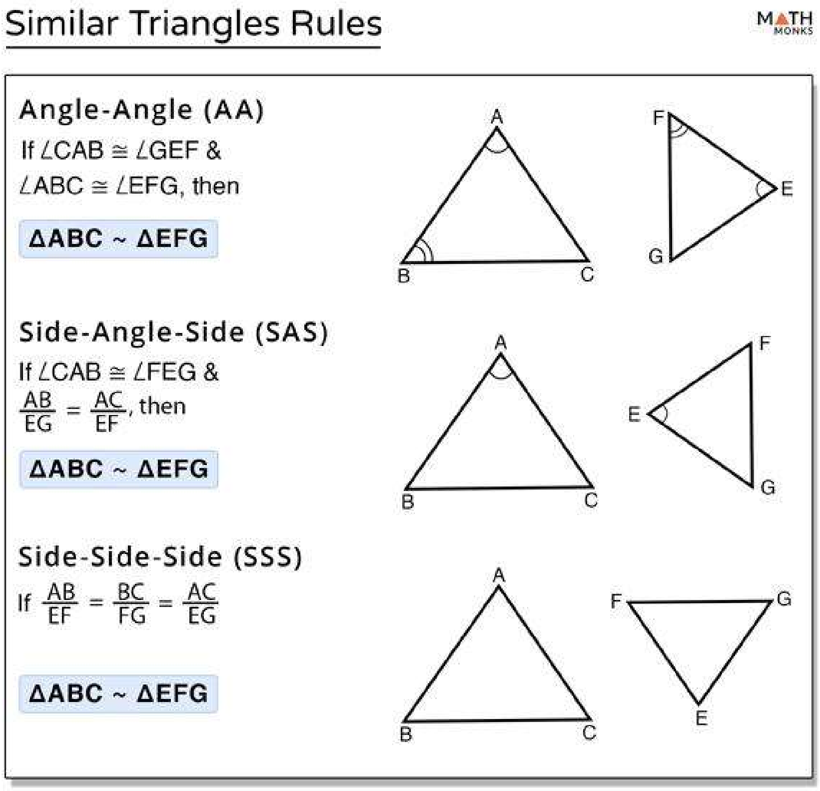
Therefore, ∠ADB = ∠ADD' + ∠D'DB = 90° + 45° = 135°.

3. Similarity

3.1.Definition:

In Euclidean geometry, two objects are similar if they have the same shape, or one has the same shape as the mirror image of the other.

3.2.Method to Prove similar triangles



3.3.Properties of similar triangles

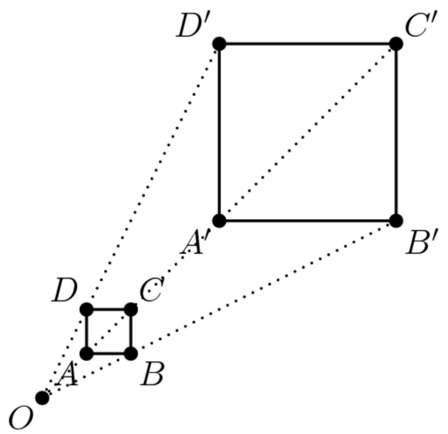
3.3.1. Ratio of corresponding sides

3.3.2. Equal angles

4. Homothety

4.1. Definition:

A transformation determined by having a point of origin (O), and a ratio of which points from the origin are scaled to.



4.2. Properties:

Shapes will be similar, not congruent, with only its size varying.

The ratio is a set value and applies to all points in relation to the origin.

(Referencing Figure 4.1) DC is parallel to D’C’. This applies to all other points and their corresponding points that have been scaled from the origin by a set factor.

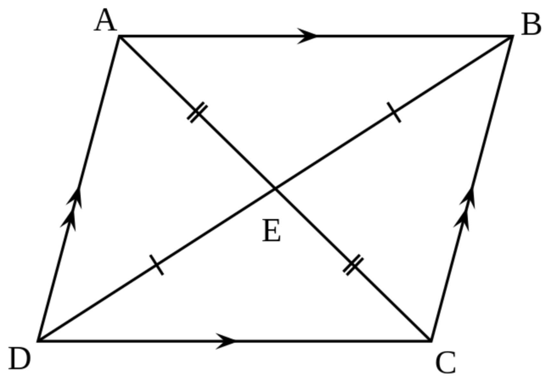
All angles remain unchanged and the ratio between two line segments stays constant.

If two triangles' sides are parallel to each other, the lines connecting the corresponding points are concurrent.

5. Quadrilaterals

5.1. Parallelogram

5.1.1. Definition:



A quadrilateral with both opposite sides parallel to each other.

5.1.2. Properties:

Opposite side lengths are congruent

When drawing a line between the opposite points, the intersections are midpoints for both lines (AE = EC, therefore E is the midpoint of AC. BE = ED, therefore…).

Angle BAC is the same as ACD (internal angle theorem). This is also the same with angles DAC and ACB.

5.2.Rhombus

5.2.1. Definition:

A parallelogram with congruent side lengths for all four sides.

5.2.2. Properties:

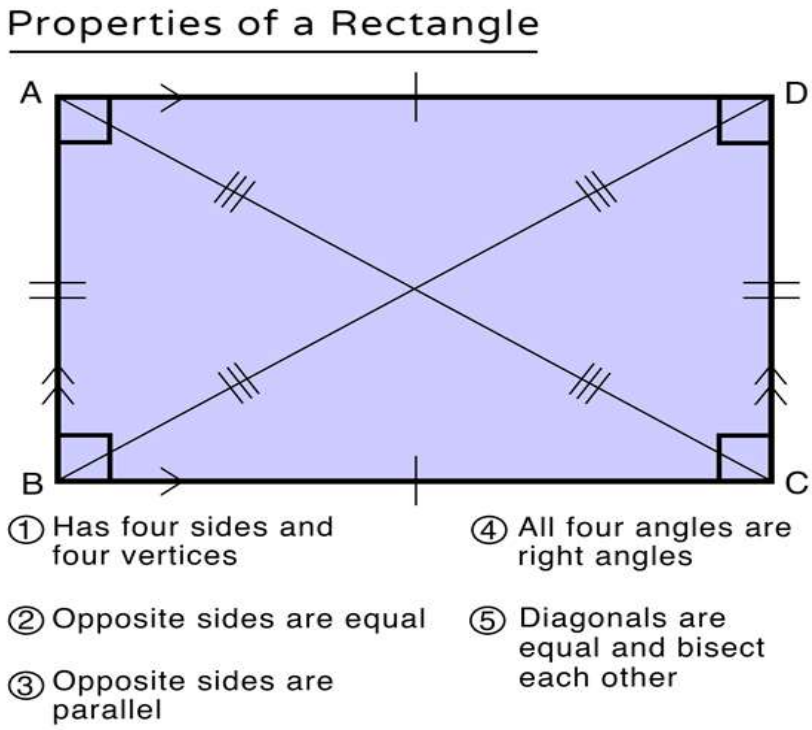
Same properties as a parallelogram, except all side lengths are the same, all the internal lines are the same, and the intersection of diagnoses forms 90 degree angles.

5.3.Rectangle

5.3.1. Definition:

A parallelogram that has all four angles being 90 degrees each.

5.3.2. Properties:

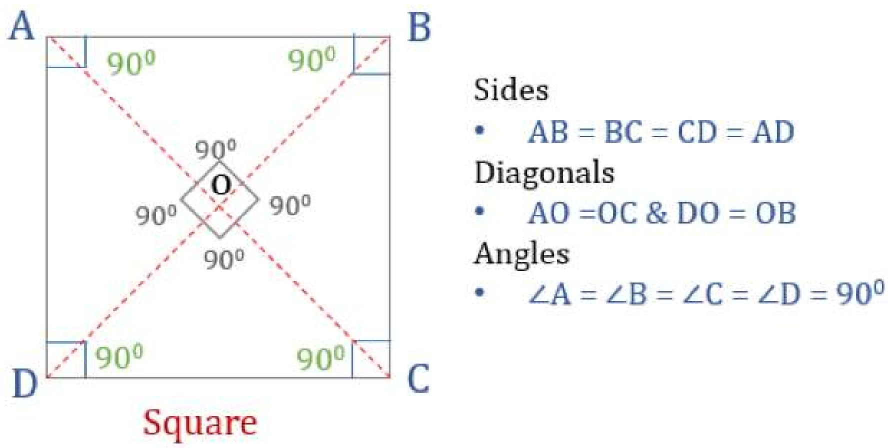


5.4.Square

5.4.1. Definition:

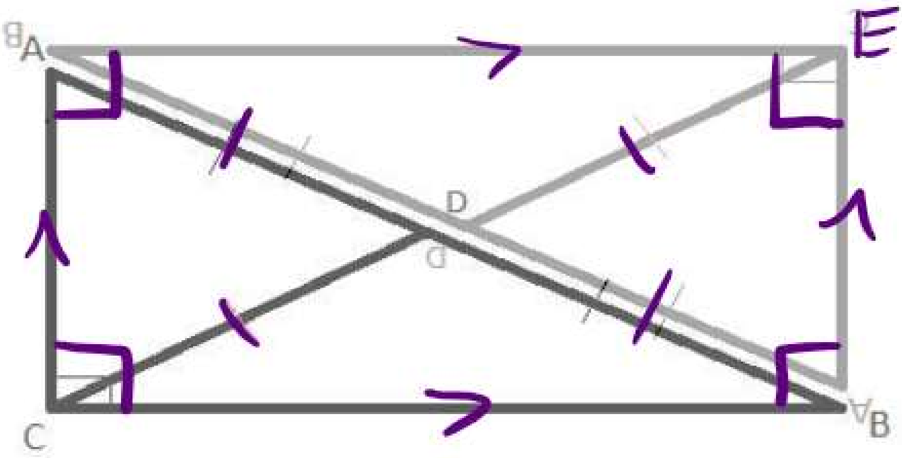
A rhombus with each angle being 90 degrees to each other.

5.4.2. Properties:



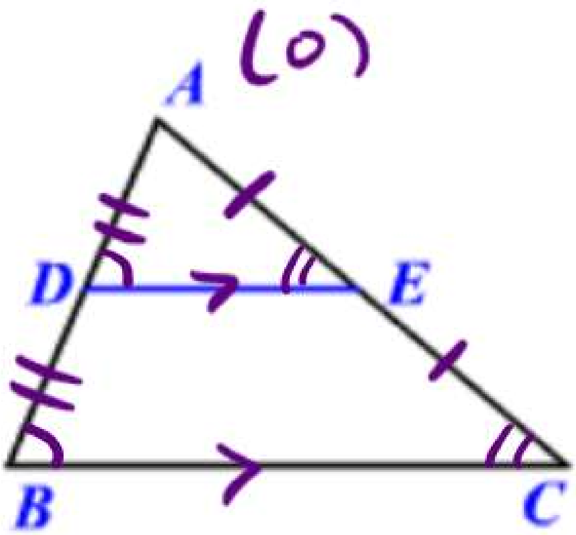
6. Midpoint

6.1. Midpoint of a right triangle



By rotating ABC 180 degrees on point D, a rectangle can be created. The properties of a rectangle states that all internal line segments (AD, CD, BD, and ED) are all congruent. Therefore, the midpoint of a right triangle creates AD, CD, and BD, which all have the same side lengths.

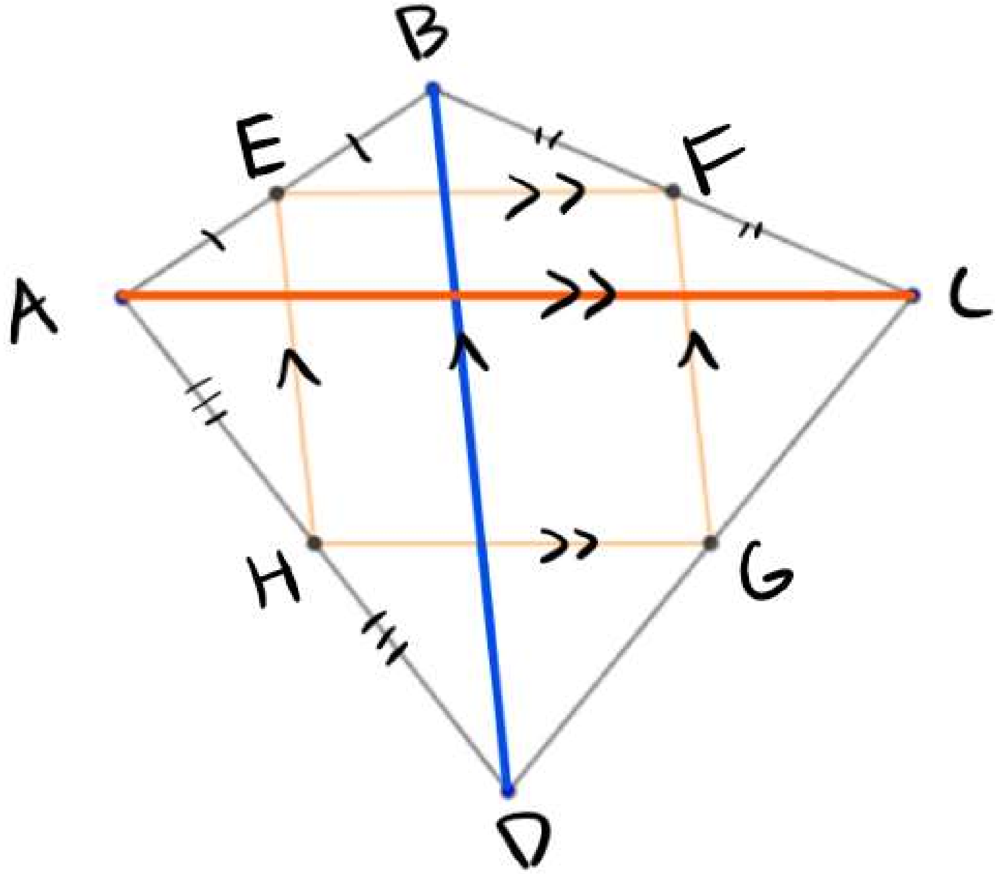
6.2. Midsegment



BC is parallel to DE since D and E are extended to B and C, respectively, by a factor of twice their original side lengths. When referring back to homothety, point A acts as the origin point, therefore making angle ADE and angle ABC the same. Therefore, DE and BC are parallel.

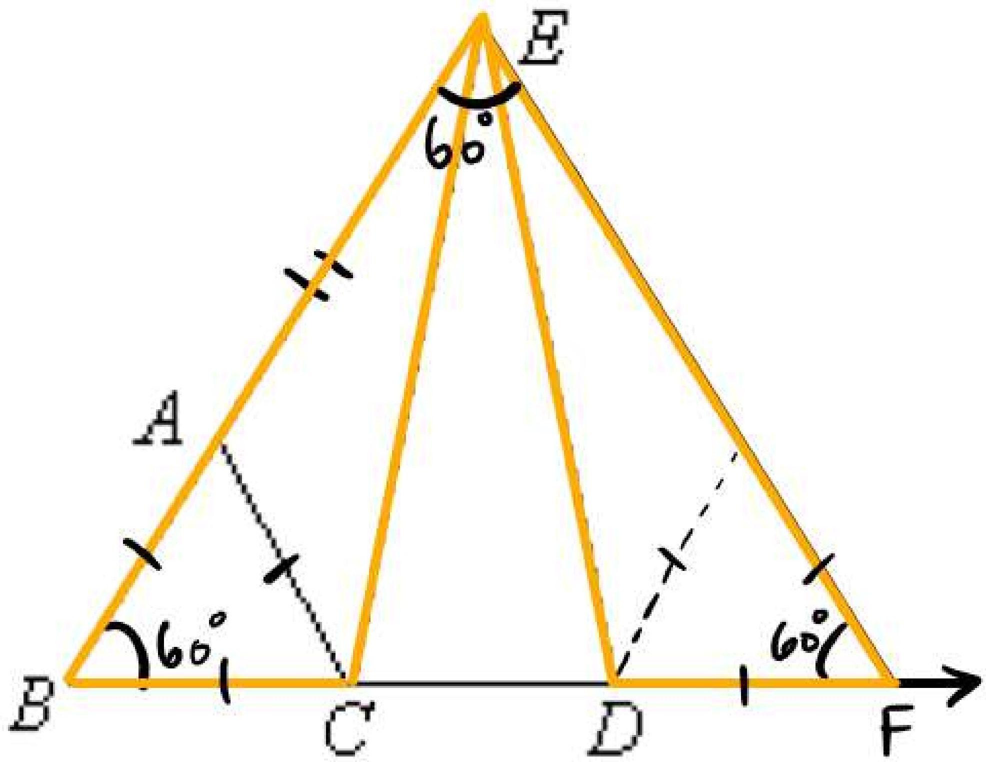
7. Problems

7.1. The quadrilateral formed by joining midpoint of a quadrilateral is a parallelogram



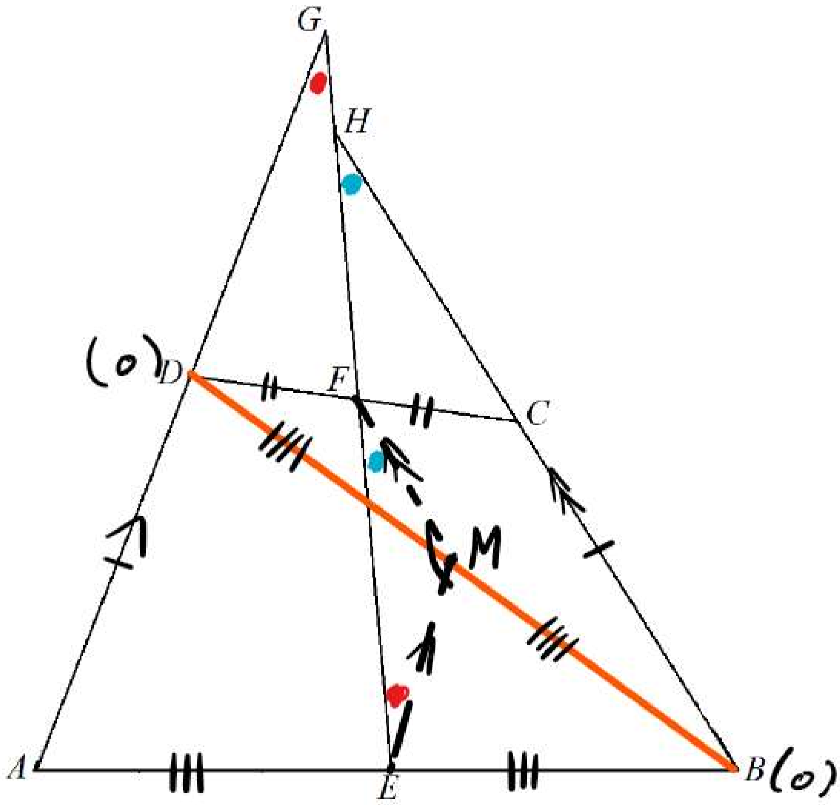
To prove that EFGH is a parallelogram, one can view point A as a point of origin for homothety. As AE is and AH are doubled to become AB and AD, respectively, angle AHE and ADB are the same (ref. 4.2, homothety). Thus, HE and DB are parallel. This analysis can be applied to the three other points (point B, C, and D), where these points are viewed as the point of origin. Therefore, EH and FG can be proven as parallel, and so can EF and HG.

7.2. Given equilateral ΔABC, extend BC to D and BA to E to let AE = BD. Join CE and DE. Prove ∠ECD =∠EDC.



To prove ∠ECD = ∠EDC, extend line BD by a length of BC, to create the new point F. Connect point E to F, to create triangle EFB. Since AE = BD, AE = CF = BD. Therefore, BE = BA + AE = BC + CF = BF. Knowing that EB and BF have the same side lengths, and intersect at an angle of 60 degrees, one can conclude that EFB is an equilateral triangle (EFB is an isosceles, therefore angle BEF and EFB must equal to (180 - 60)/2 = 120 / 2 = 60 degrees, therefore equilateral).
Once that has been proven, one can prove that BCE and FDE are the same due to SAS, therefore proving that ∠ECD and ∠EDC are congruent.

7.3. In quadrilateral ABCD with AD = BC, E and F are midpoints on AB and CD, respectively. EF meets AD at G, meets BC at H. Prove ∠DGF = ∠CHF.



To prove ∠DGF = ∠CHF, a reference line has to be drawn from point D to point B, with point M as the midpoint of DB.

Then, connect point F to M, and E to M.

When viewing point B as the point of origin (O), lines EM and AD can be viewed as parallel lines, with EM being half of the length of AD because the distance between OD is double of OM (ref. 4.2., Homothety).

Apply this analysis but onto point D as the origin point, thus proving FM is both parallel and half the length of CB.

Since FM and EM are the same lengths, FME is an isosceles triangle with ∠EFM = ∠MEF.

BH and HF are parallel lines that intersect EH, therefore ∠CHF = ∠MFE.

DG and EM are parallel lines that intersect with EG, therefore ∠DGF = ∠GEM (Alternate Interior Angle theorem).

Therefore, ∠DGF = ∠FEM = ∠EFM = ∠FHC ⟹ ∠DGF = ∠FHC