

Geometry 1 - Basis of Geometry

TSS Math Club

Oct 2022

1 Parallelism and basic geometry

1.1 Parallel lines

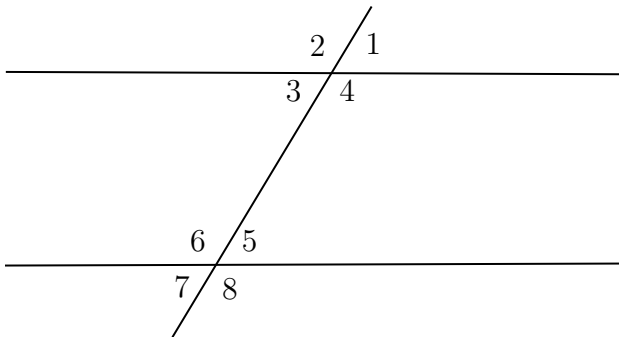
1.1.1 Parallel postulate

If a line segment intersects two straight lines forming two interior angles on the same side that are less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

1.1.2 Definition

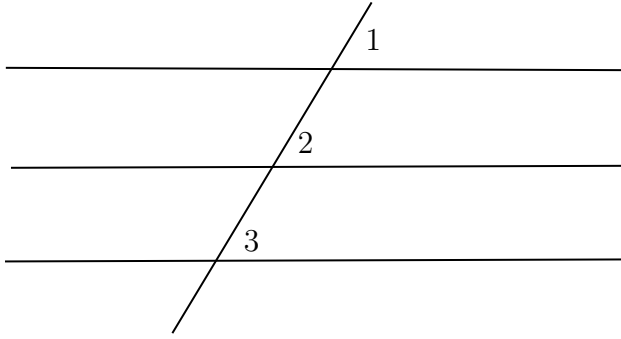
Two lines in two-dimensional Euclidean space are said to be parallel if they do not intersect.

1.1.3 Parallel Lines and Transversal



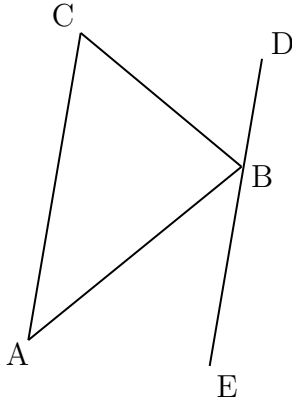
- Corresponding angles are equal ($\angle 1 = \angle 5$)
- Alternate angles are equal ($\angle 3 = \angle 5$)
- Co-interior angles are supplementary to each other ($\angle 4 + \angle 5 = 180^\circ$)

1.2 Parallel is transitive



$$\begin{aligned} &\because k \parallel l \\ &\therefore \angle 1 = \angle 2 \\ &\because l \parallel m \\ &\therefore \angle 2 = \angle 3 \\ &\therefore \angle 1 = \angle 3 \\ &\therefore k \parallel m \end{aligned}$$

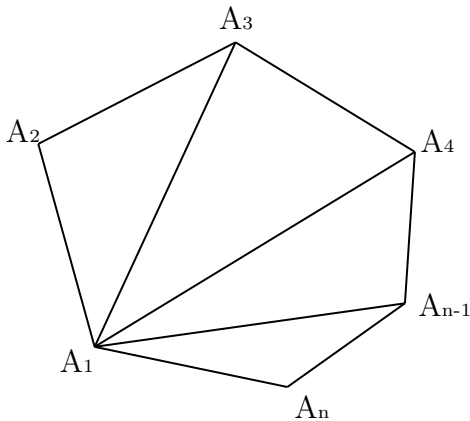
1.3 Sum of the interior angles of a triangle is 180°



Draw line DE through B such that $DE \parallel AC$

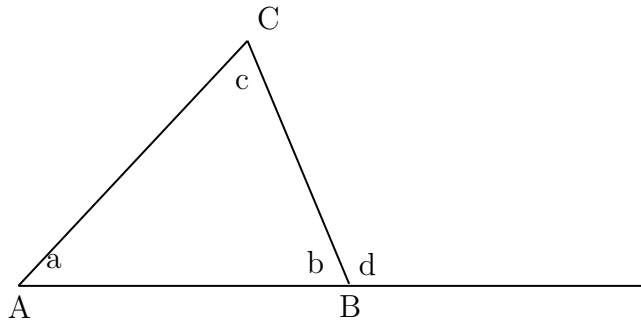
$$\begin{aligned} &\because AC \parallel DE \\ &\therefore \angle A = \angle ABE \\ &\quad \angle B = \angle CBD \\ &\therefore \angle A + \angle B + \angle C \\ &\quad = \angle CBD + \angle CBA + \angle ABE \\ &\quad = 180^\circ \end{aligned}$$

1.4 Sum of interior angles of a n-gon is $(n - 2)180^\circ$



It can be partitioned into $n-2$ triangles. Since the sum of interior angles of a n -gon is the sum of all the interior angles of the triangles, by the theorem above, the interior angles = number of triangles $\times 180^\circ$ or $(n - 2) \times 180^\circ$.

1.5 Exterior angle theorem



$$\therefore a + b + c = 180^\circ$$

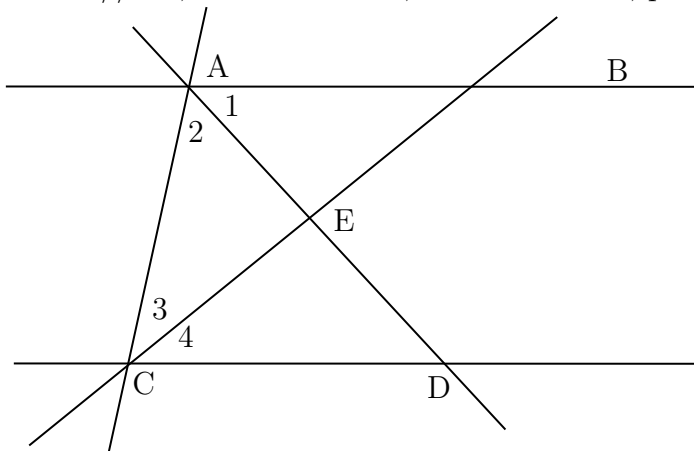
$$b + d = 180^\circ$$

$$\therefore a + c - d = 0$$

$$\text{or } a + c = d$$

1.6 Another problem

Given $AB \parallel CD$, AE bisect BAC , CE bisect ACD , prove AE perpendicular to CE .



$\therefore AE, CE$ are bisectors

$$\therefore \angle 2 = \frac{\angle BAC}{2}$$

$$\text{and } \angle 3 = \frac{\angle ACD}{2}$$

$$\therefore \angle 2 + \angle 3 = \frac{\angle BAC + \angle ACD}{2}$$

$\therefore AB \parallel CD$

$$\therefore \angle BAC + \angle ACD = 180^\circ$$

$$\therefore \angle 2 + \angle 3 = \frac{\angle BAC + \angle ACD}{2} = \frac{180^\circ}{2} = 90^\circ$$

$$\therefore \angle AEC = 180^\circ - (\angle 2 + \angle 3) = 90^\circ$$

$\therefore AE$ is perpendicular to CE

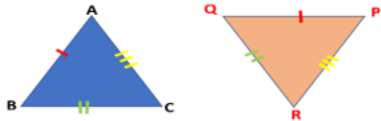
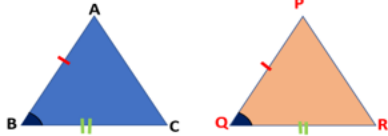
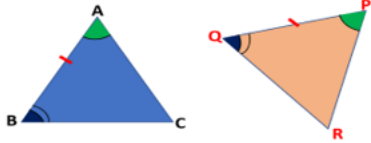
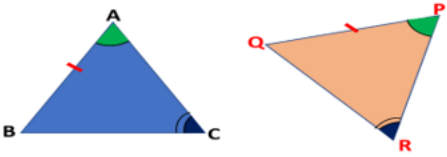
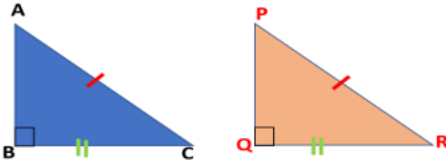
2 Congruence

2.1 Definition

The same shape and size.

2.2 Method to prove congruence

2.2.1 5 Methods:

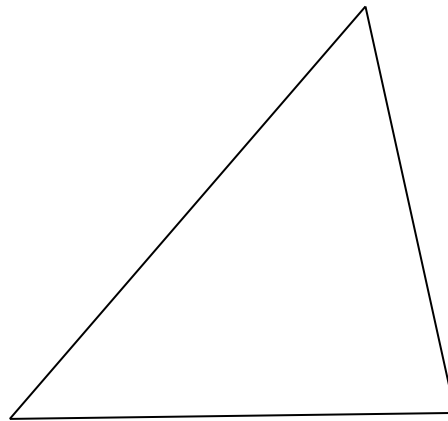
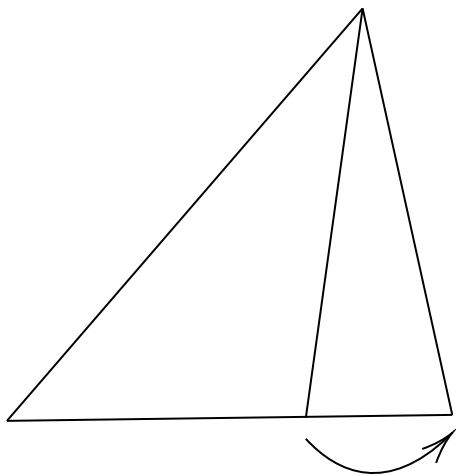
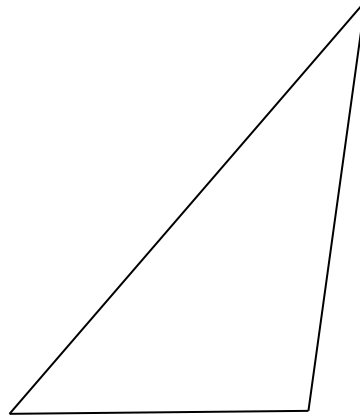
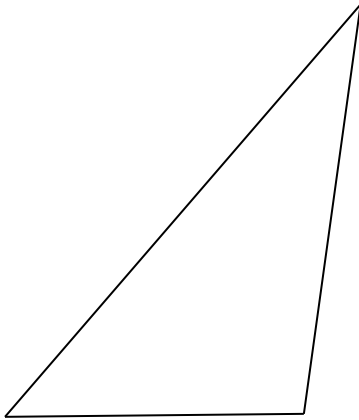
| Congruent Triangles | | |
|---|--|--|
| SSS (Side – Side – Side) Congruency |  | When the three sides of a triangles are equal to the other three sides of another triangle. $\triangle ABC \cong \triangle PQR$ |
| SAS (Side – Angle – Side) Congruency |  | When two sides and the included angle on one triangle is equal to the two sides and included angle of another triangle. $\triangle ABC \cong \triangle PQR$ |
| ASA (Angle – Side – Angle) Congruency |  | When two angles and the included side of one triangle are equal to two angles and the included side of another triangle. $\triangle ABC \cong \triangle PQR$ |
| AAS (Angle – Angle – Side) Congruency |  | When two angles and the non-included side of one triangle are equal to two angles and the non-included side of another triangle. $\triangle ABC \cong \triangle PQR$ |
| RHS (Right Angle – Hypotenuse – Side) Congruency |  | When hypotenuse and one side of a right triangle is equal to the hypotenuse and other side of another right triangle. $\triangle ABC \cong \triangle PQR$ |

2.2.2 How to write in a contest

In the $\triangle ABC$ and $\triangle A'B'C'$

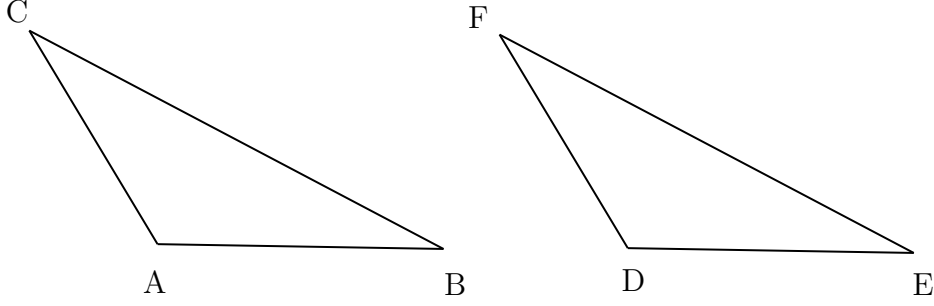
$$\left\{ \begin{array}{l} \text{Condition 1} \\ \text{Condition 2} \\ \text{Condition 3} \end{array} \right. \\ \therefore \triangle ABC \cong \triangle A'B'C'$$

2.3 Side-Side-Angle (SSA), the ambiguous case



2.3.1 Proof related to the ambiguous case

Given $AB = DE$, $BC = EF$, angle $A = \text{angle } D > 90^\circ$, prove triangle ABC congruent to triangle DEF .



Extend CA and draw BX perpendicular to CA at X
 Extend FD and draw EY perpendicular to FD at Y

\therefore the perpendicular

$$\therefore \angle X = \angle Y = 90^\circ$$

$$\therefore \angle BAC = \angle EDF$$

$$\therefore \angle BAX = \angle EDY$$

In $\triangle ABX$ and $\triangle DEY$

$$\begin{cases} \angle X = \angle Y \\ \angle BAX = \angle EDY \\ AB = DE \end{cases}$$

$$\therefore \triangle ABX \cong \triangle DEY (\text{AAS})$$

$$\therefore BX = EY$$

In $\triangle CBX$ and $\triangle FEY$

$$\begin{cases} BX = EY \\ BC = EF \end{cases}$$

$$\therefore \text{Rt } \triangle CBX \cong \text{Rt } \triangle FEY (\text{HL})$$

$$\therefore \angle C = \angle F$$

In $\triangle ABC$ and $\triangle DEF$

$$\begin{cases} \angle C = \angle F \\ \angle BAC = \angle EDF \\ AB = DE \end{cases}$$

$$\therefore \triangle ABC \cong \triangle DEF (\text{AAS})$$

2.4 Useful congruencies

- Rotation
- Translation

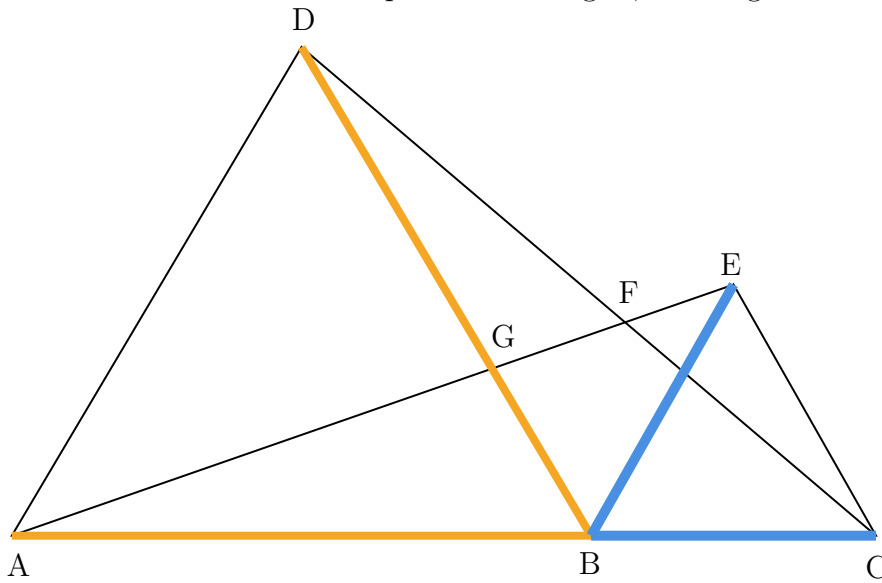
2.5 Properties of congruence

- Equal side length
- Equal angles

2.6 Problems

2.6.1 Problem

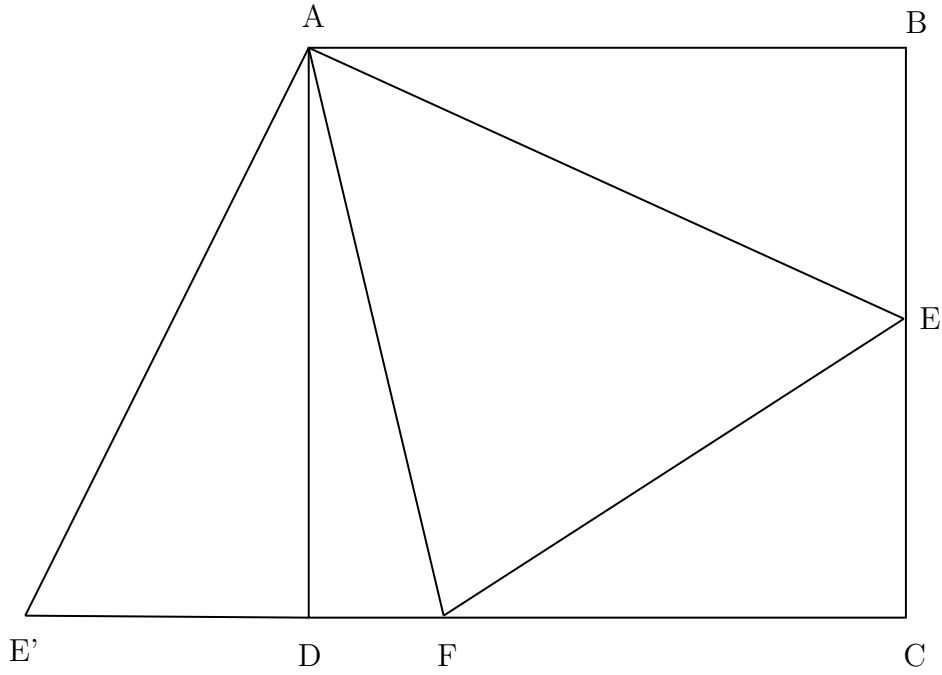
Assume ABD and BCE are equilateral triangles, find angle DFA .



$$\begin{aligned} \because ABD \text{ and } BCE \text{ are equilateral triangles} & \quad \therefore \angle EAB = \angle CDB = x \\ \therefore AB = BD, BC = BE & \quad \therefore \angle DGA = \angle EAB + \angle DBA \\ \angle ABD = \angle CBE = 60^\circ & \quad = \angle CDB + \angle DFA = 60^\circ + x \\ \therefore \angle DBE = 180^\circ - 60^\circ - 60^\circ = 60^\circ & \quad \therefore \angle DFA = 60^\circ \\ \therefore \angle ABE = \angle DBC = 120^\circ & \\ \text{In } \triangle ABE \text{ and } \triangle DBC & \\ \{ AB = DB & \\ \angle ABE = \angle DBC & \\ BE = BC & \\ \therefore ABE \cong \triangle DBC (SAS) & \end{aligned}$$

2.6.2 Problem

Given $ABCD$ is a square, and $\angle EAF = 45^\circ$, prove AE bisect BEF .



Extend CD to E' such that $E'D = BE$

Connect $E'A$

$\because ABCD$ is a square

$\therefore AD = AB$

$\angle B = \angle ADE' = 90^\circ$

In $\triangle ADE'$ and $\triangle ABE$

$$\begin{cases} AD = AB \\ \angle B = \angle ADE' \\ DE' = BE \end{cases}$$

$\therefore \triangle ADE' \cong \triangle ABE (SAS)$

$\therefore AE' = AE$

$\angle AEB = \angle E'$

$\angle E'AD = \angle EAB$

$\therefore \angle E'AF = \angle E'AD + \angle DAF$

$= \angle EAB + \angle DAF = 90^\circ - \angle EAF = 45^\circ$

In $\triangle AFE'$ and $\triangle AFE$

$$\begin{cases} AF = AF \\ \angle EAF = \angle E'AF = 45^\circ \\ AE' = AE \end{cases}$$

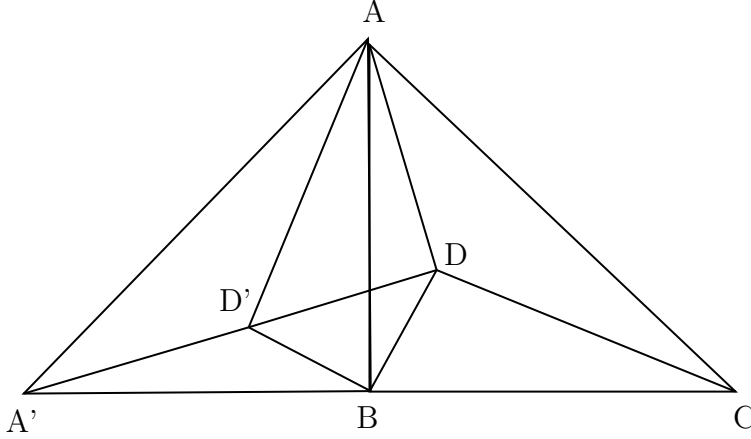
$\therefore \triangle AFE' \cong \triangle AFE (SAS)$

$\therefore \angle AEF = \angle E' = \angle AEB$

$\therefore AE$ bisect BEF

2.6.3 Problem

Given $AB = BC$ and angle $ABC = 90^\circ$, $AD = \sqrt{5}$, $BD = \sqrt{2}$, $DC = 3$, find angle ADB .



Rotate ABC 90° counterclockwise on point B , so that AB and BC overlap.

$\angle ABD + \angle DBC = 90^\circ$, therefore $\angle DBD' = \angle ABD + \angle D'BA = 90^\circ$

Since BDD' is an isosceles with $\angle DBD' = 90^\circ$.

$\angle BD'D = \angle BD'B = \frac{180^\circ - 90^\circ}{2} = 45^\circ$

$BD = BD' = \sqrt{2}$, so $DD' = \sqrt{a^2 + b^2} = c \Rightarrow \sqrt{\sqrt{2}^2 + \sqrt{2}^2} = 2$

Applying the pythagorean theorem can determine whether $\angle ADD'$ is 90° or not:

$a^2 + b^2 = c^2 \Rightarrow DD'^2 + DA^2 = D'A^2 \Rightarrow 2^2 + \sqrt{5}^2 = 3^2 \Rightarrow 9 = 9$

Therefore, $ADD'D$ is a right angle triangle, with $\angle ADD'$ being 90° .

Therefore, $\angle ADB = \angle ADD' + \angle D'DB = 90^\circ + 45^\circ = 135^\circ$.

3 Similarity

3.1 Definition

In Euclidean geometry, two objects are similar if they have the same shape, or one has the same shape as the mirror image of the other.

3.2 Method to Prove similar triangles

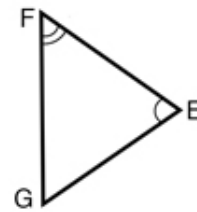
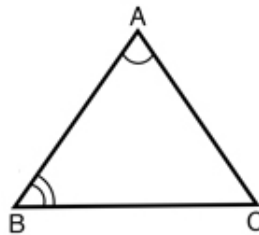
Similar Triangles Rules

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Angle-Angle (AA)

If $\angle CAB \cong \angle GEF$ &
 $\angle ABC \cong \angle EFG$, then

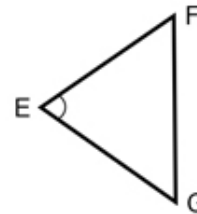
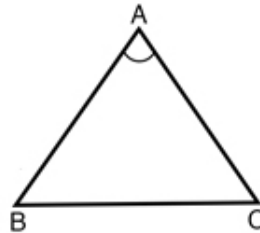
$$\triangle ABC \sim \triangle EFG$$



Side-Angle-Side (SAS)

If $\angle CAB \cong \angle FEG$ &
 $\frac{AB}{EG} = \frac{AC}{EF}$, then

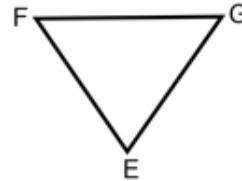
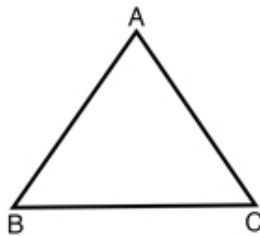
$$\triangle ABC \sim \triangle EFG$$



Side-Side-Side (SSS)

If $\frac{AB}{EF} = \frac{BC}{FG} = \frac{AC}{EG}$

$$\triangle ABC \sim \triangle EFG$$



3.3 Properties of similar triangles

- Ratio of corresponding sides
- Equal angles

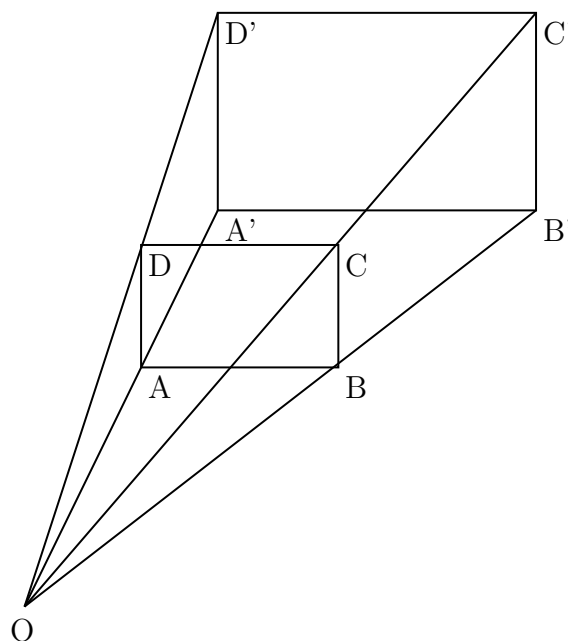
4 Homothety

4.1 Definition

A transformation determined by having a point of origin (O), and a ratio of which points from the origin are scaled to.

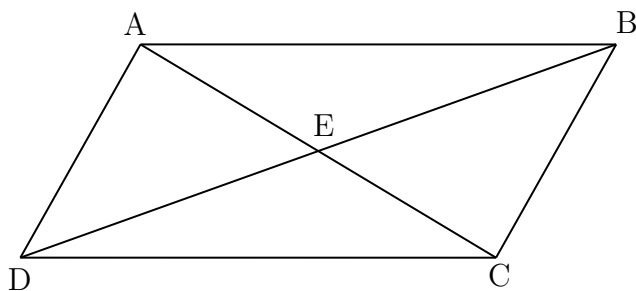
4.2 Properties

- Shapes will be similar, not congruent, with only its size varying.
- The ratio is a set value and applies to all points in relation to the origin.
- (Referencing Figure 4.1) DC is parallel to D'C'. This applies to all other points and their corresponding points that have been scaled from the origin by a set factor.
- All angles remain unchanged and the ratio between two line segments stays constant.
- If two triangles' sides are parallel to each other, the lines connecting the corresponding points are concurrent.



5 Quadrilaterals

5.1 Parallelogram



5.1.1 Definition

A quadrilateral with both opposite sides parallel to each other.

5.1.2 Properties

- Opposite side lengths are congruent
- When drawing a line between the opposite points, the intersections are mid-points for both lines ($AE = EC$, therefore E is the midpoint of AC. $BE = ED$,

therefore...).

- Angle BAC is the same as ACD (internal angle theorem). This is also the same with angles DAC and ACB.

5.2 Rhombus

5.2.1 Definition

A parallelogram with congruent side lengths for all four sides.

5.2.2 Properties

Same properties as a parallelogram, except all side lengths are the same, all the internal lines are the same, and the intersection of diagonals forms 90 degree angles.

5.3 Rectangle

5.3.1 Definition

A parallelogram that has all four angles being 90 degrees each.

5.3.2 Properties

- Has four sides and four vertices.
- Opposite sides are equal.
- Opposite sides are parallel.
- All four angles are right angles.
- Diagonals are equal and bisect each other.

5.4 Square

5.4.1 Definition

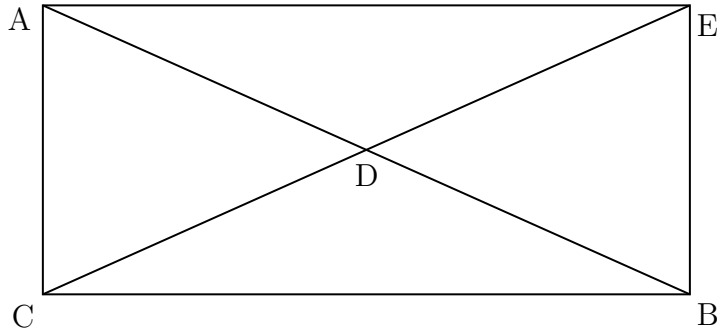
A rhombus with each angle being 90 degrees to each other.

5.4.2 Properties

- Properties of rectangle.
- Properties of rhombus.

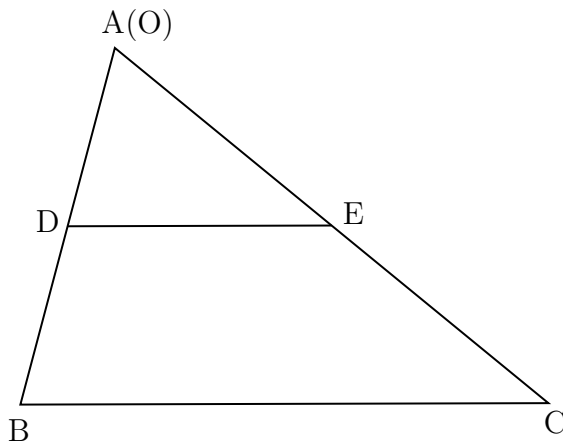
6 Midpoint

6.1 Midpoint of a right triangle



By rotating ABC 180 degrees on point D , a rectangle can be created. The properties of a rectangle states that all internal line segments (AD , CD , BD , and ED) are all congruent. Therefore, the midpoint of a right triangle creates AD , CD , and BD , which all have the same side lengths.

6.2 Midsegment

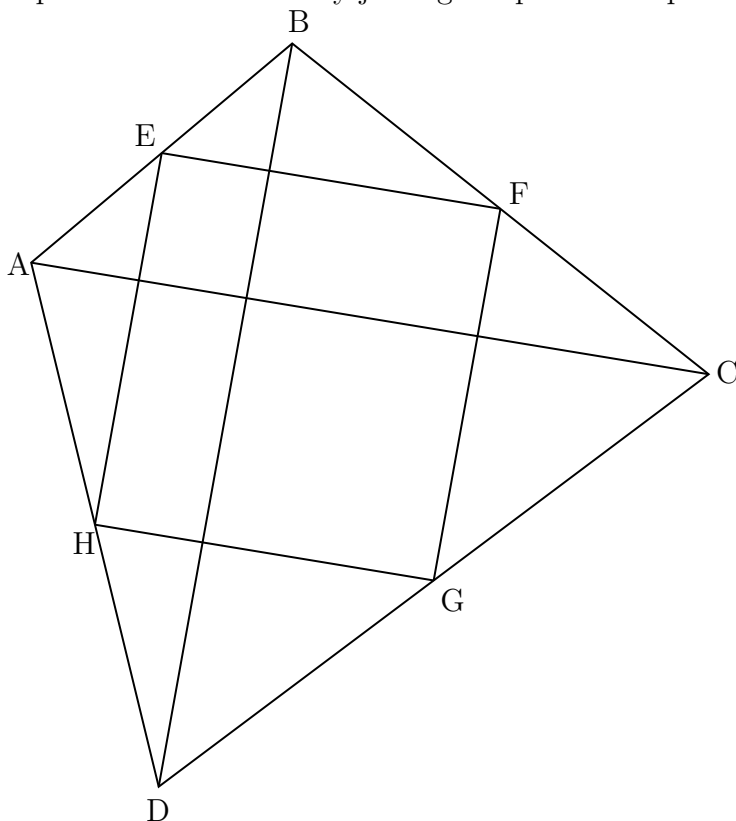


BC is parallel to DE since D and E are extended to B and C , respectively, by a factor of twice their original side lengths. When referring back to homothety, point A acts as the origin point, therefore making angle ADE and angle ABC the same. Therefore, DE and BC are parallel.

7 Problems

7.1 Problem

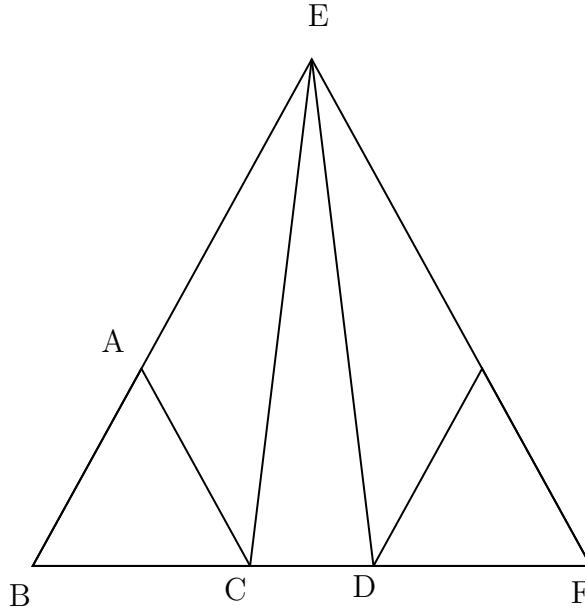
The quadrilateral formed by joining midpoint of a quadrilateral is a parallelogram.



To prove that $EFGH$ is a parallelogram, one can view point A as a point of origin for homothety. As AE is and AH are doubled to become AB and AD , respectively, angle AHE and ADB are the same (ref. 4.2, homothety). Thus, HE and DB are parallel. This analysis can be applied to the three other points (point B , C , and D), where these points are viewed as the point of origin. Therefore, EH and FG can be proven as parallel, and so can EF and HG .

7.2 Problem

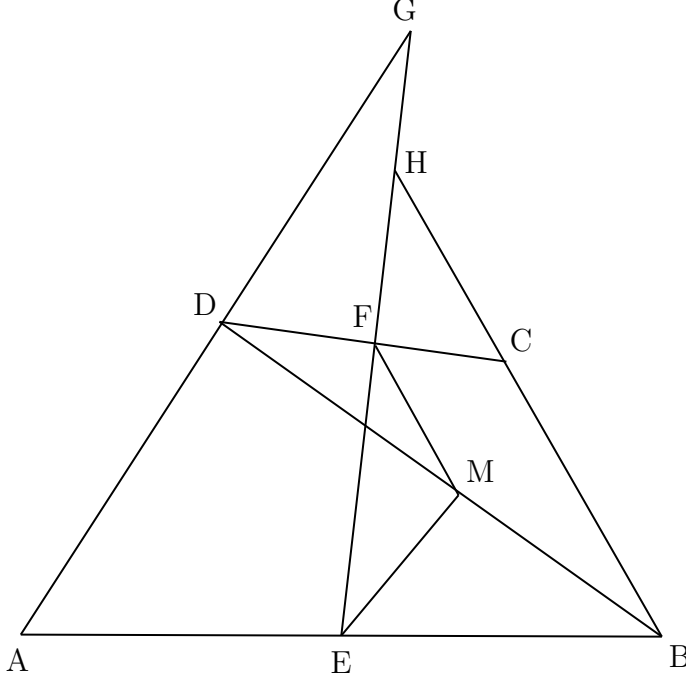
Given equilateral $\triangle ABC$, extend BC to D and BA to E to let $AE = BD$. Join CE and DE . Prove $\triangle ECD = \triangle EDC$.



To prove $\angle ECD = \angle EDC$, extend line BD by a length of BC , to create the new point F . Connect point E to F , to create triangle EFB . Since $AE = BD$, $AE = CF = BD$. Therefore, $BE = BA + AE = BC + CF = BF$. Knowing that EB and BF have the same side lengths, and intersect at an angle of 60 degrees, one can conclude that EFB is an equilateral triangle (EFB is an isosceles, therefore angle BEF and EFB must equal to $(180 - 60)/2 = 120 / 2 = 60$ degrees, therefore equilateral). Once that has been proven, one can prove that BCE and FDE are the same due to SAS, therefore proving that $\angle ECD$ and $\angle EDC$ are congruent.

7.3 Problem

In quadrilateral $ABCD$ with $AD = BC$, E and F are midpoints on AB and CD , respectively. EF meets AD at G , meets BC at H . Prove $\angle DGF = \angle CHF$.



To prove $\angle DGF = \angle CHF$, a reference line has to be drawn from point D to point B , with point M as the midpoint of DB .

Then, connect point F to M , and E to M .

When viewing point B as the point of origin (O), lines EM and AD can be viewed as parallel lines, with EM being half of the length of AD because the distance between OD is double of OM (ref. 4.2., Homothety).

Apply this analysis but onto point D as the origin point, thus proving FM is both parallel and half the length of CB .

Since FM and EM are the same lengths, FME is an isosceles triangle with $\angle EFM = \angle MEF$.

BH and HF are parallel lines that intersect EH , therefore $\angle CHF = \angle MFE$.

DG and EM are parallel lines that intersect with EG , therefore $\angle DGF = \angle GEM$ (Alternate Interior Angle theorem).

Therefore, $\angle DGF = \angle FEM = \angle EFM = \angle FHC \Rightarrow \angle DGF = \angle FHC$.