Number Theory

TSS Math Club

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1 Integers

1.1 Division with Remainder

1.1.1 Example

Find the quotient and remainder when 102 is divided 5.

1.1.2 Example

Find the quotient and remainder when 213 is divided 7.

1.2 Divisibility

1.2.1 Definition

1.2.2 Notation

$a|b$

1.2.3 Theorems

- $a|b$ and $b|c \implies a|c$
- $a|b \implies a|cb$
- $a|b$ and $a|c \implies a|mb + nc$

1.3 GCD and LCM

1.3.1 Defnition

- GCD:
- LCM:

1.3.2 Notations

- GCD:
- LCM:

1.3.3 Example

- (10,5)= [10,5]=
- (3,2)= [3,2]=
- (0,n)= [0,n]=
- (n,1)= [n,1]=

1.3.4 Theorem

If $(a, b) = d$, then $(a/d, b/d) = 1$
Proof:

1.3.5 Theorem

If $a = bq + r$, then $(a, b) = (b, r)$
Proof:

1.3.6 Euclidean Algorithm

1.3.7 Theorem

If$(a, b) = d$, then exist integers x, y such that

$$ax + by = d$$

Proof:

1.3.8 Corollary

If $d|ab$ and $(d, a) = 1$, then $d|b$
Proof:

1.4 Primes and UFD

1.4.1 Primes

Definition:

1.4.2 Lemma

If n is composite, the there is a divider d such that $d \leq n^{\frac{1}{2}}$
Proof:

1.4.3 Lemma

If n is composite, the there is a prime divider p such that $p \leq n^{\frac{1}{2}}$

1.4.4 Euclid's Lemma

If p is a prime and $p|ab$ then $p|a$ or $p|b$.
Proof:

1.4.5 Extended Euclid's Lemma 1

If p is a prime and $p|a_1a_2...a_n$ then $p|a_i$.

1.4.6 Extended Euclid's Lemma 2

If p and q_i are primes and $p|q_1q_2...q_n$ then $p = q_i$.

1.4.7 $\mathbb{Z}$ is UFD (Unique Factorization Domain)

Any positive integer can be written as a product of primes in one and only one way.
Proof:

1.4.8 GCD and LCM in Terms of Factorization

1.4.9 Theorem

$(a, b)[a, b] = ab$

1.4.10 Theorem

Number of divisor $d(n) =$

2 Diophantine Equations

2.1 Definition

2.2 Use Divisibility

2.2.1 Example

Given x, y are integers and $xy = 30$, find ordered pair (x, y).

2.2.2 Example

Given x, y are integers and

$$y = \frac{x^3 + 7x - 10}{x + 3},$$

find ordered pair (x, y).

2.2.3 Simon's Favourite Factoring Trick

Given x, y are integers and

$$3x + xy + 3y + 31 = 0,$$

find ordered pair (x, y).

2.3 Solve Linear Diophantine Equations

2.3.1 Definition

Solve $ax + by = c$ for integers x, y.

2.3.2 Theorem

For the equation above, if $(a, b)|c$, then there are infinite number of solutions. If $(a, b) \nmid c$, then there is no solution.

2.3.3 Example

Solve $3x + 4y = 10$.

2.3.4 Example

Solve $8x + 4y = 6$.

2.3.5 Example

Solve $6x + 9y = 24$.

3 Congruences and Modulo

3.1 Definition

If a is congruent to b modulo m ($a \equiv b\ (m)$) or ($a \equiv b \pmod{m}$), then $m|a - b$.

3.2 Congruences and Remainder

3.2.1 Theorem

Every integer is congruent m to exactly one of $0, 1, ..., m - 1$.

3.2.2 Theorem

$a \equiv b\ (m)$ iff a and b leave the same remainder on division by m.

3.3 Operations under modulo

3.3.1 Lemma

- $a \equiv a\ (m)$.
- If $a \equiv b\ (m)$, then $b \equiv a\ (m)$.
- If $a \equiv b\ (m)$ and $c \equiv d\ (m)$, then $a + b \equiv c + d\ (m)$.
- If $a \equiv b\ (m)$ and $c \equiv d\ (m)$, then $ab \equiv cd\ (m)$.

3.3.2 Theorem

If $ac \equiv bc\ (m)$ and $(c, m) = 1$, then $a \equiv b\ (m)$

3.3.3 Theorem

If $ac \equiv bc\ (m)$ and $(c, m) = d$, then $a \equiv b\ (m/d)$

3.4 Problems

3.4.1 Problem

Find the least residue of 1492 (mod 4), (mod 10), (mod 101).

3.4.2 Problem

Solve $2x \equiv 4\ (6)$.

3.4.3 Problem

Prove $m^2 \equiv$ 0 or 1 (4)

3.4.4 Problem

Solve $m^2 + n^2 = 1023$

3.4.5 Problem

Show every integer is congruent to (mod 9) to the sum of its digits.

4 Linear Congruences

We will try to solve the linear equation $ax \equiv b \pmod{m}$ in this section.

4.1 General Theory

4.1.1 Theorem

If $(a, m) \nmid b$, then $ax \equiv b\ (m)$ has no solutions.

4.1.2 Theorem

If $(a, m) \nmid 1$, then $ax \equiv b\ (m)$ has exactly one solution mod m.

4.1.3 Theorem

If $(a, m) \nmid d$, then $ax \equiv b\ (m)$ has exactly one solution mod m/d.

4.2 Problems

4.2.1 Problem

Solve $2x \equiv 1\ (17)$

4.2.2 Problem

Solve $3x \equiv 1\ (17)$

4.2.3 Problem

Solve $15x + 16y = 17$

4.3 Chinese Remainder Theorem (CRT)

If the n_i are pairwise coprime, and if $a_1, ..., a_k$ are any integers, then the system

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ &\vdots \\ x &\equiv a_k \pmod{n_k} \end{aligned}$$

has one solution mod $N = n_1 n_2 ... n_k$.

4.3.1 Example

Solve:

$$\begin{aligned} x &\equiv 1 \pmod 2 \\ 4x &\equiv 3 \pmod 5 \end{aligned}$$

4.3.2 Problem

Find the remainder when divided by 10 of the following:

$$4^{4^{4^{4^{4^{4^{4^{4...^{4}}}}}}}}$$

(There are 2023 4's in total).

5 Wilson's, Fermat's, Euler's Theorems

5.1 Wilson's Theorem

A natural number $n > 1$ is a prime number if and only if the product of all the positive integers less than n is one less than a multiple of n or

$$(n-1)! \equiv -1 \pmod{n}.$$

5.2 Fermat's Little Theorem

If a is not divisible by the prime p, then

$$a^{p-1} \equiv 1 \pmod p$$

5.2.1 Example

What is the least residue of $1945^8 \pmod 7$

5.2.2 Example

What is the least residue of $2025^22 \pmod{11}$

5.3 Euler's Theorem

5.3.1 Euler's Totient Function

Euler's totient function $\varphi(n)$ counts the positive integers up to a given integer n that are relatively prime to n.

5.3.2 Example

Find $\varphi(24)$

5.3.3 Euler's Totient Function is Multiplicative

If $(a, b) = 1$, then $\varphi(ab) = \varphi(a)\varphi(b)$.

5.3.4 Example

Find $\varphi(2)$, $\varphi(5)$, $\varphi(10)$.

5.3.5 Euler's Totient Function for p^n

$\varphi(p^n) = p^n - p^{n-1}$

5.3.6 Euler's Totient Function General Formula

5.3.7 Euler's Theorem

If $(a, m) = 1$, then

$$a^{\varphi(m)} \equiv 1 \pmod{m}$$

5.3.8 Example

What is the least residue of $2023^{41} \pmod{100}$