Geometry 4 - Analytic Geomtry

TSS Math Club

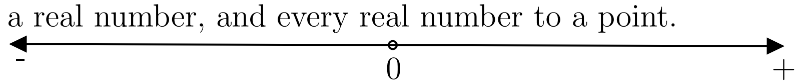
Nov 2022

1 Preliminary

1.1 Real Line

1.1.1 Definition

A number line is a picture of a graduated straight line that serves as visual representation of the real numbers. Every point of a number line is assumed to correspond to a real number, and every real number to a point.



1.2 Ordered Pair

1.2.1 Definition

Informal:

> For any two objects a and b, the ordered pair (a, b) is a notation specifying the two objects a and b, in that order.

Formal:

> (a,b)={{a},{a,b}}

1.2.2 Property

$$(a, b) = (c, d) \iff a = c \wedge b = d$$

1.3 Cartesian Product

1.3.1 Definition

The Cartesian product of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where a is in A and b is in B.

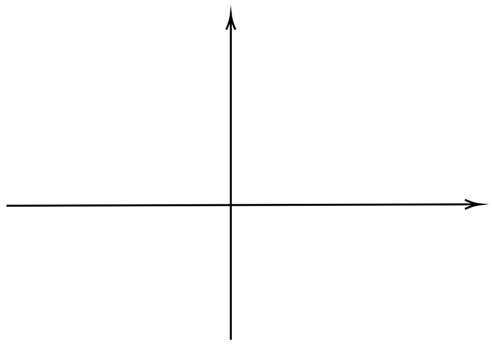
$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

2 Cartesian Plane

2.1 Definition

In Mathematics, the cartesian plane is defined as a two-dimensional coordinate plane, which is formed by the intersection of the x-axis and y-axis. The x-axis and y-axis intersect perpendicular to each other at the point called the origin.

2.2 Visual Representation

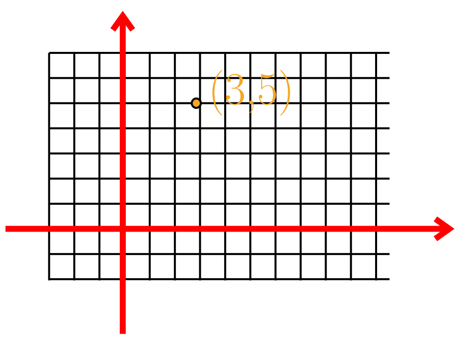


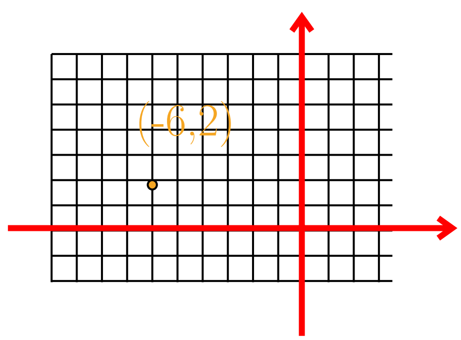
2.3 Point

2.3.1 Definition

A point is a primitive notion that models an exact location in space, and has no length, width, or thickness.

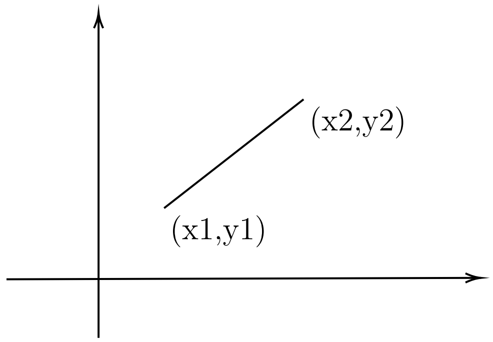
2.3.2 Plot points





2.4 Metric on the Plane

2.4.1 Distance formula



$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

2.4.2 Example

Find the distance between (1,3) and (6,7).

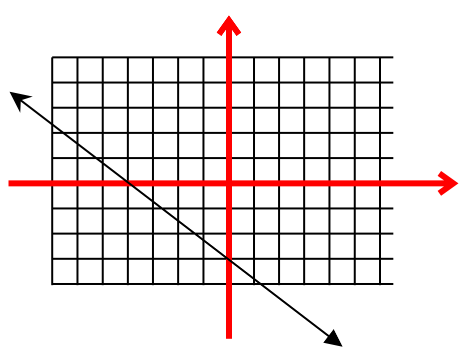
$$d = \sqrt{(1-6)^2 + (3-7)^2} = \sqrt{5^2 + 4^2} = \sqrt{41}$$

2.5 Line

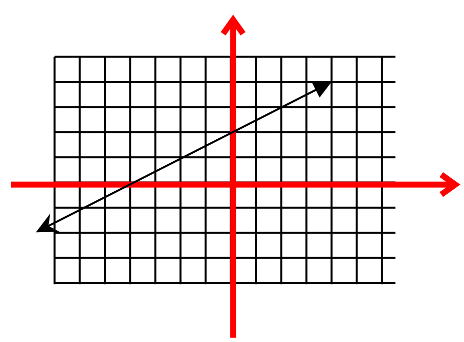
2.5.1 General Formula

$$ax + by + c = 0$$

2.5.2 Examples



3x+4y+12=0



x-2y+4=0

2.6 Circle

2.6.1 General Formula

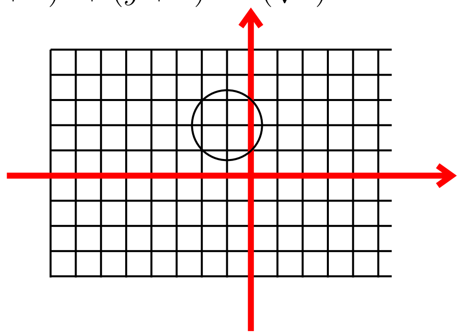
$$(x-a)^2 + (y-b)^2 = r^2$$

or

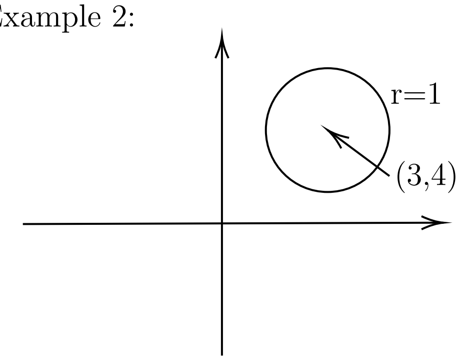
$$x^2 + y^2 + ax + by + c = 0$$

2.6.2 Examples

Example 1:
$x^2 + 2x + y^2 + 4x + 2 = 0$
$x^2 + 2x + 1 + y^2 + 4x + 4 = 3$
$(x+1)^2 + (y+2)^2 = (\sqrt{3})^2$



Example 2:



$(x-3)^2 + (y-4)^2 = 1$ or $x^2 - 6x + y^2 - 8y + 24 = 0$

2.7 Point to Line Distance Formula

The distance between the line $ax + by + c = 0$ and point (x_1, y_1) is

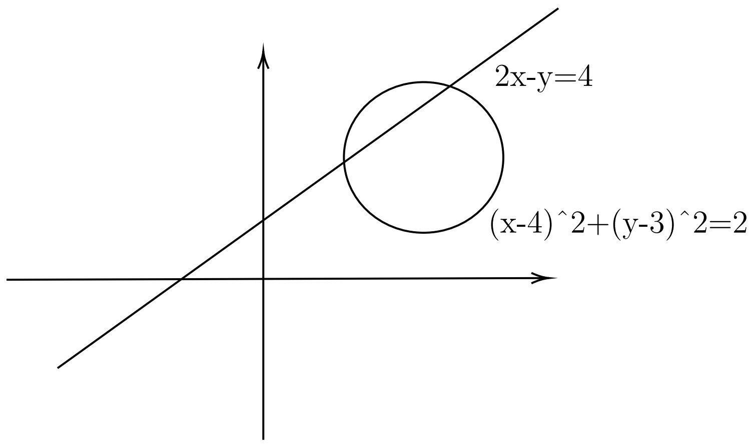
$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

2.8 Intersection

2.8.1 How to find intersection between two curve?

Solve the system of equations.

2.8.2 Example



Sub $y = 2x - 4$ into $(x-4)^2 + (y-3)^2 = 2$,
we get $(x-4)^2 + (2x-4-3)^2 = 2$.
After solving, we get $x = 3$ or $\frac{21}{5}$.
Therefore, the intersections are (3,2) and (21/5,22/5).

2.8.3 Find the Radical Axis of Two Circles

Definition: The line that passes through the intersections of the circles.
Find the radical axis bewteen $x^2 + y^2 = 5$ and $x^2 + 3x + y^2 - 7y + 3 = 0$.

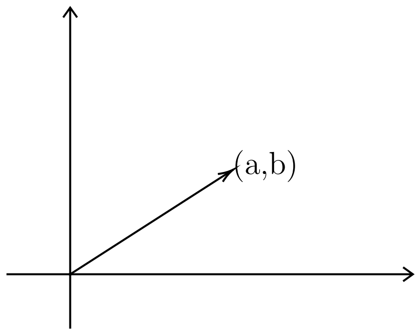
Let P and Q be the intersection, Q and P must satisfy both $x^2 + y^2 = 5$ and $x^2 + 3x + y^2 - 7y + 3 = 0$, therefore the difference $3x - 7y + 3 = -5$.
Since this line $3x - 7y + 8 = 0$ passes through P and Q, it must be the radical axis of two circles.

3 Vector

3.1 Definition

"A quality that has both magnitude and direction."-physicist.
or an ordered pair (a,b)

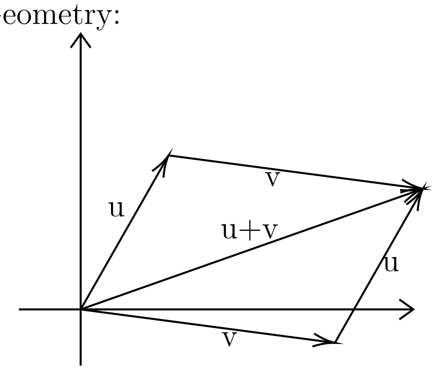
3.2 Visual Representation



3.3 Addition, Substraction and Scalar Multiplication of Vectors

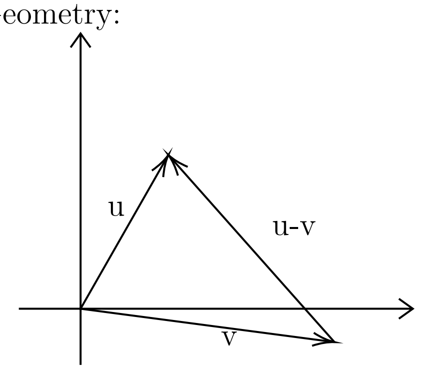
3.3.1 Addition of Vectors

Algebra: $(a, b) + (c, d) = (a + c, b + d)$
Geometry:



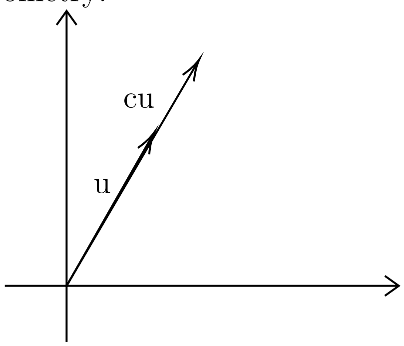
3.3.2 Substraction of Vectors

Algebra:$(a, b) - (c, d) = (a - c, b - d)$
Geometry:



3.3.3 Scalar Multiplication of Vector

Algebra: $c(a,b) = (ac, bc)$
Geometry:



3.4 Dot Product

3.4.1 Definition: Dot Product on 2D

If $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then

$$x \cdot y = x_1 y_1 + x_2 y_2$$

3.4.2 Property: Dot Product

- positivity: $v \cdot v \geq 0$
- definiteness: $v \cdot v = 0$ iff $v = 0$
- additivity: $(v + u) \cdot (w) = v \cdot w + u \cdot w$ or
- homogeneity: $c(u \cdot v) = (cu) \cdot v$
- symmetry: $u \cdot v = v \cdot u$

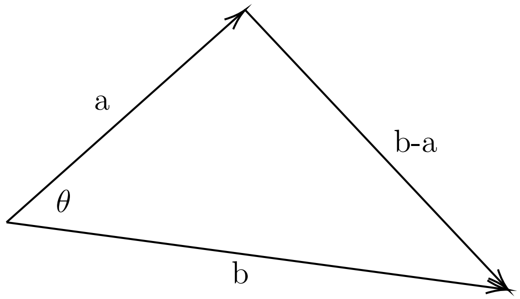
3.4.3 Dot Product and Metric

$v \cdot v = |v|^2$

3.4.4 Penpendicularity

$v \cdot u = 0$ if v and u are perpendicular to each other.

3.4.5 Dot Product and Cosine Law



$$(b-a)\cdot(b-a) = b\cdot b + a\cdot a - 2a\cdot b$$

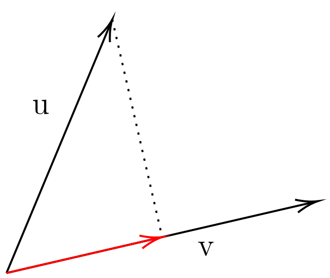
Compare with cosine law:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Therefore,

$$a\cdot b = |a||b|\cos(\theta)$$

3.4.6 Dot Product as Projection



The projection of u on v is

$$\frac{u\cdot v}{|v|}$$

3.4.7 Problem (1975 USAMO Q2)

Let A, B, C, D denote four points in space and AB the distance between A and B, and so on. Show that

$$AC^2 + BD^2 + AD^2 + BC^2 \geq AB^2 + CD^2.$$

Let a, b, c, d correspond to the position vectors of points A, B, C, and D, respectively, with respect to an arbitrary origin O. Let us also for simplicity define $a^2 = a{\cdot}a = ||a||^2$, where $||a||$ is the magnitude of vector a. Because squares are non-negative, a^2 is non-negative for all vectors a. Thus,

$$(a + b - c - d)^2 \geq 0$$

Because dot product is linear, we expand to obtain

$$a^2 + b^2 + c^2 + d^2 + 2a \cdot b + 2c \cdot d - 2a \cdot c - 2a \cdot d - 2b \cdot c - 2c \cdot d \geq 0,$$

from which we add $a^2 + b^2 + c^2 + d^2$ to both sides, rearrange, and complete the square to get

$$(a - c)^2 + (a - d)^2 + (b - c)^2 + (b - d)^2 \geq (a - b)^2 + (c - d)^2.$$

As $(a - b)^2 = ||a - b||^2 = ||AB||^2 = AB^2$ and likewise for the others,

$$AC^2 + AD^2 + BC^2 + BD^2 \geq AB^2 + CD^2,$$

which is what we wanted to prove.

3.5 Determinant

3.5.1 Definition

The area of the parallelogram formed by 2 vectors, namely (a, c) and (b, d).

3.5.2 Formula

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3.5.3 3D Determinant and Area of a Triangle

Definition: The volume of the parallelepiped formed by 3 vectors, namely $(a, d, h), (b, e, i), (c, f, j)$.
Formula:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ h & i & j \end{vmatrix} = a \begin{vmatrix} e & f \\ i & j \end{vmatrix} - b \begin{vmatrix} d & f \\ h & j \end{vmatrix} + c \begin{vmatrix} d & e \\ h & i \end{vmatrix}$$

Area of a Triangle with Vertex $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ is

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

3.5.4 Shoelace Theorem

Suppose the polygon P has vertices (a_1, b_1), (a_2, b_2), ... , (a_n, b_n), listed in clockwise order. Then the area (A) of P is

$$A = \frac{1}{2} \left| \sum_{i=1}^{n} \det \begin{pmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{pmatrix} \right|$$