Number Theory

TSS Math Club

March 2023

1 Integers

1.1 Division with Remainder

1.1.1 Example

Find the quotient and remainder when 102 is divided 5.

1.1.2 Example

Find the quotient and remainder when 213 is divided 7.

1.2 Divisibility

1.2.1 Definition

1.2.2 Notation

$a|b$

1.2.3 Theorems

- $a|b$ and $b|c \implies a|c$
- $a|b \implies a|cb$
- $a|b$ and $a|c \implies a|mb+nc$

1.3 GCD and LCM

1.3.1 Defnition

- GCD:
- LCM:

1.3.2 Notations

- GCD:
- LCM:

1.3.3 Example

- (10,5)= [10,5]=
- (3,2)= [3,2]=
- (0,n)= [0,n]=
- (n,1)= [n,1]=

1.3.4 Theorem

If $(a, b) = d$, then $(a/d, b/d) = 1$
Proof:

1.3.5 Theorem

If $a = bq + r$, then $(a, b) = (b, r)$
Proof:

1.3.6 Euclidean Algorithm

1.3.7 Theorem

If$(a, b) = d$, then exist integers x, y such that

$$ax + by = d$$

Proof:

1.3.8 Corollary

If $d|ab$ and $(d, a) = 1$, then $d|b$
Proof:

1.4 Primes and UFD

1.4.1 Primes

Definition:

1.4.2 Lemma

If n is composite, the there is a divider d such that $d \leq n^{\frac{1}{2}}$
Proof:

1.4.3 Lemma

If n is composite, the there is a prime divider p such that $p \leq n^{\frac{1}{2}}$

1.4.4 Euclid's Lemma

If p is a prime and $p|ab$ then $p|a$ or $p|b$.
Proof:

1.4.5 Extended Euclid's Lemma 1

If p is a prime and $p|a_1a_2...a_n$ then $p|a_i$.

1.4.6 Extended Euclid's Lemma 2

If p and q_i are primes and $p|q_1q_2...q_n$ then $p = q_i$.

1.4.7 $\mathbb{Z}$ is UFD (Unique Factorization Domain)

Any positive integer can be written as a product of primes in one and only one way.
Proof:

1.4.8 GCD and LCM in Terms of Factorization

1.4.9 Theorem

$(a, b)[a, b] = ab$

1.4.10 Theorem

Number of divisor $d(n) =$

2 Diophantine Equations

2.1 Definition

2.2 Use Divisibility

2.2.1 Example

Given x, y are integers and $xy = 30$, find ordered pair (x, y).

2.2.2 Example

Given x, y are integers and

$$y = \frac{x^3 + 7x - 10}{x + 3},$$

find ordered pair (x, y).

2.2.3 Simon's Favourite Factoring Trick

Given x, y are integers and

$$3x + xy + 3y + 31 = 0,$$

find ordered pair (x, y).

2.3 Solve Linear Diophantine Equations

2.3.1 Definition

Solve $ax + by = c$ for integers x, y.

2.3.2 Theorem

For the equation above, if $(a, b)|c$, then there are infinite number of solutions. If $(a, b) \nmid c$, then there is no solution.

2.3.3 Example

Solve $3x + 4y = 10$.

2.3.4 Example

Solve $8x + 4y = 6$.

2.3.5 Example

Solve $6x + 9y = 24$.

3 Congruences and Modulo

3.1 Definition

If a is congruent to b modulo m ($a \equiv b\ (m)$) or ($a \equiv b \pmod{m}$), then $m | a - b$.

3.2 Congruences and Remainder

3.2.1 Theorem

Every integer is congruent (m) to exactly one of $0, 1, ..., m - 1$.

3.2.2 Theorem

$a \equiv b\ (m)$ iff a and b leave the same remainder on division by m.

3.3 Operations under modulo

3.3.1 Lemma

- $a \equiv a\ (m)$.
- If $a \equiv b\ (m)$, then $b \equiv a\ (m)$.
- If $a \equiv b\ (m)$ and $c \equiv d\ (m)$, then $a + b \equiv c + d\ (m)$.
- If $a \equiv b\ (m)$ and $c \equiv d\ (m)$, then $ab \equiv cd\ (m)$.

4 Linear Congruences