

# Algebra 1 - Set Theory and Logic

TSS Math Club

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## 1 Four Fundamental Proof Techniques:

### 1.1 Direct Proof (Proof by Construction)

In a constructive proof one attempts to demonstrate  $P \implies Q$  directly. This is the simplest and easiest method of proof available to us. There are only two steps to a direct proof (the second step is, of course, the tricky part):

- Assume that  $P$  is true.
- Use  $P$  to show that  $Q$  must be true.

#### 1.1.1 Example

Prove odd number + odd number = even number.

### 1.2 Proof by Contradiction

The proof by contradiction is grounded in the fact that any proposition must be either true or false, but not both true and false at the same time. The method of proof by contradiction:

- Assume that  $P$  is true.
- Assume that  $\neg Q$  is true
- Use  $P$  and  $\neg Q$  to demonstrate a contradiction.

#### 1.2.1 Example

Prove  $\sqrt{2}$  is irrational number.

### 1.3 Proof by Induction

Proof by induction is a very powerful method in which we use recursion to demonstrate an infinite number of facts in a finite amount of space. In order to prove by induction, you need to:

- Show that a propositional form  $P(x)$  is true for some basis case.
- Assume that  $P(n)$  is true for some  $n$ , and show that this implies that  $P(n + 1)$  is true.
- Then, by the principle of induction, the propositional form  $P(x)$  is true for all  $n$  greater or equal to the basis case.

#### 1.3.1 Example

Prove  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$

### 1.4 Proof by Contrapositive

From first-order logic we know that the implication  $P \implies Q$  is equivalent to  $\neg P \implies \neg Q$ . The second proposition is called the contrapositive of the first proposition. By saying that the two propositions are equivalent we mean that if one can prove  $P \implies Q$  then they have also proved  $\neg P \implies \neg Q$ , and vice versa.

#### 1.4.1 Example

Prove  $n^2$  even implies  $n$  even.

## 2 Set Theory

### 2.1 Definition of a Set:

### 2.2 Construct a Set

#### 2.2.1 Construct a Set from a List of Objects:

#### 2.2.2 Construct a Set from a Given Set by Setting Restrictions:

### 2.3 Basic Sets:

### 2.4 Relations:

#### 2.4.1 Definition: $x \in A$

#### 2.4.2 Definition: $A \subset B$

#### 2.4.3 Definition: $A = B$

#### 2.4.4 Lemma: $A = B \iff A \subset B \wedge B \subset A$

## **2.5 Operations:**

**2.5.1 Definition:**  $A - B$

**2.5.2 Definition:**  $A \cup B$

**2.5.3 Definition:**  $A \cap B$

## **2.6 Ordered Pair:**

**2.6.1 Definition:**

**2.6.2 Property:**

## **2.7 Cartesian Product**

**2.7.1 Definition:**

## **3 Relation and Function**

### **3.1 Relation**

#### **3.1.1 Definition:**

### **3.2 Function**

#### **3.2.1 Definition:**

- Function
- Domain
- Codomain
- Range

#### **3.2.2 Injection (one-to-one):**

#### **3.2.3 Surjection (onto):**

#### **3.2.4 Bijection (one-to-one and onto):**

#### **3.2.5 Inverse Function:**

## 4 Cardinality

### 4.1 Index Set $J_n$

### 4.2 Cardinality of Finite Set

### 4.3 Infinite Countable Set

### 4.4 Infinite Uncountable Set

### 4.5 Two Sets with the Same Cardinality