Geometry 4 - Analytic Geomtry

TSS Math Club

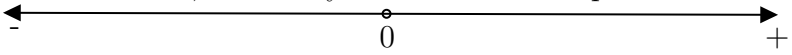
Nov 2022

1 Preliminary

1.1 Real Line

1.1.1 Definition

A number line is a picture of a graduated straight line that serves as visual representation of the real numbers. Every point of a number line is assumed to correspond to a real number, and every real number to a point.



1.2 Ordered Pair

1.2.1 Definition

Informal:

> For any two objects a and b, the ordered pair (a, b) is a notation specifying the two objects a and b, in that order.

Formal:

> (a,b)={{a},{a,b}}

1.2.2 Property

$$(a, b) = (c, d) \iff a = c \wedge b = d$$

1.3 Cartesian Product

1.3.1 Definition

The Cartesian product of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where a is in A and b is in B.

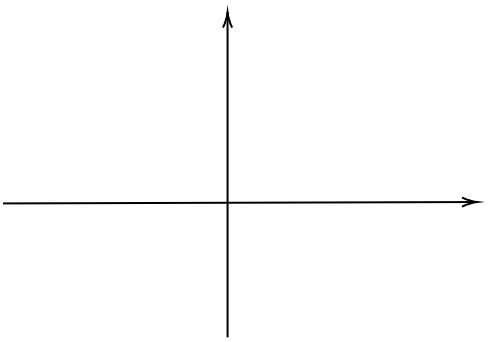
$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

2 Cartesian Plane

2.1 Definition

In Mathematics, the cartesian plane is defined as a two-dimensional coordinate plane, which is formed by the intersection of the x-axis and y-axis. The x-axis and y-axis intersect perpendicular to each other at the point called the origin.

2.2 Visual Representation

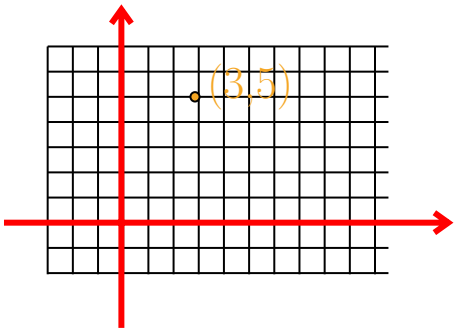


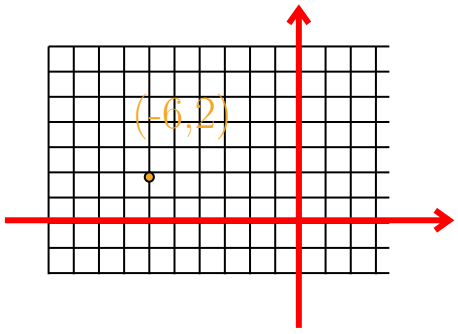
2.3 Point

2.3.1 Definition

A point is a primitive notion that models an exact location in space, and has no length, width, or thickness.

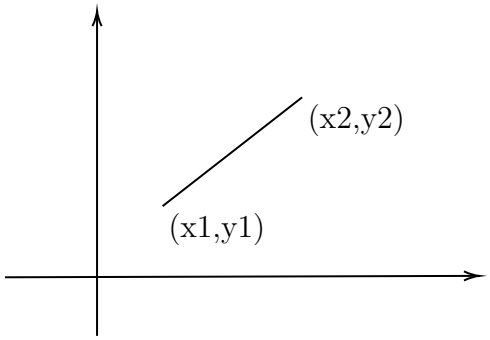
2.3.2 Plot points





2.4 Metric on the Plane

2.4.1 Distance formula



$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

2.4.2 Example

Find the distance between (1,3) and (6,7).

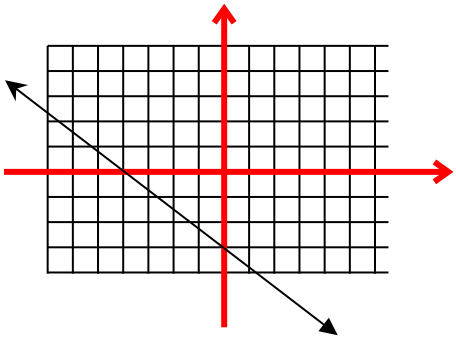
$$d = \sqrt{(1-6)^2 + (3-7)^2} = \sqrt{5^2 + 4^2} = \sqrt{41}$$

2.5 Line

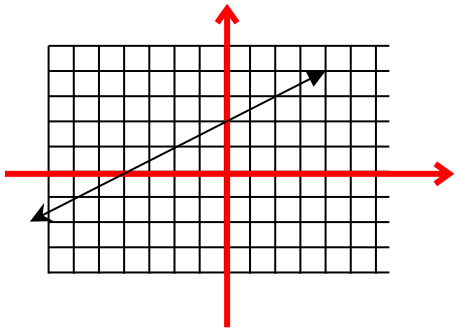
2.5.1 General Formula

$$ax + by + c = 0$$

2.5.2 Examples



3x+4y+12=0



x-2y+4=0

2.6 Circle

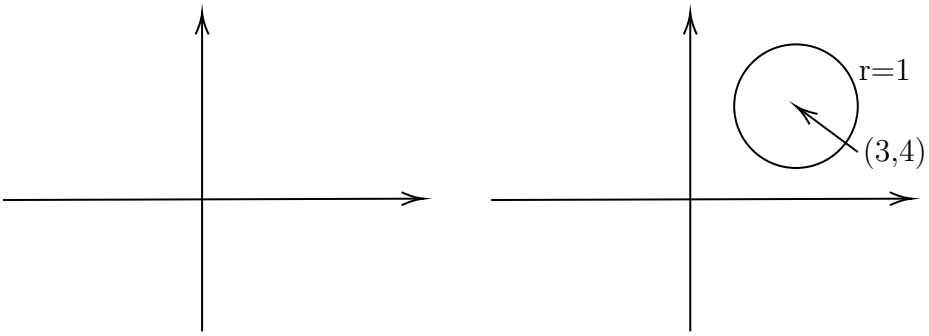
2.6.1 General Formula

$$(x - a)^2 + (y - b)^2 = r^2$$

or

$$x^2 + y^2 + ax + by + c = 0$$

2.6.2 Examples



xˆ2+2x+yˆ2+4x+2=0

2.7 Point to Line Distance Formula

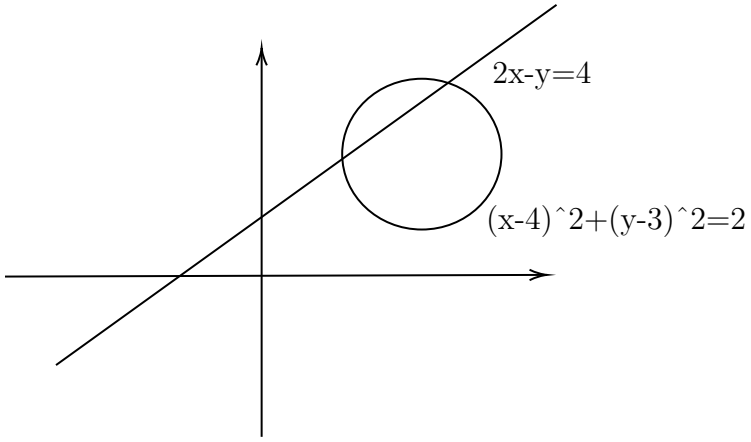
The distance between the line $ax + by + c = 0$ and point (x_1, y_1) is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

2.8 Intersection

2.8.1 How to find intersection between two curve?

2.8.2 Example



2.8.3 Find the Radical Axis of Two Circles

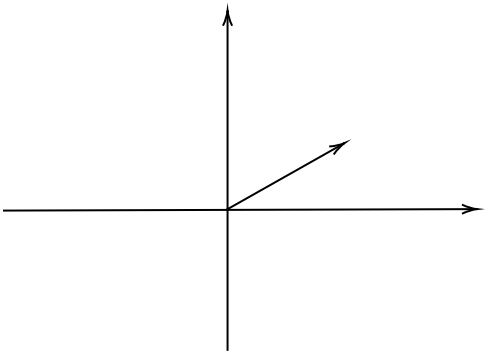
Definition:

Find the radical axis bewteen $x^2 + y^2 = 5$ and $x^2 + 3x + y^2 - 7y + 3 = 0$.

3 Vector

3.1 Definition

3.2 Visual Representation

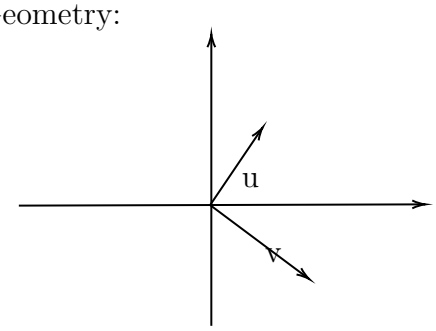


3.3 Addition, Substraction and Scalar Multiplication of Vectors

3.3.1 Addition of Vectors

Algebra:

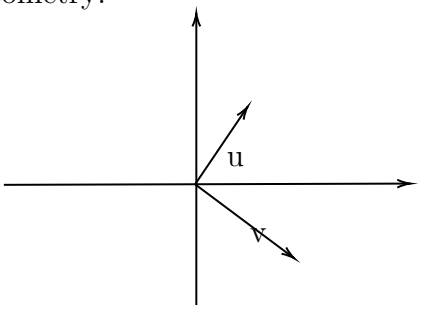
Geometry:



3.3.2 Substraction of Vectors

Algebra:

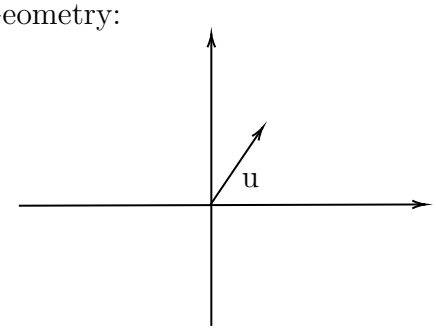
Geometry:



3.3.3 Scalar Multiplication of Vector

Algebra:

Geometry:



3.4 Dot Product

3.4.1 Definition: Dot Product on 2D

If $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then

$$x \cdot y = x_1 y_1 + x_2 y_2$$

3.4.2 Property: Dot Product

- positivity:
- definiteness:
- additivity:
- homogeneity:
- symmetry:

3.4.3 Dot Product and Metric

3.4.4 Penpendicularity

3.4.5 Dot Product and Cosine Law

3.4.6 Dot Product as Projection

3.4.7 Problem (1975 USAMO Q2)

Let A, B, C, D denote four points in space and AB the distance between A and B, and so on. Show that

$$AC^2 + BD^2 + AD^2 + BC^2 \geq AB^2 + CD^2.$$

3.5 Determinant

3.5.1 Definition

3.5.2 Formula

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} =$$

3.5.3 3D Determinant and Area of a Triangle

Definition:

Formula:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ h & i & j \end{vmatrix} = a \begin{vmatrix} e & f \\ i & j \end{vmatrix} - b \begin{vmatrix} d & f \\ h & j \end{vmatrix} + c \begin{vmatrix} d & e \\ h & i \end{vmatrix}$$

Area of a Triangle with Vertex $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ is

3.5.4 Shoelace Theorem

Suppose the polygon P has vertices (a_1, b_1), (a_2, b_2), ... , (a_n, b_n), listed in clockwise order. Then the area (A) of P is

$$A = \frac{1}{2} \left| \sum_{i=1}^{n} \det \begin{pmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{pmatrix} \right|$$

Proof

Appendix: Mathematical Induction

Mathematical Induction is a special way of proving things. It has only 2 steps:

- Step 1. Show it is true for the first one
- Step 2. Show that if any one is true then the next one is true

Then all are true

Example

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$