Algebra 3 - Sequence and Series

TSS Math Club

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1 Sequence

1.1 Definition

A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the first term, a_2 is the second term, and in general a_n is the n-th term.

1.2 Notation

The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n \geq 1}$

1.3 Examples

- $a_n = \frac{1}{n}$
- $a_1 = 1, a_{n+1} = a_n + 2$
- $a_1 = 1, a_2 = 1, a_{n+2} = a_{n+1} + a_n$

1.4 Arithmetic Sequence

1.4.1 Definition

An arithmetic sequence is a sequence where each term increases by adding/subtracting some constant d (the common difference).

$$a_{n+1} = a_n + d$$

1.4.2 Useful Formulae

- $a_n = a_1 + (n-1)d$
- $2a_n = a_{n-1} + a_{n+1}$

1.5 Geometric Sequence

1.5.1 Definition

A geometric sequence is a sequence in which each term is found by multiplying the preceding term by the same value r (common ratio).

$$a_{n+1} = r a_n$$

1.5.2 Useful Formulae

- $a_n = a_1 r^{n-1}$

- $a_n^2 = a_{n-1} a_{n+1}$

1.6 Arithmetico-Geometric Sequence

1.6.1 Definition

In mathematics, arithmetico-geometric sequence is the result of term-by-term multiplication of a geometric progression with the corresponding terms of an arithmetic progression.

$$a_n = [a + (n-1)d] b r^{n-1}$$

1.7 Recursive Sequence

A recursive sequence is a sequence in which terms are defined using one or more previous terms which are given.

1.7.1 Periodic Sequence

Example : $a_1 = x, a_2 = y, a_n = \frac{a_{n-1}+1}{a_{n-2}}$

$a_1 = x, a_2 = y, a_3 = \frac{y+1}{x}, a_4 = \frac{x+y+1}{xy}, a_5 = \frac{x+1}{y}, a_6 = x, a_7 = y, \ldots$

1.7.2 Linear Difference Equation

Example : $a_1 = 1, a_2 = 1, a_{n+2} = a_{n+1} + a_n$
How to solve:

- Step 1: find the characteristic equation $x^2 = x + 1$

- Step 2: Solve the characteristic equation $x_{1,2} = \phi, \frac{1}{\phi}$

- Step 3: the general term $a_n = c_1 x_1^n + c_2 a_2^n$ for some constant c_i

- Step 4: sub a_1 and a_2 in and get c_i

2 Series

2.1 Definition

A series is the cumulative sum of a given sequence of terms.

$$S_n = \sum_{i=1}^{n} a_i, \text{for some} \{a_i\}$$

2.2 Arithmetic Series

Let $\{a_n\}_{n\geq 1}$ be a arithmetic sequence, then

$$S_n = \sum_{i=1}^{n} a_i$$

is the arithmetic series.

2.2.1 General Term

$a_n = \frac{(a_1 + a_n)}{2}$

2.2.2 Prove for $a_n = n$

consider:
1,2,3,4,5,...,n-2,n-1,n
n,n-1,n-2,...,5,4,3,2,1
  If we add vertically, we'll get n+1 for n terms. But because this is for $2S_n$,

$$S_n = \frac{(n+1)n}{2}$$

2.2.3 Property of sum operator

IT IS A LINEAR OPERATOR:

- $\sum(a_i + b_i) = \sum a_i + \sum b_i$
- $\sum ca_i = c \sum a_i$ for constant c

2.2.4 Proof of general term

2.3 Geometric Series

Let $\{a_n\}_{n \geq 1}$ be a geometric sequence, then

$$S_n = \sum_{i=1}^{n} a_i$$

is the geometric series.

2.3.1 General Term

$a_n = \frac{a_1(r^{n+1}-1)}{r-1}$

2.3.2 Prove for $a_n = r^{n-1}$

By definition, we have

$$S_n = 1 + r + r^2 + \cdots + r^{n-1}$$

Now, consider the identity

$$x^n - 1 = (x-1)(x^{n-1} + \cdots + x^2 + x + 1)$$

Therefore,

$$S_n = 1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}$$

2.3.3 Proof of general term

2.3.4 Push to infinity

Note if $-1 < r < 1$, we have $\lim_{n \to \infty} r^n = 0$ Therefore, we have

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a_1(r^{n+1}-1)}{r-1} = \frac{a_1(\lim_{n \to \infty} r^{n+1}-1)}{r-1} = \frac{-a_1}{r-1} = \frac{a_1}{1-r}$$

2.4 Arithmetico-Geometric Series

Let $\{a_n\}_{n \geq 1}$ be a arithmetico-geometric sequence, then

$$S_n = \sum_{i=1}^{n} a_i$$

is the arithmetico-geometric series.

2.5 General term

$x_n = (a_1 + d(n-1))(g_1 \cdot r^{n-1})$

Let S_n represent the sum of the first n terms.

$S_n = a_1 g_1 + (a_1 + d)(g_1 r) + (a_1 + 2d)(g_1 r^2) + \ldots + (a_1 + (n-1)d)(g_1 r^{n-1})$

$S_n = a_1 g_1 + (a_1 g_1 + dg_1)r + (a_1 g_1 + 2dg_1)r^2 + \ldots + (a_1 g_1 + (n-1)dg_1)r^{n-1}$

$rS_n = a_1 g_1 r + (a_1 g_1 + dg_1)r^2 + (a_1 g_1 + 2dg_1)r^3 + \ldots + (a_1 g_1 + (n-1)dg_1)r^n$

$rS_n - S_n = -a_1 g_1 - dg_1 r - dg_1 r^2 - dg_1 r^3 - \ldots - dg_1 r^{n-1} + (a_1 g_1 + (n-1)dg_1)r^n$

$S_n(r-1) = (a_1 + (n-1)d)g_1 r^n - a_1 g_1 - \frac{dg_1 r(r^{n-1}-1)}{r-1}$

$S_n = \frac{(a_1 + (n-1)d)g_1 r^n}{r-1} - \frac{a_1 g_1}{r-1} - \frac{dg_1 r(r^{n-1}-1)}{(r-1)^2} = \frac{a_n g_{n+1}}{r-1} - \frac{x_1}{r-1} - \frac{d(g_{n+1}-g_2)}{(r-1)^2}$

2.6 Telescoping Series

2.6.1 Example

What's the value of

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n}$$

3 Practice Problems

3.1 2021 CIMC Part A Q5

A list of numbers is created using the following rules:

- The first number is 3 and the second number is 4.

- Each number after the second is the result of adding 1 to the previous number and then dividing by the number before that. In other words, for any three consecutive numbers in the list, a, b, c, we have $\frac{b+1}{a}$

What is the smallest positive integer N for which the sum of the first N numbers in the list is equal to an odd integer that is greater than 2021?

3.2   2022 Euclid Q5

A list a_1, a_2, a_3, a_4 of rational numbers is defined so that if one term is equal to r, then the next term is equal to $1 + \frac{1}{1+r}$. If $a_3 = \frac{41}{29}$, what is the value of a1?

3.3   2005 AIME II Q3

An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. The common ratio of the original series is $\frac{m}{n}$ where m and n are relatively prime integers. Find $m + n$.

3.3.1   2022 AMC 12A Q8

The infinite product

$$\sqrt[3]{10} \cdot \sqrt[3]{\sqrt[3]{10}} \cdot \sqrt[3]{\sqrt[3]{\sqrt[3]{10}}} \ldots$$

evaluates to a real number. What is that number?

3.4   2020 AMC 10A Q21

There exists a unique strictly increasing sequence of non-negative integers $a_1 < a_2 < \ldots < a_k$ such that

$$\frac{2^{289} + 1}{2^{17} + 1} = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_k}.$$

What is k?