Geometry 3 - Miscellaneous

TSS Math Club

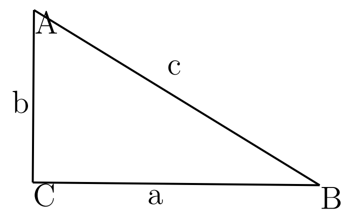
Nov 2022

1 Pythagorean Theorem

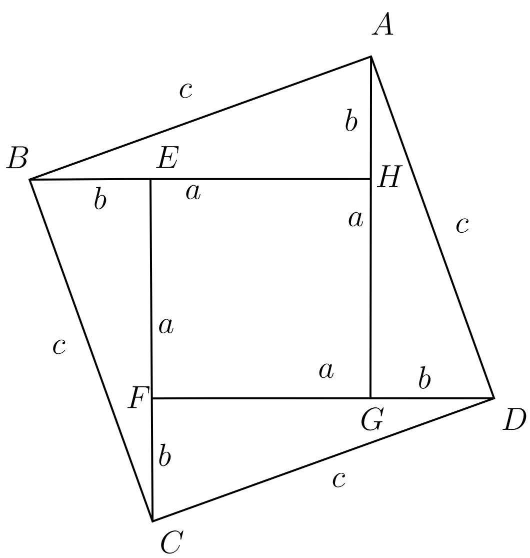
In a right-triangle,

$$a^2 + b^2 = c^2$$

where a and b are two sides and c is the hypotenuse.



1.1 Proof



$[ABCD] = c^2$
$[ABCD] = [EFGH] + 4[AEB]$
$[ABCD] = (a-b)^2 + 4\frac{ab}{2}$
$[ABCD] = a^2 - 2ab + b^2 + 2ab$
$[ABCD] = a^2 + b^2$
Therefore, $a^2 + b^2 = c^2$

2 Trigonometry

2.1 Definitions

Sine or $\sin(\theta)$: A ratio between the opposite side length and the hypotenuse of a triangle.

Cosine or $\cos(\theta)$: A ratio between the adjacent side length and the hypotenuse of a triangle.

Tangent or $\tan(\theta)$: A ratio between the opposite side length and the adjacent side of a triangle.

2.2 Pythagorean Theorem

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

2.3 Triangle Area Formula with Sine

$$S = \frac{ab\sin C}{2}$$

2.3.1 Proof

Since

$$\sin C = \frac{h}{b} \longrightarrow h = b\sin C$$

Therefore,

$$\begin{aligned} S &= \frac{h \times a}{2} \\ &= \frac{\sin C \times b \times a}{2} \\ &= \frac{ab\sin C}{2} \end{aligned}$$

2.4 Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = d$$

2.4.1 Proof

$$S = \frac{ab\sin C}{2} = \frac{bc\sin A}{2}$$
$$a\sin C = c\sin A$$
$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$abc = 4RS$$
$$abd = \frac{4ab\sin C}{2}R$$
$$c = 2\sin C R$$
$$\frac{c}{\sin C} = 2R$$

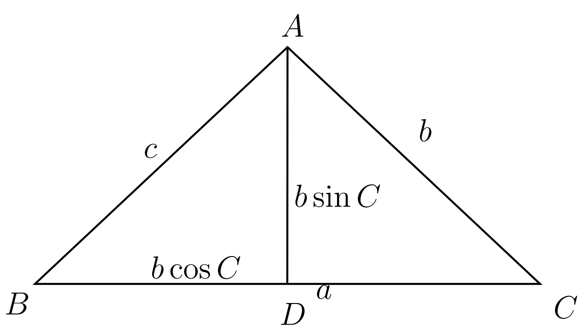
Therefore,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = d$$

2.5 Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

2.5.1 Proof



$$\frac{AD}{AC} = \sin C \longrightarrow AD = b\sin C$$
$$\frac{BC}{AC} = \cos C \longrightarrow BC = b\cos C$$
$$DC = BC - BD = a - b\cos C$$

$$\begin{aligned} c^2 &= (b\sin C)^2 + (a - b\cos C)^2 \\ &= b^2\sin^2 C + a^2 - 2ab\cos C + b^2\cos^2 C \\ &= b^2(\sin^2 C + \cos^2 C) + a^2 - 2ab\cos C \\ &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

2.6 Problem

2.6.1 Heron's Formula

$$S = \sqrt{s(s-a)(s-b)(s-c)}, s = \frac{a+b+c}{2}$$

1:

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

2:

$$\sin^2 C + \cos^2 C = 1$$
$$\sin C = \sqrt{1-\cos^2 C}$$

Substitute 1 into 2:

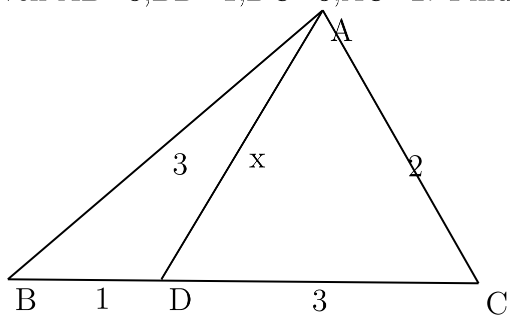
$$\begin{aligned}
\sin C &= \sqrt{1-(\frac{a^2+b^2-c^2}{2ab})^2} \\
&= \frac{\sqrt{(2ab+a^2+b^2-c^2)\times(2ab-a^2-b^2+c^2}}{2ab} \\
&= \frac{\sqrt{(a+b-c)\times(a+b+c)\times(c-a+b)\times(c+a-b)}}{2ab}
\end{aligned}$$

Substitute into $S = \frac{ab\sin C}{2}$:

$$\begin{aligned}
S &= \frac{ab\sqrt{(a+b-c)\times(a+b+c)\times(c-a+b)\times(c+a-b)}}{2ab} \\
&= \sqrt{\frac{(a+b-c)}{2}\times\frac{(a+b+c)}{2}\times\frac{(c-a+b)}{2}\times\frac{(c+a-b)}{2}} \\
&= \sqrt{\frac{(a+b+c-2c)}{2}\times\frac{(a+b+c)}{2}\times\frac{(c+a+b-2a)}{2}\times\frac{(c+a+b-2b)}{2}} \\
&= \sqrt{(\frac{a+b+c}{2}-c)\times(\frac{a+b+c}{2})\times(\frac{a+b+c}{2}-a)\times(\frac{a+b+c}{2}-b)} \\
&= \sqrt{s\times(s-a)\times(s-b)\times(s-c)}
\end{aligned}$$

2.6.2 Problem

Given AB=3,BD=1,DC=3,AC=2. Find AD.



2.6.3 Problem, Euclid 2022 Q8 b)

Consider the following statement:

> There is a triangle that is not equilateral whose side lengths form a geometric sequence, and the measures of whose angles form an arithmetic sequence.

Show that this statement is true by finding such a triangle or prove that it is false by demonstrating that there cannot be such a triangle.

3 Transversals

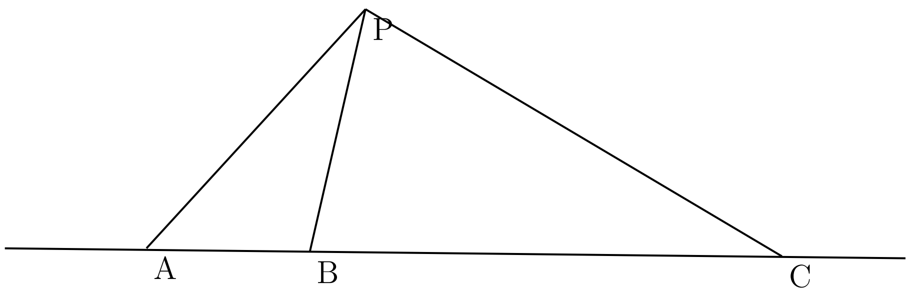
3.1 Directed Segments

Definition

3.2 Stewart's Theorem

If A,B,C collinear and P is any other point, then

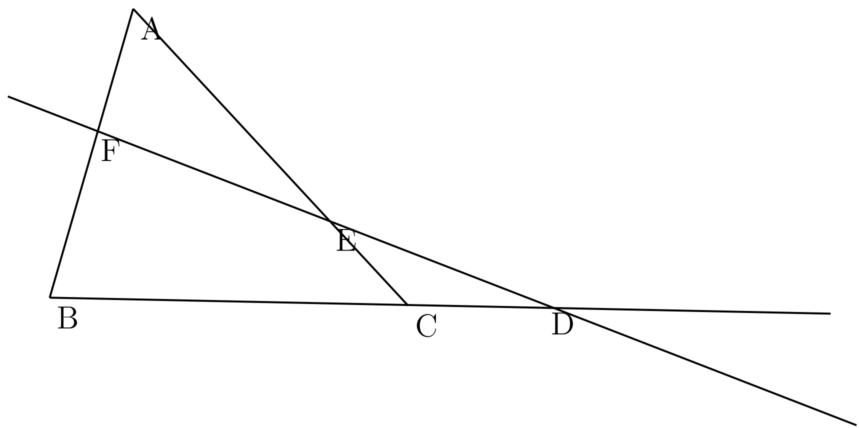
$$PA^2 \cdot BC + PB^2 \cdot CA + PC^2 \cdot AB + BC \cdot CA \cdot AB = 0$$



3.3 Menelaus' Theorem

Suppose we have a triangle ABC, and a transversal line that crosses BC, AC, and AB at points D, E, and F respectively, with D, E, and F distinct from A, B, and C, then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$$



3.4 Menelaus' Inverse Theorem

Suppose we have a triangle ABC with D on BC, E on AC, F on AB, such that,

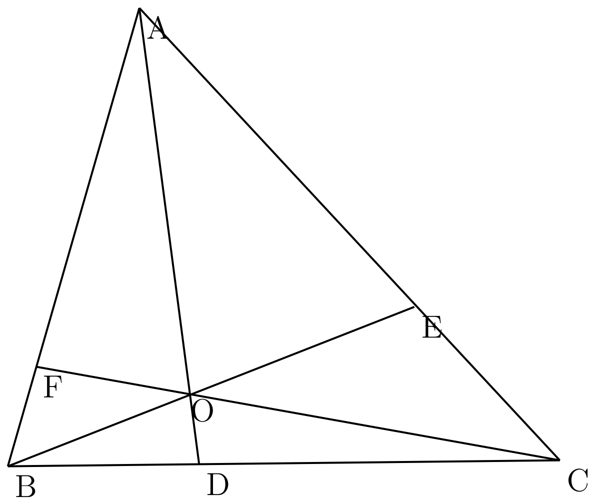
$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$$

then D,E, F collinear.

3.5 Ceva's Theorem

Given a triangle ABC, let the lines AO, BO and CO be drawn from the vertices to a common point O (not on one of the sides of ABC), to meet opposite sides at D, E and F respectively, then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$



3.6 Ceva's Inverse Theorem

Suppose we have a triangle ABC with D on BC, E on AC, F on AB, such that,

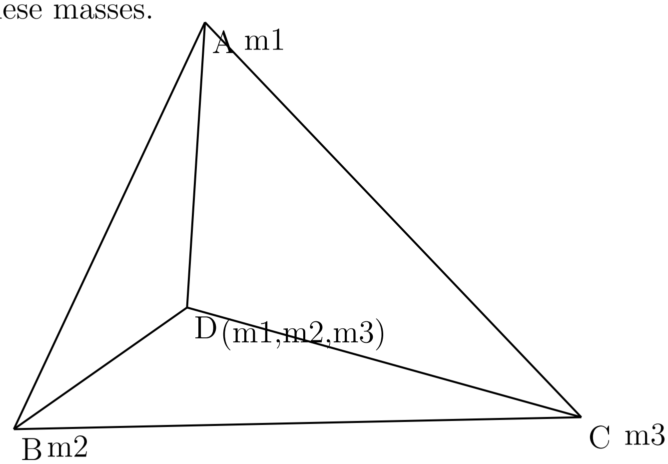
$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

then AD,BE,CF concurrent.

4 Barycentric Coordinate

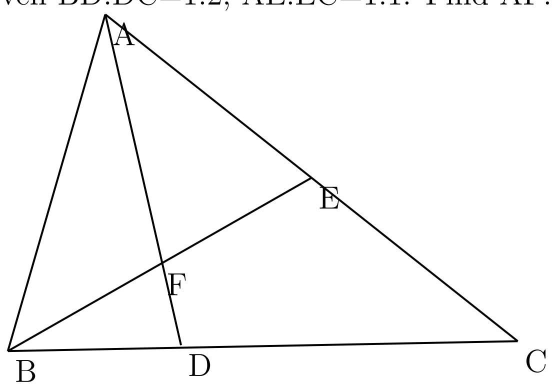
4.1 Definition

The barycentric coordinates of a point can be interpreted as masses placed at the vertices of the simplex, such that the point is the center of mass (or barycenter) of these masses.



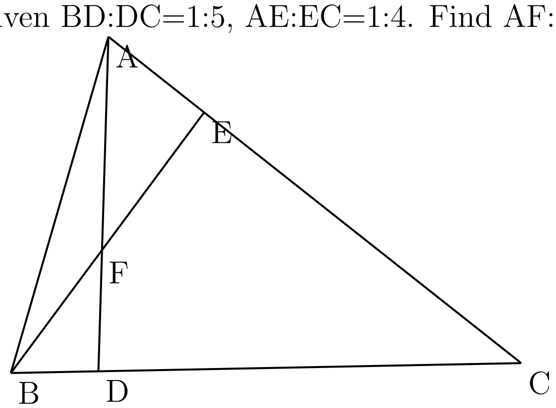
4.2 Example

Given BD:DC=1:2, AE:EC=1:1. Find AF:FD.



4.3 Problem

Given BD:DC=1:5, AE:EC=1:4. Find AF:FD.



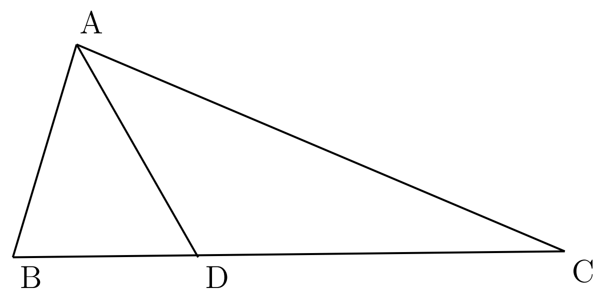
5 Angle Bisector

5.1 Definition

5.2 Angle Bisector Theorem

If AD bisects $\angle A$, then

$$\frac{BD}{CD} = \frac{AB}{AC}$$

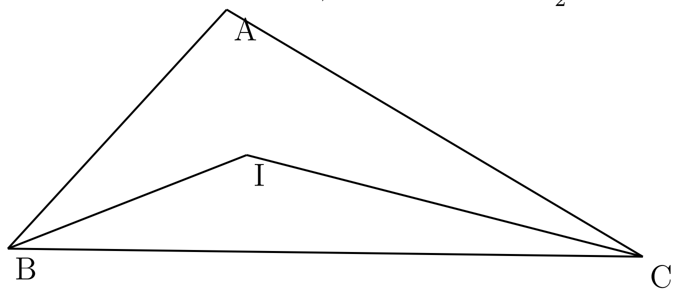


5.3 Theorem

Angle bisectors of a trinagle are concurrent, the point is called the incenter of the triangle

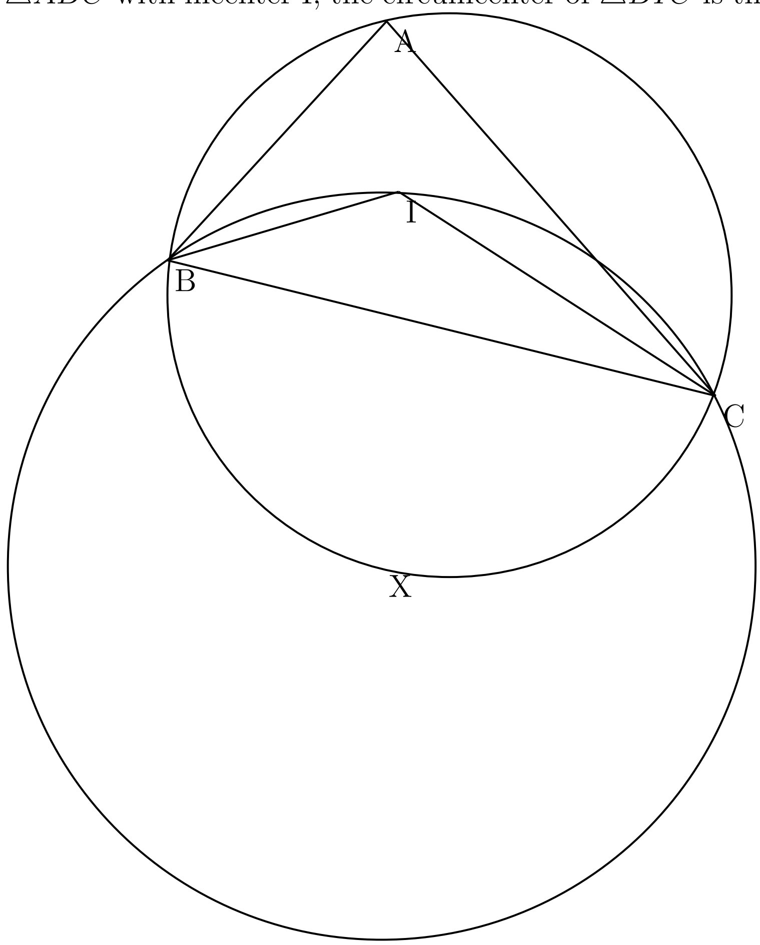
5.4 Theorem

In $\triangle ABC$ with incenter I, $\angle BIC = 90° + \frac{1}{2}\angle A$



5.5 Theorem

In $\triangle ABC$ with incenter I, the circumcenter of $\triangle BIC$ is the mid point of the arc $\overset{\frown}{BC}$.



6 Median

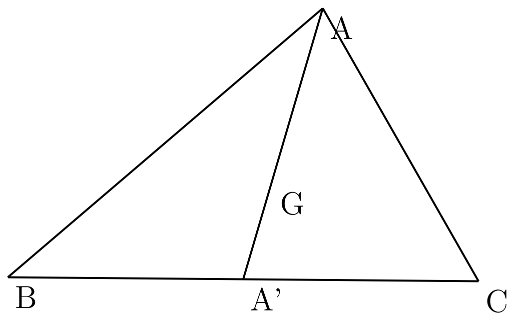
6.1 Definition

6.2 Theorem

Medians of triangle are concurrent. The point is called the centroid of the triangle.

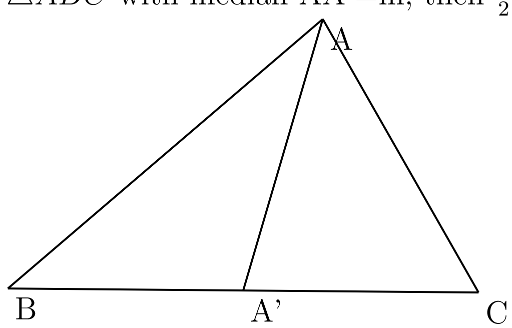
6.3 Theorem

In $\triangle ABC$ with centroid G and A' as the midpoint of BC, AG=2GA'.



6.4 Median Length Formula

In $\triangle ABC$ with median AA'=m, then $\frac{1}{2}m^2 = b^2 + c^2 - \frac{1}{2}a^2$

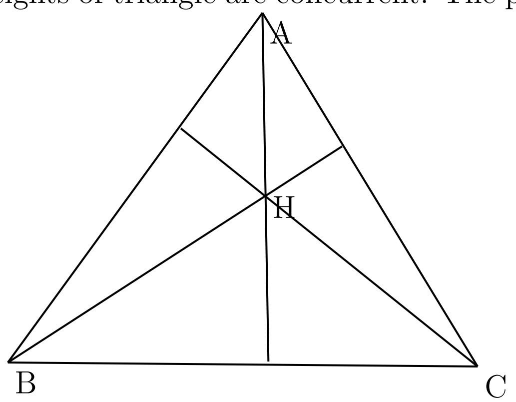


7 Height

7.1 Definition

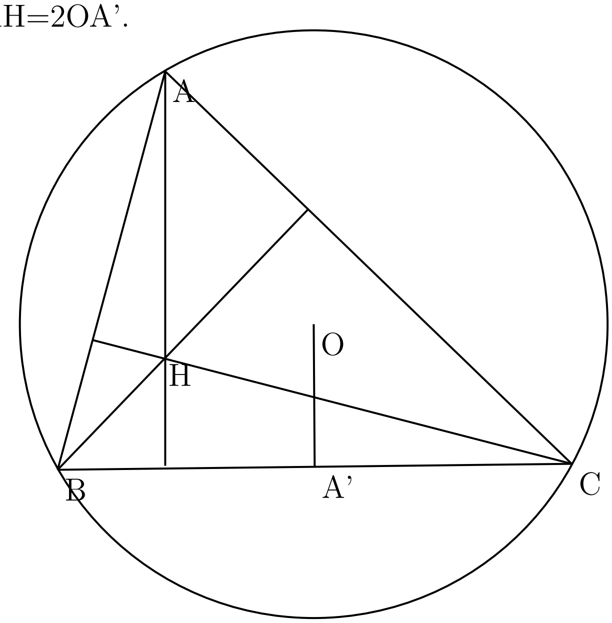
7.2 Theorem

Heights of triangle are concurrent. The point is called the orthocenter of the triangle.



7.3 Theorem

In $\triangle ABC$ with orthocenter H, A' the midpoint of BC, and the circumcenter O, AH=2OA'.



7.4 Theorem:

O the circumcenter,G the centroid,H the orthocenter are collinear. This line is called the Euler line of the triangle.