Error in centered numerical derivative definition

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Starting from Taylor's series definition for a real value a (arbitrarily selected):

$$f(x) = \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 + \frac{f^{(2)}(a)}{2!}(x-a)^2 + ...$$

which can be written simpler as

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + ...$$

In our case it's important to remember that h is a very small number, so the bigger exponent is over h, the smaller resulting value.

Now let's add some substitutions:

1. Our function will be f(x+h) 2. Expansion at point zero

$$f(x+h) = f(0) + \frac{f^{(1)}(0)}{1!}(x+h) + \frac{f^{(2)}(0)}{2!}(x+h)^2 + ...$$

Same thing for f(x-h):

$$f(x-h) = f(0) + \frac{f^{(1)}(0)}{1!}(x-h) + \frac{f^{(2)}(0)}{2!}(x-h)^2 + ...$$

Let's make a substitution:

$$\phi_i = \frac{f^{(i)}(0)}{i!}$$

Then

$$f(x+h) = \phi_0 + \phi_1(x+h) + \phi_2(x+h)^2 + \phi_3(x+h)^3 + ...$$

And

$$f(x-h) = \phi_0 + \phi_1(x-h) + \phi_2(x-h)^2 + \phi_3(x-h)^3 + ...$$

Let's subtract:

$$f(x+h) - f(x-h) = \phi_1(2h) + \phi_2(4hx) + \phi_3(6hx^2 + 2h^3) + ...$$

Let's replace dots with big O:

$$f(x+h) - f(x-h) = \phi_1(2h) + \phi_2(4hx) + \phi_3(6hx^2 + 2h^3) + O(h^4)$$

It's useful to extract factor 2h since that's the denominator in final equation:

$$f(x+h) - f(x-h) = 2h(\phi_1 + \phi_2(2x) + \phi_3(3x^2 + h^2) + O(h^3))$$

So:

$$\frac{f(x+h) - f(x-h)}{2h} = \phi_1 + \phi_2(2x) + \phi_3(3x^2 + h^2) + O(h^3)$$

Since on the right side we still have squared h:

$$\frac{f(x+h) - f(x-h)}{2h} = \phi_1 + \phi_2(2x) + O(h^2)$$

Since h is a very small value, big O for smaller exponent dominates over one with larger exponent.

I guess that's a verbose answer to the question.

Question's importance may come from the fact that standard form of numerical derivative has error of O(h) order. Discussed under [this wiki page.][1]

[1]: https://en.wikipedia.org/wiki/Numerical_differentiation