Combinatorial Game Theory
Sprague - Grundy Theorem

Project Category - Game Theory

Program Name/ Purpose- Implementing Sprague-Grundy Theorem to solve any impartial game.

Explanation-

We have already seen that we can find who wins in a game of Nim without actually playing the game. [See all the Projects on Combinatorial Game Theory]

Suppose we change the classic Nim game a bit. This time each player can only remove 1, 2 or 3 stones only (and not any number of stones as in the classic game of Nim) . Can we predict who will win?

Yes, we can using Sprague-Grundy Theorem.

So what is Sprague-Grundy Theorem ?

Suppose there is a composite game (more than one sub-game) made up of N sub-games and two players – A and B. Then Sprague-Grundy Theorem says that if both A and B play optimally (i.e- they don't make any mistakes), then the player starting first is guaranteed to win if the XOR of the grundy numbers of position in each sub-games at the beginning of the game is non-zero. Otherwise, if the XOR evaluates to zero, then player A will lose definitely, no matter what.

How to apply Sprague Grundy Theorem ?

We can apply Sprague-Grundy Theorem in any impartial game and solve it. The basic steps are listed as follows-

1) Firstly break the composite game into sub-games.
2) Then for each sub-game calculate the Grundy Number at that position.
3) Then calculate the XOR of all the calculated Grundy Numbers.
4) If the XOR value is non-zero, then the player who is going to make the turn (First Player) will win else he is destined to lose, no matter what.

Example-

The game is as follows-

"The game starts with 3 piles having 3, 4 and 5 stones, and the player to move may take any positive number of stones upto 3 only from any of the piles [Provided that the pile has that much amount of stones]. The last player to move wins. Which player wins the game assuming that both players play optimally ?"

Can we tell who will win?

Yes, by applying Sprague-Grundy Theorem.

How?

As, we can see that this game is itself composed of several sub-games.

First Step- The sub-games can be considered as each piles.

Second Step-

We see from the below table that

Grundy (3) = 3
Grundy (4) = 0
Grundy (5) = 1

N	0	1	2	3	4	5	6	7	8	9	10
Grundy(n)	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**3**	**0**	**1**	**2**

[We have already seen how to calculate the Grundy Numbers of this game in this previous projects on Combinatorial Game Theory]

Third Step-

The XOR of 3, 4, 5 is – 3 ^ 0 ^ 1 = 2

Fourth Step-

And since this is a non-negative number, so we can say that the first player will win.

References-

https://en.wikipedia.org/wiki/Sprague%E2%80%93Grundy_theorem

Exercise to the Readers-

Suppose we have a game like this-

"A game is played by two players with N integers A1,A2,…,AN. On his/her turn, a player selects an integer, divides it by 2, 3, or 6, and then takes the floor. If the integer becomes 0, it is removed. The last player to move wins. Which player wins the game if both players play optimally?"