MATHS

* Maxima & minima for a function with 2variables → $u = f(x,y)$.

1- Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$,

2- $\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0$ → Get stationary points.

3- $r = \frac{\partial^2 u}{\partial x^2}, s = \frac{\partial^2 u}{\partial x \partial y}, t = \frac{\partial^2 u}{\partial y^2}$. | at each stationary point.

4- If $rt - s^2 > 0$, $r > 0$ ⇒ Pt. of R. minima.
 $rt - s^2 > 0$, $r < 0$ ⇒ Pt. of R. maxima.
 $rt - s^2 < 0$ ⇒ Pt. of inflection (saddle point).
 $rt - s^2 = 0$ ⇒ No conclusion can be drawn.

* $\int u\,dv = uv - \int v\,du$.

eg., $\int t\cos t\,dt$ ⇒ diff. $+t$ -1 $+0$ $= t\sin t + \cos t + C$.
 Int. $\cos t$ $\sin t$ $-\cos t$

* Fourier series -

(i) $f(x)$ in $[c, c+2l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

(ii) $f(x)$ → even function $[-l, l]$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx, \quad a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad b_n = 0.$$

(iii) $f(x)$ → odd function $[-l, l]$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right).$$

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

* $\int_0^{\pi/2} \sin^m x\, dx = \int_0^{\pi/2} \cos^m x\, dx = \begin{cases} \left(\frac{m-1}{m}\right)\left(\frac{m-3}{m-2}\right)\left(\frac{m-5}{m-4}\right) \cdots \frac{1}{2} \cdot \frac{\pi}{2} \rightarrow m \text{ is even} \\ \left(\frac{m-1}{m}\right)\left(\frac{m-3}{m-2}\right)\left(\frac{m-5}{m-4}\right) \cdots \frac{2}{3} \cdot (1) \rightarrow m \text{ is odd.} \end{cases}$

$\int_0^{\pi/2} \sin^m x \cdot \cos^n x\, dx = \frac{(m-1)(m-3)(m-5) \cdots (2 \text{ or } 1)\,(n-1) \cdots (2 \text{ or } 1)}{(m+n)(m+n-2)(m+n-4) \cdots (2 \text{ or } 1)} \times K.$

where $K = \begin{cases} \pi/2 & \text{when both } m \,\&\, n \text{ are even} \\ 1 & \text{otherwise} \end{cases}$

* $\int_0^a x f(x)\, dx = a \int_0^{a/2} f(x)\, dx$, if $f(a-x) = f(x)$.

* $\int_0^{2a} f(x)\, dx = 2 \int_0^a f(x) dx$, if $f(2a-x) = f(x)$.

$= 0$, if $f(2a-x) = -f(x)$.

* $\int_0^a f(x) dx = \int_0^a f(a-x) dx.$

* Gamma function-

$\Gamma n = \int_0^\infty e^{-t} t^{n-1} dt$

$\Gamma n+1 = n \Gamma n$ or $n!$

$\Gamma \frac{1}{2} = \sqrt{\pi}$.

* Beta function-

$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx.$

$\beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n}.$

* Length of arc $y = f(x)$ b/w $x=a$ & $x=b$ $\rightarrow l = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$

* Length of arc $x = f(y)$ b/w $y = c$ & $y = d$ $\rightarrow l = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy.$

* —— " —— $r = f(\theta)$ —— " —— $\theta = \theta_1$ & $\theta = \theta_2 \rightarrow l = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$

* —— " —— $x = \phi(t)$, $y = \psi(t)$ b/w $t = t_1$ & $t = t_2$

$l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

* Volume of solid revolution, $y = f(x)$ around x-axis b/w $x = a$ to b.

$V = \int_a^b \pi y^2 dx.$

—— " —— $x = f(y)$ —— " —— y-axis b/w $y = c$ to d.

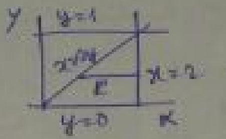
$V = \int_c^d \pi x^2 dy.$

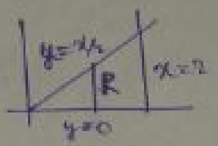
* To find area bounded by curves-

1- Area bounded by curve on x-axis $= \int_{x=a}^{x=b} y\,dx$.

2- Area ——"—————— y-axis $= \int_{y=c}^{y=d} x\,dy$.

3- Area bounded by two curves $= \iint dy\,dx$.

* $\int_0^1 \int_{2y}^{2} (e^{x^2} dx)\,dy \longrightarrow$ 

$\Rightarrow$ 

$\int_0^2 \int_0^{x/2} (e^{x^2} dy)\,dx$.

* Euler's Theorem -

1- $u = f(x,y) \rightarrow$ homogeneous of degree 'n'.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = n(n-1)u.$$

2- $u = f(x,y) + g(x,y). \rightarrow$ homogeneous of degree m & n resp.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = mf + ng.$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = m(m-1)f + n(n-1)g.$$

3- $u = f(x,y) \rightarrow$ Not homogeneous but $F(u) \rightarrow$ homogeneous of 'n'.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nF(u)}{F'(u)} \text{ or } g(u) \text{ (say)}.$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\cdot(g'(u)-1).$$

* Vector Calculus-

* Gradient of scalar point function $\phi(x,y,z)$.

$$\text{grad}\,\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi.$$

$$= i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}.$$

* Unit normal vector to the surface $\phi(x,y,z) \rightarrow \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

* Angle b/w two surfaces, $\cos\theta = \frac{\nabla f \text{ at } P \cdot \nabla g \text{ at } P}{|\nabla f \text{ at } P|\,|\nabla g \text{ at } P|}$.

* Directional derivative of surface $\phi(x,y,z)$ at point P in the dirⁿ of a vector $\vec{a}$ is given by $\nabla\phi_{\text{at } P} \cdot \frac{\vec{a}}{|\vec{a}|}$.

* Max. value of directional derivative $|\nabla\phi_{\text{at } P}|$.

* Divergence of a vector point $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$.

$$\text{div.}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z},$$

$\nabla\cdot\vec{F} = 0 \Leftrightarrow \vec{F}$ is a solenoidal function.

* $$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}.$$

$\vec{\nabla}\times\vec{F} = 0 \Leftrightarrow F$ is an irrotational vector.

* $\text{Curl}(\text{grad}\,\phi) = 0$.
* $\text{Div}(\text{curl}\,\vec{F}) = 0$.

* Green's Theorem -

$$\oint_C f_1\,dx + f_2\,dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy.$$

→ for simple closed curves.
→ $(f_1\hat{i} + f_2\hat{j})$

* Stoke's Theorem -

$$\oint_C \vec{F}\cdot\vec{dr} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds.$$

→ For simple closed curves.
$F_1\hat{i} + f_2\hat{j} + f_3\hat{k}$.
→ For open surface.

* Gauss Divergence Theorem -

$$\underbrace{\oiint_S \vec{F}\cdot\hat{n}\,ds}_{\text{flux}} = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV.$$

→ Closed surface.

* $L(1) = \frac{1}{s}$ $\qquad L(\sin at) = \frac{a}{s^2+a^2}$.

$L(t) = \frac{1}{s^2}$. $\qquad L(\cos at) = \frac{s}{s^2+a^2}$. $\qquad L(\delta(t)) = 1$.

$L(t^n) = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$. $\qquad L(\sinh at) = \frac{a}{s^2-a^2}$.

$L(e^{at}) = \frac{1}{s-a}$. $\qquad L(\cosh at) = \frac{s}{s^2-a^2}$.

$L(e^{-at}) = \frac{1}{s+a}$. $\qquad L(u(t-a)) = \frac{e^{-as}}{s}$.

→ Differential Equations -

* $\frac{dy}{dx} + Py = Q.$

$IF = e^{\int P dx}.$

$y(IF) = \int Q(IF)dx.$

* $Mdx + Ndy = 0.$

1- If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. → Exact DE.

Sol^n is $\int M dx + \int (\text{Terms in } N \text{ free from } x) dy = c$

2- Reducible exact D.E -

(i) If M & N are homogeneous function of same degree then,

$IF = \frac{1}{Mx + Ny}.$

(ii) M & N → Not homogeneous but $M = y \cdot f_1(x,y)$, $N = x \cdot f_2(x,y)$.

$IF = \frac{1}{Mx - Ny}.$

(iii) $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ or constant, then $IF = e^{\int f(x)dx}.$

(iv) $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{N} = f(y)$ or constant, then $IF = e^{\int f(y)dy}.$

After calculating IF, multiply it with DE and find sol^n as an exact DE.

* $(D^3 - D^2 + D + 1)y = 0.$ → Sol^n; $\underline{y = CF}$

↓

		CF
$m^3 - m^2 + m + 1$ →	Real & distinct sol^n m_1, m_2, m_3	$c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x}$.
	Real & equal m, m, m	$(c_1 + c_2x + c_3x^2)e^{mx}$
	Imaginary $\alpha \pm i\beta$	$e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$.
	$\alpha \pm i\beta$, $\alpha \pm i\beta$.	$e^{\alpha x}[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x]$.

* $(D^3 - D^2 + D + 1)y = f(x) \rightarrow$ Sol^n is $y = CF + PI$.

$\cancel{PI = \frac{1}{D^3 - D^2 +}}$

1- If $f(x) = e^{\alpha x}$.

$$PI = \frac{1}{D^3 - D^2 + D + 1} e^{\alpha x}$$

← Replace D by α.

2- $f(x) = \cos \alpha x$ or $\sin \alpha x$.

Replace D^2 by $(-\alpha^2)$.

If D^2 comes to be $= 0$.

$$x \cdot \frac{1}{2D} \cos 2x.$$

$$\frac{x}{2} \frac{1}{D} \cos 2x = \frac{x}{2} \cdot \frac{\sin 2x}{2} = \frac{x}{4} \sin 2x.$$

3- $f(x) = x^k$.

Write $f(D)$ in the form of $(1+t)^{-1}$ or $(1-t)^{-1}$.

$$\frac{1}{D^2 - 3D + 2} x^2.$$

$$\frac{1}{(1-D)(2-D)} x^2.$$

* Gamma Function -

$$\Gamma n = \int_0^\infty e^{-t} t^{n-1} \, dt$$

$\Gamma(n+1) = n\Gamma n$ or $n!$

$\Gamma\frac{1}{2} = \sqrt{\pi}$.

* Beta Function -

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, dx$$

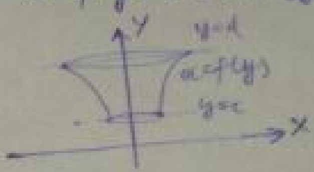
$$= \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}.$$

- The length of the arc $y=f(x)$ b/w $x=a$ & $x=b$ is given by, $l = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$.
- The length of the arc $x=f(y)$ b/w $y=c$ & $y=d$ is given by, $l = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2} \, dy$.
- The length of the arc $r=f(\theta)$ b/w $\theta=\theta_1$ & $\theta=\theta_2$ is given by, $l = \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \, d\theta$.
- The length of the arc, $x=\phi(t)$, $y=\psi(t)$ b/w $t=t_1$ & $t=t_2$ is given by, $l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} \, dt$
- The volume of solid revolution of the arc $y=f(x)$ around x-axis b/w $x=a$ & $x=b$ is given by,

$$V = \int_a^b \pi y^2 \, dx.$$

- The volume of solid revolution of the arc $x=f(y)$ around y-axis b/w $y=c$ & $y=d$ is given by

$$V = \int_c^d \pi x^2 \, dy.$$



* Change of Variables -

Jacobian - $J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{2v}{u}$.

Area $= \iint dy\,dx$.

Cartesian to polar form -

$x = r\cos\theta$, $y = r\sin\theta$.

$$J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = r$$

Area $= \iint |r| \, dr \, d\theta$.

Cartesian to cylindrical -

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$$J\left(\frac{x,y,z}{r,\theta,z}\right) = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = r$$

Volume $= \iiint |r| \, dr \, d\theta \, dz$.

Cartesian to spherical -

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$$J\left(\frac{x,y,z}{\rho,\phi,\theta}\right) = \rho^2\sin\phi.$$

* Total Derivative -

$u = f(x,y) \rightarrow x = \phi(t)$, $y = \psi(t)$.

$$\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} + \frac{du}{dy}\cdot\frac{dy}{dt}.$$

* Chain rule in Partial Differentiation -

$u = f(x,y)$; $x = \phi(r,s)$, $y = \psi(r,s)$

$u_r = u_x x_r + u_y y_r$.

$u_s = u_x x_s + u_y y_s$.

* Euler's theorem -

$\rightarrow u = f(x,y)$ is homogeneous of degree 'n'.

i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$, ii)

* Homogeneous function-

$f(kx, ky) = k^n f(x,y)$. → degree 'n'.

* Euler's Theorem-

→ $u = f(x,y)$ is homogeneous of degree n.

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$.

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u$.

→ $u = f(x,y) + g(x,y)$, f & g → homogeneous of degree m & n resp.

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = mf + ng$.

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = m(m-1)f + n(n-1)g$.

→ $u = f(x,y)$ is not homogeneous but $F(u)$ is homogeneous of degree 'n'.

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \boxed{\dfrac{nF(u)}{F'(u)}}$ $\underline{g(u)}$ (say)

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)(g'(u)-1)$.

* Some formulae-

$\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{d}{dx}(\cos^{-1}x) = \dfrac{-1}{\sqrt{1-x^2}}$

$\tan^{-1}x = \dfrac{1}{1+x^2}$

$\cot^{-1}x = \dfrac{-1}{1+x^2}$

$\sec^{-1}x = \dfrac{1}{|x|\sqrt{x^2-1}}$

$\operatorname{cosec}^{-1}x = \dfrac{-1}{|x|\sqrt{x^2-1}}$.

$\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c$.

$\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$.

$\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$.

Linear Algebra

* When diagonal elements are same and all other elements are same.

$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x+3a)(x-a)(x-a)(x-a).$$

Properties of Determinant-

1- $|A^T| = |A|$

2- $|AB| = |A||B|$

3- $|A+B| \neq |A|+|B|$

4- The determinant value of a triangular or a diagonal matrix is the product of its leading diagonal elements.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 8 \end{bmatrix} = 2\times4\times8 = 64.$$

5- Square matrix, if each element of a row (column) is zero then the value of its determinant is zero.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 7 & 6 \\ \boxed{0 \quad 0 \quad 0} \end{bmatrix} = 0.$$

6- Two rows or columns are identical, then determinant is zero.
(or proportional)

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 9 & 8 \\ 6 & 9 & 8 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 3 & 5 \\ 6 & 9 & 8 \\ 12 & 18 & 16 \end{bmatrix} = 0.$$

7- Skew symmetric $= 0$.

8- Orthogonal matrix $= 1$ or -1.

9- Square matrix, order $n \Rightarrow |kA| = k^n|A|$.

10- (i) $A(\text{adj}A) = |A| I$.

(ii) $A^{-1} = \dfrac{\text{Adj}A}{|A|}$

(iii) $|\text{Adj } A| = |A|^{n-1}$

(iv) $|\text{Adj}(\text{Adj}A)| = |A|^{(n-1)^2}$

(v) $|A^{-1}| = \dfrac{1}{|A|}$.

* Symmetric: $A^T = A$

Skew-sym.: $A^T = -A$

Orthogonal: $A^T A = A A^T = I$

* $A = \left(\dfrac{A+A^T}{2}\right) + \left(\dfrac{A-A^T}{2}\right)$.

* $A_{m\times n}$ & $B_{n\times p}$ → AB [Multiplication : mnp
Addition : mp(n-1)]

* $\underset{3\times4}{D}\underset{}{A}\underset{}{B}\underset{3\times3}{E}\underset{}{C} = I \Rightarrow B^{-1} = ECDA$.

Rank of the Matrix-

1- Null Matrix = 0.
2- Non-singular matrix = order.
3- Singular matrix = < less than order.
4- m×n matrix, Rank ≤ min.(m, n).
5- A & B → 2 matrices of same order, $\rho(A+B) \leq \rho(A) + \rho(B)$.
6- $\rho(A) = \rho(A^T)$.
7- $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$.
8- All rows or columns are identical or proportional ⇒ 1.
9- n×n matrix, rank n ⇒ $\rho(\text{Adj } A) = n$.
10- —"—, rank n-1 ⇒ $\rho(\text{Adj } A) = 1$.
11- —"— rank n-2 ⇒ $\rho(\text{Adj } A) = 0$.

Properties of Eigen Values -

1- Sum of eigen values = Trace of the matrix. → Sum of diagonal elements.
Product of eigen values = Determinant of matrix
2- Eigen values of A^T = Eigen value of A.
3- Eigen values of a triangular matrix or diagonal matrix = Leading Diagonal elements
4- λ → eigen value of non-singular matrix,
*** $\frac{1}{\lambda}$ = eigen value of A^{-1}.
*** $\frac{|A|}{\lambda}$ = eigen value of Adj |A|.

* In Square matrix if n rows are identical then compulsory (n-1) eigen values will be zero.

5- λ → eigen value of matrix A,
(i) $A^2 \rightarrow \lambda^2$
(ii) $A^n \rightarrow \lambda^n$
(iii) $kA \rightarrow k\lambda$
(iv) $A + kI \rightarrow \lambda + k$
(v) $A - kI \rightarrow \lambda - k$
(vi) $A^2 + C_1A + C_2I \rightarrow \lambda^2 + C_1\lambda + C_2$

* Eigen vectors of $A, A^2, A^3, \ldots A^n$, Adj A are always same.

System of linear Equations

- homogeneous → $AX = 0$ (Always consistent)
  - Trivial → Zero solution
    - $\rho(A)$ = No. of variables
    - $|A| \neq 0$
  - Non-trivial → Non-zero solⁿ
    - $\rho(A)$ < No. of variables
    - $|A| = 0$.
- Non-homogeneous → $AX = B,\ B \neq 0$
  - Inconsistent → No solution
    - $\rho(A) \neq \rho(AB)$
  - consistent → $\rho(A) = \rho(AB)$
    - Unique solⁿ → $\rho(A) = \rho(AB)$ = No. of variables
    - Infinitely many solⁿ → $\rho(A) = \rho(AB)$ < No. of variables

$\rho(AB)$ and $(A{:}B)$ here.
$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ → $AB = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

* Free variables or no. of independent variables or dimension of null space or dimension of space of solution or nullity = Total no. of variables − Rank
= Total no. of columns − Rank

* If $\rho(A) = n$ or $|A| \neq 0$ then set of vectors are <u>linearly independent</u>.
* If $\rho(A) < n$ or $|A| = 0$ then set of vectors are <u>linearly dependent</u>.
* *** Orthogonal vectors : $x_1^T \cdot x_2 = 0$.
* Ortho normal Vectors (i) $x_1^T \cdot x_2 = 0$; (ii) $||x_1|| = 1$ & $||x_2|| = 1$.
* Normalised Vectors: Normalised vector of $X_1 \to \dfrac{X_1}{||X_1||}$.

  Eg. $X_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$
  $||X_1|| = \sqrt{1^2 + (-2)^2 + 1^2}$
  $= \sqrt{6}$.

* *** No. of linearly independent eigen vectors = No. of distinct eigen values.
* *** The eigen ~~values~~ vectors corresponding to the distinct eigen values of a real symmetric matrix are always orthogonal (perpendicular).
* If all leading minors of a <u>real</u> <u>symmetric</u> are positive than all its eigen values are positive.
* The eigen vectors corresponding to the distinct eigen values of any square matrix are always linearly independent.

Maths - Improvements

* When 'ω' is cube root of unity is given then,

  $1+\omega+\omega^2=0$.

  $\omega^4=\omega$.

  $\omega^3=1$.

  $\omega^6=(\omega^3)^2=1$.

* Random Variable → Discrete R.V ⇒ $\sum P(x)=1$.

  → Continuous R.V ⇒ $\int_{-\infty}^{\infty} f(x)\,dx=1$.

* $$PI = \frac{96x^2}{D^2(D^2+4)}$$

$$= \frac{\cancel{96}^{24}x^2}{\cancel{4}D^2\left(1+\frac{D^2}{4}\right)}$$

$$= \frac{24}{D^2}\left(1+\frac{D^2}{4}\right)^{-1}x^2$$

$$= \frac{24}{D^2}\left(1-\frac{D^2}{4}\right)x^2.$$

$$= 24\left[\frac{1}{D^2}\left(x^2-\frac{\cancel{2}^1}{\cancel{4}_2}\right)\right]$$

$$= 24\left(\frac{x^4}{12}-\frac{x^2}{4}\right).$$

$$= 2(x^4-3x^2).$$

$$= 2x^2(x^2-3).$$

* Residue at $z=z_0$ → coeff. of $\frac{1}{z-z_0}$.

* $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}---$

  $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}---$

  $\sinh x = x+\frac{x^3}{3!}+\frac{x^5}{5!}---$

  $\cosh x = 1+\frac{x^2}{2!}+\frac{x^4}{4!}---$

  $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}---$

  $e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}---$

  $\ln(1+x) = x-\frac{x^2}{2!}+\frac{x^3}{3}-\frac{x^4}{4}---$

  $\ln(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}---$

* $\int_0^{1+i}(x^2-iy)\,dz$. along the line $y=x$.

  $z=x+iy$.

  $z=x+ix$.

  $dz=dx+i\,dx$.

  $$I=\int_0^1 (x^2-ix)(dx+i\,dx)$$

  $$=(1+i)\int_0^1 (x^2-ix)\,dx.$$

* Q. $y=e^{\sin^{-1}x}$, $z=e^{-\cos^{-1}x}$.

  Find $\left.\frac{d^2y}{dz^2}\right|_{x=1/\sqrt{2}}$.

* $\left(\frac{d^2y}{dx^2}\right)^1$ → Order = 2, Degree = 1.

* Newton-Raphson fails for x_0 when $f'(x)=0$.

* Apply Binomial distribution whenever only 2 outcomes are present. Don't apply Poisson by seeing large n & small p.

  mean = np.
  variance = npq.

* Euler method for differentiation -

$\frac{dy}{dx} = x+y$

$f(x,y) = x+y$

x_0, y_0 } → will be given in question

h → equal interval to get the desired value.

$y_1 = y_0 + hf(x_0, y_0)$.

$= y_0 + h(x_0 + y_0)$

* Apply $\oint_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$ only when $(z-z_0)$ is present in D^r.

eg., $\oint \frac{e^{2z}}{1-e^z} dz$ → Here $(z-z_0)$ is not present in D^r, Hence apply [$2\pi i$ (sum of all residues at poles)]

* $\phi(x,y,z) = x^2yz + 4xz^2$.

greatest rate of increase of ϕ at pt. $(1,-2,1)$ is magnitude of directional derivative at that point.

$\nabla\phi|_{at\ (1,-2,1)} = j + 6\hat{k}$

$|\nabla\phi| = \sqrt{1+6^2} = \sqrt{37} = 6.08$.

* Sometimes probability is asked in %. So read carefully.