COMP90051 **Statistical Machine Learning**

Semester 2, 2016

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24. Hidden Markov Models & message passing

THE UNIVERSITY OF MELBOURNE

Looking back…

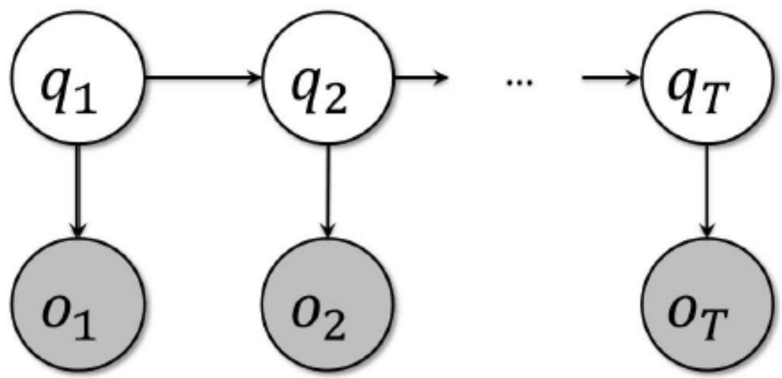
- Representation of joint distributions
- Conditional/marginal independence
  - Directed vs undirected
- Probabilistic inference
  - Computing other distributions from joint
- Statistical inference
  - Learn parameters from (missing) data
- Today: putting these all into practice…

Hidden Markov Models

Model of choice for sequential data. A form of clustering (or dimensionality reduction) for discrete time series.

The HMM (and Kalman Filter)

- Sequential observed outputs from hidden state
  - states take discrete values (i.e., clusters)
  - assumes discrete time steps 1, 2, …, *T*



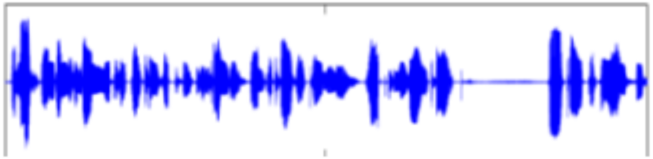
- The Kalman filter same with continuous Gaussian r.v.'s
  - i.e., dimensionality reduction, but with temporal dynamic

HMM Applications

- NLP – part of speech tagging: given words in sentence, infer hidden parts of speech

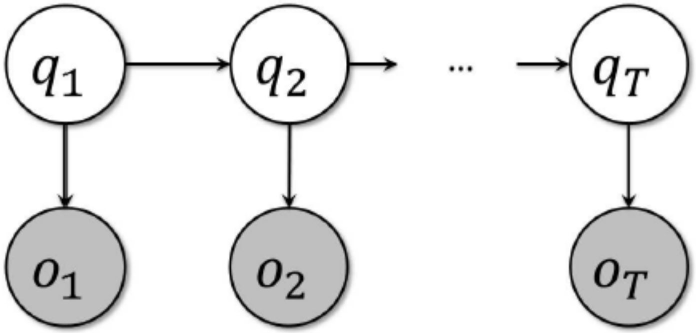
  "I love Machine Learning" → noun, verb, noun, noun

- Speech recognition: given waveform, determine phonemes



- Biological sequences: classification, search, alignment
- Computer vision: identify who's walking in video, tracking

Formulation



- Formulated as directed PGM
  - therefore joint expressed as

$$P(\mathbf{o}, \mathbf{q}) = P(q_1)P(o_1|q_1)\prod_{i=2}^{T} P(q_i|q_{i-1})P(o_i|q_i)$$

  - **bold** variables are shorthand for vector of *T* values

- Parameters (for *homogenous* HMM)

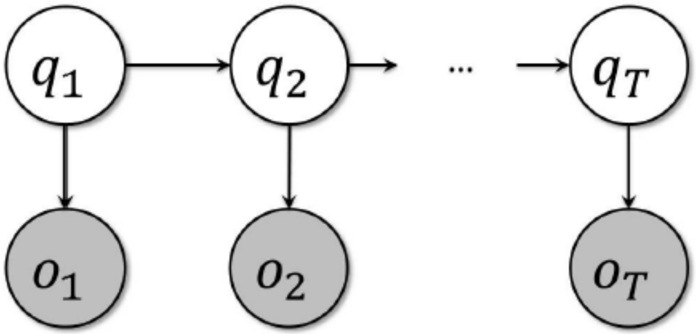
$A = \{a_{ij}\}$	transition probability matrix; $\forall i : \sum_j a_{ij} = 1$
$B = \{b_i(o_k)\}$	output probability matrix; $\forall i : \sum_k b_i(o_k) = 1$
$\Pi = \{\pi_i\}$	the initial state distribution; $\sum_i \pi_i = 1$

Independence

- Graph encodes independence btw RVs
  - conditional independence:

    $$o_i \perp \mathbf{o}_{\setminus i} \mid q_i$$

    all other o's excluding i
  - state q_i must encode all sequential context
- Markov blanket is local
  - for o_i blanket is q_i
  - for q_i blanket is $\{o_i, q_{i-1}, q_{i+1}\}$



Fundamental HMM Tasks

HMM Task	PGM Task
Evaluation. Given an HMM μ and observation sequence $\boldsymbol{o}$, determine likelihood $\Pr(\boldsymbol{o}\|\mu)$	Probabilistic inference
Decoding. Given an HMM μ and observation sequence $\boldsymbol{o}$, determine most probable hidden state sequence $\boldsymbol{q}$	MAP point estimate
Learning. Given an observation sequence $\boldsymbol{o}$ and set of states, learn parameters A, B, Π	Statistical inference

“Evaluation” a.k.a. marginalisation

- Compute prob. of observations **o** by summing out **q**

$$P(\mathbf{o}|\mu) = \sum_{\mathbf{q}} P(\mathbf{o}, \mathbf{q}|\mu)$$
$$= \sum_{q_1} \sum_{q_2} \ldots \sum_{q_T} P(q_1)P(o_1|q_1)P(q_2|q_1)P(o_2|q_2)\ldots P(q_T|q_{T-1})P(o_T|q_T)$$

- Make this more efficient by moving the sums

$$P(\mathbf{o}|\mu) = \sum_{q_1} P(q_1)P(o_1|q_1) \sum_{q_2} P(q_2|q_1)P(o_2|q_2) \ldots \sum_{q_T} P(q_T|q_{T-1})P(o_T|q_T)$$

- Deja vu? Maybe we could do var. elimation…

Elimination = Backward Algorithm

$$P(\mathbf{o}|\mu) = \sum_{q_1} P(q_1)P(o_1|q_1) \underbrace{\sum_{q_2} P(q_2|q_1)P(o_2|q_2) \ldots \underbrace{\sum_{q_T} P(q_T|q_{T-1})P(o_T|q_T)}_{m_{T\to T-1}(q_{T-1})}}_{m_{2\to 1}(q_1)}$$

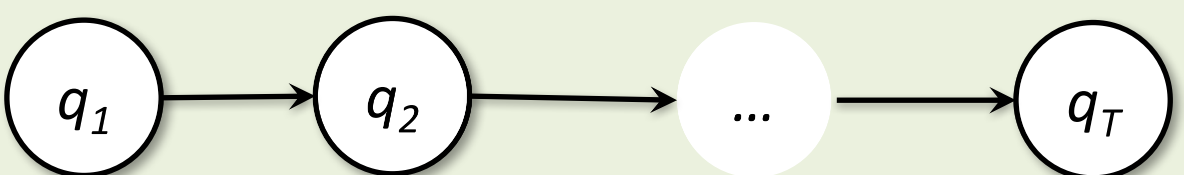
Eliminate q_T

...

Eliminate q_2

"Eliminate" q_1

$$P(\mathbf{o}|\mu) = \sum_{q_1} P(q_1)P(o_1|q_1)m_{2\to 1}(q_1)$$



Elimination = Forward Algorithm

$$P(\mathbf{o}|\mu) = \sum_{q_T} P(o_T|q_T) \underbrace{\sum_{q_{T-1}} P(q_T|q_{T-1}) P(o_T|q_T) \ldots \underbrace{\sum_{q_1} P(q_2|q_1)P(q_1)P(o_1|q_1)}_{m_{1\to 2}(q_2)}}_{m_{T-1\to T}(q_T)}$$

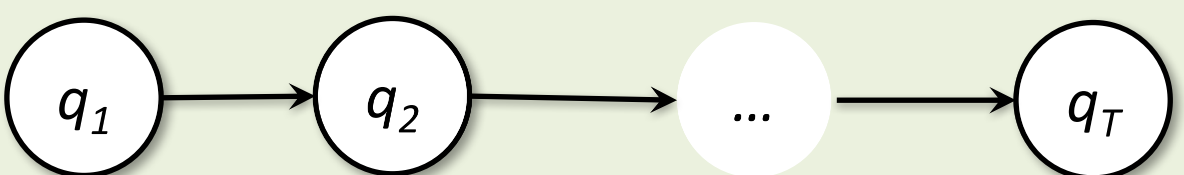
Eliminate q_1

…

Eliminate q_{T-1}

“Eliminate” q_T

$$P(\mathbf{o}|\mu) = \sum_{q_1} P(o_T|q_T) m_{T-1\to T}(q_T)$$



Forward-Backward

- Both algorithms are just *variable elimination* using different orderings
  - $q_T \ldots q_1 \rightarrow$ backward algorithm
  - $q_1 \ldots q_T \rightarrow$ forward algorithm
  - both have time complexity $O(Tl^2)$ where l is the label set
- Can use either to compute P(**o**)
  - but even better, can use the *m* values to compute marginals (and pairwise marginals over q_i, q_{i+1})

$$P(q_i|\mathbf{o}) = \frac{1}{P(\mathbf{o})} \underbrace{m_{i-1\to i}(q_i)}_{\text{forward}} P(o_i|q_i) \underbrace{m_{i+1\to i}(q_i)}_{\text{backward}}$$

Statistical Inference (Learning)

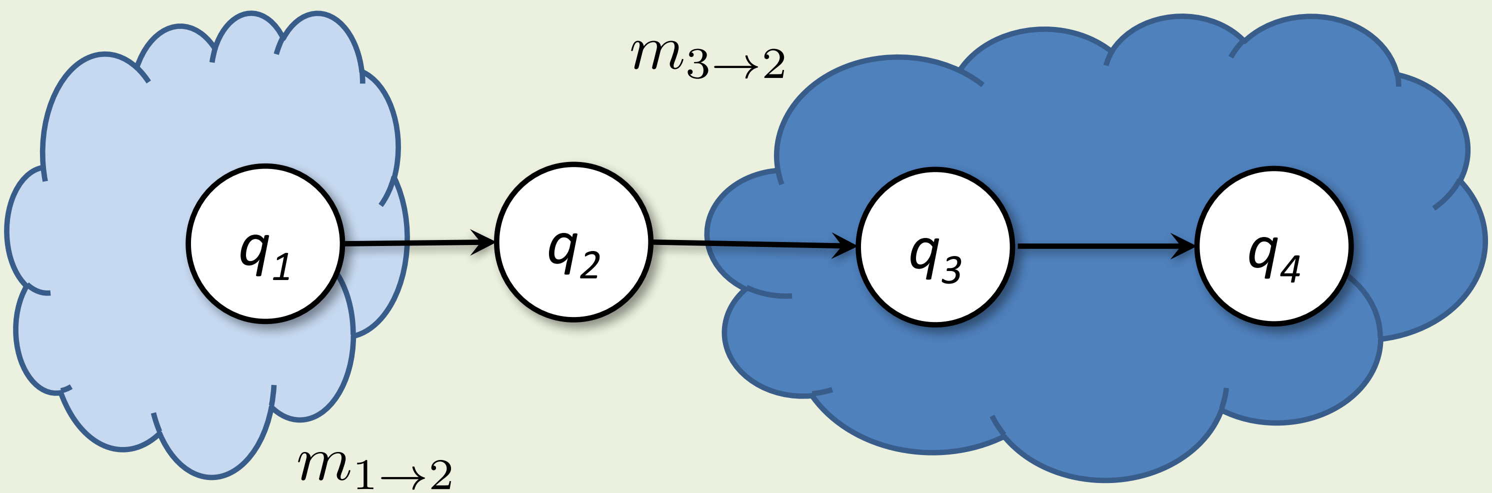
- Learn parameters $\boldsymbol{\mu} = (A, B, \boldsymbol{\pi})$, given observation sequence $\mathbf{o}$
- Called "Baum Welch" algorithm which uses EM to approximate MLE, $\mathrm{argmax}_{\boldsymbol{\mu}}\, P(\mathbf{o}|\boldsymbol{\mu})$:
  1. initialise $\boldsymbol{\mu}^1$, let $i=1$
  2. compute expected marginal distributions $P(q_t|\mathbf{o}, \boldsymbol{\mu}^i)$ for all t; and $P(q_{t-1}, q_t|\mathbf{o}, \boldsymbol{\mu}^i)$ for $t=2..T$
  3. fit model $\boldsymbol{\mu}^{i+1}$ based on expectations
  4. repeat from step 2, with $i=i+1$
- Expectations computed using forward-backward

Message Passing

Sum-product algorithm for efficiently computing marginal distributions over trees. An extension of variable elimination algorithm.

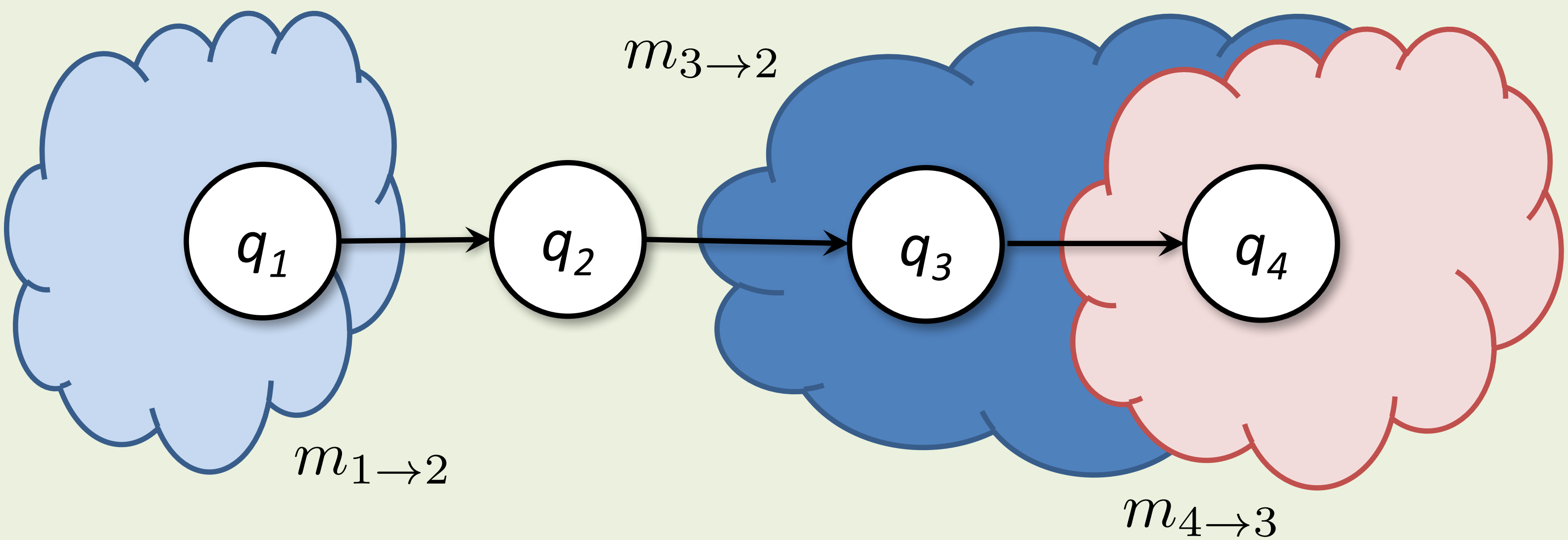
Inference as message passing

- Each *m* can be considered as a **message** which summarises the effect of the rest of the graph on the current node marginal.
  - *Inference = passing messages between all nodes*



Inference as message passing

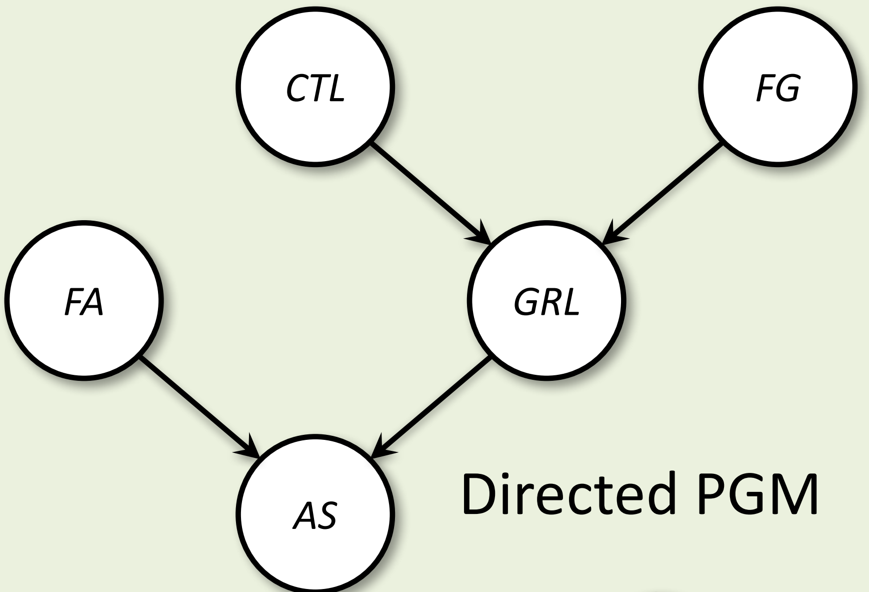
- Messages vector valued, i.e., function of target label
- Messages defined recursively: left to right, or right to left



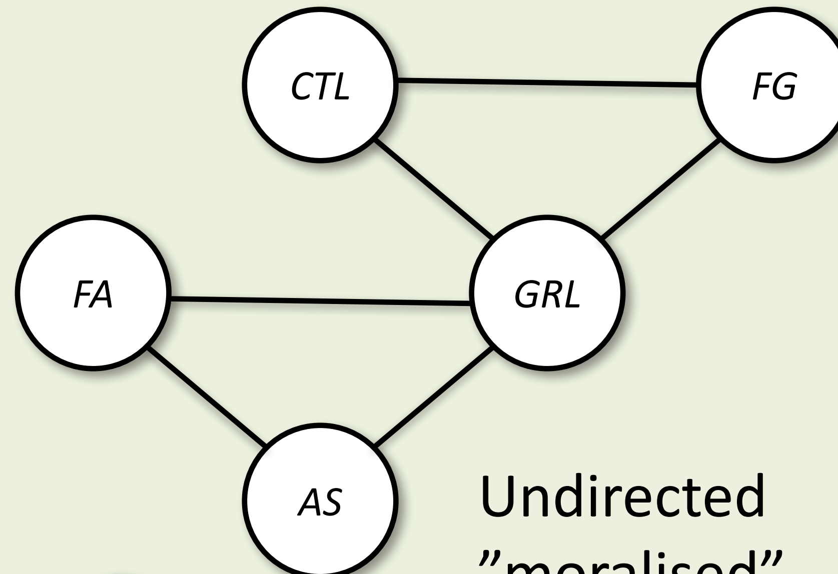
Sum-product algorithm

- Application of message passing to more general graphs
  - applies to chains, trees and poly-trees (directed PGMs with >1 parent)
  - 'sum-product' derives from:
    - **product** = product of incoming messages
    - **sum** = summing out the effect of RV(s) *aka* elimination
- Algorithm supports other operations (semi-rings)
  - e.g., max-product, swapping **sum** for **max**
  - Viterbi algorithm is the max-product variant of the forward algorithm for HMMs, solves the $\text{argmax}_{\boldsymbol{q}} P(\boldsymbol{q}|\boldsymbol{o})$

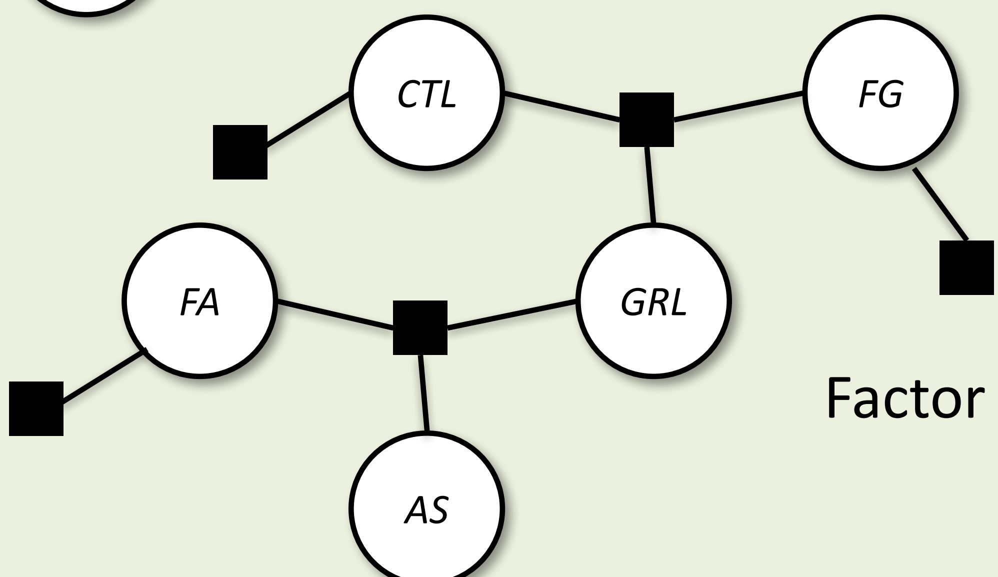
Application to Directed PGMS



Directed PGM



Undirected ”moralised” PGM

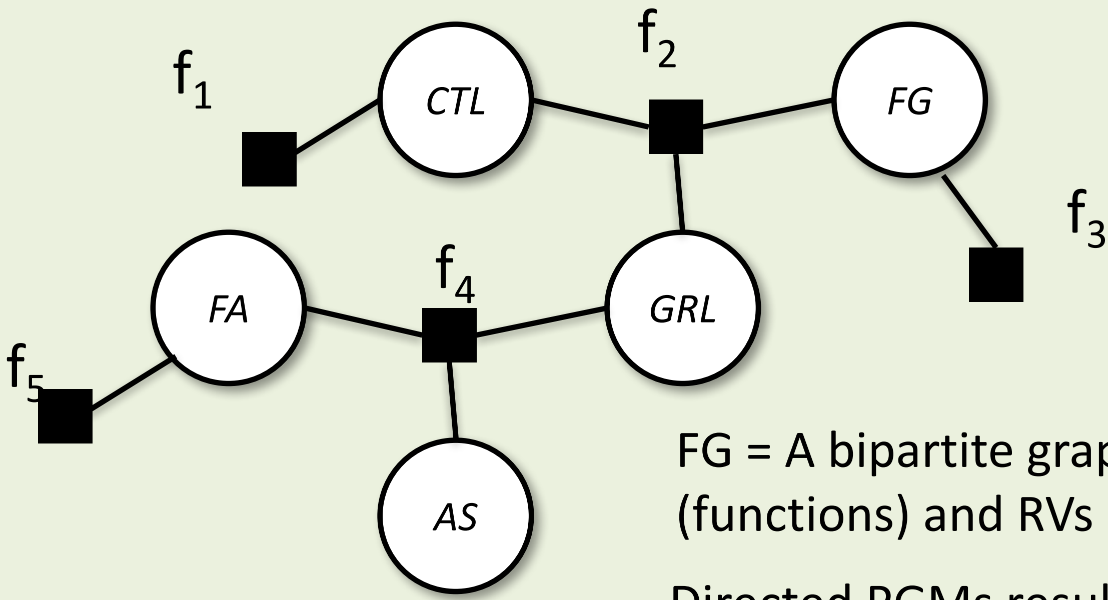


Factor graph

Factor graphs

$$f_1(CTL) = P(CTL)$$

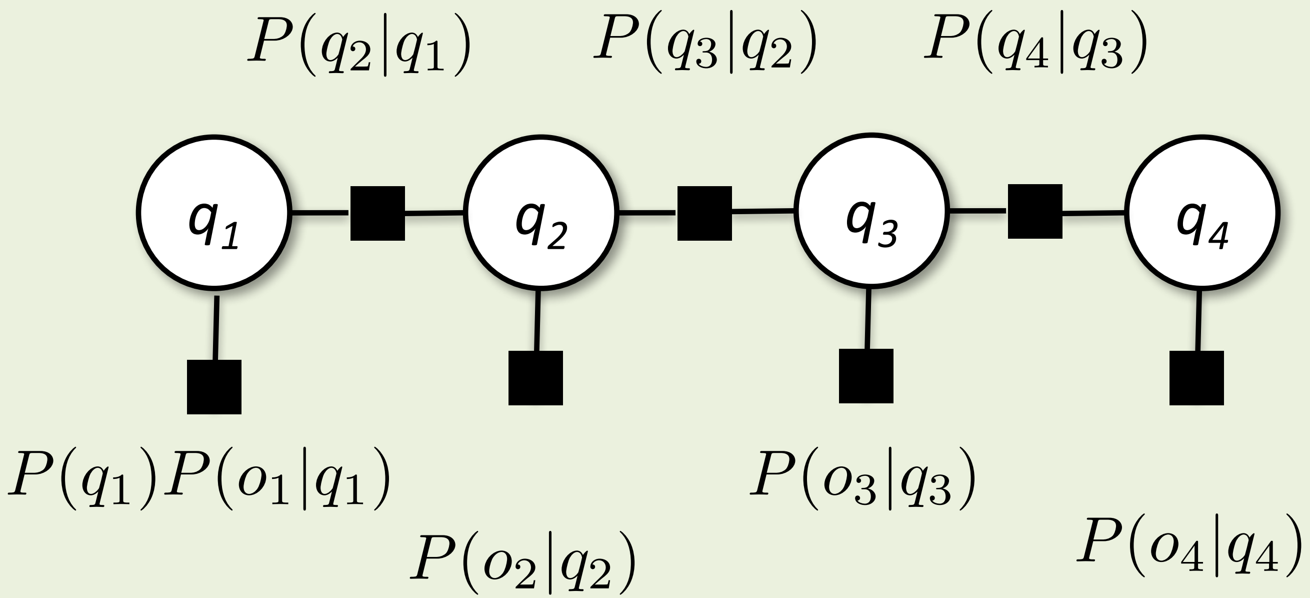
$$f_2(CTL, GRL, FG) = P(GRL|CTL, FG)$$



FG = A bipartite graph, with factors (functions) and RVs

Directed PGMs result in tree-structured FG

Factor graph for the HMM



Effect of observed nodes incorporated into unary factors

Sum-Product over Factor Graphs

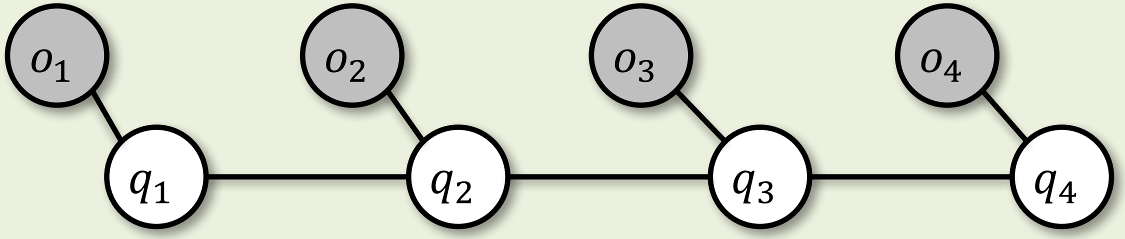
- Two types of messages :
  - between factors and RVs; and between RVs and factors
  - summarise a complete sub-graph
- E.g.,

$$m_{f_2 \to GRL}(GRL) = \sum_{CTL} \sum_{FG} f_2(GRL, CTL, FG) m_{CTL \to f_2}(CTL) m_{FG \to f_2}(FG)$$

- Structure inference as “gather-and-distribute”
  - gather messages from leaves of tree towards root
  - then propagate message back down from root to leaves

Undirected PGM analogue: CRFs

- Conditional Random Field: Same model applied to sequences
  - observed outputs are words, speech, amino acids etc
  - states are tags: part-of-speech, phone, alignment…
  - shared inference algo., i.e., sum-product / max-product
- CRFs are discriminative, model $P(\mathbf{q}|\mathbf{o})$
  - versus HMMs which are generative, $P(\mathbf{q},\mathbf{o})$
  - undirected PGM more general and expressive



Summary

- HMMs as example PGMs
  - formulation as PGM
  - independence assumptions
  - probabilistic inference using forward-backward
  - statistical inference using expectation maximisation
- Generalised inference method for U-PGMs
  - sum-product & max-product
  - factor graphs