# Pt. II A Concise Introduction to Categories (Notes)

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## 1 Categories

In order to properly understand how Alexandroff Spaces interact with the world of classical algebraic topology, we must introduce the language of category theory. The language of categories is utilized in many fields of mathematics, but in this part of the book, we will primarily explore how category theory is used in the study of finite spaces.

**Definition 1.1.** A *Category* $\mathcal{C}$ is a collection of *objects*, denoted $Ob(\mathcal{C})$ and a collection of *morphisms*, denoted $Mor(\mathcal{C})$, such that:

1. Every morphism has exactly one object as its **domain** and exactly one object as its **codomain** (they may be potentially the same object). If $f$ is a morphism with domain $X \in Ob(\mathcal{C})$ and codomain $Y \in Ob(\mathcal{C})$, we write this information as $f : X \to Y$.

2. If there exist morphisms $f : X \to Y$ and $g : Y \to Z$ for some $X, Y, Z \in Ob(\mathcal{C})$, then the there exists a morphism $gf : X \to Z$. $gf$ is called the **composite morphism** of $g$ and $f$.

3. Composition is *associative*. In other words, if there exist morphisms $f, g, h \in Mor(\mathcal{C})$ such that $f : A \to B$, $g : B \to C$ and $h : C \to D$, where $A, B, C, D \in Ob(\mathcal{C})$, then the morphism $h(gf)$ is equal to $g(hf)$, and so we simply denote this morphism $hgf$.

4. Every object $X$ has a unique morphism called the **identity morphism** whose domain and codomain are $X$, and for any morphism $f$ whose codomain is $X$ and any morphism $g$ whose domain is $X$, $1_X f$ is equal to $f$ and $g 1_X$ is equal to $g$.

The objects of the category can be thought of as atoms of information, while the morphisms between objects give information about how the objects are related. Some morphisms are of particular interest.

**Definition 1.2.** Given a category $\mathcal{C}$, a morphism $f : x \to y$ in $\mathcal{C}$ is an *isomorphism* if there exists $g : y \to x$ such that $fg = 1_y$ and $gf = 1_x$.

Many fundamental mathematical objects can be viewed in a categorical context.

**Example 1.3.** We denote **Set** to be the category where the objects are sets, and the morphisms are set maps. Objects of **Set** have no additional structure such as topology, binary operations, etc. For example, $\mathbb{R} \in Ob(\mathbf{Set})$ is just the set of all real numbers, without any addition operation, multiplication operation or any topology. A morphism in **Set** does not have to be continuous at all, it simply must be well-defined for every point in its domain. Note that the isomorphisms in **Set** are simply the bijective maps between sets.

One might gather that **Set** is quite large, and incredibly complicated. To make matters precise, we call a category $\mathcal{C}$ **large** if either $Ob(\mathcal{C})$ or $Mor(\mathcal{C})$ is not a set. Clearly, **Set** is large in this context. Vice versa, a category $\mathcal{C}$ is **small** if $Ob(\mathcal{C})$ and $Mor(\mathcal{C})$ *is* in fact a set. For example, the category **FSet**, whose objects are *finite* sets and morphisms are set maps, is a small category. A nice, and often necessary, property is the following:

**Definition 1.4.** A category $\mathcal{C}$ is *locally small* if for any $X, Y \in Ob(\mathcal{C})$ the collection of morphisms with domain $X$ and codomain $Y$, denoted $\mathcal{C}(X, Y)$, is a *set*.

Here are some more examples of categories that we have secretly been working with in the previous chapters:

1. A *Group* can be understood as a one-object category $\mathcal{C}$ where every morphism is an isomorphism. In any group $G$, we can construct its categorical version $\mathcal{C}$ by letting $Ob(\mathcal{C})$ be one object, namely $\{x\}$, and each element $g \in G$ correspond to a unique morphism $g : x \to x$ such that if $gh = k$ where $g, h, k \in G$, then $gh = k$ as compositions of morphisms. Observe that for any $g \in G$, its corresponding morphism in $\mathcal{C}$ has an inverse, namely the morphism corresponding to $g^{-1}$. Also note that this is another example of a small category.

2. The objects of the category **Top** are topological spaces, and the morphisms between spaces correspond to continuous maps. Note that the isomorphisms in this category are the homeomorphisms between spaces. Since the number of continuous maps between two spaces $X$ and $Y$ cannot exceed $|X|^{|Y|}$, we see that $\mathbf{Top}(X, Y)$ is a set, implying that **Top** is locally small.

3. A *Poset* $P$ can also be understood as a category such that each object in the category corresponds to a unique element of $P$, and a morphism $f : x \to y$ exists, where $x, y \in P$ iff $x \leq y$. In other words, the morphisms encode all the information about the order relation on the poset. Since the number of morphisms between two objects can be at most 2 and a poset is a set, we see that $P$ is a small category.

4. We can also define the category of posets, denoted $\mathcal{P}$. $Ob(\mathcal{P})$ is the collection of all posets (this is a large category). A morphism $f : P \to Q$ for any two posets $P, Q \in Ob(\mathcal{P})$ is an *order preserving function*. In order words a morphism $f : P \to Q$ corresponds to a function

$f : P \to Q$ such that for any $x, y \in P$ where $x \leq y$, $f(x) \leq f(y)$. Since the number of morphisms between $P$ and $Q$ cannot be larger than $|P|^{|Q|}$, we see that $\mathcal{P}(X, Y)$ is a set, implying that $P$ is locally small.

5. Moving in to the world of classical algebraic topology, it is clear that we can turn the collection of simplicial complexes into a category, denoted $\mathcal{SC}$. It objects are simplicial complexes, and its maps are simplicial maps. By an argument made in the previous example, this is also a locally small category.

6. Similarly, we can define the locally small category of *Ordered Simplicial Complexes*, denoted $\mathcal{OSC}$, where the objects are ordered simplicial complexes, and the morphisms are simplicial maps that preserve the ordering on the vertices, i.e. for a such a map $f : K \to L$, where $K$ and $L$ are ordered simplicial complexes, if $x \leq y$, where $x, y \in V(K)$, then $f(x) \leq f(y)$.

7. The category **FinTop** is the category whose objects are finite $T_0$ topological spaces and morphisms are continuous maps between them. Observe that **FinTop** is clearly included in **Top**, in the sense that every object and morphism that is in **FinTop** is also in **Top**. In such a case, we see that **FinTop** is a *subcategory* of **Top**.

8. Another notable subcategory of **Top** is that of *A-Spaces*, denoted $\mathcal{A}$, where we restrict our objects to just $T_0$ Alexandroff spaces, and morphisms are only continuous maps between them. Since every finite space is an Alexandroff Space, we can see that **FinTop** is a subcategory of $\mathcal{A}$.

This is by no means an exhaustive list of categories. In many instances, it becomes necessary to determine whether certain objects in a category were constructed from other objects, and in particular, if it is possible to generalize such constructions that exist in a particular category to an arbitrary category. For example, in **Set**, the Cartesian product of two sets $S_1, S_2$ is defined, namely, $S_1 \times S_2 = \{(s_1, s_2) | s_1 \in S_2, s_2 \in S_2$. One might ask if such a notion can be defined in an arbitrary category?

**Definition 1.5.** Given any category $\mathcal{C}$, the *product* of two objects $X, Y$ in $\mathcal{C}$ is an object denoted $Z$ such that the following holds: There exist morphisms $\pi_1 : Z \to X, \pi_2 : Z \to Y$, and for any other object $A$ with morphisms $f : A \to X$ and $g : A \to Y$, there exists a unique map $h : A \to Z$ such that $\pi_1 h = f$ and $\pi_2 h = y$. In other words, the following diagram commutes:

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, dashed, "\exists! h"] \arrow[dr, "g"] & \\ X & Z \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

If such a $Z$ exists (it might not), we denote it as $X \times Y$

It is important to solve following exercises to check for understanding.

**Exercise 1.6.**

1. In **Top**, the product of two topological spaces is nothing but the Cartesian product equipped with the product topology. (lemma 1.5.7).
2. In **Poset**, the product of two posets $P, Q$ exists and is the same product that is defined in definition 5.6.2.
3. In the category $\mathcal{SC}$, the product exists and is the same as in definition 5.6.3

**Exercise 1.7.** For a poset $P$ regarded as a category, what is the product of two elements $x, y \in P$?

# 2 Functors and Natural Transformations

"The purpose of inventing categories was to define functors, and the purpose of defining functors was to define natural transformations" - Samuel Eilenberg

## 2.1 Functors

In the previous chapters, it was shown that $A$-spaces were in bijective correspondence with posets. Could we make a similar statement about their respective categories?

**Definition 2.1.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, a *Functor* is made up of the following information:

1. A function $Ob(F) : Ob(\mathcal{C}) \to Ob(\mathcal{D})$
2. A function $Mor(F) : Mor(\mathcal{C}) \to Mor(\mathcal{D})$ that maps identity morphisms to identity morphisms and compositions to compositions. To be precise, for any two morphisms $f : X \to Y$ and $g : Y \to Z$ in $Mor(\mathcal{C})$, $Mor(F)(fg) = [Mor(F)(f)][Mor(F)(g)]$, and $Mor(F)(1_X) = 1_{F(X)}$.

From now on, we will drop the notation $Mor(F)$ and $Ob(F)$, as it is usually clear from the context as to which function is being discussed. The following commutative diagram illustrates the fact that functors preserve compositions of morphisms:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "F"] & Y \arrow[r, "g"] \arrow[d, "F"] & Z \arrow[d, "F"] \\
F(X) \arrow[r, "F(f)"'] & F(Y) \arrow[r, "F(g)"'] & F(Z)
\end{tikzcd}$$

Note that an identity functor $1_{\mathcal{C}}$ is simply the functor $1_{\mathcal{C}} : C \to C$ that is the identity map on both objects and morphisms. We now ask the reader to summarize results found in previous chapters using the language of category theory. The following exercises check for understanding.

**Exercise 2.2.** In **Cat**, the category where the objects are locally small categories and the morphisms are functors, prove the following:

1. **Poset** $\cong \mathcal{A}$
2. **FinTop** $\cong$ **FinPos** (**FinPos** is the category of finite posets)

$\cong$ denotes an isomorphism in **Cat**.

**Exercise 2.3.** Given two posets $\mathcal{P}$ and $\mathcal{Q}$ regarded as categories, describe a functor $F : \mathcal{P} \to \mathcal{Q}$. What properties must it satisfy?

**Exercise 2.4.** Consider a functor $F$ from a group $G$, thought of as a one-object category, to **Set**. In terms of language from standard group theory, what is $F$?

Note that in **Cat**, we can define the product of two categories $\mathcal{C} \times \mathcal{D}$ such that $Ob(\mathcal{C} \times \mathcal{D})$ is the collection of pairs $(c, d)$, where $c \in Ob(\mathcal{C})$ and $d \in Ob(\mathcal{D})$, and $Mor(\mathcal{C} \times \mathcal{D})$ is the collection of pairs $(f, g)$, where $f \in Mor(\mathcal{C})$ and $d \in Mor(\mathcal{D})$.

The statements of the previous exercise are deceptively simple, and give us a glimpse of the conceptual power that a categorical framework can allow.

## 2.2 Natural Transformations

When studying the collection of functors between two categories, it is often necessary to understand the relations between them. Natural Transformations are important in this regard.

**Definition 2.5.** Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* from $F$ to $G$, denoted $\eta : F \to G$, is made up of the following information:

1. For all $X \in Ob(\mathcal{C})$, there exists a morphism $\eta_X : F(X) \to G(X)$, called a *coordinate map*.
2. For any morphism $f : X \to Y$, the following diagram of objects $\mathcal{D}$ commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

   i.e $\eta_Y F(f) = \eta_X G(f)$.

An example of a natural transformation arises when considering functors between posets when viewed as categories.

**Proposition 2.6.** *Given two posets $\mathcal{P}$ and $\mathcal{Q}$ regarded as categories, and two functors $F, G : \mathcal{P} \to \mathcal{Q}$ (order-preserving functions from $P$ to $Q$), there exists a natural transformation $\eta : F \to G$ iff $F \leq G$, i.e. for all $x \in P$, $F(x) \leq G(x)$.*

*Proof.* Suppose there exists a natural transformation $\eta : F \to G$. Then for $x \in Ob(\mathcal{P})$, there exists $\eta_X : F(x) \to G(x)$, which implies that $F(x) \leq G(x)$. Conversely, suppose that for all $x \in P$, $F(x) \leq G(x)$, where $F$ and $G$ are now viewed as order preserving functions. As functors, this implies that for all $x \in Ob(\mathcal{C})$, there always exists a morphism (let us call it $\eta_x$) from $F(x) \to G(x)$. Now, we can define $\eta : F \to G$ using the $\eta_x$ as coordinate maps. Further, If $x \leq y$, where $x, y \in Ob(\mathcal{C})$, then we combine the fact that $F(x) \leq F(y)$ and $G(x) \leq G(y)$ with $F(x) \leq G(x)$ and $F(y) \leq G(y)$ via $\eta_x$ and $\eta_y$ to get the commutative diagram:

$$\begin{tikzcd}
F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"] & F(y) \arrow[d, "\eta_y"] \\
G(x) \arrow[r, "G(f)"'] & G(y)
\end{tikzcd}$$

Therefore, $\eta$ is a natural transformation from $F$ to $G$. ☐

Similar to how we can define isomorphisms between categories, we can define a similar concept for functors using natural transformations.

**Definition 2.7.** Two functors $F, G : \mathcal{C} \to \mathcal{D}$ are *naturally isomorphic* if there exists a natural transformation $\eta : F \to G$, and $\rho : F \to G$ such that for all objects $x \in \mathcal{C}$, $\eta_X$ is an isomorphism, with $\rho_X = \eta^{-1}$

Simply by interpreting the results for theorem 5.1.2 we observe another example of a natural transformation:

**Proposition 2.8.** *(via 5.1.2) There exists a natural transformation $\eta : |K(-)| \to id_{\mathcal{A}}$, where $|K(-)|, id_{\mathcal{A}} : \mathcal{A} \to$ **Top***

By Proposition 2.2.12, we observe the following:

**Proposition 2.9.** *Given two finite posets $\mathcal{P}$ and $\mathcal{Q}$ regarded as categories, there exists a natural transformation from $F$ to $G$, where $F, G : \mathcal{P} \to \mathcal{Q}$ iff $F \leq G$.*

This proposition suggests that the concept of a natural transformation "generalizes" the concept of a homotopy between two maps between spaces. To make this more precise, we introduce an equivalent definition of natural transformations.

**Definition 2.10.** Let $\mathcal{I}$ denote the interval category $\{0 \to 1\}$; or the category with the objects 0 and 1, along with only one non-trivial morphism $0 \to 1$. A natural transformation between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a functor $H : \mathcal{C} \times \mathcal{I} \to \mathcal{D}$ such that, on objects $H(x, 1) = F(x)$ and $H(x, 0) = G(x)$, where $x \in Ob(\mathcal{C})$, and on morphisms $H(f, 1) = F(f)$ and $H(f, 0) = G(f)$, where $f \in Mor(\mathcal{C})$.

Checking that these definitions are equivalent is left as an exercise to the reader.

## 2.3 Equivalences of Categories

Often, it is the case that two categories $\mathcal{C}$ and $\mathcal{D}$ are not isomorphic, but are very closely related. Relations between categories are understood via functors, so by more precisely classifying functors, we find that we can describe a larger variety of relationships between categories.

**Definition 2.11.** Given the functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, we say that $G$ is a *right adjoint* of $F$ iff $GF \simeq id_{\mathcal{C}}$, or in other words, $GF : \mathcal{C} \to \mathcal{C}$ is naturally isomorphic to the identity. Similarly, we say that $G$ is is a *left adjoint* of $F$ iff $FG \simeq id_{\mathcal{D}}$. We say that $\mathcal{C}$ *is equivalent to* $\mathcal{D}$ if there exists a pair of adjoint functors $F : \mathcal{C} \to \mathcal{D} : G$ such that $F$ is left adjoint to $G$ and $G$ is adjoint to $F$.

**Definition 2.12.** For locally small categories $\mathcal{C}, \mathcal{D}$, A functor $F : \mathcal{C} \to \mathcal{D}$ is a *full embedding* if, for any pair of objects $X, Y$ in $\mathcal{C}$, the map $F^* : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ given by $(f : X \to Y) \mapsto (Ff : F(X) \to F(Y))$ is bijective.

**Definition 2.13.** For locally small categories $\mathcal{C}, \mathcal{D}$, A functor $F : \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if for any object $d \in \mathcal{D}$, there exists $c \in \mathcal{C}$ such that $d \simeq Fc$, or $d$ is *isomorphic* to $Fc$.

**Theorem 2.14.** *For locally small categories $\mathcal{C}, \mathcal{D}$, if a functor $F : \mathcal{C} \to \mathcal{D}$ is a full embedding and essentially surjective, then it defines an equivalence of categories.*

*Proof.* In order to define a right adjoint $G : \mathcal{D} \to \mathcal{C}$, for an object $d \in \mathcal{D}$, we choose $c_d$ such that $\eta_d : d \simeq Fc_d$, and define $G(d) = c_d$. As for any morphism $f : d \to e$ in $\mathcal{D}$, we first choose $c_d$ such that $\eta_e : e \simeq Fc_e$, and define $G(d) = c_e$, and since $F$ is full and faithful, there exists a unique morphism $h : c_d \to c_e$ such that

$$\eta_e^{-1} \circ (Fh) \circ \eta_d = f$$

So, define $G(f) = h$. $G$ can be seen by the commutative diagram:

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[d, shift left, "\eta_c"] & e \arrow[d, shift left, "\eta_e"] \\
F(G(c)) \arrow[r, "F(G(f))"'] \arrow[u, shift left, "\eta_c^{-1}"] & F(G(e)) \arrow[u, shift left, "\eta_e^{-1}"]
\end{tikzcd}$$

This shows us that, in fact, the morphisms $\eta_x$ define a natural transformation between $FG$ and $id_{\mathcal{D}}$, and further, that the morphisms $\eta_x^{-1}$ define the inverse natural transformation between $id_{\mathcal{D}}$ and $FG$.

Now we show that $GF \simeq id_{\mathcal{C}}$ by the natural transformation given by the maps

$$\epsilon_c := G(\eta_{F(c)})$$

Here, $c \in \mathcal{C}$. Observe that $G(\eta_{F(c)}) : G(Fc) \to G(c_{F(c)}) = c$. Further, since $\eta^{-1}_{F(c)}$ is an isomorphism, so is $\epsilon_c^{-1}$ is by functoriality of $G$. Further, one can check that the following diagram commutes for any morphism $f : c \to d$ in $\mathcal{C}$:

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[d, shift left, "\epsilon_c"] & d \arrow[d, shift left, "\epsilon_d"] \\
GF(c) \arrow[r, "G(F(f))"'] \arrow[u, shift left, "\epsilon_c^{-1}"] & GF(d) \arrow[u, shift left, "\epsilon_d^{-1}"]
\end{tikzcd}$$

This tells us that $GF \simeq id_{\mathcal{C}}$. Along with the information that $FG \simeq id_{\mathcal{D}}$, we see that $\mathcal{C} \simeq \mathcal{D}$. $\square$

We have already seen an example of such an equivalence, and we leave this revelation as an exercise:

**Exercise 2.15.** Use (2.14) to prove the natural equivalence $\mathcal{OSC} \simeq \mathcal{SC}$ via the inclusion functor $i : \mathcal{OSC} \to \mathcal{SC}$, where $\mathcal{OSC}$ is the category of ordered simplicial complexes and $\mathcal{SC}$ is the category of simplicial complexes.

## 2.4 The Yoneda Lemma

**Definition 2.16.** Given an element $C \in \mathcal{C}$ for a locally small category $\mathcal{C}$, we can define a functor $h_C : \mathcal{C} \to \mathbf{Set}$ such that for any object $D$ in $\mathcal{C}$), $h_C(D) = Hom(C, D)$. For a morphism $f : D \to E$, we can define a set map $h_C(f) : Hom(C, D) \to Hom(C, E)$ by post-composition, i.e. $h_c(f)$ sends a morphism $g : C \to D$ to $f \circ g : C \to E$.

**Exercise 2.17.** Show that for all all objects $C \in \mathcal{C}$, the functor $h_C$ is well defined (it maps identities to identities, and so on).

**Definition 2.18.** A functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* if there exists an object $C \in \mathcal{C}$, such that $F$ is naturally isomorphic to $h_C$.

How many representable functors can there be, given a certain $h_C$? The Yoneda Lemma gives us a "upper bound" of sorts.

**Theorem 2.19** (The Yoneda Lemma)**.** *For any object $C \in \mathcal{C}$, and any functor $F : \mathcal{C} \to$ **Set**, the set of natural transformations from $h_C$ to $F$, denoted $Nat(h_C, F)$, is in bijective correspondence with $F(C) \in$ **Set**. So, $Nat(h_C, C) \simeq F(C)$ in **Set**.*

*Proof.* Consdier a natural transformation in $Nat(h_C, C)$ defined by maps $\eta_A : Hom(C, A) \to F(A)$. We now define the set map $\Phi : Nat(h^C, C) \to F(C)$ such that

$$\Phi(\eta) = \eta_C(id_C)$$

Further, we define a set map $\Psi : F(C) \to Nat(h^C, C)$(which will eventually become $\Phi^{-1}$) in the following way: For $d \in F(C)$, we define a natural transformation $\Psi(d) := \eta^d : h^C \to C$ given by the coordinate maps

$$\eta^d_X : (f : C \to X) \mapsto F(f)(d)$$

where $F(f) : F(C) \to F(X)$. Now, to show that they are inverses, taking a natural transformation $\eta \in Nat(h_C, C)$, we get:

$$\Psi(\Phi(\eta)) = \Psi(\eta_C(id_C)) = \eta^{\eta_C(id_C)}$$

Now, given a morphism $f : C \to X$, $\eta_X^{\eta_C(id_C)}$ sends $f$ to $F(f)(\eta_C(id_C))$. However, since $\eta$ is a natural transformation from $h_C$ to $F$, we have the following commutative diagram:

$$\begin{array}{ccc} Hom(C,C) & \xrightarrow{\eta_C} & F(C) \\ \big\downarrow{\scriptstyle h_C(f)} & & \big\downarrow{\scriptstyle F(f)} \\ Hom(C,X) & \xrightarrow{\eta_X} & F(X) \end{array}$$

The diagram tells us that $\eta_X\Big(h_C(f)(id_C))\Big) = F(f)(\eta_C(id_C))$, but by definition $h_C$:

$$\eta_X\Big(h_C(f)(id_C)\Big) = \eta_X(f \circ id_C) = \eta_X(f)$$

Therefore, $\eta_X^{\eta_C(id_C)} = \eta_X$, giving us that $\Psi$ is a left inverse of $\Phi$. Now, for any $d \in F(C)$:

$$\begin{aligned} \Phi(\Psi(d)) &= \Phi(\eta^d) \\ &= \eta^d_C(id_C) \\ &= F(id_C)(d) = d \end{aligned}$$

Therefore, $\Psi$ is a left inverse of $\Phi$, allowing us to conclude that $\Psi$ is a set inverse of $\Phi$ and giving us the set isomorphism $Nat(h^C, C) \simeq F(C)$. □

A similar statement can be made for contravariant functors.