A BRIEF INTRODUCTION TO DIFFERENTIAL FORMS IN $\mathbb{R}^n$ AND THE GAUSS-BONNET THEOREM IN $\mathbb{R}^3$

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Abstract. We first introduce basic concepts in differential geometry, including manifolds and differential forms, to discuss the proof and applications of Stokes' Theorem in $\mathbb{R}^n$, such as Brouwer's fixed point theorem. We then slightly shift gears to discuss geometry in $\mathbb{R}^3$ in order to discuss Chern's proof of the Gauss-Bonnet Theorem in $\mathbb{R}^3$. This paper serves as an educational tool; knowledge of multivariable calculus is needed, while all other concepts are introduced.

Contents

1. An Introduction to Differential Forms and Manifolds

We begin with a few introductory definitions. In particular, the concept of a differentiable manifold. A generic theme in differential geometry is that we associate an seemingly 'unknown' objects, such as manifolds, with 'known' objects, such as $\mathbb{R}^n$, so that we can study the local behavior of the object using concepts such as differential forms.

1.1. Differentiable Manifolds.

Definition 1.1. A differentiable manifold is a set M along with a set of injective maps $f_\alpha : U_\alpha \to M$, $U_\alpha \subset \mathbb{R}^n$ such that:

(1) $M = \bigcup_\alpha f_\alpha(U_\alpha)$

(2) For all α, β such that $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) = W \neq \emptyset$, then $f_\alpha^{-1}(W), f_\beta(W)$ are both open in $\mathbb{R}^n$. Further, $f_\alpha^{-1} \circ f_\beta$ and $f_\beta^{-1} \circ f_\alpha$ are differentiable.
(3) $\{U_\alpha, f_\alpha\}$, called the set of charts or coordinate systems is maximal in regards to both (1) (2)

In other words, an n dimensional manifold, denoted M^n is a set that locally 'looks' like $\mathbb{R}^n$. The second condition makes sure that if two coordinate systems overlap, then points that are 'close together' in one system are mapped to points that are also 'close together' in the other system. The last condition serves the purpose of inducing a topology on M^n. In brief, open sets on M can now be defined using open sets in $\mathbb{R}^n$. From this definition, it is clear that $\mathbb{R}^n$ is a manifold (as we can just map the topology on $\mathbb{R}^n$ to itself), but there are examples of manifolds whose global properties are drastically different from $\mathbb{R}^n$. For the sake of simplicity, we only look at the cases where the manifold satisfies Haussdorf's axiom and has a countable basis.

It is now useful to discuss the concept of orientability.

Definition 1.2. A differentiable manifold is *orientable* if there exists a differentiable structure $\{f_\alpha, U_\alpha\}$ such that for all α, β such that $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) = W \neq \emptyset$, the determinant of the differential map (known as the jacobian J) of $f^{-1} \circ g$ is positive.

We will now describe the concept of the tangent space of a manifold. In $\mathbb{R}^n$, a manifold in which we can straightforwardly compute derivatives, we can describe the tangent vectors of a curve $\alpha : I = [a, b] \to \mathbb{R}^n$ by taking the derivative of that vector. However, we cannot do such a thing on an arbitrary manifold. As a result:

Definition 1.3. Take $p \in M^n$, then let $\alpha : I \to M^n$ be smooth such that $\alpha(0) = p \in M$, then consider $D = \{\varphi : M \to \mathbb{R}\}$. Then the *tangent vector* to α at p is the map $\alpha'(0) : D \to \mathbb{R}$ such that $\alpha'(0)(\varphi) = \frac{d}{dt}(\varphi \circ \alpha)|_{t=0}$

We can associate the function $\alpha'(0)$ with the derivative of α at 0, because, given $\varphi : M \to \mathbb{R}$, we have that $\alpha'(0)(\varphi) = (\frac{d\varphi}{dx})\frac{d\alpha}{dt}$. In other words, $\alpha'(0)$ is dependent on $\frac{d\alpha}{dt}$, while allowing us to use the calculus in $\mathbb{R}^n$ to compute the derivative.

Definition 1.4. For M^n, the tangent space at a point $p \in M^n$ is denoted as $T_pM = \{\alpha'(0) \mid \alpha : I \to M^n, \alpha(0) = p\}$

In other words, the tangent space is the set of all tangent vectors. This next proposition allows for us to better understand the tangent space.

Proposition 1.5. *For $p \in M^n$, $dim(T_pM^n) = n$. Further, if $p \in f_\alpha(U_\alpha)$, then $span(\{\frac{\partial}{\partial x_j} \mid j \in [n]\}) = T_pM^n$, where $\{\frac{\partial}{\partial x_j}\}$ is the set of partial derivative operations of $\mathbb{R}^n$ with respect to the parametrization f.*

Proof. If $p \in f_\alpha(U_\alpha)$, where $U_\alpha \subset \mathbb{R}^n$ and $p = f(0, \ldots, 0)$, then for a curve $\alpha : I \to M^n$ with $\alpha(0) = p$ we have that $f^{-1}(\alpha(0)) = (x_1(t), \ldots, x_n(t))$. For simplicity's sake[1], we conflate $f^{-1} \circ \alpha$ with f, so that $f(x) = (x_1(t), \ldots, x_n(t)), x \in \mathbb{R}^n$.

[1]This is a standard practice in differential geometry

Then, it follows that for $\varphi : M \to \mathbb{R}$, that $\varphi \circ f(x) = \varphi(x_1(t), \ldots, x_n(t))$, and so

$$a'(0)\varphi = \frac{d}{dt}(\varphi \circ \alpha)|_{t=0} \tag{1.6}$$

$$= \frac{d}{dt}\varphi(x_1(t), \ldots, x_n(t))|_{t=0} \tag{1.7}$$

$$= \sum_{i=0}^{n} \left(\frac{\partial \psi}{\partial x_i}\right)_0 x_i'(0) \tag{1.8}$$

Therefore, we can write $a'(0)$ as:

$$a'(0) = \sum_{i=0}^{n} x_i'(0)\left(\frac{\partial}{\partial x_i}\right)_0 \tag{1.9}$$

Consequently, it follows that the tangent space is n-dimensional and $\{\frac{\partial}{\partial x_i}\}$ is a basis for the space. □

Proposition 1.10. *If $M = \mathbb{R}^n$, then for $p \in \mathbb{R}^n$, $T_p(\mathbb{R}^n) = \{q - p \mid q \in \mathbb{R}^n\}$. In other words, the tangent space is the set of all vectors whose origins are at p. We leave this proof to the reader*

1.2. **Differentiable Forms.** Now, we can slightly shift gears to discuss the concept of forms on manifolds.

Definition 1.11. The *dual space* at p is denoted as the set of linear functions $(T_pM)^* = \{\varphi : T_pM \to \mathbb{R}\}$, where $T_pM = \{q - p \mid q \in M^n\}$ is the tangent space at p. Further we define $\Lambda^k(T_pM)^* = \{\varphi : T_pM \times T_pM \times \ldots (k-times) \times T_pM \to \mathbb{R}\}$, or the set of all real linear functions that take k elements of T_pM that are k-linear and alternate, meaning that if (for example) $\varphi \in \Lambda^2(T_pM)^*$, then $\varphi(v_1, v_2) = -\varphi(v_2, v_1)$.

Then, the definition of a 1-form and 0-form follows.

Definition 1.12. An *exterior 1-form* is a function $\omega : M^n \to (T_pM)^*$.

In other words, the 1-form assigns each point p to a real linear function on M^n. Further, we can find a basis for these forms in the following manner. Consider the set of derivatives $\{d(x_i)_p \mid 1 \leq i \leq n\}$, where $x_i|_p : M^n \to \mathbb{R}$ is the projection of the i'th component of p in the following way: Locally, p is parametrized by some $U \subset \mathbb{R}^n$, so we can assign p, along with its parametrized neighborhood, coordinates of $\mathbb{R}^n$ and then construct $\{dx_i\}$.

We can see that

$$d(x_i)_p(e_j) = \frac{\partial x_j}{\partial x_i} = \delta_{ij}$$

So, the set serves as a basis for $(T_pM)^*$, implying that we can write the 1-form as

$$\omega = \sum_{i=1}^{n} a_i dx_i \tag{1.13}$$

Where $a_i : M^n \to \mathbb{R}$ are functions.
a_i is called a *0-form.* If each a_i is differentiable, then ω is called a *differentiable 1- form.* In order to define forms of higher degree, we need to introduce a new concept.

Definition 1.14. A *wedge product* is denoted by $\wedge$ such that $(\varphi_1 \wedge \varphi_2)(v_1, v_2) = \det(\varphi_i(v_j))$, where $\varphi_1, \varphi_2 \in (T_pM)^*$ and $v_1, v_2 \in T_pM$. By properties of the determinant, we can see that $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$ if $i \neq j$.

In M^n, one can think of the wedge product as the area of the parallelogram formed between $(\varphi_1(v_1), \varphi_1(v_2))$ and $(\varphi_2(v_1), \varphi_2(v_2))$. Now, for the generic definition of a $k - form$

Definition 1.15. An *exterior differentiable k-form* is a function $\omega : M^n \to (\Lambda^k T_pM)^*$

It also follows that we can write

$$\omega = \sum_{i_1 < \cdots < i_k}^{n} a_{i_1 < \cdots < i_k} (dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k}) \quad \text{where } i_1, \ldots i_k \in [n] \tag{1.16}$$

Where $(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p$ is a wedge product of 1-forms, and each $a_I : M^n \to \mathbb{R}$ is a differentiable map. Similarly to the basis for the 1-form, one can check that $\{(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p \mid i_1, \ldots i_k \in [n]\}$ serves as a basis for $(\Lambda^k T_pM)^*$

Example 1.17. If ω is a 2-form , then locally in most generic form,

$$\omega = a_{12}(dx_1 \wedge dx_2) + a_{13}(dx_1 \wedge dx_3) + a_{23}(dx_2 \wedge dx_3)$$

We can more easily write 1.15 as

$$\omega = \sum_{I}^{n} a_I dx_I$$

We can also naturally define wedge product between two forms

Definition 1.18. If $\omega = \sum_I^n a_I dx_I$ is a k-form and $\varphi = \sum_J^n a_J dx_J$ is an s-form, then we define the k+s-form to be:

$$\omega \wedge \varphi = \sum_{IJ} a_I b_J (dx_I \wedge dx_J)$$

Where the IJ sums of the product $dx_I \wedge dx_J$ are the sums of the $\binom{n}{k}$ indices of $\{1, \ldots, n\}$ corresponding to ω and φ.

Now that we have an understanding of forms, we can now relate a k-form on one manifold to another k-from on another manifold if given a differentiable map between the two manifolds. In particular:

Definition 1.19. Given manifolds M^n and N^m and differentiable map $f : N^m \to M^n$, then the *pull-back map* is defined as $f^* : \bigcup_{k \in [n]} \Lambda^k (T_pM^n)^* \to \bigcup_{k \in [n]} \Lambda^k (T_pN^m)^*$, such that if ω is a k-form in M^n, then $f^*\omega$ is a k-form in N^m given by

$$(f^*\omega)(p)(v_1, \ldots, v_k) = \omega(f(p))(df_p(v_1), \ldots, df_p(v_k))$$

Where $v_1, \ldots, v_k \in T_pN^m$, $p \in N^m$, and $df_p : T_pN^m \to T_pM^n$ is the differential map. If g is a 0-from in M^n then we define

$$f^*g = g \circ f$$

Proposition 1.20. *If ω and φ are k-forms in M^n, g is a zero form in M^n, and we are given $f : N^m \to M^n$, then*

(1) $f^*(\omega + \varphi) = f^*\omega + f^*\varphi$
(2) $f^*(g\omega) = f^*(g) f^*(\omega)$

(3) *If $\varphi_1, \dots, \varphi_k$ are 1-forms on M^n then $f^*(\varphi_1 \wedge \cdots \wedge \varphi_k) = f^*(\varphi_1) \wedge \cdots \wedge f^*(\varphi_k)$*

We leave these proofs to the reader. It should be noted, however, that the pull-back map seems to be a natural definition due to these properties.

1.3. **Exterior Derivatives.** Now comes an important concept, central to Stokes' theorem lemma:

Definition 1.21. An *exterior differential* of a k-form ω is a $k+1$ form $d\omega$, such that if $\omega = \sum_I^n a_I dx_I$, then $d\omega = \sum_I^n d(a_I) \wedge dx_I$

From multivariable calculus, we know that for any 0-form $a_I : \mathbb{R}^n \to \mathbb{R}$, the derivative of a is given as

$$da = \sum_{i=0}^{n} \frac{\partial a}{\partial x_i} dx_i$$

which is a 1-form.

Example 1.22. If the 1-form $\omega = xydx + x^2y^2dy$, then:

$$\begin{aligned}
d\omega &= d(xy) \wedge dx + d(x^2y^2) \wedge dy \\
&= (ydx + xdy) \wedge dx + (2xy^2dx + 2yx^2dy) \wedge dy \\
&= x(dy \wedge dx) + 2xy^2(dx \wedge dy) \quad \text{(because } dx \wedge dx = 0\text{)} \\
&= (2xy^2 - x)(dx \wedge dy) \quad \text{(because } dx \wedge dy = -dy \wedge dx\text{)}
\end{aligned}$$

We will now prove some minor properties of the exterior derivative to arrive at a significant conclusion.

Proposition 1.23. (1) *For any k-forms, ω_1, ω_2, $d(\omega_1) + d(\omega_2) = d(\omega_1 + \omega_2)$*
(2) *If ω is a k-form and φ is an s-form, then $d(\omega \wedge \varphi) = d\omega \wedge \varphi + (-1)^k \omega \wedge d\varphi$*
(3) *For any k-form ω, $d^2(\omega) = d(d\omega) = 0$.*

Proof. (1) We we know from multivariable calculus that for 0-forms $a, b : \mathbb{R}^n \to \mathbb{R}$, that by distributivity of the derivative operator, $d(a+b) = da+db$. Now, let $\omega_1 = \sum_I a_I dx_I$ and $\omega_2 = \sum_I b_I dx_I$. Then,

$$\begin{aligned}
d(\omega_1 + \omega_2) &= d\Big(\sum_I a_I dx_I + \sum_I b_I dx_I\Big) \\
&= d\Big(\sum_I (a_I + b_I) dx_I\Big) \\
&= \sum_I d(a_I + b_I) \wedge dx_I \\
&= \sum_I d(a_I) \wedge dx_I + \sum_I d(b_I) \wedge dx_I = d(\omega_1) + d(\omega_2)
\end{aligned}$$

Proving (1).

(2) Let ω and φ be as described. Then, by definition 1.17

$$\begin{aligned}
d(\omega\wedge\varphi) &= d\Big(\sum_{IJ} a_I b_J (dx_I \wedge dx_J)\Big)\\
&= \sum_{IJ} d(a_I b_J)\wedge dx_I \wedge dx_J\\
&= \sum_{IJ} d(a_I) b_J \wedge dx_I \wedge dx_J + \sum_{IJ} a_I d(b_J)\wedge dx_I \wedge dx_J\\
&= (-1)^s \sum_{IJ} b_J dx_J \wedge (d(a_I)\wedge dx_I) + (-1)^k \sum_{IJ} a_I d(x_I)\wedge (db_J \wedge dx_J)\\
&= \sum_{IJ} (d(a_I)\wedge dx_I)\wedge b_J dx_J + (-1)^k \sum_{IJ} a_I d(x_I)\wedge (db_J\wedge dx_J)\\
&= d\omega\wedge\varphi + (-1)^k \omega\wedge d\varphi
\end{aligned}$$

Thus proving (2).

(3) We first prove the proposition for a 0-form. Suppose $f:\mathbb{R}^n\to\mathbb{R}$ is a 0-form. Then,

$$\begin{aligned}
d(f) &= \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} dx_i\\
d(d(f)) &= \sum_{j=0}^{n} \frac{\partial}{\partial x_j}\Big[\sum_{i=0}^{n}\frac{\partial f}{\partial x_i} dx_i\Big]\wedge dx_j\\
&= \sum_{j=0}^{n}\sum_{i=0}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j\\
&= \sum_{i<j} \frac{\partial^2 f}{\partial x_i\partial x_j} - \frac{\partial^2 f}{\partial x_j x_i} dx_i\wedge dx_j \quad (\text{because } dx\wedge dy = -dy\wedge dx)\\
&= 0 \quad \Big(\text{because } \frac{\partial^2 f}{\partial x_j x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}\Big)
\end{aligned}$$

Now, for the general case k-form, and due to (1), we only need to consider when $\omega = a_I dx_I$. Due to (2), we have that

$$d\omega = da_I \wedge dx_I - (-1)^k a_I \wedge d^2 x_I$$

However,

$$d^2 x_I = d(1)\wedge dx_I = 0 \wedge dx_I = 0$$

By definition $d\omega = da_I \wedge dx_I$, and so

$$\begin{aligned}
d^2\omega &= d(da_I \wedge dx_I)\\
&= d^2 a_I \wedge dx_I + (-1) da_I \wedge d^2 x_I \quad \text{from(2)}\\
&= 0
\end{aligned}$$

$d^2 a_I = 0$ from the case for 0-forms. So $d^2\omega = 0$ for all ω.

□

Note that the exterior derivative is the generalized form of the derivative in multivariable calculus which we can use for in wider variety of cases in the following sense: In multivariable calculus, the derivative of a 0-form in $\mathbb{R}^n$ is a $1\times n$ matrix, which can correspond to a 1-form. Now, we can take derivatives of higher forms. it is also interesting to note that the pullback map and the exterior derivative are related in an important way.

Proposition 1.24. *If $f : M^n \to N^m$, and ω is a k-form in N^m, then $d(f^*\omega) = f^*(d\omega)$. In other words, if two forms are related by a pullback map, then their derivatives are similarly related.*

Proof. Let $\omega : N^m \to \mathbb{R}$ be a 0-form in N^m. Then,

$$\begin{aligned} f^*(d\omega) &= f^*\Big(\sum_{i\in m} \frac{\partial \omega}{\partial x_i} dx_i\Big) \\ &= \sum_{ij} \frac{\partial \omega}{\partial x_i}\frac{\partial f}{\partial y_i} dy_i \quad \text{(by definition of pullback)} \\ &= \sum_i \frac{\partial(\omega \circ f)}{\partial x_i} dx_i \\ &= d(f^*(\omega)) \end{aligned}$$

Now let $\omega = \sum_I a_I dx_I$ be a k-from in N^m. Then,

$$\begin{aligned} f^*(d\omega) &= f^*(\sum_I d(a_I) \wedge dx_I) \\ &= \sum_I f^*(d(a_I) \wedge dx_I) \quad \text{(prop. 1.19)} \\ &= \sum_I f^*(d(a_I)) \wedge f^*(dx_I) \\ &= \sum_I d(f^* a_I) \wedge f^*(dx_I) \quad \text{(from the 0-form result)} \\ &= d(f^*(\sum_I a_I dx_I)) = d(f^*\omega) \end{aligned}$$

Which concludes the proof. □

1.4. **Integration of Forms.** Forms are also useful because we can integrate them along manifolds. Before we discuss integration, we discuss representation of forms.

Definition 1.25. If $f_\alpha : U_\alpha \subset \mathbb{R}^n \to M^n$, and ω is a k-from in M^n defined at $p \in f_\alpha(U_\alpha)$, then the *representation* of ω at U_α is ω_α such that

$$\omega_\alpha = f_\alpha^* \omega$$

Proposition 1.26. *If $f_\alpha : U_\alpha \subset \mathbb{R}^n \to M^n$, $f_\beta : U_\beta \subset \mathbb{R}^n \to M^n$ such that $p \in f_\alpha(U_\alpha) \cap f_\beta(U_\beta)$, then for any k-form ω defined at p,*

$$(f_\beta^{-1} \circ f_\alpha)^* \omega_\beta = \omega_\alpha$$

Proof. We use the definition of a pull-back map.

$$\begin{aligned} (f_\beta^{-1} \circ f_\alpha)^* \omega_\beta(v_1, ..., v_k) &= \omega_\beta(df_\beta^{-1}(p))\big(d(f_\beta^{-1})_p(v_1), \dots, d(f_\beta^{-1})_p(v_k)\big) \\ &= \omega(p)\big((df_\beta \circ df_\beta^{-1})_p(v_1), \dots, (df_\beta \circ df_\beta^{-1})_p(v_k)\big) \\ &= \omega_\alpha(v_1, ..., v_k) \end{aligned}$$

□

Definition 1.27. Let M^n be a compact oriented manifold, and $\omega = a(x_1 \dots x_n)dx_1 \wedge \dots \wedge dx_n$ be a n-form defined on $A \subset M^n$, and define K to be the *support* of ω given as $K = \{p \in M^n \mid \omega(p) \neq 0\}$ such that K is compact. Then, if $K \subset V_\alpha$, where $V_\alpha = f_\alpha(U_\alpha)$ for some α, we then denote $\omega_\alpha = f_\alpha^*(\omega) = a_\alpha dx_1 \wedge \dots \wedge dx_n$, and define

$$\int_A \omega = \int_{V_\alpha} \omega = \int_{U_\alpha} \omega_\alpha = \int a_\alpha \; dx_1 \dots dx_n$$

Note that M^n must be compact and K closed due to convergence problems that will arise in later cases of integration.

Proposition 1.28. *The definition of integration of forms on manifolds is well-defined. In particular, if the support K of ω is contained in both $V_\alpha = f_\alpha(U_\alpha)$ and $V_\beta = f_\beta(U_\beta)$, then*

$$\int_{U_\alpha} \omega_\alpha = \int_{U_\beta} \omega_\beta$$

Proof. Due to the fact that $K \subset V_\alpha \cap V_\beta$ we can shrink V_β so that $V_\beta = V_\alpha$. Now, we know by 1.25 that $(f_\beta^{-1} \circ f_\alpha)^* \omega_\beta = \omega_\alpha$, so by definition of pull-back maps, let $F = f_\beta^{-1} \circ f_\alpha$, then we have

$$\begin{aligned} F^* \omega_\beta &= \omega_\alpha \\ det(dF) a_\alpha dx_1 \wedge \dots \wedge dx_n &= \omega_\alpha \end{aligned}$$

Where $a_\alpha = a_\beta(F_1(y_1, \dots y_n), \dots, F_n(y_1, \dots, y_n))$, and $y_i \in U_\beta$, $y_i \in U_\alpha$. We also know that by the substitution of variables formula given in multivariable calculus, it follows that

$$\int_{U_\alpha} a_\alpha dx_1 \wedge \dots \wedge dx_n = \int_{U_\beta} det(dF) a_\beta dx_1 \wedge \dots \wedge dx_n$$

Since M^n is oriented, $det(dF) > 0$, so we have that

$$\int_{U_\alpha} \omega_\alpha = \int_{U_\beta} \omega_\beta$$

□

Suppose K of ω lies in multiple surface patches, meaning that there does not exist $f_\alpha(U_\alpha)$ such that $K \subset f_\alpha(U_\alpha)$. In this case, we know that since K is closed and M^n is compact, K must also be compact. Given an open covering of patches $\{f_\alpha(U_\alpha)\}_{\alpha \in \Lambda}$ of K, it follows by definition of compactness that there exists a finite open subcover of K given by $\{f_\alpha(U_\alpha)\}_{\alpha \in [n]}$. We use this fact as motivation for the following definition.

Definition 1.29. Given a finite covering $\{V_\alpha\}$ of a compact manifold M, a *partition of unity* subordinate to $\{V_i\}$ is a finite family of differentiable real functions $\{\varphi_1, \dots, \varphi_n\}$ on M such that:

(1) $\sum_i^n \varphi_i(x) = 1$, for all $x \in M$

(2) $0 \leq \varphi_i(x) \leq 1$, for all $x \in M$, $i \in [n]$

(3) For all $i \in [n]$, there exists $V_i \in \{V_\alpha\}$ such that the support K_i of φ_i is contained in V_i.

Note that if ω is defined on M, the support of $\varphi_i \omega$ is contained in V_i. This allows us to define the integration of a form across multiple surface patches in the following way:

Definition 1.30. Given a finite covering $\{V_\alpha\}$ of a compact manifold M, and a subordinate partition of unity $\{\varphi_1, \ldots, \varphi_n\}$, we define

$$\int_M \omega = \sum_i^n \int_{V_i} \varphi_i \omega$$

It follows in straightforward manner that this integral is well defined, and does not depend on the covering. The proof of the existence of a partition of unity given a compact manifold is beyond the scope of this paper.

We have at present introduced and discussed most of the language needed to understand Stokes' Theorem.

2. Stokes' Theorem

Stokes' Theorem is about manifolds with boundaries, which is larger than the class of manifolds. For example, a cylinder of radius r and length d oriented in the $z-axis$ is not a manifold, because, roughly speaking, the edges of the cylinder do not locally 'look' like R^n. However, the area around the edges does look like the half plane, $H^2 = \{(x_1, x_2) \mid x_1 \leq 0\}$. This motivates the following definition:

Definition 2.1. A differentiable manifold with a regular boundary is a set M along with a set of injective maps $f_\alpha : U_\alpha \subset H^n \to M$, $H^n = \{(x_1, x_2, \ldots, x_n) \mid x_1 \leq 0\}$ (H^n is the n-dimensional half-plane) such that each U_α is open in H^n. Further,

(1) $M = \bigcup_\alpha f_\alpha(U_\alpha)$
(2) For all α, β such that $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) = W \neq \emptyset$, then $f_\alpha^{-1}(W), f_\beta(W)$ are both open in $\mathbb{R}^n$. Further, $f_\alpha^{-1} \circ f_\beta$ and $f_\beta^{-1} \circ f_\alpha$ are differentiable.
(3) $\{U_\alpha, f_\alpha\}$, is maximal in regards to both (1) (2)

The above definition is similar to that of a differential manifold, except for the fact that $\mathbb{R}^n$ is replaced by H^n. Intuitively, the edge of the half plane is what results in a boundary. We can further explore this in the following proposition

Lemma 2.2. *For a manifold M and $p \in M$, if there exists α such that $p = f_\alpha(0, x_2, \ldots, x_n)$ where $x_i \in \mathbb{R}$, then p is on the boundary of M, denoted ∂M. Further, if there exists β such that $p \in f_\beta(U_\beta)$. Then 'p' is still on the boundary, or $p = f_\beta(0, x_{2'}, \ldots, x_{n'})$ for some $x_{i'} \in \mathbb{R}$.*

Proof. Assume, for the sake of contradiction that there where exists α such that $p = f_\alpha(0, x_2, \ldots, x_n)$, but there also there exists β such that $p = f_\beta(x_{1'}, x_{2'}, \ldots, x_{n'})$ such that $x_1 \neq 0$. Then, if $W = f_\alpha(U_\alpha) \cap f_\beta(U_\beta)$, define $f : f_\beta^{-1}(W) \to f_\alpha^{-1}(W)$ such that $f(x) = f_\alpha^{-1} \circ f_\beta(x)$, for $x \in f_\beta^{-1}(W)$. It follows that f is both bijective and differentiable. By the inverse function theorem, f^{-1} will take a neighborhood $V \subset U_\alpha$ such that $f_\alpha^{-1}(p) \in V$ to another neighborhood $U \subset U_\beta$ such that $f_\beta^{-1}(p) \in U$. This in turn implies that there exists points $(x_1, \ldots, x_n) \in U_\alpha$ such that $x_1 > 0$, which is a contradiction because $U_\alpha \subset H^n$. Therefore, the boundary is well defined, because it does not change with parametrization. $\square$

The above proof suggests that the boundary of a manifold is also a manifold. To be precise:

Lemma 2.3. *Given M^n with a boundary, ∂M, it follows that ∂M is a manifold of dimension (n-1). Also, if M^n is oriented, so is ∂M*

Proof. If $\{f_\alpha, U_\alpha\}$ is a differentiable structure on M^n, then consider $\{f'_\alpha, U'_\alpha\}$ where $U'_\alpha = U_\alpha \cap \{(x_1, \dots, x_n) \mid x_1 = 0\}$, and $f'_\alpha : U'_\alpha \to M^n$ such that $f'_\alpha(x) = f_\alpha(x)$ for all $x \in U'_\alpha$. Due to $\{f_\alpha, U_\alpha\}$ being a differentiable structure, it must be that $\{f'_\alpha, U'_\alpha\}$ is a differentiable structure on ∂M, and since $x_1 = 0$, it follows that ∂M is $n-1$ dimensional.

To see the orientation, we know that, by definition of orientation on M there exists a differentiable structure $\{f_\alpha, U_\alpha\}$ such that for α, β where $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) = W \neq \emptyset$,

$$det(f_\alpha^{-1} \circ f_\beta) > 0 \tag{2.4}$$

As in the first part of the proof, consider $\{f'_\alpha, U'_\alpha\}$. Then, due to (2.4), for all α, β where $f'_\alpha(U'_\alpha) \cap f'_\beta(U'_\beta) = W' \neq \emptyset$, it must be that $det(f'^{-1}_\alpha \circ f'_\beta) > 0$, so ∂M is also oriented due to the orientation of M. $\square$

Now, we can finally state and prove Stokes' Theorem.

Theorem 2.5. *Consider a differentiable compact manifold M with boundary ∂M, and a n-1 form ω defined on M. Let $i : \partial M \to M$ be the inclusion map, defined as $i(x) = x$, for all $x \in \partial M$. Then,*[2]

$$\int_M d\omega = \int_{\partial M} i^*\omega$$

Proof. Two major cases arise with regards to the closed support K (of ω).

(1) If there exists $V = f_\alpha(U)$ such that $K \subset V$.

Let

$$\omega = \sum_j a_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n$$

Then,

$$d\omega = \Big(\sum_{j=1}^{n} (-1)^{j-1} \frac{\partial a_j}{\partial x_j}\Big) dx_1 \wedge \cdots \wedge dx_n$$

Now, suppose that $K \cap \partial M = \emptyset$. It follows that by definition of the inclusion map i, $\int_{\partial M} i^*\omega = 0$. Since $a_j : U \to \mathbb{R}$, we extend each a_j to H^n in the following manner.

$$a_j(\vec{x}) = \begin{cases} a_j(\vec{x}) & \text{if } \vec{x} \in U \\ 0 & \text{if } \vec{x} \in H^n \setminus U \end{cases}$$

Consider a parallelepiped $Q \subset H^n$, such that

$Q = \{[x_1^1, x_1^2] \times \cdots \times [x_n^1, x_n^2]\}$ where $x_i^1 \leq x_i \leq x_i^2$, for $(x_1, \dots, x_i, \dots, x_n) \in f^{-1}(K)$

[2]There is a slight abuse of notation here. In this case, ω is actually the representation of the $n-1$ form on the manifold. This was done because we can only integrate the representation of a form (we can only integrate ω_α)

In other words, Q is the smallest parallelepiped containing $f^{-1}(K)$. Note that Q exists because K is compact (and so is $f^{-1}(K)$). Therefore, we have:

$$\begin{aligned}
\int_M d\omega &= \int_U \Big(\sum_{j=1}^{n}(-1)^{j-1}\frac{\partial a_j}{\partial x_j}\Big)dx_1\dots dx_n \\
&= \sum_{j=1}^{n}(-1)^{j-1}\int_Q \Big(\frac{\partial a_j}{\partial x_j}\Big)dx_1\dots dx_n \\
&= \sum_{j=1}^{n}(-1)^{j-1}\int_{x_1^1}^{x_1^2}\cdots\int_{x_j^1}^{x_j^2}\cdots\int_{x_n^1}^{x_n^2}\Big(\frac{\partial a_j}{\partial x_j}\Big)dx_1\dots dx_n \\
&= \sum_{j=1}^{n}(-1)^{j-1}\int_{x_1^1}^{x_1^2}\cdots\int_{x_n^1}^{x_n^2}\cdots\int_{x_j^1}^{x_j^2}\Big(\frac{\partial a_j}{\partial x_j}\Big)dx_j dx_1\dots dx_{j-1}dx_{j+1}\dots dx_n \\
&= \sum_{j=1}^{n}(-1)^{j-1}\int_Q \Big[a_j(x_1,\dots,x_j^1,\dots,x_n) - a_j(x_1,\dots,x_j^0,\dots,x_n)\Big]dx_1\dots dx_{j-1}dx_{j+1}\dots dx_n \\
&= 0
\end{aligned}$$

For the last step, it was realized that

$$a_j(x_1,\dots,x_j^1,\dots,x_n) = a_j(x_1,\dots,x_j^0,\dots,x_n) = 0$$

Now, consider when $K\cap\partial M \neq \emptyset$. In this case, we know that if $p\in M$ is on the boundary, then $p = f(0,x_1,\dots,x_n)$, for all parametrization maps f. Thus, if

$$\omega = \sum_j a_j(x_1,\dots,x_n)dx_1\wedge\dots\wedge dx_{j-1}\wedge dx_{j+1}\wedge\dots\wedge dx_n$$

Then,

$$i^*\omega = a_1(0,\dots,x_n)dx_2\wedge\dots\wedge dx_n$$

We construct a parallelepiped $Q\subset H^n$ similar to the previous subcase, except that

$$x_1^1 \le x_1 \le 0, \text{ while } x_i^1 \le x_i \le x_i^2 \text{ for } i\ge 2 \text{ where } (x_1,\dots,x_i,\dots,x_n)\in f^{-1}(K)$$

So, as in the previous subcase, Q is the smallest parallelepiped that contains $f^{-1}(K)$. Now,

$$\begin{aligned}
\int_M d\omega &= \sum_{j=1}^{n}(-1)^{j-1}\int_Q \Big(\frac{\partial a_j}{\partial x_j}\Big)dx_1\dots dx_n \\
&= \int_Q \Big[a_1(0,\dots,x_n) - a_1(x_1^1,\dots,x_n)\Big]dx_2\dots dx_n \\
&+ \sum_{j=2}^{n}(-1)^{j-1}\int_Q \Big[a_j(x_1,\dots,x_j^1,\dots,x_n) - a_j(x_1,\dots,x_j^0,\dots,x_n)\Big]dx_1\dots dx_{j-1}dx_{j+1}\dots dx_n \\
&= \int_Q [a_1(0,\dots,x_n)]dx_2\dots dx_n \quad (\text{as } a_j(x_1,\dots,x_j^i,\dots,x_n) = 0 \text{ for } j\ge 1, i\in\{1,2\}) \\
&= \int_{\partial M} i^*\omega
\end{aligned}$$

(2) We can finally prove the general case. Given the differential structure $\{f_\alpha, U_\alpha\}$ on a compact manifold M, take an open covering of M $\{V_\beta\}_{\beta\in\Lambda}$, where $V_\beta = f_\beta(U_\beta)$ for some β. Then, there exists a finite subcover $\{V_\alpha\} \subset \{V_\beta\}_{\beta\in\Lambda}$ of M. Now let $\{\varphi_1 \dots, \varphi_m\}$ be a differentiable partition of unity subordinate to $\{V_\alpha\}$. For an $n-1$ form ω, we have that $\varphi_j\omega$ is an $n-1$ form completely contained in V_j, which is the case first discussed. Since $\sum_j \varphi_j = 1$, differentiating both sides gives us $\sum_j d\varphi_j = 0$. So,

$$\sum_j \varphi_j\omega = \omega$$

Further,

$$\begin{aligned}\sum_j d(\varphi_j\omega) &= \sum_j d\varphi_j\omega + \sum_j \varphi_j d\omega \\ &= \sum_j \varphi_j d\omega = d\omega\end{aligned}$$

As a result,

$$\begin{aligned}\int_M d\omega &= \sum_{j=1}^{m} \int_M \varphi_j\omega \\ &= \sum_{j=1}^{m} \int_{\partial M} i^*(\varphi_j\omega) \\ &= \int_{\partial M} \sum_{j=1}^{m} i^*(\varphi_j\omega) \\ &= \int_{\partial M} i^*\omega\end{aligned}$$

□

2.1. **Applications.** Stokes' Theorem appears in many forms. It is, in fact, the generalized form of the fundamental theorem of calculus. This can be seen in the first case of the proof, when we compute $\int_M d\omega$. This theorem will be central to proving the Gauss-Bonnet theorem, but we now look at some of its other applications.

Corollary 2.6. *Gauss Theorem. Take $M = \mathbb{R}^2$, $\omega = Pdx + Qdy$ and consider some region R bounded by the closed curve $\partial R = C$. Then,*

$$\int\int_R \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dxdy = \int_C (Pdx + Qdy)$$

Proof. By straightforward calculation,

$$d(Pdx + Qdy) = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dx \wedge dy$$

The result follows by Stokes' theorem. □

Corollary 2.7. Brower's Fixed Point Theorem. *Let $S^n \subset \mathbb{R}^n$ be the ball $S^n = \{p \in \mathbb{R}^n \mid |p| \le 1\}$, where $|p|$ is the euclidean norm. Then, for every differentiable map $f : S^n \to S^n$ there exists fixed point, $q \in S^n$ such that $f(q) = q$.*

Proof. We must first prove two lemmas:

Lemma 2.8. *If M^n is a compact oriented differentiable manifold, then there exists a differential n-from ω that is non-zero everywhere on M*

Proof. Since M is oriented, there exists a differentiable structure $\{f_\alpha, U_\alpha\}$ such that for all α, β where $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) = W \neq \emptyset$, the determinant of the differential map of $f^{-1} \circ g$ is positive. Further, since M is compact, for any open cover, there exists a finite open subcover of M, which we denote as $\{V_i\}$. Subordinate to $\{V_i\}$ there exists a partition of unity $\{\varphi_i\}$.

Now, on each V_i we define $\omega_i = 1dx_1 \wedge \cdots \wedge dx_n$, which is always non-zero. Then, due to orientablility, if any elements in the covering overlap, the form will still be positive due to the positive determinant. Since $\{V_i\}$ is a covering of M, $\omega = dx_1 \wedge \cdots \wedge dx_n$ is positive and defined globally on M. $\square$

Lemma 2.9. *For any compact, oriented, differentiable manifold M^n with boundary ∂M, there does not exist a differentiable map $f : M \to \partial M$ such that $f|_{\partial M}$ is the identity.*

Proof. By contradiction. Suppose that $f : M \to \partial M$ exists such that $f|_{\partial M}$ is the identity. Then, by Lemma 2.3, ∂M is also oriented and has dimension n-1. Take the (n-1)-form given in the previous lemma, $\omega = dx_1 \wedge \cdots \wedge dx_{n-1}$. Then, since $d\omega = 0$, and $f|_{\partial M}$ is the identity, $df^*(\omega) = f^*(d\omega) = 0$. Also, ω is non-zero everywhere, so $\omega = i^* f^*(\omega)$ and

$$\int_{\partial M} \omega = \int_{\partial M} i^*(f^*\omega) \neq 0$$

But, by Stokes' Theorem,

$$\int_{\partial M} i^*(f^*\omega) = \int_M d(f^*\omega) = \int_M f^*(d\omega) = 0$$

So $\int_{\partial M} \omega = 0$ which is a contradiction. Therefore, no such f exists. $\square$

Now, we can prove the full theorem. Suppose, for the sake of contradiction, that there exists such a function $f : S^n \to S^n$ such that $f(q) \neq q$. Then, consider the half-line (or a ray) starting from $f(q)$ and passing through q. We know that only one such line exists for each q, because $f(q) \neq q$. We also know that this ray will intersect ∂S^n at a unique point r (notice that if $q \in \partial S^n$, then $q = r$). Define $g : S^n \to \partial S^n$ such that $g(q) = r$. It then follows that $g|_{\partial S^n}$ is the identity mapping, which is a contradiction by 2.9. Therefore there must exist a fixed point. $\square$

Stokes' Theorem, as we can see, can be used to prove some important theorems. Now, in order to discuss the Gauss-Bonnet theorem in $\mathbb{R}^3$, we must first discuss important concepts related to geometry in $\mathbb{R}^3$

3. Riemannian Manifolds and Geometry in $\mathbb{R}^3$

We first need to introduce vector fields, a concept similar to that of forms:

Definition 3.1. Given a differentiable mandifold M, A differentiable vector field is a function $X : M \to T_p(M)$ such that for all linear $\varphi : M \to \mathbb{R}$, $X\varphi : M \to \mathbb{R}$ is differentiable. It naturally follows that if $f_\alpha : U_\alpha \to M$, and $X_i = \frac{\partial}{\partial x_i}$, $i \in \{1, \ldots, n\}$ is the associated basis of the parametrization, then a vector field X in $f_\alpha(U_\alpha)$ can be written as

$$X = \sum_i a_i X_i$$

Note that each vector field corresponds to a 1-form. In other words, $X = \sum_i a_i X_i$ corresponds to $\omega = \sum_i a_i dx_i$. The difference, however, is that X operates on 0-forms, while ω operates on elements in M.

Definition 3.2. A *Riemannian manifold* is a manifold M, along with a choice, for each $p \in M$, of a *Riemannian metric* inner product $\langle \ , \ \rangle_p$, such that for any differentiable vector fields X, Y we have that $p \mapsto \langle X(p), Y(p) \rangle$ is differentiable in M.

From now on, we write $\langle X(p), Y(p) \rangle = \langle X, Y \rangle_p$. If we take $M = \mathbb{R}^n$, we define, for $p \in \mathbb{R}^n$, that if $x, y \in T_p\mathbb{R}^n$ where $x = (x_1, \dots, x_n)$ and $y = (y_1 \dots, y_n)$, then $\langle x, y \rangle = \sum_i x_i y_i$, which is known as the dot product. The concept of the inner product is useful, because we can use it to quantify similarity between the positions of two vectors, which in turn can give us a frame of reference. The next few definitions further this notion.

3.1. **Cartan's Structure of Equations in $\mathbb{R}^n$.**

Definition 3.3. Given $U \subset \mathbb{R}^n$, an *orthonormal moving frame* is a set of vector fields $\{e_1, \dots, e_n\}$ such that for $p \in U$, we have that $\langle e_i, e_j \rangle_p = \delta_{ij}$. Further, we define the set of forms $\{\omega_1 \dots \omega_n\}$ such that $\omega_i(p)\Big(e_j(p)\Big) = \delta_{ij}$ as the *coframe* associated with $\{e_i\}$. So, $\{(w_i)_p\}$ is the dual basis of $\{(e_1)_p\}$

Recall that $T_p\mathbb{R}^n \subset \mathbb{R}^n$. So, $e_i : \mathbb{R}^n \to \mathbb{R}^n$, and since it's differentiable map $d(e_i) : \mathbb{R}^n \to \mathbb{R}^n$ is linear. Intuitively, this map describes how the frame is rotated as it moves from point to point in $\mathbb{R}^n$. It would now be useful to create forms that relate to the differential map.

Definition 3.4. Given the orthonormal moving frame $\{e_i\}$ and its coframe $\{\omega_1\}$ we define the *connection forms* ω_{ij} of $d(e_i) : \mathbb{R}^n \to \mathbb{R}^n$ such that for $p, v \in \mathbb{R}^n$,

$$d(e_i)_p(v) = \sum_j (\omega_{ij})_p(v) e_j$$

In more general terms,

$$de_i = \sum_j \omega_{ij} e_j$$

Because de_i is a linear map, it follows that w_{ij} depends linearly on v.

Proposition 3.5. *In the indices i, j, the connection forms are* anti-symmetric, *meaning that $\omega_{ij} = -\omega_{ji}$.*

Proof. We know that $\langle e_i, e_j \rangle = \delta_{ij}$. Differentiating both sides, we get that:

$$\begin{aligned} \langle de_i, e_j \rangle + \langle e_i, de_j \rangle &= 0 \\ \Big\langle \sum_k \omega_{ik} e_k, e_j \Big\rangle + \Big\langle e_i, \sum_k \omega_{jk} e_k \Big\rangle &= 0 \\ \omega_{ij} + \omega_{ji} &= 0 \\ \omega_{ij} &= -\omega_{ji} \end{aligned}$$

□

Now we can state and prove Elie Cartan's structure of equations in $\mathbb{R}^n$

Theorem 3.6. *Let $\{e_i\}$ be a moving frame in $U \subset \mathbb{R}^n$, let $\{w_i\}$ be its coframe, and let $\{\omega_{ij}\}$ be the connection forms. Then:*

$$d\omega_i = \sum_k \omega_k \wedge \omega_{ki}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$$

Where $i, j, k \in \{1, \dots n\}$.

Proof. Let $a_1 = (1, 0, \dots, 0), \dots a_n = (0, \dots, 1)$ be the canonical basis in $\mathbb{R}^n$, and let $x_i : U \to \mathbb{R}$ such that for all $y = (y_1, \dots, y_n) \in U$, $x_i(y) = y_i$. Since each x_i is a 0-form, each dx_i is a 1-form. In other words, x_i projects the i'th component of the vector. Further, $dx_i(a_j) = \delta_{ij}$. Therefore $\{dx_i\}$ is the coframe of $\{a_i\}$. Then, for some arbitrary orthonormal moving frame $\{e_i\}$ and its coframe $\{\omega_i\}$, we have that

$$e_i = \sum_j \beta_{ij} a_i \tag{3.7}$$

Further,

$$\omega_i = \sum_j \beta_{ij} dx_j \tag{3.8}$$

Note that

$$de_i = \sum_k \omega_{ik} e_k \tag{3.9}$$

$$= \sum_k \omega_{ik} (\sum_j \beta_{kj} a_i) = \sum_{jk} \omega_{ik} \beta_{kj} a_j \quad (3.7) \tag{3.10}$$

Since $de_i = \sum_j d\beta_{ij} a_j$, we have that

$$d\beta_{ij} = \sum_k \omega_{ik} \beta_k \tag{3.11}$$

Now differentiating 3.8,

$$\begin{aligned} d\omega_i &= \sum_j d\beta_{ij} \wedge dx_j \\ &= \sum_k \omega_{ik} \beta_{kj} \wedge dx_j \\ &= \sum_k \beta_{kj} dx_j \wedge \omega_{ki} \\ &= \sum_k \omega_k \wedge \omega_{ki} \end{aligned}$$

Differentiating 3.11, we we get the second equation:

$$\begin{aligned} d(d\beta_{ij}) = 0 &= \sum_k d\omega_{ik} \beta_{jk} - \sum_k \omega_{ik} \wedge d\beta_{jk} \\ \sum_k d\omega_{ik} \beta_{jk} &= \sum_k \omega_{ik} \wedge \sum_s \omega_{ks} \beta_{sj} \\ \omega_{ir} &= \sum_k \omega_{ik} \wedge \omega_{kr} \quad \text{(multiplying by the inverse matrix of } \beta_{ij}\text{)} \end{aligned}$$

$\square$

Therefore, in $\mathbb{R}^3$, the structure of equations are as follows:

$$d\omega_1 = \omega_{12} \wedge \omega_2 \tag{3.12}$$

$$d\omega_2 = \omega_{21} \wedge \omega_1 \tag{3.13}$$

$$d\omega_3 = \omega_{13} \wedge \omega_{32} \tag{3.14}$$

$$d\omega_{12} = \omega_{13} \wedge \omega_{23} \tag{3.15}$$

$$d\omega_{13} = \omega_{12} \wedge \omega_{21} \tag{3.16}$$

$$d\omega_{23} = \omega_{21} \wedge \omega_{13} \tag{3.17}$$

Now, before we can properly discuss manifolds in $\mathbb{R}^3$, we must state and prove a lemma that will be important in defining curvature in three dimensional space.

Lemma 3.18. *(Cartan's Lemma) If V^n is a vector space (such as $\mathbb{R}^n$), and $\{\omega_1, \ldots, \omega_r \mid r \leq n\}$ are linearly independent 0-forms. Then, if there exists another set of 0-forms $\{\varphi_1, \ldots, \varphi_r\}$ for which $\sum_j \omega_j \wedge \varphi_j = 0$, then $\varphi_i = \sum_j a_{ij}\omega_j$, such that $a_{ji} = a_{ij}$*

Proof. From linear algebra, we know that we can extend the set of linearly independent forms to have a complete basis $\{\omega_1, \ldots, \omega_n\}$ of V^n. So, for any form φ_i,

$$\varphi_i = \sum_{j \leq r} a_{ij}\omega_j + \sum_{j > r} b_{ij}\omega_j$$

Let $l = \{r, r+1, \ldots, n\}$. Then if $\sum_j \omega_j \wedge \varphi_j = 0$, then

$$\begin{aligned} 0 &= \sum_{j \leq r} a_{ij}\omega_i \wedge \omega_j + \sum_{j > r} a_{ij} o_i \wedge \omega_j \\ &= \sum_{i \leq j} (a_{ij} - a_{ji})\omega_i \wedge \omega_j + \sum_{i > l} b_{il}\omega_i \wedge \omega_l \end{aligned}$$

However, since $\omega_s \wedge \omega_t$ is linearly independent for $s < t$, it must be that $b_{ij} = 0$. Further, this implies that $a_{ij} = a_{ji}$. $\square$

We omit the proof that given a moving frame, the structure equations are unique.

3.2. **Curvature in $\mathbb{R}^3$.** Now, the manifolds that we will explore exist in $\mathbb{R}^3$, but locally 'look' like $\mathbb{R}^2$. The concept of immersion crystallizes this idea.

Definition 3.19. Given a manifold M^n an *immersion* of a manifold $M^n \to \mathbb{R}^m$ for some $m \geq n$ is a smooth bijective map $x : M^n \to \mathbb{R}^m$.

We will only be dealing with the immersion of 2-dimensional manifolds into $\mathbb{R}^3$. For M^2 let $x : M^2 \to \mathbb{R}^3$ be an immersion. We then define the metric at $p \in M^2$ for $y, z \in T_pM^2$ as:

$$\langle y, z \rangle_p := \langle dx_p(y), dx_p(z) \rangle_{x(p)}$$

The left hand side is defined by the right hand side, which is the inner product of $\mathbb{R}^3$. Now imagine that $x : M^2 \to \mathbb{R}^3$ is a local immersion, meaning that for each $p \in M^2$, there exists neighborhood U containing p such that $x|_U$ is an immersion. Let $x(U) = V$. Then, we can find a moving frame $\{e_1, e_2, e_3\}$ for V, such that e_1, e_2 are tangent to V, and e_3 is normal to V. We can accordingly generate the coframe and the connected forms as given in (3.12-7).

We can also create pull-back maps, $x * (\omega_i)$ and $x^*(\omega_{ij})$ and create a new structure of equations of M^2. Observe that $x^*\omega_3 = 0$, because, for $y \in T_pM$,

$$x^*\omega_3(y) = \omega_3(dx(y)) = \omega_3(a_1e_1 + a_2e_2) = 0$$

Notice, that we can now use both 3.14 to satisfy the conditions necessary for Cartan's lemma, giving us[3]

$$\omega_3 = 0 = \omega_{13} \wedge \omega_{32}$$
$$d(0) = 0 = \omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23}$$

Therefore,

$$\omega_{13} = h_{11}\omega_1 + h_{12}\omega_2$$
$$\omega_{23} = h_{21}\omega_1 + h_{22}\omega_2$$

with h_{ij} being differentiable real functions on U. Now, we know that $de_3 = \omega_{13}e_1 + \omega_{32}e_2$, which implies that

$$de_3 = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

We can think about the map e_3 as the orientation of the immersion, as each vector in the field points perpendicularly to the manifold. Therefore, de_3 is a map which describes the orientation of the planes that are tangent to the manifold. This is what motivates definitions of curvature.

Definition 3.20. Given $p \in M$, such that $de_3(p) = -\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$, the *Gaussian Curvature* K of M in p is defined as

$$det(de_3) = h_{11}h_{22} - h_{12}^2$$

It also follows that

$$\omega_{12} = -(h_{11}h_{22} - h_{12}^2)(\omega_1 \wedge \omega_2) = -K(\omega_1 \wedge \omega_2)$$

From the definition of curvature, we can prove an important theorem:

Theorem 3.21. *Given M^2, if $x, x' : M^2 \to \mathbb{R}^3$ both have the same induced metrics with respective curvature functions of K and K', then for all $p \in M$, $K = K'$. In other words, the curvature is only dependent on the metric, and not the immersion function.*

Proof. We denote with a prime all of the entities related to x'. Now, for $p \in U \subset M$ we can find a moving frame $\{e_1, e_2\}$. Then $\{dx(e_1), dx(e_2)\}$ is the moving frame of $x(U) = V$, and $\{dx'(e_1), dx'(e_2)\}$ for $x'(U) = V'$. Due to the uniqueness argument, $\omega_1 = \omega_1'$,$\omega_2 = \omega_2'$ and $\omega_{12} = \omega_{12}'$. Therefore

$$-K(\omega_1 \wedge \omega_2) = -K'(\omega_1' \wedge \omega_2')$$

□

[3]For simplicity's sake, we will now write $x^*(\omega_i) = \omega_i$ because U is imbedded in $\mathbb{R}^3$ by x, so essentially $U \subset \mathbb{R}^3$.

Notice that if v_1, v_2 are linearly independent vectors at M, then $\omega_1 \wedge \omega_2(v_1, v_2) = area(v_1, v_2)$. That is why $\omega_1 \wedge \omega_2$ is known as the area element, as it generates the area of the parallelogram generated by the two vectors.

Note that, given a moving frame, there must exist a unique antisymmetric 1-form ω_{12} that obeys the structure of equations in $\mathbb{R}^3$. Just take

$$\omega_{12}(e_1) = d\omega_1(e_1, e_2)$$

$$\omega_{12}(e_2) = d\omega_2(e_1, e_2)$$

One can check that it satisfies the necessary properties.

Our next goal is to show that this concept of curvature is intrinsic to the properties of the manifold, in particular, that it is not dependent on the choice of moving frame. First, we must consider the choice of moving frames. if $\{e_1, e_2\}$ and $\{\overline{e_1}, \overline{e_2}\}$ are both frames with the same orientation, then

$$\overline{e_1} = fe_1 + ge_2$$

$$\overline{e_2} = -ge_1 + fe_2$$

Where $f, g : M \to \mathbb{R}$ are differential functions such that $f^2 + g^2 = 1$. This result comes from both linear algebra and analysis.

Lemma 3.22. *At a point $p \in M$, if $\{e_1, e_2\}$ and $\{\overline{e_1}, \overline{e_2}\}$ are both frames with the same orientation, then*

$$\omega_{12} = \overline{\omega_{12}} - \tau$$

Where $\tau = fdg - gdf$. Further, if K and $\overline{K}$ are the respective curvatures of the frames, then $K = \overline{K}$. This shows that the curvature does not depend on the choice of frame.

Proof. As a result of f, g as defined before the lemma,

$$\overline{\omega_1} = f\omega_1 - g\omega_2$$

$$\overline{\omega_2} = g\omega_1 + f\omega_2$$

Differentiating ω_1, we have that

$$d\overline{\omega_1} = df \wedge \omega_1 + fd\omega_1 - dg \wedge \omega_2 - g\omega_2$$

By structure of equations in $\mathbb{R}^3$, and because $\overline{\omega}_{12} = -\overline{\omega}_{21}$,

$$d\omega_1 = \overline{\omega_{12}} \wedge \omega_2 + (fdf + gdg) \wedge \omega_1 + (gdf - fdg) \wedge \omega_2$$

But since $f^2 + g^2 = 1$, differentiating gives us that $fdf + gdg = 0$ and therefore:

$$d\omega_1 = (\overline{\omega}_{12} - \tau) \wedge \omega_2$$

Similarly for ω_2, we find that

$$d\omega_2 = -(\overline{\omega}_{12} - \tau) \wedge \omega_1$$

Combining these equations gives us:

$$\omega_{12} = \overline{\omega}_{12} - \tau$$

Now, we can calculate that $d\tau = 0$, which gives us that $d\omega_{12} = d\overline{\omega}_{12}$. Therefore,

$$\overline{K}(\overline{\omega}_1 \wedge \overline{\omega}_2) = K(\omega_1 \wedge \omega_2)$$

and so $K = \overline{K}$. □

Technically τ is the differential map of the angle function between the two frames e_i and $\overline{e}_i$. Intuitively, τ measures the rate of change of the angle between the two frames. Now that we have shown that the Gaussian curvature of a manifold is remarkably independent, we can discuss the Gauss-Bonnet Theorem, along with Morse's Theorem

4. The Gauss-Bonnet Theorem

We will only prove the Gauss-Bonnet theorem for two-dimensional manifolds immersed in $\mathbb{R}^3$. This theorem relates topological properties of the manifold to its geometric properties. With regards to topology, we would like to define the index of a vector field at a point on a manifold.

Definition 4.1. Consider a differential vector field X defined on M. Then, $p \in M$ is a *singular* point if $X(p) = 0$. Further, p is *isolated* if there exists a neighborhood $V_p \subset M$ containing p that contains no other singular point.

The number of such isolated points is finite, since M is compact. We also choose V to be homeomorphic to a disk in $\mathbb{R}^2$, because integration is easier. We will now develop a topological property corresponding to an isolated point.

Proposition 4.2. *Consider a differential vector field X defined on M and consider the set of isolated points $\{q \in M \mid X(q) = 0\}$. Now, we define the moving frame at $V_q \setminus q$ such that $(\overline{e}_1)_p = \frac{X(p)}{\|X(p)\|}$ and $(\overline{e}_2)_p$ as orthogonal to e_1 and preserving the orientation of M. Now, arbitrarily choose another moving frame $\{e_1, e_2\}$. Then, there will exist two sets of connection forms and coframes for each of the moving frames. From lemma 3.22, $\overline{\omega}_{12} - \omega_{12} = \tau$. Consider a closed curve C bounding a compact region of V containing p. Then,*

$$\int_C \tau = 2\pi I$$

Where I is known as the index *of X at p.*

In order to see why this is true, note that as we start at p and travel around the curve, each $\overline{e}_i$ and e_i must end up in the same place after any full rotation. This implies that each moving frame performs a rotation of some integer of 2π. Since $\overline{e}_i$ always points in the direction of the vector field, as the vector field 'rotates,' so does $\overline{e}_i$. Now, since τ is the differential of the angle between $\overline{e}_i$ and e_i, integrating τ along a closed curve would have to give us some integer multiple of 2π, because it would be the difference in rotations between $\overline{e}_i$ and e_i, which are each multiples of 2π.

Note that, when defining I, we chose the frame $\{e_i\}$, a Riemannian metric, and the closed curve C. We will now show that I does not depend on these choices.

Lemma 4.3. *The index of X at an isolated point $p \in M$ does not depend on the closed curve C that contains a compact subset of V_p.*

Proof. Take two such closed curves, label them C_1, C_2. Let $\int_{C_1} \tau = 2\pi I_1$ and $\int_{C_2} \tau = 2\pi I_2$, and denote Δ as the region bounded by the two curves. Then,

$$\begin{aligned} 2\pi(I_1 - I_2) &= \int_{C_1} \tau - \int_{C_2} \tau \\ &= \int_{\partial\Delta} \tau \\ &= \int_{\Delta} d\tau \quad \text{(Stokes' Theorem)} \\ &= 0 \quad (d\tau = 0) \end{aligned}$$

Therefore, $I_1 = I_2$. If C_1 and C_2 intersect, we can choose another curve C_3 that does not intersect either curve, and apply the above method. □

Now, we show that the index is independent of the choice of frame $\{e_1, e_2\}$ in the following way.

Lemma 4.4. *Given X and $\{\overline{e}_i\}$, consider $B(r,p)$, or in other words, a disk of radius r centered at an isolated point p. Let $S(r,p) = \partial B(r,p)$. Then,*

$$\lim_{r\to 0} \frac{1}{2\pi} \int_{S(r,p)} \overline{\omega}_{12} = I$$

Proof. First, we must prove that such a limit exists. Choose an arbitrary sequence

$$\int_{S(r_1,p)} \overline{\omega}_{12}, \ldots, \int_{S(r_n,p)} \overline{\omega}_{12} \ldots$$

such that $\lim\limits_{n\to 0} \{r_n\} = 0$. By Stokes' theorem,

$$\int_{S(r_i,p)} \overline{\omega}_{12} = \int_{B(r_i,p)} d\overline{\omega}_{12}$$

For $r_k \leq r_i$, we have that $B(r_i,p) \supset B(r_k,p)$, and so

$$\int_{B(r_i,p)} d\overline{\omega}_{12} \geq \int_{B(r_k,p)} d\overline{\omega}_{12}$$

Implying by Stokes' theorem that

$$\int_{S(r_i,p)} \overline{\omega}_{12} \geq \int_{S(r_k,p)} \overline{\omega}_{12}$$

Therefore $\{\int_{S(r_i,p)} \overline{\omega}_{12}\}$ is a decreasing sequence. Further we know that $\int_{S(r_i,p)} \overline{\omega}_{12} \geq 0$ by definition of an integral. It follows that the sequence is decreasing and bounded, so it converges. Since an arbitrary sequence was chosen, it must be that $\lim\limits_{r\to 0} \frac{1}{2\pi} \int_{S(r,p)} \overline{\omega}_{12}$ exists. Let $\overline{I}$ denote this limit. Now, consider

$$\int_{S(r_1,p)} \overline{\omega}_{12} - \int_{S(r_2,p)} \overline{\omega}_{12}$$

for $r_1, r_2 > 0$. Fix r_1 and let $r_2 \to 0$. Then $\int_{S(r_2,p)} \overline{\omega}_{12} = 2\pi\overline{I}$, and so by Stokes' Theorem

$$\int_{S(r_1,p)} \overline{\omega}_{12} - 2\pi\overline{I} = \int_{B(r_1,p)} d\overline{\omega}_{12}$$
$$\int_{S(r_1,p)} \overline{\omega}_{12} = -\int_{B(r_1,p)} \overline{K}(\overline{\omega}_1 \wedge \overline{\omega}_2) + 2\pi\overline{I}$$

From Lemma 3.22, we have that $\overline{\omega}_{12} = \omega_{12} + \tau$, and so

$$\begin{aligned} \int_{S(r_1,p)} \overline{\omega}_{12} &= \int_{S(r_1,p)} \omega_{12} + \int_{S(r_1,p)} \tau \\ &= \int_{B(r_1,p)} d\omega_{12} + 2\pi I \\ -\int_{B(r_1,p)} \overline{K}(\overline{\omega}_1 \wedge \omega_2) + 2\pi\overline{I} &= -\int_{B(r_1,p)} K(\omega_1 \wedge \omega_2) + 2\pi I \end{aligned}$$

Lemma 3.22 also tells us that $\overline{K}(\overline{\omega}_1 \wedge \overline{\omega}_2) = K(\omega_1 \wedge \omega_2)$, and so $\overline{I} = I$. □

Thus, Lemma 3.22 is heavily responsible for proving the independence of the index from the choice of moving frame.

Lemma 4.5. *The index is not dependent on the metric of M*

Proof. Let $\langle,\rangle_1$ and $\langle,\rangle_2$ be two arbitrarily chosen metrics on M. Then define a function dependent on $t \in [0,1]$ such that

$$\langle,\rangle_t = t\langle,\rangle_1 + (1-t)\langle,\rangle_0$$

It can be seen that $\langle,\rangle$ is a valid inner product on M that varies smoothly with p. Let I_0, I_t, I_1 be the respective indices. Then, from the previous two lemmas, we can see that I_t is a smooth function. Since I_t can only be an integer, it must be that $I_t = c$ for all $t \in (0,1)$. Therefore, due to continuity, $I_0 = I_t = I_1$. □

Now we are ready to state and prove the Gauss-Bonnet Theorem in $\mathbb{R}^3$.

Theorem 4.6. *Consider an oriented differentiable compact manifold M^2. Let X be a differential vector defined field on M with isolated singularities $p_1, \ldots, p_k$, whose respective indices are $I_1, \ldots, I_k$. Then for all Reimannian metrics on M,*

$$\int_M K\omega_1 \wedge \omega_2 = \sum_i^k 2\pi I_i$$

Proof. As previously discussed, in $M \setminus \bigcup_i \{p_i\}$ consider the frame where $(e_1)_p = \frac{X(p)}{\|X(p)\|}$ and e_2 is perpendicular to e_1. Now consider the collection of balls B_i, where each $p_i \in B_i$, and B_i contains no other isolated point. Then,

$$\begin{aligned} \int_{M\setminus(\bigcup_i B_i)} K\overline{\omega}_1 \wedge \overline{\omega}_2 &= -\int_{M\setminus(\bigcup_i B_i)} d\overline{\omega}_{12} \quad \text{(`$-$' is due to a change in orientation)} \\ &= \int_{\bigcup_i \partial B_i} \overline{\omega}_{12} \\ &= \sum_i^k \int_{\partial B_i} \overline{\omega}_{12} \end{aligned}$$

Now, from lemma 3, we know that for any frame and coframe $\{\omega_1, \omega_2\}$ that $\int_{M\setminus(\bigcup_i B_i)} K\overline{\omega}_1 \wedge \overline{\omega}_2 = \int_{M\setminus(\bigcup_i B_i)} K\omega_1 \wedge \omega_2$. Let r_i denote the radius of each B_i. Then, due to lemma 4.4, it must be that

$$\lim_{r_i \to 0} \int_{\partial B_i} \overline{\omega}_{12} = 2\pi I_i$$

Therefore, as all of the radii approach 0 we have that $M \setminus (\bigcup_i B_i) = M$, and so

$$\int_M K\overline{\omega}_1 \wedge \overline{\omega}_2 = \int_M K\omega_1 \wedge \omega_2 = \sum_i^k 2\pi I_i$$

$\square$

5. Discussion

Given M^2, the integer $\sum_i^k 2\pi I_i$ is called *the Euler-Poincare characteristic* of M and is denoted $\chi(M)$. This theorem surprisingly implies that the characteristic does not depend on the vector field X, and the integral of the curvature with respect to the area element does not depend on the metric. Therefore, each of these concepts are inherent to the structure of the manifold. One theorem states for two manifolds M and M', that $\chi(M) = \chi(M')$ if and only if they are *diffeomorphic*, in other words, there exists a smooth bijective function with a smooth inverse between M and M'. So, if we can smoothly deform one manifold into another, then their characteristic is the same. This proof is primarily due to Shiing Shen Chern, who in fact used this technique to prove a generalized Gauss-Bonnet theorem that holds for higher dimensional manifolds.

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