An Introduction to Path Integral Formalism

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1 Classical Action

Given a particle of mass m subject to a potential $V(x,t) : \mathbb{R}^2 \to \mathbb{R}$ constrained to one spatial dimension, how does machinery from classical mechanics produce the equations of motion? First, one can define the *Lagrangian*:

$$\mathcal{L}(x,\dot{x},t) = \frac{m}{2}\dot{x}^2 - V(x,t) \tag{1.0.1}$$

This is a function whose inputs are a path $x(t) : [t_a, t_b] \to \mathbb{R}$, its derivative, or velocity, $\dot{x}(t) : (t_a, t_b) \to \mathbb{R}$ and t, or time. Its output is a function, that gives at any time the potential energy subtracted from the kinetic energy for the path $x(t)$. The *action* of the Lagrangian is the functional:

$$S = \int_{t_a}^{t_b} \mathcal{L}(x,\dot{x},t)dt$$

The equation of motions can be found by the following proposition:

Proposition 1.1. *For a given Lagrangian $\mathcal{L}(x,\dot{x},t)$, the equations of motion for a physical system will be a path $x(t)$ such that for any small variation of the path δx with the endpoints fixed, i.e. $\delta x(t_b) = x(t_b), \delta x(t_a) = x(t_a)$, then:*

$$\delta S := S[x + \delta x] - S[\delta x] = 0$$

In other words, $x(t)$ is an extreme value of the action S.

To make matters simpler we have a way of finding the equations of motion just given by the $\mathcal{L}$ via the Euler Lagrange equations:

Theorem 1.2. *For a given Lagrangian $\mathcal{L}$, a path $x(t)$ is an extreme value of its action S iff:*

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

We do not prove this, as this was already done in intermediate mechanics and is primarily review. Observe that if we set $F(x) = -V'(x)$, then the E-L equations give us:

$$m\ddot{x} - F(x) = 0$$

Which is Newton's first law. When an extreme value of S is achieved we denote the value as S_{cl}.

2 Quantum Action

One of the major differences between quantum and classical mechanics is that the particles in the theory no longer take definite paths in space. Further, Feynman notes that unless the particle interacts with a macroscopic system (i.e. a measuring apparatus) one has no information regarding which path the particle takes. As a result, one can only speak in terms of probabilities that a particle was measured to be found at one point, and later on, was measured to be found at another.

To make matters precise, one can speak of a *probability amplitude*, a complex number $K(a,b)$, for the quantum system to go from x_a at time t_a to x_b at time t_b. The *probability* that the system actually travels from $a = (x_a, t_a)$ to $b = (x_b, t_b)$, $P(a,b)$ is found by taking the norm square of the probability amplitude, i.e.

$$P(a,b) = |K(a,b)|^2$$

How does one calculate $K(a,b)$? The core idea of the path integral is that *every path from* (x_a, t_a) *to* (x_b, t_b) *contributes in some way to* $K(a,b)$. For each such path $x(t)$ we can associate a some path amplitude $\varphi[x(t)]$ and combine them to get $K(a,b)$:

$$K(a,b) = \sum_{\text{all paths } x(t) \text{ from a to b}} \varphi[x(t)] \tag{2.0.1}$$

Now, the summation is just to capture the idea that all paths are taken together to get the total sum. A further question is how should we determine the phase of each path? The answer is that it should depend somehow on the quantum forces acting on it, and therefore the action of the Lagrangian, i.e.:

$$\varphi[x(t)] = Ke^{\frac{i}{\hbar}S(x(t))} \tag{2.0.2}$$

This idea arises from the proposition that the total change in time of the probability amplitude of a path should be proportional to the probability amplitude itself, i.e.

$$\frac{d}{dt}\varphi[x(t)] = C\varphi[x(t)]$$

And noting that if $x(t)$ gives a higher action S, and if S is large compared to the quantum scale then the change of amplitude should be larger in time. So, Setting $C = \frac{i}{\hbar}S(x(t))$, we get the expression for $\varphi[x(t)]$ in (2.0.12).

3 The Path Integral

In Riemannian integration, if we have a function $f : \mathbb{R} \to \mathbb{R}$, we find $\int_a^b f(x)dx$ by taking a limit as follows[1]:
We break up the interval $I = [a, b]$ into equally spaced intervals of length ϵ, i.e. $I = \bigcup_{i=0}^{N-1}[x_i, x_{i+1}]$ where $x_i \in I$, $x_{i+1} - x_i = \epsilon$, and view the lower sum:

$$\sum_{i=0}^{N-1} f(x_i)(x_{i+1} - x_i) = (\epsilon)\sum_{i=0}^{N-1} f(x_i) \tag{3.0.1}$$

as an approximation of the integral of f over (a, b). Then we can define the integral (if the limit exists) as:

$$\int_a^b f(x)dx = \lim_{\epsilon\to 0}\epsilon\sum_{i=0}^{N-1} f(x_i)$$

How can we do something similar with equation (2.0.1) and turn it into an integral? The logic for its contstruction is as follows

1. First, setting the interval $I = [t_a, t_b]$, we divide I into intervals $[t_i, t_{i+1}]$, each of length ϵ, and $t_0 := t_a$ and $t_N := t_b$.

[1] Technically speaking, we are just considering a lower sum, but this is enough to make the necessary analogy

2. For each t_i, select a point $x_i \in \mathbb{R}$. Construct a path $x(t)$ made up of straight lines such that $x(t_a) = x_a$, $x(t_b) = x_b$, and $x(t_i) = x_i$. This means that the path in-between $x(t_i)$ and $x(t_{i+1})$ is a line given by $x(t) = \frac{x_{i+1}-x_i}{t_{i+1}-t_i}$.

3. The path $x(t)$ constructed in the last step has an associated phase $\varphi[x(t)]$ given in (2.0.2), and since $x_1, \ldots, x_{N-1}$ were arbitrary, we could integrate over all such piece-wise linear paths, given the choice of $x_1, \ldots, x_{N-1}$, which gives us the a first guess at what an integral should be:

$$\int \cdots \int \int \varphi[x(t)]\ dx_1 \ldots dx_{N-1}$$

Note that $x_0 = x_a$ and $x_N = x_b$ are fixed, because we are trying to calculate $K(a,b)$.

4. The integral above does not converge as we take $N \to \infty$. Why? Note that in the Riemannian lower sum (3.0.1) there is an ϵ multiplied with the sum, which was difference $x_{i+1} - x_i$. If we were strictly following the analogy, one would multiply the integral with $t_{i+1} - t_i = \epsilon$, but this does not work, and we need to instead be more creative and find a "normalization constant" $A(\epsilon)$. It turns out that there is no way to find $A(\epsilon)$.

5. Supposing we have the normalizing factor $C(\epsilon)$. Then, the true formula for the path integral reads:

$$K(a,b) = \lim_{\epsilon \to 0} C(\epsilon) \int \cdots \int \int \int e^{\frac{i}{\hbar} S[b,a]} dx_1\ dx_2 \ldots dx_{N-1}$$

Where, given a choice of points $\{(x_i, t_i)\}$,

$$S[b,a] = \int_{t_a}^{t_b} \mathcal{L}(x, \dot{x}, t) dt$$

is the action for the piece-wise linear path $\{(x_i, t_i)\}$ constructed in step 2.

If we have the Lagrangian given in (1.0.1) then it can be shown (in chapter 4 of the textbook) that $C(\epsilon) = A_\epsilon^{-N}$, where

$$A_\epsilon = \left(\frac{2\pi i \hbar \epsilon}{m}\right)^{\frac{1}{2}}$$

And so the path integral for the chosen $\mathcal{L}$ will read:

$$K(a,b) = \lim_{\epsilon\to 0} \frac{1}{A_\epsilon} \int \cdots \int \int e^{\frac{i}{\hbar} S[b,a]} \frac{dx_1}{A_\epsilon} \frac{dx_2}{A_\epsilon} \cdots \frac{dx_{N-1}}{A_\epsilon} \tag{3.0.2}$$

The choice of taking limits piece-wise straight lines (i.e. in step 2) will be problematic if the Lagrangian is dependent on acceleration, or $\ddot{x}(t)$, because at the points x_i, $\dot{x}(t)$ is discontinuous, sending $\ddot{x}(t) \to \pm\infty$. There can be some substitutions made, but perhaps there is some other way of taking the limit. For future modifications to the path integral formalism we denote a generic path integral by the equation:

$$K(a,b) = \int_a^b e^{-\frac{i}{\hbar} S(a,b)} \mathcal{D}x(t)$$

4 Breaking Up the Amplitudes

Before we make computations, it is useful to make some observations about path integrals. Suppose we want to find $K(a,b)$, for some events $(t_a, x_a), (t_b, x_b)$ and action S. For some other event (t_c, x_c) where $t_a < t_b < t_c$, we first observe that by properties of integration:

$$S[a,b] = \int_{t_a}^{t_b} \mathcal{L}(x,\dot{x},t)dt = \int_{t_a}^{t_c} \mathcal{L}(x,\dot{x},t)dt + \int_{t_c}^{t_b} \mathcal{L}(x,\dot{x},t)dt = S[a,c] + S[c,b]$$

Therefore, we find that in terms of the path integral:

$$\begin{aligned} K(a,b) &= \int_a^b e^{-\frac{i}{\hbar} S[a,b]} \mathcal{D}(x(t)) \\ &= \int_a^b e^{-\frac{i}{\hbar} (S[a,c] + S[c,b])} \mathcal{D}(x(t)) \\ &= \int_a^b e^{-\frac{i}{\hbar} S[a,c]} e^{-\frac{i}{\hbar} S[c,b]} \mathcal{D}(x(t)) \end{aligned}$$

Now, our agnosticism about how we define the differential, i.e. $\mathcal{D}(x(t))$ pays off. Instead of thinking about paths from a to b, one can pick fix a certain point x_c, and first integrate over all paths from a to c, then c to b. Since this was done for a particular choice, we then integrate over all choices x_c. So, our path integral now is:

$$\begin{aligned}
K(a,b) &= \int_{-\infty}^{\infty}\int_a^c\int_c^b e^{-\frac{i}{\hbar}S[a,c]}e^{-\frac{i}{\hbar}S[c,b]}\mathcal{D}_a^c(x(t))\mathcal{D}_c^b(x(t))dx_c \\
&= \int_{-\infty}^{\infty}\int_a^c e^{-\frac{i}{\hbar}S[a,c]}\mathcal{D}_c^b(x(t))\int_c^b e^{-\frac{i}{\hbar}S[c,b]}\mathcal{D}_a^c(x(t))dx_c \\
&= \int_{-\infty}^{\infty} K(a,c)K(c,b)dx_c
\end{aligned}$$

Just to summarize this argument, from any path $x(t)$ a to b, we observe that this can also be thought of as a path from a to $(t_c, x(t_c))$ and a path from $(t_c, x(t_c))$ to b, and such a path is included because we integrate over dx_c above. Therefore, we see that amplitudes of successive events multiply. Since we were able to show this for an intermediate event, we can inductively argue the same if we split up $[t_a, t_b] = \bigcup_{i=0}^{N}[t_i, t_{i+1}]$ into N pieces, we would have the integral:

$$K(a,b) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} K(b,t_{n-1})K(t_{n-1},t_{n-2})\ldots K(t_1,a)dx_1\ \ldots dx_{n-1}$$

5 The Free particle

Noting that for a free particle, $V(x,t) = 0$, and so we have the Lagrangian $\mathcal{L}(x,\dot{x},t) = \frac{m\dot{x}^2}{2}$. Observe that if we choose $(t_i, x_i), (t_{i+1}, x_{i+1})$ then the line connecting them will have the line equation will have a constant slope (and therefore derivative)

$$\dot{x(t)} = \frac{x_{i+1} - x_i}{t_{i+1} - t_i}$$

And the action will be

$$S(t_i, t_{i+1}) = \int_{t_i}^{t_{i+1}} \frac{m}{2}\Big(\frac{x_{i+1} - x_i}{t_{i+1} - t_i}\Big)^2 dt = \frac{m}{2}\frac{(x_{i+1} - x_i)^2}{t_{i+1} - t_i} = \frac{m}{2\epsilon}(x_{i+1} - x_i)^2$$

Plugging the action into (3.0.2) we get the path integral

$$
\begin{aligned}
K_0(b,a) &= \lim_{\epsilon\to 0} \frac{1}{A_\epsilon} \int \cdots \int e^{\frac{i}{\hbar}S[b,a]} \frac{dx_1}{A_\epsilon}\frac{dx_2}{A_\epsilon}\dots\frac{dx_{N-1}}{A_\epsilon} \\
&= \lim_{\epsilon\to 0} A_\epsilon^{-N} \int \cdots \int e^{\frac{i}{\hbar}\sum_{i=0}^{N-1} S(t_i,t_{i+1})} dx_1\ dx_2\dots dx_{N-1} \\
&= \lim_{\epsilon\to 0} A_\epsilon^{-N} \int \cdots \int e^{\frac{mi}{2\epsilon\hbar}\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2} dx_1\ dx_2\dots dx_{N-1}
\end{aligned}
$$

This integral will converge, so we can just integrate variable-by-variable. First noting that $\frac{1}{A_\epsilon} = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{\frac{1}{2}}$, we take the first piece of the integral:

$$
\begin{aligned}
A_\epsilon^{-2}\int e^{\frac{mi}{2\epsilon\hbar}(x_1-x_0)^2+(x_2-x_1)^2}dx_1 &= \frac{m}{2\pi i\hbar\epsilon}\int e^{\frac{mi}{2\epsilon\hbar}[(x_1-x_0)^2+(x_2-x_1)^2]}dx_1 \\
&= \frac{m}{2\pi i\hbar\epsilon}\int \exp\Big(\frac{mi}{2\epsilon\hbar}(x_2^2+x_0^2+2[-x_1x_0-x_2x_1+x_1^2]\Big)dx_1 \\
&= \frac{m}{2\pi i\hbar\epsilon}e^{\left(\frac{mi}{2\epsilon\hbar}(x_2^2+x_0^2)\right)}\int e^{\left(\frac{mi}{\epsilon\hbar}[x_1^2-x_1(x_0+x_2)]\right)}dx_1 \\
&= \frac{m}{2\pi i\hbar\epsilon}e^{\left(\frac{mi}{2\epsilon\hbar}(x_2^2+x_0^2)\right)} i\sqrt{\frac{\pi}{\frac{mi}{\epsilon\hbar}}}\exp\left\{\frac{(x_0+x_2)^2}{-4\frac{\epsilon\hbar}{mi}}\right\} \\
&= \sqrt{\frac{m}{2\pi i\hbar\cdot 2\epsilon}}\exp\left\{\frac{mi}{2\epsilon\hbar}(x_2^2+x_0^2)-\frac{mi}{4\epsilon\hbar}(x_0+x_2)^2\right\} \\
&= \sqrt{\frac{m}{2\pi i\hbar\cdot 2\epsilon}}\exp\left\{\frac{mi(x_2-x_0)^2}{2\hbar\cdot 2\epsilon}\right\}
\end{aligned}
$$

Now note that we have a similar formula when we multiply by the other x_2 dependent term and A_ϵ^{-1}, i.e.

$$
A^{-3}\int\int e^{\frac{mi}{2\epsilon\hbar}(x_1-x_0)^2+(x_2-x_1)^2+(x_3-x_2)^2}dx_1dx_2 = \left(\sqrt{\frac{m}{2\pi i\hbar\cdot 3\epsilon}}\right)\exp\left\{\frac{mi(x_3-x_0)^2}{2\hbar\cdot 3\epsilon}\right\}
$$

We see that in fact, we get an expression for the whole integral:

$$
A^{-N}\int\cdots\int e^{\frac{mi}{2\epsilon\hbar}\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2}dx_1\dots dx_{N-1} = \left(\sqrt{\frac{m}{2\pi i\hbar\cdot N\epsilon}}\right)\exp\left\{\frac{mi(x_N-x_0)^2}{2\hbar\cdot N\epsilon}\right\}
$$

Since $N\epsilon = t_b - t_a$, and $x_N = x_b$ and $x_0 = x_a$ we get the final expression for $K_0(a,b)$, i.e.

$$K_0(a,b) = \sqrt{\frac{m}{2\pi i\hbar \cdot (t_b - t_a)}} \exp\left\{\frac{mi(x_b - x_a)^2}{2\hbar \cdot (t_b - t_a)}\right\}$$

So, we have a complex Gaussian. What does this tell us about probability?

$$P_0(a,b) = K_0(a,b)^* K_0(a,b) = \frac{m}{2\pi i\hbar \cdot (t_b - t_a)}$$

Observe that then $t_b \to t_a$ then the quantity diverges to ∞, and the probability is inversely proportional to the distance between t_b and t_a, and not at all dependent on the positions at those times!