Exercise Class Solutions 8

4 Vector Geometry

4.2 Projections and Planes

4.2.1

Compute $\vec{u} \cdot \vec{v}$ where:

d) $\vec{u} = \begin{bmatrix} 3 & -1 & 5 \end{bmatrix}^T$, $\vec{v} = \begin{bmatrix} 6 & -7 & -5 \end{bmatrix}^T$

Solution

d)

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= \begin{bmatrix} 3 & -1 & 5 \end{bmatrix}^T \cdot \begin{bmatrix} 6 & -7 & -5 \end{bmatrix}^T \\
&= 3 \cdot 6 + (-1) \cdot (-7) + 5 \cdot (-5) \\
&= 18 + 7 + (-25) \\
&= 0
\end{aligned}$$

4.2.2

Find the angle between the following pairs of vectors.

b) $\vec{u} = \begin{bmatrix}3 & -1 & 0\end{bmatrix}^T$, $\vec{v} = \begin{bmatrix}-6 & 2 & 0\end{bmatrix}^T$

Solution

b) Let θ be the angle. Compute the dot product of the vectors

$$\vec{u} \cdot \vec{v} = 3 \cdot (-6) + (-1) \cdot 2 + 0 \cdot 0 = -18 - 2 = -20$$

and observe that as a consequence of it being negative, the angle between them is obtuse. Now compute the lengths of the vectors

$$\|\vec{u}\| = \sqrt{3^2 + (-1)^2 + 0^2} = \sqrt{9 + 1 + 0} = \sqrt{10}$$

$$\|\vec{v}\| = \sqrt{(-6)^2 + 2^2 + 0^2} = \sqrt{36 + 40 + 0} = \sqrt{40}$$

and use these values to calculate the angle

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) \\ &= \cos^{-1}\left(\frac{-20}{\sqrt{10}\sqrt{40}}\right) \\ &= \cos^{-1}\left(\frac{-20}{\sqrt{400}}\right) \\ &= \cos^{-1}\left(\frac{-20}{20}\right) \\ &= \cos^{-1}(-1)\end{aligned}$$

which tells us that the angle between the two vectors is π. This is easily confirmed, as $\vec{v} = -2\vec{u}$.

4.2.8

Find the three internal angles of the triangle with vertices:

b) $A(3, 1, -2)$, $B(5, 2, -1)$, and $C(4, 3, -3)$.

Solution

b) Compute the vectors of the triangle's three edges

$$\overrightarrow{AB} = \begin{bmatrix} 5-3 \\ 2-1 \\ (-1)-(-2) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\overrightarrow{AC} = \begin{bmatrix} 4-3 \\ 3-1 \\ (-3)-(-2) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 4-5 \\ 3-2 \\ (-3)-(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

and observe they all have length $\sqrt{6}$. Thus it is an equilateral triangle and all the angles are an equal $\pi/3$.

4.2.10

In each case, compute the projection of $\vec{u}$ on $\vec{v}$.

b) $\vec{u} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}^T$, $\vec{v} = \begin{bmatrix} 4 & 1 & 1 \end{bmatrix}^T$

Solution

b) We need the dot product

$$\vec{u} \cdot \vec{v} = 3 \cdot 4 + (-2) \cdot 1 + 1 \cdot 1 = 12 + (-2) + 1 = 11$$

and the length of $\vec{v}$

$$\|\vec{v}\| = \sqrt{4^2 + 1^2 + 1^2} = \sqrt{16 + 1 + 1} = \sqrt{18}$$

and then we can compute

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = \frac{11}{\sqrt{18}^2} \vec{v} = \frac{11}{18} \vec{v}$$

4.2.12

Calculate the distance from the point P to the line in each case and find the point Q on the line closest to P.

b) $P(1, -1, 3)$
line: $\begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T + t \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}^T$

Solution

b) Let $\vec{u} = \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}^T - \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 4 \end{bmatrix}^T$ denote the vector from $P_0(1, 0, -1)$, which is a point on the line, to P; and let $\vec{d}$ denote the direction vector of the line. Then

$$\text{proj}_{\vec{d}}\vec{u} = \frac{\vec{u} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} = \frac{0 \cdot 3 + (-1) \cdot 1 + 4 \cdot 4}{3^2 + 1^2 + 4^2}\vec{d} = \frac{0 + 1 + 16}{9 + (-1) + 16}\vec{d} = \frac{15}{26}\vec{d}$$

and

$$\begin{aligned}
\overrightarrow{OQ} &= P_0 + \frac{15}{26}\vec{d} \\
&= \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T + \frac{15}{26} \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}^T \\
&= \frac{1}{26} \begin{bmatrix} 26 & 0 & -26 \end{bmatrix}^T + \frac{1}{26} \begin{bmatrix} 45 & 15 & 60 \end{bmatrix}^T \\
&= \frac{1}{26} \begin{bmatrix} 26 + 45 & 15 & 60 - 26 \end{bmatrix}^T \\
&= \frac{1}{26} \begin{bmatrix} 71 & 15 & 34 \end{bmatrix}^T
\end{aligned}$$

is the position vector of Q so $Q(\frac{71}{26}, \frac{15}{26}, \frac{34}{26})$ is the required point. The distance from P to Q

$$\begin{aligned}
&\sqrt{\left(\frac{71}{26} - 1\right)^2 + \left(\frac{15}{26} - (-1)\right)^2 + \left(\frac{34}{26} - 3\right)^2} \\
=&\frac{1}{26}\sqrt{(71 - 26)^2 + (15 + 26)^2 + (34 - 78)^2} \\
=&\frac{1}{26}\sqrt{45^2 + 41^2 + (-44)^2} \\
=&\frac{1}{26}\sqrt{5642}
\end{aligned}$$

is the distance from P to the line.

4.2.13

Compute $\vec{u} \times \vec{v}$ where:

b) $\vec{u} = \begin{bmatrix} 3 & -1 & 0 \end{bmatrix}^T$, $\vec{v} = \begin{bmatrix} -6 & 2 & 0 \end{bmatrix}^T$

Solution

b) The cross product is

$$\vec{u} \times \vec{v} = \begin{bmatrix} (-1) \cdot 0 - 0 \cdot 2 \\ -(3 \cdot 0 - 0 \cdot (-6)) \\ 3 \cdot 2 - (-1) \cdot (-6) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which makes sense because $\vec{v} = -2\vec{u}$.

4.2.14

Find the equation of each of the following planes.

b) Passing through $A(1,-1,6)$, $B(0,0,1)$, and $C(4,7,-11)$.

d) Passing through $P(3,0,-1)$ and parallel to the plane with equation $2x-y+z=3$.

f) Containing $P(2,1,0)$ and the line $\begin{bmatrix}x & y & z\end{bmatrix}^T = \begin{bmatrix}3 & -1 & 2\end{bmatrix}^T + t\begin{bmatrix}1 & 0 & -1\end{bmatrix}^T$

h) Containing the lines

$$\begin{bmatrix}x & y & z\end{bmatrix}^T = \begin{bmatrix}3 & 1 & 0\end{bmatrix}^T + t\begin{bmatrix}1 & -1 & 3\end{bmatrix}^T$$

and

$$\begin{bmatrix}x & y & z\end{bmatrix}^T = \begin{bmatrix}0 & -2 & 5\end{bmatrix}^T + t\begin{bmatrix}2 & 1 & -1\end{bmatrix}^T$$

j) Each point of which is equidistant from $P(0,1,-1)$ and $Q(2,-1,-3)$.

Solution

b) The normal of the plane is the cross product of any two vectors making use of all three points

$$\begin{aligned}\overrightarrow{BA}\times\overrightarrow{BC} &= \begin{bmatrix}1\\-1\\5\end{bmatrix}\times\begin{bmatrix}4\\7\\-12\end{bmatrix}\\ &= \begin{bmatrix}(-1)\cdot(-12)-5\cdot 7\\ -(1\cdot(-12)-5\cdot 4)\\ 1\cdot 7-(-1)\cdot 4\end{bmatrix}\\ &= \begin{bmatrix}12-35\\ -((-12)-20)\\ 7+4\end{bmatrix}\\ &= \begin{bmatrix}-23\\32\\11\end{bmatrix}\end{aligned}$$

and the plane is given by

$$\begin{bmatrix}-23\\32\\11\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-23\\32\\11\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow -23x+32y+11z=11$$

d) The plane has normal $\begin{bmatrix}2 & -1 & 1\end{bmatrix}^T$ so it is given by

$$\begin{bmatrix}2\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}3\\0\\-1\end{bmatrix} \Rightarrow 2x-y+z=5$$

f) The normal of the plane can be found with a cross product of the direction vector of the line and a second vector defined by P and a point on the line:

$$\begin{aligned}
\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2-3 \\ 1-(-1) \\ 0-2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \cdot (-2) - (-1) \cdot 2 \\ -(1 \cdot (-2) - (-1) \cdot (-1)) \\ 1 \cdot 2 - 0 \cdot (-1) \end{bmatrix} \\
&= \begin{bmatrix} 0+2 \\ -((-2)-1) \\ 2-0 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}
\end{aligned}$$

Thus the plane is given by

$$\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow 2x + 3y + 2z = 7$$

h) The normal of the plane is the cross product of the two direction vectors

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} (-1) \cdot (-1) - 3 \cdot 1 \\ -(1 \cdot (-1) - 3 \cdot 2) \\ 1 \cdot 1 - (-1) \cdot 2 \end{bmatrix} = \begin{bmatrix} 1-3 \\ -((-1)-6) \\ 1-(-2) \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix}$$

and, using a point on either line, the plane is given by

$$\begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \Rightarrow -2x + 7y + 3z = 1$$

j) The normal of the plane is

$$\overrightarrow{PQ} = Q - P = \begin{bmatrix} 2-0 & -1-1 & -3-(-1) \end{bmatrix}^T = \begin{bmatrix} 2 & -2 & -2 \end{bmatrix}^T$$

and the mid point between P and Q

$$P + \frac{1}{2}\|\overrightarrow{PQ}\| = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

is on the plane. Thus the plane is given by

$$\begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Rightarrow 2x - 2y - 2z = 6$$

4.2.15

In each case, find the equation of the line.

b) Passing through $P(2, -1, 3)$ and perpendicular to the plane $2x + y = 1$.

Solution

b) We find that the normal vector of the plane is

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

then we find that the equation for the line is

$$r(t) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

4.2.16

In each case, find the shortest distance from the point P to the plane and find the point Q on the plane closest to P.

b) $P(3, 1, -1)$; plane with equation $2x + y - z = 6$.

Solution

b) Find that the normal vector for the plane is

$$\vec{n} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Let $\vec{q}$ be the position vector of the point Q and $\vec{p}$ be the position vector of the point P. We know that $\vec{q} = \vec{p} + t\vec{n}$ for some t. Since Q lies on the plane we have that

$$\vec{n} \cdot \vec{q} = 2x + y - z = 6$$

and

$$\begin{aligned} 6 &= \vec{n} \cdot \vec{q} \\ &= \vec{n} \cdot (\vec{p} + t\vec{n}) \\ &= \vec{n} \cdot \vec{p} + t\|\vec{n}\|^2 \\ &= 2 \cdot 3 + 1 \cdot 1 + (-1) \cdot (-1) + t \cdot (2 \cdot 2 + 1 \cdot 1 + (-1) \cdot (-1)) \\ &= 8 + 6t \end{aligned}$$

solving for t gives us

$$t = -\frac{2}{6} = -\frac{1}{3}$$

we can then plug in t to find the value for Q and get

$$\begin{aligned} \vec{q} &= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{7}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} \end{aligned}$$

so $Q = (7/3, 2/3, -2/3)$. We can find the distance by calculating

$$\|\overrightarrow{PQ}\| = \sqrt{(-2/3)^2 + (-1/3)^2 + (1/3)^2} = \frac{\sqrt{6}}{3}$$

4.2.19

Find the equations of the line of intersection of the following planes.

b) $3x + y - 2z = 1$ and $x + y + z = 5$.

Solution

b) Subtract the second equation from the first to get

$$2x - 3z = -4$$

then solve for x to get

$$x = \frac{3z}{2} - 2$$

and let z be a free variable so $z = t$ where $t \in \mathbb{R}$ so we have

$$x = \frac{3t}{2} - 2.$$

We can then subsitute x and z into one of the equations to get the value for y

$$\frac{3t}{2} - 2 + y + t = 5 \Rightarrow y = 7 - \frac{5t}{2}$$

So the parametric equation for the line is then

$$\begin{aligned} x &= \frac{3t}{2} - 2 \\ y &= 7 - \frac{5t}{2} \\ z &= t \end{aligned}$$