Exercise Class Solutions 7

2 Matrix Algebra

2.5 Matrix Transformations

2.5.1

Give the matrix of the transformation T in each case:

b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $y = -x$.

d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is rotation through $-\pi/2$.

Solution

b) This transformation carries the vector $\begin{bmatrix} x & y \end{bmatrix}^T$ to $\begin{bmatrix} -y & -x \end{bmatrix}^T$. Now observe that

$$\begin{bmatrix} -y \\ -x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so the 2×2 matrix is the matrix of the transformation T.

d) By Example 5 we have that

$$R_{-\pi/2} = \begin{bmatrix} \cos -\pi/2 & -\sin -\pi/2 \\ \sin -\pi/2 & \cos -\pi/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

2.5.2

In each case show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is not a linear transformation:

b) $T\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) = \begin{bmatrix} 0 & y^2 \end{bmatrix}^T$.

Solution

b) It is not a linear transformation because it does not preserve scalar multiplication; e.g.,

$$T\left(2\begin{bmatrix} 0 & 1 \end{bmatrix}^T\right) = T\left(\begin{bmatrix} 0 & 2 \end{bmatrix}^T\right) = \begin{bmatrix} 0 & 4 \end{bmatrix}^T$$

$$2T\left(\begin{bmatrix} 0 & 1 \end{bmatrix}^T\right) = 2\begin{bmatrix} 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 \end{bmatrix}^T$$

2.5.3

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is as in (b) and (d) of Exercise 1, find $T\left(\begin{bmatrix}1 & 1\end{bmatrix}^T\right)$ and $T\left(\begin{bmatrix}2 & -1\end{bmatrix}^T\right)$.

Solution

b)

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + (-1) \cdot 1 \\ (-1) \cdot 1 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 - 1 \\ (-1) + 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 2 + (-1) \cdot (-1) \\ (-1) \cdot 2 + 0 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 + 1 \\ (-2) + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

d)

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 1 \\ (-1) \cdot 1 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 + 1 \\ (-1) + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 2 + 1 \cdot (-1) \\ (-1) \cdot 2 + 0 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 - 1 \\ (-2) + 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

2.5.4

If $a > 0$ is a fixed real number, define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(X) = aX$ for all X in R^n. Show that T is a linear transformation, and find its matrix. [T is called a *dilation* if $a > 0$, and a *contraction* if $a < 0$].

Solution

Let A and B be vectors in $\mathbb{R}^n$ and b a scalar. Then T preserves both addition

$$T(A + B) = a(A + B) = aA + aB = T(A) + T(B)$$

and scalar multiplication

$$T(bA) = abA = baA = bT(A)$$

and is thus a linear transformation whose matrix is aI_n.

2.5.12

In each case find a rotation or reflection that equals the given transformation.

b) Rotation through π followed by reflection in the X axis.

d) Reflection in the X axis followed by rotation through $\pi/2$.

f) Reflection in the X axis followed by reflection in the line $y = x$.

Solution

b) It is equal to reflection in the Y axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \cdots = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

d) It is equal to the reflection in the line $y = x$:

$$\begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \cdots = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

f) It is equal to rotation through $\pi/2$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \cdots = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

2.5.14

Find the inverse of the shear transformation in Example 4 and describe it geometrically.

Solution

Since the shear adds a scalar multiple of the y component to the x component, its inverse subtracts that same scalar multiple; i.e., the inverse of a positive shear is a negative shear by the same scalar, and vice versa.

4 Vector Geometry

4.1 Vectors and Lines

4.1.1

Compute $\|\vec{v}\|$ if $\vec{v}$ equals:

b) $\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^T$

Solution

b) $\sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6}$

4.1.2

Find a unit vector in the direction of:

b) $\begin{bmatrix} -2 & -1 & 2 \end{bmatrix}^T$

Solution

To do this we use the inverse of the length of the vector as a scalar:

b) The length of our vector is $\sqrt{(-2)^2 + (-1)^2 + 2^2} = \sqrt{4+1+4} = \sqrt{9} = 3$ so the unit vector in its direction is $\frac{1}{3}\begin{bmatrix} -2 & -1 & 2 \end{bmatrix}^T$.

4.1.4

Find the distance between the following pairs of points.

d) $\begin{bmatrix} 4 & 0 & -2 \end{bmatrix}$ and $\begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T$

Solution

d)

$$\begin{aligned} \sqrt{(4-3)^2 + (0-2)^2 + ((-2)-0)^2} &= \sqrt{1^2 + (-2)^2 + (-2)^2} \\ &= \sqrt{1+4+4} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

4.1.7

Determine whether $\vec{u}$ and $\vec{v}$ are parallel in each of the following cases.

b) $\vec{u} = \begin{bmatrix} 3 & -6 & 3 \end{bmatrix}^T$; $\vec{v} = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}^T$;

Solution

b) They are parallel by Theorem 4 because $\vec{u} = -3\vec{v}$.

4.1.9

In each case, find $\overrightarrow{PQ}$ and $\|\overrightarrow{PQ}\|$.

b) $P(2, 0, 1)$, $Q(1, -1, 6)$

Solution

b)

$$\overrightarrow{PQ} = \begin{bmatrix} 1-2 \\ (-1)-0 \\ 6-1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}, \quad \|\overrightarrow{PQ}\| = \sqrt{(-1)^2 + (-1)^2 + 5^2} = \sqrt{27}$$

4.1.11

Let $\vec{u} = \begin{bmatrix}3 & -1 & 0\end{bmatrix}^T$, $\vec{v} = \begin{bmatrix}4 & 0 & 1\end{bmatrix}^T$, and $\vec{w} = \begin{bmatrix}-1 & 1 & 5\end{bmatrix}^T$. In each case, find $\vec{x}$ such that:

b) $2(3\vec{v} - \vec{x}) = 5\vec{w} + \vec{u} - 3\vec{x}$

Solution

b)

$$\begin{aligned}
& & 2(3\vec{v} - \vec{x}) &= 5\vec{w} + \vec{u} - 3\vec{x} \\
&\Rightarrow & 6\vec{v} - 2\vec{x} &= 5\vec{w} + \vec{u} - 3\vec{x} \\
&\Rightarrow & 3\vec{x} - 2\vec{x} &= 5\vec{w} + \vec{u} - 6\vec{v} \\
&\Rightarrow & \vec{x} &= 5\vec{w} + \vec{u} - 6\vec{v} \\
&\Rightarrow & \vec{x} &= 5\begin{bmatrix}-1\\1\\5\end{bmatrix} + \begin{bmatrix}3\\-1\\0\end{bmatrix} - 6\begin{bmatrix}4\\0\\1\end{bmatrix} \\
&\Rightarrow & \vec{x} &= \begin{bmatrix}-5\\5\\25\end{bmatrix} + \begin{bmatrix}3\\-1\\0\end{bmatrix} - \begin{bmatrix}24\\0\\6\end{bmatrix} \\
&\Rightarrow & \vec{x} &= \begin{bmatrix}-5+3-24\\5-1\\25-6\end{bmatrix} \\
&\Rightarrow & \vec{x} &= \begin{bmatrix}-26\\4\\19\end{bmatrix}
\end{aligned}$$

4.1.22

Find all vector and parametric equations of the following lines.

b) The line passing through $P(3, -1, 4)$ and $Q(1, 0, -1)$.

d) The line parallel to $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ and passing through $P(1, 1, 1)$.

f) The line passing through $P(2, -1, 1)$ and parallel to the line with parametric equations $x = 2 - t$, $y = 1$, and $z = t$.

Solution

b) The vector

$$\overrightarrow{QP} = \begin{bmatrix} 3-1 \\ -1-0 \\ 4-(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

is the direction vector of the line. The vector equation for the line is

$$\vec{p} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

The parametric equation for the line is

$$\begin{aligned} x &= 3 + 2t \\ y &= -1 - t \\ z &= 4 + 5t \end{aligned}$$

d) The vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is parallel to our line so it is also the direction vector of our line. The vector equation for the line is

$$\vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The parametric equation for the line is

$$\begin{aligned} x &= 1 + t \\ y &= 1 + t \\ z &= 1 + t \end{aligned}$$

f) The direction vector for the given line is

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

and since it is parallel to our line it is also the direction vector for our line. The vector equation for the line is

$$\vec{p} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The parametric equation for the line is

$$\begin{aligned} x &= 2 - t \\ y &= -1 \\ z &= 1 + t \end{aligned}$$

4.1.23

In each case, verify that the points P and Q lie on the line.

b) $P(2, 3, -3)$, $Q(-1, 3, -9)$, $\begin{aligned} x &= 4 - t \\ y &= 3 \\ z &= 1 - 2t \end{aligned}$

Solution

b) For a point to be on the line it needs to satisfy the system of equations so we can just plug in the values into the system and see if it holds.
$2 = 4 - t$
$3 = 3$
$-3 = 1 - 2t$
So if $t = 2$ the system holds and therefore the point P is on the line.

c) Same as before,
$-1 = 4 - t$
$3 = 3$
$-9 = 1 - 2t$
So if $t = 5$ the system holds and therefore the point Q is on the line.