Exercise Class Solutions 9

4 Vector Geometry

4.5 An Application to Computer Graphics

4.5.1

Consider the letter A described in Figure 4.41. Find the data matrix for the letter obtained by:

a) Rotating the letter through $\frac{\pi}{4}$ about the origin.

b) Rotating the letter through $\frac{\pi}{4}$ about the point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution

a) The data matrix for the letter is

$$D = \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix}$$

so we rotate it as we have done before

$$
\begin{aligned}
R_{\pi/4}D &= \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} D \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1\cdot 0+(-1)\cdot 0 & 1\cdot 0+1\cdot 0 \\ 1\cdot 6+(-1)\cdot 0 & 1\cdot 6+1\cdot 0 \\ 1\cdot 5+(-1)\cdot 3 & 1\cdot 5+1\cdot 3 \\ 1\cdot 1+(-1)\cdot 3 & 1\cdot 1+1\cdot 3 \\ 1\cdot 3+(-1)\cdot 9 & 1\cdot 3+1\cdot 9 \end{bmatrix}^T \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 0+0 & 0+0 \\ 6+0 & 6+0 \\ 5-3 & 5+3 \\ 1-3 & 1+3 \\ 3-9 & 3+9 \end{bmatrix}^T \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 6 & -3 & -2 & -6 \\ 0 & 6 & 8 & 4 & 12 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 6 & -3 & -2 & -6 \\ 0 & 6 & 8 & 4 & 12 \end{bmatrix}
\end{aligned}
$$

b) We use homogeneous coordinates for the data matrix of the letter

$$
K_D = \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}
$$

and since the coordinate we want to rotate around is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ we get

$$\begin{aligned}
&\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} R_{\pi/4} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} K_D \\
=& \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} K_D \\
=& \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} K_D \\
=& \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -3 \\ 0 & 0 & \sqrt{2} \end{bmatrix} K_D \\
=& \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 1+\sqrt{2} \\ 1 & 1 & 2\sqrt{2}-3 \\ 0 & 0 & \sqrt{2} \end{bmatrix} K_D \\
=& \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 1+\sqrt{2} \\ 1 & 1 & 2\sqrt{2}-3 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}$$

This last matrix multiplication is left to the reader.

4.5.2

Find the matrix for turning the letter A in figure 4.41 upside down in place.

Solution

To turn the letter A upside down in place we need to reflect the vertices around the line that is in the middle of the letter. We calculate the y value of the line by taking the average of the maximum and minimum y values.

$$y_{mid} = (y_{max} + y_{min})/2 = (9+0)/2 = 4.5$$

To reflect it we will use homogenous coordinates, we will translate everything so $P(0, 4.5)$ will become the origin and pulling the line down to the x-axis and then reflecting around the x-axis before translating everything back.

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4.5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$$

4.5.4

Find the 3×3 matrix for rotating through the angle θ about the point $P(a, b)$.

Solution

We will need to use homogenous coordinates and translate the point $P(a, b)$ to be the origin, rotate the origin by θ and then translate back by $P(a, b)$.

$$\begin{aligned} R &= \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta & a(-\cos\theta + 1) + b\sin\theta \\ \sin\theta & \cos\theta & b(-\cos\theta + 1) - a\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

2 Matrix Algebra

2.5 Matrix Transformations

2.5.18

(*Modified from book.*) Let L denote the line through the origin in $\mathbb{R}^2$ with slope m. The reflection in L has matrix

$$A = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Show that $m = \sin\theta/\cos\theta = \tan\theta$ yields the formula of Example 10.

Solution

We start by replacing m with $\sin\theta/\cos\theta$. Notice that the slope for a line going through the origin with angle θ is given by $\sin\theta/\cos\theta$.

$$\begin{aligned}
Q &= \frac{1}{1+\frac{\sin^2\theta}{\cos^2\theta}}\begin{bmatrix} 1-\frac{\sin^2\theta}{\cos^2\theta} & 2\cdot\frac{\sin\theta}{\cos\theta} \\ 2\cdot\frac{\sin\theta}{\cos\theta} & \frac{\sin^2\theta}{\cos^2\theta}-1 \end{bmatrix} \\
&= \frac{1}{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}\begin{bmatrix} \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta} & 2\cdot\frac{\sin\theta}{\cos\theta} \\ 2\cdot\frac{\sin\theta}{\cos\theta} & \frac{\sin^2\theta-\cos^2\theta}{\cos^2\theta} \end{bmatrix} \\
&= \frac{1}{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}\begin{bmatrix} \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta} & 2\cdot\frac{\sin\theta}{\cos\theta} \\ 2\cdot\frac{\sin\theta}{\cos\theta} & \frac{\sin^2\theta-\cos^2\theta}{\cos^2\theta} \end{bmatrix} \\
&= \cos^2\theta\begin{bmatrix} \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta} & 2\cdot\frac{\sin\theta}{\cos\theta} \\ 2\cdot\frac{\sin\theta}{\cos\theta} & \frac{\sin^2\theta-\cos^2\theta}{\cos^2\theta} \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta-\sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta-\cos^2\theta \end{bmatrix}
\end{aligned}$$

which is the same as the formula given in Example 10.

4 Vector Geometry

4.5 An Application to Computer Graphics

4.5.5

(*Modified from book.*) Find the reflection of the point P in the line $y = 1+2x$ in $\mathbb{R}^2$ using the formula derived in Problem 2.5.18 if:

a) $P = P(1, 1)$

b) $P = P(1, 4)$

c) What about $P = P(1, 3)$? Explain.
[Hint: Example 1.]

Solution

Let's find the matrix of the transformation which we will use in all parts of the problem

$$\begin{aligned}
B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1-2^2}{1+2^2} & \frac{2\cdot 2}{1+2^2} & 0 \\ \frac{2\cdot 2}{1+2^2} & \frac{2^2-1}{1+2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \cdots \\
&= \begin{bmatrix} -3/5 & 4/5 & -4/5 \\ 4/5 & 3/5 & 2/5 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

a)

$$B \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 9/5 \\ 1 \end{bmatrix}$$

b)

$$B \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 9/5 \\ 18/5 \\ 1 \end{bmatrix}$$

c)

$$B \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

This is due to the point being on the line.

4.5.3

(*Modified from book.*) Find the 3×3 matrix for reflecting in the line $y = mx + b$ using the formula derived in Problem 2.5.18.

Solution

Using the equation given in Problem 2.5.18 we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} & 0 \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m & -2mb \\ 2m & m^2-1 & 2b \\ 0 & 0 & 1+m^2 \end{bmatrix}$$

4.2 Projections and Planes

4.2.17

a) Does the line through $P(1,2,-3)$ with direction vector $\vec{d} = \begin{bmatrix}1 & 2 & -3\end{bmatrix}^T$ lie in the plane $2x - y - z = 3$? Explain.

b) Does the plane through $P(4,0,5)$, $Q(2,2,1)$, and $R(1,-1,2)$ pass through the origin? Explain.

Solution

a) No, even though the point $P(1,2,-3)$ is on the plane since $2\cdot 1 - 1\cdot 2 - 1\cdot(-3) = 2 - 2 + 3 = 3$, we see that the dot product of the line's direction vector with the normal of the plane is

$$\begin{bmatrix}1 & 2 & -3\end{bmatrix}^T \cdot \begin{bmatrix}2 & -1 & -1\end{bmatrix}^T = 1\cdot 2 + 2\cdot(-1) + (-3)\cdot(-1) = 1 - 2 + 3 = 2$$

so they are not orthogonal and thus the line cannot lie in the plane.

b) Start by computing the normal of the plane

$$\overrightarrow{QP} = \begin{bmatrix}4-2\\0-2\\5-1\end{bmatrix} = \begin{bmatrix}2\\-2\\4\end{bmatrix}, \overrightarrow{RP} = \begin{bmatrix}4-1\\0-(-1)\\5-2\end{bmatrix} = \begin{bmatrix}3\\1\\3\end{bmatrix}$$

$$\vec{n} = \overrightarrow{QP} \times \overrightarrow{RP} = \begin{bmatrix}(-2)\cdot 3 - 4\cdot 1\\-(2\cdot 3 - 4\cdot 3)\\2\cdot 1 - (-2)\cdot 3\end{bmatrix} = \begin{bmatrix}-6-4\\-(6-12)\\2-(-6)\end{bmatrix} = \begin{bmatrix}-10\\6\\8\end{bmatrix}$$

which gives us the equation of the plane

$$\begin{bmatrix}-10\\6\\8\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-10\\6\\8\end{bmatrix} \begin{bmatrix}2\\2\\1\end{bmatrix} \Rightarrow -10x + 6y + 8z = 0.$$

When we plug the origin into the equation of the plane we get

$$-10\cdot 0 + 6\cdot 0 + 8\cdot 0 = 0 + 0 + 0 = 0$$

so the plane does indeed pass through the origin.

4.2.20

In each case, find all points of intersection of the given plane and the line $\begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^T + t \begin{bmatrix} 2 & 5 & -1 \end{bmatrix}^T$.

b) $2x - y - z = 5$

d) $-x - 4y - 3z = 6$

Solution

b) Plugging the equation of the line into the equation of the plane makes it evident that the line does not intersect the plane:

$$2(1+2t) - (-2+5t) - (3-t) = 2 + 4t + 2 - 5t - 3 + t = 1 \neq 5$$

d) Observe that the line is not orthogonal to the normal of the plane

$$\begin{aligned} \begin{bmatrix} 2 & 5 & -1 \end{bmatrix}^T \cdot \begin{bmatrix} -1 & -4 & -3 \end{bmatrix}^T &= 2 \cdot (-1) + 5 \cdot (-4) + (-1) \cdot (-3) \\ &= -2 - 20 + 3 \\ &= -19 \end{aligned}$$

so it does not lie in it and must intersect the plane in exactly one point. Plug the equation of the line into the equation of the plane

$$\begin{aligned} & & -(1+2t) - 4(-2+5t) - 3(3-t) &= 6 \\ &\Rightarrow & -1 - 2t + 8 - 20t - 9 + 3t &= 6 \\ &\Rightarrow & -19t - 2 &= 6 \\ &\Rightarrow & t &= -\frac{8}{19} \end{aligned}$$

and plug the arrived at value of t into the equation of the line

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^T - \frac{8}{19} \begin{bmatrix} 2 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} \frac{3}{19} & -\frac{78}{19} & \frac{65}{19} \end{bmatrix}^T$$

from which we see that the line intersects the plane in the point $(\frac{3}{19}, -\frac{78}{19}, \frac{65}{19})$.