

## Exercise Class Solutions 8

### 4 Vector Geometry

#### 4.2 Projections and Planes

##### 4.2.1

Compute  $\vec{u} \cdot \vec{v}$  where:

d)  $\vec{u} = [3 \quad -1 \quad 5]^T, \vec{v} = [6 \quad -7 \quad -5]^T$

**Solution**

d)

$$\begin{aligned}\vec{u} \cdot \vec{v} &= [3 \quad -1 \quad 5]^T \cdot [6 \quad -7 \quad -5]^T \\ &= 3 \cdot 6 + (-1) \cdot (-7) + 5 \cdot (-5) \\ &= 18 + 7 + (-25) \\ &= 0\end{aligned}$$

### 4.2.2

Find the angle between the following pairs of vectors.

b)  $\vec{u} = [3 \ -1 \ 0]^T$ ,  $\vec{v} = [-6 \ 2 \ 0]^T$

#### Solution

b) Let  $\theta$  be the angle. Compute the dot product of the vectors

$$\vec{u} \cdot \vec{v} = 3 \cdot (-6) + (-1) \cdot 2 + 0 \cdot 0 = -18 - 2 = -20$$

and observe that as a consequence of it being negative, the angle between them is obtuse. Now compute the lengths of the vectors

$$\|\vec{u}\| = \sqrt{3^2 + (-1)^2 + 0^2} = \sqrt{9 + 1 + 0} = \sqrt{10}$$

$$\|\vec{v}\| = \sqrt{(-6)^2 + 2^2 + 0^2} = \sqrt{36 + 4 + 0} = \sqrt{40}$$

and use these values to calculate the angle

$$\begin{aligned}\theta &= \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right) \\ &= \cos^{-1} \left( \frac{-20}{\sqrt{10} \sqrt{40}} \right) \\ &= \cos^{-1} \left( \frac{-20}{\sqrt{400}} \right) \\ &= \cos^{-1} \left( \frac{-20}{20} \right) \\ &= \cos^{-1}(-1)\end{aligned}$$

which tells us that the angle between the two vectors is  $\pi$ . This is easily confirmed, as  $\vec{v} = -2\vec{u}$ .

### 4.2.8

Find the three internal angles of the triangle with vertices:

b)  $A(3, 1, -2)$ ,  $B(5, 2, -1)$ , and  $C(4, 3, -3)$ .

### Solution

b) Compute the vectors of the triangle's three edges

$$\overrightarrow{AB} = \begin{bmatrix} 5 - 3 \\ 2 - 1 \\ (-1) - (-2) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\overrightarrow{AC} = \begin{bmatrix} 4 - 3 \\ 3 - 1 \\ (-3) - (-2) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 4 - 5 \\ 3 - 2 \\ (-3) - (-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

and observe they all have length  $\sqrt{6}$ . Thus it is an equilateral triangle and all the angles are an equal  $\pi/3$ .

### 4.2.10

In each case, compute the projection of  $\vec{u}$  on  $\vec{v}$ .

b)  $\vec{u} = [3 \quad -2 \quad 1]^T$ ,  $\vec{v} = [4 \quad 1 \quad 1]^T$

### Solution

b) We need the dot product

$$\vec{u} \cdot \vec{v} = 3 \cdot 4 + (-2) \cdot 1 + 1 \cdot 1 = 12 + (-2) + 1 = 11$$

and the length of  $\vec{v}$

$$\|\vec{v}\| = \sqrt{4^2 + 1^2 + 1^2} = \sqrt{16 + 1 + 1} = \sqrt{18}$$

and then we can compute

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = \frac{11}{\sqrt{18}^2} \vec{v} = \frac{11}{18} \vec{v}$$

#### 4.2.12

Calculate the distance from the point  $P$  to the line in each case and find the point  $Q$  on the line closest to  $P$ .

b)  $P(1, -1, 3)$

$$\text{line: } \begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T + t \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}^T$$

#### Solution

b) Let  $\vec{u} = \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}^T - \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 4 \end{bmatrix}^T$  denote the vector from  $P_0(1, 0, -1)$ , which is a point on the line, to  $P$ ; and let  $\vec{d}$  denote the direction vector of the line. Then

$$\text{proj}_{\vec{d}} \vec{u} = \frac{\vec{u} \cdot \vec{d}}{\|\vec{d}\|^2} \vec{d} = \frac{0 \cdot 3 + (-1) \cdot 1 + 4 \cdot 4}{3^2 + 1^2 + 4^2} \vec{d} = \frac{0 + 1 + 16}{9 + 1 + 16} \vec{d} = \frac{15}{26} \vec{d}$$

and

$$\begin{aligned} \overrightarrow{OQ} &= P_0 + \frac{15}{26} \vec{d} \\ &= \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T + \frac{15}{26} \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}^T \\ &= \frac{1}{26} \begin{bmatrix} 26 & 0 & -26 \end{bmatrix}^T + \frac{1}{26} \begin{bmatrix} 45 & 15 & 60 \end{bmatrix}^T \\ &= \frac{1}{26} \begin{bmatrix} 26 + 45 & 15 & 60 - 26 \end{bmatrix}^T \\ &= \frac{1}{26} \begin{bmatrix} 71 & 15 & 34 \end{bmatrix}^T \end{aligned}$$

is the position vector of  $Q$  so  $Q(\frac{71}{26}, \frac{15}{26}, \frac{34}{26})$  is the required point. The distance from  $P$  to  $Q$

$$\begin{aligned} &\sqrt{\left(\frac{71}{26} - 1\right)^2 + \left(\frac{15}{26} - (-1)\right)^2 + \left(\frac{34}{26} - 3\right)^2} \\ &= \frac{1}{26} \sqrt{(71 - 26)^2 + (15 + 26)^2 + (34 - 78)^2} \\ &= \frac{1}{26} \sqrt{45^2 + 41^2 + (-44)^2} \\ &= \frac{1}{26} \sqrt{5642} \end{aligned}$$

is the distance from  $P$  to the line.

**4.2.13**

Compute  $\vec{u} \times \vec{v}$  where:

b)  $\vec{u} = [3 \quad -1 \quad 0]^T$ ,  $\vec{v} = [-6 \quad 2 \quad 0]^T$

**Solution**

b) The cross product is

$$\vec{u} \times \vec{v} = \begin{bmatrix} (-1) \cdot 0 - 0 \cdot 2 \\ -(3 \cdot 0 - 0 \cdot (-6)) \\ 3 \cdot 2 - (-1) \cdot (-6) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which makes sense because  $\vec{v} = -2\vec{u}$ .

#### 4.2.14

Find the equation of each of the following planes.

b) Passing through  $A(1, -1, 6)$ ,  $B(0, 0, 1)$ , and  $C(4, 7, -11)$ .

d) Passing through  $P(3, 0, -1)$  and parallel to the plane with equation  $2x - y + z = 3$ .

f) Containing  $P(2, 1, 0)$  and the line  $\begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^T + t \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$

h) Containing the lines

$$\begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix}^T + t \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}^T$$

and

$$\begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} 0 & -2 & 5 \end{bmatrix}^T + t \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}^T$$

j) Each point of which is equidistant from  $P(0, 1, -1)$  and  $Q(2, -1, -3)$ .

#### Solution

b) The normal of the plane is the cross product of any two vectors making use of all three points

$$\begin{aligned} \overrightarrow{BA} \times \overrightarrow{BC} &= \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \times \begin{bmatrix} 4 \\ 7 \\ -12 \end{bmatrix} \\ &= \begin{bmatrix} (-1) \cdot (-12) - 5 \cdot 7 \\ -(1 \cdot (-12) - 5 \cdot 4) \\ 1 \cdot 7 - (-1) \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 12 - 35 \\ -((-12) - 20) \\ 7 + 4 \end{bmatrix} \\ &= \begin{bmatrix} -23 \\ 32 \\ 11 \end{bmatrix} \end{aligned}$$

and the plane is given by

$$\begin{bmatrix} -23 \\ 32 \\ 11 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -23 \\ 32 \\ 11 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 11 \end{bmatrix} \Rightarrow -23x + 32y + 11z = 11$$

d) The plane has normal  $\begin{bmatrix} 2 & -1 & 1 \end{bmatrix}^T$  so it is given by

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \Rightarrow 2x - y + z = 5$$

- f) The normal of the plane can be found with a cross product of the direction vector of the line and a second vector defined by  $P$  and a point on the line:

$$\begin{aligned} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2-3 \\ 1-(-1) \\ 0-2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot (-2) - (-1) \cdot 2 \\ -(1 \cdot (-2) - (-1) \cdot (-1)) \\ 1 \cdot 2 - 0 \cdot (-1) \end{bmatrix} \\ &= \begin{bmatrix} 0+2 \\ -((-2)-1) \\ 2-0 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \end{aligned}$$

Thus the plane is given by

$$\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow 2x + 3y + 2z = 7$$

- h) The normal of the plane is the cross product of the two direction vectors

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} (-1) \cdot (-1) - 3 \cdot 1 \\ -(1 \cdot (-1) - 3 \cdot 2) \\ 1 \cdot 1 - (-1) \cdot 2 \end{bmatrix} = \begin{bmatrix} 1-3 \\ -((-1)-6) \\ 1-(-2) \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix}$$

and, using a point on either line, the plane is given by

$$\begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \Rightarrow -2x + 7y + 3z = 1$$

- j) The normal of the plane is

$$\overrightarrow{PQ} = Q - P = [2-0 \quad -1-1 \quad -3-(-1)]^T = [2 \quad -2 \quad -2]^T$$

and the mid point between  $P$  and  $Q$

$$P + \frac{1}{2} \|\overrightarrow{PQ}\| = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

is on the plane. Thus the plane is given by

$$\begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Rightarrow 2x - 2y - 2z = 6$$

**4.2.15**

In each case, find the equation of the line.

- b) Passing through  $P(2, -1, 3)$  and perpendicular to the plane  $2x + y = 1$ .

**Solution**

- b) We find that the normal vector of the plane is

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

then we find that the equation for the line is

$$r(t) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$



#### 4.2.16

In each case, find the shortest distance from the point  $P$  to the plane and find the point  $Q$  on the plane closest to  $P$ .

b)  $P(3, 1, -1)$ ; plane with equation  $2x + y - z = 6$ .

#### Solution

b) Find that the normal vector for the plane is

$$\vec{n} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Let  $\vec{q}$  be the position vector of the point  $Q$  and  $\vec{p}$  be the position vector of the point  $P$ . We know that  $\vec{q} = \vec{p} + t\vec{n}$  for some  $t$ . Since  $Q$  lies on the plane we have that

$$\vec{n} \cdot \vec{q} = 2x + y - z = 6$$

and

$$\begin{aligned} 6 &= \vec{n} \cdot \vec{q} \\ &= \vec{n} \cdot (\vec{p} + t\vec{n}) \\ &= \vec{n} \cdot \vec{p} + t\|\vec{n}\|^2 \\ &= 2 \cdot 3 + 1 \cdot 1 + (-1) \cdot (-1) + t \cdot (2 \cdot 2 + 1 \cdot 1 + (-1) \cdot (-1)) \\ &= 8 + 6t \end{aligned}$$

solving for  $t$  gives us

$$t = -\frac{2}{6} = -\frac{1}{3}$$

we can then plug in  $t$  to find the value for  $Q$  and get

$$\begin{aligned} \vec{q} &= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{7}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} \end{aligned}$$

so  $Q = (7/3, 2/3, -2/3)$ . We can find the distance by calculating

$$\|\vec{PQ}\| = \sqrt{(-2/3)^2 + (-1/3)^2 + (1/3)^2} = \frac{\sqrt{6}}{3}$$

**4.2.19**

Find the equations of the line of intersection of the following planes.

- b)  $3x + y - 2z = 1$  and  $x + y + z = 5$ .

**Solution**

- b) Subtract the second equation from the first to get

$$2x - 3z = -4$$

then solve for  $x$  to get

$$x = \frac{3z}{2} - 2$$

and let  $z$  be a free variable so  $z = t$  where  $t \in \mathbb{R}$  so we have

$$x = \frac{3t}{2} - 2.$$

We can then substitute  $x$  and  $z$  into one of the equations to get the value for  $y$

$$\begin{aligned}\frac{3t}{2} - 2 + y + t &= 5 \\ y &= 7 - \frac{5t}{2}\end{aligned}$$

So the parametric equation for the line is then

$$\begin{aligned}x &= \frac{3t}{2} - 2 \\ y &= 7 - \frac{5t}{2} \\ z &= t\end{aligned}$$