Exercise Class Solutions 5

Additional Exercises

1

We have the following language over the alphabet $\Sigma = \{a, b\}$

$$A = \{w | w \text{ starts with the symbol } \mathbf{a} \text{ OR } w \text{ ends with the symbol } \mathbf{b}\}$$

Note that this is an "inclusive OR".

a) Construct a state diagram for a DFA ("Deterministic Finite-state Automaton") which recognizes A.

b) Express A using a regular expression.

c) Find a context-free grammar that generates A.

d) We have the following language over the alphabet $\Sigma = \{a, b\}$

$$B = \{w | w \text{ contains an odd number of } \mathbf{a} \text{ AND does NOT contain two consecutive } \mathbf{b}\}$$

Construct a state diagram for a DFA ("Deterministic Finite-state Automaton") which recognizes B.

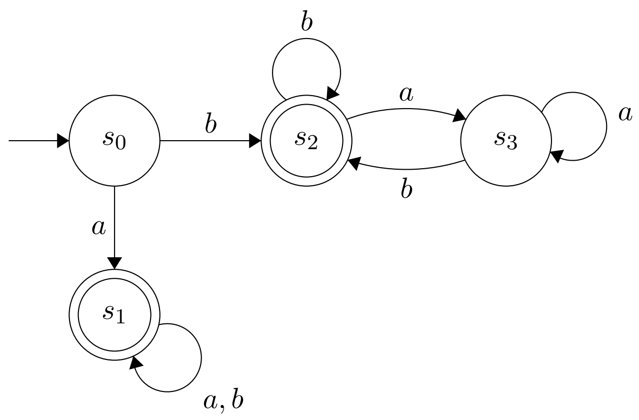
e) Construct a state diagram for a NFA ("Nondeterministic Finite-state Automaton") which recognizes the language AB.

f) Construct a state diagram for a NFA ("Nondeterministic Finite-state Automaton") which recognizes the language $A \cup B$.

g) Construct a state diagram for a NFA ("Nondeterministic Finite-state Automaton") which recognizes the language B^*.

Solution

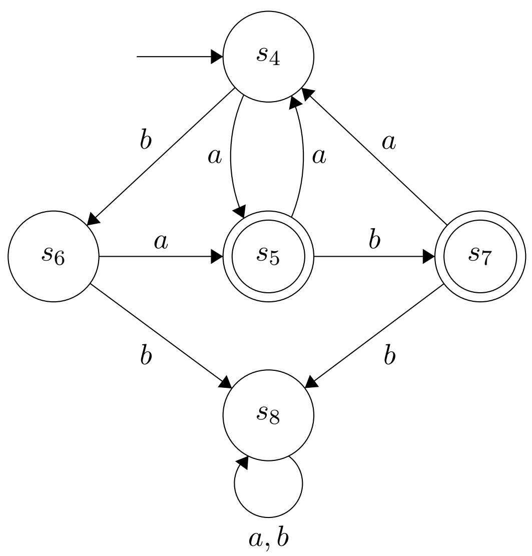
a)



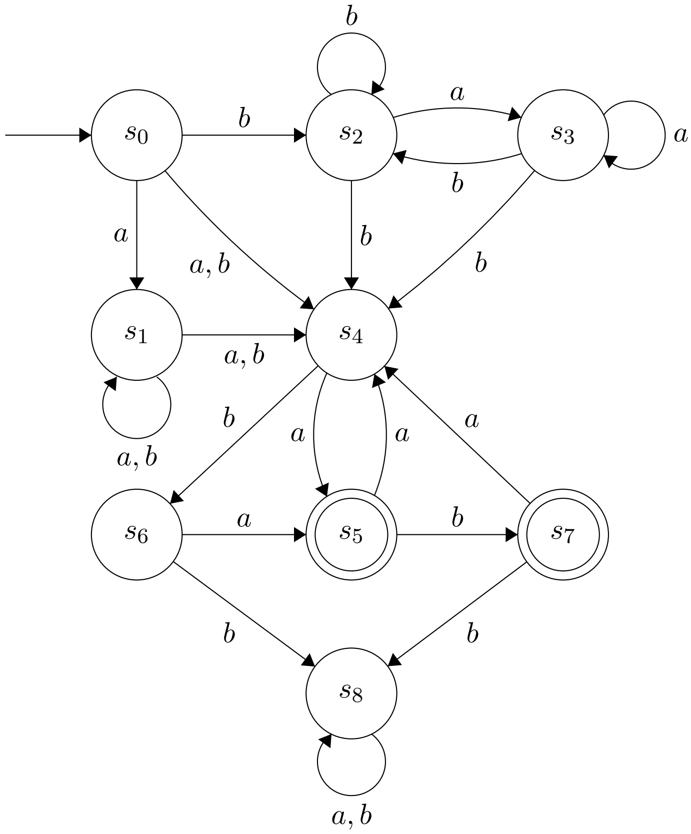
b) $(\mathbf{a}(\mathbf{a} \cup \mathbf{b})^*) \cup ((\mathbf{a} \cup \mathbf{b})^* \mathbf{b})$

c) $S \to aR$
$S \to Rb$
$R \to aR$
$R \to bR$
$R \to \lambda$

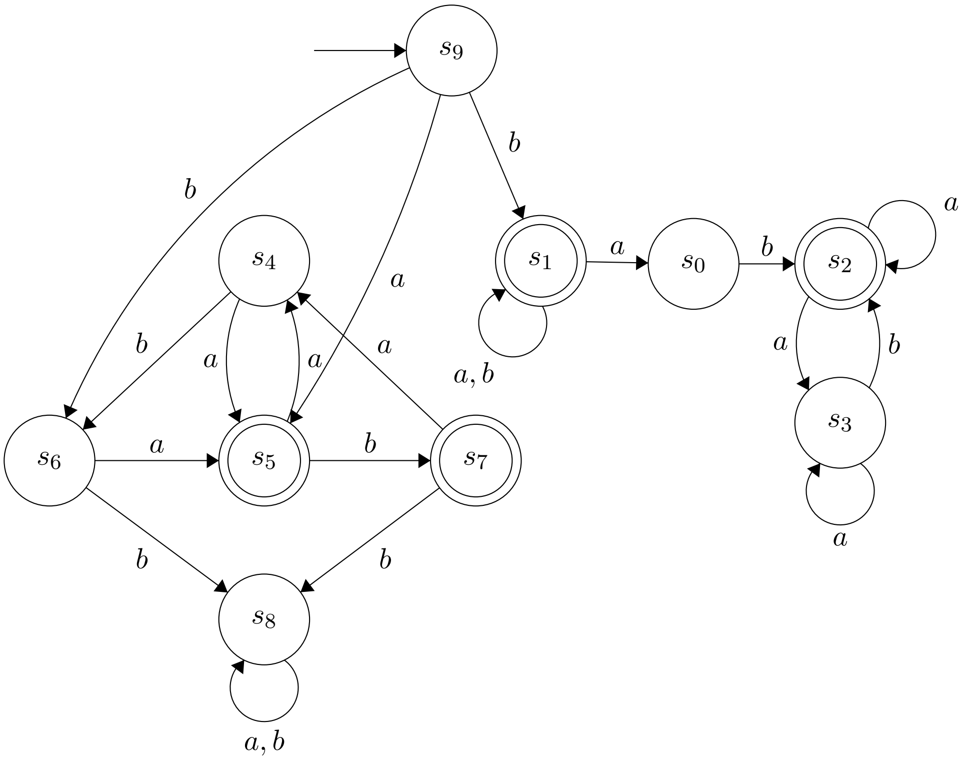
d)



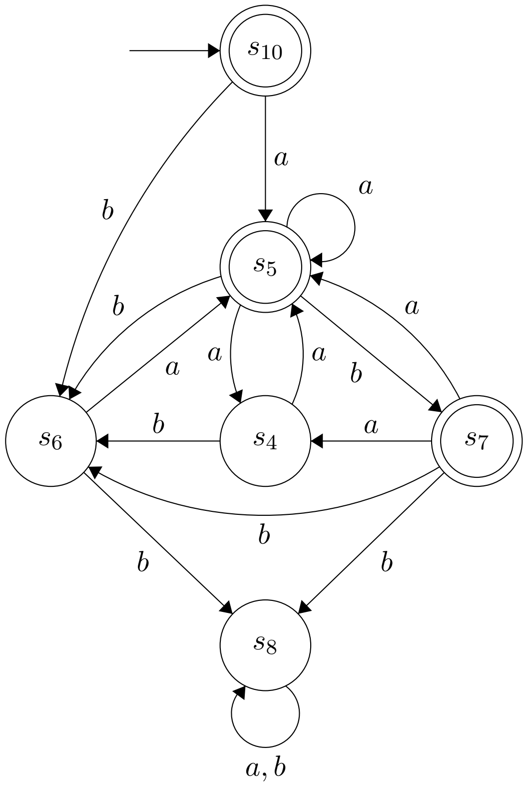
e)



f)



g)



1 Systems of Linear Equations

1.2 Gaussian Elimination

1.2.4

Find all solutions (if any) to each of the following systems of linear equations.

b)

$$\begin{aligned} 3x - y &= 0 \\ 2x - 3y &= 1 \end{aligned}$$

d)

$$\begin{aligned} 3x - y &= 2 \\ 2y - 6x &= -4 \end{aligned}$$

f)

$$\begin{aligned} 2x - 3y &= 5 \\ 3y - 2x &= 2 \end{aligned}$$

Solution

b) Subtract the second equation from the first

$$\begin{aligned} 1x + 2y &= -1 \\ 2x - 3y &= 1 \end{aligned}$$

Subtract the first equation twice from the second

$$\begin{aligned} 1x + 2y &= -1 \\ 0x - 7y &= 3 \end{aligned}$$

Multiply the second equation by a negative seventh

$$\begin{aligned} 1x + 2y &= -1 \\ 0x + 1y &= -3/7 \end{aligned}$$

Subtract the second equation twice from the first

$$\begin{aligned} 1x + 0y &= -1/7 \\ 0x + 1y &= -3/7 \end{aligned}$$

From this system of equations we can readily read that $x = -1/7$ and $y = -3/7$.

d) Take the system of equations

$$3x - y = 2$$
$$2y - 6x = -4$$

Translate them to augmented matrix

$$\left[\begin{array}{cc|c} 3 & -1 & 2 \\ -6 & 2 & -4 \end{array}\right]$$

Add the first row to the second twice

$$\left[\begin{array}{cc|c} 3 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

We see that $3x - y = 2$. Then $y = s$ and $x = (2 + s)/3$ is a solution for all $s \in \mathbb{R}$.

f) Take the system of equations

$$2x - 3y = 5$$
$$3y - 2x = 2$$

Translate them to augmented matrix

$$\left[\begin{array}{cc|c} 2 & -3 & 5 \\ -2 & 3 & 2 \end{array}\right]$$

Add the first row to the second

$$\left[\begin{array}{cc|c} 2 & -3 & 5 \\ 0 & 0 & 7 \end{array}\right]$$

Multiply the second equation by a seventh

$$\left[\begin{array}{cc|c} 2 & -3 & 5 \\ 0 & 0 & 1 \end{array}\right]$$

Since we have encountered a row of the form $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right]$, the system of equations is inconsistent and therefore there is no solution.

1.2.5

b)

$$\begin{aligned} -2x + 3y + 3z &= -9 \\ 3x - 4y + z &= 5 \\ -5x + 7y + 2z &= -14 \end{aligned}$$

d)

$$\begin{aligned} x + 2y - z &= 2 \\ 2x + 5y - 3z &= 1 \\ x + 4y - 3z &= 3 \end{aligned}$$

f)

$$\begin{aligned} 3x - 2y + z &= -2 \\ x - y + 3z &= 5 \\ -x + y + z &= -1 \end{aligned}$$

Solution

b) Take the system of equations

$$\begin{aligned} -2x + 3y + 3z &= -9 \\ 3x - 4y + z &= 5 \\ -5x + 7y + 2z &= -14 \end{aligned}$$

Translate them to augmented matrix

$$\left[\begin{array}{ccc|c} -2 & 3 & 3 & -9 \\ 3 & -4 & 1 & 5 \\ -5 & 7 & 2 & -14 \end{array}\right]$$

Add the first row to the second

$$\left[\begin{array}{ccc|c} -2 & 3 & 3 & -9 \\ 1 & -1 & 4 & -4 \\ -5 & 7 & 2 & -14 \end{array}\right]$$

Swap the first and second row

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & -4 \\ -2 & 3 & 3 & -9 \\ -5 & 7 & 2 & -14 \end{array}\right]$$

Add the first row to the second twice

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & -4 \\ 0 & 1 & 11 & -17 \\ -5 & 7 & 2 & -14 \end{array}\right]$$

Add the first row to the third five times

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & -4 \\ 0 & 1 & 11 & -17 \\ 0 & 2 & 22 & -34 \end{array}\right]$$

Subtract the second row from the third twice

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & -4 \\ 0 & 1 & 11 & -17 \\ 0 & 0 & 00 & 0 \end{array}\right]$$

Add the second row to the first

$$\left[\begin{array}{ccc|c} 1 & 0 & 15 & -21 \\ 0 & 1 & 11 & -17 \\ 0 & 0 & 00 & 0 \end{array}\right]$$

Let $z = t$, then it along with $x = -15t - 21$ and $y = -11t - 17$ form infinite solutions to the system of equations.

d) Take the system of equations

$$\begin{aligned} x + 2y - z &= 2 \\ 2x + 5y - 3z &= 1 \\ x + 4y - 3z &= 3 \end{aligned}$$

Translate them to augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 5 & -3 & 1 \\ 1 & 4 & -3 & 3 \end{array}\right]$$

Subtract the first row from the second row twice

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 4 & -3 & 3 \end{array}\right]$$

Subtract the first row from the third row

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

Subtract the second row from the third row twice

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

Multiply the third row by a fifth

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Since we have encountered a row of the form $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right]$, the system of equations is inconsistent and therefore there is no solution.

f) Take the system of equations

$$\begin{aligned} 3x - 2y + z &= -2 \\ x - y + 3z &= 5 \\ -x + y + z &= -1 \end{aligned}$$

Translate them to augmented matrix

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{array}\right]$$

Swap the first and second row

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 5 \\ 3 & -2 & 1 & -2 \\ -1 & 1 & 1 & -1 \end{array}\right]$$

Subtract the first row from the second thrice

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 5 \\ 0 & 1 & -8 & -17 \\ -1 & 1 & 1 & -1 \end{array}\right]$$

Add the first row to the third

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 5 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

Multiply the third row by a fourth

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 5 \\ 0 & 1 & -8 & -17 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

From this we have that $z = 1$. We also see that

$$\begin{aligned} && y - 8z &= -17 \\ \Rightarrow && y &= 8z - 17 \\ \Rightarrow && y &= 8 - 17 \\ \Rightarrow && y &= -9 \end{aligned}$$

and

$$\begin{aligned} && x - y + 3z &= 5 \\ \Rightarrow && x &= y - 3z + 5 \\ \Rightarrow && x &= -9 - 3 + 5 \\ \Rightarrow && x &= -7 \end{aligned}$$

so $x = -7$, $y = -9$, and $z = 1$ form a unique solution to the system of equations.