Exercise Class Solutions 11

7 Discrete Probability

7.2 Probability Theory

7.2.1

What probability should be assigned to the outcome of heads when a biased coin is tossed, if heads is three times as likely to come up as tails? What probability should be assigned to the outcome of tails?

Solution

We have $p(H) = 3p(T)$. Because $p(H) + p(T) = 1$, it follows that $3p(T) + p(T) = 4p(T) = 1$. We conclude that $p(T) = 1/4$ and $p(H) = 3/4$.

7.2.5

A pair of dice is loaded. The probability that a 4 appears on the first die is 2/7, and the probability that a 3 appears on the second die is 2/7. Other outcomes for each die appear with probability 1/7. What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled?

Solution

As with a pair of fair dice, the sample space has 36 outcomes. Those six having a sum of 7 each have $(1/7)(1/7)$ probability bar the outcome where 4 appears on the first die and a 3 appears on the second die; it has probability of $(2/7)(2/7)$. We conclude that the probability of 7 appearing as the sum of the numbers on these loaded dice is the sum of those probabilities: $9/49$.

7.2.23

What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

Solution

These flips are independent Bernoulli trials. By Theorem 2, the probability of three successes in the remaining four trials/flips (and thus the probability of four heads appearing) is $C(4,3)(1/2)^3(1/2)^1 = 4(1/2)^4 = 1/4$.

7.2.25

What is the conditional probability that a randomly generated bit string of length four contains at least two consecutive 0s, given that the first bit is a 1? (Assume the probabilities of a 0 and a 1 are the same.)

Solution

Let $p(E)$ and $p(F)$ be the respective probabilities. We have

$$E = \{0000, 0001, 0010, 0011, 0100, 1000, 1001, 1100\}$$

$$F = \{1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$$

$$E \cup F = \{1000, 1001, 1100\}$$

so

$$p(E|F) = p(E \cup F)/p(F) = 3/8.$$

7.2.28

Assume that the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has

a) exactly three boys?

b) at least one boy?

c) at least one girl?

d) all children of the same sex?

Solution

For this problem, we model the bearing of children as Bernoulli trials where success—as controversial as it may seem—is having a boy.

a) $C(5,3)(0.51)^3(0.49)^2$

b) $1 - C(5,0)(0.51)^0(0.49)^5 = 1 - 0.49^5$

c) $1 - C(5,5)(0.51)^5(0.49)^0 = 1 - 0.51^5$

d) $C(5,0)(0.51)^0(0.49)^5 + C(5,5)(0.51)^5(0.49)^0 = 0.49^5 + 0.51^5$

7.2.29

A group of six people play the game of "odd person out" to determine who will buy refreshments. Each person flips a fair coin. If there is a person whose outcome is not the same as that of any other member of the group, this person has to buy the refreshments. What is the probability that there is an odd person out after the coins are flipped once?

Solution

This is a Bernoulli trial where we are concerned with the probability of exactly one or five successes:

$$C(6,1)(1/2)^1(1/2)^5 + C(6,5)(1/2)^5(1/2)^1 = 6/2^6 + 6/2^6 = 3/2^4$$

7.2.35

Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p.

a) the probability of no failures

b) the probability of at least one failure

c) the probability of at most one failure

d) the probability of at least two failures

Solution

a) $C(n, n)p^n(1-p)^{n-n} = p^n$

b) $1 - p^n$

c) $p^n + C(n, n-1)p^{n-1}(1-p)^{n-(n-1)} = p^n + np^{n-1}(1-p)$

d) $1 - (p^n + np^{n-1}(1-p))$

7.3 Bayes' Theorem

7.3.1

Suppose that E and F are events in a sample space and $p(E) = 1/3$, $p(F) = 1/2$, and $p(E|F) = 2/5$. Find $p(F|E)$.

Solution

Use Bayes' Theorem to get

$$p(F|E) = \frac{p(E|F) \cdot p(F)}{p(E)} = \frac{(2/5)(1/2)}{1/3} = \frac{3}{5}$$

7.3.3

Suppose that Frida selects a ball by first picking one of two boxes at random and then selecting a ball from this box at random. The first box contains two white balls and three blue balls, and the second box contains four white balls and one blue ball. What is the probability that Frida picked a ball from the first box if she has selected a blue ball?

Solution

Let A be the event that Frida selected a blue ball and let B be the event that she picked a ball from the first box. From the problem statement we have $p(A) = \frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{1}{5} = 2/5$, $p(B) = 1/2$ and $p(A|B) = 3/5$. Now we can apply Bayes' Theorem to get $p(B|A)$.

$$p(B|A) = \frac{(3/5)(1/2)}{2/5} = 3/4$$

7.3.5

Suppose that 8% of all bicycle racers use steroids, that a bicyclist who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of the time. What is the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids?

Solution

Let A be the event that a randomly selected bicyclist uses steroids and $+$ be the event that a randomly selected bicyclist tests positive for steroids. From the problem description we have $p(A) = 0.08$, $p(+) = 0.08 \cdot 0.96 + (1 - 0.08) \cdot 0.09 = 0.1596$ and $p(+|A) = 0.96$. Applying Bayes' Theorem yields

$$p(A|+) = \frac{0.96 \cdot 0.08}{0.1596} = 0.481.$$

7.3.9

Suppose that 8% of the patients tested in a clinic are infected with HIV. Furthermore, suppose that when a blood test for HIV is given, 98% of the patients infected with HIV test positive and that 3% of the patients not infected with HIV test positive. What is the probability that

a) a patient testing positive for HIV with this test is infected with it?

b) a patient testing positive for HIV with this test is not infected with it?

c) a patient testing negative for HIV with this test is infected with it?

d) a patient testing negative for HIV with this test is not infected with it?

Solution

Let A be the event that a patient is infected with HIV. Let $+$ be the event that a patient tests positive for HIV. From the problem description we have $p(A) = 0.08$, $p(\bar{A}) = 1 - 0.08 = 0.92$, $p(+) = 0.08 \cdot 0.98 + 0.92 \cdot 0.03 = 0.106$, $p(\bar{+}) = 1 - 0.106 = 0.894$, $p(+|A) = 0.98$, $p(+|\bar{A}) = 0.03$, $p(\bar{+}|A) = 1 - 0.98 = 0.02$, and $p(\bar{+}|\bar{A}) = 1 - 0.03 = 0.97$.

a) $p(A|+) = \frac{0.98 \cdot 0.08}{0.106} = 0.74$

b) $p(\bar{A}|+) = \frac{0.03 \cdot 0.92}{0.106} = 0.26$

c) $p(A|\bar{+}) = \frac{0.02 \cdot 0.08}{0.894} = 0.002$

d) $p(\bar{A}|\bar{+}) = \frac{0.97 \cdot 0.92}{0.894} = 0.998$

7.4 Expected Value and Variance

7.4.3

What is the expected number of times a 6 appears when a fair die is rolled 10 times?

Solution

Since rolling a die and getting a 6 can be observed as a Bernoulli trial with $p = 1/6$ we can use Theorem 2 to find that $E(X) = 10 \cdot 1/6 = 5/3$.

7.4.7

The final exam of a discrete mathematics course consists of 50 true/false questions, each worth two points, and 25 multiple-choice questions, each worth four points. The probability that Linda answers a true/false question correctly is 0.9, and the probability that she answers a multiple-choice question correctly is 0.8. What is her expected score on the final?

Solution

$$2 \cdot 50 \cdot 0.9 + 4 \cdot 25 \cdot 0.8 = 170$$