Exercise Class Solutions 10

6 Counting

6.4 Binomial Coefficients and Identities

6.4.3

Find the expansion of $(x+y)^6$.

Solution

Look at the row

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

of Pascal's triangle. It tells us that the expansion is

$$x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.$$

6.4.5

How many terms are there in the expansion of $(x+y)^{100}$ after like terms are collected?

Solution

There are 101 terms.

6.4.9

What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$?

Solution

$$\binom{200}{101} \cdot 2^{101} \cdot (-3)^{99} = -2^{101} \cdot 3^{99} \cdot \binom{200}{99}$$

6.4.13

What is the row of Pascal's triangle containing the binomial coefficients $\binom{9}{k}$, $0 \leq k \leq 9$?

Solution

$$\binom{9}{0} \quad \binom{9}{1} \quad \binom{9}{2} \quad \binom{9}{3} \quad \binom{9}{4} \quad \binom{9}{5} \quad \binom{9}{6} \quad \binom{9}{7} \quad \binom{9}{8} \quad \binom{9}{9}$$

or

$$1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1$$

6.4.15

Show that $\binom{n}{k} \leq 2^n$ for all positive integers n and all integers k with $0 \leq k \leq n$.

Solution

$$\binom{n}{k} \leq \sum_{i=0}^{n} \binom{n}{i} = 2^n$$

by Corollary 1.

6.5 Generalized Permutations and Combinations

6.5.1

In how many different ways can five elements be selected in order from a set with three elements when repetition is allowed?

Solution

$3^5 = 243$

6.5.3

How many strings of six letters are there?

Solution

$26^6 = 308915776$

6.5.5

How many ways are there to assign three jobs to five employees if each employee can be given more than one job?

Solution

5^3, or 125. For each job we have a choice between the five employees.

6.5.7

How many ways are there to select three unordered elements from a set with five elements when repetition is allowed?

Solution

We know by Theorem 2 that the answer is

$$C(n + r - 1, r) = C(5 + 3 - 1, 3) = C(7, 3) = 35$$

6.5.9

A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose

a) six bagels?

b) a dozen bagels?

c) two dozen bagels?

d) a dozen bagels with at least one of each kind?

e) a dozen bagels with at least three egg bagels and no more than two salty bagels?

Solution

a) Using Theorem 2 we have that this equals to $C(8+6-1,6) = 1716$

b) Using Theorem 2 again we have that this equals to

$$C(8+12-1,12) = 50388$$

c) Again we use Theorem 2 to get $C(8+24-1,24) = 2629575$

d) Since we have to have at least one of each kind we can just start by assuming we have already picked one of each kind, leaving us with 4 bagels that we can choose any way we like. We can then use Theorem 2 again to get $C(8+4-1,4) = 330$

e) Start by assuming we already picked three egg bagels. That leaves us with 9 bagels that we can choose but we can pick at most two salty bagels so we will sum over the possible amount of salty bagels we pick and use Theorem 2 again

$$C(7+9-1,9) + C(7+8-1,8) + C(7+7-1,7) = 9724$$

6.5.11

How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?

Solution

$$C(n+r-1,r) = C(2+8-1,8) = C(9,8) = 9$$

6.5.13

A book publisher has 3000 copies of a discrete mathematics book. How many ways are there to store these books in their three warehouses if the copies of the book are indistinguishable?

Solution

By Example 9 we have that placing r indistinguishable objects into n distinguishable boxes is

$$C(n + r - 1, r) = C(n + r - 1, n - 1) = C(3002, 2) = 4504501$$

6.5.21

How many ways are there to distribute six indistinguishable balls into nine distinguishable bins?

Solution

$C(14, 6) = 3003$

6.5.23

How many ways are there to distribute 12 distinguishable objects into six distinguishable boxes so that two objects are placed in each box?

Solution

By Theorem 4 we have that the number of ways is

$$\frac{n!}{n_1!n_2!\dots n_k!} = \frac{12!}{2!^6} = 7484400$$

6.5.31

How many different strings can be made from the letters in $ABRACADABRA$, using all the letters?

Solution

Need to use the product rule, count the number of occurences of each letter.
A: 5, B: 2, C: 1, D: 1, R: 2

$$C(11,5) \cdot C(6,2) \cdot C(4,1) \cdot C(3,1) \cdot C(2,2) = 83160$$

6.5.41

How many ways are there to deal hands of seven cards to each of five players from a standard deck of 52 cards?

Solution

By Example 8 we have that this equals to

$$\frac{52!}{7!^5 17!}$$