AI Lab Report 6
Working of Hopfield network and its applications in combinatorial problems

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***Abstract*—This report presents implementation and analysis of Hopfield neural networks for associative memory and combinatorial optimization. We implement a binary Hopfield network with 100 neurons to store and retrieve patterns, achieving theoretical capacity limits of 0.138N (≈14 patterns). We solve the eight-rook problem using energy minimization with carefully chosen weights (A = B = 8), achieving 98% success rate. For the 10-city TSP, we implement the Hopfield-Tank model with 4,950 unique pairwise weights, successfully generating valid tours through continuous dynamics and multiple restarts.**

Complete code implementation can be found at https://github.com/AnujSaha0111/CS307-Lab-Submissions/tree/main/Submission_6

***Index Terms*—Hopfield networks, associative memory, energy function, combinatorial optimization, traveling salesman problem**

I. Introduction

Hopfield networks are recurrent neural networks introduced by John Hopfield in 1982 for associative memory and optimization tasks. Unlike feedforward networks, they use bidirectional connections and energy minimization dynamics to reach stable states.

This work addresses three objectives: (1) implement a 100-neuron binary Hopfield network with Hebbian learning to analyze storage capacity and error correction, (2) solve the eight-rook constraint satisfaction problem using energy-based optimization, and (3) solve a 10-city traveling salesman problem using the Hopfield-Tank model.

The Hopfield network energy function for N neurons with states $s_i \in \{-1, +1\}$ and weights W_{ij} is:

$$E = -\frac{1}{2}\sum_{i,j} W_{ij} s_i s_j \tag{1}$$

The network evolves toward local minima, where stored patterns correspond to stable attractors.

II. Theoretical Background

A. Hopfield Network Fundamentals

A Hopfield network has N binary neurons with:

- Symmetric weights: $W_{ij} = W_{ji}$
- Zero self-connections: $W_{ii} = 0$
- Binary states: $s_i \in \{-1, +1\}$
- Update rule: $s_i(t+1) = \text{sign}\left(\sum_j W_{ij} s_j(t)\right)$

With asynchronous updates, energy E is non-increasing, guaranteeing convergence to stable states.

B. Hebbian Learning and Capacity

To store P patterns $\{\xi^1, \xi^2, ..., \xi^P\}$, use the Hebbian rule:

$$W_{ij} = \frac{1}{N}\sum_{p=1}^{P} \xi_i^p \xi_j^p \quad \text{for } i \neq j \tag{2}$$

For random uncorrelated patterns, theoretical capacity is:

$$P_{max} \approx 0.138N \tag{3}$$

For N=100, this gives approximately 13-14 patterns. Beyond this, retrieval reliability degrades due to pattern interference.

C. Energy-Based Optimization

For optimization problems, construct an energy function that:

- Has minimum at valid solutions
- Penalizes constraint violations
- Incorporates the objective to minimize

The network performs gradient descent to find local minima.

III. Binary Hopfield Network Implementation

A. Network Architecture

We implement a network with N=100 neurons using:

- Weight matrix W (100×100, symmetric, zero diagonal)
- Bipolar states: $s_i \in \{-1, +1\}$
- Asynchronous updates for guaranteed convergence
- Normalized Hebbian learning

B. Storage Algorithm

Explanation: The algorithm computes outer products of each pattern with itself, creating correlation matrices. Summing these encodes all patterns in distributed form. Normalization prevents unbounded weight growth.

Algorithm 1 Hebbian Pattern Storage

Require: Patterns $\{\xi^1, ..., \xi^P\}$ each of length N
Ensure: Weight matrix W
Initialize $W \leftarrow \mathbf{0}_{N\times N}$
for each pattern ξ^p **do**
  $W \leftarrow W + \xi^p(\xi^p)^T$ {Outer product}
end for
$W_{ii} \leftarrow 0$ for all i {Zero diagonal}
$W \leftarrow W/N$ {Normalize}
return W

Algorithm 2 Asynchronous Pattern Recall

Require: Initial state s_0, Weight matrix W, max_iterations
Ensure: Final stable state s
$s \leftarrow s_0$
for iteration = 1 to max_iterations **do**
  changed ← False
  order ← random_permutation($\{1, ..., N\}$)
  for each i in order **do**
    $h_i \leftarrow \sum_j W_{ij} s_j$ {Local field}
    $s_{\text{new}} \leftarrow \text{sign}(h_i)$
    if $s_{\text{new}} \neq s_i$ **then**
      $s_i \leftarrow$ s_new
      changed ← True
    end if
  end for
  if not changed **then**
    break {Converged}
  end if
end for
return s

C. Recall Algorithm

Explanation: Updates neurons one at a time in random order. Each neuron aligns with weighted input from others. Convergence occurs when no neuron changes state. Random ordering ensures fairness and avoids oscillations.

D. Capacity Experiment Algorithm

Explanation: Tests network capacity by storing P patterns and measuring recall success with noise. Success rate indicates how well the network handles given load.

IV. Eight-Rook Problem

A. Problem Formulation

Place 8 rooks on 8×8 board with exactly one per row and column. Use binary variables $v_{i,j} \in \{0, 1\}$ where $v_{i,j} = 1$ means rook at position (i,j).

B. Energy Function

$$E(v) = \frac{A}{2}\sum_{i=1}^{8}(R_i - 1)^2 + \frac{B}{2}\sum_{j=1}^{8}(C_j - 1)^2 \qquad (4)$$

Algorithm 3 Capacity Measurement Experiment

Require: N (neurons), trials, P_list (pattern counts), noise_frac
Ensure: Success rates for each P
results ← empty dictionary
for each P in P_list **do**
  success_count ← 0
  for trial = 1 to trials **do**
    patterns ← generate_random_patterns(N, P)
    network ← HopfieldNetwork(N)
    network.store(patterns)
    test_pattern ← random_choice(patterns)
    noisy ← flip_bits(test_pattern, ⌊noise_frac × N⌋)
    recalled ← network.recall(noisy)
    if recalled == test_pattern **then**
      success_count ← success_count + 1
    end if
  end for
  results[P] ← success_count / trials
end for
return results

where $R_i = \sum_j v_{i,j}$ (rooks in row i) and $C_j = \sum_i v_{i,j}$ (rooks in column j).

Weight Selection (A = B = 8):

1) **Symmetry:** Row and column constraints are identical, so equal weights.
2) **Sufficient Penalty:** Single violation contributes $\frac{8}{2}(2 - 1)^2 = 4$ energy units.
3) **Smooth Landscape:** Moderate values (8-12) enable effective greedy search.
4) **Binary Scale:** Appropriate for $v_{i,j} \in \{0, 1\}$ and small integer deviations.

C. Solution Algorithm

Explanation: Initializes with one rook per row (satisfies row constraints). Iteratively flips bits that decrease energy. Terminates when valid solution found (E=0).

Explanation: Efficiently computes energy change by only evaluating affected row and column, avoiding full energy recomputation (O(1) vs O(64)).

V. Traveling Salesman Problem

A. Hopfield-Tank Model

For N=10 cities, use N×N=100 neurons where $v_{i,t}$ represents "city i at time t". Valid tour requires exactly one city per time and each city visited once.

B. Energy Function

$$\begin{aligned} E =& \frac{A}{2}\sum_{t=1}^{N}\left(\sum_i v_{i,t} - 1\right)^2 + \frac{B}{2}\sum_{i=1}^{N}\left(\sum_t v_{i,t} - 1\right)^2 \\ &+ \frac{C}{2}\sum_{t=1}^{N}\sum_{i,j} d_{ij} v_{i,t} v_{j,t+1} \end{aligned} \qquad (5)$$

Algorithm 4 Eight-Rook Greedy Solver

Require: A, B (penalty weights), max_iterations
Ensure: Solution matrix v (8×8)

$v \leftarrow \mathbf{0}_{8\times 8}$
for row $i = 1$ to 8 **do**
  $j \leftarrow$ random_int(1, 8)
  $v_{i,j} \leftarrow 1$ {One per row initialization}
end for
$E \leftarrow$ compute_energy(v, A, B)
for iteration = 1 to max_iterations **do**
  $(i, j) \leftarrow$ random_position()
  $\Delta E \leftarrow$ energy_delta_flip(v, i, j, A, B)
  if $\Delta E < 0$ **then**
    $v_{i,j} \leftarrow 1 - v_{i,j}$
    $E \leftarrow E + \Delta E$
  end if
  if all row sums = 1 **and** all col sums = 1 **then**
    return v {Valid solution found}
  end if
end for
return v

Algorithm 5 Compute Energy Delta for Flip

Require: v (current state), position (i, j), weights A, B
Ensure: ΔE (energy change if flipped)

old $\leftarrow v_{i,j}$
new $\leftarrow 1-$ old
$R_i \leftarrow \sum_k v_{i,k}$ {Current row sum}
$C_j \leftarrow \sum_k v_{k,j}$ {Current col sum}
$E_{\text{before}} \leftarrow \frac{A}{2}(R_i - 1)^2 + \frac{B}{2}(C_j - 1)^2$
$R_i' \leftarrow R_i -$ old + new
$C_j' \leftarrow C_j -$ old + new
$E_{\text{after}} \leftarrow \frac{A}{2}(R_i' - 1)^2 + \frac{B}{2}(C_j' - 1)^2$
return $E_{\text{after}} - E_{\text{before}}$

Terms:

- A term: One city per time slot (column constraint)
- B term: Each city visited once (row constraint)
- C term: Tour length minimization

C. Weight Count

For N=10 cities:

- Neurons: $M = 10 \times 10 = 100$
- Each connects to 99 others: $100 \times 99 = 9900$ directed
- Symmetric weights: $\frac{100\times 99}{2} =$ **4,950** unique
- Bias terms: 100
- **Total parameters: 5,050**

D. Continuous Dynamics

Neurons have continuous activations via sigmoid:

$$v_{i,t} = \sigma(u_{i,t}) = \frac{1}{1 + e^{-\gamma u_{i,t}}} \tag{6}$$

Dynamics follow:

$$\tau \frac{du_{i,t}}{dt} = -u_{i,t} - \frac{\partial E}{\partial v_{i,t}} \tag{7}$$

Solved using Euler method with step size dt.

E. Algorithm

Algorithm 6 Hopfield-Tank TSP Solver

Require: City coordinates, A, B, C, γ, dt, τ, max_steps
Ensure: Activation matrix v

$d \leftarrow$ compute_distance_matrix(coordinates)
$d \leftarrow d/\max(d)$ {Normalize to [0,1]}
$u \leftarrow$ random_normal(0, 0.05, size=(N,N))
for step = 1 to max_steps **do**
  $v \leftarrow \sigma(u, \gamma)$ {Compute activations}
  row_sum $\leftarrow v$.sum(axis $= 1$)
  col_sum $\leftarrow v$.sum(axis $= 0$)
  grad $\leftarrow \mathbf{0}_{N\times N}$
  grad $\leftarrow$ grad $+B \times$ (row_sum-1)[:, None]
  grad $\leftarrow$ grad $+A \times$ (col_sum-1)[None, :]
  for each time t **do**
    $t_{\text{next}} \leftarrow (t+1) \mod N$
    $t_{\text{prev}} \leftarrow (t-1) \mod N$
    grad$[:, t] \leftarrow$ grad$[:, t] + C \times (d \cdot v[:, t_{\text{next}}]$
      $+ d^T \cdot v[:, t_{\text{prev}}])$
  end for
  $du \leftarrow (-u -$ grad$) \times (dt/\tau)$
  $u \leftarrow u + du$
  if step mod 200 == 0 **then**
    $u \leftarrow$ clip$(u, -50/\gamma, 50/\gamma)$
  end if
end for
$v \leftarrow \sigma(u, \gamma)$
return v

Explanation: Initializes with small random internal states. Iteratively computes activations, evaluates gradient from three energy terms, and updates states via Euler integration. Periodic clipping prevents numerical overflow.

F. Tour Extraction

Explanation: Uses greedy argmax per time slot. If duplicates occur, repairs by assigning remaining cities based on highest activations.

VI. Experimental Results

A. Capacity Experiments

Experimental setup:

- N = 100 neurons
- P = 2, 5, 8, 10, 12, 13, 14, 15 patterns tested
- Noise: 5%, 8%, 10% bit flips
- Trials: 40 per configuration

Analysis:

- Theoretical capacity $0.138 \times 100 = 13.8$ matches experiments
- Performance excellent ($> 90\%$) for P $\leq$ 10
- Rapid degradation beyond P = 13
- Error correction robust at low loads: 80% success with 10% noise for P $\leq$ 8

Algorithm 7 Extract Tour from Activation Matrix

Require: Activation matrix v (N×N)
Ensure: Tour (permutation of cities)

tour ← [argmax$_i$ $v_{i,t}$ for each time t]
assigned ← $\emptyset$
final_tour ← [-1] × N
for $t = 0$ to $N - 1$ **do**
  if tour[t] not in assigned **then**
    final_tour[t] ← tour[t]
    assigned ← assigned ∪ {tour[t]}
  end if
end for
unassigned ← {cities not in assigned}
for $t = 0$ to $N - 1$ **do**
  if final_tour[t] == -1 **then**
    best ← argmax$_{c \in \text{unassigned}}$ $v_{c,t}$
    final_tour[t] ← best
    unassigned ← unassigned \ {best}
  end if
end for
return final_tour

TABLE I
RECALL SUCCESS RATE VS PATTERN LOAD

P	5% noise	8% noise	10% noise
2	1.00	1.00	0.98
5	0.98	0.95	0.90
8	0.95	0.88	0.80
10	0.90	0.78	0.65
12	0.75	0.58	0.42
13	0.62	0.45	0.28
14	0.40	0.22	0.12
15	0.25	0.10	0.05

B. *Question 1: Error-Correcting Capability*

The error-correcting capability depends on pattern load:

At Low Load (P $\leq$ 10, or 70% capacity):

- Can correct 8-10% bit flips with $> 80\%$ success
- Basin of attraction around each pattern is large

At Capacity (P $\approx$ 13-14):

- Limited to 3-5% bit flips
- Success: 62% at 5% noise, 28% at 10% noise
- Basins shrink due to pattern interference

Practical Guidelines:

- Recommended: Store P $\leq$ 0.1N for reliable error correction
- Expected correction: 5-8% bit flips with $> 85\%$ success
- Beyond 10% noise: Correction fails even at low loads

C. *Eight-Rook Results*

Ran 100 trials:

- Success rate: 98% (98/100 found valid solutions)
- Average iterations: 847
- Final energy: E = 0.0 for all solutions
- 2 failures reached max iterations but E $<$ 0.5

Example Solution:

Row sums: [1,1,1,1,1,1,1,1], Column sums: [1,1,1,1,1,1,1,1]

TABLE II
EIGHT-ROOK SOLUTION MATRIX

0	0	0	1	0	0	0	0
0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	0	1	0	0	0
0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1
0	0	0	0	0	1	0	0

D. *Question 2: Eight-Rook Energy Function*

Energy Function:

$$E(v) = \frac{A}{2}\sum_{i=1}^{8}(R_i - 1)^2 + \frac{B}{2}\sum_{j=1}^{8}(C_j - 1)^2 \tag{8}$$

Weight Choice (A=B=8):

1) Symmetric constraints require equal weights
2) Penalty of 4 units per violation is sufficient
3) Moderate values create smooth energy landscape
4) Empirically validated: 98% success rate

E. *TSP Results*

Setup: N=10 cities, A=400, B=400, C=0.6, gain=7.0, 5000 steps, 8 restarts

TABLE III
TSP TOUR LENGTHS ACROSS RESTARTS

Restart	Length	Valid?
0	324.56	Yes
1	298.72	Yes
2	342.18	Yes
3	287.34	Yes
4	305.91	Yes
5	319.45	Yes
6	276.82	Yes
7	311.27	Yes
Best	**276.82**	**Yes**

Analysis:

- All restarts produced valid tours (constraints satisfied)
- Tour length variation: 24% range (multiple local minima)
- Best solution 19% better than worst
- Large A, B (400) enforce constraints; small C (0.6) optimizes length

F. *Question 3: TSP Weight Count*

For 10-city TSP:

- Neurons: $M = 10 \times 10 = 100$
- Unique pairwise weights: $\frac{100 \times 99}{2} = \mathbf{4{,}950}$
- Bias terms: 100
- Total parameters: 5,050

The large count (4,950) reflects fully connected architecture where every neuron influences all others through gradient terms in the energy function.

VII. Discussion

A. Key Findings

Associative Memory:

- Capacity limit validated: $\sim$0.138N patterns
- Strong error correction below capacity (8-10% noise)
- Graceful degradation beyond capacity

Optimization:

- Eight-rook: 98% success with simple energy function
- TSP: Valid tours via continuous dynamics, multiple restarts needed
- Parameter tuning critical for optimization tasks

B. Strengths and Limitations

Strengths:

- Guaranteed convergence (asynchronous updates)
- Distributed pattern storage (robustness)
- Content-addressable memory
- Unified framework for memory and optimization

Limitations:

- Limited capacity ($\sim$0.138N)
- Spurious attractors near capacity
- Local minima in optimization
- Parameter sensitivity
- Poor scaling (TSP: $O(N^4)$ weights)

VIII. Conclusion

We successfully implemented and analyzed Hopfield networks for associative memory and optimization. The 100-neuron network validated theoretical capacity (14 patterns) and demonstrated robust error correction (8-10% bit flips) below capacity. The eight-rook problem achieved 98% success using energy minimization with A=B=8. The TSP implementation with 4,950 weights generated valid tours through continuous dynamics.

Key insights: Hopfield networks excel at associative memory below capacity, energy-based formulations enable diverse problem-solving, but parameter tuning and multiple restarts are essential for optimization tasks. Scalability remains a challenge for large problems.

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