CS201A:Mathematics for Computer Science - I
Assignment 1

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Question 1: *[5 points]* Prove that contrapositivity works. i.e. show the equivalence of the two statements: $p \implies q$ and $\neg q \implies \neg p$.

One way we can prove the above result is using truth-tables

p	q	$\neg p$	$\neg q$	$p \implies q$	$\neg q \implies \neg p$	$(p \implies q) \iff (\neg q \implies \neg p)$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	1
1	1	0	0	1	1	1

Therefore, the given expression: $p \implies q$ and $\neg q \implies \neg p$ is a **tautology**.
Alternately, We know from the rules of inference that,

$$(p \implies q) \iff \neg p \vee q$$

Also, we know that the logical or operator is commutative, So,

$$(p \implies q) \iff q \vee \neg p$$

Writing q as $\neg(\neg q)$ gives,

$$(p \implies q) \iff \neg(\neg q) \vee (\neg p)$$

which is, again from the same rule of inference,

$$(p \implies q) \iff (\neg q \implies \neg p)$$

Hence proved, every proposition is equivalent to its contrapositive.

Question 2: ***[5+4 points]*** **Twenty five boys and twenty five girls sit around a circular table. Prove that it is always possible to find a person both of whose neighbors are girls. What happens when there are 24 boys and girls each?**

Part 1: 25 boys and 25 girls
Say $\mathbb{P}$ be the set of all cyclic permutations of 25 boys and girls.
Let us first define a property p for any given cyclic permutation of boys and girls such that:
A cyclic permutation satisfies p iff it does not contain any person both of whose neighbors are girls
Then, the statement we want to prove can be written as

$$\forall \sigma \in \mathbb{P} : \sigma \; does \; not \; satisfy \; p$$

Which is equivalent to saying that:

$$\neg(\exists \; \sigma \in \mathbb{P} : \sigma \; satisfies \; p)$$

We will try to prove the above statement in order to prove the given statement.
As all boys and all girls can be considered identical for our problem, each cyclic permutation can be fully described by just the number of boys between 2 consecutive girls.
So for 25 girls, we will need 25 integers, say $x_1, x_2, \ldots x_{25}$ ($x_i \geq 0 \; \forall i$) to denote the no. of boys between consecutive girls.
Clearly, for any permuation if two consecutive x_i's are zero, then there would exist a girl having girls as both the neighbors. So any such permutation will not satisfy p.
Also, if any of the x_i is 1, then there will be a boy with both neighbors as girls, again failing to satisfy p.
Therefore, any permutation satisfying p must contain **atleast 13** $\mathbf{x_i \; : \; x_i > 1}$
But, that makes the minimum no. of boys to make such a permutation = $13 \times 2 = 26 (> 25)$
Hence, there **does not exist** any cyclic permutation for 25 boys and girls which can satisfy p.

Hence proved, it is always possible to find a person both of whose neighbors are girls.
Part 2: 24 boys and 24 girls
We, can make similar types of arguments as we made in part 1 for this case also except that, in this case,
Any permutation satisfying p must contain **atleast 12** $\mathbf{x_i \ : \ x_i > 1}$
Thus, the minimum no. of boys to make such a permutation = $12 \times 2 = 24$ which is possible
Now, based on above analysis, we can simply give the example of the cyclic permutation:
-GBBGGBBG . . . GBBG-
One can easily check that this permutation satisfies p.
So for this part, we **can not** say that there is always a person both of whose neighbors are girls, as is clear from the counter-example stated above.

Question 3: *[5+4 points]* Show that $|\mathbb{N}| = |\mathbb{Q}| = |\mathbb{Q}^2|$

Cardinality of 2 sets are equal iff there exists a bijection from 1 set to other. We start by making a bijection from $\mathbb{Z}$ *to* $\mathbb{N}$ as:

$$f : \mathbb{Z} \to \mathbb{N} : f(x) = \left\{ \begin{array}{cc} -2x - 1 & , x < 0 \\ 2x & , x \geq 0 \end{array} \right\}$$

Now, we try to make a bijection from $\mathbb{Q}$ *to* $\mathbb{Z}$
We know that every rational number r can be represented in terms of 2 co-prime integers, p and q where $q \neq 0$ either p=0 or p can be expressed in terms of its prime factors as:

$$p = p_1^{\alpha_1} \times p_2^{\alpha_2} \times \ldots \times p_n^{\alpha_n} \ ; \ \alpha_i > 0$$

Similarly, q can be expressed as:

$$q = q_1^{\beta_1} \times q_2^{\beta_2} \times \ldots \times q_m^{\beta_m} \ ; \ \beta_i > 0$$

Now, we define a bijection $g : \mathbb{Q} \to \mathbb{Z}$ as

$$g : \mathbb{Q} \to \mathbb{Z} : g(x) = \left\{ \begin{array}{cc} 0 & , x = 0 \\ p^2 \frac{q^2}{q_1 q_2 \cdots q_m} & , x = p/q, p, q > 0 \\ -p^2 \frac{q^2}{q_1 q_2 \cdots q_m} & , x = -p/q, p, q > 0 \end{array} \right\}$$

We can also prove that g is a bijection by 2 arguments:

1. Every rational number has a unique image in g.
   This is obviously true as p and q are taken to be co-prime and every number has unique prime factors.

2. Every integer has a pre-image in g.
   We know that power of prime factors in p in image (g(p/q)) is even and power of factors of q is odd. So, if we consider any arbitrary integer $z > 0$ (similarly can be proved for $z < 0$ as well)
   Let $z = z_1^{\gamma_1} z_1^{\gamma_1} \dots z_k^{\gamma_k}$, if γ_i is even, then the term $z_i^{(\gamma_i)/2}$ belongs to p, else the term $z_i^{(\gamma_i+1)/2}$ belongs to q, and p/q is a rational pre-image of z.

Now, from bijections f and g, we can easily derive a bijection h such that:

$$h : \mathbb{Q} \to \mathbb{N} : h(x) = f(g(x)) \ \forall x \in \mathbb{Q}$$

That proves, $|\mathbb{N}| = |\mathbb{Q}|$
Now, for the second part, as we have already proven a bijection from $\mathbb{Q}$ *to* $\mathbb{N}$, we can simply define a bijection h from $\mathbb{Q}^2$ *to* $\mathbb{N}^2$ as:

$$j : \mathbb{Q}^2 \to \mathbb{N}^2 : j(q_1, q_2) = (h(q_1), h(q_2)) \ \forall q_1, q_2 \in \mathbb{Q}$$

The only remaining task is to find a bijection l from $\mathbb{N}^2$ *to* $\mathbb{N}$, that we will define as:

$$l : \mathbb{N}^2 \to \mathbb{N} : l(n_1, n_2) = 2^{n_1}(2n_2 + 1) - 1 \ \forall n_1, n_2 \in \mathbb{N}$$

The -1 is done so that 0 has a preimage in l.
We can again check that l is a bijection using similar arguments as we made for g.
So, we have created a bijection from $\mathbb{Q}^2$ *to* $\mathbb{N}$ which is $l(j(q_1, q_2)$
Thus, proving the second part, $|\mathbb{N}| = |\mathbb{Q}^2|$
Hence, $|\mathbb{N}| = |\mathbb{Q}| = |\mathbb{Q}^2|$

Question 4: *[5+4 points]* Prove the classical result: $|\mathbb{N}| < |2^{\mathbb{N}}| = |\mathbb{R}|$

We know $|S| < |2^S| \ \forall S$ **(derived in lecture)**. Therefore,

$$|\mathbb{N}| < |2^{\mathbb{N}}|$$

Now, we need to prove $|2^{\mathbb{N}}| = |\mathbb{R}|$
The set $2^{\mathbb{N}}$ is in bijection with the set of all infinitely long {0,1} (binary) strings.
Where each binary string corresponds to a set in $2^{\mathbb{N}}$, such that the n^{th} digit in the string denotes whether the natural number n is present in the set it corresponds to.
Now, consider a function f from the interval [0,1] to set of all binary strings as: f(x) = binary representation of x.
We know every real from 0 to 1 has a unique binary representation. (We can not claim it directly for R because we can't know where exactly to put the decimal point)
The binary representation of every number is unique, and every binary string has to correspond to some real in [0,1], so f is injective as well as surjective, and hence a bijection.
Now, take consider $g(x) = tan(\pi(x - 1/2))\ \forall x \in (0, 1)$
As tan is one-one, onto in $(-\pi/2\ to\ \pi/2)$, g(x) is a bijection.
So, using the results derived above, it is proved that:

$$|2^{\mathbb{N}}| = |[0, 1]| = |\mathbb{R}|$$

Question 5: *[6 points]* Let $f_n = f_{n-1} + f_{n-2}$ and $f_1 = f_2 = 1.\{f_n\}_n$ are called *Fibonacci numbers.* Show that: $f_1 + f_2 + \ldots + f_n = f_{n+2} - 1.$

The given result can be proved with the help of mathematical induction
Let $P(n)$ be the proposition that $f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$.
First, we verify the result for the base case, here $n_0 = 1$ is the base case.
We know by definition of Fibonacci numbers that

$$f_3 = f_2 + f_1 = 1 + 1 = 2$$

$$f_1 = 1\ and\ f_3 - 1 = 1$$

Hence verified that $p(1)$ is **true**.
Now we can assume that $p(r)$ is true.

$$i.e.\ f_1 + f_2 + \ldots + f_r = f_{r+2} - 1$$

Adding f_{r+1} on both sides

$$f_1 + f_2 + \ldots + f_r + f_{r+1} = f_{r+2} + f_{r+1} - 1$$

$$\implies f_1 + f_2 + \ldots + f_r + f_{r+1} = f_{r+3} - 1$$

$$\implies P(r+1)$$

So, we have derived that $P(r) \implies P(r+1)$
Hence proved that $\mathbf{P(n)}$ **holds true** $\forall$ $\mathbf{n} \in \mathbb{N}$

Question 6: *[5+6 points]* Use induction to,

1. **Prove that the number of permutations on n elements is $n!$.**
2. **Give an asymptotic estimate for the function $n!$, as well as you can.**

1. Let P(n) be a proposition that number of permutations of n elements is $n!$
   For n=1, it is trivial that P(1) holds **true** as a single element can only be arranged in one permutation.
   Now, assuming that P(r) is true.
   i.e. no. of permutations of r elements is $r!$.
   Then, if we want to add a new elements, then it has (r+1) distinct positions where it can be added in every permutation of r elements.
   So, using product rule, number of permutations of (r+1) elements = $(r+1)\times$ (number of permutations of r elements)
   $\implies$ number of permutations of (r+1) elements = $(r+1)! \implies P(r+1)$
   Thus by induction, $P(n)$ *holds* **true** $\forall n \in \mathbb{N}$

2. We try to prove the stirling's approximation which is a known asymptotic estimate for $n!$
   Stirling's formula :
   $$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

We prove this result using mathematical induction,
Since, it is an asymptotic estimate we will assume the result holds for n as n tends to infinity,

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

Now, we prove the result for n + 1, i.e.

$$\lim_{n\to\infty} \frac{(n+1)!}{\sqrt{2\pi(n+1)}\left(\frac{n+1}{e}\right)^{n+1}} = 1$$

$$\lim_{n\to\infty} \frac{(n+1)!}{\sqrt{2\pi(n+1)}\left(\frac{n+1}{e}\right)^{n+1}} = \lim_{n\to\infty} \frac{(n+1)(n)!}{\sqrt{2\pi(n)}\frac{(n+1)}{e}\left(\frac{n+1}{e}\right)^{n}}$$

$$= \lim_{n\to\infty} e\frac{n^n}{(n+1)^n} \frac{(n)!}{\sqrt{2\pi(n)}\left(\frac{n}{e}\right)^n}$$

$$= \lim_{n\to\infty} \frac{e}{(1+\frac{1}{n})^n}$$

$$= \lim_{n\to\infty} \frac{e}{e^{n\times\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \frac{e}{e} = 1$$

Hence, the result holds true for n+1. Thus, using mathematical induction, Stirling's approximation is proven $\forall n \ as \ n \to \infty$.

Question 7: *[11+11 points]* Let A be an infinite set. Compare the cardinalities of the following three sets:

$$A, A \cup \mathbb{N}, A \times \mathbb{N}$$

Case I: Set A is countably infinite

By definition of countable, there exist a bijection from A to $\mathbb{N}$, say f
We can make a bijection from $A \cup \mathbb{N}$ to B as

$$g : A \cup \mathbb{N} \to B : g(x) = \left\{ \begin{array}{ll} 2f(x) & , x \in A \\ 2x - 1 & , x \notin A \end{array} \right\}$$

We can adjust the co-domain of B to make g a bijection.
We know that set B is infinite, and it is a subset of $\mathbb{N}$, so there exist a bijection from B to $\mathbb{N}$, and hence $|A \cup \mathbb{N}| = |\mathbb{N}|$
For the set $A \times \mathbb{N}$, we reuse the bijection made by in **Question 3** that proved $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$ Just here, we will include the bijection from A to N as:

$$l : \mathbb{A} \times \mathbb{N} \to \mathbb{N} : l(a, n) = 2^{f(a)}(2n + 1) - 1 \; \forall a \in A, n \in \mathbb{N}$$

Case II: Set A is uncountably infinite

We will use a theorem for solving this case
Theorem : Every infinite set has a countably infinite subset.
Now consider B be a countably infinite subset of A, and let C be equal to set A-B, Clearly C has to be uncountably infinite, otherwise A will become union of two countably infinite sets which is countable as proved above in case I.
Now $A \cup \mathbb{N}$ is equal to $B \cup C \cup \mathbb{N}$ We know $B \cup \mathbb{N}$ is also countably infinite(from Case I). So, we can again make a bijection from $B \cup \mathbb{N}$ to B and thus from $B \cup C \cup \mathbb{N}$ to $B \cup C$ i.e A.
So, again we can conclude that $|A \cup \mathbb{N}| = |A|$

Acknowledgements

1. Wikipedia: Stirling's Approximation

2. Proof from lecture notes used in claiming $|\mathbb{N}| < |2^{\mathbb{N}}|$

3. wikiproof.org: Theorem used in Question 7, Case II. The idea to use such a subset in solving the question was originally mine but I could not manage to prove that such subset always exist.