CS201A:Mathematics for Computer Science - I
Assignment 2

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Question 1: *[9 points]* Write a pseudocode to find the next natural number in base b.

We assume that the given number as well as its next natural number can be stored in k digits in b base notation.
Since b can be greater than 10, we might not be able to write it as a decimal number, therefore, we use an array 'd' to input the k digits of n, where d[0] will contain the least significant digit of n in base b and d[k-1] store the most significant digit of n.
Following is the psuedocode of my algorithm to find the next natural number in base b.

$find\_next\_number(d, b)$
$next[k]$
$incremented \leftarrow 0$
$i \leftarrow 0$
while $i < k$ **do**
    if not (incremented) **then**
        if $d[i] = b - 1$ **then**
            $next[i] \leftarrow 0$
        else
            $next[i] \leftarrow d[i] + 1$
            $incremented \leftarrow 1$
        end if

else
$next[i] \leftarrow d[i]$
end if
$i \leftarrow i + 1$
end while
return next

Question 2: *[6 points]* Learn what a graph is. Then, show that the sum of degree of each vertex is twice the number of edges.

Consider a Graph G with vertices V,
Consider that initially

$$E(G) = \phi,$$

Therefore,

$$deg(V) = 0 \qquad \forall v \in V,$$

Also,

$$|E| = 0$$

Hence the give proposition holds for an arbitrary edge set $E \subseteq V \times V$.
Then if we add a new edge (u, v) where $u, v \in V$ the number of edges increase by 1.
Also, $deg(u)$ and $deg(v)$ also increase by 1 each.
So the following holds

$$\sum_{v \in V} = 2|E|$$

Thus by the principle of mathematical induction, the proposition is proved.

Question 3: *[6+9 points]*

1. **Find the total number of r-ary (i.e. in r arguments) functions from $[n]$ to $[k]$.**

2. Show that the number of surjective (or, *onto*) maps from $[n]$ to $[k]$ is,

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

1. **Total number of r-ary functions from [n] to [k]**
   First we find the number of single argument functions from [m] to [k]
   As there is no condition on the type of function, for each element in [m], we just need an element belonging to [k] as the image for the element
   Therefore, each element of [m] has exactly k choices.
   Hence, we can apply product rule in this situation which will give us the total number of distinct functions to be: $k \times k \times \ldots \times k(m\ times) = k^m$
   Now each r-argument function can be seen as a single argument function where the argument is a r-length tuple.
   Again using product rule, the number of r-tuples that can be made using elements from [n] is equal to n^r. Therefore, the set of all the possible r-tuples from [n] is in bijection with the set$[n^r]$
   Implies the number of r-ary functions from [n] to [k] will be equal to the number of single argument functions from $[n^r]$ to [k] which we can calculate using the formula derived above to be k^{n^r}

2. **Number of surjections from [n] to [k]**
   Consider $A_1, A_2, A_3 \ldots A_k$ denote the set of functions in which range of A_i does not contain i
   Using the principle of inclusion-exclusion on set A_i's gives

$$\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{S\subseteq[k],S\neq\phi}(-1)^{|S|+1}\left|\bigcap_{j\in S} A_j\right| \tag{1}$$

   Now if we simplify the summation in RHS by taking —S— = 1,2,3 . . . k.
   For $|S| = p$, the number of terms will be $\binom{k}{p}$
   Also, for any given S with $|S| = p$, the term $\left|\bigcap_{j\in S} A_j\right|$ can be calculated

as follows:
We know that range of any function in $\bigcap_{j \in S} A_j$ will not contain atleast p elements (the elements which are there in S)
So each element in domain ([n]) will have will be left with only $(k-p)$ choices.
Now, using product rule,

$$\left|\bigcap_{j \in S} A_j\right| = (k-p)^n$$

Therefore, the summation in RHS of equation 1 can be simplified to

$$\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{p=1}^{k} (-1)^{p+1} \binom{k}{p} (k-p)^n$$

Now, we see that the LHS of above equation denotes the number of functions which have do not have atleast one element in [k] in their range.
So, the number of surjections will be just total number of functions from [n] to [k] - LHS of above eqn

$$\#Surjections = k^n - \left|\bigcup_{i=1}^{k} A_i\right| = k^n - \sum_{p=1}^{k} (-1)^{p+1} \binom{k}{p} (k-p)^n$$

Which on some rearrangement gives ...

$$\#Surjections = (-1)^0 \binom{k}{0} (k-0)^n + \sum_{i=1}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

$$\#Surjections = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

Hence proved,

$$\#Surjections\ from\ [n]\ to\ [k] = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

Question 4: *[7 points]* Find all possible solutions for a sequence S_n which satisfies,

$$S_n = S_{n-1} + 6S_{n-2}$$

Let S_0 and S_1 be respectively the first and second term of the sequence. (As we are starting indexing from zero, here the n^{th} term of sequence will be denoted by S_{n-1})
Now assume the generating function for this sequence as $G(t)$ given by

$$G(t) = \sum_{i=0}^{\infty} S_i t^i$$

$$\implies tG(t) = \sum_{i=0}^{\infty} S_i t^{i+1}$$

$$\implies tG(t) = \sum_{i=1}^{\infty} S_{i-1} x^i \tag{2}$$

Further,

$$t^2G(t) = \sum_{i=0}^{\infty} S_i t^{i+2}$$

$$\implies t^2G(t) = \sum_{i=2}^{\infty} S_{i-2} x^i \tag{3}$$

From (1)+6×(2), we get ...

$$(t + 6t^2)G(t) = \sum_{i=1}^{\infty} 6S_{i-1}t^i + \sum_{i=2}^{\infty} S_{i-2}t^i$$

$$\implies (t + 6t^2)G(t) = S_0 t + \sum_{i=2}^{\infty}(S_{i-1} + 6S_{i-2})t^i$$

$$\implies (t + 6t^2)G(t) = S_0 t + \sum_{i=2}^{\infty}(S_i)t^i$$

$$\implies (t+6t^2)G(t) = S_0t - S_0 - S_1t + \sum_{i=0}^{\infty}(S_i)t^i = (S_0 - S_1)t - S_0 + G(t)$$

Which on simplifying gives,

$$G(t) = \frac{S_0 + (S_1 - S_0)t}{1 - t - 6t^2} \tag{4}$$

Now as we have got the generating function, we can simplify it using partial fractions as:

$$G(t) = \frac{S_0 + (S_1 - S_0)t}{(1-3t)(1+2t)}$$

$$G(t) = S_0\left[\frac{\frac{3}{5}(1+2t) + \frac{2}{5}(1-3t)}{(1+2t)(1-3t)}\right] + (S_1 - S_0)\left[\frac{\frac{1}{5}(1+2t) - \frac{1}{5}(1-3t)}{(1+2t)(1-3t)}\right]$$

$$G(t) = \frac{S_0}{5}\left[\frac{3}{1-3t} + \frac{2}{1+2t}\right] + \frac{S_1 - S_0}{5}\left[\frac{1}{1-3t} - \frac{1}{1+2t}\right]$$

$$G(t) = \frac{2S_0 + S_1}{5}\left[\frac{1}{1-3t}\right] + \frac{3S_0 - S_1}{5}\left[\frac{1}{1+2t}\right]$$

Now, using binomial theorem

$$G(t) = \frac{2S_0 + S_1}{5}\sum_{i=0}^{\infty}(3t)^i + \frac{3S_0 - S_1}{5}\sum_{i=0}^{\infty}(-2t)^i$$

$$\implies G(t) = \sum_{i=0}^{\infty}\left[\frac{2S_0 + S_1}{5}(3)^i + \frac{3S_0 - S_1}{5}(-2)^i\right]t^i = \sum_{i=0}^{\infty}S_it^i$$

Now comparing the coefficients of t^n on both sides,

$$\mathbf{S_n = \frac{2S_0 + S_1}{5}(3)^n + \frac{3S_0 - S_1}{5}(-2)^n}$$

Hence, there can be infinitely many sequences be made by varying the values of S_0 and S_1

Question 5: *[8 points]* Given 9 vertices, join all pairs of vertices by either red or blue edge. Show that there is always either a red *triangle* or a blue *quadrilateral.*

For 9 vertices and all possible edges, each vertex has 8 edges connected to it. By pigeonhole principle, comparing edges to pigeons and colour of edge to holes, we can claim that for every vertex, either there exists at least 4 red edges or 4 blue edges connected to it.
Consider the following 2 possibilities :

1. **There exists a vertex with at least 4 red edges.**
   In this case, let this vertex be v_1 and let $v_2,\ v_3,\ v_4,\ v_5$ are 4 vertices such that edges $(v_1, v_2),\ (v_1, v_3),\ (v_1, v_4),\ (v_1, v_5)$ are red.
   Now if,
   $$\exists\ (u, v)\ :\ u, v \in \{v_2, v_3, v_4, v_5\}\ \&\ (u, v)\ \textit{is red},$$
   then, there exists a red triangle with vertices (v_1, u, v).
   Otherwise, if there exists no such red edge, which implies all these edges are blue thus making a blue quadrilateral with vertices (v_2, v_3, v_4, v_5).
   Hence, in case 1, there will always exist a red triangle or blue quadrilateral.

2. **There is no such vertex with 4 or more red edges.**
   $\implies$ Every vertex has at least 5 blue edges connected to it.
   Now consider any vertex v_1, and say $v_2,\ v_3,\ v_4,\ v_5,\ v_6$ are blue.
   Now take another vertex v_7.
   Then since there are only 9 vertices and at least 5 vertices from v_7 must be blue, there has to be 2 distinct vertices,
   $$u, v \in \{v_2, v_3, v_4, v_5, v_6\}\ :\ (u, v_7)\ \&\ (v, v_7)\ \textit{are blue}.$$
   But then (v_1, u, v_7, v) form a blue quadrilateral.
   Hence in case 2, there always exists a blue quadrilateral.

So in all the possible situations, we have seen that there either exists a red triangle or a blue quadrilateral.

Question 6: ***[5+8+2 points]***

1. **Suppose α is a rational. Show that there exists an $n_0 \in \mathbb{N}$ such that for every rational number $\frac{p}{q}$ with $1 \leq q < n_0$,**

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{n_0 q}$$

2. **Suppose α is an irrational real. Show that for any $n \in \mathbb{N}$, there is a rational number $\frac{p}{q}$ with $1 \leq q \leq n$, s.t.,**

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{nq}$$

Finally, what do above two properties characterize ?

1. Since α is given to be rational, we can consider it of the form a/b where a and b are co-prime integers and $b > 0$
   Multiplying by bq on both sides on given inequality reduces to

$$|qa - pb| \geq \frac{b}{n_0}$$

   Now, if we take $n_0 = b$, then RHS of inequality becomes 1
   And, $q < n_0 \implies q < b$, so $\frac{a}{b} \neq \frac{p}{q} \implies |aq - bp| \neq 0$ (because a and b are co-prime, so for a/b = p/q, we need a=p and b=q)
   Further, $|qa - bp|$ has to be an integer, We can claim that

$$|qa - pb| \geq 1$$

   Therefore, we can say for any rational α, where $\alpha = \frac{a}{b}$, a,b are co-prime, $\exists n_0 = b$ such that for every rational p/q with $1 \leq q < n_0$, $\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{n_0 q}$

2. Multiplying by q gives the simplified inequality

$$|q\alpha - p| < \frac{1}{n}$$

Now, consider the fractional part of $q\alpha, 1 \leq q \leq n$
If there is some q for which fractional part of $q\alpha < 1/n$, then we have proven the inequality, we can assign $p = [q\alpha]$ where [.] represents greatest integer function.
Now, we prove that there always exist such a q, we will use method of contradiction
Assuming $\nexists q : q\alpha - [q\alpha] < 1/n$
Now consider the intervals of form $\left(\frac{i}{n}, \frac{i+1}{n}\right), 1 \leq i < n$ as (n-1) holes and the fractional parts of $q\alpha$ as n pigeons
Applying pigeonholes principle, we get atleast one interval in which there are two fractional parts lie for distinct q, say q_1 and q_2
But then $|(q_1 - q_2)\alpha| < 1/n$ because the length of each interval is 1/n, which is a contradiction.
Hence there always exist a $q_0, 1 \leq q_0 \leq n$ for which $q\alpha - [q\alpha] < 1/n$
Hence assigning, $\frac{p}{q} = \frac{[q_0\alpha]}{q_0}$ will ensure that,

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{nq}$$

Property 1 characterizes that we can always quantify the difference between any two rationals using a natural number.
Property 2 characterizes that we can have a rational approximation for any irrational which can be made arbitrarily close to the irrational, as in the inequality we can take n to be arbitrarily large.

1 Acknowledgements

1. Professor Rajat Mittal's notes uploaded on Course Website, used in combining ideas of question 6.2.