GROUP WORK PROJECT # 2
GROUP NUMBER: 11902

MScFE 692: Capstone Review

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Statement of integrity: By typing the names of all group members in the text boxes below, you confirm that the assignment submitted is original work produced by the group (excluding any non-contributing members identified with an "X" above).	
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1. **OBJECTIVE**

Assess the cause of volatility by 2 properties:
i) Stochastic volatility (Heston Model).
ii) Jump in the underlying stock prices (Merton Model).

1.1.GENERAL PARAMETERS

- - S_0 = \$ 80 (Initial underlying stock price)
  - r = 5.5% (Risk free rate)
  - σ = 35% (Volatility)
  - T = 3 months (Time to maturity)

1.2.**HESTON PARAMETERS (Stochastic Volatility Modeller)**

- - V_0 = 3.2%
  - K_v = 1.85
  - θ_v = 0.045
  - Correlation values = [-0.30,-0.70]

1.3.MERTON PARAMETERS (Jump Volatility Modeller)

- - μ = -0.5
  - δ = 0.22
  - λ = [0.75,0.25]

2. **THEORY**

2.1.HESTON MODEL

a) **Introduction to Heston Model**
The Heston Model to assess the stochastic volatility presumes few assumptions, that consists the asset prices (**S_t**), and its variance **(v_t)** follows a risk neutral stochastic differential equation's (SDEs):

$$\mathbf{dS_t = rS_t + (\sqrt{v_t})\, S_t\, dW_t^{(1)}} \text{ -----(1).}$$

$$\mathbf{dv_t = \varkappa(\theta - v_t) + (\sigma\sqrt{v_t})\, S_t\, dW_t^{(2)}} \text{ -----(2).}$$

where,
$d(W_t^{(1)},W_t^{(2)}) = \rho\, dt$ ---- (3),
r is the risk free rate, **$\varkappa > 0$** the rate of mean reversion, **($\theta > 0$)** long term variance, (**$\sigma > 0$**) vol-of-vol (volatility), **ρ** is the spot variance calculations 1.

$S_t\, dW_t^{(1)} \sim dZ^{(1)}, S_t\, dW_t^{(2)} \sim dZ^{(2)}$ are Wiener processes with a correlation **ρ.**
Equation (1) indicates (SDE for asset), under risk neutral setting, **(μ = r)**.
Equation (2) indicates (SDE for variance) identical to the vesicek model.

b) **Simulating the Heston Model**
The Heston Model has been simulated under the Monte-Carlo simulation methods.

$$\mathbf{S_t = S_{t-1}\, e^{\alpha}} \text{ ----(4)}$$

$$\mathbf{\alpha = (r - (v_t/2))dt + (\sigma\sqrt{v_t})\, S_t\, dW_t^{(2)}} \text{ ----(4.1)}$$

$$\mathbf{v_t = v_{t-1} + \varkappa(\theta - v_{t-1})dt + (\sigma\sqrt{v_t})\, S_t\, dW_t^{(2)}} \text{ ----(5)}$$

The variances largely follows a CIR process, therefore the Feller condition, **$2\varkappa\theta > \sigma^2$** ensures **($v_t > 0$)** 1.

Heston key result is a semi-closed form for the European option prices, where call prices can be written as:

$$C(S_0, K, T) = S_0 P_1 - K e^{(-rT)} P_2 \text{ ----(6)}$$

$$C_0 = e^{-rT} E^Q[\max(S_T - K),0] \text{ ----(6.1)}$$

where,
P_j are risk-neutral probabilities.
Equation (6) is the Heston Formula 1.
Equation (6.1) is the risk-neutral valuation form of the Heston Formula.

2.2.MERTON MODEL

a) Introduction to Merton Model

Initially, the presumption is about the normally distributed world (Black-Scholes), and then cater to the non-constant volatility under stochastic processes (Heston). However, the drift component has its own substantial impact, as the GBM model (even in Heston) produces a relative “smooth” stock price valuation. Whereas, this is not holds true to the real prices scenarios where it experiences sudden jumps and drops impacting the earnings surprises and profit warnings like fundamental reasons.

One of the crucial jump-diffusion models for pricing options is Merton (1976).
The Merton (1976) jump diffusion model augments GBM with a Poisson jump process so that asset prices S_t under the real-world dynamics satisfies:

$$(dS_t / S_{t-1}) = \mu \, dt + \sigma \, dW_t + (e^Y - 1) \, dN_t \text{ ---- (7)}$$

Where,

W_t is the Brownian motion,

N_t is the Poisson process, with intensity (λ), and jump size (Y) are i.i.d.
Merton assumes $Y \sim N(\alpha, \delta^2)$, and the term S_{t-1} denotes the prices just before the jump.

Under risk-neutral pricing the drift is adjusted to $r - \lambda(E[e^Y] - 1)$, so the discounted asset prices are martingales 2.

In addition, Merton (1976) model SDE in eq(7) can also be described as:

$$dS_t = (r - r_j) S_t \, dt + \sigma S_t \, dZ_t + J_t S_t \, dN_t \text{ ---- (7.1)}$$

where,
$r_j = \lambda (e^\beta) - 1$, and $\beta = (\mu_j + (\delta^2/2))$ **---- (8)** is “correction” to the drift term in order to maintain the risk-neutral measures.

J_t is a jump at time t, which follows a distribution,

$$\text{Log}(1+J_t) \sim N(\text{Log}(1+\mu_j) - (\delta^2/2), \delta^2) \text{ ----(9)}$$

b) Simulating the Merton Model

The consideration for the simulation in Merton Model in this study were primarily dependent upon the Monte Carlo simulation on the discretised model for pricing eq(10).

$$S_t = S_{t-1} (e^\alpha + (e^\gamma - 1) Y_t) \text{ ---- (10)}$$

$$\alpha = (r - r_j - (\delta^2/2))dt + (\sigma\sqrt{dt}) Z_t^{(1)} \text{ ---- (10.1)}$$

$$\gamma = \mu_j + \delta\, Z_t^{(2)} \text{---- (10.2)}$$

In equation(10), there 3 sources of randomness here $Z_t^{(1)}$ stock price diffusion (standard normal), $Z_t^{(2)}$ size of the jump (standard normal), and Y_t timing of the jump (poisson)2.

2.3.GREEKS (DELTA and GAMMA)

a) **Finite Difference (Central) Method for Option Greeks.**

In many option pricing models especially those without the closed-form sensitivities, the greek must be calculated numerically.

Central Finite Difference method approximates derivatives of the option price with respect to input parameters by evaluating the pricing model at small, symmetric perturbations 3.

The Delta and Gamma is defined as:

$$\Delta = \partial V / \partial S_0\ ,\ \ \Gamma = \partial^2 V / \partial S_0^2 \text{ ---- (11)}$$

where **$V(S_0)$** denote the model price of an option as a function computed using the small perturbing the underlying by **(h)** 3.

- **Central Difference Delta**

$$\Delta_{FD} \approx (V(S_0 + h) - V(S_0 - h)) / 2h \text{ ---- (12)}$$

The symmetric estimator cancels first-order bias and is more accurate than the forward and backward difference approximations 3.

- **Central Difference Gamma**

$$\Gamma_{FD} \approx [V(S_0 + h) - 2V(S_0) + V(S_0 - h)] / h^2 \text{ ----(13)}$$

The second-order central difference provides a stable numerical estimate of curvature 3.

2. PUT-CALL PARITY

- The put-call parity showcases a fundamental arbitrage between call option and put option with the same strike price K and maturity T.

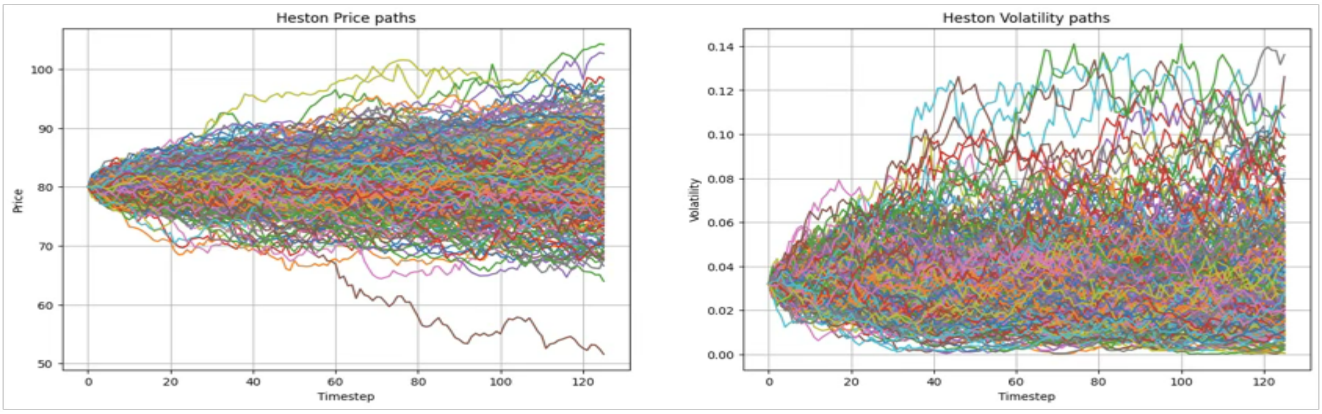
- In continuous-time finance, the idea exercised whenever the underlying asset can be traded continuously and the payoff structure is linear. The key idea is that the distribution of the underlying has no impact only the absence of arbitrage has some weight 12.

$$C(S_0, K, T) - P(S_0, K, T) = S_0 - K\, e^{(-rT)} \text{ ---- (14)}$$

3. RESULTS AND CONCLUSION

3.1. HESTON MODEL

a) Heston Price and Volatility path.



b) Q5. Pricing ATM European Call and Put at correlation value (-0.30).

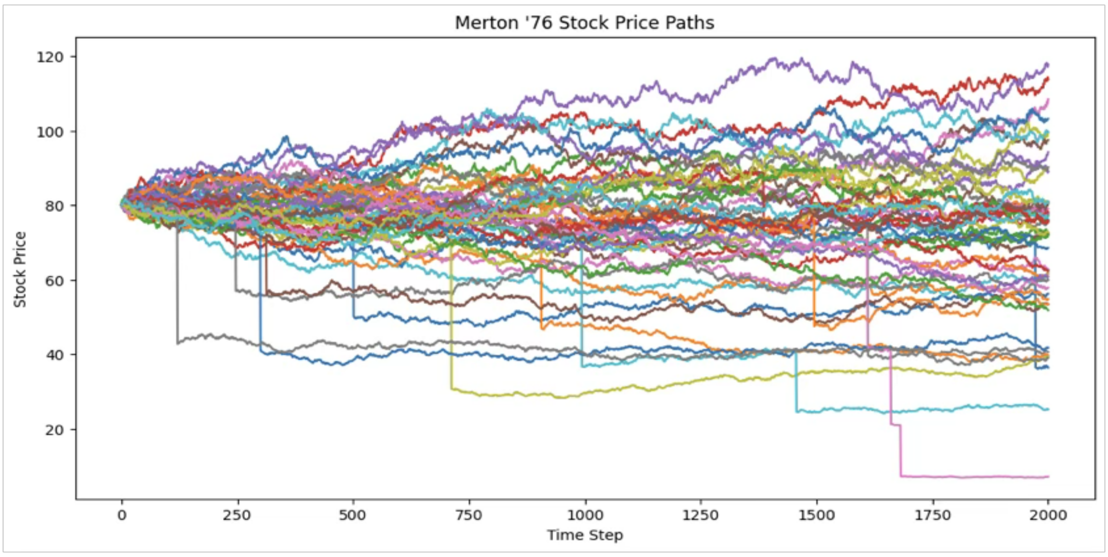
- ```
  === Q5: Heston (rho = -0.30) ===
  European Call Price under Heston: 3.47
  European Put Price under Heston: 2.37
  ```

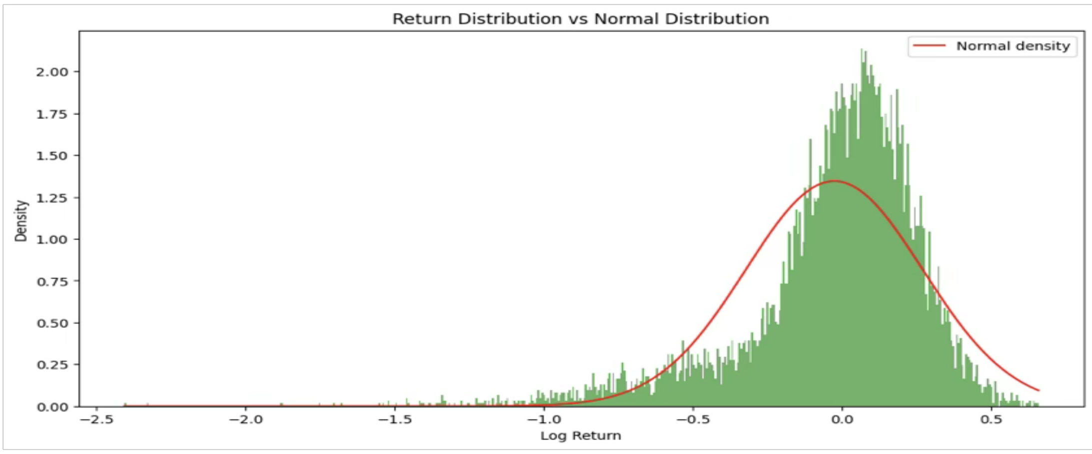
c) Q6. Pricing ATM European Call and Put at correlation value (-0.70).

- ```
  === Q6: Heston (rho = -0.70) ===

  European Call Price under Heston: 3.4955

  European Put  Price under Heston: 2.3998
  ```

3.2.MERTON MODEL

a) **Q8. Merton (1976) Stock Price Path and Return vs Normal Distribution and Using Merton model price an ATM European options call and put with jump intensity parameter = 0.75.**

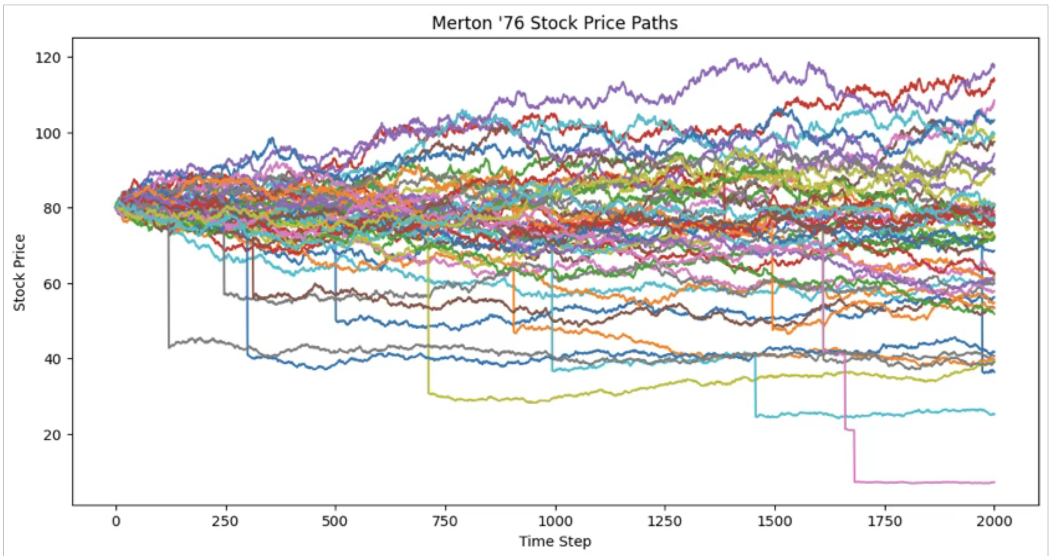


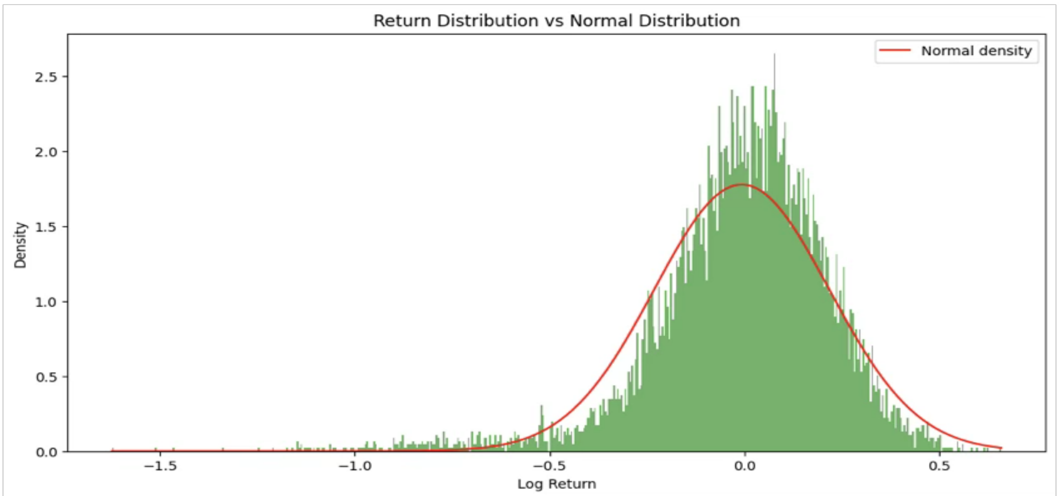


- Q8 – λ=0.75 Return's Mean: -0.0269, Std Dev: 0.2967, Skewness: -1.4713, Kurtosis: 3.7074

```
=== Q8: Merton (lambda = 0.75) ===

ATM Call Option Price : 8.21
ATM Put  Option Price : 7.06
```

b) **Q8. Merton (1976) Stock Price Path and Return vs Normal Distribution and Using Merton model price an ATM European options call and put with jump intensity parameter = 0.25.**






```
Q9 – λ=0.25 Return's Mean: -0.0072, Std Dev: 0.2245, Skew-
ness: -1.1810, Kurtosis: 3.6441

=== Q9: Merton (lambda = 0.25) ===

ATM Call Option Price : 6.99

ATM Put  Option Price : 5.89
```

3.3.GREEKS (Δ and Γ)

a) **Q7. Calculate delta and gamma for each of the options in Questions 5 and 6.**

```
delta is : 0.631

gamma is : 0.131
```

b) **Q10. Calculate delta and gamma for each of the options in Questions 9 and 9.**

```
---GREEKS at European Call option at lambda = 0.75 ---

Delta : 0.6419
Gamma : 0.0262

---GREEKS at European Put option at lambda = 0.75 ---

Delta : -0.3541

Gamma : 0.0261

---GREEKS at European Call option at lambda = 0.25 ---

Delta : 0.5973
Gamma : 0.0281

---GREEKS at European Put option at lambda = 0.25 ---

Delta : -0.4021
Gamma : 0.0281
```

3.4.PUT CALL PARITY

a) **Q11. For Questions 5, 6, 8, and 9, determine if the prices of the put and call from the Heston Model and Merton Model satisfy put-call parity.**

	Q	Model	Scenario	K	Call	Put	C - P	S_0 - K*exp(-rT)	Difference
1	**5**	Heston	rho = -0.30	80	3.48	2.40	1.08	1.09	-0.0154
2	**6**	Heston	rho = -0.70	80	3.50	2.40	1.1	1.09	0.0031
3	**8**	Merton	Lambda = 0.75	80	8.28	7.23	1.05	1.09	-0.049
4	**9**	Merton	Lambda = 0.25	80	6.86	5.77	1.09	1.09	0.0006

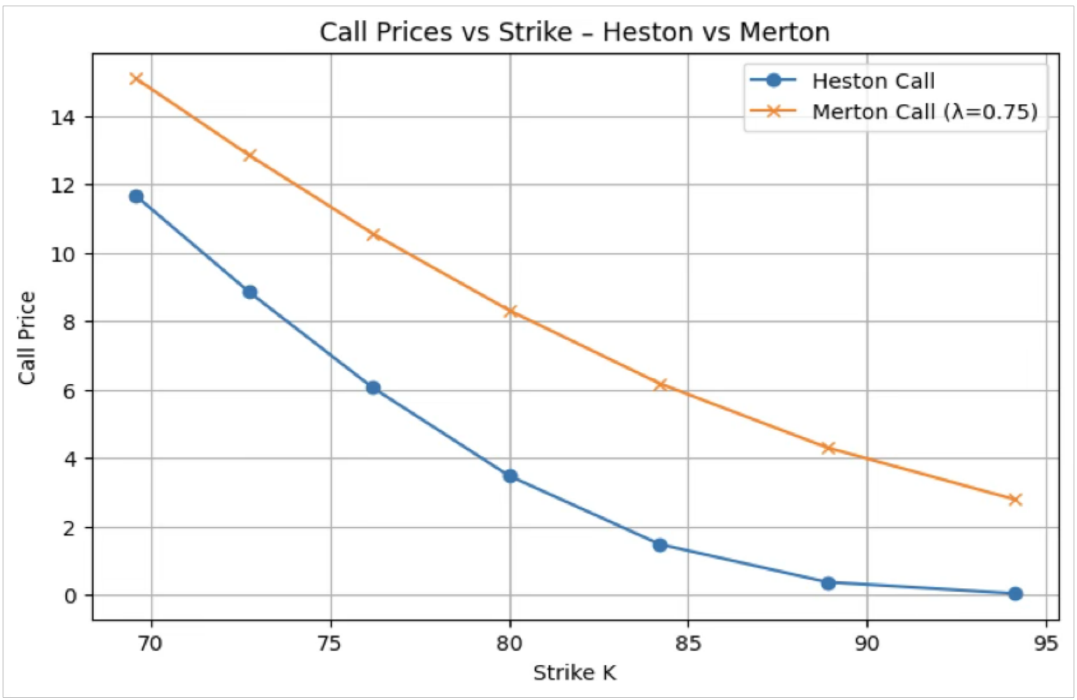
Overall, in all four questions, the calculated value of C−P stays very close to the theoretical benchmark. This means none of the models violate put-call parity, and the small differences seen in the table can be fully explained by simulation noise rather than any theoretical failure.

b) **Q12. Run the Heston Model and Merton Model for 7 different strikes: 3 OTM calls; 1 ATM call; and 3 ITM calls. The strikes should be equally spaced. Try to use the following APPROXIMATE moneyness values: 0.85, 0.90, 0.95, 1, 1.05, 1.10, and 1.15. Recall that moneyness = stock/strike.**

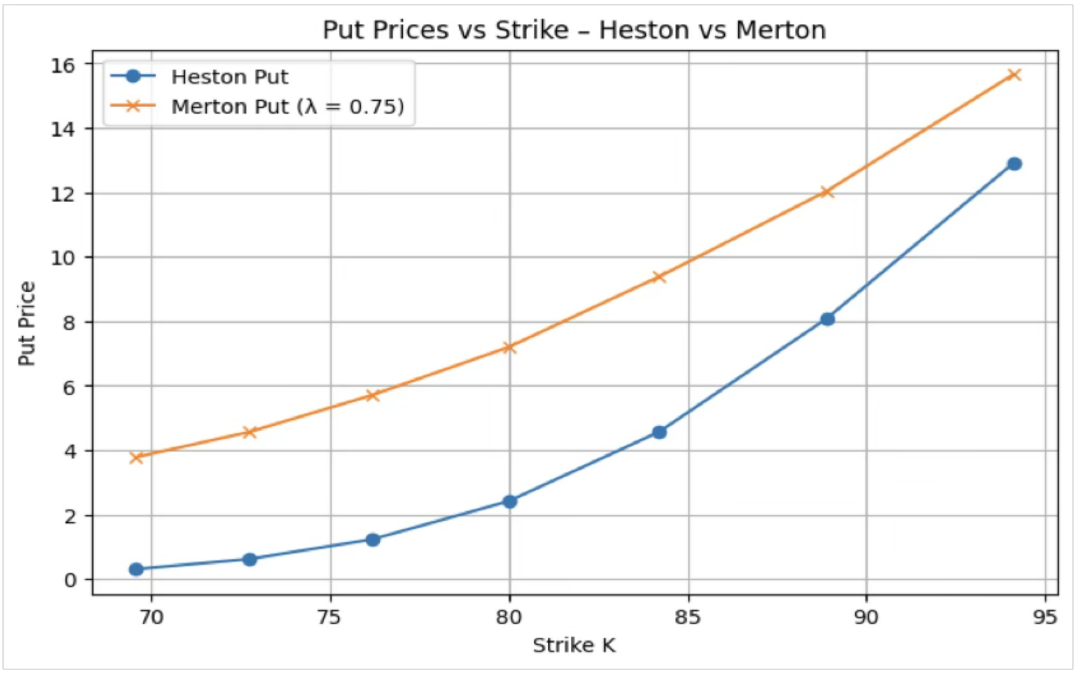
Moneyness and Strikes:

Moneyness	Strikes
m = 0.85	K ≈ 94.1176
m = 0.90	K ≈ 88.8889
m = 0.95	K ≈ 84.2105
m = 1.00	K ≈ 80.0000
m = 1.05	K ≈ 76.1905
m = 1.10	K ≈ 72.7273
m = 1.15	K ≈ 69.5652

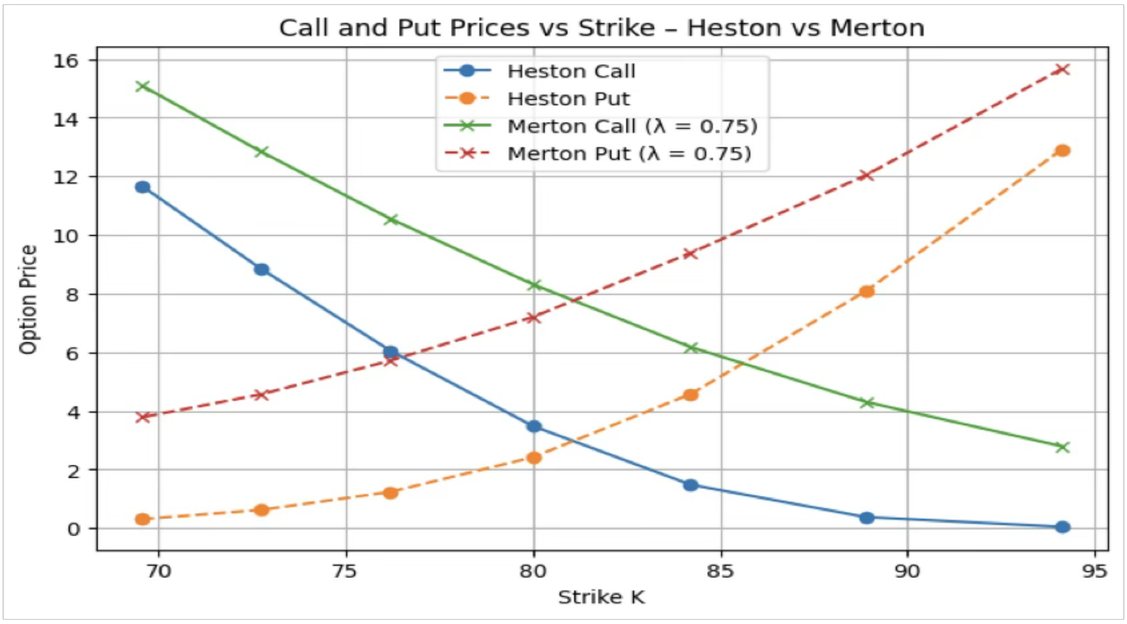
\	Moneyness (S0/K)	Strike K	Heston Call	Heston Put	Merton Call (λ=0.75)	Merton Put (λ=0.75)
1	0.85	94.18	0.047	12.90	2.83	15.68
2	0.90	88.89	0.38	8.08	4.31	12.02
3	0.95	84.21	1.49	4.57	6.20	9.28
4	1.00	80.00	3.50	2.42	8.34	7.24
5	1.05	76.19	6.07	1.24	10.57	5.65
6	1.10	72.73	8.86	0.62	12.87	4.56
7	1.15	69.57	11.67	0.31	15.13	3.72



=== Call Prices vs Strike: Heston vs Merton (λ = 0.75) ===



=== Put Prices vs Strike: Heston vs Merton (λ = 0.75) ===



3.5. AMERICAN OPTION (Case)

a) **Q13. Repeat Questions 5 and 7 for the case of an American call option (no need to price the put). Comment on the differences you observe from original Questions 5 and 7.**

```
American Call Price under Heston = 3.439, SE = 0.0113
```

3.6. BARRIER OPTIONS – UP-AND-IN CALL (CUI)

a) **Q14. Using Heston model data from Question 6 to price a European up-and-in call option (CUI) with a barrier level of 95 and a strike price of 95.**

```
=== Up-and-In Call vs Vanilla Call under Heston (rho = -0.70) ===
Vanilla European Call (K=95): 0.0290
Up-and-In Call (K=95, H=95): 0.0290
Difference (Vanilla - Up-and-In): 0.0000
Ratio (Up-and-In / Vanilla): 1.0000
```

3.7. BARRIER OPTIONS – DOWN-AND-IN PUT (PDI)

a) **Q15. Using Merton model data from Question 8 with a price a European down-and-in put option (PDI) with a barrier level of $65 and a strike price of $65 as well.**

```
Barrier Down-and-In put option (PDI) with barrier $65.0 and strike
price $65

Down-and-In Put under lambda=0.75: 2.801535

Vanilla Put with strike K=65.0: 2.715882

Percentage of paths that hit the barrier: 27.29%

Avg payoff if knocked-in: 10.407922
------------------------------------------------------------

Down-and-In Put under lambda=0.25: 1.424718

Vanilla Put with strike K=65: 1.336904

Percentage of paths that hit the barrier: 24.60%

Avg payoff if knocked-in: 5.871722
------------------------------------------------------------
```

Bibliography

1: Heston, Steven L., A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, 1993
2: Merton, Robert C., Option Pricing When Underlying Stock Returns Are Discontinuous, 1976
3: Broadie, Mark; Glasserman, Paul, Estimating Security Price Derivatives Using Simulation, 1996