Anujith K
1811031

CS-460

Assignment, 10th Sept-2021

Minimum Background Test

Vectors and Matrices

1: $y \cdot z = (1\times2)+(3\times3) = 11$

2. $$Xy = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} (2\times1)+(4\times3) \\ (1\times1)+(3\times3) \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} 14 \\ 10 \end{bmatrix}} \begin{bmatrix} 14 \\ 10 \end{bmatrix}.$$

3. $X = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$

$$|X| = (2 \times 3) - (1 \times 4)$$
$$= 6 - 4 = 2. \neq 0.$$

As X is non-singular, X is invertible

$$\text{adj } X = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}.$$

So, $X^{-1} = \dfrac{\text{adj } X}{|X|}$

$$= \frac{1}{2} \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & -2 \\ \frac{-1}{2} & 1 \end{bmatrix}$$

4. The rank of X is 2.

Because: $[2 \quad 4] \neq c[1 \quad 3] \quad \forall c \in \mathbb{R}$.

i.e., both rows are linearly independend of each othe

So. The rank is 2

Calculus

1. $y = x^3 + x - 5$

So; $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^3 + x - 5)$

$= 3x^2 + 1$

2. $f(x_1, x_2) = x_1 \sin(x_2)\, e^{-x_1}$

$\dfrac{\partial f}{\partial x_1} = \dfrac{\partial}{\partial x_1}\left(x_1 \sin x_2\, e^{-x_1}\right)$

$= \sin x_2 \left[e^{-x_1} \dfrac{\partial x_1}{\partial x_1} + x_1 \dfrac{\partial e^{-x_1}}{\partial x_1} \right]$

$= \sin x_2 \left[e^{-x_1} + x_1 \times -1 \times e^{-x_1} \right]$

$= e^{-x_1}(1 - x_1) \sin x_2$

$$\partial_{x_2} f = \frac{\partial}{\partial x_2} \left(x_1 \sin x_2 \, e^{-x_1} \right)$$

$$= x_1 e^{-x_1} \frac{\partial \sin x_2}{\partial x_2}$$

$$= x_1 e^{-x_1} \cos x_2$$

$$\nabla f(x) = \begin{pmatrix} \partial_{x_1} f \\ \partial_{x_2} f \end{pmatrix}$$

$$= \begin{pmatrix} (1-x_1) e^{-x_1} \sin x_2 \\ x_1 e^{-x_1} \cos x_2 \end{pmatrix}$$

Probability and Statistics

1. The sample mean $= \dfrac{1+1+0+1+0}{8}$

$$\text{Sample mean}(\mu) = \frac{3}{5}$$

2. Sample variance $= \dfrac{\sum_{i=1}^{5} (\mu - x_i)^2}{n-1}$

$$\text{Sample variance}(S^2) = \left[\left(1-\frac{3}{5}\right)^2 + \left(1-\frac{3}{5}\right)^2 + 0\left(\frac{-3}{5}\right)^2 + \left(1-\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right)^2\right]\frac{1}{4}$$

$$= \left[\left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2\right]\frac{1}{4}$$

$$= \frac{4+4+9+4+9}{5^2} \times \frac{1}{4}$$

$$= \frac{30}{5 \times 4 \times 5}$$

$$= \frac{3}{10}$$

3. Probability of head $= P_h = \frac{1}{2}$

Probability of tail $= P_t = \frac{1}{2}$

Probability of $S = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$$= \frac{1}{32}$$

4. Let the $P_t = p$ be the probability of getting tail ($P(x=1)$)

& $P_h = (1-p)$ would be the probability of getting head ($P(x=0)$)

As $P_h + P_t = 1$

So, the probability of the sample S be P_s.

$$P_s = \prod_{i=1}^{5} p^{x_i}(1-p)^{(1-x_i)}$$

$$= p^{\Sigma x_i}(1-p)^{5-\Sigma x_i}$$

Taking the log on both side;

$$\log(P_s) = \left(\sum_{i=1}^{5} x_i\right)\log p + \left(5-\sum_{i=1}^{5} x_i\right)\log(1-p)$$

$$= 3\log p + 2\log(1-p)$$

$$\frac{d(\log P_s)}{dp} = \frac{3}{p} + \frac{2}{1-p}\,x-1$$

$$= \frac{3}{p} - \frac{2}{1-p}$$

As we are taking the maximum probability of S, maximise the probability of $\log P_s$. So take LHS = 0

$$0 = \frac{3}{p} - \frac{2}{1-p}$$

$$\frac{3}{p} = \frac{2}{1-p}$$

$$3(1-p) = 2p$$

$$3 - 3p = 2p$$

$$5p = 3$$

$$p = 3/5$$

So the value that maximizes the probability of Sample S

so should have $p(x=1) = 3/5$ & $p(x=0) = 2/5$

5.

- What is $p(z=T \text{ AND } y=b)$?

Sol:- 0.1

- What is $P(z=T \mid y=b)$

Sol; $$P(z=T \mid y=b) = \frac{P(z=T \text{ AND } y=b)}{P(y=b)}$$

$$= \frac{0.1}{0.1+0.15} = \frac{10}{25}$$

$$= 0.4$$

Big-O Notation

1. $f(n) = \ln(n)$ & $g(n) = \lg(n)$

Here ln & lg are base with e & 2 respectively

i.e, $f(n) = \frac{\log n}{\log e}$ & $g(n) = \frac{\log n}{\log 2}$

i.e, both the functions are equivalent so;

$f(n) = O(g(n))$ & $g(n) = O(f(n))$

2. $f(n) = 3^n$ & $g(n) = n^{100}$

f(n) grows more rapidly as n becomes large

i.e, $g(n) = O(f(n))$.

3. $f(n) = 3^n$, $g(n) = 2^n$

~~goes~~ f(n) grows rapidly as n becomes large

So, $g(n) = O(f(n))$

4. $f(n) = 1000n^2 + 2000n + 4000$, $g(n) = 3n^3 + 1$

$f(n) = O(g(n))$, since g(n) grows rapidly as n becomes large

Medium Background Test

Divide and Conquer

Algorithm;

find-transition (i, j)

let mid = i + floor ((j-1)/2)

let a = array [mid] and b = array [mid + 1]

if (a == b == 1) return find-transition (i, mid)

else if (a == b == 0) return find transition (mid +1, j)

else return (last occurrence of 0)

=

For each recursive call we know that the array [i] = = 0 & array [j] = = 1. When we stop, we know that a == 0 & b == 1, so array[mid] is the last entry with 0

The running time can be analyzed by the recurrence $T(n) = T(n/2) + O(1)$, $T(n) = 6$ for $n \leq 4$, which solves to $O(\log n)$. The algorithm is based on the idea of binary search tree

Probability and Random Variables

Probability

(a) $P(A \cup B) = P(A \cap (B \cap A^c))$

Sol - False

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Sol - True

(c) $P(A) = P(A \cap B) + P(A^c \cap B)$

Sol - False

(d) $P(A|B) = P(B|A)$

Sol - False

(e) $P(A_1 \cap A_2 \cap A_3) = P(A_3 | A_2 \cap A_1) P(A_2 | A_1) P(A_1)$

Sol - True

Discrete and Continuous Distributions

Multivariate Gaussian - $\frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left(-\frac{1}{2} -(X-\mu)^T \Sigma^{-1} (x-\mu)\right)$

Bernoulli - $p^x (1-p)^{1-x}$

Uniform - $\frac{1}{b-a}$ when $a \le x \le b$: 0 otherwise

Binomial - $\binom{n}{x} p^x (1-p)^{n-x}$

Mean, Variance and Entropy

(a) It's given that $Var(X) = E[(X - E(X))^2]$

$$(X - E(X))^2 = X^2 - 2X E[X] + (E[X])^2$$

$$E[(X - E(X))^2] = E[X^2] - 2 E[2X E X] + (E[X])^2$$

Given => $$Var X = E[X^2] - 2(E[X])^2 + E[X]^2$$

$$Var X = (E[X])^2 - E[X^2]$$

$$=$$

(b) mean = p , variance = $p(1-p)$

entropy = $-(1-p)\log(1-p) - p \log p$

Law of Large Numbers and Central limit Theorm

(a) Due to the law of large numbers, if a Fair die is rolled the number of times 3 shows up should be close to 1000

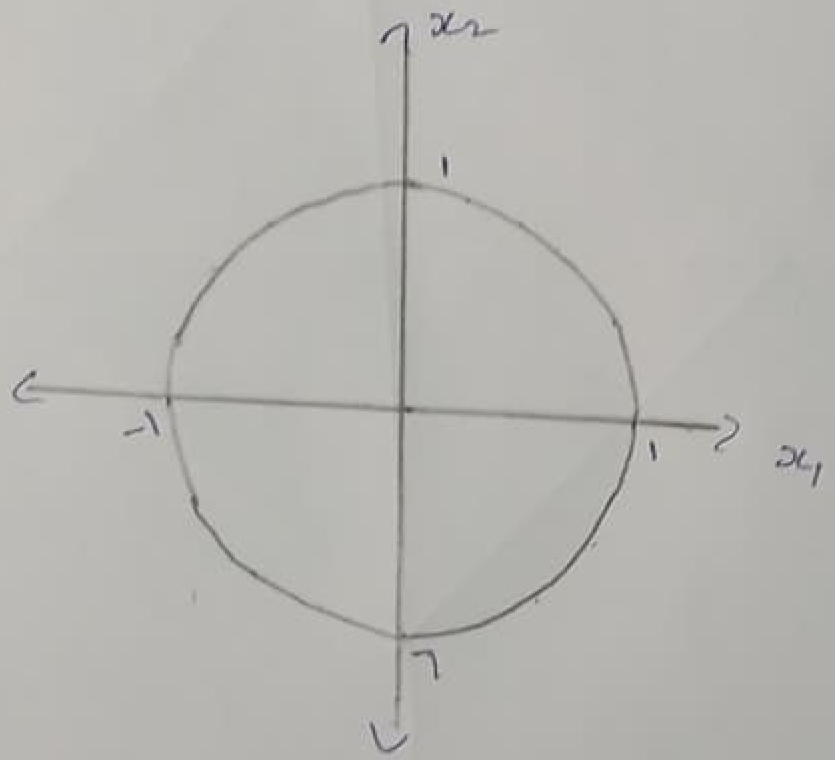
(b) The LHS should tend to the RHS as $n \to \infty$ due to Central limit Theorm.

Linear Algebra

Vector norms

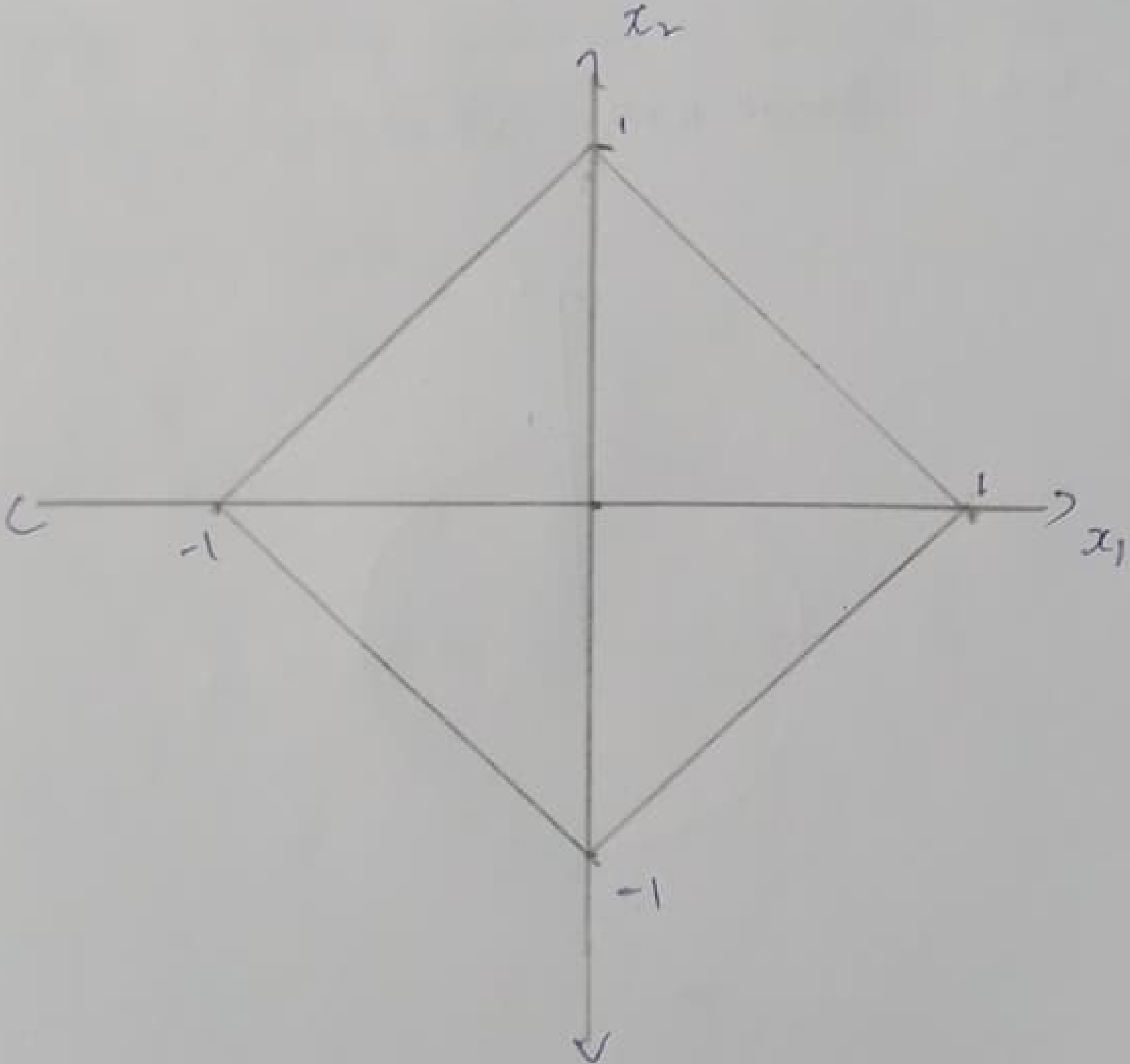
(a) $\|x\|_2 \leq 1$ (Recall $\|x\|_2 = \sqrt{\sum_i x_i^2}$

Sol



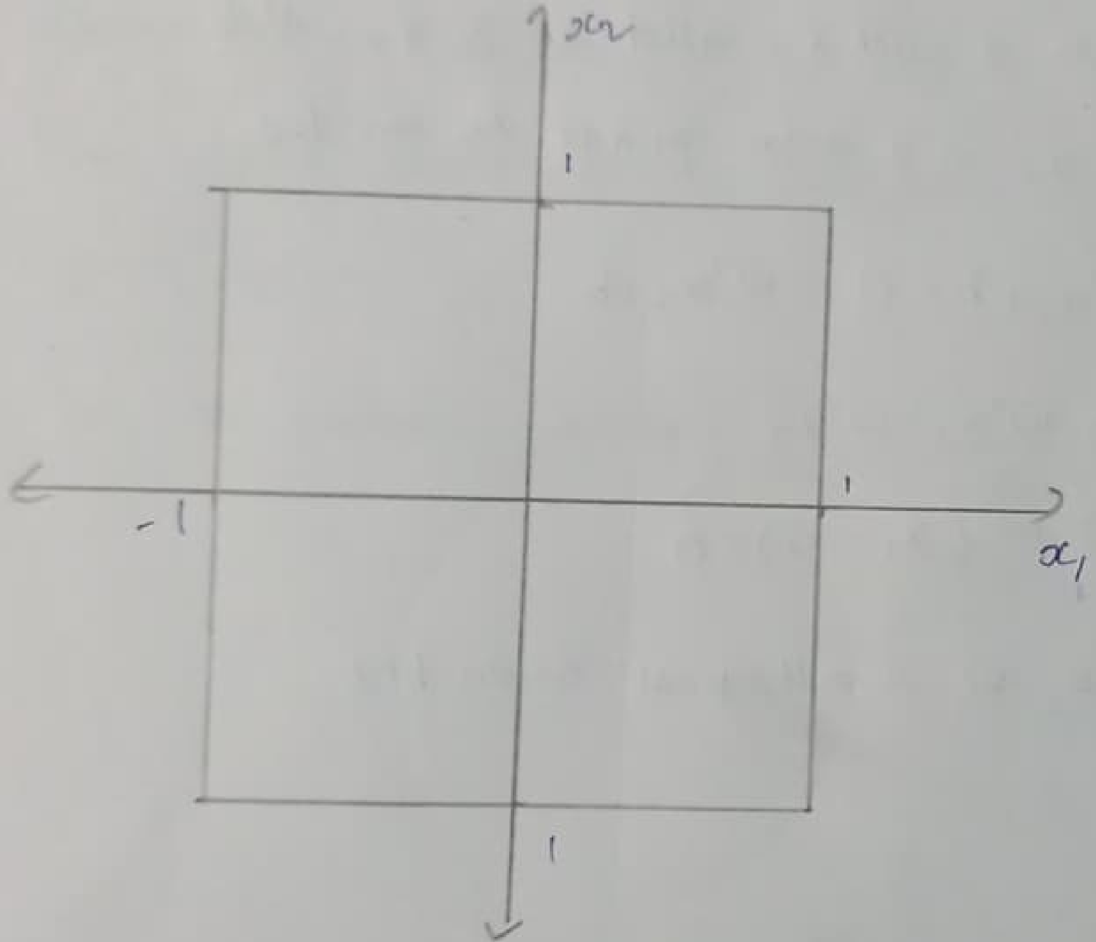
(b) $\|x_0\|_1 \leq 1$ (Recall $\|x\|_1 = \sum_i |x_i|$

Sol



(e) $\|x\|_\infty \leq 1$ (Recall $\|x\|_\infty = \max_i |x_i|$)

Sol



Geometry

(a) This line is all x such that $w^T x + b = 0$.

Consider 2 such x, called x_1 & x_2. Note

$x_1 - x_2$ is a vector parallel to our line.

$$w^T x_1 + b = 0 = w^T x_2 + b$$

$$\Rightarrow w^T x_1 = w^T x_2$$

$$\Rightarrow w^T (x_1 - x_2) = 0$$

i.e, w is orthogonal to our line.

(b) Let x be any point on the hyperplane $w^T x + b = 0$.

The Euclidean distance from the origin to the hyperplane can be computed by projected x onto the normal vector of the hyperplane, which is given by w.

i.e, the distance given by $\frac{|w^T x|}{\|w\|_2} = \frac{|-b|}{\|w\|_2}$

$$= \frac{|b|}{\|w\|_2}$$