with m being the bead mass. In this case, the swimmer should achieve propulsion and fulfill the requirement of the *scallop theorem* by relying on an asymmetry in coasting times of the constitutive beads.

To elaborate on this idea, we first build an analytic theory that relates the swimming velocity and coasting times. We successfully compare the model to experiments and lattice Boltzmann simulations in which no assumptions are imposed, thereby verifying the hypothesis that there is a swimming regime in which the inertial effects in the swimming dynamics can be separated and the swimmer coasting time harnessed for propulsion.

Our modeling efforts revolve around a dumbbell (Fig. 1a), that consists of two submerged beads of mass m_i and radii a_i. The beads are linked by a linear spring with stiffness k and natural length L, capturing, within the harmonic approximation $\vec{G}_{i,j} = -k(|\vec{x}_i - \vec{x}_j| - L)$, possible direct interactions between beads. The external forcing $\vec{F}_i$ is a sinusoidal force applied to each bead with the same intensity F and frequency ω in opposite directions to satisfy the force-free condition. The swimming dynamics of this object is studied using the equations of motion

$$\frac{\partial \boldsymbol{x}}{\partial t} = \hat{M}(\boldsymbol{x}) \left[\boldsymbol{F}(t) + \boldsymbol{G}(\boldsymbol{x}) - \hat{m}\frac{\partial^2 \boldsymbol{x}}{\partial t^2} \right], \tag{1}$$

where bold symbols account for concatenated vectors, e.g. $\boldsymbol{x} = (\vec{x}_1, \vec{x}_2)$. In this equation, we assume a low-$\mathrm{Re_f}$ dynamics by using the mobility matrix $\hat{M}(\boldsymbol{x})$. This matrix models hydrodynamic interactions with the Stokes drag (diagonal elements) and the Oseen tensor (off diagonal elements). The inertia of the beads is explicitly taken into account by a force $-\hat{m}(\partial^2 \boldsymbol{x}/\partial t^2)$. The matrix $\hat{m}$ has m_1 and m_2 in its diagonal elements.

This equation is solved (see [30] (Sect. I.A)) using a perturbative scheme [7]. Assuming $F/(ka_i) \ll 1$ and $a_i/L \ll 1$, one obtains the period-averaged swimming speed

$$\begin{aligned} \overline{U}^F = & \frac{3F^2\omega_0^4}{2k^2\overline{\theta}^2} \frac{a_1^2 a_2^2}{(a_1+a_2)^3 L^2} \\ & \times \frac{\omega(\theta_2 - \theta_1)}{\left(\left(\omega_0^2 - \omega^2 + \frac{\omega^2}{\theta_1\theta_2} \right)^2 + \left(\frac{\omega_0^2}{\overline{\theta}} - \frac{\omega^2}{\theta_1} - \frac{\omega^2}{\theta_2} \right)^2 \right)}, \end{aligned} \tag{2}$$

where $\theta_i = m_i\omega/(6\pi\eta a_i) = \tau_i\omega$ is the ratio of the coasting time to the external forcing period, $\overline{\theta} = (m_1 + m_2)\omega/(6\pi\eta(a_1 + a_2))$ is the swimmer coasting time, and $\omega_0^2 = k(m_1 + m_2)/(m_1 m_2)$. The solution for arbitrary separation (within the limit of the validity of the Oseen tensor) is provided in [30] (Sect. I.A). The superscript F in Eq.(2) refers to a force-based approach [6, 7, 31] where the stroke of the beads is known only *a posteriori*.

Alternatively, one can impose a stroke *a priori* and calculate the swimming velocity $\overline{U}^S$ [2, 3]. Now $\boldsymbol{G}(\boldsymbol{x})$ is removed from Eq.(1). Assuming $a_i/L \ll 1$ and a stroke $|\vec{x}_2(t) - \vec{x}_1(t)| = L + d\sin(\omega t)$ (see [30] (Sect. I.B)), one obtains

$$\overline{U}^S = \frac{3d^2}{2} \frac{a_1^2 a_2^2}{(a_1+a_2)^3 L^2} \frac{\omega(\theta_2 - \theta_1)}{1 + \overline{\theta}^2}. \tag{3}$$

Notably, there is a unique mapping between the two approaches (see [30] (Sect. I.C)).

In both force-based and stroke-based protocols, the analytical model described with Eq. (1) predicts a translation of the device in the direction of the beads with the smallest coasting time. This result may be sensitive to the a_i/L conditions, as it can be seen in [30] where $\overline{U}^F$ is calculated without approximation beyond the use of the Oseen tensor.

One can relax the assumptions made on the hydrodynamic flows and study the asymmetric dumbbell with lattice Boltzmann (LB) simulations (Fig. 1b) [26, 32, 33]. This algorithm solves a discrete version of the Boltzmann equation and recovers solutions of the Navier-Stokes equations in the limit of low Mach and low Knudsen numbers. For the bead dynamics, a leap-frog algorithm is used to solve Newton's equation of motion. The beads are discretized on the fluid lattice and their dynamics is coupled to the fluid by a mid-grid bounce-back boundary condition [26, 30, 34, 35]. As such, both the fluid and the spring-connected beads are simulated without any dynamical assumptions (see [30] (Sect. II.A)). For further comparison, we choose the numerical parameters to recover the expected Reynolds numbers of the beads and the fluid. We first confirm that there is no net flow responsible for the swimmer's displacement, and that a symmetric dumbbell does not swim. Finally, we show that a reciprocal deformation of an asymmetric pair results in a translational motion of the device in the direction of the small bead, as predicted by the theory.

Finally, we perform experiments using magneto-capillary swimmers (see [30] (Sect. II.B)) established previously [27–29, 36] (Fig. 1c). In short, the paramagnetic beads with a radius of 397, 500 or 793 μm are deposited on an air-water interface. When placed in a magnetic field B_z perpendicular to the interface, their capillary attraction is balanced by magnetic dipole repulsion [28]. Imposing a small oscillating field ($B_0 + b\sin(\omega t)$) in the direction parallel to the interface induces oscillations in the relative distance between the beads. The homogeneity of B_z and the flatness of the interface away from the beads ensure force-free conditions at all times. Consequently, symmetric dumbbells with two identical beads show no self-propulsion. However, a translation of the device is observed for two beads of different sizes. The swimmer moves towards the small bead, as shown in Fig. 2a (see also [30] SI Movie 1), in agreement with simulations and theoretical predictions. The swimmer is typically slow, reaching speeds up to $15\,\mu$m/s, i.e. $4 \times 10^{-3} L/T$ body-length L per period T, which gives a flow dominated by viscous drag instead of inertia, as quantified with $\mathrm{Re_f} \sim 10^{-2}$. Similar speeds and Reynolds