UNIT 5

Z-TRANSFORM

PROFESSIONAL'S
The Destination Towards Knowledge & Success

SYLLABUS

Fundamental Difference between Continuous and Discrete Time Signals, Discrete Time Signal Representation using Complex Exponential and Sinusoidal Components, Periodicity of Discrete Time using Complex Exponential Signal, Concept of Z-transform of a Discrete Sequence, Distinction between Laplace, Fourier and Z transforms, Region of Convergence in Z-transform, Constraints on ROC for Various Classes of Signals, Inverse Z-transform, Properties of Z-transform.

PART - A
SHORT QUESTIONS WITH ANSWERS

Q1) Find whether the function $e^{j6\pi n}$ is periodic or aperiodic.

Ans. : e^{j6pn}

Given that,

$$x[n] = e^{j6\pi n}$$

Comparing with,

$$x[n] = e^{j\omega_0 n}$$

We have,

$$\omega_0 = 6\pi$$

And

$$N = \frac{2\pi}{\omega_0}$$

$$= \frac{2\pi}{6\pi} M$$

$$= \frac{1}{3} M$$

The smallest value of M for which N is an integer is 3.

$$\therefore \quad N = \frac{1}{3}(3) = 1, N = 1$$

The given signal is periodic.

Q2) What do you mean by continuous time signals and discrete time signals?

Ans. : (1) A continuous time signal is defined for each and every value of the independent variable 't'. A continuous time signal is also called as analog signal.

(2) A discrete-time signal is defined only at discrete instants of time. Consequently for discrete-time signals, the independent variable takes on only a discrete values. The amplitude of discrete time signals between two instants of time is not defined.

Examples

(i) A speech signal as a function of time and atmospheric pressure as a function of an atitude are examples of continuous time signals.

(ii) The weekly Bombay stock market index is an example of discrete-time signal.

Q3) What is the condition for periodicity of a discrete time signals?

Ans. : A Discrete time signal is said to be periodic if it repeats itself after certain 'N' samples. Consider a discrete-time complex exponential signal defined by,

$$x[n] = e^{j\omega_0 n}$$

According to Periodicity $x[n] = x[n + N]$

$$x[n + N] = e^{j\omega_0 (n + N)}$$

$$\Rightarrow \quad e^{j\omega_0 (n)} e^{j\omega_0 N} = e^{j\omega_0 n}$$

$$\therefore \quad e^{j\omega_0 N} = 1$$

$\omega_0 N$ must be a multiple of 2π inorder to satisfy the condition $e^{j\omega_0 N} = 1$

$$\therefore \quad \omega_0 N = 2\pi M$$

Where, m is an integer.

$$\omega_0 = \frac{2\pi M}{N}$$

(or) $$f_o = \frac{M}{N} \text{ (A rational number)}$$

This is the condition for periodicity.

If the lower limit of summation is taken as '0' then it is called as one-sided or unilateral z-transform and is defined as,

$$Z[x[n]] = X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Which is also known as Unilateral z-transform.

Q4) Define DTFT and IDTFT.

Ans. : The Discrete-time Fourier Transform (DTFT) of finite energy aperiodic signal x[n] is a representation of signal in terms of complex exponential sequence $e^{j\omega n}$, where ω is a real frequency variable.

The discrete-time Fourier transform (DTFT) of x[n] is defined as,

$$X(e^{j\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \quad \text{... (5.4.1)}$$

The inverse discrete-time Fourier transform is defined as,

$$\left.\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega \\ \text{(or)} \quad x[n] &= \frac{1}{2\pi}\int_{0}^{2\pi} X(\omega)\, e^{j\omega n}\, d\omega \end{aligned}\right\} \quad \text{... (5.4.2)}$$

Eqs. (5.4.1) and (5.4.2) constitute a discrete-time Fourier transform pair for the sequence x[n].

Q5) What is the concept of Z-transform of a discrete sequence?

Ans. : z-transform is used to analyze discrete-time signals that do not have a Discrete-time Fourier transform (DTFT).

The z-transform comes into two varieties,

(1) bilateral, (or two-sided) z-transform.

(2) Unilateral (or one-sided) z-transform.

The bilateral z-transform offers clear understanding of the system characteristics such as stability, causality and frequency response. The unilateral z-transform is a convenient tool for solving difference equations with initial conditions.

The z-transform of a discrete-time signal x[n] is defined as,

$$Z[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Which is known as Bilaterila z-transform.

Q6) What is the relationship between z-transform and DTFT?

Ans. : The z-transform of a discrete-time signal x[n] is defined as,

$$Z[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Substituting $z = re^{j\omega}$ in Eq. (5.5.4), we get,

$$Z[x[n]] = X(z)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\,(re^{j\omega})^{-n}$$

$$\Rightarrow \quad Z[x[n]] = \sum_{n=-\infty}^{\infty} [x[n]\, r^{-n}]\, e^{-j\omega n}$$

$$\therefore \quad Z[x[n]] = F[x[n]\, r^{-n}]$$

Thus, the z-transform of x[n] is the Fourier transform of $x[n]r^{-n}$.

Now, if r = 1, we obtain by using Eq. (8.4.1),

$$Z[x[n]] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \text{DTFT}\,\{x[n]\} \quad ; \; r = 1$$

Q7) What are the advantages and limitation of z-transform?

Ans. : Advantages

(1) They convert difference equations of a system into a linear algebraic equation so that the system analysis becomes easy.

(2) Convolution in discrete-time domain is converted to multiplication in z-domain.

(3) Almost all signals are converged into z-transform which are not convergent in DTFT. Therefore, in z-transform it is not a question of convergence but it is a question of region of convergence (ROC).

Limitations

(1) z-transform cannot apply in continuous signal.

(2) Frequency domain response cannot be achieved and be plotted.

Q8) Define z-plane?

Ans. : It is convenient to represent the complex number z as a location in a complex plane termed as the z-plane as shown in figure. The point $z = re^{j\omega}$ is located at a distance "r" from the origin and at an angle ω from the positive real axis.

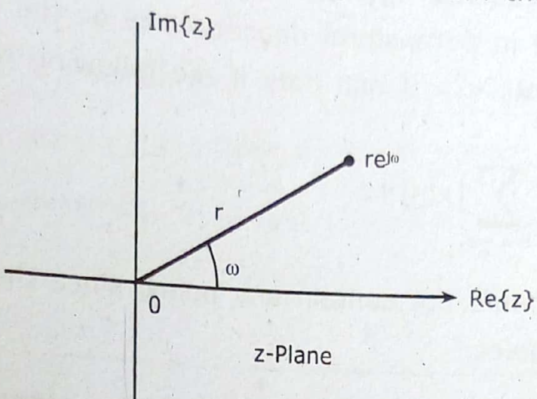

Figure The z-Plane, a Point is Located at a Distance r from the Origin and an Angle ω Relative to the Real Axis

Q9) Define poles and zeroes of a z-transform.

Ans. : The z-transform is most useful when the infinite sum can be expressed as a two polynomial of z, such as,

$$X(z) = \frac{P(z)}{Q(z)}$$

The values of z for which the z-transform $X(z) = 0$, are called the zeros of X(z). The value of z for which the z-transform $X(z) = \infty$, are called the poles of X(z). We denote the locations of zeros in the z-plane with the 'O' symbol and the location of poles with the 'X' symbols.

Q10) Give the differences between Fourier transform and Laplace transform.

Ans. :

Table Comparison between various Transform Forms

S.No.	Laplace Transform	Fourier Transform
(1)	$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$	$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$
(2)	Used as a tool for the analysis of continuous time signals wherever Fourier transform is not applicable	Used as tool for the analysis of continuous-time periodic and a periodic signals.
(3)	Laplace transform can be used to analyze signals and systems that are not stable.	Fourier transform cannot be used to analyze signals and systems that are not stable.

Q11) What is the region of convergence in z-transform?

Ans. : The range of values of complex number 'z' for which the z-transform converges to a finite value is called as Region of Convergence (ROC) In other words, the z-transform of sequence x[n] converges absolutely only for values of z in its ROC. The convergence in z-transform depends only on the magnitude of z, because the z-transform $|X(z)| < \infty$ if and only if the following equation holds,

$$\sum_{n=-\infty}^{\infty} |x(n)||z^{-n}| < \infty$$

The ROC should not contain any poles, since the z-transform, X(z) becomes infinite at the poles.

Q12) Determine the z-transform of the unit impulse signal and draw its ROC.

Ans. : An unit impulse function is defined as,

$$x[n] = \delta[n] = \begin{cases} 1 & ; n = 0 \\ 0 & ; n \neq 0 \end{cases}$$

z-transform of x[n] is defined as,

$$X[z] = Z[x(n)] = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= 1 \qquad [\because X[n] = \delta[n] = 0 \text{ for } n \neq 0]$$

$$\therefore \quad \delta(n) \xleftrightarrow{Z.T} 1$$

The ROC for X(z) is shown in the shaded areas in figure.

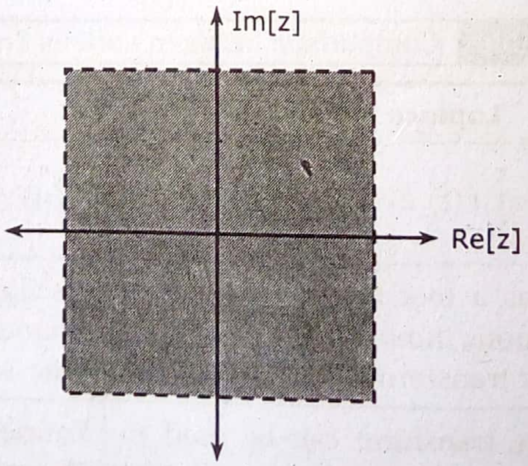

Figure ROC of Unit Impulse Function

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Q13) State any three properties of region of convergence (RC) in z-transform.

Ans. : **PROPERTY I :** The ROC of X(z) consists of a ring in the z-plane centered around the origin $(\sigma, j\omega) = (0, 0)$.

PROPERTY II : The ROC of z-transform of a discrete time signal does not contains any poles.

PROPERTY III : If X(z) is rational, then the ROC does not include any poles of X(z).

Q14) Give the ROC condition for the infinite duration, causal signals.

Ans. : Let x[n] be infinite duration, right-sided (causal) signal and is as shown in figure.

Let, $x(n) = a^n \; ; n \geq 0$

Now, the z-transform of x(n) is,

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n} = \sum_{n=0}^{\infty} a^n\, z^{-n} = \sum_{n=0}^{\infty} (a\, z^{-1})^n$$

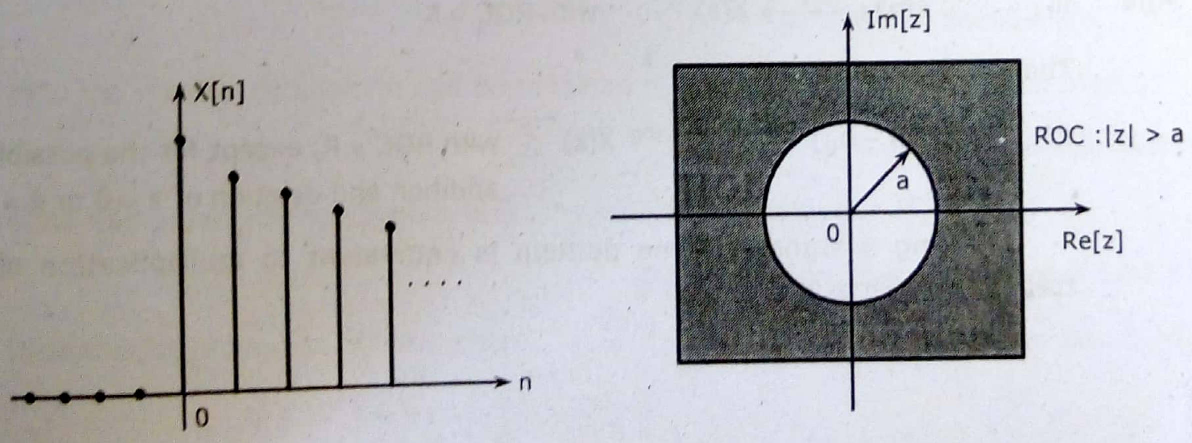
Using infinite geometric series sum formula,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad ; \text{ if } 0 < |a| < 1$$

Here the condition to be satisfied for the convergence of X(z) is,

$$0 < \left|az^{-1}\right| < 1$$

$$\therefore \quad \left|a\, z^{-1}\right| < 1 \Rightarrow \frac{|a|}{|z|} < 1 \Rightarrow |z| > |a|$$



(a) Infinite Duration Causal Signal

(b) ROC of a Infinite Duration Causal Signal

Figure

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Q15) Define inverse z-transform. What are the methods to obtain an inverse z-transform?

Ans. : The inverse z-transform of X(z) is defined as,

$$Z^{-1}[X(z) = x[n] = \frac{1}{2\pi j}\oint X(z)\, z^{n-1}\, dz$$

The transform relation between x[n] and X[z] is denoted as,

$$x[n] \xleftrightarrow{Z.T} X(z)$$

The inverse z-transforms can be obtained using any of the following three methods,

(1) Contour integration (or residue method).

(2) Power series expansion method.

(3) Partial fraction expansion.

Q16) What is the linearity property of z-transform?

Ans. : If,

$$x_1[n] \xleftrightarrow{Z.T} X_1(z) \quad ; \quad \text{with ROC} = R_1$$

And,

$$x_2[n] \xleftrightarrow{Z.T} X_2(z) \quad ; \quad \text{with ROC} = R_2$$

Then,

$$ax_1[n] + bx_2[n] \xleftrightarrow{Z.T} aX_1(z) + bX_2(z) \;\; ; \text{ with ROC contaning } R_1 \cap R_2$$

The z-transform of a weighted sum of two signals is equal to the weighted sum of individual z-transform.

Q17) What do you mean by time shifting property of z-transform?

Ans. : If, $x[n] \xleftrightarrow{Z.T} X(z) \quad ; \quad \text{with ROC} = R$

Then,

$$x[n-n_0] \xleftrightarrow{Z.T} z^{-n_0}\, X(z) \quad ; \quad \text{with ROC} = R\text{, except for the possible addition and deletion of } z = 0 \text{ or } z = \infty.$$

Shifting a signal in time domain is equivalent to multiplication of the z-transform with z^{-n_0}.

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PART - B
ESSAY QUESTIONS WITH REFERENCES

Q1) What are the major differences between continuous time signal and discrete time signal?

[Refer Section No. 5.2]

Q2) Explain discrete time representation by using exponential and the sinusoidal components?

[Refer Section No. 5.3]

Q3) When a signal is called as periodic signal? How its periodicity is calculated by using complex exponential signal? [Refer Section No. 5.4]

Q4) Give a relationship between z-transform and DTFT? What are the advantages and limitations of z-transform? [Refer Section Nos. 5.5.1 and 5.5.2]

Q5) State and explain z-transform of discrete sequenced? What are its advantages and limitations? [Refer Section Nos. 5.5 and 5.5.2]

Q6) Distinguish between Fourier transform and z-transform? [Refer Section No. 5.6]

Q7) State and explain Region of Convergence (ROC) in z-transform? What are its properties?

[Refer Section Nos. 5.7 and 5.7.1]

Q8) Draw and explain the ROC for finite duration anti-causal sequence X(n), and give an example. [Refer Section No. 5.8.2 and Example Problem : 5.9]

Q9) Explain the constraints on ROC for various classes of signals?

[Refer Section No. 5.8]

Q10) Write the steps required to find the inverse z-transform partial fraction expansion method?

[Refer Section No. 5.9.3]

Q11) How the inverse z-transform can be obtained by contour interaction fraction method?

[Refer Section No. 5.9.1]

Q12) State and prove scaling property and time reversal property of z-transform?

[Refer Section Nos. 5.10.4 and 5.10.5]

Q13) Describe the properties of z-transform? [Refer Section No. 5.10]

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5.1 INTRODUCTION

In a manner similar to that the Laplace transform is used for analyzing continuous-time signals for which the Fourier transform does not exist, z-transform enables to analyze discrete-time signals that do not have a Discrete-Time Fourier Transform (DTFT).

The z-transform comes in two varieties,

(1) Bilateral or two-sided z-transform

(2) Unilateral or one-sided z-transform.

The z-transform of a discrete-time signal x[n] is defined as,

$$Z[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \qquad \dots (5.1.1)$$

Where, $z = re^{j\omega}$ is a complex variable. The values of z for which the sum converges define a region in the z-plane referred to as the Region of Convergence (ROC).

Generally, if x(n) has a z-transform X(z), we write,

$$x(n) \xleftrightarrow{Z} X(z)$$

5.2 FUNDAMENTAL DIFFERENCE BETWEEN CONTINUOUS AND DISCRETE TIME SIGNALS

DEFINITION

(1) A continuous time signal is defined for each and every value of the independent variable 't'. A continuous time signal is also called as analog signal.

(2) A discrete-time signal is defined only at discrete instants of time. Consequently for discrete-time signals, the independent variable takes on only a discrete values. The amplitude of discrete time signals between two instants of time is not defined.

Examples

(i) A speech signal as a function of time and atmospheric pressure as a function of an atitude are examples of continuous time signals.

(ii) The weekly Bombay stock market index is an example of discrete-time signal.

MATHEMATICAL REPRESENTATION

The way in which continuous and discrete-time signals are mathematically represented are also different.

(1) A one-dimensional continuous-time signal that varies in time is represented as x(t). In this representation, t can take on any real value.

(2) Discrete-time signals are represented as an indexed sequence of number, mathematically denoted by x[n], where n can hold only integer values (..., −2, −1, 0, 1, 2, ...).

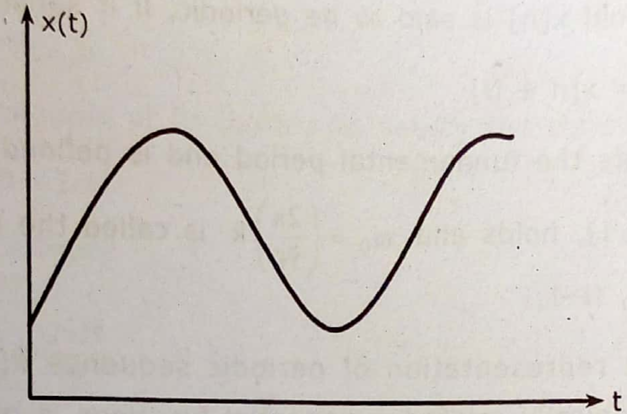
Discrete-time signals can often be considered as a result of sampling a continuous-time signal by using an analog to digital (A/D) converter. Assume that a continuous-time signal x(t) is sampled at a rate of $f_s = \frac{1}{T_s}$ samples per second. The sampled discrete-time signal x[n] is then equal to the value of the analog signal x(t) at time nT_s as follows,

$$x[n] = x(t)\big|_{t = nT_s} = x(nT_s) \quad ; \; -\infty < n < \infty$$

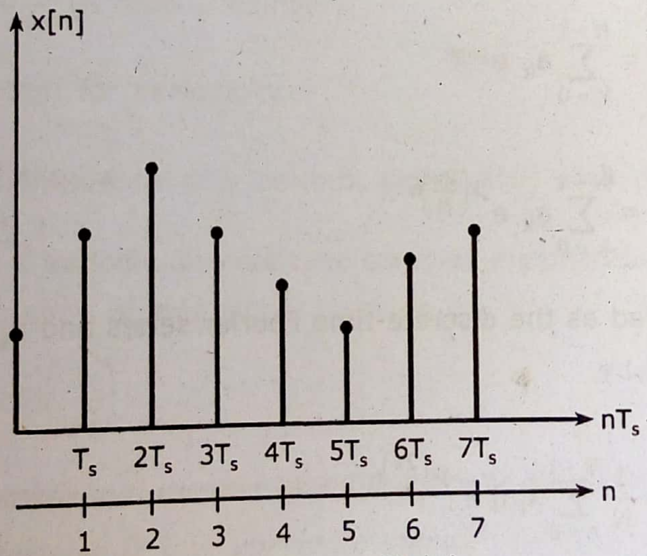
Where, T_s represents the sampling period.

GRAPHICAL REPRESENTATION

Fig. 5.2.1 illustrates the graphical representation of continuous and discrete-time signals.



(a) Continuous Time Signal



(b) Discrete-time Signal

Fig. 5.2.1 Graphical Representation

REVIEW QUESTIONS

(1) *What are the major differences between continuous time signal and discrete time signal?*

(2) *How continuous time signal is different from discrete time signal in both mathematical and graphical representations?*

5.3 DISCRETE TIME SIGNAL REPRESENTATION USING COMPLEX EXPONENTIAL AND SINUSOIDAL COMPONENTS

In a manner similar to that employed to represent any periodic continuous time signal using Fourier series representation, we can show that any periodic discrete-time signal can be represented by linear combination of harmonically related discrete-time exponential and sinusoidal functions such representation of discrete-time signals in terms of a set of discrete-time sinusoids and complex exponential functions is described by the Fourier series formulae.

A discrete-time signal x[n] is said to be periodic, if it satisfies the condition,

$$x[n] = x[n + N] \qquad \text{... (5.3.1)}$$

Where, N represents the fundamental period and is defined as the smallest value of N for which Eq. (5.3.1), holds and $\omega_0 = \left(\frac{2\pi}{N}\right) k$ is called the fundamental frequency where, k = 0, 1, 2, ... , N−1.

The Fourier series representation of periodic sequence x[n] in terms of linear combination of N harmonically related exponential functions is given by,

$$x[n] = \sum_{k=0}^{N-1} a_k \, e^{j\omega_0 n}$$

$$= \sum_{k=0}^{N-1} a_k \, e^{jk\left(\frac{2\pi}{N}\right)n} \qquad \text{... (5.3.2)}$$

Eq (5.3.2) is defined as the discrete-time Fourier series and a_k represents the Fourier coefficients, is defined by,

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n} \qquad \text{... (5.3.3)}$$

REVIEW QUESTIONS

(1) *How a discrete time signal is represented by using both exponential and sinusoidal components?*

(2) *What is the relationship between discrete time signal and exponential components?*

5.4 PERIODICITY OF DISCRETE TIME SIGNAL USING COMPLEX EXPONENTIAL SIGNAL

A Discrete time signal is said to be periodic if it repeats itself after certain 'N' samples. Consider a discrete-time complex exponential signal defined by,

$$x[n] = e^{j\omega_0 n}$$

According to Periodicity $x[n] = x[n + N]$

$$x[n + N] = e^{j\omega_0 (n + N)}$$

$$\Rightarrow \quad e^{j\omega_0 (n)} e^{j\omega_0 N} = e^{j\omega_0 n}$$

$$\therefore \quad e^{j\omega_0 N} = 1$$

$\omega_0 N$ must be a multiple of 2π inorder to satisfy the condition $e^{j\omega_0 N} = 1$

$$\omega_0 N = 2\pi M$$

Where, m is an integer.

$$\omega_0 = \frac{2\pi M}{N}$$

(or) $$f_o = \frac{M}{N} \text{ (A rational number)}$$

This is the condition for periodicity.

The fundamental frequency of a periodic signal $x[n]$ with period N is $\frac{2\pi}{N}$.

If $x[n] = e^{j\omega_0 n}$ is a periodic discrete time complex exponential signal with $\omega_0 \neq 0$ then the fundamental frequency of this signal is given as,

$$N = M\left(\frac{2\pi}{\omega_0}\right)$$

COMMENT : *If M/N is a rational number then the discrete-time complex exponential signal $e^{j\omega_0 n}$ is periodic, otherwise it is a aperiodic signal.*

EXAMPLE PROBLEM 1

Find whether the following are periodic or aperiodic,

(i) $e^{j6\pi n}$

(ii) $e^{j\left(\frac{2\pi}{3}\right)n} + e^{j\left(\frac{3\pi}{4}\right)n}$

SOLUTION

(i) e^{j6pn}

Given that,

$$x[n] = e^{j6\pi n}$$

Comparing with,

$$x[n] = e^{j\omega_0 n}$$

We have,

$$\omega_0 = 6\pi$$

And $$N = \frac{2\pi}{\omega_0} M = \frac{2\pi}{6\pi} M = \frac{1}{3} M$$

The smallest value of M for which N is an integer is 3.

$$\therefore \quad N = \frac{1}{3}(3) = 1, N = 1$$

The given signal is periodic.

(ii) $e^{j\left(\frac{2\pi}{3}\right)n} + e^{j\left(\frac{3\pi}{4}\right)n}$

The time period of $e^{j\left(\frac{2\pi}{3}\right)n}$ is,

$$N_1 = \left(\frac{2\pi}{2\pi / 3}\right) M = 3M$$

For $M = 1$

$\Rightarrow \quad N_1 = 3$

The time period of $e^{j\frac{3\pi}{4}}$ is,

$$N_2 = \left(\frac{2\pi}{3\pi / 4}\right) M = \frac{8}{3} M$$

For $M = 3$

$\Rightarrow \quad N_2 = 8$

The ratio of time periods,

$$\frac{N_1}{N_2} = \frac{3}{8} \Rightarrow 8N_1 = 3N_2$$

$$N = 8N_1 = 3N_2 = 24$$

The given signal is periodic with time period 24.

5.4.1 Representation of Aperiodic Discrete-time Signal : Discrete-time Fourier Transform (DTFT)

The Discrete-time Fourier Transform (DTFT) of finite energy aperiodic signal x[n] is a representation of signal in terms of complex exponential sequence $e^{j\omega n}$, where ω is a real frequency variable.

The discrete-time Fourier transform (DTFT) of x[n] is defined as,

$$X(e^{j\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad \dots (5.4.1)$$

The inverse discrete-time Fourier transform is defined as,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

(or)

$$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(\omega)\, e^{j\omega n}\, d\omega \qquad \dots (5.4.2)$$

Eqs. (5.4.1) and (5.4.2) constitute a discrete-time Fourier transform pair for the sequence x[n].

REVIEW QUESTIONS

(1) *Derive the condition for periodicity of discrete time signal?*

(2) *When a signal is said to be periodic? How the periodicity of discrete time signal is calculated by using complex exponential signal?*

5.5 CONCEPT OF Z-TRANSFORM OF A DISCRETE SEQUENCE

z-transform is used to analyze discrete-time signals that do not have a Discrete-time Fourier transform (DTFT).

The z-transform comes into two varieties,

(1) bilateral, (or two-sided) z-transform.

(2) Unilateral (or one-sided) z-transform.

The bilateral z-transform offers clear understanding of the system characteristics such as stability, causality and frequency response. The unilateral z-transform is a convenient tool for solving difference equations with initial conditions.

The z-transform of a discrete-time signal x[n] is defined as,

$$Z[x[n]] = X(z) \doteq \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \qquad \dots (5.5.1)$$

Which is known as Bilaterila z-transform.

Where, z represents a complex variable, given by $z = re^{j\omega}$ (r represents magnitude of Z and ω represents angle of z).

The definition of z-transform given by Eq. (5.5.1) is called as two-sided or bilateral z-transform. In the bilateral z-transfer, the range of summation is taken from $-\infty$ to $+\infty$. If the lower limit of summation is taken as '0' then it is called as one-sided or unilateral z-transform and is defined as,

$$Z[x[n]] = X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \qquad \text{... (5.5.2)}$$

Which is also known as Unilateral z-transform.

From Eqs. (5.5.1) and (5.5.2), it can be noticed that for any signal x[n] that is zero for n < 0 has same bilateral and unilateral z-transforms.

INVERSE Z-TRANSFORM

The inverse Z-transform of X(z) is defined as,

$$Z^{-1}[X(z) = x[n] = \frac{1}{2\pi j}\oint X(z)\, z^{n-1}\, dz \qquad \text{... (5.5.3)}$$

The transform relation between x[n] and X[z] is denoted as,

$$x[n] \xleftrightarrow{Z.T} X(z)$$

Where, X(z) represents the z-transform of discrete-time signal x[n].

5.5.1 Relationship between z-Transform and DTFT

The z-transform of a discrete-time signal x[n] is defined as,

$$Z[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \qquad \text{... (5.5.4)}$$

Substituting $z = re^{j\omega}$ in Eq. (5.5.4), we get,

$$Z[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, (re^{j\omega})^{-n}$$

$$\Rightarrow \qquad Z[x[n]] = \sum_{n=-\infty}^{\infty} [x[n]\, r^{-n}]\, e^{-j\omega n}$$

$$\therefore \qquad Z[x[n]] = F[x[n]\, r^{-n}] \qquad \text{... (5.5.5)}$$

Thus, the z-transform of x[n] is the Fourier transform of $x[n]r^{-n}$.

Now, if r = 1, we obtain by using Eq. (8.4.1),

$$Z[x[n]] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \text{DTFT}\,\{x[n]\} \qquad ; \qquad r = 1$$

5.5.2 Advantages and Limitations of z-Transform

Advantages

(1) They convert difference equations of a system into a linear algebraic equation so that the system analysis becomes easy.

(2) Convolution in discrete-time domain is converted to multiplication in z-domain.

(3) Almost all signals are converged into z-transform which are not convergent in DTFT. Therefore, in z-transform it is not a question of convergence but it is a question of region of convergence (ROC).

Limitations

(1) z-transform cannot apply in continuous signal.

(2) Frequency domain response cannot be achieved and be plotted.

5.5.3 z-Plane

It is convenient to represent the complex number z as a location in a complex plane termed as the z-plane as shown in Fig. 5.5.1. The point $z = re^{j\omega}$ is located at a distance "r" from the origin and at an angle ω from the positive real axis.

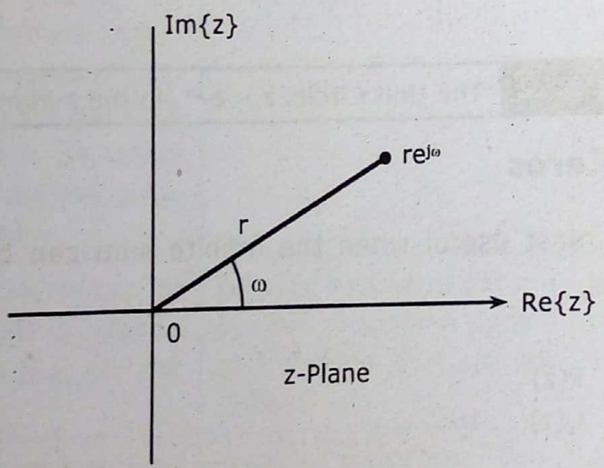


Fig. 5.5.1 The z-Plane, a Point is Located at a Distance r from the Origin and an Angle ω Relative to the Real Axis

From the definition of z-transform, we have,

$$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \qquad \dots (5.5.6)$$

Substituting $z = re^{j\omega}$ in Eq. (5.5.6), we get,

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\, r^{-n} \qquad \dots (5.5.7)$$

If r = 1, then Eq. (5.5.7) becomes,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \text{DTFT}[x[n]]$$

Therefore, the z-transform of a discrete-time signal x[n] reduce to the DTFT on the contour in the complex z-plane, which corresponds to a circle of radius equal to unity. The circle of radius unity in the z-plane is called unit circle and is shown in Fig. 5.5.2.

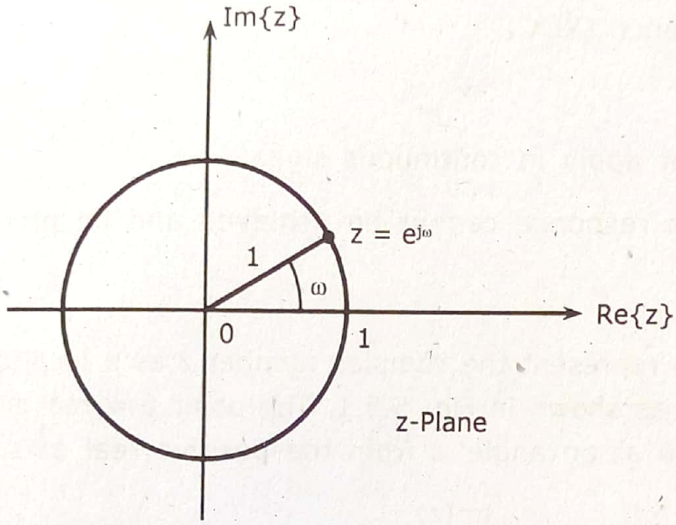


Fig. 5.5.2 The Unit Circle, $z = e^{j\omega}$, in the z-Plane

5.5.4 Poles and Zeros

The z-transform is most useful when the infinite sum can be expressed as a two polynomial of z, such as,

$$X(z) = \frac{P(z)}{Q(z)}$$

The values of z for which the z-transform X(z) = 0, are called the zeros of X(z). The value of z for which the z-transform X(z) = ∞, are called the poles of X(z). We denote the locations of zeros in the z-plane with the 'O' symbol and the location of poles with the 'X' symbols.

REVIEW QUESTIONS

(1) *State and explain z-transform of discrete sequence write its advantages and limitations?*

(2) *Give a relationship between z-transform and DTFT?*

5.6 DISTINCTION BETWEEN LAPLACE, FOURIER AND Z-TRANSFORM

Table 5.6.1 Comparison between various Transform Forms

S.No.	Laplace Transform	Fourier Transform	z-transform
(1)	$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$	$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$	$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$
(2)	Used as a tool for the analysis of continuous time signals wherever Fourier transform is not applicable	Used as tool for the analysis of continuous-time periodic and a periodic signals.	Used as a tool for the analysis of discrete-time signals.
(3)	Frequency response of a system cannot be drawn and estimated using Laplace transform.	Frequency response of a system can be drawn and estimated using Fourier transform.	Frequency response of a system cannot be drawn and estimated using z-transform.
(4)	Complex exponentials are more general signals than complex sinusoids. So, Laplace transform represents a more general class of signals than the Fourier transform including signals that are not absolutely integrable.	Fourier transform exists only for a restricted class of signals.	Complex exponentials are a more general signals class than complex sinusoids, so the z-transform can represent a broader class of signals than the DTFT, including signals that are not absolutely summable.
(5)	Laplace transform can be used to analyze signals and systems that are not stable.	Fourier transform cannot be used to analyze signals and systems that are not stable.	z-transforms can be used to analyze signals and systems that are not stable.
(6)	Laplace transform is used in transient and stability analysis of control systems.	Fourier transform provides an important set of tools for the study of communication systems. They also have extensive applications in the context of filtering.	z-transform is used for transient and stability analysis of sampled data control systems.

REVIEW QUESTIONS

(1) *Distinguish between Laplace, Fourier and Z-transform?*

(2) *Write the differences between Fourier transform and Z-transform?*

5.7 REGION OF CONVERGENCE IN Z-TRANSFORM

The computation of X(z) involves summation of infinite terms which are functions of z. Hence it is possible that the infinite series may not converge to finite value for certain values of z. Therefore for every X(z) there will be a set of values of z for which X(z) can be computed.

The range of values of complex number 'z' for which the z-transform converges to a finite value is called as Region of Convergence (ROC) In other words, the z-transform of sequence x[n] converges absolutely only for values of z in its ROC. The convergence in z-transform depends only on the magnitude of z, because the z-transform $|X(z)| < \infty$ if and only if the following equation holds,

$$\sum_{n=-\infty}^{\infty} |x(n)|\left|z^{-n}\right| < \infty$$

The ROC should not contain any poles, since the z-transform, X(z) becomes infinite at the poles. Basically, the ROC can be classified into four configuration regions for the z-transform, including the interior of a circle, the exterior of a circle, the annulus and the entire z-plane. To understand the z-transform and its associated ROC let us consider some examples,

EXAMPLE PROBLEM 1

Determine the z-transform and ROC of the following finite-duration signals,

(i) $x_1(n) = \left\{ \underset{\uparrow}{1}, 2, 6, -2, 0, 3 \right\}$

(ii) $x_2(n) = \left\{ 1, 2, \underset{\uparrow}{6}, -2, 0, 3 \right\}$

(iii) $x_3(n) = \left\{ 1, 2, 6, -2, 0, \underset{\uparrow}{3} \right\}$

SOLUTION

(i) By definition,

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n)z^{-n} = \sum_{n=0}^{5} x_1(n)z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5}$$

$$X_1(z) = 1 + 2z^{-1} + 6z^{-2} - 2z^{-3} + 3z^{-5}$$

ROC is the entire z-plane except z = 0 [since X(z) becomes unbounded for z = 0]. For a finite-duration right-sided signal [i.e., x(n) = 0 for n < 0 and n > N_1 for some finite N_1], the ROC will be the entire z-plane except z = 0.

(ii) By definition,

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n)z^{-n} = \sum_{n=-2}^{3} x_2(n)z^{-n}$$

$$= x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$X_2(z) = z^2 + 2z + 6 - 2z^{-1} + 3z^{-3}$$

ROC is the entire z-plane except $z = 0$ and $z = \infty$ [since X(z) becomes unbounded for $z = 0$ and $z = \infty$]. For a finite-duration two-sided signal (i.e., the signal is of finite extent for both $n < 0$ and $n > 0$), the ROC will be the entire z-plane except $z = 0$ and $z = \infty$.

(ii) By definition,

$$X_3(z) = \sum_{n=-\infty}^{\infty} x_4(n)z^{-n} = \sum_{n=-5}^{0} x(n)z^{-n}$$

$$= x(-5)z^5 + x(-4)z^4 + x(-3)z^3 + x(-2)z^{-2} + x(-1)z + x(0)$$

$$X_3(z) = z^5 + 2z^4 + 6z^3 - 2z^2 + 3$$

ROC is the entire z-plane except $z = \infty$ [since X(z) becomes unbounded for $z = \infty$]. For a finite-duration left-sided signal (i.e., $x(n) = 0$ for $n > 0$ and $n < -N_1$ for some finite positive N_1], the ROC will be the entire z-plane except $z = \infty$.

EXAMPLE PROBLEM 2

Determine the z-transform of the unit impulse signal and draws its ROC?

SOLUTION

An unit impulse function is defined as,

$$x[n] = \delta[n] = \begin{cases} 1 & ; n = 0 \\ 0 & ; n \neq 0 \end{cases}$$

z-transform of x[n] is defined as,

$$X[z] = Z[x(n)] = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= 1 \qquad [\because X[n] = \delta[n] = 0 \text{ for } n \neq 0]$$

$$\therefore \quad \delta(n) \xleftrightarrow{Z.T} 1 \qquad \dots (5.7.1)$$

The ROC for X(z) is shown in the shaded areas in Fig. 5.7.1,

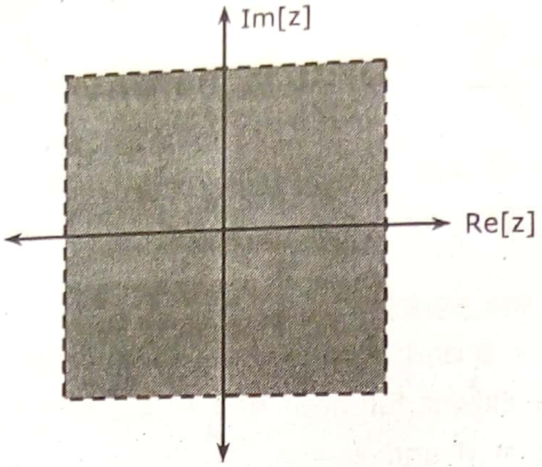


Fig. 5.7.1 ROC of Unit Impulse Function

EXAMPLE PROBLEM 3

Determine the z-transform of the unit step signal and plot its ROC?

SOLUTION

An unit step signal is defined as,

$$x[n] = u[n]$$

$$= \begin{cases} 1 & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$$

z-transform of x[n] is defined as,

$$X[z] = Z[x(n)]$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} u[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} 1 z^{-n}$$

Using infinite geometric series sum formula, that is,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \text{ ; if } 0 < |a| < 1$$

We have,

$$X(z) = \frac{1}{1 - z^{-1}} \; ; \text{ if } 0 < \left|z^{-1}\right| < 1$$

$$= \frac{z}{z-1} \; ; \text{ if } |z| > 1$$

$$\therefore \quad u(n) \xleftrightarrow{Z.T} \frac{z}{z-1} \quad ; \quad |z| > 1 \qquad \text{... (5.7.2)}$$

ROC for X(z) is $|Z| > 1$, as shown in the shaded areas in Fig. 5.7.2,

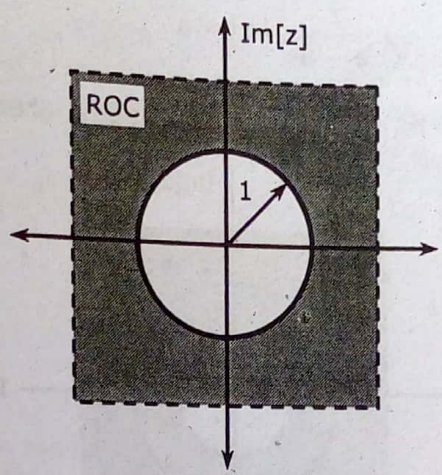


Fig. 5.7.2 ROC for Sequence x(n) = u(n)

EXAMPLE PROBLEM 4

Determine the z-transform of causal signal defined by, $x[n] = a^n u[n]$ where $0 < a < 1$ and plot its ROC?

SOLUTION

The z-transform of x[n] is defined as,

$$Z[x[n]] = X(z) = \sum_{n=-\infty}^{\infty} X(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(a^n\, u(n)\right) z^{-n} \qquad [\because \text{ causal signal } = 0 \text{ for } n < 0]$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

Using infinite geometric series formula,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \ ; \text{ if } 0 < |a| < 1$$

$$\therefore \quad X(z) = \frac{1}{1-az^{-1}} \ ; \text{ if } 0 < |az^{-1}| < 1$$

$$= \frac{z}{z-a} \ ; \text{ if } |z| > a$$

$$\therefore \quad a^n\, u(n) \xleftrightarrow{Z.T} \frac{z}{z-a} \ ; |z| > a \qquad \dots (5.7.3)$$

ROC for X(z) is |z| > a, as shown in the shaded areas in Fig. 5.7.3,

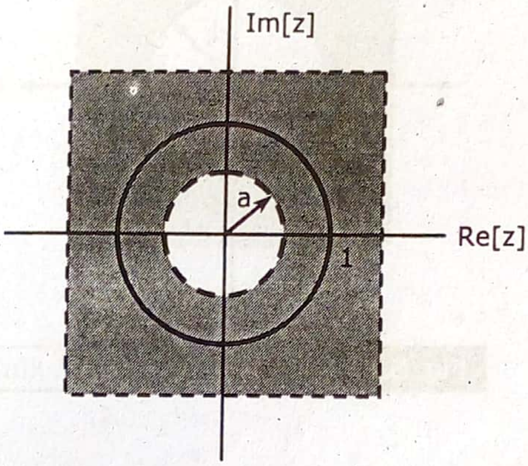


Fig. 5.7.3 ROC for the Sequence $x(n) = a^n$

EXAMPLE PROBLEM 5

Determine the z-transform of the anti-causal signal shown in Fig. 5.7.4(a).

$x(n) = -a^n u(-n-1)$ and depict the ROC. Here a > 1.

SOLUTION

By definition,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n\, u(-n-1)\, z^{-n}$$

We know that,

$$u(-n-1) = \begin{cases} 1, & (-n-1) \geq 0 \rightarrow n \leq -1 \\ 0, & (-n-1) < 0 \rightarrow n > -1 \end{cases} \quad \ldots (5.7.4)$$

By using Eq. (5.7.4), we get,

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = -\sum_{n=1}^{\infty} (a^{-1} z)^n = -\sum_{n-1=0}^{\infty} (a^{-1} z)^n$$

Let m = n − 1, which gives n = m + 1, m = 0 as n = 1 and m = ∞ as n = ∞.

$$X(z) = -\sum_{m=0}^{\infty} (a^{-1} z)^m = -(a^{-1} z)\sum_{m=0}^{\infty} (a^{-1} z)^m$$

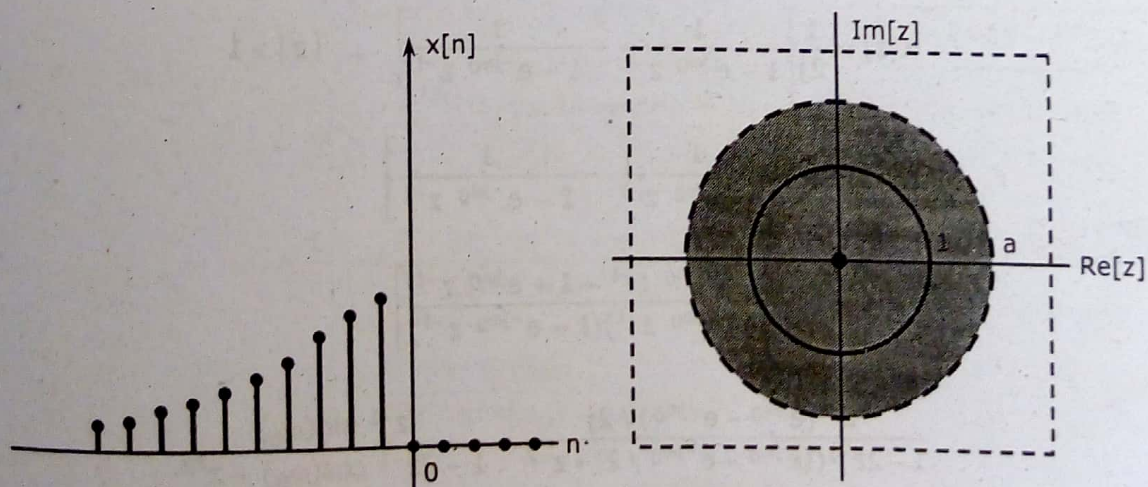
$$= -(a^{-1} z)\frac{1}{1 - a^{-1} z} \quad ; \quad |a^{-1} z| < 1$$

$$= \left(\frac{-z}{a}\right)\left(\frac{a}{a-z}\right) \quad ; \quad |z| < |a|$$

$$X(z) = \frac{z}{z-a}; |z| < |a|$$

$$\therefore \quad -a^n u(-n-1) \xleftrightarrow{Z.T} \frac{z}{z-a}, |z| < |a| \quad \ldots (5.7.5)$$

The ROC for X(z) is |z| < |a|, as shown in the shaded area in Fig. 5.7.4(b).



(a) Anticausal Signal

(b) ROC Plot of Anticausal signal x(n)

Fig. 5.7.4

Scanned by CamScanner

EXAMPLE PROBLEM 6

Determine the z-transform of the following signals,

(i) $x(n) = \sin(\omega_0 n)\, u(n)$

(ii) $x(n) = \cos(\omega_0 n)\, u(n)$

SOLUTION

(i) $x(n) = \sin(\omega_0 n)\, u(n)$

$$x(n) = \sin(\omega_0 n)\, u(n) = \frac{1}{2j}[e^{j\omega_0 n} - e^{-j\omega_0 n}]\, u(n)$$

$$\therefore \quad Z[x(n)] = \frac{1}{2j} Z[e^{j\omega_0 n} u(n) - e^{-j\omega_0 n} u(n)]$$

$$\Rightarrow \quad X(z) = \frac{1}{2j} Z[e^{j\omega_0 n} u(n)] - \frac{1}{2j} Z[e^{-j\omega_0 n} u(n)]$$

We have,

$$a^n u(n) \xleftrightarrow{Z.T} \frac{z}{z-a}, \; |z| > |a|$$

$$\therefore \quad \left.\begin{aligned} (e^{j\omega_0})^n u(n) &\xleftrightarrow{Z.T} \frac{z}{z - e^{j\omega_0}} \quad ; \quad |z| > |e^{j\omega_0}| \rightarrow |z| > 1 \\ (e^{-j\omega_0})^n u(n) &\xleftrightarrow{Z.T} \frac{z}{z - e^{-j\omega_0}} \quad ; \quad |z| > |e^{-j\omega_0}| \rightarrow |z| > 1 \end{aligned}\right\} \quad \ldots (5.7.6)$$

The set of values of $|z|$ for which the z-transforms of both terms converge is $|z| > 1$ and thus, we obtain,

$$X(z) = \frac{1}{2j}\left[\frac{1}{1 - e^{j\omega_0} z^{-1}} - \frac{1}{1 - e^{-j\omega_0} z^{-1}}\right] \quad ; \quad |z| > 1$$

$$X(z) = \frac{1}{2j}\left[\frac{1}{1 - e^{j\omega_0} z^{-1}} - \frac{1}{1 - e^{-j\omega_0} z^{-1}}\right]$$

$$= \frac{1}{2j}\left[\frac{1 - e^{-j\omega_0} z^{-1} - 1 + e^{j\omega_0} z^{-1}}{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}\right]$$

$$\frac{z^{-1}(e^{j\omega_0} - e^{-j\omega_0})/2j}{1 - 2z^{-1}((e^{j\omega_0} + e^{-j\omega_0}/2) + z^{-2}} = \frac{z^{-1}\sin(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + z^{-2}}$$

$$\therefore \quad \sin(\omega_0 n)\, u(n) \xleftrightarrow{Z.T} \frac{z^{-1}\sin(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + z^{-2}} \; ; \; |z| > 1 \quad \ldots (5.7.7)$$

The ROC for X(z) is $|z| > 1$, as shown in the shaded areas in Fig. 5.7.5.

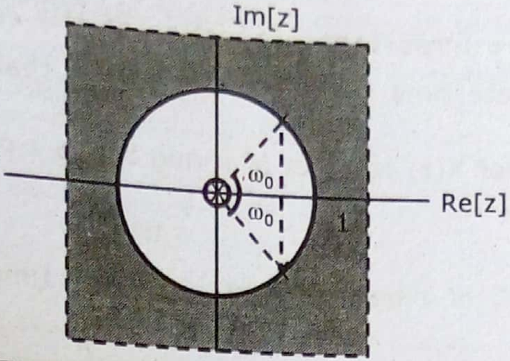


Fig. 5.7.5 ROC $\sin(\omega_0 n)$ and $\cos(\omega_0 n)$

(ii) $x(n) = \cos(\omega_0 n)\, u(n)$

$$x(n) = \cos(\omega_0 n)\, u(n)$$

$$\Rightarrow \qquad x(n) = \frac{1}{2}[e^{j\omega_0 n} + e^{-j\omega_0 n}]\, u(n)$$

$$\therefore \qquad Z[x(n)] = \frac{1}{2} Z[e^{j\omega_0 n} u(n) + e^{-j\omega_0 n} u(n)]$$

$$= \frac{1}{2} Z[e^{j\omega_0 n} u(n)] + \frac{1}{2} Z[e^{-j\omega_0 n} u(n)] \qquad \ldots (5.7.8)$$

The set of values of $|z|$ for which the z-transforms of both terms converge is $|z| > 1$ and using Eqs. (5.7.6) in Eq. (5.7.8), we get,

$$X(z) = \frac{1}{2}\left[\frac{1}{1 - e^{j\omega_0} z^{-1}} + \frac{1}{1 - e^{-j\omega_0} z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1 - e^{-j\omega_0} z^{-1} + 1 - e^{j\omega_0} z^{-1}}{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}\right]$$

$$= \frac{1}{2}\left[\frac{2 - 2z^{-1}(e^{j\omega_0} + e^{-j\omega_0})/2}{1 - 2z^{-1}((e^{j\omega_0} + e^{-j\omega_0})/2) + z^{-2})}\right]$$

$$= \frac{1}{2}\left[\frac{2 - 2z^{-1}\cos(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + z^{-2})}\right]$$

$$\therefore \qquad \cos(\omega_0 n)\, u(n) \xleftrightarrow{Z.T} \frac{1 - z^{-1}\cos(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + z^{-2}} \; ; \; |z| > 1 \qquad \ldots (5.7.9)$$

The ROC for X(z) is $|z| > 1$, as shown in the shaded areas in Fig. 5.7.5.

5.7.1 Properties of ROC in z-Transform

Following are the important properties of the region of convergence for the z-transform of a discrete time signal, let us assume that X(z) is a z-transform of x[n],

Property I : The ROC of X(z) consists of a ring in the z-plane centered around the origin $(\sigma, j\omega) = (0, 0)$.

Property II : The ROC of z-transform of a discrete time signal does not contains any poles.

Property III : : If discrete time signal x[n] is of finite duration then the ROC of X(z) will be entirez-plane except possibly $z = 0$ and/or $z = \infty$.

(1) If x(n) is finite duration right sided (causal) signal, then the ROC is entire z-plane.

(2) If x(n) is finite duration left sided (anticausal) signal, then the ROC is entire z-plane except $z = \infty$.

(3) If x(n) is finite duration two sided (non-causal) signal, then the ROC is entire z-plane except $z = 0$ and $z = \infty$.

Property IV : If x[n] is infinite duration right sided (causal) signal, then the ROC is exterior of the circle of radius (see Fig. 5.8.4(b).

Property V : If x(n) is infinite duration left sided (anticausal) signal, then the ROC is interior of a circle of radius b (see Fig. 5.8.5(b)).

Property VI : If x(n) is infinite duration two sided (non-causal) signal, then the ROC is the region in between two circles of radius a and b here $a < b$ (see Fig. 5.8.6(b)).

Property VII : If X(z) is rational, then the ROC does not include any poles of X(z).

Property VIII : If X(z) is rational and if x(n) is right sided, then the ROC is exterior of a circle whose radius corresponds to the pole with largest magnitude.

Property IX : If X(z) is rational and if x(n) is left sided, then the ROC is interior of a circle whose radius corresponds to the pole with smallest magnitude

Property X : If X(z) is rational and if x(n) is two sided, then the ROC is region in between two circles whose radius corresponds to the pole of causal part with largest magnitude and the pole of anticausal part with smallest magnitude.

Table 5.7.1 shows the z-transforms of some standard signals with their ROC's.

Table 5.7.1 z-transform of Standard Signals with their ROC's

S.No.	Discrete-time Signal x(n)	z-transform X(z)	ROC
(1)	$\delta(n)$	1	Entire z-plane
(2)	$\delta(n-k)$	z^{-k}	$\lvert z\rvert > 0, k > 0$; $\lvert z\rvert > \infty, k > 0$
(3)	$u(n)$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert > 1$
(4)	$-u(-n-1)$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert < 1$
(5)	$n\,u(n)$	$\frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
(6)	$a^n u(n)$	$\frac{1}{1-az^{-1}} = \frac{z}{(z-a)}$	$\lvert z\rvert > \lvert a\rvert$
(7)	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}} = \frac{z}{(z-a)}$	$\lvert z\rvert < \lvert a\rvert$
(8)	$n\,a^n u(n)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
(9)	$-n\,a^n u(-n-1)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert < \lvert a\rvert$
(10)	e^{-an}	$\frac{z}{z-e^{-a}}$	$\lvert z\rvert < \lvert e^{-a}\rvert$
(11)	$n^2 u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
(12)	ne^{-an}	$\frac{ze^{-a}}{(z-e^{-a})^2}$	$\lvert z\rvert > \lvert e^{-a}\rvert$
(13)	$\sin\omega_0 n$	$\frac{z\sin\omega_0}{z^2-2z\cos\omega_0+1}$	$\lvert z\rvert > 1$
(14)	$\cos\omega_0 n$	$\frac{z(z-\cos\omega_0)}{z^2-2z\cos\omega_0+1}$	$\lvert z\rvert > 1$

(15)	$\sin(h\omega_0 n)$	$\dfrac{z \sinh\omega_0}{z^2 - 2z \cosh\omega_0 + 1}$	$\|z\| > 1$
(16)	$\cos(h\omega_0 n)$	$\dfrac{z(z - \cosh\omega_0)}{z^2 - 2z \cosh\omega_0 + 1}$	$\|z\| > 1$
(17)	$e^{-an} \sin\omega_0 n$	$\dfrac{ze^{-a} \sin\omega_0}{z^2 - 2ze^{-a} \cos\omega_0 + e^{-2a}}$	$\|z\| > \|e^{-a}\|$
(18)	$e^{-an} \cos\omega_0 n$	$\dfrac{z(z - e^{-a} \cos\omega_0)}{z^2 - 2ze^{-a} \cos\omega_0 + e^{-2a}}$	$\|z\| > \|e^{-a}\|$
(19)	$a^n \sin\omega_0 n$	$\dfrac{za \sin\omega_0}{z^2 - 2za \cos\omega_0 + a^2}$	$\|z\| > \|a\|$
(20)	$a^n \cos\omega_0 n$	$\dfrac{z(z - a \cos\omega_0)}{z^2 - 2za \cos\omega_0 + a^2}$	$\|z\| > \|a\|$

REVIEW QUESTIONS

(1) *Explain the region of convergence in Z-transform?*

(2) *Describe the properties of ROC in Z-transform?*

(3) *How the region of convergence is described in Z-transform? Explain with an example.*

5.8 CONSTRAINTS ON ROC FOR VARIOUS CLASSES OF SIGNALS

We discuss the constraints on ROC for six types of signals and they are,

(1) Finite duration, right sided (causal) signal.

(2) Finite duration, left sided (anticausal) signal.

(3) Finite duration, two sided (non-causal) signal.

(4) Infinite duration, right sided (causal) signal.

(5) Infinite duration, left sided (anticausal) signal.

(6) Infinite duration, two sided (non-causal) signal.

5.8.1 Type I : Finite Duration, Right Sided (Causal) Signal

Let, x[n] be a finite duration right-sided (causal) signal with N-samples, defined in the range $0 \le n \le (N - 1)$ as shown in Fig. 5.8.1(a).

From the Fig. 5.8.1(a), x[n] is defined as,

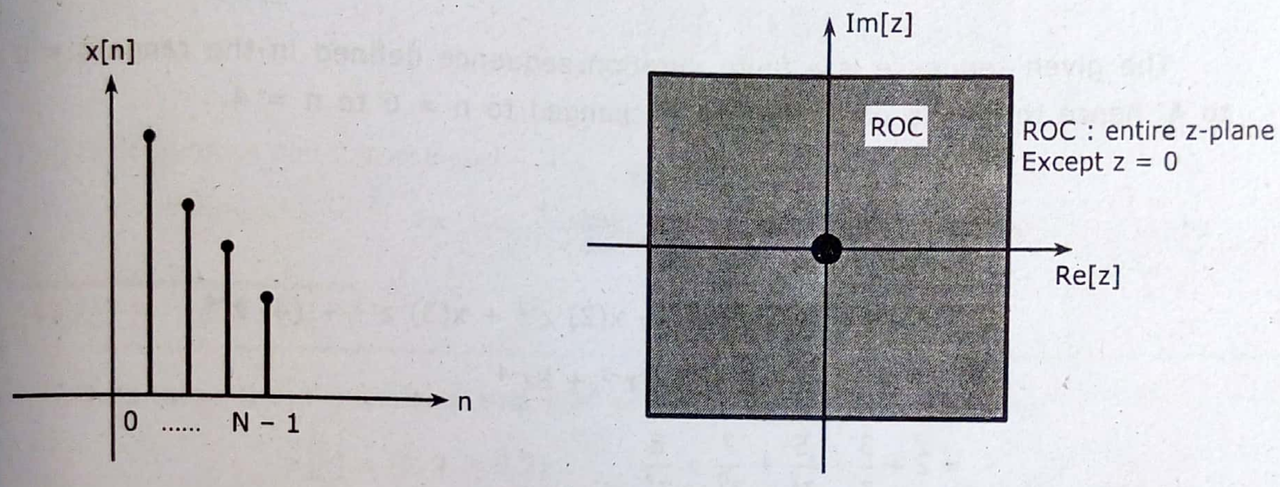
$$\therefore \quad x[n] = \{x(0), x(1), x(2), \ldots . x(N - 1)\}$$

Now, the Z - transform of x(n) is,

$$X(z) = \sum_{n=0}^{N-1} x[n]\, z^{-n} = x(0) + x(1)z^{-1} + x(2)\, z^{-2} + \ldots\ldots\ x(N-1)\, z^{-(N-1)}$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \ldots\ldots + \frac{x(N-1)}{z^{N-1}}$$

In the above summation, when z = 0, all the terms except the first term become infinite. Hence the X(z) exists for all values of z, except z = 0. Therefore, the ROC for finite duration right sided (or causal signal) is entire z-plane except z = 0 as shown in Fig. 5.8.1(b).



(a) Finite Duration Causal Signal

(b) ROC of a Causal Signal

Fig. 5.8.1

EXAMPLE PROBLEM 1

Determine the Z-transform and ROC of the discrete-time sequence,

$$x[n] = \{2, 3, 5, 7, 8\}$$
$$\uparrow$$
$$n = 0$$

SOLUTION

Given that,

$$x(n) = \{2, 3, 5, 7, 8\}$$
$$\uparrow$$
$$n = 0$$

i.e.,
$$x[0] = 1$$
$$x(1) = 3$$
$$x(2) = 5$$
$$x(3) = 7$$
$$x(4) = 8$$

And $x[n] = 0$ for $n < 0$ and for $n > 4$.

By the definition of z-transform,

$$z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

The given sequence is a finite duration sequence defined in the range n = 0 to 4, hence the limits of summation is changed to n = 0 to n = 4.

$$\therefore \quad X(z) = \sum_{n=0}^{4} x(n)\, z^{-n}$$

$$= x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + (4)\, z^{-4}$$

$$= 2 + 3z^{-1} + 5z^{-2} + 7z^{-3} + 8z^{-4}$$

$$= 2 + \frac{3}{z} + \frac{5}{z^2} + \frac{7}{z^3} + \frac{8}{z^4}$$

In X(z), when z = 0, except the first term all other terms will become infinite. Hence X(z) will be finite all values of z, except z = 0. Therefore, the ROC is entire z-plane except z = 0.

5.8.2 Type II : Finite Duration, Left Sided (Anticausal) Signal

Let, x[n] be a finite duration, left-sided anti-causal signal with N-samples, defined in the range $-(N - 1) \leq n \leq 0$, as shown in Fig. 5.8.2(a).

From the Fig. 5.8.2(a), x(n) can be defined as,

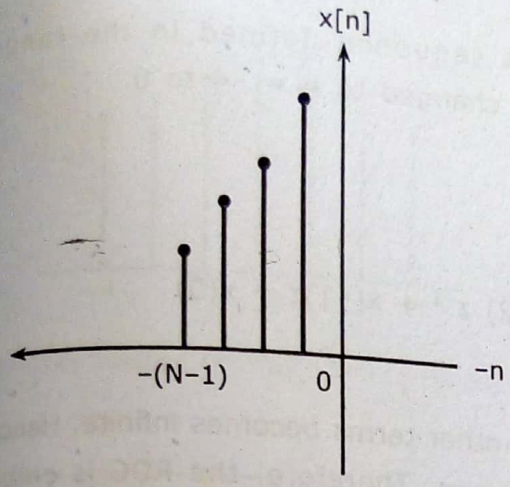
$$\therefore \quad x[n] = \{x(-(N-1)), \ldots\ldots, x(-2), x(-1), x(0)\}$$

Now, the z-transform of x(n) is,

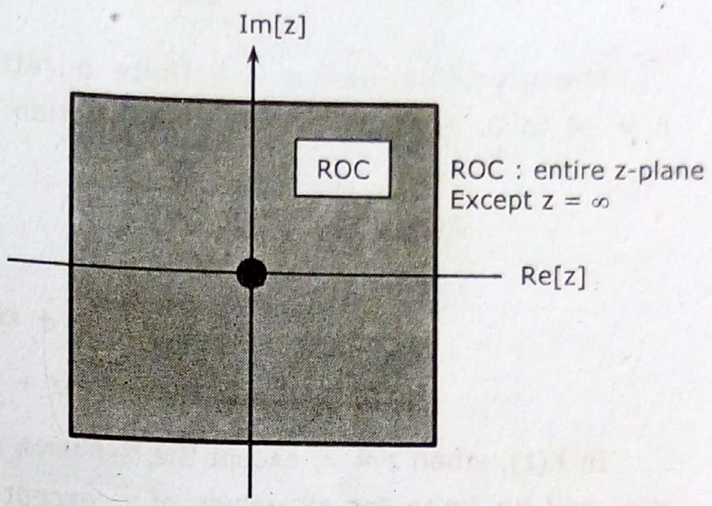
$$X(z) = \sum_{n=-(N-1)}^{0} x[n]\, z^{-n}$$

$$= x(-(N - 1))\, z^{(N-1)} + \ldots\ldots + x(-2)z^2 + x(-1)z + x(0)$$

In the above summation, when $z = \infty$, all the terms except the last term become infinite. Hence the X(z) exists for all values of z, except, $z = \infty$. Therefore, the ROC of X(z) is entire z-plane, except $z = \infty$.



(a) Finite Duration Anti-Causal Signal



(b) ROC of a Anti-Causal Signal

Fig. 5.8.2

EXAMPLE PROBLEM 1

Determine the z-transform and ROC of the discrete time sequence,

$$x[n] = \{6, 2, 3, 5, \underset{\substack{\uparrow \\ n=0}}{7}\}$$

SOLUTION

Given that,

$$x[n] = \{6, 2, 3, 5, \underset{\uparrow}{7}\}$$

i.e, $x(-4) = 6$

$x(-3) = 2$

$x(-2) = 3$

$x(-1) = 5$

$x(0) = 7$

And $x[n] = 0$ for $n < -4$ and for $n > 0$.

By the definition of z-transform,

$$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

The given sequence is a finite duration sequence defined in the range n = -4 to 0, hence the limits of summation is changed to n = -4 to 0.

$$X(z) = \sum_{n=-4}^{0} x[n]\, z^{-n}$$

$$= x(-4)\, z^4 + x\,(-3)\, z^3 + x(-2)\, z^{-2} + x(-1)z + x(0)$$

$$\therefore \quad = 6z^4 + 2z^3 + 3z^2 + 5z + 7$$

In X(z), when z = ∞, except the last term all other terms becomes infinite. Hence X(z) will be finite for all values of z, except z = ∞. Therefore, the ROC is entire z-plane except z = ∞.

5.8.3 Type III : Finite Duration, Two Sided (Non-causal) Signal

Let, x[n] be a finite duration, two-sided non-causal signal with N-samples, defined in the range $-M \le n \le +M$, where, $M = \frac{N-1}{2}$ as shown in Fig. 5.8.3(a),

From Fig. 5.8.3(a), we define x[n] as

$$x[n] = \{x(-M), \ldots\ldots, x(-2), x(-1), x(0), x(1), x(2), \ldots\ldots x(M)\}$$

Now, the z-transform for x[n] is,

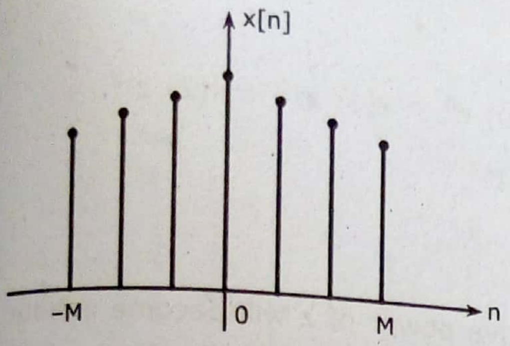
$$X(z) = \sum_{n=-M}^{+M} x[n]\, z^{-n}$$

$$= x(-M)\, z^M + \ldots + x(-2)z^2 + x(-1)\, z + x(0) + x(1)\, z^{-1}$$
$$+ \; x(2)\, z^{-2} + \ldots\ldots \; + x(M)\, z^{-M}$$

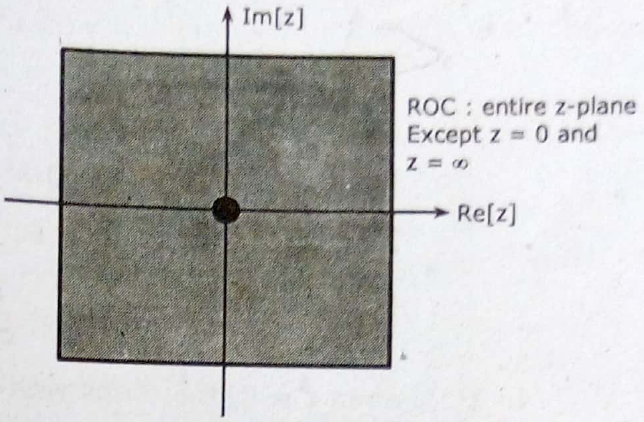
$$= x(-M)\, z^M + \ldots\ldots + x(-2)\, z^2 + x(-1)z + x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \ldots\ldots + \frac{x(M)}{z^M}$$

In the above summation, when z = 0, the terms with negative power of z attain infinity and when z = ∞, the terms with positive power of z attain infinity. Hence X(z) converges for all values of z, except z = 0 and z = ∞.

Therefore, the ROC is entire z-plane, except $z = 0$ and $z = \infty$, as shown in Fig. 5.8.3(b).



(a) Finite Duration Two-Sided Non-Causal Signal



(b) ROC of a Non-causal Signal

Fig. 5.8.3

EXAMPLE PROBLEM 1

Determine the Z-transform and ROC for the discrete-time sequence,

$$x[n] = \{2, 3, \underset{\substack{\uparrow \\ n=0}}{5}, 6, 7\}$$

SOLUTION

Given that,

$$x[n] = \{2, 3, \underset{\substack{\uparrow \\ n=0}}{5}, 6, 7\}$$

i.e.,

$$x(-2) = 2$$

$$x(-1) = 3$$

$$x(0) = 5$$

$$x(1) = 6$$

$$x(2) = 7$$

And $x[n] = 0$ for $n < -2$ and for $n > 2$.

By the definition of z-transform,

$$z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

The given sequence is a finite duration sequence defined in the range n = −2 to +2, hence the limits of summation is changed to n = −2 to n = 2.

$$X(z) = \sum_{n=-2}^{2} x[n]z^{-n}$$

$$= x(-2)\, z^2 + x(-1)\, z^1 + x(0)\, z^0 + x(1)\, z^{-1} + x(2)\, z^{-2}$$

$$= 2z^2 + 3z + 5 + 6z^{-1} + 7z^{-2}$$

$$\therefore \quad = 2z^2 + 3z + 5 + \frac{6}{z} + \frac{7}{z^2}$$

In X(z), when z = 0, the terms with negative power of z will become infinite and when z = ∞, the terms with positive power of z will become infinite. Hence X(z) will be finite for all values of z except when z = 0 and z = ∞. Therefore, the ROC is entire z-plane except z = 0 and z = ∞.

5.8.4 Type IV : Infinite Duration, Right Sided (Causal) Signal

Let x[n] be infinite duration, right-sided (causal) signal and is as shown in Fig. 5.8.4(a).

Let, $x(n) = a^n \; ; n \geq 0$

Now, the z-transform of x(n) is,

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n} = \sum_{n=0}^{\infty} a^n\, z^{-n} = \sum_{n=0}^{\infty} (a\, z^{-1})^n$$

Using infinite geometric series sum formula,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad ; \text{ if } 0 < |a| < 1$$

We get,

$$X(z) = \frac{1}{1 - az^{-1}}$$

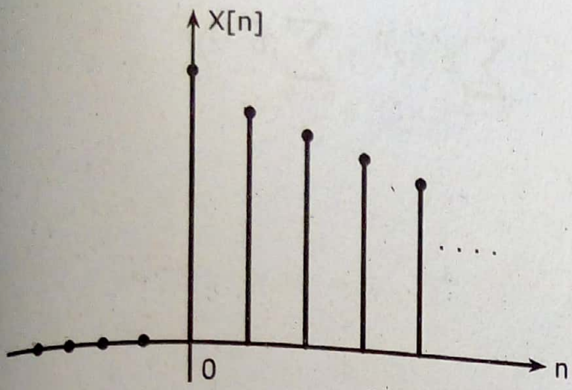
$$\therefore \quad X(z) = \frac{1}{1 - az^{-1}} = \frac{1}{1 - \frac{a}{z}} = \frac{1}{\frac{z-a}{z}} = \frac{z}{z-a}$$

Here the condition to be satisfied for the convergence of X(z) is,

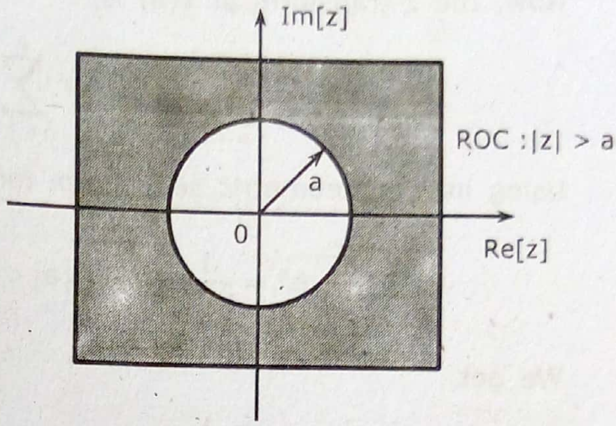
$$0 < |az^{-1}| < 1$$

$$\therefore \quad |a\, z^{-1}| < 1 \Rightarrow \frac{|a|}{|z|} < 1 \Rightarrow |z| > |a|$$

The term |a|, represent a circle of radius a in z-plane as shown in Fig. 5.8.4(b). From the above analysis we can say that, X(z) converges for all points external to the circle of radius "a" in z-plane. Therefore, the ROC of X(z) is exterior of the circle of radius a in z-plane as shown in Fig. 5.8.4(b).



(a) Infinite Duration Causal Signal



(b) ROC of a Infinite Duration Causal Signal

Fig. 5.8.4

EXAMPLE PROBLEM 1

Determine the Z-transform and its ROC for a unit-step sequence u(n).

SOLUTION

Unit step sequence u[n] is defined by,

$$u[n] = \begin{cases} 1 \; ; & \text{for } n \geq 0 \\ 0 \; ; & \text{for } n < 0 \end{cases}$$

$$z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=0}^{\infty} u[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = \frac{1}{1 - z^{-1}}$$

$$= \frac{1}{1 - 1/z} = \frac{z}{z - 1}$$

Here the condition for convergence is, $0 < |z^{-1}| < 1$.

$$\therefore \quad \left|z^{-1}\right| < 1 \Rightarrow \frac{1}{|z|} < 1 \Rightarrow |z| > 1$$

The term |z| = 1 represents a circle of unit radius in z-plane. Therefore, the ROC is exterior of unit circle in z-plane.

5.8.5 Type V : Infinite Duration, Left Sided (Anticausal) Signal

Let x[n] be infinite duration, left-sided anti-causal signal and is as shown in Fig. 5.8.5(a).

Let, $x(n) = b^n \; ; \quad n \leq 0$

Now, the z-transform of x(n) is,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{0} b^n z^{-n} = \sum_{n=0}^{\infty} b^{-n} z^n = \sum_{n=0}^{+\infty} (b^{-1}z)^n$$

Using infinite geometric series sum formula,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \text{ if } 0 < |a| < 1$$

We get,

$$X(z) = \frac{1}{1 - b^{-1} z}$$

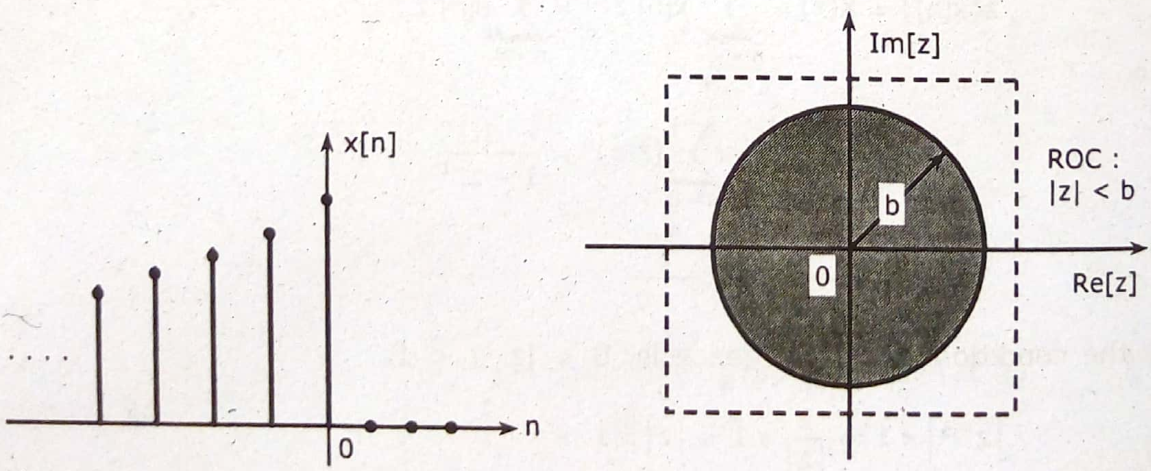
$$\therefore \quad X(z) = \frac{1}{1 - b^{-1} z} = \frac{1}{1 - \frac{z}{b}} = \frac{1}{\frac{b-z}{b}} = \frac{b}{b-z} = -\frac{b}{z-b}$$

Here the condition to be satisfied for the convergence of X(z) is,

$$0 < |b^{-1} z^{-1}| < 1$$

$$\therefore \quad |b^{-1} z| < 1 \Rightarrow \frac{|z|}{|b|} < 1 \Rightarrow |z| < |b|$$

The term |b| represent a circle of radius b in z-plane as shown in Fig 5.8.5. From the above analysis we can say that X(z) converges for all points internal to the circle of radius b in z-plane. Therefore, the ROC of X(z) is interior of the circle of radius b as shown in Fig. 5.8.5(b).



(a) Infinite Duration Anti-Causal Signal

(b) ROC of Infinite Duration Anti-Causal Signal

Fig. 5.8.4

EXAMPLE PROBLEM 1

Determine the z-transform and its ROC for a sequence $x[n] = (0.3)^n u[-n-1]$

SOLUTION

The $u[-n-1]$ is a discrete unit step signal, which is defined as,

$$u[-n-1] = 0 \quad ; \quad \text{for } n \geq 0$$

$$= 1 \quad ; \quad \text{for } n \leq -1$$

$$\therefore \quad x[n] = 0 \quad ; \quad \text{for } n \geq 0$$

$$= 0.3^n \quad ; \quad \text{for } n \leq -1$$

By the definition of z-transform,

$$z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=-\infty}^{-1} 0.3^n z^{-n}$$

$$= \sum_{n=1}^{\infty} 0.3^{-n} z^n = \sum_{n=1}^{\infty} (0.3^{-1} z)^n$$

$$= \sum_{n=0}^{\infty} (0.3^{-1} z)^n - 1 \qquad \left[\because (0.3^{-1} z)^0 = 1\right]$$

$$= \frac{1}{1 - (0.3^{-1} z)} - 1$$

Using infinite geometric series sum formula,

$$= \frac{1}{1 - \dfrac{z}{0.3}} - 1 = \frac{0.3}{0.3 - z} - 1 = \frac{0.3 - 0.3 + z}{0.3 - z} = \frac{z}{0.3 - z} = \frac{z}{z - 0.3}$$

Here the condition for convergence is, $0 < |0.3^{-1} z| < 1$.

$$\therefore \quad |0.3^{-1}| < 1$$

$$\Rightarrow \quad \frac{|z|}{0.3} < 1$$

$$\Rightarrow \quad |z| < 0.3$$

The term $|z| = 0.3$, represents a circle of radius 0.3 in z-plane. Therefore, the ROC is interior of the circle of radius 0.8 in z-plane.

5.8.6 Type VI : Infinite Duration, Two-Sided (Non-Causal) Signal

Let x[n] be infinite duration, two-sided (non-causal) signal and is as shown in Fig. 5.8.6(a).

Let, $$x[n] = a^n u[n] + b^n u[-n] \quad (\text{Let } a > b)$$

Now, the z-transform of x(n) is,

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{0} b^n z^{-n} + \sum_{n=0}^{+\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{+\infty} b^{-n} z^n + \sum_{n=0}^{+\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{+\infty} (b^{-1} z)^n + \sum_{n=0}^{+\infty} (a\, z^{-1})^n$$

Using Infinite geometric series sum formula (IGSS),

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}; \text{ if, } 0 < |a| < 1$$

$$\therefore \quad X(z) = \frac{1}{1 - b^{-1} z} + \frac{1}{1 - a\, z^{-1}} \quad ; \quad \text{if } |b^{-1} z| < 1 \text{ and } |a z^{-1}| < 1$$

z-transform converges, if

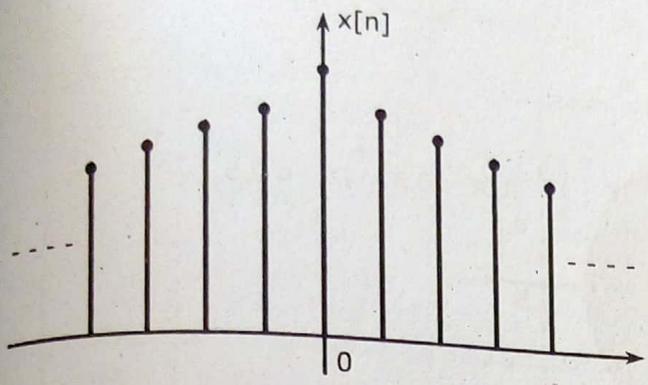
$$\therefore \quad |z| < |b|$$

And $$|a| < |z|$$

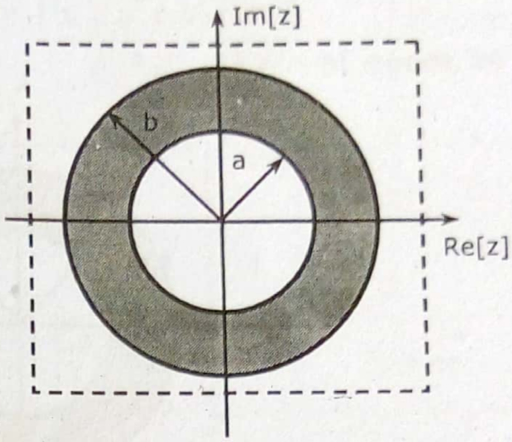
$$\Rightarrow \quad |a| < |z| < |b|$$

The term |b| and |a| represents a circle of radius r_1 in z-plane. If |b| > |a| then there will be region between two circles as shown in Fig. 5.8.6(b). Now the X(z) will converge for all points in the region between two circles (because the first term of X(z) converges for |z| < |b| and the second term of X(z) converges for |z| > |a|).

Hence the ROC is the region between two circles of radius a and b as shown in Fig. 5.8.6(b).



(a) Infinite Duration Non-causal Signal



(b) ROC of Infinite Duration Non-causal Signal

Fig. 5.8.6

EXAMPLE PROBLEM 1

Determine the z-transform and its ROC for a sequence,

$x[n] = (0.3)^n u[n] + (0.5)^n u[-n-1]$

SOLUTION

Given Data : $x[n] = (0.3)^n u[n] + (0.5)^n u[-n-1]$

$$X(z) = Z\{x[n]\}$$

$$= Z\{0.3^n u[n] + 0.5^n u[-n-1]\}$$

$$= Z\{0.3^n u[n]\} + Z\{0.5^n u[-n-1]\}$$

We have,

$$a^n u(n) \xleftrightarrow{z} \frac{1}{1-az^{-1}}, |z| > |a|$$

And $$-a^n u(-n-1) \xleftrightarrow{z} \frac{1}{1-az^{-1}}, |z| < |a|$$

$$X(z) = \frac{1}{1-0.3z^{-1}} + \frac{1}{1-0.5z^{-1}}$$

$$= \frac{z}{z-0.3} - \frac{z}{z-0.5} = \frac{z(z-0.5)-z(z-0.3)}{(z-0.3)(z-0.5)}$$

$$= \frac{z^2-0.5z-z^2+0.3z}{z^2-0.5z-0.3z+0.15} = \frac{-0.2z}{z^2-0.8z+0.15}$$

The condition for convergence of X(z) is,

$$| z | > 0.3 \text{ and } | z | < 0.5 \Rightarrow 0.3 > | z | > 0.5$$

$$\therefore \quad \text{ROC} : 0.3 < | z | < 0.5$$

As shown in the Fig. 5.8.7,

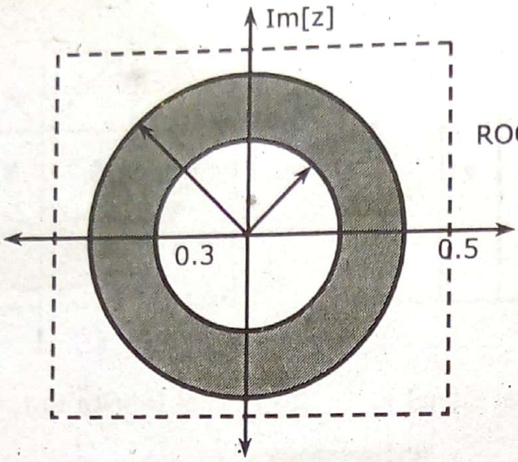
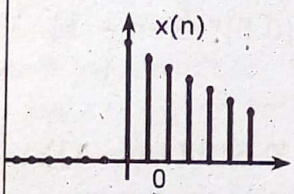
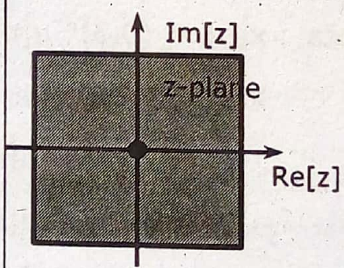
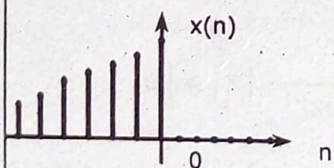
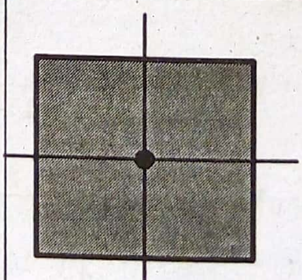
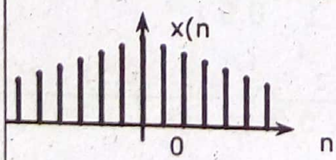
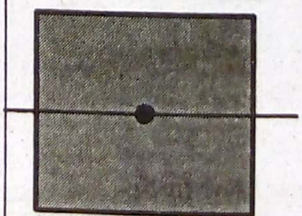
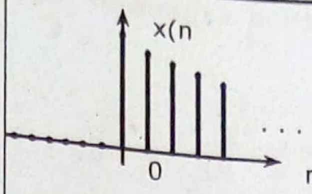
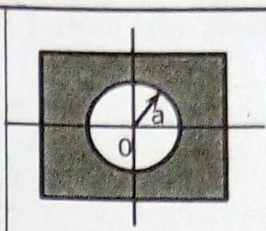
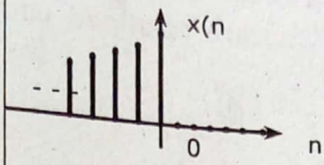
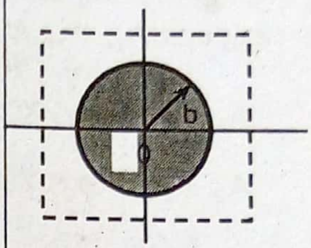
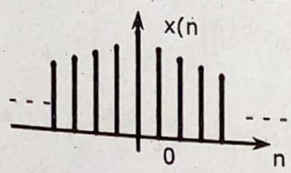
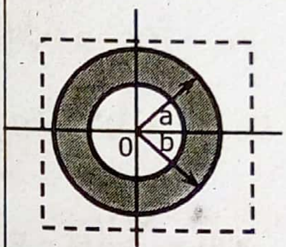


Fig. 5.8.7 ROC for Sequence x(n)

The Table 5.8.1 summarizes the ROC of six types of discrete time signals.

Table 5.8.1 Finite/Infinite Duration Causal, Anti-causal and Two-sided Signals with their ROCs

Sl.No.	Class of Signal	Sequence Waveform	ROC's	ROC Condition
(1)	Finite duration causal.			Entire z-plane except z = 0
(2)	Finite duration anti-causal.			Entire z-plane except z = ∞
(3)	Finite duration two-sided.			Entire z-plane except z = 0 and z = ∞.

(4)	Infinite duration causal.			Exterior of z-plane, $|z| > a$
(5)	Infinite duration non-causal.			Interior of z-plane, $|z| < b$.
(6)	Infinite duration two-sided.			ROC, $a < |z| < b$

REVIEW QUESTIONS

(1) *Explain all the constraints on ROC for various classes of signals?*

(2) *Describe the constraints for right sided and left sided finite duration signals?*

(3) *Find the z-transform of infinite and draw its ROC in the z-plane.*

5.9 INVERSE Z-TRANSFORM

The inverse z-transforms can be obtained using any of the following three methods,

(1) Contour integration (or residue method).

(2) Power series expansion method.

(3) Partial fraction expansion.

5.9.1 Contour Integration Method

The inverse z-transform of X(z) is given by,

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz \qquad \dots (5.9.1)$$

$$= \sum \left[\text{Residues of } X(z)\, z^{n-1} \text{ at poles inside closed path C}\right] \dots (5.9.2)$$

In general, using the partial fraction expansion method, the function $x(z)\, z^{n-1}$ is defined as,

$$X(z)\, z^{n-1} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + \ldots.. + \frac{A_N}{z-p_N}$$

Where, $A_1, A_2, A_3, \ldots.. A_N$ are the residues at poles $p_1, p_2, p_3, \ldots.. p_N$ respectively of $X(z)\, z^{n-1}$. The residues $A_1, A_2, A_3, \ldots..$ are obtained using the following equations,

$$\left.\begin{aligned} A_1 &= \left[(z-p_1)\, X(z)\, z^{n-1}\right]\Big|_{z=p_1} \\ A_2 &= \left[(z-p_2)\, X(z)\, z^{n-1}\right]\Big|_{z=p_2} \\ &\;\vdots \\ A_N &= \left[(z-p_N)\, X(z)\, z^{n-1}\right]\Big|_{z=p_N} \end{aligned}\right\} \quad \ldots (5.9.3)$$

Substituting Eq. (5.9.3) in Eq. (5.9.2), we get,

$$x[n] = \sum_{i=1}^{N} \left[(z-p_i)\, X(z)\, z^{n-1}\right]\Big|_{z=p_i}$$

where, N represents the number of poles lying inside the contour C. If the function $X(z)\, z^{n-1}$ contains poles of order m, i.e.,

$$X(z)\, z^{n-1} = \frac{A}{(z-p)^m}$$

then,

$$\mathrm{Res}\left\{X(z)\, z^{n-1}\right\}\Big|_{z=p} = \underset{z \to p}{\mathrm{Lt}} \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}} (z-p)^m\, X(z)\, z^{n-1}\right]$$

EXAMPLE PROBLEM 1

Find the inverse z-transform for the following z-transforms using contour integral method,

(a) $X(z) = \dfrac{z(3z-1)}{2(z-1)(z+1/2)}$; **ROC :** $|z| > 1$

(b) $X(z) = \dfrac{z(z+2)}{z^2 - 4z + 3}$,

(i) $|z| > 3$

(ii) $|z| < 1$

(iii) $1 < |z| < 3$

SOLUTION

(a) $X(z) = \dfrac{z(3z-1)}{2(z-1)(z+1/2)}$; ROC : $|z| > 1$

Given that,

$$X(z) = \frac{z(3z-1)}{z(z-1)(z+1/2)}$$

Its poles are at,

$z = 1$

And $z = \dfrac{-1}{2}$

Using contour integration method,

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1}$$

$$= \sum \text{Residues of } X(z)\, z^{n-1} \text{ at poles inside } |z| > 1$$

$$\text{Res}\left[X(z)\, z^{n-1}\right]\Big|_{z=1} = \underset{z \to 1}{\text{Lt}} \left\{(z-1)\frac{z(3z-1)}{2(z-1)(z+1/2)}\, z^{n-1}\right\}$$

$$= \underset{z \to 1}{\text{Lt}} \left\{\frac{z^n(3z-1)}{2(z+1/2)}\right\} = \left(\frac{2}{3}\right)$$

$$\text{Res}\left[X(z)\, z^{n-1}\right]\Big|_{z=-1/2} = \underset{z \to 1/2}{\text{Lt}} \left\{(z+1/2)\frac{(3z-1)}{2(z-1)(z+1/2)}\, z^{n-1}\right\}$$

$$= \underset{z \to 1/2}{\text{Lt}} \left\{\frac{z^n(3z-1)}{2(z-1)}\right\} = \left(\frac{5}{6}\right)\left(\frac{-1}{2}\right)^n$$

$$\therefore \quad x[n] = \left(\frac{2}{3}\right) + \left(\frac{5}{6}\right)\left(\frac{-1}{2}\right)^n$$

(b) $X(z) = \dfrac{z(z+2)}{z^2 - 4z + 3}$

Given that,

$$X(z) = \frac{z(z+2)}{z^2 - 4z + 3} = \frac{z(z+2)}{(z-1)(z-3)}$$

x(z) has poles located at z = 1 and z = 3.

$$\therefore \quad x[n] = \sum \text{Residues of } X(z)\, z^{n-1} \text{ at poles inside closed integral C}$$

(i) $|z| > 3$, both the poles are interior to C.

$$\therefore \quad X(z)\, z^{n-1} = \frac{z(z+2)}{z^2 - 4z + 3} \cdot z^{n-1} = \frac{(z+2)z^n}{z^2 - 4z + 3}$$

$$x[n] = \lim_{z \to 1} \frac{(z+2)z^n}{(z-3)} + \lim_{z \to 3} \frac{z^n(z+2)}{(z-1)}$$

$$= \left[\frac{-3}{2} + \frac{5}{2}(3)^n\right] u(n)$$

$$\therefore \quad x[n] = \left[\frac{5}{2}(3)^n - \frac{3}{2}\right] u(n)$$

(ii) $|z| < 1$, both the poles are exterior to C, so the residues are negative.

$$x[n] = \lim_{z \to 1} (-1) \frac{z^n(z+2)}{(z-3)} + \lim_{z \to 3} \frac{(-1)\, z^n(z+2)}{(Z-1)}$$

$$\therefore \quad x[n] = \frac{1}{2}(3 - (3)^n 5)\, u(-n-1)$$

(iii) $1 < |z| < 3$, the residue at z = 1 is positive and the residue at z = 3 is negative.

$$x[n] = \lim_{z \to 1} \frac{z^n(z+2)}{(z-2)} + \lim_{z \to 3} (-1) \frac{z^n(z+2)}{(Z-1)}$$

$$\therefore \quad x[n] = \frac{-3}{2} u(n) - \left[\frac{5}{2}(3)^n\right] u(-n-1)$$

5.9.2 Power Series Expansion Method

Any given z-transform, X(z) can be expressed as a rational function of z as shown below,

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \ldots\ldots + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \ldots\ldots + a_D z^{-D}}$$

On dividing the numerator polynomial N(z) by denominator polynomial D(z), we can express X(z) as a power series in z to obtain,

$$X(z) = C_0 + C_1 z^{-1} + C_2 z^{-2} + C_3 z^{-3} + \ldots + C_{-1} z^1 + C_{-2} z^2 + \ldots \qquad \ldots (5.9.4)$$

From the definition of z-transform, we have,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$X(z) = \ldots + x[-3]\, z^{3} + x[-2]\, z^{2} + x[-1]\, z^{1} + x[0] + x[1]\, z^{-1} + x[2]\, z^{-2} + \ldots\ldots \quad \ldots (5.9.5)$$

On comparing Eqs. (5.9.4) and (5.9.5), we notice that there is one-to-one correspondence between power series co-efficients C_n and the desired sequence values $x[0]$, $x[1]$, ... etc., that is,

$$x[n] = \begin{cases} C_{-n} & ; n < 0 \\ C_0 & ; n = 0 \\ C_n & ; n > 0 \end{cases}$$

In Eq. (5.9.4), X(z) is expressed as both positive and negative powers of z, in which the ROC is in between two circles of radius r_1 and r_2 in z-plane (i.e., ROC $r_1 < |z| < r_2$). However it is also possible to express X(z) as positive powers of z or as negative power of z.

Case I : When ROC is exterior of a circle of radius r ($|z| > r$).

In this case, the X(z) is expressed as a negative powers of z. That is,

$$X(z) = C_0 + C_1\, z^{-1} + C_2\, z^{-2} + C_3\, z^{-3} + \ldots\ldots \quad \ldots (5.9.6)$$

Case II : When ROC is interior of a circle of radius r ($|z| < r$).

In this case, the X(z) is expressed as a positive powers of z. That is,

$$X(z) = C_0 + C_{-1}\, z^{1} + C_{-2}\, z^{2} + \ldots\ldots \quad \ldots (5.9.7)$$

EXAMPLE PROBLEM 1

Determine the inverse z-transform of following z-transforms using power series method.

(i) $X(z) = \dfrac{4z}{z^2 - 3z + 2}$ **; ROC : $|z| > 2$**

(ii) $X(z) = \dfrac{4z}{z^2 - 3z + 2}$ **; ROC : $|z| < 2$**

SOLUTION

(i) $X(z) = \dfrac{4z}{z^2 - 3z + 2}$; ROC : $|z| > 2$

Given that,

$$X(z) = \frac{4z}{z^2 - 3z + 2}$$

$$X(z) = \frac{4z}{(z^2 - 3z + 2)} = \frac{4z}{(z-1)(z-2)}$$

For ROC : $|z| > 2$, $x[n]$ is a right-sided sequence where $n \geq 0$. Hence, the long division is done in such a way that $X[z]$ is expressed in negative powers of z i.e., z^{-1}.

$$\begin{array}{r l}
 & 4z^{-1} + 12z^{-2} + 28z^{-3} + \ldots \\
z^2 - 3z + 2\,\big) & 4z \\
 & 4z - 12 + 8z^{-1} \\
 & \quad 12 - 8z^{-1} \\
 & \quad 12 - 36z^{-1} + 24z^{-2} \\
 & \qquad 28z^{-1} - 24z^{-2} \\
 & \qquad 28z^{-1} - 84z^{-2} + 56z^{-3}
\end{array}$$

$$X(z) = 4z^{-1} + 12z^{-2} + 28z^{-3} + \ldots \qquad \ldots (5.9.8)$$

Comparing Eq. (5.9.8) with Eq. (5.9.6), we get,

$$X[n] = \{0, 4, 12, 28, \ldots\}$$
$$\uparrow$$
$$n = 0$$

(ii) $X(z) = \dfrac{4z}{z^2 - 3z + 2}$; ROC : $|z| < 2$

Given that,

$$X(z) = \frac{4z}{(z^2 - 3z + 2)}$$

For ROC : $|z| < 2$, $x[n]$ sequence is left-handed sequence where $n \leq 0$. The long division is done in such a way that $X(z)$ is expressed in positive powers of z.

$$\begin{array}{r l}
 & 2z + 3z^2 + \frac{7}{2}z^3 \\
2 - 3z + z^2\,\big) & 4z \\
 & 4z - 6z^2 + 2z^3 \\
 & \quad 6z^2 - 2z^3 \\
 & \quad 6z^2 - 9z^3 + 3z^4 \\
 & \qquad 7z^3 - 3z^4 \\
 & \qquad 7z^3 - \frac{21}{2}z^4 + \frac{7}{2}z^5
\end{array}$$

$$X[z] = 2z + 3z^2 + \frac{7}{2}z^3 + \ldots \qquad \ldots (5.9.9)$$

Comparing Eq. (5.9.9) with Eq. (5.9.7), we get,

$$x[n] = \left\{\ldots \frac{7}{2}, 3, 2, 0\right\}$$
$$\uparrow$$
$$n = 0$$

5.9.3 Partial Fraction Expansion Method

Inverse z-transform can be obtained using partial fraction expansion method by the following step,

Step I : Represent both numerator and denominator polynomials in z of X(z) as positive powers of z.

Step II : Factorize the denominator polynomial into factors that contain the poles $P_1, P_2, \ldots P_n$ of X(z),

$$\frac{X(z)}{z} = \frac{N(z)}{D(z)} = \frac{N(z)}{(z - P_1)(z - P_2)(z - P_2)\ldots(z - P_N)}$$

Step III : In this step the residues are evaluated as poles.

In this step consider two cases,

(1) **Distinct Poles :** Suppose that the poles $p_1, p_2, \ldots p_N$ are all different (distinct), then using the partial fraction expansion, we may write,

$$\frac{X(z)}{z} = \frac{A_1}{(z - p_1)} + \frac{A_2}{(z - p_2)} + \ldots + \frac{A_N}{(z - p_N)} \qquad \ldots (5.9.10)$$

Where, $A_1, A_2, \ldots A_N$ is residues of poles $p_1, p_2, \ldots p_N$ and are defined as,

$$A_k = \left[(z - p_k)\frac{X(z)}{z}\right]_{z = p_k} \qquad (k = 1, 2, \ldots N)$$

(2) **Multiple Order Poles) :** If X(z) has a pole of multiplicity m, i.e., if it contains the factor (z k)^m in its denominator, using the partial fraction expansion, we may write,

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{(z - p_2)^2} + \ldots + \frac{A_N}{(z - p_k)^m}$$

The co-efficient A_k are computed using the following formula,

$$A_k = \frac{1}{(m - k)!}\left[\frac{d^{m-k}}{dz^{m-k}}\left((z - p_k)^m \frac{X(z)}{z}\right)\right]_{z = p_k}$$

Step IV : Here, the evaluation of inverse z-transform is performed.

The partial fraction expansion in Eq. (5.9.10), may be rewritten as,

$$X(z) = \frac{A_1 z}{z(1 - p_1 z^{-1})} + \frac{A_2 z}{z(1 - p_2 z^{-1})} + \ldots + \frac{A_N z}{z(1 - p_N z^{-1})}$$

$$X(z) = A_1\frac{1}{1 - p_1 z^{-1}} + A_2\frac{1}{1 - p_2 z^{-1}} + \ldots + A_N\frac{1}{1 - p_N z^{-1}} \qquad \ldots (5.9.11)$$

The inverse z-transform, $x[n] = Z^{-1}[X(z)]$, can be obtained by inverting each term in Eq. (5.9.10) by using the formula,

$$Z^{-1}\left[\frac{1}{1-p_k z^{-1}}\right] = \begin{cases} (p_k)^n u(n) & ; \text{ if ROC } |z| > |p_k| \text{ (causal signals)} \\ -(p_k)^n u(-n-1) & ; \text{ if ROC } |z| < |p_k| \text{ (anticausal signals)} \end{cases} \quad \ldots (5.9.12)$$

EXAMPLE PROBLEM 1

Find the inverse z-transform of,

$$X(z) = \frac{1 - z^{-1} + z^{-2}}{(1-(1/2)z^{-1})(1-2z^{-1})(1-z^{-1})} \text{ with ROC } 1 < |z| < 2.$$

SOLUTION

Given that,

$$X(z) = \frac{1 - z^{-1} + z^{-2}}{(1-(1/2)z^{-1})(1-2z^{-1})(1-z^{-1})}$$

STEP I : Represent polynomials in positive powers of z ; Negative power of z are eliminated by multiplying both the numerator and denominator of the above equation by z^3. This results in,

$$X(z) = \frac{z^3 - z^2 + z}{(z-1/2)(z-2)(z-1)}$$

STEP II : $$\frac{X(z)}{z} = \frac{z^2 - z + 1}{(z-1/2)(z-2)(z-1)}$$

STEP III : Evaluate residues at their respective poles.

We use partial fraction expansion to write,

$$\frac{X(z)}{z} = \frac{A_1}{(z-1/2)} + \frac{A_2}{(z-2)} + \frac{A_3}{(z-1)}$$

Where,

$$A_1 = \left(z - \frac{1}{2}\right)\frac{X(z)}{z}\bigg|_{z=1/2} = \frac{\left(\frac{1/2}{2}\right)^2 - \left(\frac{1}{2}\right) + 1}{\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-1\right)} = \frac{\left(\frac{3}{4}\right)}{\left(\frac{3}{4}\right)} = 1$$

$$A_2 = (z-2)\frac{X(z)}{z}\bigg|_{z=2} = \frac{4-2+1}{\left(2-\frac{1}{2}\right)(2-1)} = \frac{3}{(3/2)} = 2$$

$$A_3 = (z-1)\left.\frac{X(z)}{z}\right|_{z=1}$$

$$= \frac{1-1+1}{\left(\frac{1}{2}\right)(-1)}$$

$$A_3 = -2$$

$$\therefore \quad \frac{X(z)}{z} = \frac{1}{z-1/2} + \frac{2}{z-2} - \frac{2}{z-1}$$

Step IV : Evaluate inverse z-transform,

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{2z}{z-2} - \frac{2z}{z-1}$$

$$X(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}} + \frac{2}{1-2z^{-1}} - \frac{2}{1-z^{-1}} \quad \text{... (5.9.13)}$$

The ROC, $1 < |z| < 2$, is a ring in the z-plane. The first term in Eq. (5.9.13) has pole at $z = 1/2$. Here the ROC has a radius greater than this pole, so this corresponds to causal (right-sided) signal. Hence, by using Eq. (5.9.12), we get,

$$\therefore \quad \frac{1}{1-(1/2)z^{-1}} \xleftrightarrow{z^{-1}} \left(\frac{1}{2}\right)^n u(n)$$

The second term in Eq. (5.9.13) has a pole at $z = 2$. Here the ROC has a radius less than this pole, so this pole corresponds to the anticausal (left-sided) signal. Hence, by using Eq. (5.9.12), we get,

$$\therefore \quad \frac{2}{1-2z^{-1}} \xleftrightarrow{z^{-1}} -2(2)^n u(-n-1)$$

The third term in Eq. (5.9.13) has a pole at $z = 1$. Here the ROC has a radius greater than this pole, so this pole corresponds to causal (right-sided) signal. Hence by using Eq. (5.9.12), we get,

$$\frac{2}{1-z^{-1}} \xleftrightarrow{z^{-1}} 2u(n)$$

$$\therefore \quad x(n) = \left(\frac{1}{2}\right)^n u(n) - 2(2)^n u(-n-1) - 2u(n)$$

EXAMPLE PROBLEM 2

Determine the inverse z-transform of,

$$X(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

If,

(i) ROC : $|z| > 3$

(ii) ROC : $|z| < 0.5$

(iii) ROC : $0.5 < |z| < 3$

SOLUTION

Given that,

$$X(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

STEP I : Convert X(z) to positive powers of z,

$$X(z) = \frac{3 - \left(\frac{4}{z}\right)}{1 - 3.5\left(\frac{1}{z}\right) + 1.5\left(\frac{1}{z^2}\right)}$$

$$= \frac{\frac{(3z-4)}{z}}{\frac{(z^2 - 3.5z + 1.5)}{z^2}}$$

$$= \frac{3z^2 - 4z}{z^2 - 3.5z + 1.5}$$

$$\Rightarrow \quad \frac{X(z)}{z} = \frac{3z - 4}{z^2 - 3.5z + 1.5}$$

STEP II : Factorize the denominator polynomial,

$$\frac{X(z)}{z} = \frac{3z - 4}{(z - 0.5)(z - 3)}$$

STEP III : Evaluate residues at their respective poles.

Using partial fraction expansion, the equation in step (2) can be written as,

$$\frac{X(z)}{z} = \frac{A_1}{(z-0.5)} + \frac{A_2}{(z-3)}$$

Where,

$$A_1 = (z-0.5)\left.\frac{X(z)}{z}\right|_{z=0.5} = \frac{3(0.5)-4}{0.5-3} = \frac{-2.5}{-2.5} = 1$$

$$A_2 = (z-3)\left.\frac{X(z)}{z}\right|_{z=3} = \frac{3(3)-4}{3-0.5} = \frac{5}{2.5} = 2$$

$$\therefore \quad \frac{X(z)}{z} = \frac{1}{(z-0.5)} + \frac{2}{z-3}$$

STEP IV : Evaluate inverse z-transforms

$$X(z) = \frac{z}{z-0.5} + \frac{2z}{z-3} = \frac{1}{1-0.5z^{-1}} + \frac{2}{1-3z^{-1}} \qquad \text{... (5.9.14)}$$

X(z) has poles located at z = 0.5 and z = 3

(i) The ROC : $|z| > 3$, is the region in the z-plane outside the outermost pole located at z = 3, so both poles correspond to causal (right-sided) signals. Therefore,

$$\frac{1}{1-0.5z^{-1}} \xleftrightarrow{z^{-1}} (0.5)^n\, u(n)$$

And

$$\frac{2}{1-3z^{-1}} \xleftrightarrow{z^{-1}} 2(3)^n\, u(n)$$

$$\therefore \quad x[n] = 2(3)^n\, u(n) + (0.5)^n\, u(n) = [2(3)^n + (0.5)^n]\, u(n)$$

(ii) The ROC : $|z| < 0.5$, is the region in the z-plane inside the innermost pole located at z = 0.5, so both poles correspond to anticausal (left-sided) signals. Therefore,

$$\frac{1}{1-0.5z^{-1}} \xleftrightarrow{z^{-1}} -(0.5)^n\, u(-n-1)$$

$$\frac{2}{1-3z^{-1}} \xleftrightarrow{z^{-1}} -2(3)^n\, u(-n-1)$$

$$\therefore \quad x[n] = -2(3)^n\, u(-n-1) - (0.5)^n\, u(-n-1)$$

$$= [-2(3)^n - (0.5)^n]\, u(-n-1)$$

(iii) The ROC : $0.5 < |z| < 3$, is a ring in the z-plane. The pole of the first term in Eq. (5.9.14) is at $z = 0.5$. The ROC lies outside of this pole, so this pole corresponds to causal (right-sided) signal. Therefore,

$$\frac{1}{1 - 0.5z^{-1}} \xleftrightarrow{z^{-1}} (0.5)^n\, u(n)$$

The second term in Eq. (5.9.14) has a pole at $z = 3$. Here the ROC lies outside of this pole, so this pole corresponds to the anticausal (left-sided) signal. Therefore,

$$\frac{2}{1 - 3z^{-1}} \xleftrightarrow{z^{-1}} -2(3)^n\, u(-n-1)$$

$$\therefore \quad x[n] = -2(3)^n\, u(-n-1) + (0.5)^n\, u(n)$$

EXAMPLE PROBLEM 3

Find x[n] using partial fraction method,

$$X(z) = \frac{z(z+10)}{(z-1)(z^2-8z+20)}$$

SOLUTION

Given that, $X(z) = \dfrac{z(z+10)}{(z-1)(z^2-8z+20)}$

$$\frac{X(z)}{z} = \frac{(z+10)}{(z-1)(z-4+j2)(z-4-j2)}$$

$$= \frac{A_1}{(z-1)} + \frac{A_2}{(z-4+j2)} + \frac{A_3}{(z-4-j2)}$$

Where,

$$A_1 = (z-1)\left.\frac{X(z)}{z}\right|_{z=1} = \frac{(1+10)}{(1-4+j2)(1-4-j2)} = \frac{11}{(-3+j2)(-3-j2)}$$

$$= \frac{11}{3^2+2^2} = \frac{11}{13}$$

$$A_2 = (z-4+j2)\left.\frac{X(z)}{z}\right|_{z=4-j2} = \frac{4-j2+10}{(4-j2-1)(4-j2-4-j2)} = \frac{14-j2}{(3-j2)(-4j)}$$

$$= 0.98\ \angle 115.56° = 0.98\ e^{j115.56°}$$

$$A_3 = \text{Conjugate of } A_2 = 0.98\angle{-115.56^\circ} = 0.98\, e^{j115.56^\circ}$$

$$\therefore \quad X(z) = \frac{11\, z}{13\,(z-1)} + \frac{0.98e^{j115.56}\, z}{(z-4+j2)} + \frac{0.98e^{-j115.56^\circ}\, z}{(z-4-j2)}$$

Using z-transform pair we get the following inverse z-transform,

$$x[n] = \frac{11}{13}u(n) + [0.98e^{115.56^\circ}(4-j2)^n + 0.98e^{-j115.56^\circ}(4+j2)^n]\, u(n)$$

But, $115.56^\circ = 2$ radian

$$(4+j2)^n = (4.47)^n\, e^{j0.463n}$$

And $(4-j2)^n = (4.47)^n\, e^{-j0.463n}$

$$\therefore \quad x[n] = \frac{11}{13}u(n) + [0.98e^{j2}\, e^{-j0.463n}\,(4.47)^n + 0.98(4.47)^n\, e^{-j2}\, e^{0.4636n}]\, u(n)$$

$$= \frac{11}{13}u(n) + 0.98\times(4.47)^n[e^{j(2-0.4636n)} + e^{-j(2-0.4636n}]\, u(n)$$

$$\therefore \quad X[n] = \left[\frac{11}{13} + 1.96(4.47)^n \cos(2-0.4763n)\right] u(n)$$

EXAMPLE PROBLEM 4

Find the inverse z-transform of,

$$\mathbf{X(z) = \frac{z^3 - 6z^2 - 5z + 3}{z^2 - z - 2} \text{ with ROC } |z| < 1}$$

SOLUTION

Given that,

$$X(z) = \frac{z^3 - 6z^2 - 5z + 3}{z^2 - z - 2}$$

Since the order of the numerator polynomial is greater than the order of the denominator polynomial, the given rational z-transform is improper. We use long division to express X(z) as the sum of a proper rational function and a polynomial in z,

$$\begin{array}{r|l} & z - 5 \\ z^2 - z - 2 & z^3 - 6z^2 - 5z + 3 \\ & \underline{z^3 - z^2 - 2z} \\ & -5z^2 - 3z + 3 \\ & \underline{-5z^2 + 5z + 10} \\ & -8z - 7 \end{array}$$

Thus we may write,

$$X(z) = (z-5) - \frac{(8z+7)}{z^2 - z - 2} = (z-5) - \left(\frac{8z+7}{(z+1)(z-2)}\right)$$

$$= (z-5) - \left[\frac{A_1}{(z+1)} + \frac{A_2}{(z-2)}\right]$$

Where,

$$A_1 = (z+1)\left.\frac{8z+7}{(z+1)(z-2)}\right|_{z=-1} = \frac{-8+7}{-3} = \frac{1}{3}$$

$$A_2 = (z-2)\left.\frac{8z+7}{(z+1)(z-2)}\right|_{z=2} = \frac{16+7}{3} = \frac{23}{3}$$

$$\therefore \quad X(z) = (z-5) - \left[\frac{(1/3)}{z+1} + \frac{(23/3)}{z-2}\right]$$

$$= (z-5) - \frac{(1/3)z^{-1}}{1+z^{-1}} - \frac{(23/3)z^{-1}}{1-2z^{-1}}$$

X(z) has poles at z = −1 and z = 2.

The ROC : |z| < 1, is the region in the z-plane inside z = 1, so the poles corresponds to anticausal (left-sided) signals. We know that,

$$\frac{1}{1+z^{-1}} \xleftrightarrow{z^{-1}} -(-1)^n u(-n-1) \qquad (\text{for } |z| < 1)$$

Using time shifting property,

$$-(1/3)(-1)^{n-1} u(-n) \xleftrightarrow{Z.T} \frac{(1/3)z^{-1}}{1+z^{-1}}$$

Similarly, we get,

$$-(23/3)(2)^{n-1} u(-n) \xleftrightarrow{Z.T} \frac{(23/3)z^{-1}}{1-2z^{-1}}$$

$$\therefore \quad Z^{-1}[X(z)] = Z^{-1}\left[(z-5) - \frac{(1/3)z^{-1}}{1+z^{-1}} - \frac{(23/3)z^{-1}}{1-2z^{-1}}\right]$$

$$X[n] = \delta(n+1) - 5\delta(n) + \left(\frac{1}{3}\right)(-1)^{n-1} u(-n) + \left(\frac{23}{3}\right)(2)^{n-1} u(-n)$$

REVIEW QUESTIONS

(1) *Write the steps required to find the inverse z-transform using contour integration method? Give one example.*

(2) *How the inverse z-transform can be obtained by using partial fraction expansion.*

5.10 PROPERTIES OF Z-TRANSFORMS

The z-transform possess a number of properties which are often useful for determining the transforms of certain discrete time signals which cannot be evaluated easily. These properties are also useful for obtaining the inverse z-transform of more complicated expressions. We will use a shorthand notation.

$$x[n] \xleftrightarrow{Z.T} X(z)$$

to indicate the relationship between discrete-time signal x[n] and its z-transform X[z].

5.10.1 Linearity

If, $x_1[n] \xleftrightarrow{Z.T} X_1(z)$; with ROC = R_1

And, $x_2[n] \xleftrightarrow{Z.T} X_2(z)$; with ROC = R_2

Then, $ax_1[n] + bx_2[n] \xleftrightarrow{Z.T} aX_1(z) + bX_2(z)$; with ROC contaning $R_1 \cap R_2$

The z-transform of a weighted sum of two signals is equal to the weighted sum of individual z-transform.

PROOF

The z-transform of $ax_1[n] + bx_2[n]$ is given by,

$$Z[ax_1[n] + bx_2[n]] = \sum_{n=-\infty}^{\infty} [ax_1[n] + bx_2[n]]\, z^{-n}$$

$$= a\sum_{n=-\infty}^{\infty} x_1[n]\, z^{-n} + b\sum_{n=-\infty}^{\infty} x_2[n]\, z^{-n}$$

$$\therefore \quad Z[ax[n] + bx_2[n]] = aX_1(z) + bX_2(z) \qquad \ldots (5.10.1)$$

5.10.2 Time Shifting

If, $x[n] \xleftrightarrow{Z.T} X(z)$; with ROC = R

Then, $x[n - n_0] \xleftrightarrow{Z.T} z^{-n_0} X(z)$; with ROC = R, except for the possible addition and deletion of z = 0 or z = ∞.

Shifting a signal in time domain is equivalent to multiplication of the z-transform with z^{-n_0}.

PROOF

The z-transform of $x[n - n_0]$ is given by,

$$Z[x[n-n_o]] = \sum_{n=-\infty}^{\infty} x[n-n_0]\, z^{-n}$$

Let, $k = n - n_0$

Which gives,

$$n = k + n_0$$

$$k = -\infty \qquad [\because n = -\infty]$$

And $$k = \infty \qquad [\because n = \infty]$$

$$Z[x[n-n_o]] = \sum_{k=-\infty}^{\infty} x(k)\, z^{-(k+n_0)}$$

$$= z^{-n_0} \sum_{k=-\infty}^{\infty} x(k)\, z^{-k} = X(z) z^{-n_0}$$

$\therefore$ $$x[n-n_0] \xleftrightarrow{Z.T} X(z)\, z^{-n_0} \qquad \text{... (5.10.2)}$$

Similarly, $$x[n+n_0] \xleftrightarrow{Z.T} X(z)\, z^{n_0} \qquad \text{... (5.10.3)}$$

5.10.3 Time Expansion

The concept of time scaling in continuous-time cannot be extended directly to discrete time, since the discrete-time index is defined only for integer values. However, the discrete-time concept of time expansion can be defined. Let k be a positive integer and define the signal,

$$x_{<k>}[n] = \begin{cases} x\left(\dfrac{n}{k}\right) & ;\text{ if n is a multiple of k} \\ 0, & ;\text{ otherwise.} \end{cases}$$

If $$x[n] \xleftrightarrow{Z.T} X(z) \quad ; \quad \text{with ROC} = R$$

Then, $$x_{<k>}(n) \xleftrightarrow{Z.T} X(z^k) \quad ; \quad \text{with ROC} = R^{\frac{1}{k}}$$

That is, if R is $R_1 < |z| < R_2$ then the new ROC is $R_1 < |z|^k < R_2$ or $R_1^{1/k} < |z| < R_2^{1/k}$. Also, if X(z) has a pole or zero at $z = R_1$, then $X(z^k)$ has a pole (or zero) at $z = R_1^{1/k}$

PROOF

The z-transform of $x_{<k>}[n]$ is given by,

$$Z[x_{<k>}[n]] = \sum_{n=-\infty}^{\infty} x_{<k>}[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{k}\right) z^{-n}$$

Let, $q = n/k$, which gives $q = -\infty$ as $n = -\infty$ and $q = \infty$ as $n = \infty$. Therefore,

$$Z[x_{<k>}[n]] = \sum_{q=-\infty}^{\infty} x[q] z^{-kq}$$

$$= \sum_{q=-\infty}^{\infty} x[q]\,(z^k)^{-q} = X(z^k)$$

$$\therefore \quad x_{<k>}[n] \xleftrightarrow{Z.T} X(z^k) \qquad \ldots (5.10.4)$$

EXAMPLE PROBLEM 1

Find the Z-transform and ROC of the signal,

$$x[n] = a^{n/3}\, u\left(\frac{n}{3}\right) = \begin{cases} a^{n/3}, & n = 0, 3, 6, \ldots \\ 0, & \text{elsewhere} \end{cases}$$

where $|a| < 1$.

SOLUTION

Consider the given signal,

$$x[n] = a^{n/3}\, u\left(\frac{n}{3}\right) = f\left(\frac{n}{3}\right) = f_{(3)}(n)$$

Where, $f(n) = a^n\, u(n)$ and its z-transform is given by,

$$a^n\, u(n) \longleftrightarrow \frac{1}{1 - az^{-1}}, \; |z| > |a|$$

Now, using the time-expansion property, we obtain,

$$f_{(3)}(n) \longleftrightarrow F(z^3)$$

$$a^{n/3}\, u\left(\frac{n}{3}\right) \longleftrightarrow \frac{1}{1 - az^{-3}}, \; |z| > |a|^{1/3}$$

5.10.4 Scaling in the Z-Domain

If, $x[n] \xleftrightarrow{Z.T} X(z)$; with ROC = R

Then, $(z_0)^n x[n] \xleftrightarrow{Z.T} X\left(\frac{z}{z_0}\right)$; with ROC = $|z_0|R$

Scaling in the z-domain corresponds to multiplying the time domain sequence by $(z_0)^n$.

PROOF

The z-transform of $(z_0)^n x(n)$ is given by,

$$Z[(z_0)^n x(n)] = \sum_{n=-\infty}^{\infty} Z[(z_0)^n x(n)] z^{-n} = \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{z_0}\right)^{-n}$$

$$\therefore \quad Z[(z_0)^n x(n)] = X\left(\frac{z}{z_0}\right) \quad \text{... (5.10.5)}$$

Similarly, $$Z[(z_0)^{-n} x(n)] = x(z.z_0) \quad \text{... (5.10.6)}$$

5.10.5 Time Reversal

If, $x[n] \xleftrightarrow{Z.T} X(z)$; with ROC = R

Then, $x[-n] \xleftrightarrow{Z.T} X\left(\frac{1}{z}\right) = X(z^{-1})$; with ROC = $\frac{1}{R}$

The time-reversal (or reflection) corresponds to replacing z by z^{-1}. Hence, if R is of the form $R_1 < |z| < R_2$, the ROC of the time-reversal signal is $R_1 < 1/|z| < R_2$ or $1/R_2 < |z| < 1/R_1$.

PROOF

The z-transform of x[−n] is given by,

$$Z[x[-n]] = \sum_{n=-\infty}^{\infty} x[-n] z^{-n}$$

Let m = −n, which gives m = ∞ as n = − ∞ and m = −∞ as n = ∞. Therefore,

$$Z[x(-n)] = \sum_{m=-\infty}^{-\infty} x(m) z^{m} = \sum_{m=-\infty}^{\infty} x(m) \left(\frac{1}{z}\right)^{-m}$$

$$\therefore \quad Z[x(-n)] = X\left(\frac{1}{z}\right) = X(z^{-1}) \quad \text{... (5.10.7)}$$

An interesting consequence of the time-reversal property is that,

If x(n) is real and even, i.e., $x(n) = x(-n)$. Then,

$$X(z) = X(z^{-1}) = X\left(\frac{1}{z}\right) \quad \text{... (5.10.8)}$$

If x(n) is real and odd, i.e., $x(n) = -x(-n)$. Then,

$$X(z) = -X(z^{-1}) = -X\left(\frac{1}{z}\right) \quad \text{... (5.10.9)}$$

5.10.6 Differentiation in the Z-Domain

If, $x(n) \xleftrightarrow{Z.T} X(z)$; with ROC = R

Then, $nx(n) \xleftrightarrow{Z.T} -z\frac{dX(z)}{dz}$; with ROC = R

This property states that the Multiplication by n in the time domain corresponds to differentiation in z-domain. This operation does not change the location of the poles and hence, the ROC remains same.

PROOF

The z-transform of x(n) is given by,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Differentiating both sides of the above equation, w.r.t 'z' we get,

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dz}(z^{-n})$$

$$= \sum_{n=-\infty}^{\infty} [-nx(n)]\, z^{-n-1}$$

$$\frac{dX(z)}{dz} = -\frac{1}{z}\sum_{n=-\infty}^{\infty} [nx(n)]\, z^{-n}$$

$$\Rightarrow \quad -z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} [nx(n)]\, z^{-n} \quad \text{... (5.10.10)}$$

$$\therefore \quad Z[nx[n]] = -z\frac{dx(z)}{dz} \quad \text{... (5.10.11)}$$

Differentiating Eq. (5.10.10) again w.r.t 'z' we get,

$$\frac{d^2X(z)}{dz^2} = \frac{-1}{z}\sum_{n=-\infty}^{\infty}[n\,x[n]]\,(-nz^{-n-1})$$

$$= \frac{-1}{z}\sum[n^2\,x[n]]\,z^{-n}(z^{-1}) = \frac{1}{z^2}\sum_{n=-\infty}^{\infty}[n^2\,x(n)]z^{-n}$$

$$\Rightarrow \quad \frac{z^2 d^2X(z)}{dz^2} = z[n^2\,x[n]]$$

$$\therefore \quad z[n^k\,x[n]] = (-z)^k\frac{d^k\,x(z)}{dz^k} \qquad \ldots (5.10.12)$$

5.10.7 Convolution Property

If, $x_1[n] \xleftrightarrow{Z.T} X_1(z)$; with ROC = R_1

And $x_2[n] \xleftrightarrow{Z.T} X_2(z)$; with ROC = R_2

Then, $x_1[n] * x_2[n] \xleftrightarrow{Z.T} X_1(z)\,X_2(z)$ with ROC = $R_1 \cap R_2$

This property states that the convolution of two discrete-time signals is equivalent to product of their respective z-transforms.

PROOF

The z-transform of $(x_1[n] * x_2[n])$ is given by,

$$Z[x_1[n] * x_2[n]] = \sum_{n=-\infty}^{\infty}[x_1[n] * x_2[n]]\,z^{-n} = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}x_1[k]\,x_2[n-k]\right)z^{-n}$$

By interchanging the order of summation, we have,

$$Z[x_1[n] * x_2[n]] = \sum_{k=-\infty}^{\infty}x_1[k]\left(\sum_{n=-\infty}^{\infty}x_2[n-k]\,z^{-n}\right)$$

Applying the time-shifting property, the bracketed term becomes $X_2(z)\,z^{-k}$. Substituting this into the above equation gives,

$$Z[x_1[n] * x_2[n]] = \sum_{k=-\infty}^{\infty}x_1[k]\,(X_2(z)\,z^{-k}) = X_2(z)\sum_{k=-\infty}^{\infty}x_1[k]\,z^{-k}$$

$$\therefore \quad Z[x_1[n] * x_2[n]] = X_1(z)\,X_2(z) \qquad \ldots (5.10.13)$$

EXAMPLE PROBLEM 1

Find the Z-transform of the following signal using convolution property,

$$x[n] = \left(\frac{1}{3}\right)^n u(n) * \left(\frac{1}{5}\right)^n u(n)$$

SOLUTION

Let, $x_1[n] = (1/3)^n u(n)$

And $x_2(n) = (1/5)^n u(n)$

With $X_1(z)$ and $X_2(z)$ as their z-transforms respectively,

$$X_1(z) = Z[x_1(n)]$$

$$= Z\left[\left(\frac{1}{3}\right)^n u(n)\right]$$

$$= \frac{z}{z - \left(\frac{1}{3}\right)}$$

$$X_2(z) = Z[x_2(n)]$$

$$= Z\left[\left(\frac{1}{5}\right)^n u(n)\right]$$

$$= \frac{z}{z - \left(\frac{1}{5}\right)}$$

$$Z[x(n)] = X(z)$$

$$= Z[x_1(n) * x_2(n)]$$

$$= X_1(z)X_2(z)$$

$$\therefore \quad = \left[\frac{z}{z - \left(\frac{1}{3}\right)}\right]\left[\frac{z}{z - \left(\frac{1}{5}\right)}\right]$$

$$\therefore \quad X(z) = \frac{z^2}{z^2 - \left(\frac{8}{15}\right)z + \left(\frac{1}{15}\right)}.$$

5.10.8 Accumulation Property

If, $x[n] \xleftrightarrow{Z.T} X(z)$

Then, $\sum_{k=-\infty}^{n} x(k) \xleftrightarrow{Z.T} \frac{1}{1-z^{-1}} X(z)$

PROOF

From the definition of z-transform, we have,

$$Z\left[\sum_{k=-\infty}^{n} x[k]\right] = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{n} x(k)\, z^{-n}\right]$$

Substituting, $n - k = r$ (or $n = k + r$ or $k = n - r$) at RHS, we get,

$$Z\left[\sum_{k=-\infty}^{n} x[k]\right] = \sum_{k=-\infty-r}^{k=\infty-r} \sum_{r=n-(-\infty)}^{n-n} x[k]\, z^{-(k+r)}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{r=\infty}^{0} x[k]\, z^{-k}\, z^{-r}$$

$$= \left(\sum_{r=0}^{\infty} z^{-r}\right) \sum_{k=-\infty}^{\infty} x[k]\, z^{-k}$$

$$= \left(\sum_{r=0}^{\infty} z^{-r}\right) X(z)$$

Using infinite geometric series sum formula,

i.e., $\sum_{n=0}^{\infty} c_n = \frac{1}{1-c}$

We get,

$$Z\left[\sum_{k=-\infty}^{\infty} X[k]\right] = \frac{1}{1-z^{-1}} X(z) \qquad \text{... (5.10.14)}$$

5.10.9 Conjugation Property

If, $x[n] \xleftrightarrow{Z.T} X(z)$; with ROC = R

Then, $x^*[n] \xleftrightarrow{Z.T} X^*(z^*)$; with ROC = R

PROOF

The z-transform of $x^*(n)$ is,

$$Z[x^*[n]] = \sum_{n=-\infty}^{\infty} x^*[n]\, z^{-n} = \left[\sum_{n=-\infty}^{\infty} x[n]\,(z^*)^{-n}\right]^*$$

$$= [X(z^*)]^*$$

$$Z[x^*[n]] = X^*(z^*) \qquad \text{... (5.10.15)}$$

CASE I : If $x[n]$ is real and even, i.e.,

$$x[n] = x^*[n] = x[-n]$$

Then from Eqs. (5.10.8) and (5.10.15), we have,

$$X^*(z^*) = X(z) = X\left(\frac{1}{z}\right) \qquad \text{... (5.10.16)}$$

CASE II : If $x[n]$ is real and odd, i.e.,

$$x[n] = x^*[n] = -x[-n]$$

Then from Eqs. (5.10.9) and (5.10.15), we have,

$$X(z) = X^*(z^*) = -X\left(\frac{1}{z}\right) \qquad \text{... (5.10.17)}$$

5.10.10 Initial Value Theorem

If, $\quad x[n] = 0$ for $n < 0$ [i.e., x(n) is causal]

Then, $\quad \underset{n \to 0}{\text{Lt}}\; x[n] = x[0] = \underset{z \to \infty}{\text{Lt}}\; X(z)$

The initial value of a function x(t) i.e., x(0) directly from its z-transform x(z) without the need for finding the inverse z-transform of x(z).

PROOF

We know that,

$$Z[x[n]] = X[z] = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=0}^{\infty} x[n]\, z^{-n} \qquad [\because\ x(n) \text{ is causal}]$$

$$= x[0] + x[1]z^{-1} + x[2]\, z^{-2} + \ldots\ldots$$

Taking limit $z \to \infty$ on both the sides, we get,

$$\underset{z\to\infty}{\text{Lt}}\ X(z) = x[0] + \underset{z\to\infty}{\text{Lt}}\ x[1]z^{-1} + \underset{z\to\infty}{\text{Lt}}\ X[2]z^{-2} + \ldots..$$

$$\therefore \quad \underset{n\to 0}{\text{Lt}}\ x[n] = x[0] = \underset{z\to\infty}{\text{Lt}}\ X(z) \quad \ldots (5.10.18)$$

5.10.11 Final Value Theorem

If, $x[n] \xleftrightarrow{Z.T} X(z)$

Find, X(z) exists and no poles outside the unit circle and it has no double or higher order poles on the unit circle centered at the origin of the Z-plane, then,

$$\underset{n\to\infty}{\text{Lt}}\ x[n] = x[\infty]$$

$$= \underset{z\to 1}{\text{Lt}}\ (z-1)\,X(z)$$

PROOF

We know that,

$$Z[x[n]] = X[z] \quad \ldots (5.10.19)$$

$$Z[x[n+1]] = zX[z] - zx[0] \quad \ldots (5.10.20)$$

Substracting Eq. (5.10.19) from Eq. (5.10.20) we get,

$$Z[x[n+1]] - Z[x[n]] = z\,X(z) - z\,x(0) - X(z)$$

$$\sum_{n=0}^{\infty} x[n+1]\,z^{-n} - \sum_{n=0}^{\infty} x(n)\,z^{-n} = (z-1)\,X(z) - z\,x(0)$$

Taking limit $z \to 1$ on both the sides, we get,

$$\underset{z\to 1}{\text{Lt}}\ [(z-1)\,X(z) - z\,X(0)] = \underset{z\to 1}{\text{Lt}} \sum_{n=0}^{\infty} \{x[n+1] - x[n]\}\,z^{-n}$$

$$\underset{n\to\infty}{\text{Lt}}\ x[n] = x[\infty]$$

$$\therefore \quad = \underset{z\to 1}{\text{Lt}}\ (z-1)\,X(z)$$

The Table 5.10.1 lists various properties of z-transform,

Table 5.10.1 Properties of z-Transform

S.No.	Property	Signal	z-transform
(1)	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$
(2)	Time shifting	$x[n - n_0]$ $x[n + n_0]$	$z^{-n_0} X(z)$ $z^{n_0} X(z)$
(3)	Time expansion	$x_k[n] = \begin{cases} x(r), & n = rk \\ 0, & n \neq rk \end{cases}$ Where, r = integer	$X(z^k)$
(4)	Scaling in the z-domain	$(z_0)^n x[n]$	$X\left(\frac{z}{z_0}\right)$
(5)	Time reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$
(6)	Differentiation in z-domain	$n x[n]$	$-z\frac{dX(z)}{dz}$
(7)	Convolution	$x_1[n] * x_2[n]$	$X_1(z) X_2(z)$
(8)	Accumulation	$\sum_{k=-\infty}^{n} x(k)$	$\frac{1}{1-z^{-1}} X(z)$
(9)	Conjugation	$x^*[n]$	$X^*(z^*)$
(10)	Initial value theorem	$\underset{n \to 0}{Lt} \; x[n]$	$\underset{z \to \infty}{Lt} \; X(z)$
(11)	Final value theorem	$\underset{n \to \infty}{Lt} \; x[n]$	$\underset{z \to 1}{Lt} \; (z - 1) X(z)$

5.10.12 Solved Problems Using Properties of Z-Transform

EXAMPLE PROBLEM 1

Determine the Z-transform of the following signal,

$$x[n] = \frac{1}{5}(n + n^2)\left(\frac{1}{3}\right)^{n-1} u(n-1)$$

SOLUTION

Given,
$$x[n] = \frac{1}{5}(n + n^2)\left(\frac{1}{3}\right)^{n-1} u(n-1)$$

$$= \frac{1}{5}n\left(\frac{1}{3}\right)^{n-1} u(n-1) + \frac{1}{5}n^2\left(\frac{1}{3}\right)^{n-1} u(n-1)$$

We know that,

$$\left(\frac{1}{3}\right)^{n} u(n) \xleftrightarrow{Z.T} \frac{z}{z-(1/3)}$$

Using the time-shifting property, we get,

$$z\left[\left(\frac{1}{3}\right)^{n-1} u(n-1)\right] = z^{-1}\left[\frac{z}{z-(1/3)}\right] = \frac{1}{z-(1/3)}$$

Using the multiplication by n property, we have,

$$z\left[n\left(\frac{1}{3}\right)^{n-1} u(n-1)\right] = -z\frac{d}{dz}\left\{\frac{1}{z-(1/3)}\right\} = -z\left\{\frac{-1}{[z-1/3]^2}\right\} = \frac{z}{[z-(1/3)]^2}$$

Again using the multiplication by n property, we get,

$$z\left[n^2\left(\frac{1}{3}\right)^{n-1} u(n-1)\right] = -z\frac{d}{dz}\left\{\frac{z}{[z-(1/3)]^2}\right\} = -z\left\{\frac{[z-(1/3)]^2(1) - z.2\,[z-(1/3)]}{[z-(1/3)]^4}\right\}$$

$$= -z\left\{\frac{\left[z-\left(\frac{1}{3}\right)\right]\left[\left[z-\left(\frac{1}{3}\right)\right]-2z\right]}{[z-(1/3)]^4}\right\}$$

$$= \frac{-z[-z-(1/3)]}{[z-(1/3)]^3} = \frac{z[z+(1/3)]}{[z-(1/3)]^3}$$

Using the linearity property, we get,

$$z\left[\frac{1}{5}n\left(\frac{1}{3}\right)^{n-1} u(n-1) + \frac{1}{5}n^2\left(\frac{1}{3}\right)^{n-1} u(n-1)\right] = \frac{z}{[z-(1/3)]^2} + \frac{z[z+(1/3)]}{[z-(1/3)]^3}$$

$$= \frac{2z^2}{[z-(1/3)]^3}$$

EXAMPLE PROBLEM 2

Using the properties of Z-transform, find the Z-transform of the sequence,

$$x[n] = \begin{cases} 1 & ; 0 \le n \le N-1 \\ 0 & ; \text{elsewhere} \end{cases}$$

SOLUTION

Given, $$x[n] = \begin{cases} 1 & ; 0 \le n \le N-1 \\ 0 & ; \text{elsewhere} \end{cases}$$

x[n] can be represented as,

$$x[n] = u(n) - u(n-N)$$

We know that,

$$Z[u(n)] = \frac{z}{z-1}$$

Using the time shifting property, we have,

$$Z[u(n-N)] = z^{-N}\, z[u(n)] = z^{-N}\left(\frac{z}{z-1}\right)$$

Using the linearity property, we have,

$$Z[u(n) - u(n-N)] = Z[u(n)] - Z[u(n-N)]$$

$$= \frac{z}{z-1} - z^{N}\left(\frac{z}{z-1}\right) = \frac{z}{z-1}\left[1 - z^{-N}\right]$$

EXAMPLE PROBLEM 3

Using appropriate properties, find the Z-transform of the signal,

$$x[n] = 3(5)^n\, u(-n)$$

SOLUTION

Given, $$x[n] = 3(5)^n\, u(-n)$$

We know that,

$$Z[u(n)] = \frac{z}{z-1}\ ;\ \text{ROC} : |z| > 1$$

Using the time reversal property, we have,

$$Z[u(-n)] = Z[u(n)]_{z \to 1/z}$$

$$= \frac{(1/z)}{(1/z)-1} = \frac{1}{1-z}\ ;\ \text{ROC} : |z| < 1$$

Using scaling in z-domain property, we have,

$$Z\left[5^n\, u(-n)\right] = Z\left[u(-n)\right]\Big|_{z \to z/5} = \left.\frac{1}{1-z}\right|_{z \to z/5}$$

$$= \frac{1}{1-(z/5)}\,;\ \text{ROC} : |z| < 5$$

Using the linearity property, we have,

$$Z\left[3(5)^n\, u(-n)\right] = 3\, Z\left[5^n\, u(-n)\right] = 3\left[\frac{1}{1-(z/5)}\right] = \frac{15}{5-z}$$

$$= \frac{15}{5-z}\,;\ \text{ROC} : |z| < 5$$

REVIEW QUESTIONS

(1) *State and prove scaling property of z-transform?*

(2) *State and prove initial value theorem of z-transom.*

(3) *Discuss about differentiation in Z-domain?*

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