## 2.1 CARTESIAN PRODUCTS AND RELATIONS

Let A and B be any two sets. The cartesian product of A and B is a set defined by,

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

**Example**

**Ex. 1)** Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$

$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Clearly $A \times B \neq B \times A$.

If A has 'm' elements and B has 'n' elements, then, $A \times B$ will have 'mn' elements.

### EXAMPLE PROBLEM 1

**Let $A = \{a, b\}$, $B = \{p, q\}$ and $C = \{q, r\}$. Then, find**

**(i) $A \times (B \cup C)$**

**(ii) $(A \times B) \cup (A \times C)$**

**(iii) $A \times (B \cap C)$**

**(iv) $(A \times B) \cap (A \times C)$**

**SOLUTION**

(i) $B \cup C = \{p, q, r\}$, $A = \{a, b\}$

$A \times (B \cup C) = \{(a, p), (a, q), (a, r), (b, p), (b, q), (b, r)\}$

(ii) $A = \{a, b\}$, $B = \{p, q\}$, $C = \{q, r\}$

$\Rightarrow \quad A \times B = \{(a, p), (a, q), (b, p), (b, q)\}$

$A \times C = \{(a, q), (a, r), (b, q), (b, r)\}$

$(A \times B) \cup (A \times C) = \{(a, p), (a, q), (a, r), (b, p), (b, q), (b, r)\}$

From (i), (ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(iii) $B \cap C = \{q\}$, $A = \{a, b\}$

$A \times (B \cap C) = \{(a, q), (b, q)\}$

(iv) $(A \times B) \cap (A \times C) = \{(a, q), (b, q)\}$

From (iii), (iv) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Some important results on cartesian products are as follows,

(1) $A \times B$ and $B \times A$ have the same number of elements, but $A \times B \neq B \times A$ unless $A = B$.

(2) If A and B are disjoint sets, then $A \times B$ and $B \times A$ are also disjoint.

(3) If either A or B is null set, then the set $A \times B$ is also a null set.

(4) If either A or B is infinite and other is a non empty set then, $A \times B$ is also an infinite set.

(5) If $A \subset B$ then $A \times C \subset B \times C$.

(6) If $A \subset B$ and $C \subset D$, then $A \times C \subset B \times D$.

(7) If $A \subseteq B$, then $A \times B \Rightarrow (A \times B) \cap (B \times A)$.

**Definition :** A relation on the set A is a relation from A to A or a subset of $A \times A$ is called a binary relation on A. If R is a relation on A and $(a, b) \in R$, then we write $a\,R\,b$.

Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

## EXAMPLE PROBLEM 2

**Let A = {0, 1, 2} and B = {a, b}. Then {(0, a), (0, b), (1, a), (2, b)} is a relation from A to B. Relations can be represented graphically as shown in Fig. 2.1.1.**

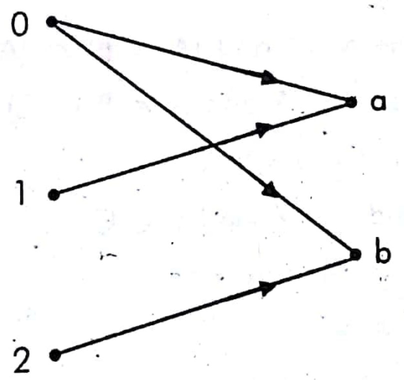



**Fig. 2.1.1 :** Relation from A to B

**SOLUTION**

**Definition :** Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

## EXAMPLE PROBLEM 3

**Let A = {1, 2, 3}, B = {a, b}, $R_1$ = {(1, a), (1, b) (2, a)}, write $R_2$ as relation on A.**

**SOLUTION**

Then $R_1$ is a relation from A to B write $R_2 = \{(1, 1), (2, 3), (3, 1)\}$. Then, $R_2$ is a relation on A.

## 2.2 FUNCTIONS

Let A and B be any two sets. A function 'f' from A to B is a rule that assigns to each element x in A exactly one element y in B. Then, f is called a function or a mapping from A to B and we write $f : A \to B$.

**Examples**

(1) Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$ and let $f = \{(1, a), (2, a), (3, d), (4, c)\}$.

Here, every element of A is associated with an element of B. So f is a function.

(2) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9\}$ and f is assign to each member in A of its square values. Then f is not a function from A to B, since no member of B is assigned to the element $4 \in A$.

**Definitions**

(1) Let f be a function from A to B. The set A is called the domain of the function f and B is called the codomain of f. The element $y \in B$ which the function f associates to an element $x \in A$ is denoted by f(x) and is called the f image of x. The element x may be referred as the pre-image of f(x). Each element of A has a unique image and each element of B need not appear as the image of an element in A. There can be more than one element of A which have the same image in B. We denote the range of $f : A \to B$ by f(A).

$f(A) = \{f(x) : x \in A\}$ clearly $f(A) \subseteq B$.

(2) Two functions f and g are said to be equal if they are defined on the same domain and if $f(a) = g(a), \forall a \in A$.

### 2.2.1 Into and One-to-One

**(1) Into Mapping :** A mapping $f : A \to B$ is said to be an into mapping, if f (A) is a proper subset of B.

**Examples**

(i) Let $f : Z \to Z$ be defined by $f(x) = 2x, x \in Z$, where Z is the set of integers. Then, f is an into mapping because f(Z) the set of even integers is a proper subset of codomain Z.

(ii) Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$, then $f = \{(a, 1), (b, 2), (c, 1), (d, 2)\}$ is an into mapping.

**(2) One to One Mapping or Injective Mapping :** A mapping $f : A \to B$ is said to be injective, if for each pair of distinct elements of A, their f-images are distinct.

$f : A \to B$ is one-one if $x_1 \neq x_2$ in A $\Rightarrow f(x_1) \neq f(x_2)$ in B.

or $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$.

**Examples**

(i) Let $f : A \rightarrow B$ defined by $f = \{(a, 1), (b, 3), (c, 2), (d, 4)\}$ where $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$.

Here distinct elements in A have distinct images in B. Therefore, f is one-one.

(ii) Let $f : R \rightarrow R$ be defined by $f(x) = 2x + 1 \ \forall \ x \in R$. Then, for $x_1, x_2 \in R$.

$x_1 \neq x_2$, we have $f(x_1) \leq f(x_2)$, so f is one one.

## 2.3 ONTO FUNCTIONS

(1) **Onto or Surjective Mapping :** A mapping $f : A \rightarrow B$ is said to be an onto mapping if $f(A) = B$. In this case, we say that f maps onto B.

**Example :** Let $f : Z \rightarrow Z$ be defined by $f(x) = x+1$. Then, every element y in the codomain z has pre-image $y - 1$ in the domain set Z. Therefore $f(z) = Z$ and f is onto.

For finite sets A, B with $|A| = m$ and $|B| = n$, there are,

$$\binom{n}{n}n^m - \binom{n}{n-1}(n-1)^m + \binom{n}{n-2}(n-2)^m - \ldots + (-1)^{n-2}\binom{n}{2}2^m + (-1)^{n-1}\binom{n}{1}1^m$$

$$= \sum_{k=0}^{n-1}(-1)^k\binom{n}{n-k}(n-k)^m$$

$$= \sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^m \text{ onto functions from A to B.}$$

**Examples**

(1) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{w, x, y, z\}$. Applying the general formula with $m = 7$ and $n = 4$, we find that there are,

$$\binom{4}{4}4^7 - \binom{4}{3}3^7 + \binom{4}{2}2^7 - \binom{4}{1}1^7$$

$$= \sum_{k=0}^{3}(-1)^k\binom{4}{4-k}(4-k)^7$$

$$= \sum_{k=0}^{4}(-1)^k\binom{4}{4-k}(4-k)^7$$

= 8400 functions from A onto B.

(2) If $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$, then there are 36 onto functions from A to B or, equivalently, 36 ways to distribute four distinct objects into three distinguishable containers, with no container empty (and no regard for the location of objects in a given container). Among these 36 distributions we find the following collection of six (one of six such possible collections of six).

(i) $\{a, b\}_1\ \{c\}_2\ \{d\}_3$

(ii) $\{a, b\}_1\ \{d\}_2\ \{c\}_3$

(iii) $\{c\}_1\ \{a, b\}_2\ \{d\}_3$

(iv) $\{c\}_1\ \{d\}_2\ \{a, b\}_3$

(v) $\{d\}_1\ \{a, b\}_2\ \{c\}_3$

(vi) $\{d\}_1\ \{c\}_2\ \{a, b\}_3$. where,

for example, the notation $\{c\}_2$ means that c is in the second container. Now if we no longer distinguish the containers, these $6 = 3!$ distributions become identical, so there are $36/(3!) = 6$ ways to distribute the distinct objects a, b, c, d among three identical containers, leaving no container empty.

## 2.3.1 Stirling Numbers of the Second Kind

For $m \geq n$ there are $\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^m$ ways to distribute m distinct objects into n numbered (but otherwise identical) containers with no container left empty. Removing the numbers on the containers, so that they are now identical in appearance, we find that one distribution into these n (nonempty) identical containers corresponds with n! such distributions into the numbered containers. So the number of ways in which it is possible to distribute the m distinct objects into n identical containers, with no container left empty, is,

$$\frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^m$$

This will be denoted by S(m, n) and is called a Stirling number of the second kind.

Number of one-one functions from A to B

$$= {}^{n}P_{m} = \frac{n!}{(n-m)!} = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 360$$

Number of onto functions from A to B = 0 $[\because |A| < |B|$ i.e., $m < n]$

(ii) Number of functions from B to A

$= |A|^{|B|}$

$= (4)^6$

$= 4096$

Number of one-one functions from B to A = 0 $[\because |B| > |A|]$

Number of onto functions from B to A

$= n!\ S(m, n)$

$$= n!\left[\frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^m\right]$$

$$= \sum_{k=0}^{n}(-1)^k\ {}^{n}C_{n-k}(n-k)^m$$

$$= \sum_{k=0}^{4}(-1)^k\ {}^{4}C_{4-k}(4-k)6$$

$\because$ for $f : B \to A$
$|B| = m = 6$
$|A| = n = 4$

$= {}^4C_4\ 4^6 - {}^4C_3\ 3^6 + {}^4C_2\ 2^6 - {}^4C_1\ 1^6$

$= 1560$

## 2.4 SPECIAL FUNCTIONS

**(1) Bijective Mapping :** A mapping $f : A \to B$ is said to be bijective, if f is both injective and surjective.

If f is injective, each element of B has atmost one-preimage. If f is surjective, each element of B has atleast one pre-image. So, if f is bijective, each element of B has exactly one pre-image.

**(2) Many to One Mapping :** A mapping $f : A \to A$ is said to be many to one, if two or more distinct elements have same image.

**Example :** Let $f : R \to R$ be defined by $f(x) = x^2$. Then, f is many to one, since $f(-3) = 9$, $f(3) = 9$.

(3) **Constant Mapping :** A function $f : A \to B$ is called a constant mapping, if the same element $b \in B$ is assigned to every element in A. In other words, $f : A \to B$ is a constant mapping, if the range of f consists of only one element.

**Example :** Let $f : R \to R$ be defined by $f(x) = k$, where $k \in R$. Then, f is a constant function.

(4) **Identity Mapping :** Let A be a set. The function $f : A \to A$ defined by $f(x) = x$ is called identity function.

## SOLVED PROBLEM 1

**Consider the function $h : N \times N \to N$, so that $h(a, b) = (2a + 1) 2^b - 1$, where $N = \{0, 1, 2, 3, . . . \}$ is the set of natural numbers. Prove that the function h is one-one and onto.**

**SOLUTION**

Consider the function $h(a, b) = (2a + 1) 2^b - 1$. If possible, suppose h is not one to one.

Let $(a_1, b_1), (a_2, b_2) \in N \times N$ where, $(a_1, b_1) \neq (a_2, b_2)$

$h(a_1, b_1) = h(a_2, b_2)$

$$\Rightarrow (2a_1 + 1) 2^{b_1} - 1 = (2a_2 + 1) 2^{b_2} - 1$$

$$\Rightarrow (2a_1 + 1) 2^{b_1} = (2a_2 + 1) 2^{b_2}$$

Without loss of generality, assume $b_2 \geq b_1$.

R.H.S. is a power of 2, i.e., even L.H.S. is the result of dividing an odd number by an odd number. Which cannot be even.

$$\frac{2a_1+1}{2a_2+1} = 1 \text{ and } 2^{b_2-b_1} = 1$$

$$\Rightarrow 2a_1 + 1 = 2a_2 + 1 \text{ and } b_2 = b_1$$

$$\Rightarrow a_1 = a_2 \text{ and } b_1 = b_2$$

which is a contradiction.

$\therefore$ h is a one-one function, we prove h to be a surjection by showing that corresponding to every natural number $n \exists (a, b) \in N \times N$ such that $h(a, b) = n$.

**Case 1 :** n is even, i.e., $n = 2k$

Then, $h(a, b) = (2a + 1) 2^b - 1 = 2k$

$\Rightarrow (2a + 1) 2^b = 2k + 1$, we can see that $h(k, 0) = 2k$

Therefore, for even n, $h\left(\frac{n}{2}, 0\right) = n$.

**Composite Mapping :** Let $f : A \to B$ and $g : B \to C$, then, the composite of functions f and g of $A \to C$ is given by $fog : A \to C$ such that $(gof)(x) = g[f(x)]$ for every $x \in A$.

**Example :** Let $f(x) = 2x + 1$ and $g(x) = 3x$ be two functions.

$gof(x) = g[f(x)] = g(2x + 1) = 3(2x + 1) = 6x + 3.$

## 2.5.1 Theorems and Problems

### SOLVED PROBLEM 1

**If A = {1, 2, 3, 4} and R, S are relations on A defined by R = {(1, 2), (1, 3), (2, 4), (4, 4)}, S = {(1, 1), (1, 2), (1, 3), (2, 3), (2, 4)} find RoS, SoR, $R^2$ and $S^2$.**

**SOLUTION**

$RoS = \{(1, 3), (1, 4)\}$

$SoR = \{(1, 2), (1, 3), (1, 4), (2, 4)\}$

$R^2 = RoR = \{(1, 4), (2, 4)\}$

$S^2 = SoS = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$

### SOLVED PROBLEM 2

**Let f : R → R where R is the set of real numbers, find fog and gof, where $f(x) = x^2 - 2$ and g : R→R, g (x) = x + 4. State whether these functions are injective, surjective and bijective.**

**SOLUTION**

Given, $f(x) = x^2 - 2$ and $g(x) = x + 4$

$fog(x) = f(g(x)) = f(x + 4) = (x + 4)^2 - 2 = x^2 + 8x + 14$

$gof(x) = g(f(x)) = g(x^2 - 2) = x^2 - 2 + 4 = x^2 + 2$

Consider $f(x) = x^2 - 2$

Let $x_1, x_2 \in R$

$f(x_1) = f(x_2)$

$\Rightarrow x_1^2 - 2 = x_2^2 - 2$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

$\Rightarrow$ f is not one-one

SOLVED PROBLEM 7

**Let $f : R \to R$ be given by $f(x) = x^2$ and $g : R - \{2\} \to R$ be given by $g(x) = \dfrac{x}{x-2}$. Find fog. Is gof defined ?**

SOLUTION

$$f(x) = x^2 \text{ and } g(x) = \frac{x}{x-2}$$

$$\Rightarrow fog(x) = f(g(x))$$

$$\Rightarrow f\left(\frac{x}{x-2}\right) = \left(\frac{x}{x-2}\right)^2 = \frac{x^2}{x^2 - 4x + 4}$$

$$gof(x) = g(f(x))$$

$$= g(x^2)$$

$$= \frac{x^2}{x^2 - 2} \text{ is defined when } x^2 \neq 2 \Rightarrow x \neq \sqrt{2}.$$

## 2.6 PIGEONHOLE PRINCIPLE

**[Nov./Dec. - 2008]**

For $m > n$, if m pigeons occupy n pigeonholes then two or more pigeons occupy the same pigeonhole or atleast one pigeonhole contain two (or) more pigeons in it.

This statement is called as Pigeonhole Principle.

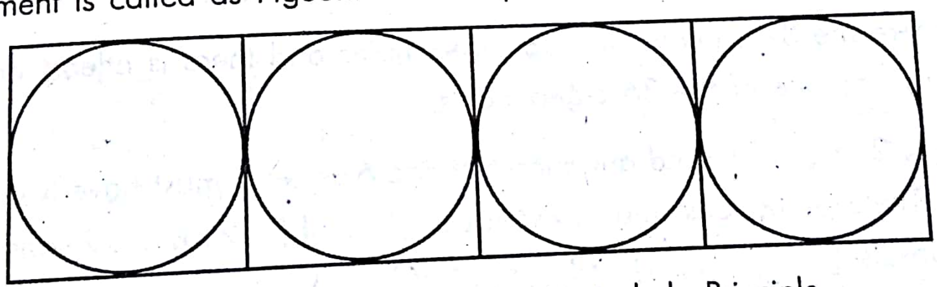


**Fig. 2.6.1 :** Illustrating the Pigeonhole Principle

Here in this example, 6 pigeons and 4 pigeonholes, atleast one pigeonhole must contain two or more pigeons in it by the definition of pigeonhole principle.

If m pigeons occupy n pigeonholes, for $P = [(m-1)/n]$ then atleast one pigeonhole must contain $P + 1$ (or) more pigeons.

**Examples**

(1) In a set of 13 children, atleast two have birthday during the same month. Here we have 13 pigeons (children) and 12 pigeonholes (the months of the year).

(2) Seven cars carry 26 passengers, atleast one car must have 4 or more passengers. Here, 26 (passengers) pigeons and 7 pigeon holes, if seven cars carry equal number of passengers 21 passengers are filled and remaining 5 passengers can sit in any one of the seven cars.

So, 4 (or) more passengers can sit in atleast one car.

(3) A magnetic tape drive contains a collections of 5,00,000 strings or words made up of four or fewer lower case letters, can it be the 5,00,000 words are all distinct?

Here in 26 ways we can fill a 'n' letter string and the $26^n$ is the possible number of strings of 'n' letters. And the total number of possible strings made up of four or fewer letters is,

$26^4 + 26^3 + 26^2 + 26 = 475{,}254$

With these 475,254 strings as pigeonholes and 5,00,000 words on the tape as pigeons and here atleast one word is repeated on the tape.

(4) For $|S| = 37$ where $S < 2^7$. Then S contains two elements which have the same remainder upon division by 36.

Here 37 positive integers are the pigeons in S. When any positive integer n is divided by 36 then there exists a unique quotient (q) and remainder (r).

For $n = 36q + r$ where $0 \leq r < 36$.

Here 36 pigeonholes are constitute by 36 possible values.

$\therefore$ There are 37 pigeons and 36 pigeonholes and there is atleast one more pigeon will occupy any one of the 36 pigeonholes.

(5) Let $S = \{1, 2, 3 \ldots 9\}$ and any subset of size 6 of set S must have two elements whose sum is 10. The pigeons constitute a six element subset of $\{1, 2, 3, \ldots\ldots 9\}$ and the pigeonholes are the subsets $\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}$.

Here 6 pigeons are subset of size '6' and 5 pigeonholes are having two elements whose sum is 10. When the six pigeons go to this respective pigeonholes, they must fill atleast one of the two element subsets whose members sum to 10.

(6) A group of 61 people, atleast 6 people were born in the same month. Here we have 61 pigeons (people) and 12 pigeonholes (months), if people born in same month is a equal number then 60 people can be filled in 12 pigeonholes by five people but there are 61 people so the remaining one person can born in any month. So minimum six people are born in same month.

When an algorithm correctly solves a certain type of problem and satisfies these five conditions, then we may find ourselves examining it further in the following ways.

(1) Can we somehow measure how long it takes the algorithm to solve a problem of a certain size ? Whether we can may very well depend, for example, on the compiler being used, so we want to develop a measure that doesn't actually depend on such considerations as compilers, execution speeds or other characteristics of a given computer.

For example, if we want to compute $a^n$ for $a \in R$ and $n \in Z^+$, is there some "function of n" that can describe how fast a given algorithm for such exponentiation accomplishes this ?

(2) Suppose we can answer questions such as the set forth at the start of item 1. Then if we have two (or more) algorithms that solve a given problem, is there perhaps a way to determine whether one algorithm is "better" than another ?

In particular, suppose we consider the problem of determining whether a certain real number x is present in the list of n numbers $a_1, a_2, \ldots a_n$. Hence, we have a problem of size n.

If there is an algorithm that solves this problem, we first determine the time taken. To measure this, we seek a function f(n) called the time complexity function of the algorithm. We expect (both here and in general) that the value of f(n) will increase as n increases. Also, our major concern is dealing with any algorithm is how the algorithm performs for large values of n.

Let f be a function of n. By the term asymptotic we mean the study of the function f as n becomes larger and larger without bound.

**(1) Big - O Notation :** Let f(x) and g(x) be real valued functions, i.e., their range is a subset of R. We say that f(x) is big-O of g(x) written $f(x) = O(g(x))$, if there exist positive constants c and $x_0$ such that,

$$|f(x)| \leq c|g(x)| \text{ for all } x \geq x_0.$$

**Examples**

(i) Let $f(n) = n^2 + 4n$ and $g(n) = n^2$; $n \geq 0$. Notice that,

$$4n \leq n^2 \text{ for all } n \geq 4.$$

This implies that,

$$n^2 + 4n \leq n^2 + n^2 \text{ for all } n \geq 4.$$

(or)

$$n^2 + 4n \leq 2n^2 \text{ for all } n \geq 4.$$

Let $c = 2$ and $n_0 = 4$. Then,

$$f(n) \leq cg(n) \text{ for all } n \geq n_0.$$

Because both f(n) and g(n) are nonnegative, $|f(n)| = f(n)$ and $|g(n)| = g(n)$. Thus, $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$

Hence $f(n) = O(g(n))$.

(ii) Let $f(n) = \frac{n(n+1)}{2}$ and $g(n) = n^2, n \geq 0$. Notice that $\frac{(n+1)}{2} \leq n$ for all $n \geq 1$.

This implies that,

$\frac{n(n+1)}{2} \leq n \cdot n \leq n^2$ for all $n \geq 1$.

Choose $c = 1$ and $n_0 = 1$. Then,

$f(n) \leq cg(n)$ for all $n \geq n_0$.

Because both f(n) and g(n) are nonnegative, $|f(n)| = f(n)$ and $|g(n)| = g(n)$. Thus, $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$.

Hence $f(n) = O(g(n))$.

(iii) Let $f(n) = n \cdot \lg n, n > 0$. Because $\lg n \leq n$ for all $n \geq 1$, it follows that,

$n \cdot \lg n \leq n \cdot n = n^2$ for all $n \geq 1$.

This shows that $n.\lg n = O(n^2)$.

**(2) Big - Ω Notation :** Let f(x) and g(x) be real valued functions. The function f(x) is omega of g(x), written $f(x) = \Omega(g(x))$, if there exist positive constants c and $x_0$ such that,

$c|g(x)| \leq |f(x)|$ for all $x \geq x_0$.

To simultaneously determine an upper bound and a lower bound on a complexity function, we introduce theta notation.

**(3) Big - Θ Notation :** Let f(x) and g(x) be real valued function. The function f(x) is theta of g(x), written $f(x) = \Theta(g(x))$, if there exist positive constants $c_1$, $c_2$ and $x_0$ such that,

$c_1|g(x)| \leq |f(x)| \leq c_2|g(x)|$ for all $x \geq x_0$.

## 2.7.1 Strictly Increasing and Nondecreasing Functions

**(1) Strictly Increasing Function :** Let f(x) be a real valued function. Then f(x) is called strictly increasing if for all $x_1$ and $x_2$,

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

In other words, f(x) gets larger and larger.

## 2.8 PROPERTIES OF RELATIONS

**Definition :** A relation on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

**Examples**

(1) Consider the set of all positive integers. Then, the relation "divides" is a reflexive relation since $a/a \in R$ whenever a is a positive integer.

(2) Consider the following relations on the set $A = \{1, 2, 3, 4\}$.

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$

$R_2 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$

$R_3 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2)\}$

Then, $R_2$ is a reflexive relation, since it has all pairs of the form (a, a) namely (1, 1), (2, 2), (3, 3) and (4, 4). The relations $R_1$ and $R_3$ are not reflexive since, (3, 3) is not in any of these relations.

**Definition :** A relation R on the set A is irreflexive if for every $a \in A$, $(a, a) \notin R$. i.e. R is irreflexive if no element in A is related to itself.

**Example :** Consider the set $A = \{1, 2, 3, 4\}$.

Then, the following relations are irreflexive relations.

$R_1 = \{(2, 4), (4, 2)\}$

$R_2 = \{(1, 2), (2, 3), (3, 4)\}$

$R_3 = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

A relation R on a set A is a subset of $A \times A$. If A contains 'n' elements, then $A \times A$ contains $n^2$ elements. There are $2^{n^2}$ relations on A, since a relation is determined by specifying whether each of the $n^2$ ordered pairs in $A \times A$ is in R.

However, if R is reflexive, each of the n ordered pairs (a, a) for $a \in A$ must be in R. Each of the other $n (n - 1)$ ordered pairs of the form (a, b) where $a \neq b$ may or may not be in R. Hence, there are $2^{n(n-1)}$ reflexive relations on A.

**Definition :** A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for $a, b \in A$. A relation R on a set A such that $(a, b) \in R$ and $(b, a) \in R$ only if $a = b$ for $a, b \in A$, is called anti symmetric. i.e., A relation is symmetric, if and only if a R b implies that b R a.

A relation is an anti-symmetric if and only if there are no pairs of distinct elements a and b with a related to b and b related to a.

**Examples**

(1) Let L be the set of all straight lines in a plane. The relation R on L defined by 'x is parallel to y' is symmetric, since if a straight line 'a' is parallel to a straight line 'b', then 'b' is also parallel to 'a'. Thus $(a, b) \in R \Rightarrow (b, a) \in R$.

(2) Let R be a relation in the natural numbers N which is defined by '$x - y > 0$'. Then, R is not symmetric since $4 - 2 > 0$, but $2 - 4 \not> 0$.

(3) Let A be a family of sets and R be the relation in A defined by 'x is a subset of y'. Then, R is antisymmetric since $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$.

(4) The relation "divides" on the set of positive integers is not symmetric since 1/2 is not equal to 2/1. It is antisymmetric since if a and b are any two positive integers with a/b and b/a then $a = b$.

(5) Let L be a set of straight lines in a plane. The relation in L defined by 'x is perpendicular to y' is not antisymmetric. Since, if straight line 'a' is perpendicular to straight line 'b', then 'b' is perpendicular to 'a', but 'a' cannot be equal to 'b'. i.e., $(a, b) \in R, (b, a) \in R$, but $a \neq b$. But, this relation is symmetric.

**Definition :** A relation R on a set A is called transitive if, whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

**Examples**

(1) Consider the relation $R_1 = \{(a, b) / a \leq b\}$ on the set of integers. Then $R_1$ is transitive, since $a \leq b$ and $b \leq c$ imply that $a \leq c$.

(2) Let L be the set of all straight lines in a plane and R be the relation in L defined by 'x is parallel to y'. If 'a' is parallel to 'b' and 'b' is parallel to 'c', then clearly 'a' is parallel to 'c'. Hence, R is transitive.

(3) Let A be the set of all Indians. Let R be the relation in A defined by 'x loves y'. If 'a' loves 'b' and 'b' loves 'c', it does not necessarily follow that 'a' loves 'c'. So, R is not a transitive relation.

(4) The relation 'divides' on the set of positive integers is transitive.

**Proof :** Suppose that 'a' divides 'b' and 'b' divides 'c'. Then, there are positive integers 'q' and 't' such that $b = aq$ and $c = bt$.

Hence $c = a\,q\,t \Rightarrow a$ divides c.

Therefore, the relation "divides" is transitive.

(5) Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations,

$R_1 = \{(1, 1), (2, 2), (3, 3)\}$, and

$R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$

$R_1 \cap R_2 = \{(1, 1)\}$

$R_1 - R_2 = \{(2, 2), (3, 3)\}$

$R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}$

**Definition :** Let R be a relation from set A to set B and S be a relation from set B to set C. The composite of R and S is the relation consisting of ordered pairs (a, c) where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by RoS.

**Example :** Consider,

$A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4\}$

$C = \{0, 1, 2\}$

Let R is a relation from A to B with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is a relation from B to C which is $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$. Then $RoS = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

**Definition :** Let R be a relation on the set A. The powers $R^n$, $n = 1, 2, 3, \ldots$ are defined inductively by,

$R^1 = R$, $R^2 = RoR$ and $R^{n+1} = R^n oR$.

## EXAMPLE PROBLEM 1

**Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers of $R^n$ for $n = 2, 3, 4 \ldots$**

**SOLUTION**

Since $R^2 = RoR$, we have,

$R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$

$R^3 = R^2 oR = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$

$R^4 = R^3 oR = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$

i.e., $R^4 = R^3$

## 2.10 PARTIAL ORDERS

### 2.10.1 POSET (or) Partial Order Sets

[Nov./Dec. - 2008]

Let S be a non empty set. A relation R on S is said to be a partial order relation if R is reflexive, anti-symmetric and transitive. A relation of partial order is often denoted by $\leq$. A non empty set together with a partial order relation $\leq$ on S is called a partial order set or POSET and is denoted by $(S, \leq)$. If R is a partial order on a set S then $R^{-1}$ is also a partial order relation on S. The poset $(S, R^{-1})$ is called the dual of (S, R).

**EXAMPLE PROBLEM 1**

**Show that the relation greater than or equal to $\geq$ is a partial ordering on the set of integers.**

**SOLUTION**

Since $a \geq a$ for every integer a, $\geq$ is reflexive.

If $a \geq b$ and $b \geq a$, then $a = b$.

Hence $\geq$ is anti-symmetric.

Also, $a \geq b, b \geq c \Rightarrow a \geq c$ for all integers a, b, c.

So, $\geq$ is transitive.

Hence $\geq$ is a partial ordering on the set of integers.

**EXAMPLE PROBLEM 2**

**Let X be a non empty set and P(X) be the power set of X, $(P(X), \leq)$ is a POSET where A M $\leq$ B means '$A \subseteq B$' for A, B $\in$ P(X), because**

**SOLUTION**

(i) $A \subseteq A$ for all $A \in P(X)$ i.e., $\leq$ is reflexive.

(ii) $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$, i.e., $\leq$ is antisymmetric.

(iii) $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$, i.e. $\leq$ is transitive.

**EXAMPLE PROBLEM 3**

**The divisibility relation is a partial ordering on the set of positive integers. Because**

**SOLUTION**

(i) $a \mid a$ for all $a \in Z$. i.e., $\mid$ is reflexive.

(ii) $a \mid b, b \mid a \Rightarrow a = b$. i.e., $\mid$ is antisymmetric.

(iii) $a \mid b, b \mid c \Rightarrow a \mid c$. i.e., $\mid$ is transitive.

## 2.11 EQUIVALENCE RELATIONS

Let R be relation in a set A. Then, R is an equivalence relation in A if and only if,

(1) R is reflexive, i.e., for all $a \in A$, $(a, a) \in R$.

(2) R is symmetric, i.e., $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.

(3) R is transitive, i.e., $(a, b) \in R$, $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

**Examples**

(i) The most trival example of an equivalence relation is that of "equality". For any elements in any set,

– $a = a$, i.e., reflexive.

– $a = b \Rightarrow b = a$, i.e., symmetric.

– $a = b, b = c \Rightarrow a = c$, i.e. transitive.

(ii) Let R be the relation in the real numbers defined by $x \leq y$. Then,

– $x \leq x$ i.e., $(x, x) \in R$. So, R is reflexive.

– $x \leq y$ but, $y \nleq x$. So, R is not symmetric.

Hence R is not an equivalence relation.

(iii) Let m be a positive integer greater than 1. Then, the relation.

$R = \{(a, b) : a \equiv b \text{ (mod m)}\}$ [where $a \equiv b$ mod m iff $m/(a - b)$]

is an equivalence relation on the set of integers.

**Proof :** $a \equiv b$ mod m if and only if m divides $a - b$.

$a - a = 0$ is divisible by m.

Hence m divides $a - a$.

i.e., $a \equiv a$ (mod m).

Therefore, R is reflexive.

Now suppose $a \equiv b$ (mod m) then,

$a - b$ is divisible by m.

$\Rightarrow a - b = km$. where $k \in z$.

$\Rightarrow b - a = (-k)\, m$

$\Rightarrow$ m divides $b - a$

Hence $b \equiv a$ mod m.

The relation R is symmetric.

Suppose, $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

$\Rightarrow$ m divides $a - b$ and m divides $b - c$

$\Rightarrow a - b = km$ and $b - c = lm$

where k, l are integers.

$\Rightarrow a - b + b - c = a - c = (k + l)\, m$

$\Rightarrow$ m divides $a - c$

Hence $a \equiv c \bmod m$.

Therefore, the relation R is transitive.

Hence R is an equivalence relation.

The relation R is called congruence modulo m.

(iv) The following relations on set {0, 1, 2, 3} are equivalence relations.

$R_1 = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$

$R_2 = \{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

## 2.11.1 Theorem and Problems

### THEOREM 1

**Show that the relation Q on set A = {(a, b) : a, b are integers and b ≠ 0} is an equivalence relation where Q is defined as (a, b) Q (c, d) iff ad = bc.**

**PROOF**

Let $(a, b) \in A$, then

(a, b) Q (a, b) since $ab = ba$

Therefore, Q is reflexive.

Suppose, $(a, b), (c, d) \in A$

Let (a, b) Q (c, d)

$\Rightarrow ad = bc$

$\Rightarrow da = cb$

$\Rightarrow$ (c, d) Q (a, b)

Therefore, Q is symmetric.

## 2.12 PARTITIONS

### 2.12.1 Equivalence Classes and Partitions

Let R be an equivalence relation on a Set A. The set of all elements that are related to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$.

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is $[a]_R = \{s : (a, s) \in R\}$. If $b \in [a]_R$, b is called a representative of this equivalence class.

**Examples**

(1) Let A be the set of all triangles in a plane and let R be an equivalence relation in A, defined by 'x is congruent to y' for every $x, y \in A$. When $a \in A$, then, the equivalence class[a] is the set of all triangles of A congruent to the triangle a. Similarly, if $b \in A$, then the equivalence class[b] is the set of all triangles of A congruent to the triangle b.

(2) Let us determine the equivalence classes in the set I of all integers with respect to equivalence relation 'congruent modulo 5'. The integers congruent to '0' modulo 5 form an equivalence class [0] i.e., [0] contains the elements which leave the remainder 0 when divided by 5 or the elements of this class are multiples of 5 i.e., integers are of the form 5 k for some integer k. The elements which leave remainder 1 when divided by 5 form another equivalence class [1] i.e., the integers are of the form $5k + 1$ for some integer k and so on. Thus, every integer is expressible as $5k$, $5k + 1$, $5k + 2$, $5k + 3$, $5k + 4$ for some integer k. (Since, when any integer x is divided by 5, the possible remainders are (0, 1, 2, 3, 4).

Hence, the set of all integers can be divided into the following equivalence classes with respect to the relation congruence modulo 5.

$[0] = \{5k : k \in Z\} = \{\ldots -15, -10, -5, 0, 5, 10 \ldots\}$

$[1] = \{5k + 1 : k \in Z\} = \{\ldots -9, -4, 1, 6, 11 \ldots\}$

$[2] = \{5k + 2 : k \in Z\} = \{\ldots -8, -3, 2, 7, 12 \ldots\}$

$[3] = \{5k + 3 : k \in Z\} = \{\ldots -7, -2, 3, 8, 13 \ldots\}$

$[4] = \{5k + 4 : k \in Z\} = \{\ldots -6, -1, 4, 9, 14 \ldots\}$

These classes have following properties.

(i) The set Z is the union of these five non empty classes.

(ii) Integers in each class have a relation of congruence modulo 5 with one another.

(iii) Integers in different classes do not have a relation of congruence modulo 5 with one another.

(iv) The classes are mutually disjoint.

**Partitions :** Let S be a non empty set. Then a partition of S is a collection of non-empty disjoint subsets of S whose union is S i.e., let $A_1, A_2, A_3 \ldots$ be the non-empty subsets of S then the set $P = \{A_1, A_2, A_3 \ldots\}$ is called a partition of S if,

(1) $S = A_1 \cup A_2 \cup \ldots$

(2) Either $A_1 = A_2$ or $A_1 \cap A_2 = \phi$ for all $A_1, A_2, A_3 \ldots$ in P.

## EXAMPLE PROBLEM 1

**Let I be the set of all integers. We know that $a \equiv b \bmod 5$ is an equivalence relation in Z. Consider the set of five equivalence class [0], [1], [2], [3], [4] where,**

**SOLUTION**

$[0] = \{\ldots -10, -5, 0, 5, 10 \ldots\}$

$[1] = \{\ldots -9, -4, 1, 6, 11 \ldots\}$

$[2] = \{\ldots -8, -3, 2, 7, 12 \ldots\}$

$[3] = \{\ldots -7, -2, 3, 8, 13 \ldots\}$

$[4] = \{\ldots -6, -1, 4, 9, 14 \ldots\}$ clearly.

These sets are non empty. The sets [0], [1], [2], [3], [4] are pair wise disjoint and $Z = [0] \cup [1] \cup [2] \cup [3] \cup [4]$.

Hence {[0], [1], [2], [3], [4]} is a partition of Z.

## 2.12.2 Procedure to Find Partitions

Suppose, A is a finite set and R is a relation on finite set A. Then $[a]_R$ is called an equivalence class of R. The partition P constructed from the above theorem consists of all equivalence classes of R and this partition will be denoted by A/R. Recall that partitions of A are also called quotient sets of A, and the notation A/R reminds us that P is the quotient set of A that is constructed and determines R. The general procedure for determining partitions A/R for a finite or countable is as follows.

**Step 1 :** Choose any element 'a' of A and compute $[a]_R$.

**Step 2 :** If $[a]_R \neq A$, then choose an element b, not included in $[a]_R$ and compute the equivalence class $[b]_R$.

**Step 3 :** If A is not the union of previously computed equivalence classes, then choose an element x of A. That is not in any of those equivalence classes and compute $[x]_R$.

**Step 4 :** Repeat Step 3, until all elements of A are included in the compound equivalence classes. If A is not countable, this process could continue indefinitely. In that case, continue until a pattern emerges that allows you to describe or give a formula for all equivalence classes.

## SOLVED PROBLEM 15

**Let the compatibility relation on a set {1, 2, 3, 4, 5, 6} be given by the matrix.**

$$\begin{array}{c|ccccc} 2 & 1 & & & & \\ 3 & 1 & 1 & & & \\ 4 & 0 & 0 & 1 & & \\ 5 & 0 & 0 & 1 & 1 & \\ 6 & 1 & 0 & 1 & 0 & 1 \\ \hline & 1 & 2 & 3 & 4 & 5 \end{array}$$

**Draw the graph and find the maximal compatibility blocks of the relation.**

**SOLUTION**

Graph of the given relation is,

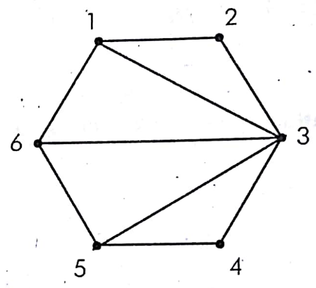


**Fig. 2.12.3 :** A Graph Obtained from Above Matrix

Maximal compatibility blocks are {1, 2, 3}, {1, 3, 6}, {3, 5, 6}, {3, 4, 5}.

## 2.13 PRINCIPLE OF INCLUSION AND EXCLUSION AND GENERALIZATIONS OF PRINCIPLE

### 2.13.1 The Principle of Inclusion and Exclusion

The sum rule which is applied to the non-disjoint sets is called 'Principle of Inclusion-Exclusion' also called as 'Sieve Method'.

#### THEOREM 1

**If A and B are two subsets of any set (universal) then,**

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

**PROOF**

Given for any set S

$A, B \subseteq S$

i.e., A and B are subsets of S.

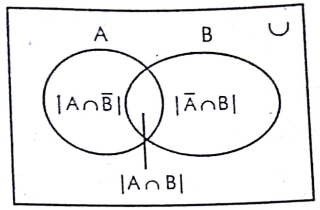



**Fig. 2.13.1 :** Venn Diagram of $|A \cup B|$

$|A \cup B|$ = All the elements of A + all the elements of B.

$|A| = |A \cap \bar{B}| + |A \cap B|$ (from Venn diagram) and

$|B| = |\bar{A} \cap B| + |A \cap B|$ (from Venn diagram)

We have to prove that $|A \cup B| = |A| + |B| - |A \cap B|$

R.H.S $= |A| + |B| - |A \cap B| = |A \cap \bar{B}| + |A \cap B| + |\bar{A} \cap B| + |A \cap B| - |A \cap B|$

$= |A \cap \bar{B}| + 2|A \cap B| + |\bar{A} \cap B| - |A \cap B|$

$= |A \cap \bar{B}| + |\bar{A} \cap B| + |A \cap B|$

$= |A \cup B|$ (from Venn diagram) = L.H.S

$\therefore \quad |A \cup B| = |A| + |B| - |A \cap B|$

Hence it is proved.

## THEOREM 2

**If A, B and C are any three subsets of set S, then**
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

**PROOF**

Given A, B and C are three subsets of set S.

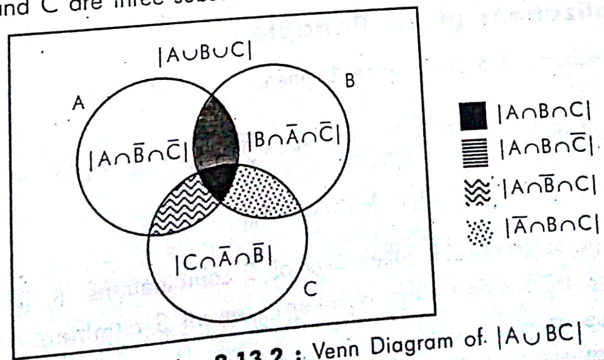



**Fig. 2.13.2 :** Venn Diagram of $|A \cup BC|$

From Venn diagram,

$$|A| = |A\cap\bar{B}\cap\bar{C}|+|A\cap B\cap\bar{C}|+|A\cap C\cap\bar{B}|+|A\cap B\cap C|,$$

$$|B| = |B\cap\bar{A}\cap\bar{C}|+|A\cap B\cap\bar{C}|+|\bar{A}\cap B\cap C|+|A\cap B\cap C|,$$

$$|C| = |C\cap\bar{A}\cap\bar{B}|+|\bar{A}\cap B\cap C|+|A\cap C\cap\bar{B}|+|A\cap B\cap C|.$$

$$\therefore\ |A|+|B|+|C| = |A\cap\bar{B}\cap\bar{C}|+|A\cap B\cap\bar{C}|+|A\cap C\cap\bar{B}|+|A\cap B\cap C|$$
$$+|B\cap\bar{A}\cap\bar{C}|+|A\cap B\cap\bar{C}|+|\bar{A}\cap B\cap C|+|A\cap B\cap C|$$
$$+|C\cap\bar{A}\cap\bar{B}|+|\bar{A}\cap B\cap C|+|A\cap C\cap\bar{B}|+|A\cap B\cap C| \quad \ldots(1)$$

From Venn diagram,

$$|A\cup B\cup C| = |A\cap\bar{B}\cap\bar{C}|+|B\cap\bar{A}\cap\bar{C}|+|C\cap\bar{A}\cap\bar{B}|+|A\cap B\cap\bar{C}|+|A\cap\bar{B}\cap C|$$
$$+|\bar{A}\cap B\cap C|+|A\cap B\cap C| \quad \ldots(2)$$

Replacing in equation (1) we get,

$$|A|+|B|+|C| = |A\cup B\cup C|+|A\cap B\cap\bar{C}|+|A\cap B\cap C|+|\bar{A}\cap B\cap C|$$
$$+|A\cap\bar{B}\cap C|+|A\cap B\cap C|$$
$$= |A\cup B\cup C|+|A\cap\bar{B}\cap C|+|A\cap B|+|B\cap C| \quad \ldots(3)$$

$$[\because |A\cap B\cap\bar{C}|+|A\cap B\cap C|=|A\cap B|+|\bar{A}\cap B\cap C|+|A\cap B\cap C|=|B\cap C|]$$

Adding $|A\cap B\cap C|$ on both sides of equation (3) we get,

$$|A|+|B|+|C|+|A\cap B\cap C| = |A\cup B\cup C|+|A\cap B|+|B\cap C| +|A\cap\bar{B}\cap C|+|A\cap B\cap C|$$

$$|A|+|B|+|C|+|A\cap B\cap C| = |A\cup B\cup C|+|A\cap B|+|B\cap C|+|C\cap A|$$

$$[\because |A\cap B\cap C|+|A\cap\bar{B}\cap C|=|C\cap A|]$$

$$\therefore\ |A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|B\cap C|-|C\cap A|+|A\cap B\cap C|$$

Hence it is proved.

## 2.13.2 Generalizations of the Principle

If $A_i$ are finite subsets of a universal set U, then,

$$|A_1\cup A_2\cup\ldots\cup A_n| = \sum_{i=0}^{n}|A_i|-\sum_{i,j}|A_i\cap A_j|+\sum_{i,j,k}|A_i\cap A_j\cap A_k|+\ldots$$
$$+(-1)^{n-1}|A_1\cap A_2\cap\ldots\cap A_n| \quad \ldots(1)$$

Where the second summation is taken over all 2-combinations (i, j) of the integers {1, 2, . . . n} , the third summation is taken over all 3-combinations {i, j, k} of {1, 2, . . . n} and so on.

**Proof :** We prove the theorem by Mathematical Induction.

Let the theorem be true for $n = 1$. Assume that theorem to be true for any n subsets of S. Suppose, then that we have $n + 1$ sets for 2 sets repeatedly in the proof.

Consider, $A_1 \cup A_2 \cup ...... \cup A_n \cup A_{n+1}$ as the union of the subsets.

$$|(A_1 \cup A_2 \cup ...... \cup A_n \cup A_{n+1})| = |(A_1 \cup A_2 \cup A_3 .... \cup A_n)| + |(A_1 \cup A_2 \cup A_3 ....... \cup A_n)|$$
$$|(A_1 \cup A_2 \cup A_3 ....... \cup A_n)| \cap |A_{n+1}| \quad ...(2)$$

Applying Induction to (2) of the (3) sets in equation (3)

$$|A_1 \cup A_2 \cup ... \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i,j \le n} |A_i \cap A_j| + \sum_{i,j,k \le n} |A_i \cap A_j \cap A_k| ...+$$
$$(-1)^{n-1} |A_1 \cap A_2 \cap ... \cap A_n| \quad ...(3)$$

$\therefore$ Intersection distributes over unions, we get,

$$|(A_1 \cup A_2 \cup ...... \cup A_n) \cap A_{n+1}| = |(A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \cup ...... \cup (A_n \cap A_{n+1})| \quad ...(4)$$

$$\Rightarrow |(A_1 \cup A_2 \cup .... A_n) \cap A_{n+1}| = \sum_{i=1}^{n} |(A_i \cap A_{n+1})| - \sum_{i,j<n} |(A_i \cap A_{n+1}) \cap (A_j \cap A_{n+1})|$$
$$+ \sum |(A_i \cap A_{n+1}) \cap (A_j \cap A_{n+1}) \cap (A_k \cap A_{n+1})| + ... + (-1)^{n-1} |(A_1 \cap A_{n+1}) \cap$$
$$(A_2 \cap A_{n+1}) \cap ..... (A_n \cap A_{n+1})| \quad ...(5)$$

Substituting equations (3) and (5) in equation (2), we get,

$$(A_i \cap A_{n+1}) \cap (A_j \cap A_{n+1}) \cap (A_k \cap A_{n+1}) = A_i \cap A_j \cap A_k \cap A_{n+1}$$

$$\Rightarrow |(A_1 \cup A_2 \cup A_3 ....... A_n) \cup A_{n+1}| = \sum_{i=1}^{n} |A_i| - \sum_{i,j \le n} |A_i \cap A_j|$$
$$+ \sum_{i,j,k \le n} |A_i \cap A_j \cap A_k| + ... + (-1)^{n-1} |A_1 \cap A_2 \cap A_3 ... A_n \cap A_{n+1}|$$
$$+ \sum_{i=1}^{n} |A_i \cap A_{n+1}| - \sum_{i,j,k \le n} |A_i \cap A_j \cap A_{n+1}| + \sum_{i,j,k \le n} |A_i \cap A_j \cap A_k \cap A_{n+1}|$$
$$+ ... + (-1)^n |A_1 \cap A_2 \cap A_3 ... A_n \cap A_{n+1}|.$$

We see that $\sum_{i,j \le n} |A_i \cap A_j| + \sum_{i=1}^{n} |A_i \cap A_{n+1}|$.

Where the first summation is taken over i and the j summation is taken over i for the 2-combinations of the form $\{i, j\} \in \{1, 2, \ldots n\}$ and $\{i, n+1\}$, where $i \in \{1, 2, 3, \ldots n\}$ respectively which can be simplified to $\Sigma A_i \cap A_j$ where the sum is taken over all 2-combinations of $\{1, 2, 3, \ldots n, n + 1\}$.

$\because$ $|U|$ is the set of all 2-combinations of $\{1, 2, 3, \ldots n, n + 1\}$ similarly by induction we can prove for $n = n + 1$.

Hence by induction the theorem is proved.

## EXAMPLE PROBLEM 1

**Suppose that 200 faculty members can speak French and 50 can speak Russian, while only 20 can speak both French and Russian. How many faculty members can speak either French or Russian?**

**SOLUTION**

Let F be the set of faculty members who can speak only French. Let R be the set of faculty members who can speak only Russian.

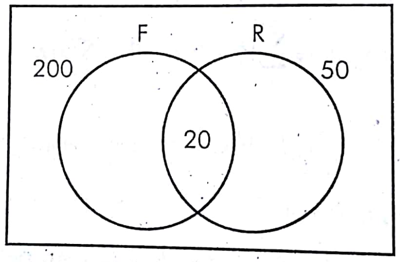



**Fig. 2.13.3 :** Illustrating F and R Sets

$n(F) = 200$

$n(R) = 50$

$n(F \cap R) = 20$

$n(F \cup R) = ?$

$\because n(F \cup R) = n(F) + n(R) - n(F \cap R) = 200 + 50 - 20$

$\therefore n(F \cup R) = 230$

## EXAMPLE PROBLEM 2

**If there are 200 faculty members that speak French, 50 can speak Russian, 100 that speak Spanish, 20 that speak French and Russian, 60 that speak French and Spanish, 35 that speak Russian and Spanish. While only 10 speak French, Russian and Spanish. How many speak either French or Russian or Spanish?**

## 2.14 DERANGEMENTS

A derangement is a permutation of objectives that leaves no object in its original positions.

No arrangement means when nothing is in its right place. Amongest the permutations of $\{1, 2, \ldots n\}$, n natural numbers, there are some, in which none of the n number appears in its natural place, such permutations are called derangements, and are denoted by $D_n$. Thus $(a_1, a_2, \ldots, a_n)$ is a derangement if $a_1 \neq 1$, $a_2 \neq 2 \ldots$ and $a_n \neq n$.

### 2.14.1 Definition

**Derangements Dn :** A permutation P of $a = \{a_1, a_2, \ldots a_n\}$ is a derangement if $P(a_i) \neq a_i$ for $i = 1, 2, \ldots n$.

**Note :** We can prove that the inverse of a derangement is a derangement.

If $P(a_i) = a_j (i \neq j)$,

then $P^{-1}(a_j) = a_i, (i \neq j)$

**Example :** If there is only one element with only one place to be filled, there cannot be any derangement. So $D_1 = 1$, $D_2 = 2$, which is (2, 1) and $D_3 = 3$, which are (2, 3, 1) and (3, 1, 2).

### 2.14.2 Derivation of Formula for $D_n$

Let us have n balls each differently coloured and n boxes each having a different colour and matching with exactly one colour of a ball. Then the problem is in how many ways can we place each ball in exactly one coloured box such that no ball is placed in a box matching its colour. Let $D_n$ designate this number.

If $A_i$ denotes the event that ball number i is matched with box number i of the same colour, then remaining $n - 1$ balls can be permuted in the remaining $n - 1$ boxes in $(n - 1)!$ ways, $i = 1, 2, \ldots n$. Let $A_i \cap A_j$ denote the event of both $i^{th}$ and $j^{th}$ balls are matched and put in the boxes bearing colours i and j respectively. Thus, denoting the cardinal number of the corresponding sets by n, we have $n(A_i) = (n - 1)!$ and $n(A_i \cap A_j) = (n - 2)!$ and in general $n(A_{i_1} \cap A_{i_2} \cap A_{i_r}) = (n-r)!$.

Let $S_n = (A_1 \cup A_2 \ldots \cup A_n)$, then $n(S_n)$

$$n(S_n) = \sum_{i=1}^{n} n(A_i) - \sum_{i_1<i_2} n(A_{i_1} \cap A_{i_2}) \ldots + (-1)^{r-1} \sum_{i_1<i_2 \ldots i_r} n(A_{i_1} \cap A_{i_2} \cap A_{i_r}) + \ldots$$

$$+ (-1)^{n-1} n(A_1 \cap A_2 \cap \ldots A_n)$$

$$= n(n-1)! - C(n, 2) \cdot (n-2)! \ldots + (-1)^{r-1} C(n, r) \cdot (n-r)! + (-1)^{n-1} C(n, n)(n-n)!$$

$$= n!\left[1 - \frac{1}{2!} + \frac{1}{3!} \ldots + (-1)^{r-1}\frac{1}{r!} + \ldots + (-1)^{n-1}\frac{1}{n!}\right]$$

Now $D_n = (A_1 \cap A_2 \cap \ldots \cap A_n)$ and $n(D_n) = n! - (S_n)$

$$= n!\left(1-\frac{1}{1!}+\frac{1}{2!}..(-1)^r\frac{1}{r!}+(-1)^n\frac{1}{n!}\right)$$

**Note :** We know that by the principle of inclusion and exclusion "If we consider a set A with |A| = N, and condition $C_i$, $1 \leq i \leq m$ satisfied by some of the elements of A. The number of elements of A that satisfy none of the conditions $C_i$, $1 \leq i \leq m$ is given by,

$$\bar{N} = \sum_{1\leq i\leq m} N(C_i) + \sum_{1\leq i\leq j\leq m} N(C_i, C_j) - \sum_{1\leq i\leq j\leq k\leq m} N(C_i, C_j, C_k) + \ldots + (-1)^m N(C_1, C_2, \ldots, Cm)$$

We have,

$$D_n = N(\bar{C}_1, \bar{C}_2, \ldots \bar{C}_n)$$

$$= |A| - \sum_{1\leq i\leq n} N(C_i) + \sum_{1\leq i\leq j\leq m} N(C_i, C_j) - \ldots + (-1)^n N(C_1, C_2, C_3 \ldots, C_n)$$

$$= n! - \left[{}^nC_1(n-1)! - {}^nC_2(n-2)! + {}^nC_3(n-3)! + (-1)^{n-1}\, {}^nC_n\right]$$

$$= n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots\frac{(-1)^n}{n!}\right]$$

$D_n = n!\ (e^{-1})$, first $(n+1)$ terms.

In particular,

$$D_5 = 5!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right) = 44$$

## 2.14.3 Solved Problems

### SOLVED PROBLEM 1

**The number of derangements of 1, 2, 3, 4 is,**

**$d_4 = 4!\ [1 - 1 + (1/2!) - (1/3!) + (1/4!)]$**

**$= 4![(1/2!) - (1/3!) + (1/4!)] = (4)\ (3) - 4 + 1 = 9.$**

### SOLUTION

These nine derangements are,

| | | |
|---|---|---|
| 2143 | 3142 | 4123 |
| 2341 | 3412 | 4312 |
| 2413 | 3421 | 4321 |

Among the 24 – 9 = 15 permutations of 1, 2, 3, 4 that are not derangements one finds 1234, 2314, 3241, 1342, 2431 and 2314.

Ram be... ... in a race to come in according to how they are favoured. In how many ways can they reach the finish line so that he loses all of his bets?

SOLUTION

This problem is simply reduce to finding derangement $D_{12}$ as Ram will lose all of his bets if not a single horse reach in according to how that is favoured. Hence, by the formula of $D_{12}$, we have,

$$D_{12} = 12!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}-\ldots+\frac{(-1)^{12}}{12!}\right]$$

SOLVED PROBLEM 6

**Let n book be distributed to n students. Suppose that the books are returned and distributed to the students again later on. In how many ways can the books be distributed so that no student will get the same book twice?**

SOLUTION

The n books are distributed for the first time in n! ways. The second time in $D_n$ ways. Hence, the total number of ways is given by,

$$n!\,.\,D_n = (n!)^2\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}-\ldots+\frac{(-1)^n}{n!}\right]$$

## 2.15 ROOK POLYNOMIALS, ARRANGEMENTS WITH FORBIDDEN POSITIONS

A rook is a piece that can capture another piece or pawn situated in the same row or same column, in the game of chess provided that there is no other piece or pawn between the two. The problem of nontaking rooks asks for the number of ways, $r_k(e)$, of partitioning k identical rooks on a chessboard, x with n squares, so that m rook can capture another rook.

The polynomial $R(x, \xi) = 1 + r_1 x + r_2 x^2 + \ldots + r_n x^n$ is called the rook polynomial of $\xi$.

EXAMPLE PROBLEM 1

Hence, for k = 1, 2, . . . 8 (and also for k = 0 and k > 8), $r_k = C(8, k)\ P(8, k)$, giving

$$R(x, \xi_{8+8}) = \sum_{k\geq 0}(k!)C(8,k)^2 x^k .$$

Let N = {1, 2, . . . n} for each i in N, let there be a specified (possibly empty) subset, $A_i$ in N. It is required the number of bijective functions f from N to N such that, for every i, f(i) does not belong to $A_i$ (f is a permutation of N, the elements of $A_i$ are called the forbidden positions for $i \in N$).

The expansion formula for the rook polynomial is given by,

$$R(x, c) = x\ R(x, d) + R(x, e)$$

where, c = chessboard in which a special square has been distinguished.

d = board obtained from c by deleting row and column of the special square.

e = board obtained from c by deleting only the special square.

**Note :** The shaded squares are not part of the chessboard.

## 2.15.1 Solved Problems

### SOLVED PROBLEM 1

**Find the rook polynomial of a 2 × 2 board by use of the expansion formula.**

SOLUTION

For any special square, that we choose,

We have,

$R(x, d) = 1 + x,$

$R(x, e) = 1 + 3x + x^2,$

Hence $R(x, c) = x(1 + x) + (1 + 3x + x^2)$

$= 1 + 4x + 2x^2$

### SOLVED PROBLEM 2

**Find the Rook Polynomial of the following board.**

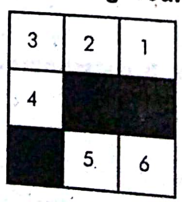



**Fig. 2.15.1 :** Board

**Compute the rook polynomial r (c, x) for this chess board.**

**SOLUTION**

If the board is given the name as c, then it will be denoted as, r(c, x).

$\therefore \quad r(c, x) = 1 + r_1x + r_2x^2 + \ldots + r_nx^n$ . . . (1)

while defining this polynomial, it has been assumed that $n \geq 2$.

In the trivial case where $n = 1$, $r_2$, $r_3$...... are identically zero.

$\therefore \quad r(c, x) = 1 + x$ . . . (2)

Equation (1) and (2) are put in the following combined form which holds for a board c with $n \geq 1$ squares.

$r(c, x) = 1 + r_1x + r_2x^2 + \ldots\ldots + r_nx^n$ . . . (3)

Here $r_1 = n =$ number of squares in the board.

$\therefore$ For the given chess board, Rook polynomial can be defined as,

$r(c,x) = 1 + 6x + 8x^2 + 2x^3$.

## SOLVED PROBLEM 3

**Find the rook polynomial of the forbidden subboard (squares marked with ×) in the Fig. 2.15.2.**

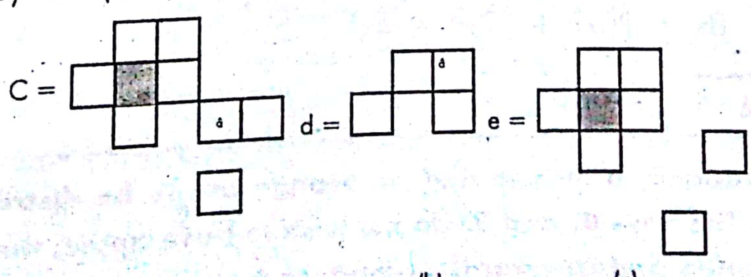
| | | | | |
|---|---|---|---|---|
| | × | × | | |
| × | | × | | |
| | × | | × | × |
| | | | | |
| | | | × | |

**Fig. 2.15.2 :** Board

**SOLUTION**

Fig. 2.15.3 shows the decomposition of the forbidden board relative to the indicated special square,

$R(x, c) = xR(x, d) + R(x, e)$ . . . (1)




(a) (b) (c)

**Fig. 2.15.3**



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Decomposition of d as in Fig. 2.15.3 is shown in the Fig. 2.15.4.

$R(x, d) = x(1 + x) + (1 + 3x + 2x^2) = 1 + 4x + 3x^2$ . . . (2)

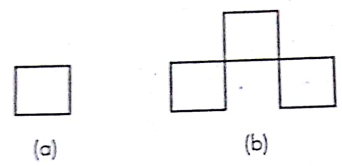

Fig. 2.15.4

The board e is the union of 2 disjoint subboards, one of which is a single square and the other is the board f of the Fig. 2.15.5.

Thus, $R(x, e) = (1 + x)\, R(x, f)$

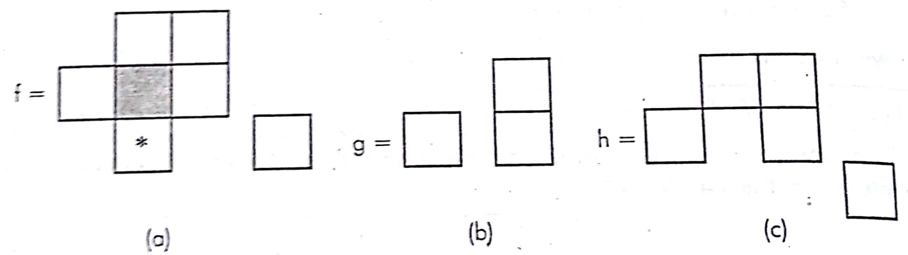

Fig. 2.15.5

Decomposition of board f into g and h

$R(x, f) = x(1 + 3x + x^2) + (1 + 4x + 3x^2)(1 + x)$

[∵ h is also the union of 2 disjoint subboards]

$= 1 + 6x + 10x^2 + 4x^3$

$R(x, e) = 1 + 7x + 16x^2 + 14x^3 + 4x^4$ . . . (3)

Putting (2) and (3) in (1), yields

$R(x, c) = 1 + 8x + 20x^2 + 17x^3 + 4x^4$ . . . (4)

## SOLVED PROBLEM 4

**An apple, a banana, a mango and an orange are to be distributed to four boys $B_1$, $B_2$, $B_3$ and $B_4$. The boys $B_1$ and $B_2$ do not wish to have apple, the boy $B_3$ does not want banana or mango and $B_4$ returns orange. In how many ways the distribution can be made so that no boy is displeased?**