UNIT 4

CONVOLUTION AND CORRELATION OF SIGNALS

SYLLABUS

Concept of Convolution in Time Domain and Frequency Domain, Graphical Representation of Convolution, Convolution Property of Fourier Transforms, Cross Correlation and Autocorrelation of Functions, Properties of Correlation Function, Energy Density Spectrum, Parseval's Theorem, Powers Density Spectrum, Relation between Autocorrelation Function and Energy/Power Spectral Density Function, Relation between Convolution and Correlation, Detection of Periodic Signals in the Presence of Noise by Correlation, Extraction of Signal from Noise by Filtering.

PART - A
SHORT QUESTIONS WITH ANSWERS

Q1) Write the properties for convolution integral.

Ans. : The properties of convolution integral are as follows,

(1) Commutative Property : The commutative property of convolution integral states that,

$$x(t) * h(t) = h(t) * x(t)$$

(2) Distributive Property : The distributive property of convolution integral states that,

$$x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$$

(3) Associative Property : The associative property of convolution integral states that,

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

Q2) State frequency convolution theorem.

Ans. : The frequency convolution theorem states that the multiplication of continuous time signals in time domain is equivalent to convolution of their respective spectra in frequency domain. Mathematically,

If, $$x_1(t) \xleftrightarrow{FT} X_1(\omega)$$

And $$x_2(t) \xleftrightarrow{FT} X_2(\omega)$$

Then, $$x_1(t) \square x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

Q3) What do you mean by convolution sum?

Ans. : In discrete-time systems, the convolution sum, relates input and impulse response of the system to output. It is mathematically expressed as,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

The relationship expressed in above equation is called the convolution sum of input x[n] and unit impulse response h(n). This operation is represented symbolically as,

$$y(n) = x(n)*h(n)$$

Q4) Define cross correlation and outcorrelation.

Ans. : Core Correlation : Cross-correlation is the measure of similarity between one signal and time-delayed version of another signal. It means that the cross-correlation explains how much one signal is related to the time delayed version of another signal.

Let x(t) and y(t) be the two different complex valued energy signals. The cross-correlation between these two energy signals is defined as,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y^*(t-\tau)\, dt = \int_{-\infty}^{\infty} x(t+\tau)\, y^*(t)\, dt$$

Autocorrelation : Autocorrelation is the measure of similarity between signal and time delayed version of same signal.

If x(t) is a complex-valued energy signal then the autocorrelation function is defined as,

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt = \int_{-\infty}^{\infty} x(t+\tau)\, x^*(t)\, dt$$

Q5) What is the correlation between the two periodic power signals?

Ans. : An important special case of correlation of power signals is the correlation between two periodic signals whose fundamental periods are such that the product of the two signals is also periodic. This will happen any time the ratio of their fundamental periods is a rational number.

For two signals whose product has a period T, the general form of the correlation function (for real power signals) is given by,

$$R_{xy}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, y(t-\tau)\, dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t+\tau)y(t)\, dt$$

Q6) What are the steps involved in the graphical representation of correlation of DTS signals?

Ans. : Cross correlation of a discrete-time signals x(n) and y(n) can be computed using graphical method by the following sequence of operations.

Step I (Shifting) : Shift the sequence y(n) by m units to obtain y(n − m).

Step II (Multiplication and Adding) : Multiply the shifted sequence y(n − m) with x(n) and add all the values to obtain $R_{xy}(m)$.

Repeat this procedure for each value of m over the interval −∞ to ∞.

Q7) What is the correlation theorem in cross correlation function?

Ans. : Fourier transform of cross correlation function is the product of two energy signals in frequency domain.

i.e.,

$$R_{xy}(\tau) \xleftrightarrow{F.T} X(\omega)\, Y^*(\omega)$$

The above is known as the correlation theorem.

Q8) List out the properties of auto correlation of power signals.

Ans. : **Property I :** The autocorrelation function of power signals exhibits *conjugate symmetry*, that is,

$$R_{xx}(\tau) = R^*_{xx}(-\tau)$$

Property II : The value of the autocorrelation function of a power signal at the origin is equal to the average power of the signal, that is,

$$R_{xx}(0) = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = P_x$$

Property III : The autocorrelation $R_{xx}(\tau)$ and power spectral density, $S(\omega)$ of a power signal forms a Fourier transform pair, i.e.,

$$R_{xx}(\tau) \xleftrightarrow{F.T} S(\omega) \text{ or } S(f)$$

Property IV : The autocorrelation of a periodic signal with period T is periodic with the same period.

If, $x(t) = x(t \pm T)$,

Then, $R_{xx}(\tau) = R_{xx}(\tau \pm T)$

Q9) Define the term "normalized energy"?

Ans. : *The normalized energy (or simply energy) of a signal x(t) is defined as energy dissipated by a voltage signal applied across a 1 ohm resistor (or alternatively by a current signal flowing through a 1 ohm resistor).* The signal x(t) may be complex or real valued.

The normalized energy of the signal is defined as,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

It should be noted that the energy of a signal defined by the above equation is only when the above integral is finite.

Q10) What is Parseval's power theorem?

Ans. : Parseval's power theorem defines the power of a signal in terms of its Fourier series coefficients, i.e., in terms of the harmonic components present in the signal. Mathematically, it is given by,

$$P = \sum_{n=-\infty}^{\infty} |C_n|^2$$

Q11) State the properties of power spectral density.

Ans. : The following are the properties of power spectral density,

PROPERTY I : The area under the PSD function is equal to the average power of that signal, i.e.,

$$P = \int_{-\infty}^{\infty} S(f) \, df$$

PROPERTY II : The input and output PSDs of an LTI system are related as,

$$S_y(f) = |H(f)|^2 S_x(f)$$

Where,

$S_y(f)$ = Power spectral density of output y(t).

$S_x(f)$ = Power spectral density of input x(t).

H(f) = Transfer function of the system.

PROPERTY III : The autocorrelation function $R_{xx}(\tau)$ and PSD S(f) form a Fourier transform pair, i.e.,

$$R_{xx}(\tau) \xleftrightarrow{F.T} S_x(f)$$

Q12) Compare the energy spectral density and power spectral density.

Ans. :

Table Comparison of ESD and PSD

Sl. No.	Energy Spectral Density (ESD)	Power Spectral Density (PSD)
(1)	It defines the distribution of energy of a signal in frequency domain.	It defines the distribution of power of a signal in frequency domain.
(2)	It is given by, $\psi(f) = \lvert X(f) \rvert^2$	It is given by, $S(f) = \lim\limits_{T \to \infty} \frac{\lvert X_T(f) \rvert^2}{T}$
(3)	The total energy is given by, $E = \int_{-\infty}^{\infty} \psi(f)\, df = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(\omega)\, d\omega$	The total power is given by, $P = \int_{-\infty}^{\infty} S(f)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, d\omega$
(4)	The autocorrelation for an energy signal and its ESD form a Fourier transform pair. $R_{xx}(\tau) \xleftrightarrow{F.T} \psi(f)$ (or) $R_{xx}(\tau) \xleftrightarrow{F.T} \psi(\omega)$	The autocorrelation for a power signal and its PSD form a Fourier transform pair. $R_{xx}(\tau) \xleftrightarrow{F.T} S(f)$ (or) $R_{xx}(\tau) \xleftrightarrow{F.T} S(\omega)$

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PART - B

ESSAY QUESTIONS WITH REFERENCES

Q1) Give some properties of convolution integral. How the convolution is expressed in time domain and frequency domain? [Refer Section Nos. 4.2.2, 4.2.3 and 4.2.4]

Q2) Obtain the convolution of the functions,

$$x_1(t) = e^{-t}\, u(t)$$

And $$x(t) = u(t + 1)$$ [Refer Section No. 4.2.4]

Q3) How the convolution of two continuous time signals are performed by using a graphical method? Give an example. [Refer Section No. 4.3]

Q4) What is the convolution property of Fourier transforms? [Refer Section No. 4.4]

Q5) Define autocorrelation and cross-correlation and give a graphical interpretation of correlation of continuous-time signals? [Refer Section Nos. 4.5.1 and 4.5.3]

Q6) Write properties of autocorrelation? [Refer Section Nos. 4.6.3 and 4.6.4]

Q7) Explain properties of cross correlation of both energy signal and power signal? [Refer Section Nos. 4.6.1 and 4.6.2]

Q8) Explain about Parsevals theorem for energy signals. Discuss about ESD. [Refer Section Nos. 4.7.1 and 4.7.2]

Q9) Explain power spectral density along with its properties? [Refer Section Nos. 4.8.2 and 4.8.3]

Q10) Differentiate between ESD and PSD? [Refer Section No. 4.8]

Q11) Discuss about power density spectrum. State and prove Parseval's power theorem? [Refer Section Nos. 4.8 and 4.8.1]

Q12) How autocorrelation is proportional to energy spectral density and power spectrum density? [Refer Section Nos. 4.9.1 and 4.9.2]

Q13) Explain how and when the cross-correlation and convolutiona re equivalent. [Refer Section No. 4.10]

Q14) Explain an effective method of detection of periodic signals in the presence of noise? [Refer Section No. 4.11]

Q15) How correlation is related to the filtering? [Refer Section No. 4.12]

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4.1 INTRODUCTION

Convolution is an important mathematical operation used to combine two signals to form a new signal. Convolution gives the relationship between the input signal and impulse response of the Linear Time Invariant (LTI) system to the output signal.

Correlation is an another important mathematical operation in the analysis of signals and systems, that is similar to convolution. Correlation is also used to combine two signals to form a new signal. Correlation between two signals is used to measure the degree to which the two signals are similar. Correlation is widely used in practice, particularly in Radar, Sonar and digital communications. Correlation is of two types,

(1) Cross-correlation

(2) Autocorrelation.

4.2 CONCEPT OF CONVOLUTION IN TIME DOMAIN AND FREQUENCY DOMAIN

Convolution is a mathematical operation which is used to express the input-output relationship of Linear Time Invariant (LTI) system.

4.2.1 Convolution Integral

In continuous-time LTI system, the convolution integral, relates input and impulse response of the system to output.

Consider an LTI continuous-time system, with x(t) and y(t) as its input and output respectively as shown in Fig. 4.2.1(a).

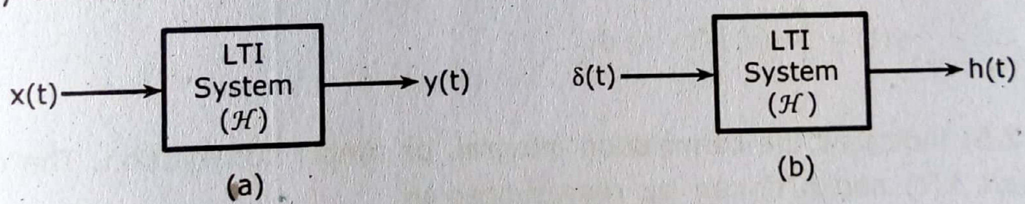


Fig. 4.2.1 An LTI System

If the input to the system is an impulse as shown in Fig. 4.2.1(b) then the output of the system is denoted by h(t) and is called the impulse response of the system.

The impulse response is defined as,

$$h(t) = \mathcal{H}[\delta(t)]$$

We know that any arbitrary signal x(t) can be represented as,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

The output of the system shown in Fig. 4.2.1 is given by,

$$y(t) = \mathcal{H}[x(t)]$$

$$\therefore \quad y(t) = \mathcal{H}\left[\int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau\right]$$

For a linear system,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, \mathcal{H}[\delta(t-\tau)]\, d\tau \quad \text{... (4.2.1)}$$

If the response of the system due to impulse $\delta(t)$ is $h(t)$, then the response of the system due to delayed impulse is,

$$h(t, \tau) = \mathcal{H}[\delta(t-\tau)] \quad \text{... (4.2.2)}$$

Substituting Eq. (4.2.2) in Eq. (4.2.1), we get,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t, \tau)\, d\tau \quad \text{... (4.2.3)}$$

For a time invariant system, the output due to input delayed by τ sec is equal to the output delayed by τ sec. That is,

$$h(t, \tau) = h(t-\tau) \quad \text{... (4.2.4)}$$

Substituting Eq. (4.2.4) in Eq. (4.2.3), we get,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad \text{... (4.2.5)}$$

Eq. (4.2.5) indicates the convolution integral, or simply convolution. The convolution of two signals $x(t)$ and $h(t)$ can be represented as,

$$y(t) = x(t) * h(t)$$

Where the symbol * denotes the convolution between two signals.

4.2.2 Properties of Convolution Integral

The properties of convolution integral are as follows,

(1) Commutative Property : The commutative property of convolution integral states that,

$$x(t) * h(t) = h(t) * x(t)$$

(2) Distributive Property : The distributive property of convolution integral states that,

$$x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$$

(3) **Associative Property :** The associative property of convolution integral states that,

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

(4) **Shift Property :** The shift property of convolution integral states that if,

$$x(t) * h(t) = y(t)$$

Then, $x(t) * h(t - T) = y(t - T)$

Similarly, $x(t - T) * h(t) = y(t - T)$

And, $x(t - T_1) * h(t - T_2) = y(t - T_1 - T_2)$

(5) **Convolution with an Impulse :** Convolution of a signal x(t) with a unit impulse is the signal itself. That is,

$$x(t) * \delta(t) = x(t)$$

(6) **Width Property :** Let the duration of x(t) and h(t) be T_1 and T_2 respectively. Then the duration of the signal obtained by convolving x(t) and h(t) is $T_1 + T_2$.

4.2.3 Time Convolution Theorem

Statement : The time convolution theorem states that convolution of two continuous-time signal in time domain is equivalent to multiplication of their respective spectra in frequency domain. Mathematically,

If, $x_1(t) \xleftrightarrow{FT} X_1(\omega)$

And $x_2(t) \xleftrightarrow{FT} X_2(\omega)$

Then, $x_1(t) * x_2(t) \xleftrightarrow{FT} X_1(\omega)X_2(\omega)$

PROOF

The Fourier transform of $[x_1(t) * x_2(t)]$ is defined as,

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-j\omega t}\, dt$$

But, $$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t - \tau)\, d\tau$$

$$\therefore \quad F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} [x_1(\tau)\, x_2(t - \tau)]\, d\tau \right\} e^{-j\omega t}\, dt$$

Let $t - \tau = k$, in the second integration then, we have $t = k + \tau$ and $dt = dk$.

$$\therefore \quad F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(k)e^{-j\omega(k+\tau)}\,dk\right]d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\underbrace{\left[\int_{-\infty}^{\infty} x_2(k)e^{-j\omega k}\,dk\right]}_{X_2(\omega)} e^{-j\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\,X_2(\omega)\,e^{-j\omega}\,d\tau = \underbrace{\left[\int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}\,d\tau\right]}_{X_1(\omega)} X_2(\omega)$$

$$= X_1(\omega)X_2(\omega)$$

$$\therefore \quad x_1(t) * x_2(t) \xleftrightarrow{FT} X_1(\omega)X_2(\omega) \qquad \ldots (4.2.6)$$

Eq. (4.2.6) represents the time convolution theorem.

4.2.4 Frequency Convolution Theorem

Statement : The frequency convolution theorem states that the multiplication of continuous time signals in time domain is equivalent to convolution of their respective spectra in frequency domain. Mathematically,

If, $\quad x_1(t) \xleftrightarrow{FT} X_1(\omega)$

And $\quad x_2(t) \xleftrightarrow{FT} X_2(\omega)$

Then, $\quad x_1(t)\Box x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$

PROOF

The Fourier transform of $x_1(t)\Box x_2(t)$ is given by,

$$F[x_1(t).x_2(t)] = \int_{-\infty}^{\infty} [x_1(t)x_2(t)]\,e^{-j\omega t}\,dt$$

Expressing $x_2(t)$ as the inverse Fourier transform of $X_2(\omega')$, we get,

$$F[x_1(t)x_2(t)] = \int_{-\infty}^{\infty} x_1(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega')e^{j\omega' t}\,d\omega'\right)e^{-j\omega t}\,dt$$

Interchanging the order of integration,

$$F[x_1(t)x_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega')\, d\omega' \left(\int_{-\infty}^{\infty} x_1(t)\, e^{-j(\omega - \omega')}\, dt\right) \quad \text{... (4.2.7)}$$

Using the frequency-shifting property, the bracketed term is $X_1(\omega - \omega')$. Substituting this into the Eq. (4.2.7) gives,

$$F[x_1(t)x_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega')\, X_1(\omega - \omega')\, d\omega' \quad \text{... (4.2.8)}$$

Putting, $\omega - \omega' = \lambda$, in Eq. (4.2.8), we get,

$$F[x_1(t)x_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega - \lambda) X_1(\lambda)\, d\lambda = \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

$$\therefore \quad x_1(t)x_2(t) \xleftrightarrow{\text{F.T}} \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

(or) $$2\pi x_1(t)x_2(t) \xleftrightarrow{\text{FT}} X_1(\omega) * X_2(\omega) \quad \text{... (4.2.9)}$$

In terms of frequency, we get,

$$F[x_1(t)x_2(t)] = X_1(f) * X_2(f) \quad \text{... (4.2.10)}$$

Eq. (4.2.9) and Eq. (4.2.10) represents the frequency convolution theorem.

EXAMPLE PROBLEM 1

Find the convolution of the following signals,

(i) $x_1(t) = e^{-t}\, u(t),\ x_2(t) = e^{-2t}\, u(t)$

(ii) $x_1(t) = e^{-t}\, u(t),\ x_2(t) = u(t + 1)$

SOLUTION

(i) $x_1(t) = e^{-t}\, u(t),\ x_2(t) = e^{-2t}\, u(t)$

Convolution of signals in time-domain is defined as,

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t - \tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\tau}\, u(\tau)\, e^{-2(t-\tau)}\, u(t - \tau)\, d\tau$$

We have,

$$u(\tau) = \begin{cases} 1 & ; \quad \tau \geq 0 \\ 0 & ; \quad \tau < 0 \end{cases} \quad \text{and} \quad u(t-\tau) = \begin{cases} 1 & ; \quad t-\tau \geq 0 \rightarrow t \geq \tau \\ 0 & ; \quad \tau - t < 0 \rightarrow t < \tau \end{cases}$$

Hence, $u(\tau)u(t - \tau) = 1$ only for $0 < \tau < t$.

For all other values of τ,

$$u(\tau)u(t - \tau) = 0.$$

$$x_1(t) * x_2(t) = \int_0^t e^{-\tau}\, e^{-2(t-\tau)}\, d\tau$$

$$= e^{-2t} \int_0^t e^{\tau}\, d\tau$$

$$= e^{-2t}\left[e^{\tau}\right]_0^t$$

$$= e^{-2t}[e^t - 1]$$

$$\therefore \quad = (e^{-t} - e^{-2t})u(t)$$

(ii) $x_1(t) = e^{-t}\, u(t)$, $x_2(t) = u(t + 1)$

Convolution of signals in time-domain is defined as,

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\tau}\, u(\tau)\, u(t+1-\tau)\, d\tau$$

In this case,

$$u(\tau) = \begin{cases} 1 & ; \quad \tau \geq 0 \\ 0 & ; \quad \tau < 0 \end{cases} \quad \text{and} \quad u(t+1-\tau) = \begin{cases} 1 & ; \quad \tau \leq t+1 \\ 0 & ; \quad \tau > t+1 \end{cases}$$

Hence $u(\tau).u(t+1-\tau) = 1$ only for $0 < \tau < t+1$. For all other values of τ, $u(\tau).u(t+1-\tau) = 0$.

$$x_1(t) * x_2(t) = \int_0^{t+1} e^{-\tau}\, d\tau = \left[-e^{-\tau}\right]_0^{t+1}$$

$$\therefore \quad = -e^{-(t+1)} + e^0 = 1 - e^{-(t+1)}$$

EXAMPLE PROBLEM 2 :

Using time-convolution theorem. Find the convolution of two signals of Fourier transform. $x_1(t) = e^{-2t}\, u(t)$ and $x^2(t) = e^{-2t}\, u(t)$

SOLUTION

Given Data : $x_1(t) = e^{-2t}\, u(t)$

$x_2(t) = e^{-2t}\, u(t)$

We have,

$$F[e^{-at}\, u(t)] = \frac{1}{a + j\omega}$$

$$\therefore \quad F[x_1(t)] = X_1(\omega)$$

$$= \frac{1}{2 + j\omega}$$

And $\quad F[x_2(t)] = X_2(\omega)$

$$= \frac{1}{2 + j\omega}$$

From the definition of time-convolution theorem,

$$x_1(t) * x_2(t) \xleftrightarrow{F.T} X_1(\omega)\, X_2(\omega)$$

$$\Rightarrow \quad F[x_1(t) * x_2(t)] = X_1(\omega)\, X_1(\omega)$$

$$\Rightarrow \quad x_1(t) * x_2(t) = F^{-1}[X_1(\omega)\, X_1(\omega)]$$

$$= F^{-1}\left[\frac{1}{2 + j\omega} \cdot \frac{1}{2 + j\omega}\right]$$

$$= F^{-1}\left[\frac{1}{(2 + j\omega)^2}\right]$$

$$= te^{-2t}u(t)$$

REVIEW QUESTIONS

(1) *Give some properties of convolution integral.*

(2) *How the convolution is expressed in time domain and frequency domain?*

(3) *State and prove frequency and time convolution theorem.*

4.3 GRAPHICAL REPRESENTATION OF CONVOLUTION

The convolution of two continuous-time signals $x_1(t)$ and $x_2(t)$ using the graphical method involves the following steps,

Step I (Change of Time Index) : Replace the independent variable t by a dummy variable τ for the given signals $x_1(t)$ and $x_2(t)$ and plot the graph for $x_1(\tau)$ and $x_2(\tau)$.

Step II (Folding) : Keep the function $x_1(\tau)$ fixed and fold (or invert) the function $x_2(\tau)$ about the vertical axis ($\tau = 0$) to get $x_2(-\tau)$.

Step III (Shifting) : Shift the function $x_2(-\tau)$ by t units. The shifted $x_2(-\tau)$ now represents $x_2(t - \tau)$. Plot the graph for $x_1(\tau)$ and $x_2(t - \tau)$ on the same axis beginning with time shift t.

Step IV (Multiplication and Integration) : The signals $x_1(\tau)$ and the shifted signal $x_2(t - \tau)$ are multiplied to get a product signal $x_1(\tau)x_2(t - \tau)$. For a particular value of $t = k$, integration of the product signal represents the area under the product curve (i.e., common area). This common area represents the convolution of $x_1(t)$ and $x_2(t)$ for a shift of $t = k$. That is,

$$\int_{-\infty}^{\infty} x_1(\tau)x_2(k - \tau)\, d\tau = [x_1(t) * x_2(t)]_{t=k}$$

Repeat this procedure for different values of t by successively progressing the frame by different amounts and find the values of the convolution function x(t)*h(t) at those values of t.

EXAMPLE PROBLEM 1

Find the convolution of given signal as shown in Fig. 4.3.1 with itself.

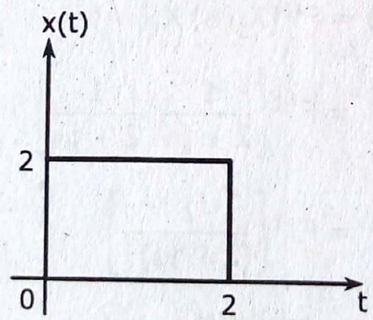


Fig. 4.3.1

SOLUTION

By definition of convolution,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau$$

Given convolution of a signal with itself, so $h(t) = x(t)$ only. The mathematical expression for $x(t)$ and $h(t)$ is,

$$x(t) = h(t) = \begin{cases} 2 & ; \quad 0 < t < 2 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

The graphical plots of $x(t)$ and $h(t)$ are shown in Fig. 4.3.2.

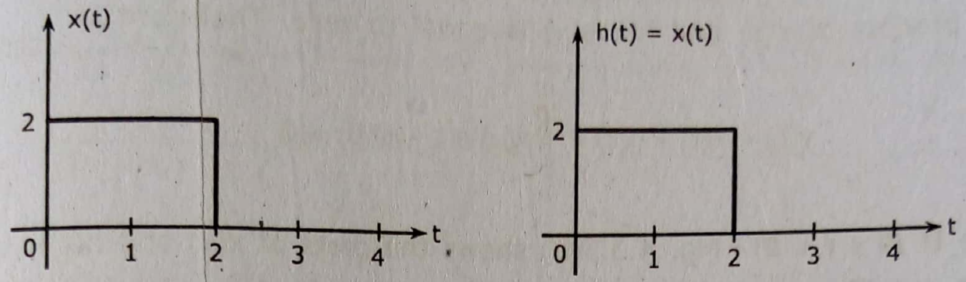


Fig. 4.3.2 Graphical Plots of x(t) and h(t)

Step I, II (Change of Time Index, Folding) : Change the variable t with independent variable τ. Thus $x(t)$ and $h(t)$ now represents $x(\tau)$ and $h(\tau)$ about $\tau = 0$. $h(\tau)$ now represents $h(-\tau)$. The graphical plots of $x(\tau)$ and $h(-\tau)$ are shown in Fig. 4.3.3.

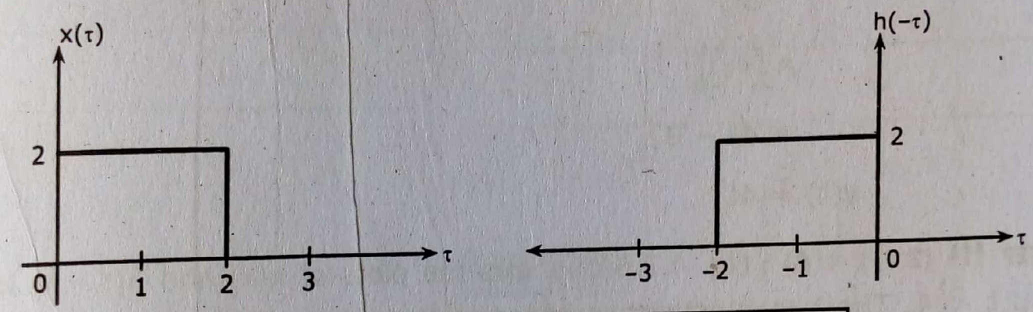


Fig. 4.3.3 Graphical Plots of x(τ) and h(−τ)

Step III (Shifting) : Shift the signal $h(-\tau)$ by t units beginning with very large negative time. The shifted $h(-\tau)$ now represents $h(t - \tau)$. The graphical plots of $x(\tau)$ and $h(t - \tau)$ are shown in Fig. 4.3.4. The Mathematical expression of function $h(t - \tau)$ is defined as,

$$h(t-\tau) = \begin{cases} 2 & ; \quad 0 < (t-\tau) < 2 \rightarrow -t < -\tau < 2 - t \rightarrow t - 2 < \tau < t \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

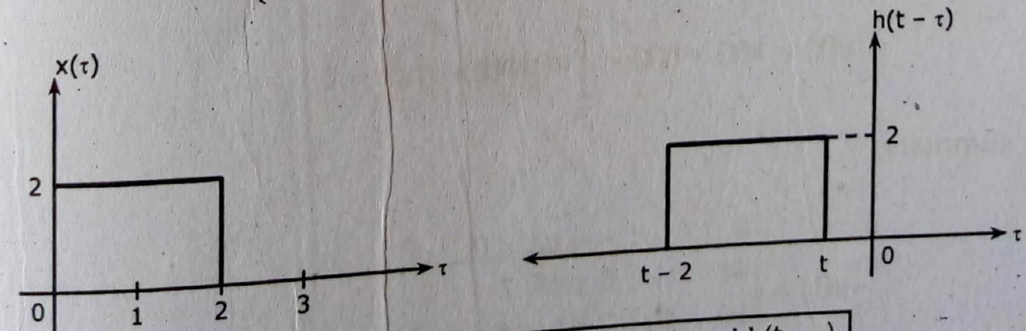


Fig. 4.3.4 Graphical Plots of x(τ) and h(t − τ)

STEP IV, V (Multiplication and Integration) : Perform multiplication and integration to find x(t)*h(t) for all values of t. The graphical plots for various intervals of t are shown in Fig. 4.3.5.

CASE I ($-\infty < t < 0$) : Fig. 4.3.5(a) shows the plots of $x(\tau)$ and $h(t - \tau)$ in the interval $-\infty < t < 0$. From Fig. 4.3.5(a), we can see that the signals do not overlap, hence the product of $x(\tau)$ and $h(t - \tau)$ is equal to zero. Therefore,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{0} x(\tau)\, h(t - \tau)\, d\tau = 0$$

STEP II ($0 \le t < 2$) : Fig. 4.3.5(b) shows the plots of $x(\tau)$ and $h(t - \tau)$ in the interval $0 \le t < 2$. The two functions overlaps in the interval $0 \le \tau \le t$ (shaded area). Therefore,

$$y(t) = x(t) * h(t)$$

$$= \int_{0}^{t} 2.2\, d\tau$$

$$= [4\tau]_0^t$$

$$= 4t - 0$$

$$y(t) = 4t$$

STEP III ($2 \le t < 4$) : Fig. 4.3.5(c) shows the plots of $x(\tau)$ and $h(t - \tau)$ in the interval $2 \le t < 4$. The two functions overlaps in the interval $t - 2 \le \tau \le 2$ (shaded area). Therefore,

$$y(t) = x(t) * h(t) = \int_{t-2}^{2} 2.2\, d\tau = [4\tau]_{t-2}^{2} = 8 - 4(t - 2) = 16 - 4t$$

STEP IV ($4 \le t < \infty$) : Fig. 4.3.5(d) shows the plots of $x(\tau)$ and $h(t - \tau)$ in the interval $4 \le t < \infty$. The two functions does not overlaps. Therefore,

$$y(t) = x(t) * h(t) = \int_{4}^{\infty} x(\tau) h(t - \tau)\, d\tau = 0$$

In summary,

$$y(t) = \begin{cases} 0 & ; -\infty < t < 0 \\ 4t & ; 0 \le t < 2 \\ 16 - 4t & ; 2 \le t < 4 \\ 0 & ; 4 \le t < \infty \end{cases}$$

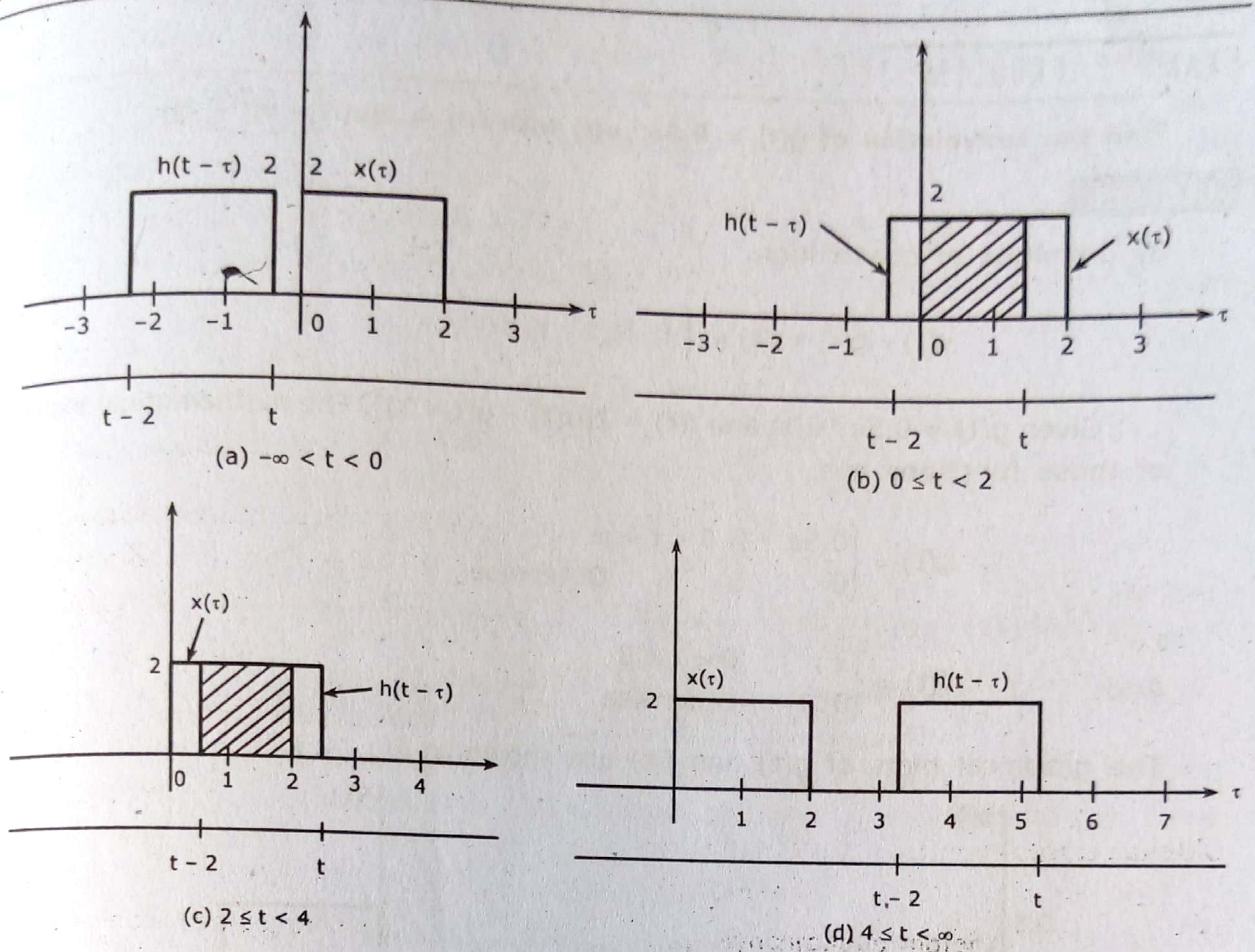


Fig. 4.3.5

The convolved signal y(t) = x(t)*h(t) is as shown in Fig. 4.3.6.

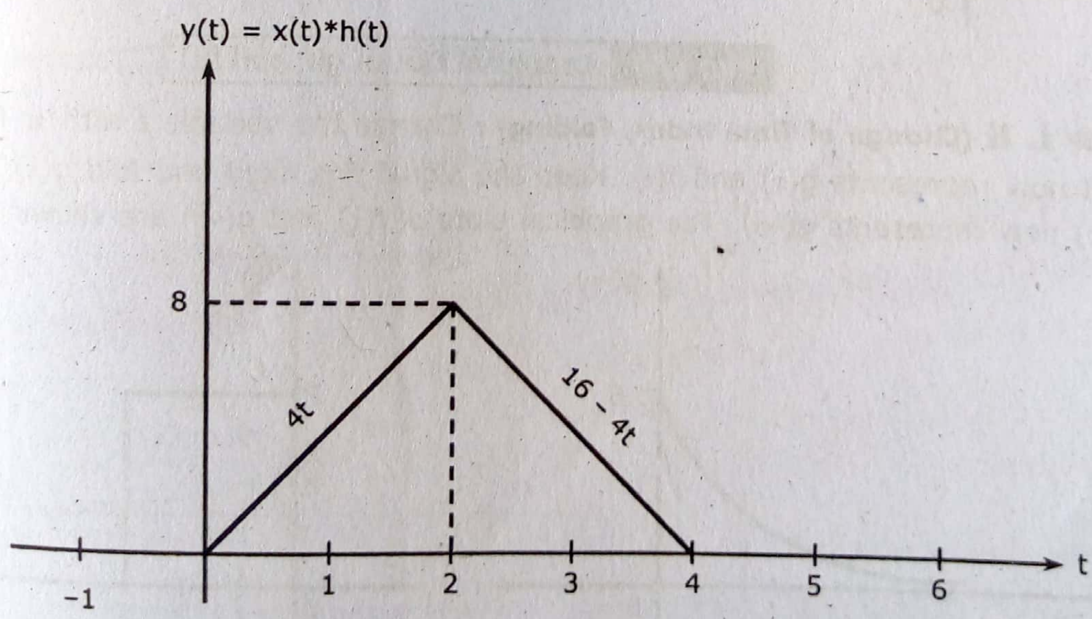


Fig. 4.3.6 Graphical Plot of y(t) = x(t)*h(t)

EXAMPLE PROBLEM 2

Find the convolution of $g(t) = 0.5e^{-t}\, u(t)$ with $f(t) = 2[u(t) - u(t - 3)]$.

SOLUTION

By definition of convolution,

$$y(t) = g(t) * f(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau$$

Given $g(t) = 0.5e^{-t}u(t)$ and $f(t) = 2[u(t) - u(t - 3)]$. The mathematical expressions of these functions are,

$$g(t) = \begin{cases} 0.5e^{-t}\ ; & 0 < t < \infty \\ 0\ ; & \text{Otherwise} \end{cases}$$

And
$$f(t) = \begin{cases} 2\ ; & 0 < t < 3 \\ 0\ ; & \text{Otherwise} \end{cases}$$

The graphical plots of g(t) and f(t) are shown in Fig. 4.3.7.

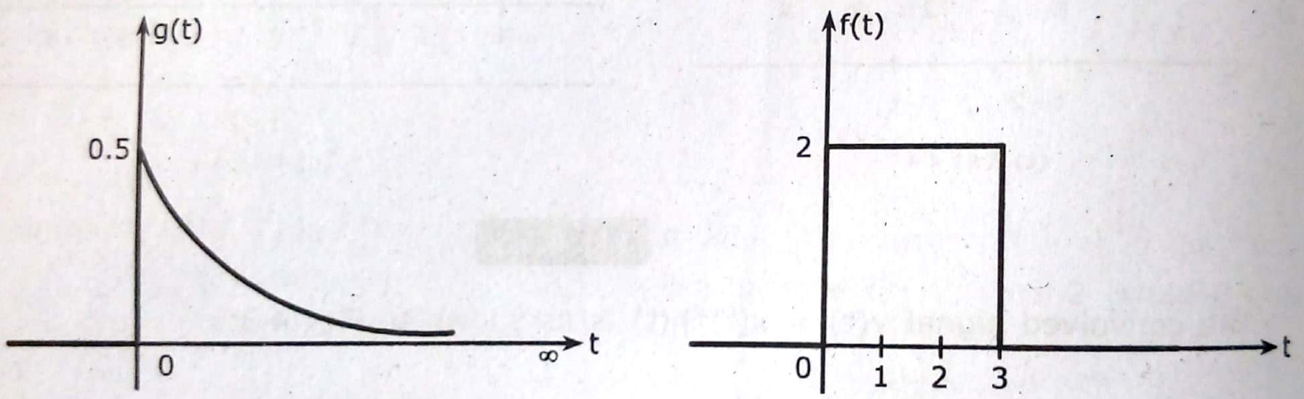


Fig. 4.3.7 Graphical Plot of g(t) and f(t)

STEP I, II (Change of Time Index, Folding) : Change the variable t with τ. Thus g(t) and f(t) now represents $g(\tau)$ and $f(\tau)$. Keep the signal $f(\tau)$ fixed and fold $g(\tau)$ about $\tau = 0$. $g(\tau)$ now represents $g(-\tau)$. The graphical plots of $f(\tau)$ and $g(-\tau)$ are shown in Fig. 4.3.8.

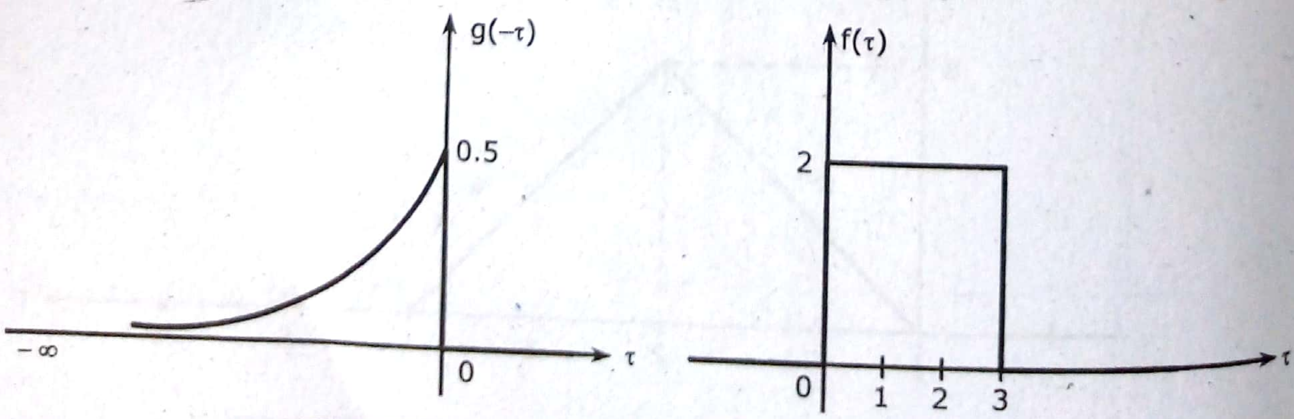


Fig. 4.3.8 Graphical Plot of $g(-\tau)$ and $f(\tau)$

STEP III (Shifting) : Shift the signal $g(-\tau)$ by t units beginning with very large negative time. The shifted $g(-\tau)$ now represents $g(t - \tau)$. The graphical plots of $f(\tau)$ and $g(t - \tau)$ are shown in Fig. 4.3.9. The mathematical expression of function $g(t - \tau)$ is defined as,

$$g(t-\tau) = \begin{cases} 0.5e^{-t} & ; \quad 0 < t-\tau < \infty \rightarrow -t < -\tau < \infty \rightarrow -\infty < \tau < t \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

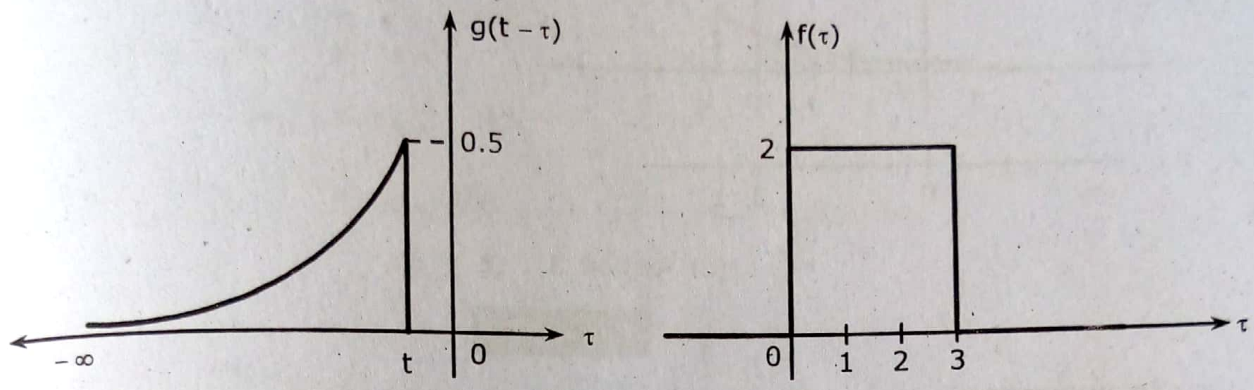
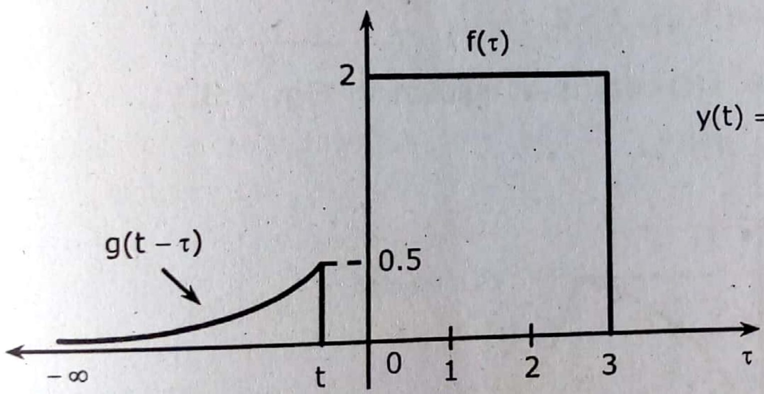


Fig. 4.3.9 Graphical Plots of $f(\tau)$ and $g(t - \tau)$

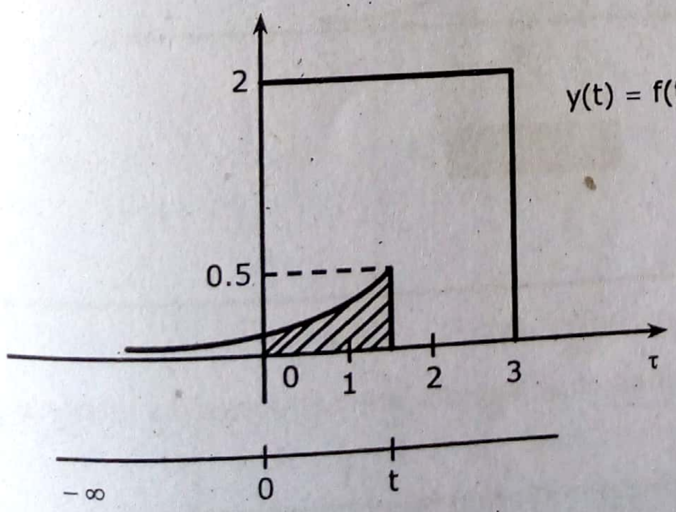
STEP IV, V (Multiplication and Integration) : Perform multiplication and integration to find $g(t)*f(t)$ for all values of t. The graphical plots for various intervals of t are shown in Fig. 4.3.10.



$$y(t) = f(t) * g(t) = \int_{-\infty}^{t} f(\tau)\, g(t-\tau)\, d\tau$$

$$= 0$$

(a) Region 1 : (t < 0)

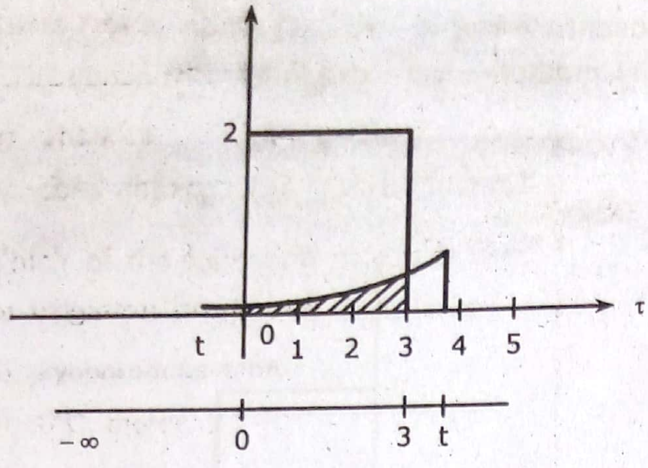


$$y(t) = f(t) * g(t) = \int_{0}^{t} f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_{0}^{t} 2(0.5)e^{-(t-\tau)}\, d\tau$$

$$= \int_{0}^{t} e^{-t} e^{\tau}\, d\tau = e^{-t}\left[e^{t} - e^{0}\right]$$

$$= e^{-t}\left[e^{t} - 1\right] = 1 - e^{-t}$$



(b) Region 2 : (0 < t < 3)

$$y(t) = f(t) * g(t) = \int_0^3 f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^3 2(0.5)e^{-(t-\tau)}\, d\tau = \int_0^3 e^{-t} e^{\tau}\, d\tau$$

$$= e^{-t}\left[e^3 - e^0\right] = e^{-t}\left[e^3 - 1\right]$$

$$= e^{3-t} - e^{-t}$$

$$y(t)_{t=3} = e^{3-3} - e^{-3} = 1 - e^{-3}$$

(c) Region 3 : (t > 3)

Fig. 4.3.10

In summary,

$$y(t) = \begin{cases} 0 & ; \quad t < 0 \\ 1 - e^{-t} & ; \quad 0 < t < 3 \\ 1 - e^{-3} & ; \quad t = 3 \\ e^{3-t} - e^{-t} & ; \quad t > 3 \end{cases}$$

The convolved signal y(t) = x(t)∗h(t) is as shown in Fig. 4.3.11.

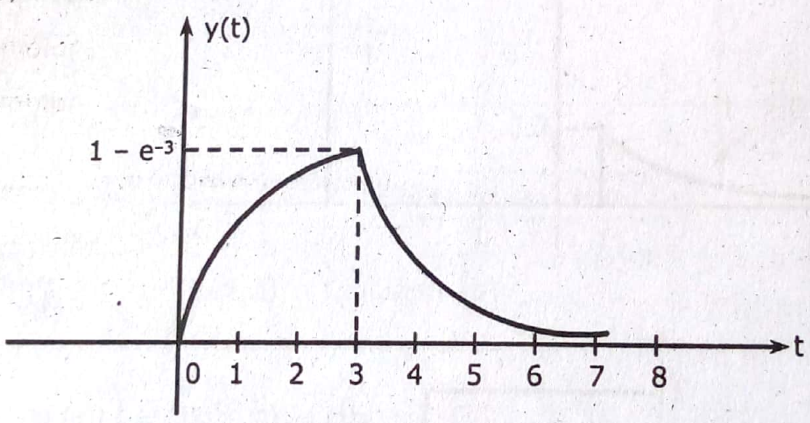


Fig. 4.3.11

REVIEW QUESTIONS

(1) Describe the graphical interpretation of convolution?

(2) How the convolution of two continuous time signals are performed by using a graphical method?

(3) What re the steps involved in finding the convolution by graphical method?

4.4 CONVOLUTION PROPERTY OF FOURIER TRANSFORMS

The convolution property is the most important property of Fourier transform. By using this convolution property the input output behaviour of linear system is analyzed in frequency domain.

Consider the convolution of two nonperiodic continuous-time signals x(t) and h(t). We define,

$$y(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Now we express $x(t - \tau)$ in terms of its Fourier transform,

$$x(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega(t-\tau)}d\omega$$

Substituting this expression into the convolution integral yields,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}e^{-j\omega\tau}d\omega\right]d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right]X(\omega)e^{j\omega t}d\omega$$

We recognize the inner integral over τ as the Fourier transform of $h(\tau)$ or $H(\omega)$. Hence, y(t) may be rewritten as,

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t}d\omega$$

And we identify $H(\omega)$ $X(\omega)$ as the Fourier transform of y(t). We conclude that convolution of h(t) and x(t) in the time domain corresponds to multiplication of their Fourier transforms, $H(\omega)$ and $X(\omega)$, in the frequency domain, that is,

$$y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(\omega) = X(\omega)H(\omega) \quad \ldots (4.4.1)$$

In discrete-time systems, the convolution sum, relates input and impulse response of the system to output. It is mathematically expressed as,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad \ldots (4.4.2)$$

The relationship expressed in Eq. (4.4.2) is called the convolution sum of input x[n] and unit impulse response h(n). This operation is represented symbolically as,

$$y(n) = x(n) * h(n)$$

EXAMPLE PROBLEM 1

Determine the convolution of the two sequences,

$$x(n) = \left(\frac{1}{2}\right)^n u(n) \text{ and } h(n) = \left(\frac{1}{4}\right)^n u(n)$$

SOLUTION

By definition,

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k) \left(\frac{1}{4}\right)^{n-k} u(n-k)$$

In this case,

$$u(k) = \begin{cases} 1 & ; \quad k \geq 0 \\ 0 & ; \quad k < 0 \end{cases} \text{ and } u(n-k) = \begin{cases} 1 & ; \quad n-k \geq 0 \rightarrow k \leq n \\ 0 & ; \quad n-k < 0 \rightarrow k > n \end{cases}$$

The product u(k).u(n−k) will be non-zero in the range $0 \leq k \leq n$. Therefore the summation index in the above equation is changed to k = 0 to n.

$$\therefore \quad x(n) * h(n) = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{n-k} = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^n \left(\frac{1}{4}\right)^{-k}$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k 4^k = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left(\frac{4}{2}\right)^k$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} 2^k$$

Using finite geometric series sum formula, that is,

$$\sum_{n=0}^{N} C^n = \frac{C^{N+1} - 1}{C - 1}$$

We have,

$$x(n) * h(n) = \left(\frac{1}{4}\right)^n \left(\frac{2^{n+1} - 1}{2 - 1}\right) = \left(\frac{1}{4}\right)^n (2^{n+1} - 1); \qquad n \geq 0$$

$$= \left(\frac{1}{4}\right)^2 (2^{n+1} - 1)\, u(n) \qquad ; \qquad \text{for all n}$$

4.4.1 Properties of Convolution Sum

The properties of convolution sum are similar to those of the convolution integral. The properties of convolution sum are as follows,

(1) Commutative Property : $x(n) * h(n) = h(n) * x(n)$

(2) Distributive Property : $x(n) * [h_1(n)+h_2(n)] = [x(n) * h_1(n)] + [x(n) * h_2(n)]$

(3) Associative Property : $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$

(4) Shift Property : Shift property of convolution sum states that,

If, $x(n) * h(n) = y(n)$

Then, $x(n) * h(n - n_0) = y(n - n_0)$

Similarly, $x(n - n_0) * h(n) = y(n - n_0)$

And $x(n - n_1) * h(n - n_2) = y(n - (n_1 + n_2))$

(5) Convolution with an Impulse : $x(n) * \delta(n) = x(n)$

(6) Width Property : Let N_1 and N_2 be the number of samples in the non-zero extents of the sequences x(n) and h(n) respectively. Then the number of samples in the non-zero extents of the sequence $x(n) * h(n)$ is $N_1 + N_2 - 1$.

4.4.2 Graphical Representation of Convolution Sum

The convolution of two discrete-time sequences x(n) and h(n) using the graphical method involves the following sequence of operations,

STEP I (Change of Discrete-Time Index) : Replace the independent variable by a dummy variable k for the given signals x(n) and h(n) and plot the graphs of x(k) and h(k) as functions of k.

STEP II (Folding) : Invert h(k) about the vertical axis (k = 0) to obtain h(−k).

STEP III (Shifting) : Shift the inverted h(k), i.e., h(−k) by n units to obtain h(n − k).

STEP IV (Multiplication and Addition) : Finally multiply x(k) and h(n − k) for particular values of n and add all the products to obtain x(n)*h(n). The procedure is repeated for each value of n over the range $-\infty$ to ∞.

Comment : *If x(n) starts at $n = n_1$ and h(n) starts at $n = n_2$, then choose $n = n_1+n_2$ as starting time for evaluating the output sequence $y(n) = x(n)*h(n)$.*

EXAMPLE PROBLEM 1

Find the convolution of two signals x(n) = {1, 1, 0, 1, 1} and h(n) = {1, -2, -3, 4} and represent them graphically.

SOLUTION

STEP I AND II (Change of Time Index and Folding) : The graphical plots of x(k) and h(k) shifted around vertical axis (k = 0) i.e., h(−k) is as shown in Fig. 4.4.1.

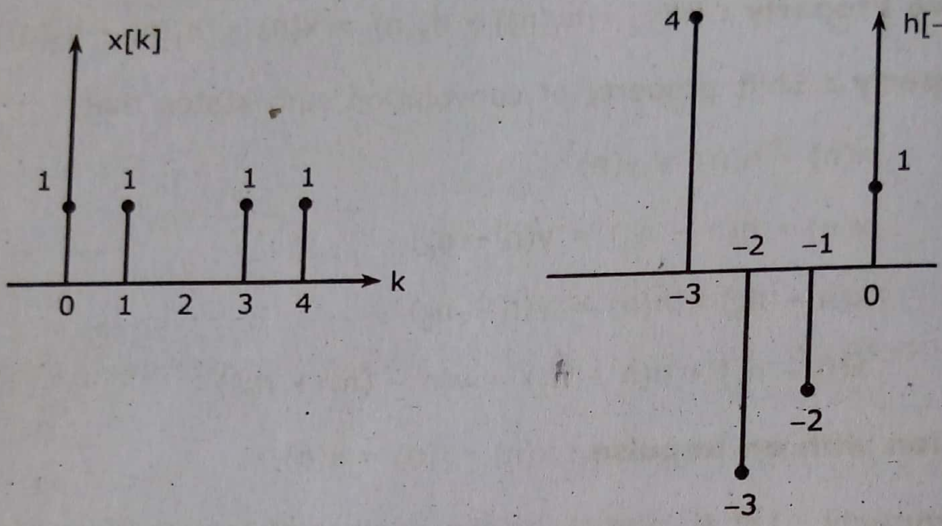
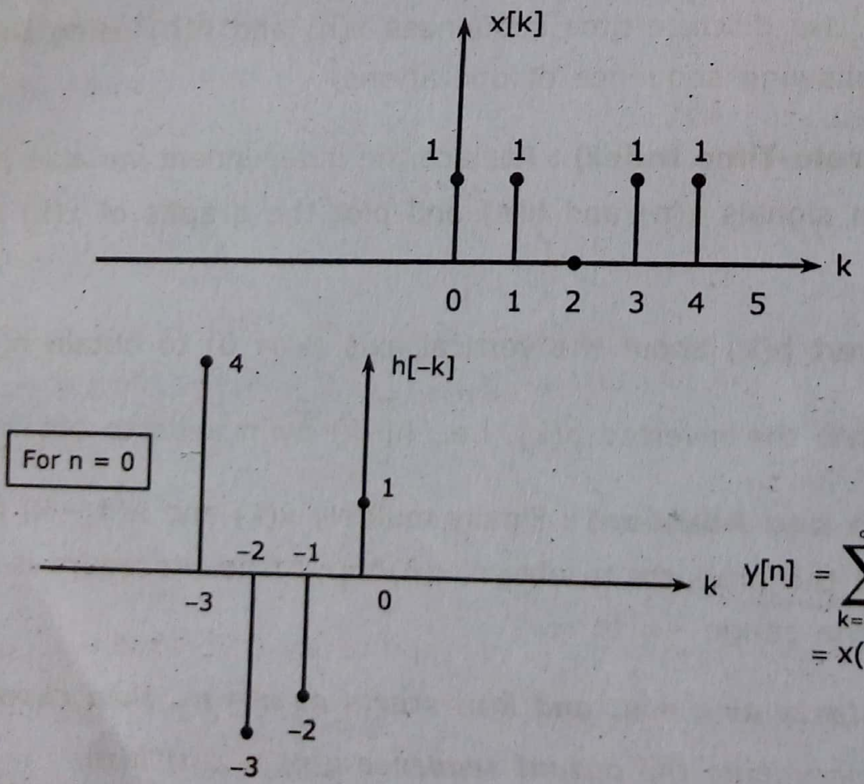


Fig. 4.4.1 Graphical Plots of x(k) and h(-k)

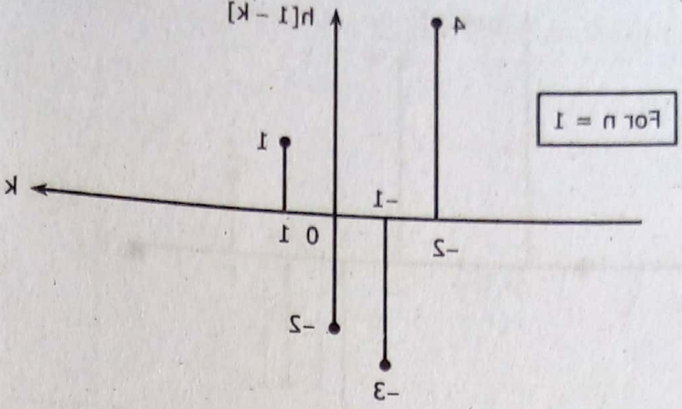
STEP III, IV AND V : Since both the sequences start at n = 0, hence the starting time for evaluating the output sequence y(n) = x(n)*h(n) is n=0. The graphical plots for evaluating y(n) for various values of n are shown in Fig. 4.4.2.



$$y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= x(0).h(0) = 1.1 = 1$$

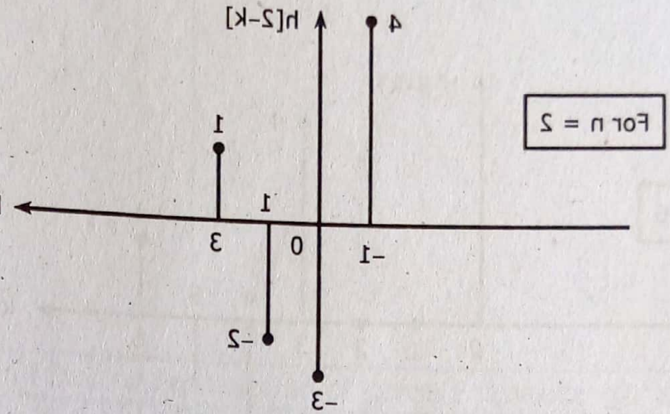
For n = 1



$$y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= 1.(-2) + 1.1 = -1$$

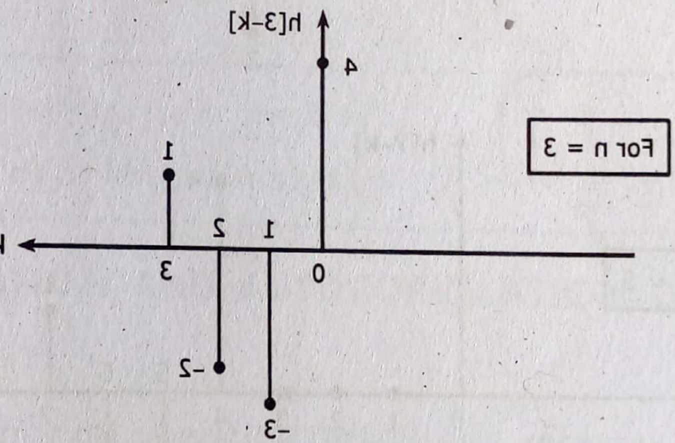
For n = 2



$$y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= (1)(-3) + 1(-2) + 0 = -5$$

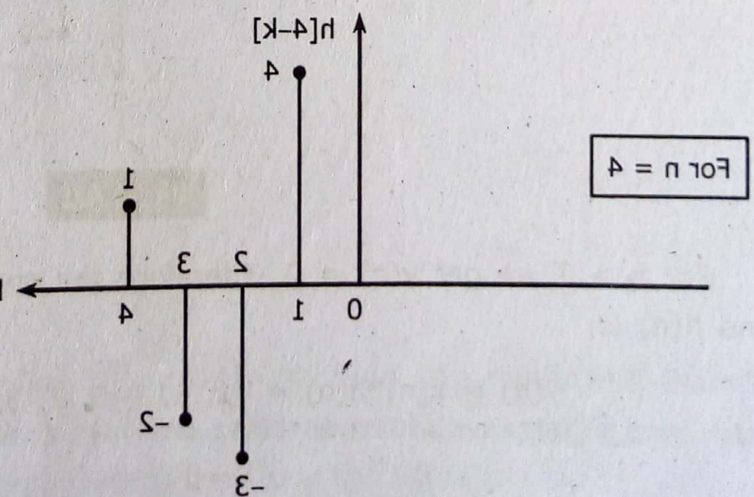
For n = 3



$$y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= 1.4 + 1(-3) + 0 + 1.1 = 2$$

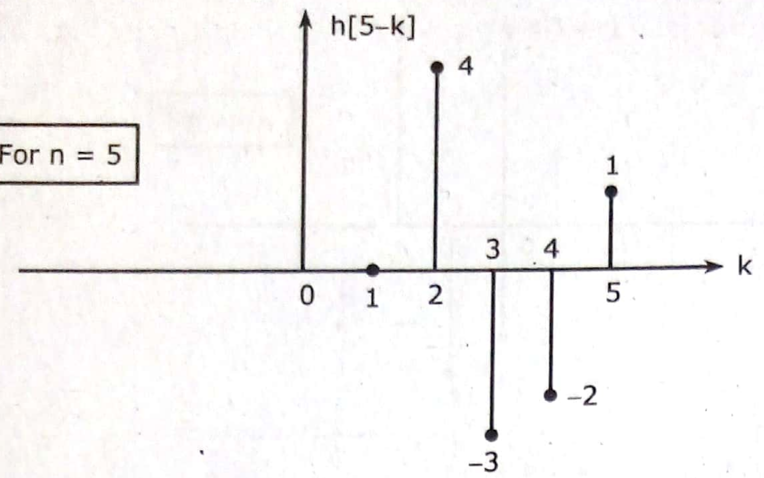
For n = 4



$$y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= 0 + 1.4 + 0 + 1(-2) + 1.1 = 3$$

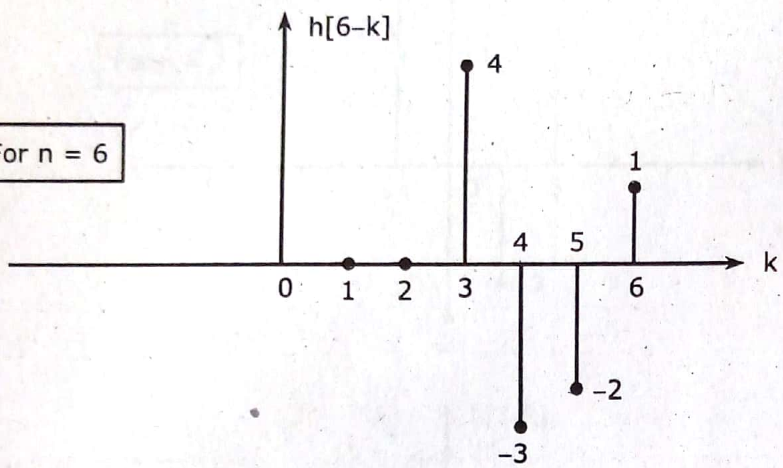
For n = 5



$$y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= 0+0+0+(1.(-3)+(1.(-2) = -5$$

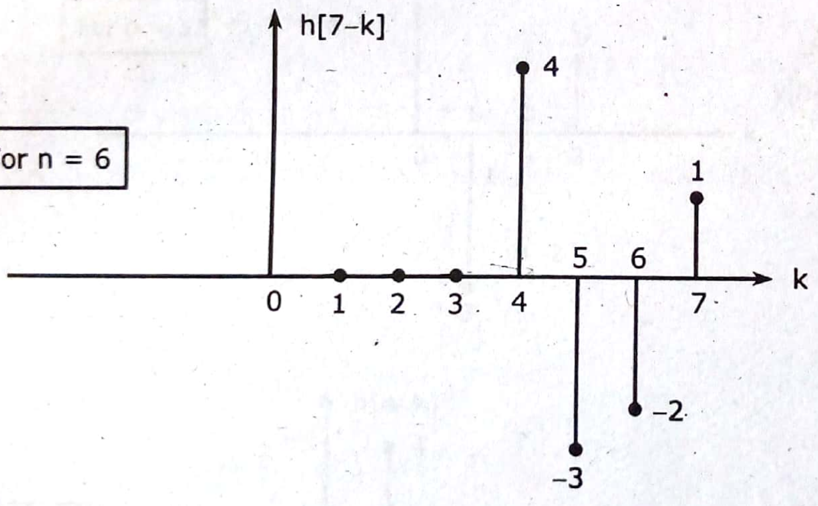
For n = 6



$$y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= 0+0+0+1.4+1.(-3) = 1$$

For n = 6



$$y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= 1.4 = 4$$

Fig. 4.4.2

For n > 7 we get y(n) = 0. Therefore the convolution of given sequences x(n) and h(n) is,

$$y(n) = x(n)*h(n) = \{1, \underset{\substack{\uparrow \\ n=0}}{-1}, -5, 2, 3, -5, 1, 4\}$$

The graphical plot of convolution of given two signals is as shown in Fig. 4.4.3.

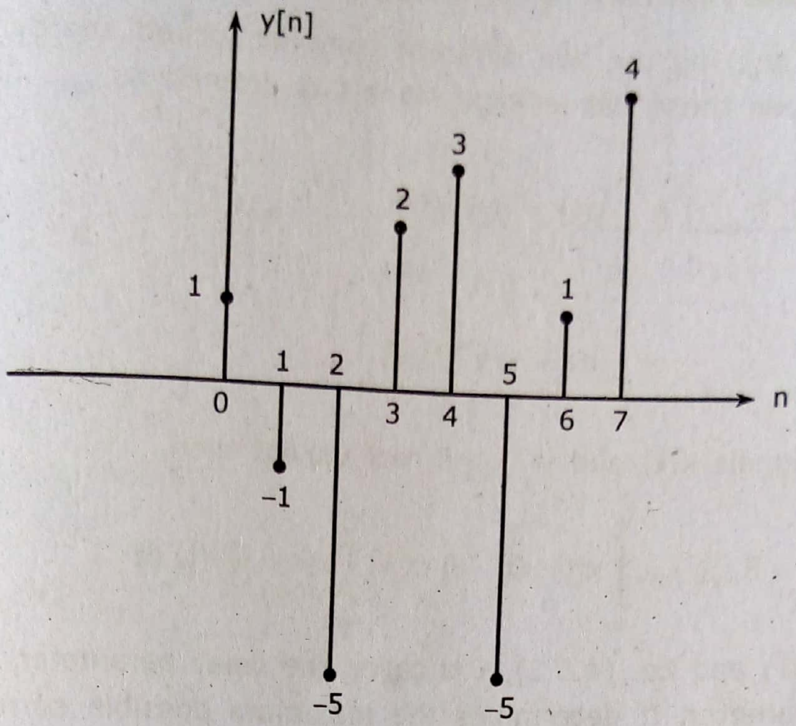


Fig. 4.4.3 Graphical Plot of Convolution of given Sequence x(n) and h(n)

REVIEW QUESTIONS

(1) *What is the convolution property of Fourier transform?*

(2) *Write the convolution property for non-periodic signals?*

4.5 CROSS CORRELATION AND AUTOCORRELATION OF FUNCTIONS

Correlation is a mathematical operation that closely resembles convolution correlation is a measure of similarity between the two signals.

There are two types of correlation,

(1) Cross correlation.

(2) Autocorrelation.

4.5.1 Cross Correlation

Cross-correlation is the measure of similarity between one signal and time-delayed version of another signal. It means that the cross-correlation explains how much one signal is related to the time delayed version of another signal.

4.5.1.1 Cross Correlation of Energy Signals

Let x(t) and y(t) be the two different complex valued energy signals. The cross-correlation between these two energy signals is defined as,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y^{*}(t - \tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t + \tau)\, y^{*}(t)\, dt \qquad \dots (4.5.1)$$

Or, if two signals x(t) and y(t) are real valued then,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y(t - \tau)\, dt = \int_{-\infty}^{\infty} x(t + \tau)\, y(t)\, dt \qquad \dots (4.5.2)$$

In Eq. (4.5.1) and Eq. (4.5.2), τ is called the delay parameter, searching parameter or scanning parameter. It determines the maximum possible correlation between two signals. The subscripts xy on the cross correlation function $R_{xy}(\tau)$ indicates the signals being correlated. The second possible cross correlation function obtained by reversing the order of subscripts xy, that is $R_{yx}(\tau)$ is defined as,

For complex valued signals,

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t)\, x^{*}(t - \tau)\, dt = \int_{-\infty}^{\infty} y(t + \tau)\, x^{*}(t)\, dt$$

For real valued signals,

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t)\, x(t - \tau)\, dt = \int_{-\infty}^{\infty} y(t + \tau)\, x(t)\, dt$$

The cross-correlation function $R_{xy}(\tau)$ will be finite over some range of τ if the signals x(t) and y(t) have some similarity. The energy signals x(t) and y(t) are said to be ***orthogonal*** i.e., (having no similarity) over the entire time interval if $R_{xy}(0)$ is zero. Substituting $\tau = 0$ in Eq. (4.5.1), the condition for orthogonality of the two energy signals can be written as,

$$R_{xy}(0) = \int_{-\infty}^{\infty} x(t)\, y^{*}(t)\, dt = 0 \qquad \dots (4.5.3)$$

(or) $$R_{yx}(0) = \int_{-\infty}^{\infty} y(t)\, x^{*}(t)\, dt = 0 \qquad \dots (4.5.4)$$

4.5.1.2 Cross Correlation of Power Signals

The cross correlation function between two different complex valued power signals x(t) and y(t) is mathematically defined by,

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t-\tau)\, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t+\tau)\, y^*(t)\, dt \quad \text{... (4.5.5)}$$

If x(t) and y(t) are real, then,

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y(t-\tau)\, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t+\tau)\, y(t)\, dt \quad \text{... (4.5.6)}$$

A second cross-correlation function $R_{yx}(\tau)$ can be defined in a similar manner. The pair of complex power signals x(t) and y(t) are said to be ***orthogonal*** over the entire interval if,

$$R_{xy}(0) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t)\, dt = 0$$

(or) $$R_{yx}(0) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} y(t)\, x^*(t)\, dt = 0$$

Cross Correlation of Periodic Signals

An important special case of correlation of power signals is the correlation between two periodic signals whose fundamental periods are such that the product of the two signals is also periodic. This will happen any time the ratio of their fundamental periods is a rational number.

For two signals whose product has a period T, the general form of the correlation function (for real power signals) is given by,

$$R_{xy}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y(t-\tau)\, dt = \frac{1}{T} \int_{-T/2}^{T/2} x(t+\tau) y(t)\, dt \quad \text{... (4.5.7)}$$

4.5.2 Autocorrelation

Autocorrelation is the measure of similarity between signal and time delayed version of same signal.

4.5.2.1 Autocorrelation of Energy Signals

If x(t) is a complex-valued energy signal then the autocorrelation function is defined as,

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt = \int_{-\infty}^{\infty} x(t+\tau)\, x^*(t)\, dt \quad \dots (4.5.8)$$

If x(t) is a real-valued signal then,

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt = \int_{-\infty}^{\infty} x(t+\tau)\, x(t)\, dt \quad \dots (4.5.9)$$

4.5.2.2 Autocorrelation of Power Signals

If x(t) is a complex-valued power signal, then its autocorrelation function is defined as,

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t+\tau)\, x^*(t)\, dt \quad \dots (4.5.10)$$

It x(t) is a real-valued signal then,

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t+\tau)\, x^*(t)\, dt \quad \dots (4.5.11)$$

AUTOCORRELATION OF PERIODIC SIGNALS

When the power signal x(t) is a periodic signal, the integrand in Eq. (4.5.11) is also periodic and the time average can be taken over one period (T, say). Thus we may express the autocorrelation function of a periodic signal $x_p(t)$ of period T as,

$$R_{xp}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x_p(t)\, x_p(t-\tau)\, dt = \frac{1}{T} \int_{-T/2}^{T/2} x_p(t+\tau)\, x_p(t)\, dt \quad \dots (4.5.12)$$

4.5.3 Graphical Interpretation of Correlation of Continuous-Time Signals

Cross correlation of a real valued signals x(t), y(t) can be obtained using the graphical method by the following sequence of operations.

Step I (Shifting) : Shift the signal y(t) by units y(t − τ). Plot the graphs for x(t) and y(t − τ) on the same axis beginning with the shift τ.

Step II (Multiplication and Integration) : This signals x(t) and y(t − τ) are multiplied to get the product signal x(t) y(t − τ). For particular value of τ = k, the integration of the product signal represents the area under the product curve. This common area represents the cross-correlation of x(t) and y(t) for a shift of τ = k. That is,

$$R_{xy}(\tau)\big|_{\tau=k} = \int_{-\infty}^{\infty} x(t)\, y(t-\tau)\Big|_{\tau=k}$$

Repeat this procedure for different values of t and find the values of cross-correlation function $R_{xy}(\tau)$ at those values of τ.

EXAMPLE PROBLEM 1

Find the cross-correlation of the signals x(t) and y(t) shown in Fig. 4.5.1.

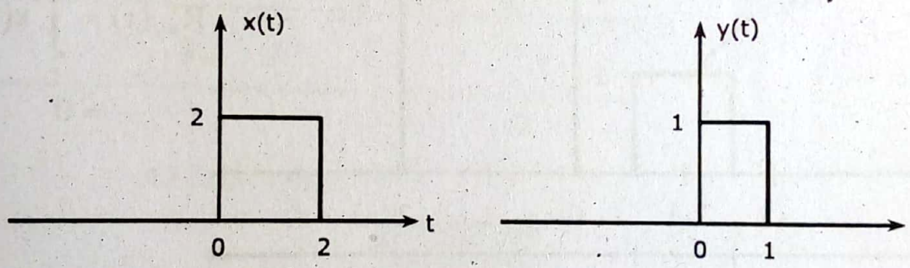


Fig. 4.5.1

SOLUTION

The definition of correlation,

$$R_{xy}(\tau)\big|_{\tau=k} = \int_{-\infty}^{\infty} x(t)\, y(t-\tau)\Big|_{\tau=k}$$

Given correlation of two signals x(t) and y(t), the mathematical expressions of these signals are,

$$x(t) = \begin{cases} 2 & ; \quad 0 < t < 2 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

And

$$y(t) = \begin{cases} 1 & ; \quad 0 < t < 1 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

STEP I (Shifting) : Shift the signal y(t) by τ units y(t − τ). Plot the graphs for x(t) and y(t − τ) on the same axis beginning with time shift τ as shown in Fig. 4.5.2. Mathematical expression of function y(t − τ) is defined as,

$$y(t-\tau) = \begin{cases} 1 & ; \quad 0 < (t-\tau) < 1 \rightarrow +\tau < t < 1+\tau \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

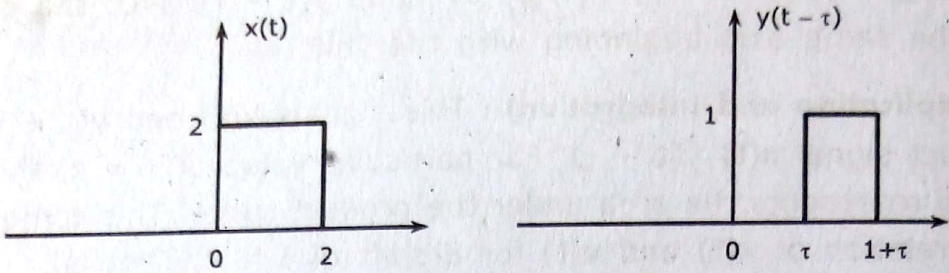
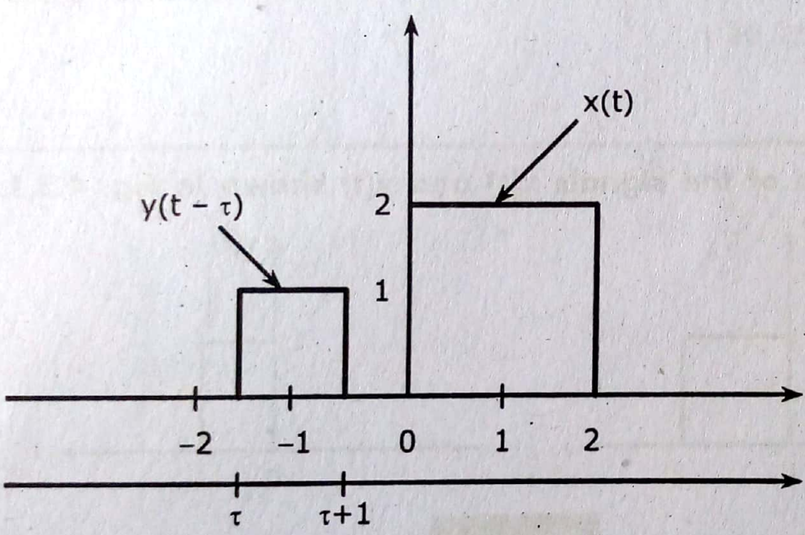


Fig. 4.5.2

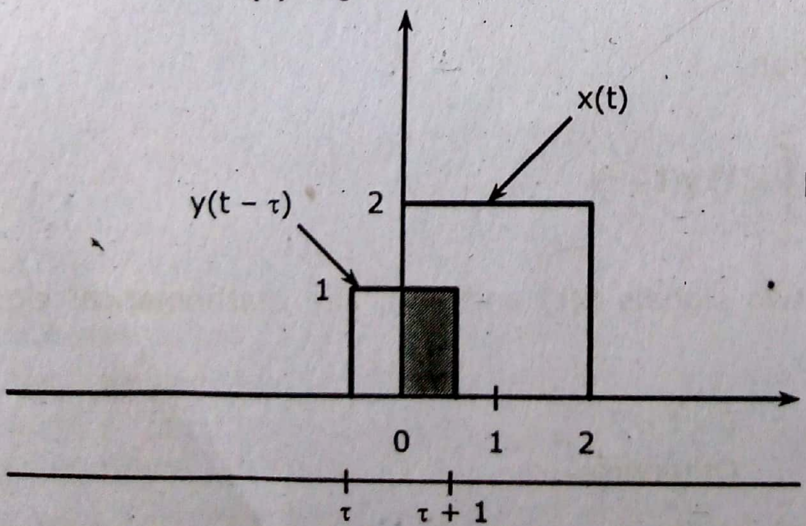
STEP II (Multiplication and Integration) : Perform multiplication and integration to find correlation of signals x(t) and y(t) for all values of τ. The graphical plots for various intervals of τ are shown in Fig. 4.5.3.



$$R_{xy}(\tau) = \int_{\tau}^{\tau+1} x(t)\, g(t-\tau)\, dt$$

$$= 0$$

(a) Region 1 : (τ < −1)



$$R_{xy}(\tau) = \int_{\tau}^{\tau+1} x(t)\, g(t-\tau)\, dt$$

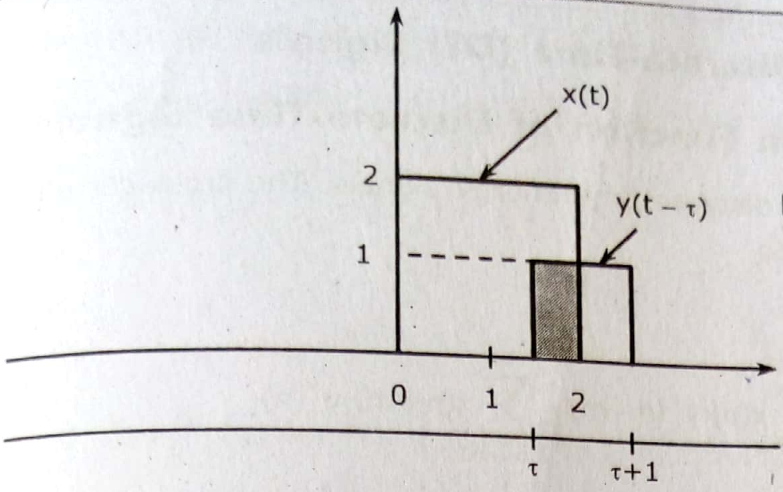
$$= \int_{0}^{\tau+1} 2.1\, dt$$

$$= 2t\Big]_0^{\tau+1}$$

$$= 2(\tau+1) = 2\tau + 2$$

$$= 2 + 2\tau$$

(b) Region 2 : (−1 < τ < 1)



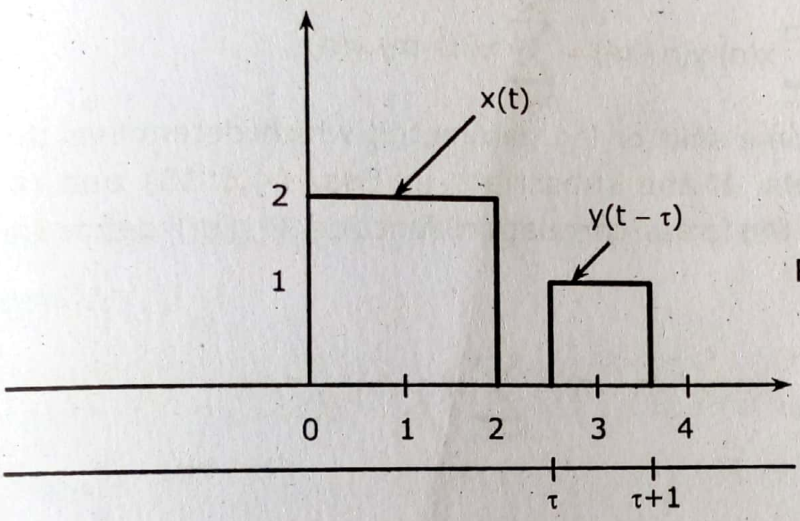
$$R_{xy}(\tau) = \int_{\tau}^{2} x(t)\, g(t-\tau)\, dt$$

$$= \int_{0}^{2} 2.1\, dt$$

$$= 2t\Big]_{\tau}^{2}$$

$$= 2(2-\tau) = 4 - 2\tau$$

(c) Region 3 : $(1 < \tau < 2)$



$$R_{xy}(\tau) = \int_{\tau}^{2} x(t)\, g(t-\tau)\, dt$$

$$= 0$$

(d) Region 4 : $(2 < \tau)$

Fig. 4.5.3

In summary,

$$R_{xy}(\tau) = \begin{cases} 0 & ; \quad \tau < -1 \\ 2 + 2\tau & ; \quad -1 < \tau < 1 \\ 4 - 2\tau & ; \quad 1 < \tau < 2 \\ 0 & ; \quad \tau < 2 \end{cases}$$

The correlated signal $R_{xy}(\tau)$ is as shown in Fig. 4.5.4.

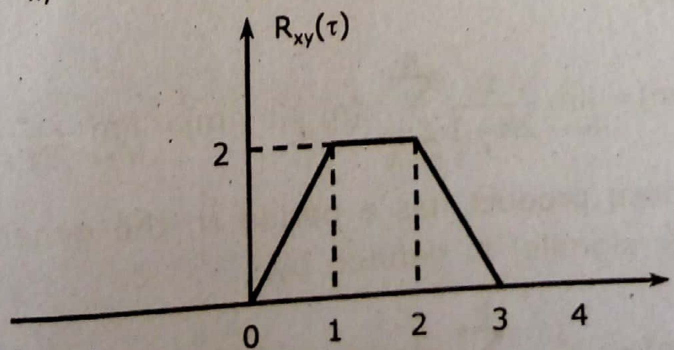


Fig. 4.5.4

4.5.4 Correlation of Discrete-Time (DT) Signals

4.5.4.1 Cross Correlation Function of Discrete-Time Signals

Let x(n) and y(n) be two discrete-time energy signals. The cross-correlation function of x(n) and y(n) is defined as,

(1) Complex-valued Signals,

$$R_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n)\, y^*(n-m) = \sum_{n=-\infty}^{\infty} x(n+m)\, y^*(n) \quad \text{... (4.5.13)}$$

(2) Real-valued signals,

$$R_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-m) = \sum_{n=-\infty}^{\infty} x(n+m)\, y(n) \quad \text{... (4.5.14)}$$

Here the index m is the time shift or lag parameter, which determines the maximum possible correlation of signals. If the subscripts in Eqs. (4.5.13) and (4.5.14) are interchanged then we obtain the cross-correlation function $R_{yx}(m)$ defined as,

(1) Complex-valued signals,

$$R_{yx}(m) = \sum_{n=-\infty}^{\infty} y(n)\, x^*(n-m) = \sum_{n=-\infty}^{\infty} x(n+m)\, y^*(n) \quad \text{... (4.5.15)}$$

(2) Real-valued signals,

$$R_{yx}(m) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-m) = \sum_{n=-\infty}^{\infty} x(n+m)\, y(n) \quad \text{... (4.5.16)}$$

Let x(n) and y(n) be two discrete-time power signals. The cross-correlation function of these signals is defined as,

(1) Complex-valued signals,

$$R_{xy}(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, y^*(n-m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n+m)\, y^*(n)$$

(2) Real-valued signals,

$$R_{xy}(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, y(n-m) = \lim_{N\to\infty} \frac{1}{2N+1} x(n+m)\, y(n)$$

For two signals whose product has a period N, the general form of the correlation function (for real power signals) is defined by,

$$R_{xy}(m) = \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, y(n-m)$$

4.5.4.2 Autocorrelation Function of Discrete-time Signals

If x(n) is an energy signal, then its autocorrelation is,

$$R_{xx}(m) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-m)$$

$$= \sum_{n=-\infty}^{\infty} x(n+m)\, x(n)$$

If x(n) is a power signal, then its autocorrelation is,

$$R_{xx}(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, x(n-m)$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n+m)\, x(n)$$

4.5.4.3 Graphical Representation of Correlation of Discrete-time Signals

Cross correlation of a discrete-time signals x(n) and y(n) can be computed using graphical method by the following sequence of operations.

Step I (Shifting) : Shift the sequence y(n) by m units to obtain y(n − m).

Step II (Multiplication and Adding) : Multiply the shifted sequence y(n − m) with x(n) and add all the values to obtain $R_{xy}(m)$.

Repeat this procedure for each value of m over the interval $-\infty$ to ∞.

EXAMPLE PROBLEM 1

Determine the cross correlation between the two sequences x(n) = {1, 0, 0, 1} and h(n) = {4, 3, 2, 1}.

SOLUTION

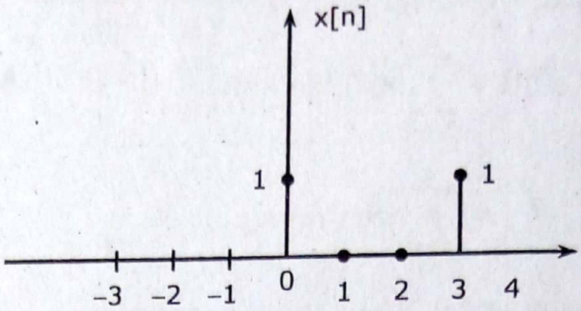
Given,

$$x(n) = \{\underset{\substack{\uparrow \\ n=0}}{1}, 0, 0, 1\} \quad \text{and} \quad h(n) = \{\underset{\substack{\uparrow \\ n=0}}{4}, 3, 2, 1\}$$

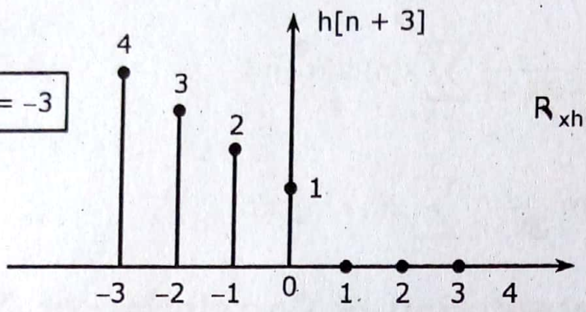
By definition, cross-correlation is given by,

$$R_{xh}(m) = \sum_{n=-\infty}^{\infty} x(n)\, h(n-m)$$

The graphical plots of x(n) and shifted signal h(n − m) for various values of m are shown in Fig. 4.5.5.

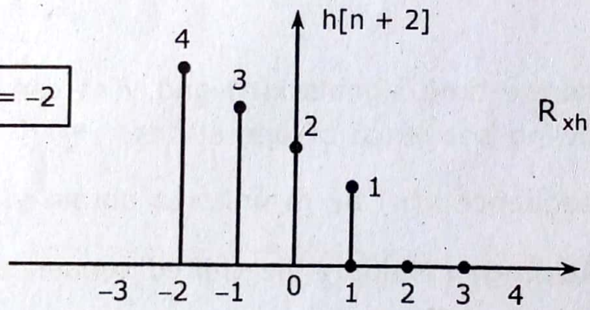


For m = −3



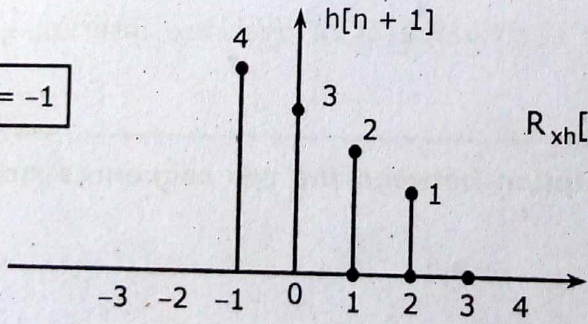
$$R_{xh}[m] = \sum_{n=-\infty}^{\infty} x(n)\, h(n-m)$$
$$= 1.1 = 1$$

For m = −2



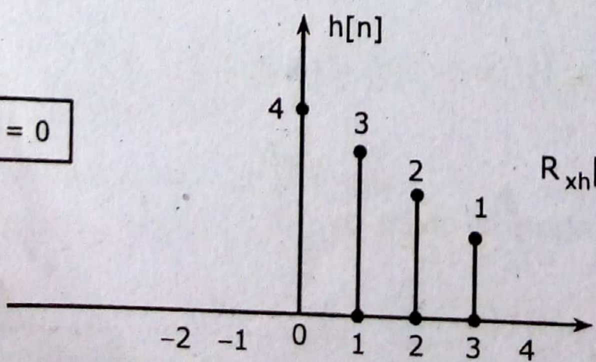
$$R_{xh}[m] = \sum_{n=-\infty}^{\infty} x(n)\, h(n-m)$$
$$= 2.1 + 1.0 = 2$$

For m = −1



$$R_{xh}[m] = \sum_{n=-\infty}^{\infty} x(n)\, h(n-m)$$
$$= 3.1 + 2.0 + 1.0 = 3$$

For m = 0



$$R_{xh}[m] = \sum_{n=-\infty}^{\infty} x(n)\, h(n-m)$$
$$= 4.1 + 1.1 = 5$$

For m = 1

$$R_{xh}[m] = \sum_{n=-\infty}^{\infty} x(n)\, h(n-m) = 2.1 = 2$$

For m = 2

$$R_{xh}[m] = \sum_{n=-\infty}^{\infty} x(n)\, h(n-m) = 3.1 = 3$$

For m = 3

$$R_{xh}[m] = \sum_{n=-\infty}^{\infty} x(n)\, h(n-m) = 4.1 = 4$$

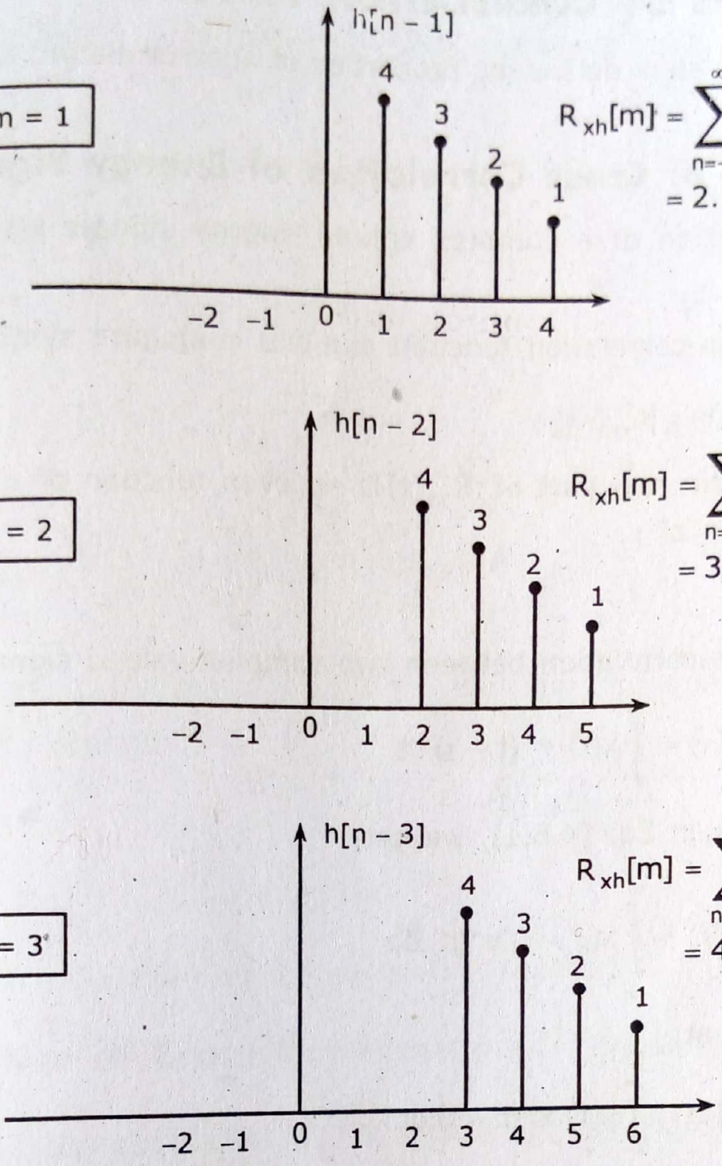


Fig. 4.5.5 Graphical Plots of Cross Correlation of given Sequences for Various Shift Values

Therefore, from Fig. 4.5.5 the cross-correlation of given sequences is,

$$R_{xh}(m) = \{1, 2, 3, 5, 2, 3, 4\}$$

$\uparrow$

$n = 0$

REVIEW QUESTIONS

(1) *How the cross correlation is obtained in energy signals and power signals?*

(2) *Define autocorrelation and cross correlation? And give a graphical interpretation of correlation of continuous-time signals?*

4.6 PROPERTIES OF CORRELATION FUNCTION

In this section, we shall define the properties of autocorrelation and cross correlation function.

4.6.1 Properties of Cross Correlation of Energy Signals

The cross correlation of a complex valued energy signals x(t) and y(t) has the following properties,

PROPERTY I : The cross-correlation function exhibits conjugate symmetry, that is,

$$R_{xy}(\tau) = R^*_{yx}(-\tau)$$

This means that the real part of $R^*_{xy}(\tau)$ is an even function of τ, whereas imaginary part is an odd function of τ.

PROOF

We have the cross-correlation between two complex valued signals x(t) and y(t) as,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y^*(t-\tau)\, dt \quad \text{... (4.6.1)}$$

Letting, $p = t - \tau$ in Eq. (4.6.1), we get,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(p+\tau)\, y^*(p)\, dp \quad \text{... (4.6.2)}$$

Also, we know that,

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t)\, x^*(t-\tau)\, dt \quad \text{... (4.6.3)}$$

Substituting $p = t$ in Eq. (4.6.3), we get,

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(p)\, x^*(p-\tau)\, dp \quad \text{... (4.6.4)}$$

$$\therefore \quad R^*_{yx}(\tau) = \left[\int_{-\infty}^{\infty} y(p) x^*(p-\tau)\, dp\right]^* = \int_{-\infty}^{\infty} y^*(p)\, x(p-\tau)\, dp \quad \text{... (4.6.5)}$$

Substituting $\tau = -\tau$ in Eq. (4.6.5) we get,

$$R^*_{yx}(-\tau) = \int_{-\infty}^{\infty} y^*(p)\, x(p+\tau)\, dp \quad \text{... (4.6.6)}$$

Comparing Eq. (4.6.2) with Eq. (4.6.6), we get,

$$R_{xy}(\tau) = R^*_{yx}(-\tau) \quad \text{... (4.6.7)}$$

PROPERTY II : The complex valued energy signals x(t) and y(t) are said to be orthogonal (i.e., having no similarity) over the entire time interval if $R_{xy}(0)$ is zero.

PROOF

By definition,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y^*(t - \tau)\, dt$$

$$\therefore \quad R_{xy}(\tau)\Big|_{\tau = 0} = R_{xy}(0)$$

$$= \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = 0$$

PROPERTY III : Fourier transform of cross correlation function is the product of two energy signals in frequency domain.

i.e.,

$$R_{xy}(\tau) \xleftrightarrow{F.T} X(\omega)\, Y^*(\omega) \quad \text{... (4.6.8)}$$

The Eq. (4.6.8) is known as the correlation theorem.

4.6.2 Properties of Cross Correlation of Power Signals

PROPERTY I : The cross-correlation function of two power signals x(t) and y(t) exhibits conjugate symmetry, that is,

$$R_{xy}(\tau) = R^*_{xy}(-\tau)$$

PROPERTY II : The complex valued power signals x(t) and y(t) are said to be orthogonal over the entire time interval if $R_{xy}(\tau)$ is zero. That is,

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t) = 0$$

$$\tau = 0$$

PROPERTY III : The cross-correlation does not satisfies commutative property, i.e.,

$$R_{xy}(\tau) \neq R_{yx}(\tau)$$

4.6.3 Properties of Autocorrelation of Energy Signals

PROPERTY I : The autocorrelation exhibits conjugate symmetry, i.e.,

$$R_{xx}(\tau) = R^*_{xx}(-\tau) \quad \text{... (4.6.9)}$$

This means that the real part of $R_{xx}(\tau)$ is an even function of τ and the imaginary part of $R_{xx}(\tau)$ is an odd function of τ.

PROOF

The autocorrelation of an energy signal x(t) is given by,

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt = \int_{-\infty}^{\infty} x(t+\tau)\, x^*(t)\, dt \quad \text{... (4.6.10)}$$

Taking the complex conjugate, we have,

$$R^*_{xx}(\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t-\tau)\, dt \quad \text{... (4.6.11)}$$

Replacing τ with $-\tau$, we get,

$$R^*_{xx}(-\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt \quad \text{... (4.6.12)}$$

Comparing Eq. (4.6.10) with Eq. (4.6.12), we get,

$$\therefore \quad R_{xx}(\tau) = R^*_{xx}(-\tau) \quad \text{... (4.6.13)}$$

PROPERTY II : The value of autocorrelation of an energy signal at the origin ($\tau = 0$) gives the total energy of that signal. i.e.,

$$R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2\, dt = E_x \quad \text{... (4.6.14)}$$

PROOF

We have,

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

Putting, $\tau = 0$ gives,

$$R_{xx}(0) = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt = E_x$$

Property III : Autocorrelation is maximum at $\tau = 0$, for all values of τ. That is,

$$|R_{xx}(\tau)| \leq |R_{xx}(0)| \text{ for all } \tau$$

PROOF

Consider the integral,

$$\int_{-\infty}^{\infty} [x(t) \pm x(t+\tau)]^2 \, dt \geq 0$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} x^2(t) + \int_{-\infty}^{\infty} x^2(t+\tau) \pm 2\int_{-\infty}^{\infty} x(t)\, x(t+\tau) \geq 0 \quad \text{... (4.6.15)}$$

Since the first two integrals in Eq. (4.6.15) are equal (i.e., $\int_{-\infty}^{\infty} x^2(t)$ is energy of the signal, also $\int_{-\infty}^{\infty} x^2(t+\tau)$ is energy of same signal shifted by an amount equal to τ). Hence Eq. (4.6.15) now becomes,

$$\left[2\int_{-\infty}^{\infty} x^2(t)\, dt\right] \pm \left[2\int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt\right] \geq 0$$

From property 2, defined by Eq. (4.6.14) and by definition of autocorrelation function, we get,

$$2R_{xx}(0) \pm 2R_{xx}(\tau) \geq 0$$

$$\Rightarrow \quad R_{xx}(0) \geq \pm R_{xx}(\tau)$$

Considering magnitude, we get,

$$|R_{xx}[0]| \geq |R_{xx}[\tau]|$$

Property IV : The autocorrelation function and the energy spectral density form a Fourier transform pair, that is,

$$R_{xx}(\tau) \xleftrightarrow{\text{F.T}} \psi(\omega) \text{ or } \psi(f)$$

4.6.4 Properties of Autocorrelation of Power Signals

Property I : The autocorrelation function of power signals exhibits *conjugate symmetry*, that is,

$$R_{xx}(\tau) = R_{xx}^{*}(-\tau)$$

PROOF

We have,

$$R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^{*}(t-\tau)\, dt = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t+\tau)\, x^{*}(t)\, dt$$

$$\therefore \quad R_{xx}^{*}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^{*}(t)\, x(t-\tau)\, dt$$

$$\Rightarrow \quad R_{xx}^{*}(-\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^{*}(t)\, x(t+\tau)\, dt = R_{xx}(\tau)$$

$$\therefore \quad R_{xx}(\tau) = R_{xx}^{*}(-\tau)$$

Property II : The value of the autocorrelation function of a power signal at the origin is equal to the average power of the signal, that is,

$$R_{xx}(0) = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^{2}\, dt = P_x \qquad \text{... (4.6.16)}$$

PROOF

We have autocorrelation of periodic power signal as,

$$R_{xx}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^{*}(t-\tau)\, dt$$

Putting $\tau = 0$, we get,

$$R_{xx}(0) = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^{*}(t)\, dt = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^{2}\, dt = P_x$$

Property III : The autocorrelation $R_{xx}(\tau)$ and power spectral density, $S(\omega)$ of a power signal forms a Fourier transform pair, i.e.,

$$R_{xx}(\tau) \xleftrightarrow{\text{F.T}} S(\omega) \text{ or } S(f)$$

PROPERTY IV : The autocorrelation of a periodic signal with period T is periodic with the same period.

If, $x(t) = x(t \pm T)$,

Then, $R_{xx}(\tau) = R_{xx}(\tau \pm T)$

PROOF

Let x(t) be a periodic signal with period T. Then,

$$x(t) = x(t + T)$$

$$x(t - \tau) = x(t - \tau + T) \text{ and } x(t + \tau) = x(t + \tau + T) \quad \text{... (4.6.17)}$$

We have autocorrelation function of (real valued) periodic signal as,

$$R_{xx}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x(t - \tau)\, dt$$

Replacing τ with $\tau - T$, we get,

$$R_{xx}(\tau - T) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x[t - (\tau - T)]\, dt$$

Using Eq. (4.6.17), we get,

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x(t - \tau + T)\, dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x(t - \tau)\, dt = R_{xx}(\tau)$$

We also have autocorrelation function of periodic signal defined as,

$$R_{xx}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t + \tau)\, x(t)\, dt$$

Replacing τ with $\tau + T$ we get,

$$R_{xx}(\tau + T) = \frac{1}{T}\int_{-T/2}^{T/2} x(t + \tau + T)\, x(t)\, dt$$

By using Eq. (4.6.17), we get,

$$R_{xx}(\tau + T) = \frac{1}{T}\int_{-T/2}^{T/2} X(t + \tau)\, x(t).dt = R_{xx}(\tau)$$

$$\therefore \quad R_{xx}(\tau) = R_{xx}(\tau \pm T)$$

REVIEW QUESTIONS

(1) *Write the properties of cross correlation of both energy signal and power signal?*

(2) *Write the properties of autocorrelation. What is the correlation theorem?*

4.7 ENERGY DENSITY SPECTRUM

Spectral density plays an important role in determining the power and energy of signal. This parameter is a function of frequency and is directly related to the amplitude spectrum of the concerned signal. In simple words, *spectral density defines the distribution of energy or power of the signal per unit bandwidth as function of frequency.*

In general we encounter two types of signals based on energy. Some signals are finite energy signals (for example, pulse signals) and others are infinite energy (for example, periodic signals). *Signals having finite energy are called energy signals* while the *signals having infinite energy are called power signals*. When you deal with energy signals it is customary to use energy spectral density as the parameter for computation of the signal. Likewise when dealing with power signals we use power spectral density of the signal as characterizing the parameter.

NORMALIZED ENERGY

The normalized energy (or simply energy) of a signal x(t) is defined as energy dissipated by a voltage signal applied across a 1 ohm resistor (or alternatively by a current signal flowing through a 1 ohm resistor). The signal x(t) may be complex or real valued.

The normalized energy of the signal is defined as,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \qquad \text{... (4.7.1)}$$

It should be noted that the energy of a signal defined by the Eq. (4.7.1) is only when the above integral is finite.

4.7.1 Parseval's Theorem for Energy Signals

Parsevals theorem for energy signals states that energy of a signal is determined the area under curve $|X(\omega)|^2$ or $|X(f)|^2$ as a function of frequency. That is,

$$E = \frac{1}{2\pi}\int |X(\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

PROOF

Consider a function x(t) such that $x(t) \xleftrightarrow{F.T} X(\omega)$. Let x*(t) be is conjugate, then,

$$x^*(t) \xleftrightarrow{F.T} X^*(-\omega)$$

The energy of a signal x(t) is given by,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t)x^*(t) \, dt = \int_{-\infty}^{\infty} x^*(t) \, dt \quad \text{... (4.7.2)}$$

Replacing x(t) in Eq. (4.7.2) using the definition of inverse Fourier transform, we have,

$$E = \int_{-\infty}^{\infty} x^*(t) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega \right\} dt$$

Interchanging the order of integration,

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left\{ \int_{-\infty}^{\infty} x^*(t) e^{j\omega t} \, d\omega \right\} d\omega$$

We have,

$$x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

$$\Rightarrow \quad x^*(\omega) = \int_{-\infty}^{\infty} x^*(t) \left(e^{-j\omega t}\right)^* dt$$

$$= \int_{-\infty}^{\infty} x^*(t) \, e^{j\omega t} \, dt$$

$$\therefore \quad E = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \, X^*(\omega) \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega \quad \text{... (4.7.3)}$$

We have $\omega = 2\pi f$. So $d\omega = 2\pi df$. Therefore,

$$E = \int_{-\infty}^{\infty} |X(f)|^2 \, df \quad \text{... (4.7.4)}$$

Eqs. (4.7.3), (4.7.4) is called ***Parseval's theorem*** for ***energy signals***, also called ***Rayleigh energy theorem***. Eqs. (4.7.3) and (4.7.4) signifies that the energy of a signal is determined by the area under $|X(\omega)|^2$ or $|X(f)|^2$ versus frequency curve.

4.7.2 Energy Spectral Density (ESD)

Energy spectral density is defined as the distribution of energy of the signal per unit bandwidth as function of frequency. ESD is denoted by $\psi(f)$.

$$\psi(f) = \frac{\text{Energy}}{\text{Unit bandwidth}} = \frac{E}{\Delta f}$$

As seen from the Rayleighs theorem $|X(f)|^2$ gives the distribution of energy of the signal x(t), in the frequency domain. Thus,

$$\psi(f) = |X(f)|^2 = |X(\omega)|^2$$

Let us consider a signal x(t) applied at the input of an ideal bandpass filter whose transfer function is shown in Fig. 4.7.1(a). This filter suppresses all frequencies except a narrow band $\Delta f(\Delta f \to 0)$ centered at frequency f_0. If Y(f) is the Fourier transform of the response y(t) of the filter, then,

$$Y(f) = X(f)\,H(f) \qquad \text{... (4.7.5)}$$

The energy E_y of the output y(t) is given by,

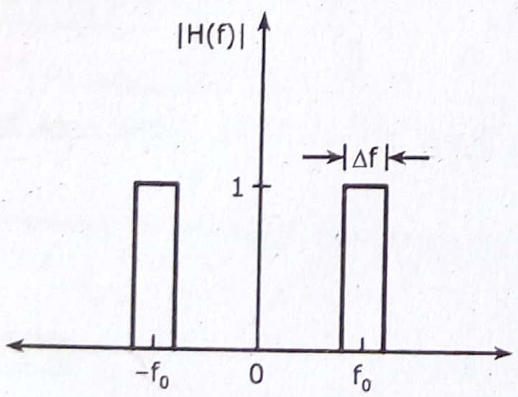
$$E_y = \int_{-\infty}^{\infty} |Y(f)|^2\, df \qquad \text{... (4.7.6)}$$

Using Eq. (4.7.5) the energy of the bandpass filter output can be written as,

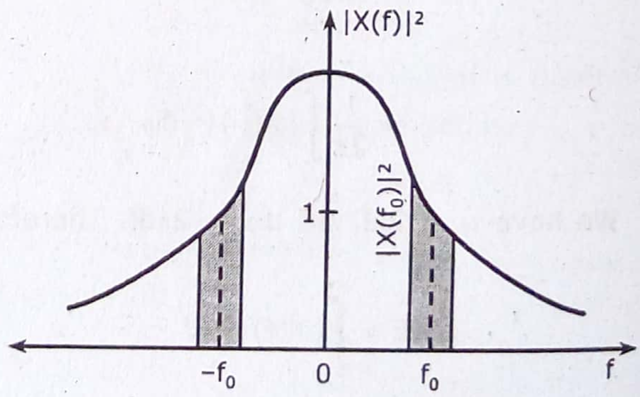
$$E_y = \int_{-\infty}^{\infty} |X(f)\,H(f)|^2\, df \qquad \text{... (5.6.7)}$$

Since $|H(f)| = 0$ everywhere except over a narrow band Δf where it is unity, we have,

$$E_y = 2|X(f_0)|^2\,\Delta f \qquad \text{... (4.7.8)}$$



(a) Transfer Function of Bandpass



(b) Spectral Distribution of the Input Signal Showing the Contribution to the Output Energy (Shaded Portion)

Fig. 4.7.1

The energy of the output signal is thus $2|X(f_0)|^2\Delta f$. This factor $2|X(f_0)|^2\Delta f$ represents the contribution to the energy of x(t) by the frequency components of x(t) lying within a narrow band Δf centered at f_0. Hence, $2|X(f_0)|^2$ is the energy per positive frequency components contributes one-half of the energy $2|X(f_0)|^2$ while the remaining half is contributed by the negative frequency components centered at f_0. Therefore, factor $2|X(f_0)|^2$ can be interpreted as the energy per unit bandwidth of the input signal which is contributed by the frequency components (positive or negative) around the frequency f.

i.e.,
$$|X(f)|^2 = \frac{E}{\Delta f} = \psi(f) \qquad \text{... (4.7.9)}$$

Therefore the total energy E of a signal can be written in terms of energy density spectrum as,

$$E = \int_{-\infty}^{\infty} |X(f)|^2 \, df = \int_{-\infty}^{\infty} \psi_x(f) \, df \qquad \text{... (4.7.10)}$$

Since $|X(f)|^2 = |X(-f)|^2$, $\psi_x(f)$ is an even function of f and we may write Eq. (4.7.10) as,

$$E = \int_{-\infty}^{\infty} |X(f)|^2 \, df = 2\int_{0}^{\infty} |X(f)|^2 \, df = \int_{0}^{\infty} \psi_x(f) \, df$$

4.7.3 Energy Spectral Densities of Input and Output

If x(t) and y(t) are the input and response respectively of a linear system (shown in Fig. 4.7.2) then,

$$Y(f) = X(f)\,H(f) \qquad \text{... (4.7.11)}$$

Where,

H(f) = The transfer function of the system.

X(f), Y(f) = The Fourier transforms of the input and the response respectively.

The energy spectral density of the input (excitation) is $|X(f)|^2$ and that of the output (response) is $|Y(f)|^2$. Therefore,

$$\psi_x(f) = |X(f)|^2$$

And
$$\psi_y(f) = |Y(f)|^2 = |H(f)\,X(f)|^2 = |H(f)|^2\,|X(f)|^2$$
$$= |H(f)|^2\,\psi_x(f) \qquad \text{... (4.7.12)}$$

Eq. (4.7.12), states that the energy density spectrum of the output (response) of a linear system is the product of the energy density spectrum of the input and square of the magnitude of the transfer function.

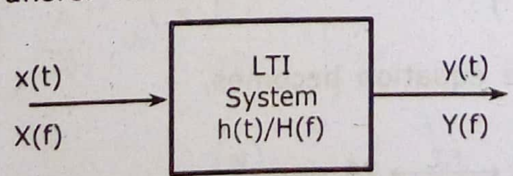


Fig. 4.7.2 An LTI System

4.7.4 Properties of Energy Spectral Density

PROPERTY I : The total area under the energy spectral density function is equal to the total energy of that signal i.e.,

$$E = \int_{-\infty}^{\infty} \psi(f)\, df$$

PROPERTY II : If x(t) is input to a linear time invariant (LTI) system with transfer function $H(\omega)$, then input and output energy spectral density functions are related as,

$$\psi_0(\omega) = |H(\omega)|^2\, \psi_i(\omega)$$

Where,

$\psi_0(\omega)$ = Output energy spectral density function.

$\psi_i(\omega)$ = Input energy spectral density function.

$|H(\omega)|^2$ = Energy gain at frequency ω.

PROPERTY III : The autocorrelation function $R_{xx}(\tau)$ and energy spectral density function $\psi_x(\omega)$ form a Fourier transform pair i.e.,

$$R_{xx}(\tau) \xleftrightarrow{F.T} \psi_x(\omega)$$

EXAMPLE PROBLEM 1

Determine the energy of the sinc pulse defined by, x(t) = A sinc(2Wt)

SOLUTION

The energy of the given signal can be written as,

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = A^2 \int_{-\infty}^{\infty} \text{sinc}^2(2Wt)\, dt \quad \text{... (4.7.13)}$$

The above integral which is difficult to evaluate can be calculated indirectly by using Rayleigh energy theorem.

For this purpose we start with the following Fourier transform pairs.

$$A\text{Rect}\left(\frac{t}{2T}\right) \xleftrightarrow{F.T} 2AT\, \text{sinc}\left(\frac{\omega T}{\pi}\right)$$

Choosing T = 1, above equation becomes,

$$A\text{Rect}\left(\frac{t}{2}\right) \xleftrightarrow{F.T} 2A\, \text{sinc}\left(\frac{\omega}{\pi}\right)$$

Applying the duality property, we get,

$$2A\,\text{sinc}\left(\frac{t}{\pi}\right) \xleftrightarrow{\text{F.T}} 2\pi A\ \text{Rect}\left(\frac{-\omega}{2}\right) = 2\pi A\ \text{Rect}\left(\frac{\omega}{2}\right)$$

Applying the time-scaling property (choose time-scaling factor equal to $2\pi W$), we get,

$$2A\,\text{sinc}(2Wt) \xleftrightarrow{\text{F.T}} \frac{1}{|2\pi W|} 2\pi A\ \text{Rect}\left(\frac{-\omega}{2(2\pi W)}\right)$$

$$\left(\because\ x(at) \xleftrightarrow{\text{F.T}} \frac{1}{|a|} x\left(\frac{\omega}{a}\right)\right)$$

Using linear property, i.e., $a\ x(t) \xleftrightarrow{\text{F.T}} a\ X(\omega)$, we get,

$$A\,\text{sinc}(2Wt) \xleftrightarrow{\text{F.T}} \frac{A}{2W}\ \text{Rect}\left(\frac{\omega}{4\pi W}\right)$$

$$\therefore \quad F[x(t)] = X(\omega) = \frac{A}{2W}\ \text{Rect}\left(\frac{\omega}{4\pi W}\right)$$

From the definition of Rayleigh's theorem, energy of a signal x(t) is given by,

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

The frequency spectrum of given function x(t) is shown in Fig. 4.7.3.

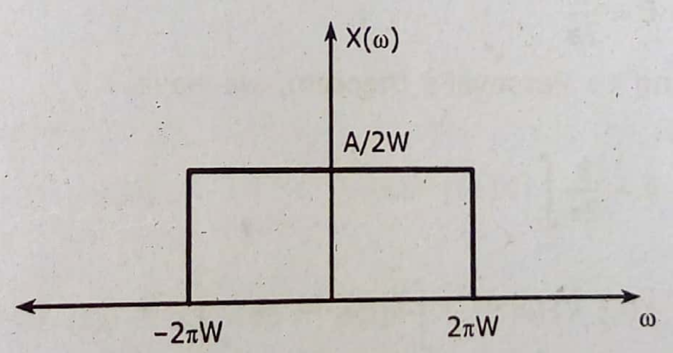


Fig. 4.7.3 Frequency Spectrum of x(t) = A sinc(2Wt)

$$E = \frac{1}{2\pi}\int_{-2\pi W}^{2\pi W} \left|\frac{A}{2W}\right|^2 d\omega = \frac{A^2}{2\pi W^2}\left[\omega\right]_{-2\pi W}^{2\pi W}$$

$$= \frac{A^2}{8\pi W^2}[2\pi W + 2\pi W] = \frac{A^2}{8\pi W^2}[4\pi W] = \frac{A^2}{2W}$$

EXAMPLE PROBLEM 2

Verify Parseval's theorem for the energy signal $x(t) = e^{-at}\,u(t)$, $a > 0$.

SOLUTION

The energy of a signal x(t) is given by,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} |e^{-at}\,u(t)|^2 \, dt$$

$$= \int_{0}^{\infty} |e^{-at}|^2 \, dt$$

$$= \int_{0}^{\infty} e^{-2at} \, dt$$

$$= \left[\frac{e^{-2at}}{-2a}\right]_0^{\infty}$$

$$= \frac{e^{-\infty} - e^{0}}{-2a}$$

$$E = \frac{0-1}{-2a}$$

$$\therefore \qquad E = \frac{1}{2a}$$

Now, according to Parseval's theorem, we have,

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

$$X(\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-at}\,u(t)\,e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\,e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \frac{1}{-(a+j\omega)}\left[e^{-(a+j\omega)t}\right]_0^{\infty} = \frac{1}{a+j\omega}$$

$$= \left(\frac{1}{a+j\omega}\right)\left(\frac{a-j\omega}{a-j\omega}\right) = \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}} \quad \text{... (4.7.14)}$$

$$\therefore \quad E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a^2+\omega^2}\, d\omega = \frac{1}{2\pi a}\tan^{-1}\left[\frac{\omega}{a}\right]_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi a}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{1}{2a} \quad \text{... (4.7.15)}$$

Thus, from Eq. (4.7.14) and Eq. (4.7.15), we see that the energy is same in both cases. Hence, Parseval's theorem is verified.

REVIEW QUESTIONS

(1) *What are energy signals, power signals and normalized energy?*

(2) *Explain about Parseval's theorem for energy signals and ESD in detail.*

(3) *State and prove Parseval's theorem.*

4.8 POWER DENSITY SPECTRUM

Signals with infinite energy (for example periodic signals) are called *power signals*. The meaningful parameter of a power signal x(t) is the *average power* P_x. *The average power (or simply power) of a signal is defined as the average power dissipated by a voltage x(t) applied across a 1 ohm resistor (or by a current signal x(t) flowing through 1 ohm resistor).*

Thus the average power P_x is given by,

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt \quad \text{... (4.8.1)}$$

The power P_x defined by Eq. (4.8.1) is actually the *mean square value* or the *time average of the squared signal.* Thus we may write.

$$P_x = \overline{x^2(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt \quad \text{... (4.8.2)}$$

4.8.1 Parseval's Theorem for Power Signals

Parseval's power theorem defines the power of a signal in terms of its Fourier series coefficients, i.e., in terms of the harmonic components present in the signal. Mathematically, it is given by,

$$P = \sum_{n=-\infty}^{\infty} |C_n|^2 \quad \text{... (4.8.3)}$$

The average power P of the function x(t) is given by,

$$P = \overline{x^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = \int_{-\infty}^{\infty} S(f)\, df \qquad \text{... (4.8.11)}$$

Where $\overline{x^2(t)}$ is the mean square value of x(t). The average power is therefore given by,

$$P = \frac{1}{\pi}\int_{0}^{\infty} S(\omega)\, d\omega = 2\int_{0}^{\infty} S(f)\, df \qquad \text{... (4.8.12)}$$

The PSD of a periodic function is given by,

$$S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2\, \delta(\omega - n\omega_0) \qquad \text{... (4.8.13)}$$

(or)

$$S(f) = \sum_{n=-\infty}^{\infty} |C_n|^2\, \delta(f - nf_0) \qquad \text{... (4.8.14)}$$

4.8.3 Properties of Power Spectral Density (PSD)

The following are the properties of power spectral density,

Property I : The area under the PSD function is equal to the average power of that signal, i.e.,

$$P = \int_{-\infty}^{\infty} S(f)\, df \qquad \text{... (4.8.15)}$$

Property II : The input and output PSDs of an LTI system are related as,

$$S_y(f) = |H(f)|^2\, S_x(f) \qquad \text{... (4.8.16)}$$

Where,

$S_y(f)$ = Power spectral density of output y(t).

$S_x(f)$ = Power spectral density of input x(t).

H(f) = Transfer function of the system.

Property III : The autocorrelation function $R_{xx}(\tau)$ and PSD S(f) form a Fourier transform pair, i.e.,

$$R_{xx}(\tau) \xleftrightarrow{F.T} S_x(f) \qquad \text{... (4.8.17)}$$

4.8.4 Comparison of ESD and PSD

Comparison of ESD and PSD of a function x(t) are depicted in Table 4.8.1.

Table 4.8.1 Comparison of ESD and PSD

Sl. No.	Energy Spectral Density (ESD)	Power Spectral Density (PSD)
(1)	It defines the distribution of energy of a signal in frequency domain.	It defines the distribution of power of a signal in frequency domain.
(2)	It is given by, $\psi(f) = \lvert X(f) \rvert^2$	It is given by, $S(f) = \lim\limits_{T \to \infty} \dfrac{\lvert X_T(f) \rvert^2}{T}$
(3)	The total energy is given by, $E = \int_{-\infty}^{\infty} \psi(f)\, df = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} \psi(\omega)\, d\omega$	The total power is given by, $P = \int_{-\infty}^{\infty} S(f)\, dt = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, d\omega$
(4)	The autocorrelation for an energy signal and its ESD form a Fourier transform pair. $R_{xx}(\tau) \xleftrightarrow{F.T} \psi(f)$ (or) $R_{xx}(\tau) \xleftrightarrow{F.T} \psi(\omega)$	The autocorrelation for a power signal and its PSD form a Fourier transform pair. $R_{xx}(\tau) \xleftrightarrow{F.T} S(f)$ (or) $R_{xx}(\tau) \xleftrightarrow{F.T} S(\omega)$

REVIEW QUESTIONS

(1) *Explain the power spectral density along with its properties?*

(2) *Differentiate between the ESD and PSD?*

(3) *Explain and derive an expression for Parseval's theorem in power signals?*

4.9 RELATION BETWEEN AUTOCORRELATION FUNCTION AND ENERGY/POWER SPECTRAL DENSITY FUNCTION

4.9.1 Relation between ESD and Autocorrelation Function

The autocorrelation function $R_{xx}(\tau)$ for an energy signal and its energy spectral density function $\psi(\omega)$ form a Fourier transform pair, i.e.,

$$R_{xx}(\tau) \xleftrightarrow{F.T} \psi(\omega)$$

PROOF

The autocorrelation of a function x(t) is given as,

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

Replacing $x^*(t - \tau)$ by its inverse Fourier transform, we have,

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega(t-\tau)}\, d\omega\right]^* dt$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\left[\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega(t-\tau)}\, d\omega\right] dt$$

Interchanging the order of integration, we have,

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt\right] e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, X(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)e^{j\omega\tau}\, d\omega \qquad \left(\because\ |X(\omega)^2| = \psi(\omega)\right)$$

$$= F^{-1}[\psi(\omega)]$$

$$\therefore \quad \psi(\omega) = F[R_{xx}(\tau)]$$

This proves that $R(\tau)$ and $\psi(\omega)$ form a Fourier transform pair,

$$R_{xx}(\tau) \xleftrightarrow{\text{F.T}} \psi(\omega)$$

4.9.2 Relation between PSD and Autocorrelation Function

The autocorrelation function $R_{xx}(\tau)$ for a power signal and its power spectral density (PSD) function $S(\omega)$ of a power signal form a Fourier transform pair, i.e.,

$$R_{xx}(\tau) \xleftrightarrow{\text{F.T}} S(\omega)$$

PROOF

The autocorrelation function of a power (periodic) signal x(t) in terms of Fourier series coefficients is given as,

$$R_{xx}(\tau) = \sum_{n=-\infty}^{\infty} C_n C_{-n} e^{jn\omega_0\tau}$$

Where, C_n and C_{-n} are the exponential Fourier series coefficients.

$$\therefore \quad R_{xx}(\tau) = \sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0\tau} \qquad \ldots (4.9.1)$$

Applying Fourier transform on both sides of Eq. (4.9.1) we get,

$$F[R_{xx}(\tau)] = \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0\tau} \right) e^{-j\omega\tau}\, d\tau$$

Interchanging the order of integration and summation, we get,

$$F[R_{xx}(\tau)] = \sum_{n=-\infty}^{\infty} |C_n|^2 \int_{-\infty}^{\infty} e^{-j\tau(\omega - n\omega_0)}\, d\tau$$

But, $\quad 1 \xleftrightarrow{F.T} 2\pi\delta(\omega)$

And $\quad 1.e^{j\omega_0 t} \xleftrightarrow{F.T} 2\pi\delta(\omega - \omega_0)$

Thus, above equation becomes,

$$= 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0)$$

The RHS is the PSD $S(\omega)$ or $S(f)$ of the periodic function x(t).

$$\therefore \quad F[R_{xx}(\tau)] = S(\omega) \text{ [or } S(f)]$$

i.e., $\quad R_{xx}(\tau) \xleftrightarrow{F.T} S(\omega)$ [or $S(f)$]

REVIEW QUESTIONS

(1) *Derive a relation between ESD and autocorrelation function?*

(2) *How autocorrelation is proportional to power spectral density?*

4.10 RELATION BETWEEN CONVOLUTION AND CORRELATION

Both the convolution and correlation mathematical tools have a striking similarity. Ofcourse the two integrals are closely related. To obtain the cross correlation of x(t) and y(t), according to the equation $R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y(t-\tau)\, dt$, we multiply x(t) with function y(t) displaced by τ sec. The area under the product curve is the cross correlation between x(t) and y(t) at $t = \tau$. On the other hand, the convolution of x(t) and y(t) at $t = \tau$ is obtained by folding y(t) backward about the vertical axis at the origin and taking the area under the product curve of x(t) and the folded function y(−t) displaced by τ. It, therefore, follows that the cross correlation of x(t) and y(t) is the same as the convolution of x(t) and y(−t).

The same conclusion can be arrived at analytically as follows.

The convolution of x(t) and y(−t) is given by,

$$x(t) * y(-t) = \int_{-\infty}^{\infty} x(\tau)\, y(\tau - t)\, d\tau$$

Replacing the dummy variable τ in the above integral by another variable k, we have,

$$x(t) * y(-t) = \int_{-\infty}^{\infty} x(k)\, y(k - t)\, dk$$

Changing the variable from t to τ, we get,

$$x(\tau) * y(-\tau) = \int_{-\infty}^{\infty} x(k)\, y(k - \tau)\, dk = R_{xy}(\tau)$$

$$\therefore \quad R_{xy}(\tau) = x(t) * y(-t)\Big|_{t=\tau}$$

And $$R_{xy}(\tau) = y(t) * x(-t)\Big|_{t=\tau}$$

All of the techniques used to evaluate the convolution of two functions can be directly applied in order to find the correlation of two functions. Similarly, all of the results derived for convolution also apply to correlation.

If one of the function is an even function of t, let us say y(t) is an even function of t, i.e.,

$$y(t) = y(-t)$$

Then the cross correlation and convolution are equivalent.

REVIEW QUESTIONS

(1) *Derive a relationship between convolution and correlation?*

(2) *Explain how and when the cross-correlation and convolution are equivalent?*

4.11 DETECTION OF PERIODIC SIGNALS IN THE PRESENCE OF NOISE BY CORRELATION

In detection theory, detecting the periodic signals in presence of random noise/AWGN is of utmost importance. It finds various applications in RADAR, SONAR detection, detection of periodic components in brain waves, the detection of cyclic components in ocean wave analysis, in meteorology, etc. The correlation techniques discussed earlier finds to be a powerful tool in the solution of the above problems.

The noise signal encountered in practice is a signal with random amplitude variations. Such a signal is uncorrelated with any periodic signal.

Let x(t) represent the periodic signal, n(t) represent the noise. Then,

$$\overline{R_{xn}(\tau)} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, n(t-\tau)\, dt = 0$$

4.11.1 Detection by Autocorrelation

Let x(t) be the periodic signal mixed with noise n(t).

So the received signal will be, additive of periodic signal and noise n(t), i.e.,

$$y(t) = x(t) + n(t)$$

Where the received signal y(t) is also periodic. Autocorrelation function of periodic signal y(t) is given by,

$$\therefore \quad \overline{R_{yy}(\tau)} = \lim_{T \to \infty} \frac{1}{T} \int_0^T y(t)\, y(t-\tau)\, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T [x(t) + n(t)]\, [x(t-\tau) + n(t-\tau)]\, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \left[\int_0^T x(t).x(t-\tau)\, dt + \int_0^T x(t)\, n(t-\tau)\, dt + \int_0^T n(t)\, x(t-\tau)\, dt + \int_0^T n(t)\, n(t-\tau)\, dt \right] \quad \ldots (4.11.1)$$

Using the definitions of autocorrelation and cross correlation functions, Eq. (4.11.1) becomes,

$$\overline{R_{yy}(\tau)} = \overline{R_x(\tau)} + \overline{R_{xn}(\tau)} + \overline{R_{nx}(\tau)} + \overline{R_n(\tau)}$$

Since s(t) and n(t) are uncorrelated, we have,

$$\overline{R_{xn}(\tau)} = \overline{R_{nx}(\tau)} = 0$$

$$\therefore \quad \overline{R_{yy}(\tau)} = \overline{R_x(\tau)} + \overline{R_n(\tau)}$$

We know that the autocorrelation function of a periodic signal is periodic and that of non-periodic function tends to zero for large values of τ.

As x(t) is periodic and n(t) is non-periodic, $\overline{R_x(\tau)}$ is also periodic and $\overline{R_n(\tau)}$ is arbitrarily small for large values of τ.

Therefore, for sufficiently large values of τ, the autocorrelation of a given signal can be calculated by the numerical techniques used for convolution, even on digital computers.

4.11.2 Detection by Crosscorrelation

The presence of noise in a periodic signal can also be carried out by cross correlating the received signal with another periodic signal of the same frequency. Detection by cross-correlation is much more effective than by the autocorrelation. However The drawback is that it is necessary to know the frequency of the signal to be detected, earlier. In many cases such as RADAR, etc., the frequency is known earlier. Let the received signal by,

$$y(t) = x(t) + n(t)$$

Let the locally generated signal of same frequency as that of x(t) be g(t).

$$\therefore \quad R_{yg}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_T [x(t) + n(t)]\, g(t - \tau)\, dt$$

i.e.,

$$\overline{R_{yg}(\tau)} = \overline{R_{xg}(\tau)} + \overline{R_{ng}(\tau)}$$

If g(t) and n(t) are uncorrelated, $R_{nl}(\tau) = 0$

$$\therefore \quad \overline{R_{yg}(\tau)} = \overline{R_{xg}(\tau)}$$

It is obvious that if the cross-correlation of the corrupted signal y(t) with g(t) gives a periodic signal, y(t) must also contain a periodic component of the same frequency as that of g(t).

$$\therefore \quad \overline{R_{yy}(\tau)} = \overline{R_x(\tau)} + \overline{R_n(\tau)}$$

Hence, we can conclude that whether a periodic component is present or not, for any value of τ, we have,

$$\overline{R_{yg}(\tau)} = \overline{R_{sg}(\tau)}$$

REVIEW QUESTIONS

(1) How the noise of a periodic signal can be detected by using auto correlation technique?

(2) Explain an effective method of detection of signals in the presence of noise?

4.12 EXTRACTION OF A SIGNAL FROM NOISE BY FILTERING

Filtering technique is nothing but extraction of a given signal in the frequency domain. If the given signal is extracted in the time domain then it is known as correlation technique.

RELATIONSHIP BETWEEN CORRELATION AND FILTERING

Consider $f_1(t)$ and $f_2(t)$ are the signals, whose cross correlation function is $R_{12}(\tau)$.

If, $$f_1(t) \xleftrightarrow{F.T} F_1(\omega)$$

$$f_2(t) \xleftrightarrow{F.T} F_2(\omega)$$

Then, $$\overline{R_{12}(\tau)} \xleftrightarrow{F.T} F_1(\omega)F_2(-\omega)$$

Multiplication of the spectra $F_1(\omega)$ and $F_2(-\omega)$ in the frequency domain is equivalent to cross correlation of $f_1(t)$ and $f_2(t)$ in the time domain. The time domain relation of cross correlation function is obtained by evaluating the integral by a cross correlator. Cross correlation in time and frequency domain is shown in following figures.

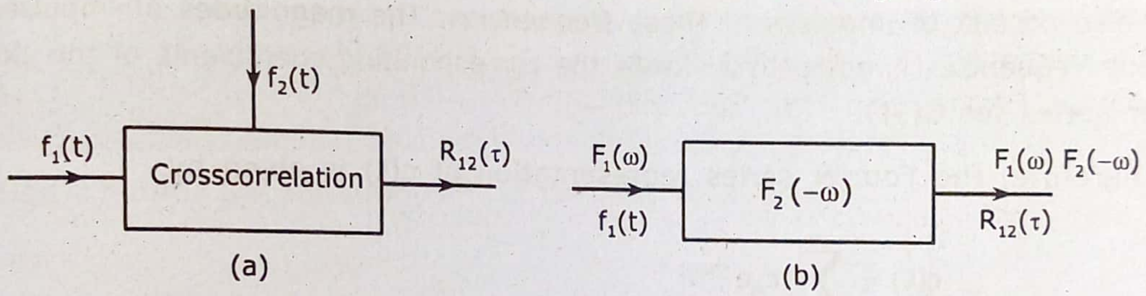


Fig. 4.12.1 Cross Correlation in Time and Frequency Domain

In the given figures,

$f_1(t)$ → Input signal

$F_2(-\omega)$ → Transfer function

$R_{12}(\tau)$ → Output

From the Fig. 4.12.1(b), it is noted that when the signal $f_1(t)$ is applied to $F_2(-\omega)$ the output will be $R_{12}(\tau)$. When the signal $f_1(\omega)$ is applied to $F_2(-\omega)$, the cross correlation between signals $f_1(t)$ and $f_2(t)$ may be effected. This operation basically represents filtering. The impulse response of a system is given by,

$$h \;(\; F^{-1} [F_2(-\omega)]$$

$$\therefore \quad h(t) = f_2(-t) \qquad \left(\begin{matrix} \because f_2(t) \leftrightarrow F_2(\omega) \\ f_2(-t) \leftrightarrow F_2(-\omega) \end{matrix}\right)$$

$f_1(t)$ and $f_2(t)$ are the signals and its cross correlation function is the response of a system with the impulse response $f_2(-t)$ when the driving function is $f_1(t)$.

If x(t) is the desired periodic signal component and n(t) is the random noise component then the received signal is given by,

$$y(t) = x(t) + n(t)$$

The cross correlating signal f(t) with another periodic signal c(t) of the same period as that of x(t), detects the desired periodic signal components x(t) which is present in y(t). The above said cross correlation function is performed by a system which has a unit impulse response c(−t) or a transfer function C(−ω)

i.e.,
$$c(t) \leftrightarrow C(\omega)$$
$$c(-t) \leftrightarrow C(\omega)$$

Where,

c(t) = Periodic signal with period T_0.

C(ω) = Fourier transform of periodic signal c(t).

Therefore, C(ω) consist of impulses located at $\omega = 0, \pm 2\omega_0, \pm 3\omega_0 \ldots \pm n\omega_0$. Similarly C(−ω) also consist of impulses at these frequencies. The magnitudes of impulses located different frequencies is equal to 2π times the corresponding coefficients of the exponential Fourier series for C(−t).

Therefore, the Fourier series representation of c(t) is given by,

$$c(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Where, $\omega_0 = \frac{2\pi}{T_0}$

$$\therefore \quad \text{F.T.}\,[c(t)] = C(\omega) = 2\pi \sum_n C_n \delta(\omega - n\omega_0)$$

And $$C(-\omega) = 2\pi \sum_n C_n * \delta(\omega - n\omega) \qquad [\because C(-\omega) = C^*(\omega)]$$

REVIEW QUESTIONS

(1) How correlation is related to the filtering?

(2) Explain how a signal can be extracted from noise by using filtering technique?