UNIT 1

SIGNAL ANALYSIS

PROFESSIONAL'S
The Destination Towards Knowledge & Success

SYLLABUS

Analogy between Vectors and Signals, Orthogonal Signal Space, Signal Approximation using Orthogonal Functions, Mean Square Error, Closed or Complete Set of Orthogonal Functions, Orthogonality in Complex Functions, Exponential and Sinusoidal Signals, Concepts of Impulse Function, Unit Step Function, Signum Function.

PART - A
SHORT QUESTIONS WITH ANSWERS

Q1) Define signal? How it is classified based on the number of sources for the signal?

Ans. : A signal is defined as "a function of one or more independent variables which contains some information".

Classification of Signals Based on the Number of Sources for the Signal

(1) **One-Channel Signals :** Signals that are generated by a single signal source are called one-channel signals.

Example : The record of room temperature with respect to time, the audio output of a mono speaker, etc.

(ii) **Multi-Channel Signals :** Signals that are generated by multiple signal sources are called multi-channel signals.

Examples

(i) The audio output of two stereo speakers is an example of two-channel signal.

(ii) The record of ECG (Electro Cardia Graph) at eight different places in a human body is an example of eight-channel signal.

Q2) Define error vector.

Ans. : Figure depicts, the component of a vector $\bar{A}_1$ along the vector $\bar{A}_2$ obtained by drawing a perpendicular from the end of vector $\bar{A}_1$ on the vector $\bar{A}_2$. Let the length of the perpendicular may be represented by another vector $\bar{A}_e$. Thus the vector $\bar{A}_1$ can now be expressed in terms of vector $\bar{A}_2$ and $\bar{A}_e$ as,

$$\bar{A}_1 = C_{12}\,\bar{A}_2 + \bar{A}_e$$

Two other alternative ways of representing the component of vector $\bar{A}_1$ in terms of vector $\bar{A}_2$.

$$\bar{A}_1 = C_1\bar{A}_2 + \bar{A}_{e1}$$

$$\bar{A}_1 = C_2\bar{A}_2 + \bar{A}_{e2}$$

In each case $\bar{A}_1$ is represented in terms of $\bar{A}_2$ plus another vector, which will be called the error vector. If we are asked to approximate $\bar{A}_1$ by $\bar{A}_2$ then $\bar{A}_e$ represents the error in their approximation.

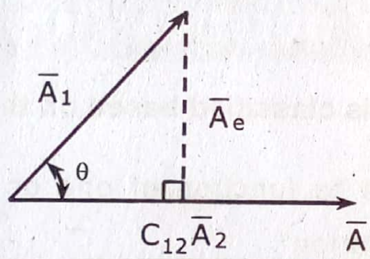


Figure Component of the Vector $\bar{A}_1$ along Vector $\bar{A}_2$

Q3) What do you mean by an orthogonal vector?

Ans. : if the component of a vector $\bar{A}_1$ along $\bar{A}_2$ is $C_{12}\bar{A}_2$ then the magnitude of C_{12} is an indication of the similarity of the two vectors. If C_{12} is zero, then the vector has no component along the other vector and hence the two vectors are mutually perpendicular. Such vectors are known as orthogonal vectors. Orthogonal vectors are thus independent vectors. If the vectors are orthogonal, then the parameter C_{12} is zero as shown in figure below,

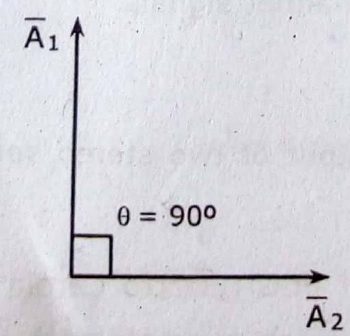


Figure Orthogonal Vectors (Here $C_{12} = 0$)

Q4) Define orthogonal signals.

Ans. : Two vectors $\bar{A}_1$ and $\bar{A}_2$ are orthogonal if their dot product is zero, i.e., $\bar{A}_1.\bar{A}_2 = 0$. Similarly two signals $x_1(t)$ and $x_2(t)$ are orthogonal if the integral of the product of those signals is zero, i.e.,

$$\int_{t_1}^{t_2} x_1(t)x_2(t)\, dt = 0$$

Q5) Define orthogonal vector space.

Ans. : The condition of orthogonality implies that the dot product of any two vectors $\bar{x}_j$ and $\bar{x}_r$ must be zero and the dot product of any vector with itself must be unity.

This can be expressed as,

$$\bar{x}_j.\bar{x}_r = \begin{cases} 0 & ; j \neq r \\ 1 & ; j = r \end{cases}$$

The dot product of both sides of the equation for $\bar{A}$ with vector $\bar{x}_r$ gives,

$$\bar{A}.\bar{x}_r = C_1\bar{x}_1.\bar{x}_r + C_2\bar{x}_2.\bar{x}_r + ... + C_r\bar{x}_r.\bar{x}_r + ... + C_n\bar{x}_n.\bar{x}_r$$

Since $\bar{x}_j.\bar{x}_r = 0$ for $j \neq r$ and equal to 1 for $j = r$ we get all the terms of the form $C_n\bar{x}_n.\bar{x}_r\,(n \neq r)$ are zero,

$$\therefore \qquad \bar{A}.\bar{x}_r = \bar{C}_r\bar{x}_r.\bar{x}_r = C_r$$

This set of vectors $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ which are mutually perpendicular to each other are called an *orthogonal vector space*. In general, the product $\bar{x}_j.\bar{x}_r$ can be some constant k_r instead of unity. When k_r is unity, the set is called normalized orthogonal set or *orthonormal vector space*. Therefore, in general, for orthogonal vector space we have,

$$\bar{x}_j.\bar{x}_r = \begin{cases} 0 & ; j \neq r \\ k_r & ; j = r \end{cases}$$

Q6) What is orthogonal signal space?

Ans. : any signal x(t) can be expressed as a sum of its components along a set of 'n' mutually orthogonal functions, provided these functions form a complete set. The space bounded by a set of 'n' mutually orthogonal functions forming the co-ordinate system is called orthogonal signal space.

Q7) Derive the formula for mean square error.

Ans. : By the definition of mean square error,

$$\varepsilon = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[x(t) - \sum_{r=1}^{n} C_r m_r(t)\right]^2 dt$$

But from the equation for C_r, we have,

$$\int_{t_1}^{t_2} x(t)m_r(t)dt = C_r\int_{t_1}^{t_2} m_r^2(t)dt = C_r k_r$$

$$\varepsilon = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} x^2(t)dt + \sum_{r=1}^{n} C_r^2 k_r - 2\sum_{r=1}^{n} C_r^2 k_r\right] = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} x^2(t)dt - \sum_{r=1}^{n} C_r^2 k_r\right]$$

The mean square error can therefore be evaluated using above fromula.

Q8) What do you mean by complete set or closed set?

Ans. : A set of mutually orthogonal functions $m_1(t)$, $m_2(t)$, ... $m_r(t)$ over the interval (t_1, t_2) is said to be a complete or a closed set if there exists no function $x(t)$ for which it is true that,

$$\int_{t_1}^{t_2} x(t)\cdot m_k(t)\,dt = 0 \qquad ; \text{ for } k = 1, 2, \ldots.$$

If a function $x(t)$ could be found such that the above integral is zero, then obviously $x(t)$ is orthogonal to each member of the set $[m_r(t)]$ and as a result, it itself a member of the set. So the set cannot be complete without $x(t)$ being its member.

Q9) Prove that the complex exponential signals are orthogonal functions.

Ans. : Consider two complex exponential signals,

$$x_1(t) = e^{jn\omega_0 t} \text{ and } x_2(t) = e^{jm\omega_0 t}$$

Let the interval be t_0 to $t_0 + T$, i.e., from t_0 to $t_0+(2\pi/\omega_0)$. $x_1(t)$ and $x_2(t)$ are orthogonal over the interval t_0 to $t_0+(2\pi/\omega_0)$, if

$$I = \int_{t_0}^{t_0+(2\pi/\omega_0)} x_1(t)x_2^*(t)dt = 0$$

Here, $x_1(t) = e^{jn\omega_0 t}$

And $x_2^*(t) = [e^{jm\omega_0 t}]^* = e^{-jm\omega_0 t}$

$$\therefore \quad I = \int_{t_0}^{t_0+(2\pi/\omega_0)} e^{jn\omega_0 t}\, e^{-jm\omega_0 t}\, dt$$

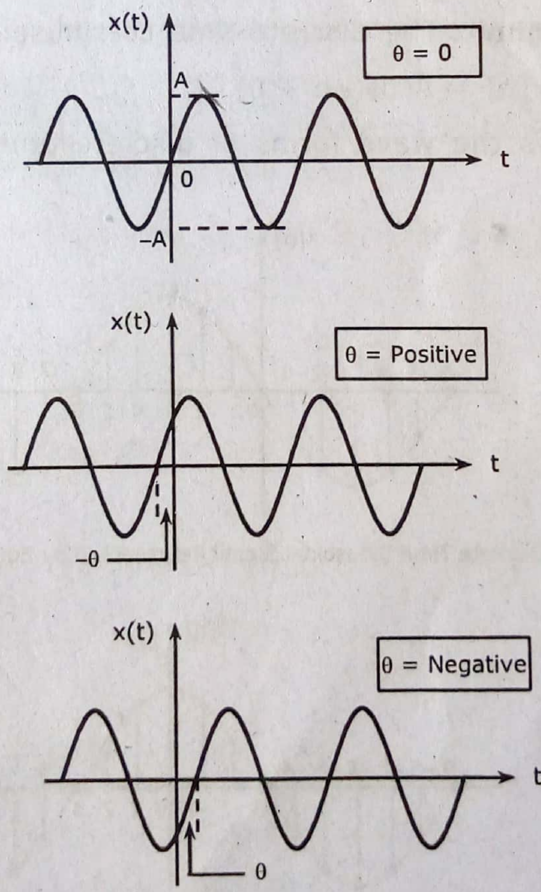
$$= \int_{t_0}^{t_0+(2\pi/\omega_0)} e^{j(n-m)\omega_0 t}\, dt = \left[\frac{e^{j(n-m)\omega_0 t}}{j(n-m)\omega_0}\right]_{t_0}^{t_0+(2\pi/\omega_0)}$$

$$= \left[\frac{e^{j(n-m)\omega_0[t_0+(2\pi/\omega_0)]} - e^{j(n-m)\omega_0 t_0}}{j(n-m)\omega_0}\right] = \frac{e^{j(n-m)\omega_0 t_0}\, e^{j(n-m)2\pi} - e^{j(n-m)\omega_0 t_0}}{j(n-m)\omega_0}$$

$$= \frac{e^{j(n-m)\omega_0 t_0}(1) - e^{j(n-m)\omega_0 t_0}}{j(n-m)\omega_0}$$

$$[\because e^{j(n-m)2\pi} = \cos(n-m)2\pi + j\sin(n-m)2\pi = 1]$$

$$= 0$$



(b) Sinusoidal Signal

Figure Continuous Time Sinusoidal Signals

Q10) How can you define a sinusoidal signal in continuous line?

Ans. : Sinusoidal Signal : A continuous-time sinusoidal signal in its most general form is given by,

$$x(t) = A\sin(\omega t + \theta)$$

Where,

A = Amplitude.

ω = Angular frequency (radians/sec) = $2\pi f$.

θ = Phase angle in radians.

When, $\theta = 0$, $x(t) = A\sin\omega t$

θ = positive, $x(t) = A\sin(\omega t + \theta)$

θ = Negative, $x(t) = A\sin(\omega t - \theta)$

Fig. 1.6.5(a) shows the various forms of a sinusoidal signal. The time period of a continuous time sinusoidal signal is given by,

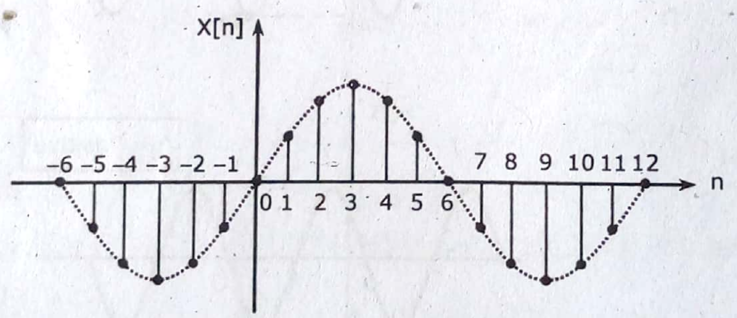
$$T = \frac{2\pi}{\omega}$$

Q11) What do you mean by co-sinusoidal signal of a discrete time?

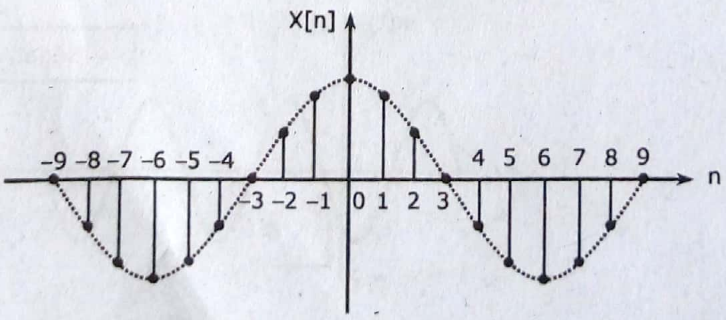
Ans. : Co-Sinusoidal Signal : The discrete-time co-sinusoidal signal is given by,

$$x[n] = A\cos(\omega n + \theta)$$

Figure shows the wave forms of discrete-time sinusoidal and co-sinusoidal signals.



(a) Discrete Time Sinusoidal Signal Represented by Equation $x(n) = A\sin(\omega_0 n)$



(b) Discrete Time Co-Sinusoidal Signal Represented by Equation $x(n) = A\cos(\omega_0 n)$

Figure Discrete Time Sinusoidal Signals and Co-sinusoidal Signals

Q12) What is unit impulse signal?

Ans. : The unit impulse signal is a signal with infinite magnitude and zero duration, but with unit area.

Mathematically, a unit imrulse signal is defined as,

$$\delta(t) = \begin{cases} \infty & ; t = 0 \\ 0 & ; t \neq 0 \end{cases}$$

And
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

An impulse with infinite magnitude and zero duration does not exist in reality. However a signal with large magnitude and short duration (when compared to time constant of a system) can be considered as an impulse signal. Practically, the magnitude of the impulse is measured by its area.

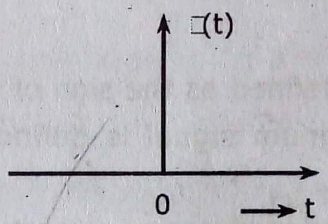


Figure Continuous Time Unit Impulse Function

Q13) What are the properties of unit impulse response?

Ans. : Properties of continuous-time unit impulse function are given below,

(1) It is an even function of time t,

i.e., $\delta(t) = \delta(-t)$

(2)
$$\int_{-\infty}^{\infty} x(t)\delta(t)\, dt = x(0)$$

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\, dt = x(t_0)$$

(3)
$$\delta(at) = \frac{1}{|a|}\delta(t)$$

(4) $x(t)\delta(t - t_0) = x(t_0)\, \delta(t - t_0) = x(t_0)$

(5) $x(t)\, \delta(t) = x(0)\, \delta(t) = x(0)$.

(6)
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

Q14) What is a continuous time unit step function?

Ans. : A continuous-time signal is said to be a unit step signal if,

$$u[t] = \begin{cases} 1 & ; \text{ for } t > 0 \\ 0 & ; \text{ for } t < 0 \end{cases}$$

Conventionally, a continuous-time step signal is represented by u(t). The graphical representation of a continuous-time unit step signal is shown in figure.

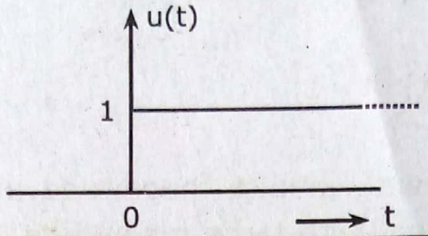


Figure Continuous Time Unit Step Signal

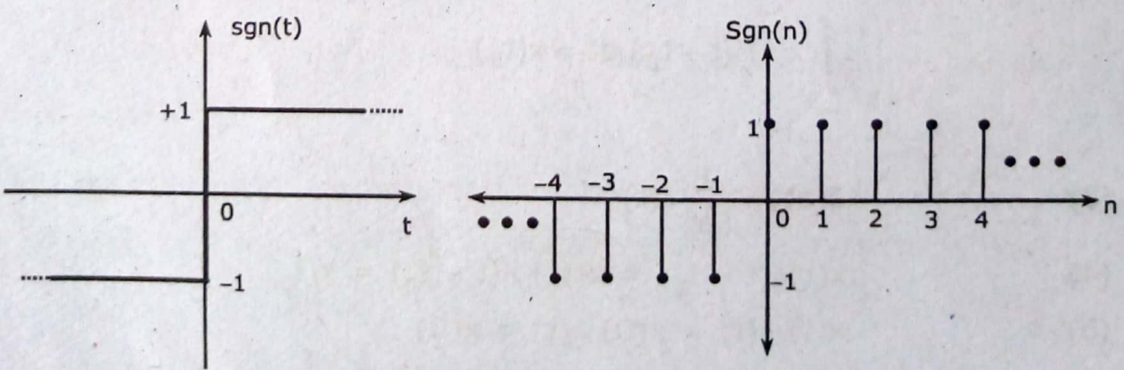
Q15) Define signum function. What it is described in the continuous-time and discrete-time format?

Ans. : The signum function is defined as the sign of the independent variable t. Therefore, the continuous-time signum signal is defined as,

$$sgn(t) = \begin{cases} +1 & ; \ t > 0 \\ 0 & ; \ t = 0 \\ -1 & ; \ t < 0 \end{cases}$$

A discrete-time signum function can be obtained by sampling the continuous time signum function. It is a train of samples of values +1 for positive n and -1 for negative n. Therefore,

$$sgn[n] = \begin{cases} +1 & ; \ n > 0 \\ 0 & ; \ n = 0 \\ -1 & ; \ n < 0 \end{cases}$$



(a) Antinuous Time Signum Function

(b) Discrete Time Signum Function

Figure Signum Function

PART - B

ESSAY QUESTIONS WITH REFERENCES

Q1) Obtain the condition under which two signals $x_1(t)$ and $x_2(t)$ are said to be orthogonal to each other. Hence prove that $\sin(n\omega_0 t)$ and $\cos(m\omega_0 t)$ are orthogonal to each other for all integer values m, n? **[Refer Section No. 1.3.2]**

Q2) A rectangular function is defined as,

$$x(t) = \begin{cases} A & \text{for } 0 < t < \frac{\pi}{2} \\ -A & \text{for } \frac{\pi}{2} < t < \frac{3\pi}{2} \\ A & \text{for } \frac{3\pi}{2} < t < 2\pi \end{cases}$$

Approximate the above function by A cost between the intervals $(0, 2\pi)$ such that the mean square error is minimum? **[Refer Section No. 1.3.2]**

Q3) Explain how a function can be approximated by a set of orthogonal functions? **[Refer Section No. 1.3.6]**

Q4) Explain when a function is said to be a complete set. **[Refer Section No. 1.4]**

Q5) Define orthogonality in complex functions? And prove that the complex exponential signals are orthogonal signals? **[Refer Section No. 1.5]**

Q6) What are exponential and sinusoidal signals in continues time? **[Refer Section No. 1.6]**

Q7) What are exponential and sinusoidal signals in discrete time? **[Refer Section No. 1.6]**

Q8) What are elementary signals (exponential and sinusoidal signals) in continuous and discrete time? **[Refer Section No. 1.6]**

Q9) Write the concept of impulse function and unit step function. **[Refer Section Nos. 1.7 and 1.7.1]**

Q10) How can you define signum function in both continuous time and discrete time sequences? Explain with graphical representation. **[Refer Section No. 1.7.3]**

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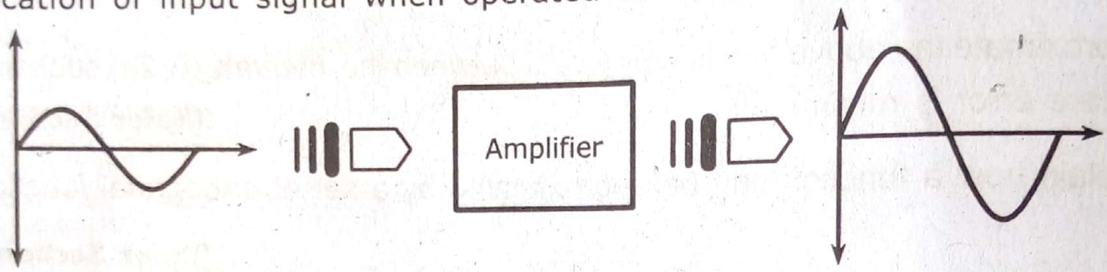
1.1 INTRODUCTION

The concept of signals and systems are widespread in many areas of science and engineering. These concepts are very extensively applied in the field of circuit analysis and design, long distance communication, bio-medical engineering, aeronautics, speech and image processing etc.

Signals represents a function of some independent variables which contain some information about the behaviour or some natural phenomenon. When these signals are operated on few devices, produces a signals in the same or modified form. Such devices are called systems. A system may thus defined as a set of elements or functional blocks which are connected together to produce an output in response to an input signals. Amplifiers, transmitters and receivers in a communication system, chemical plants, nuclear reactor, a government establishment etc., are few examples of systems. Fig. 1.1.1 shows the amplification of input signal when operated on a system, here, amplifier.



(a) Input Signal (b) System (c) Output Signal

Fig. 1.1.1 Signal and System

1.2 DEFINITION AND CLASSIFICATION

1.2.1 Definition of a Signal

A signal is defined as "a function of one or more independent variables which contains some information". The independent variable can be time, spatial co-ordinates, pressure, temperature etc. The most popular independent variable in signals is time and is denoted by the letter 't'. Mathematically, signals are represented as,

$$x(t) = 2t \quad \text{... (1.2.1)}$$

$$x(a, b) = 3a + 4b + 5a^2 \quad \text{... (1.2.2)}$$

Eq. (1.2.1) represents a signal which is a function of independent variable 't', while Eq. (1.2.2) represents a signal which is a function of two independent variables 'a' and 'b'.

Examples

(1) Voltages and currents in electrical and electronic circuits. These are called electric signals.

(2) Human speech and sounds produced by animals. These are called non-electric signals.

1.2.2 Classification of a Signal

The signals can be classified in a number of ways. Some ways of classifying the signals are explained below,

(1) Classification of Signals Based on the Number of Sources for the Signal

(i) **One-Channel Signals :** Signals that are generated by a single signal source are called one-channel signals.

Example : The record of room temperature with respect to time, the audio output of a mono speaker, etc.

(ii) **Multi-Channel Signals :** Signals that are generated by multiple signal sources are called multi-channel signals.

Examples

- The audio output of two stereo speakers is an example of two-channel signal.
- The record of Electro Cardia Graph (ECG) at eight different places in a human body is an example of eight-channel signal.

(2) Classification of Signals Based on the Number of Independent Variables

(i) **One-Dimensional Signals :** A signal which is a function of one independent variable is called one-dimensional signal.

Example : A speech signal, which is a function of time is an example of one dimensional signal.

(ii) **Multi-Dimensional Signals :** A signal which is a function of two or more independent variables is called multi-dimensional signal.

Examples

- A photograph image is an example of a two-dimensional signal. The brightness at each point of a photograph is a function of two spatial coordinates "x" and "y".
- The motion picture of a black and white TV is an example of a three-dimensional signal. The brightness at each point of a black and white motion picture is a function of two spatial coordinates "x" and "y" and time "t".

(3) Classification of Signals Based on the Nature of Independent Variables

(i) **Continuous Signals :** A signal which is defined continuously for every value of independent variable, is called continuous signal. Continuous signals are also called as analog signals.

(ii) **Discrete Signals :** A signal which is defined for discrete intervals of independent variable, is called discrete signal. Discrete signals are either sampled version of analog signals for processing by digital systems or output of digital systems.

REVIEW QUESTIONS

(1) *Define signal and give some examples? Explain its types?*
(2) *How the signals are classified based on number of source variables and nature of variables?*

1.3 ANALOGY BETWEEN VECTORS AND SIGNALS

A problem is better understood if it can be compared with some familiar phenomenon. Hence, we always search for analogies while studying a new problem. There exists a perfect analogy between vectors and signals, which leads to better understanding of signal analysis

1.3.1 Orthogonal Vectors

A vector is a physical quantity which possess both magnitude and direction. Vectors are denoted by bold face type or with arrow (→) or dash (–) over vector. Let $\bar{A}_1$ and $\bar{A}_2$ be two coplanar vectors having an angular separation θ, as shown in Fig. 1.3.1(a). The component of vector $\bar{A}_1$ along vector $\bar{A}_2$ is obtained by drawing a line from the end of $\bar{A}_1$ on to $\bar{A}_2$, in a number of ways. Let the component of vector $\bar{A}_1$ along $\bar{A}_2$ is $C_{12}\bar{A}_2$, where the magnitude of C_{12} is an indication of the similarity of vectors.

Fig. 1.3.1(a) depicts, the component of a vector $\bar{A}_1$ along the vector $\bar{A}_2$ obtained by drawing a perpendicular from the end of vector $\bar{A}_1$ on the vector $\bar{A}_2$. Let the length of the perpendicular may be represented by another vector $\bar{A}_e$. Thus the vector $\bar{A}_1$ can now be expressed in terms of vector $\bar{A}_2$ and $\bar{A}_e$ as,

$$\bar{A}_1 = C_{12}\,\bar{A}_2 + \bar{A}_e \qquad \text{... (1.3.1)}$$

Fig. 1.3.1(b) and Fig. 1.3.1(c) illustrates two other alternative ways of representing the component of vector $\bar{A}_1$ in terms of vector $\bar{A}_2$.

Thus in Fig. 1.3.1(b),

$$\bar{A}_1 = C_1\bar{A}_2 + \bar{A}_{e1} \qquad \text{... (1.3.2)}$$

And in Fig. 1.3.1(c),

$$\bar{A}_1 = C_2\bar{A}_2 + \bar{A}_{e2} \qquad \text{... (1.3.3)}$$

In each case $\bar{A}_1$ is represented in terms of $\bar{A}_2$ plus another vector, which will be called the error vector. If we are asked to approximate $\bar{A}_1$ by $\bar{A}_2$ then $\bar{A}_e$ represents the error in their approximation. It can be understood from the geometry of these figures, that the error vector is smallest in Fig. 1.3.1(a). Therefore the *component of a vector along another vector is defined as "the component of a vector $\bar{A}_1$ along the vector $\bar{A}_2$ is given by $C_{12}\bar{A}_2$, where C_{12} is chosen such that the error vector is minimum".*

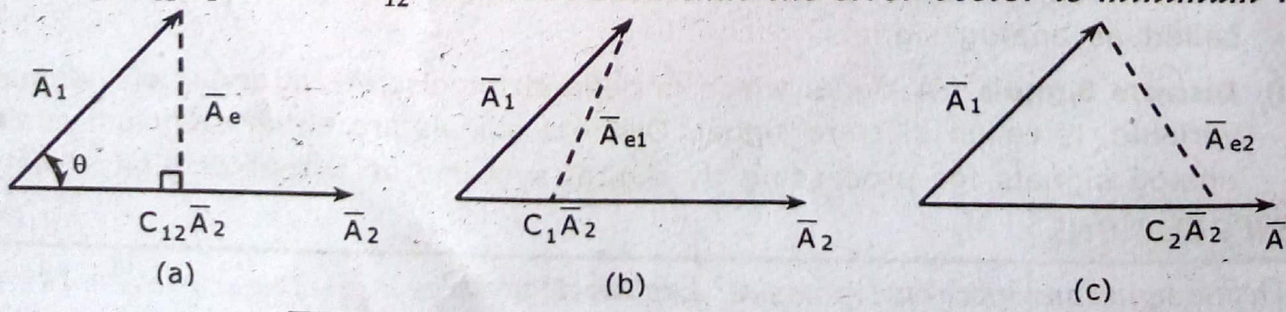


Fig. 1.3.1 Vector $\bar{A}_1$ in Terms of Vector $\bar{A}_2$

It can also be noticed from Fig. 1.3.1 that, the larger the component of one vector along the other vector, the more closely do the two vectors resemble each other in their directions and the smaller is the error vector. Therefore, if the component of a vector $\bar{A}_1$ along $\bar{A}_2$ is $C_{12}\bar{A}_2$, then the magnitude of C_{12} is an indication of the similarity of the two vectors. If C_{12} is zero, then the vector has no component along the other vector and hence the two vectors are mutually perpendicular. Such vectors are known as "orthogonal vectors". Orthogonal vectors are thus independent vectors. If the vectors are orthogonal, then the parameter C_{12} is zero as shown in Fig. 1.3.2.

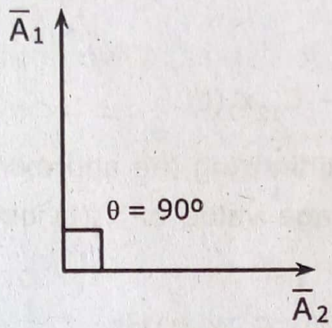


Fig. 1.3.2 Orthogonal Vectors (Here $C_{12} = 0$)

The dot product of two vectors $\bar{A}$ and $\bar{B}$ is defined as,

$$\bar{A}.\bar{B} = |\bar{A}||\bar{B}|\cos\theta$$

Where θ is angle between vectors $\bar{A}$ and $\bar{B}$. It follows from the definition, that

$$\bar{A}.\bar{B} = \bar{B}.\bar{A}$$

According to this notation, the component of $\bar{A}$ along $\bar{B} = |\bar{A}|\cos\theta = \dfrac{\bar{A}.\bar{B}}{|\bar{B}|}$

And the component of $\bar{B}$ along $\bar{A} = |\bar{B}|\cos\theta = \dfrac{\bar{A}.\bar{B}}{|\bar{A}|}$

Similarly, the component of vector $\bar{A}_1$ along $\bar{A}_2 = \dfrac{\bar{A}_1.\bar{A}_2}{|\bar{A}_2|} = C_{12}\bar{A}_2$

$$\therefore \quad C_{12} = \frac{\bar{A}_1.\bar{A}_2}{A_2^2} = \frac{\bar{A}_1.\bar{A}_2}{\bar{A}_2.\bar{A}_2} \qquad \text{... (1.3.4)}$$

If $\bar{A}_1$ and $\bar{A}_2$ are orthogonal vectors then,

$$\bar{A}_1.\bar{A}_2 = 0$$

$$\therefore \quad C_{12} = 0 \qquad \text{... (1.3.5)}$$

Therefore, *two vectors* $\bar{A}_1$ and $\bar{A}_2$ *are said to be orthogonal vectors if their dot product is zero,* i.e., $\bar{A}_1.\bar{A}_2 = 0$.

1.3.2 Orthogonal Signals

The concept of vector comparison and orthogonality can be extended to signals. Consider two signals $x_1(t)$ and $x_2(t)$. We can approximate $x_1(t)$ in terms of $x_2(t)$ over a certain interval $(t_1 < t < t_2)$ as follows,

$$x_1(t) = C_{12}x_2(t) \quad ; \quad (t_1 < t < t_2)$$

Like in vectors, the main criterion in selecting C_{12} is to minimize the error between the actual function and the approximated (error) function over the interval $(t_1 < t < t_2)$ and is defined as,

$$x_e(t) = x_1(t) - C_{12}x_2(t)$$

One possible criterion for minimizing the approximated function $x_e(t)$ over the interval t_1 to t_2 is to minimize the average value of $x_e(t)$ over this interval, that is to minimize,

$$\frac{1}{(t_2 - t_1)}\int_{t_1}^{t_2}[x_1(t) - C_{12}x_2(t)]dt$$

However, this criterion is inadequate because there can be a large positive and negative errors present that may cancel one another in the process of averaging and give the false indication that the error is zero. To overcome this we choose to minimize the mean square of the error instead of the error itself.

Let us designate the average of squared error $[x_e^2(t)]$ by ε.

$$\varepsilon = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[x_e^2(t)dt\right]$$

$$= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}[x_1(t) - C_{12}x_2(t)]^2dt$$

To find the value of C_{12} which will minimize the error, we must have, $\left(\frac{d\varepsilon}{dC_{12}}\right) = 0$.

i.e., $$\frac{d\varepsilon}{dC_{12}} = \frac{d}{dC_{12}}\left[\frac{1}{(t_2 - t_1)}\int_{t_1}^{t_2}[x_1(t) - C_{12}x_2(t)]^2dt\right] = 0$$

$$\Rightarrow \quad \frac{d}{dC_{12}}\left\{\frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2}[x_1^2(t) - 2C_{12}x_1(t)x_2(t) + C_{12}^2x_2^2(t)]\,dt\right]\right\} = 0$$

Changing the order of integration and differentiation, we get,

$$\frac{1}{(t_2-t_1)}\left[\int_{t_1}^{t_2}\frac{d}{dC_{12}}[x_1^2(t)]dt - 2\int_{t_1}^{t_2}\frac{d}{dC_{12}}[C_{12}x_1(t)x_2(t)]dt\right.$$

$$\Rightarrow \qquad \left. + \int_{t_1}^{t_2}\frac{d}{dC_{12}}[C_{12}^2x_2^2(t)]dt\right] = 0$$

$$\Rightarrow \qquad \frac{1}{(t_2-t_1)}\left[\int_{t_1}^{t_2}\frac{d}{dC_{12}}\left[x_1^2(t)\,dt\right] - 2\int_{t_1}^{t_2} x_1(t).x_2(t)\ dt + 2C_{12}\int_{t_1}^{t_2} x_2^2(t).dt\right] = 0$$

The first integral is obviously zero and hence above equation becomes,

$$\int_{t_1}^{t_2} x_1(t)x_2(t)\,dt = C_{12}\int_{t_1}^{t_2} x_2^2(t)\,dt$$

$$\therefore \qquad C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)x_2(t)\,dt}{\int_{t_1}^{t_2} x_2^2(t)\ dt} \qquad \ldots (1.3.6)$$

Observe the similarity between Eqs. (1.3.4) and (1.3.6) which express C_{12} for vectors and signals respectively. Two vectors $\bar{A}_1$ and $\bar{A}_2$ are orthogonal if their dot product is zero, i.e., $\bar{A}_1.\bar{A}_2 = 0$. Similarly two signals $x_1(t)$ and $x_2(t)$ are orthogonal if the integral of the product of those signals is zero, i.e.,

$$\int_{t_1}^{t_2} x_1(t)x_2(t)\ dt = 0 \qquad \ldots (1.3.7)$$

EXAMPLE PROBLEM 1

Show that the functions sin($n\omega_0 t$) and cos($m\omega_0 t$) are orthogonal over any interval {t0 to [t0 + ($2\pi/\omega_0$)]} for integral values of n and m.

SOLUTION

We know that two signals $x_1(t)$ and $x_2(t)$ are orthogonal over an interval (t_1, t_2), if

$$\int_{t_1}^{t_2} x_1(t)x_2(t)\ dt = 0$$

Let, $x_1(t) = \sin(n\omega_0 t)$

And $x_2(t) = \cos(m\omega_0 t)$

Also consider,

$$I = \int_{t_1}^{t_2} x_1(t)x_2(t) = \int_{t_0}^{t_0+(2\pi/\omega_0)} \sin(n\omega_0 t)\cos(m\omega_0 t)\,dt$$

$$= \int_{t_0}^{t_0+(2\pi/\omega_0)} \frac{1}{2}[\sin(n+m)\omega_0 t + \sin(n-m)\omega_0 t]\,dt$$

$$\left(\because \sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]\right)$$

$$= \frac{1}{2}\left[-\frac{\cos(n+m)\omega_0 t}{n+m} - \frac{\cos(n-m)\omega_0 t}{n-m}\right]_{t_0}^{t_0+(2\pi/\omega_0)}$$

$$= -\frac{1}{2}\left\{\left[\frac{\cos(n+m)\omega_0[t_0+(2\pi/\omega_0)] - \cos(n+m)\omega_0 t_0}{n+m}\right] + \left[\frac{\cos(n-m)\omega_0[t_0+(2\pi/\omega_0)] - \cos(n-m)\omega_0 t_0}{n-m}\right]\right\}$$

$$= -\frac{1}{2}\left\{\left[\frac{\cos(n+m)\omega_0 t_0 - \cos(n+m)\omega_0 t_0}{n+m}\right] + \left[\frac{\cos(n-m)\omega_0 t_0 - \cos(n-m)\omega_0 t_0}{n-m}\right]\right\} = 0$$

This shows that the functions $\sin(n\omega_0 t)$ and $\cos(m\omega_0 t)$ are mutually orthogonal.

EXAMPLE PROBLEM 2

A rectangular function is defined as,

$$x(t) = \begin{cases} A & \text{for } 0 < t < \frac{\pi}{2} \\ -A & \text{for } \frac{\pi}{2} < t < \frac{3\pi}{2} \\ A & \text{for } \frac{3\pi}{2} < t < 2\pi \end{cases} \quad \dots (1.3.8)$$

Approximate the above function by Acost between the intervals (0, 2π) such that the mean square error is minimum.

SOLUTION

We know that a signal $x_1(t)$ can be approximated by $x_2(t)$ over the interval (t_1, t_2) as,

$$x_1(t) = C_{12}x_2(t) \quad ; \quad (t_1 < t < t_2)$$

Let $x_1(t)$ be rectangular function defined by Eq. (1.3.8) and $x_2(t)$ = Acost. The given function $x(t) = x_1(t)$ can be approximated by Acost over the interval $(0, 2\pi)$ as,

$$x_1(t) = C_{12}\,[A\cos t]$$

C_{12} has to be found such that the mean square error is minimum over the interval $(0, 2\pi)$ given by,

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)x_2(t)\,dt}{\int_{t_1}^{t_2} x_2^2(t).dt} = \frac{\int_0^{2\pi} x(t)A\cos t\,dt}{\int_0^{2\pi}[A\cos t]^2\,dt}$$

$$= \frac{\int_0^{\pi/2}(A)A\cos t\,dt + \int_{\pi/2}^{3\pi/2}(-A)A\cos t\,dt + \int_{3\pi/2}^{2\pi}(A)A\cos t\,dt}{\int_0^{2\pi} A^2\left[\frac{1+\cos 2t}{2}\right]dt} \qquad \left(\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right)$$

$$= \frac{A^2\left[\int_0^{\pi/2}\cos t\,dt - \int_{\pi/2}^{3\pi/2}\cos t\,dt + \int_{3\pi/2}^{2\pi}\cos t\,dt\right]}{\frac{A^2}{2}\left[\int_0^{2\pi}(1+\cos 2t)\,dt\right]}$$

$$= \frac{[\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^{3\pi/2} + [\sin t]_{3\pi/2}^{2\pi}}{\frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}}$$

$$= \frac{\left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right] + \left[\sin 2\pi - \sin\frac{3\pi}{2}\right]}{\frac{1}{2}\left[(2\pi - 0) + \frac{\sin 4\pi - \sin 0}{2}\right]}$$

$$= \frac{(1-0) - (-1-1) + [0-(-1)]}{\pi + \frac{0-0}{2}} = \frac{4}{\pi}$$

$$\therefore \quad C_{12} = \frac{4}{\pi}$$

So the given function x(t) can be approximated by Acost as $x(t) = (4/\pi)A\cos t$ over the interval $(0,2\pi)$ such that the mean square error is minimum. The approximation of the rectangular function in terms of the cosine waveform is shown in Fig. 1.3.3.

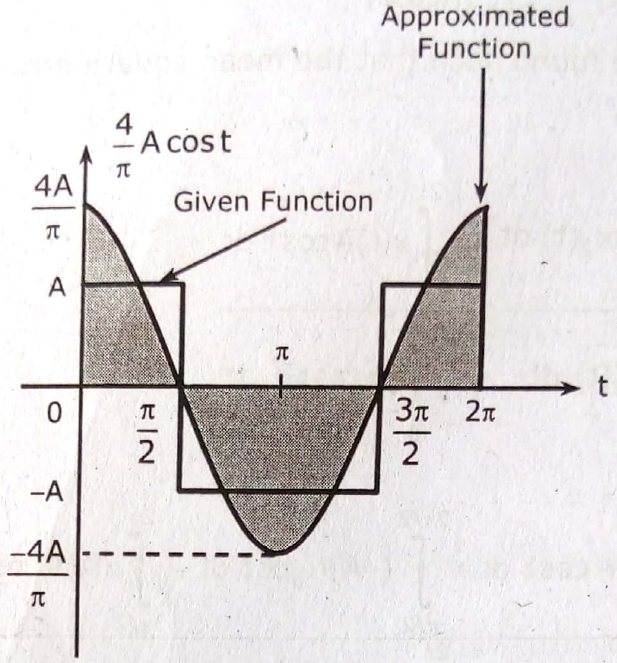


Fig. 1.3.3 Approximation of Rectangular Function by Acost over $(0, 2\pi)$

1.3.3 Graphical Evaluation of Component of One Function in the Other

In section 1.3.2, we have seen how to approximate a one function in terms of other function mathematically. Let us now study the graphical method of evaluating the component of a function in the other function, using Eq. (1.3.6).

Consider two functions $x_1(t)$ and $x_2(t)$ shown in Fig. 1.3.4 (a) and (c) respectively. Suppose it is desired to find the component of signal $x_2(t)$ contained in signal $x_1(t)$ over a period (0, T). This component is given by $C_{12}x_2(t)$, where the magnitude C_{12} is given as,

$$C_{12} = \frac{\int_0^T x_1(t)\, x_2(t)\, dt}{\int_0^T x_2^2(t)\, dt} \quad \text{... (1.3.9)}$$

The numerator integral in Eq. (1.3.9) can be evaluated by multiplying the two functions and finding the area under the product curve shown in Fig. 1.3.4(e). In a similar way the denominator integral can be found by evaluating the area under the function $[x_2(t)]^2$.

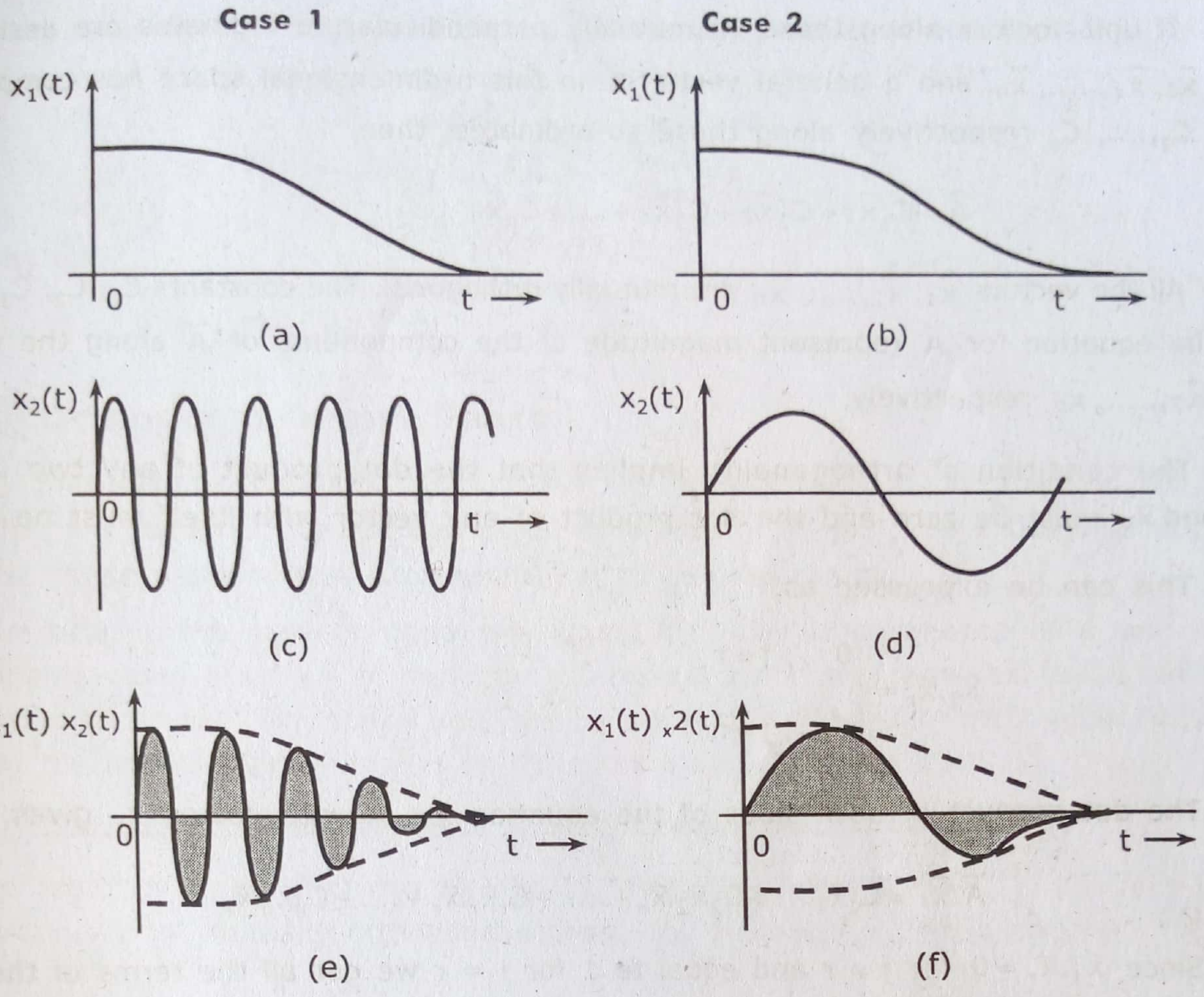


Fig. 1.3.4 Graphical Evaluation of the Component of Waveform $x_2(t)$ in a Signal $x_1(t)$

The two possible cases that can be observed while evaluating the component of one function in the other are,

CASE I (The Function $x_1(t)$ is Varying Much More Slowly Than $x_2(t)$) : In this case, the area under the curve $x_1(t).x_2(t)$ will be very small since the positive and negative areas will be approximately equal and will tend to cancel each other as shown in Fig. 1.3.4(e). Therefore $x_1(t)$ will contain a small component of the function $x_2(t)$.

CASE II (The Function $x_1(t)$ Varies At About The Same Rate As $x_2(t)$) : In this case, the area under the curve $x_1(t) \cdot x_2(t)$ will be much larger as shown in Fig. 1.3.4(f), since there is a great possibility of similarity between the two functions. Hence, $x_1(t)$ contains a large component of the function $x_2(t)$.

1.3.4 Orthogonal Vector Space

The space bounded by a set of 'n' orthogonal vectors forming the co-ordinate axes is called orthogonal vector space. Generally a linear equation with n-independent variables may be viewed as a vector. This vector is expressed in terms of its components along 'n' mutually perpendicular co-ordinates.

If unit vectors along these 'n' mutually perpendicular co-ordinates are designated as $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ and a general vector $\bar{A}$ in this n-dimensional space has components $C_1, C_2, \ldots, C_n$ respectively along these co-ordinates, then,

$$\bar{A} = C_1\bar{x}_1 + C_2\bar{x}_2 + C_3\bar{x}_3 + \ldots + C_n\bar{x}_n$$

All the vectors $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ are mutually orthogonal. The constants $C_1, C_2, C_3, \ldots C_n$ in the equation for $\bar{A}$ represent magnitude of the components of $\bar{A}$ along the vectors $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ respectively.

The condition of orthogonality implies that the dot product of any two vectors $\bar{x}_j$ and $\bar{x}_r$ must be zero and the dot product of any vector with itself must be unity.

This can be expressed as,

$$\bar{x}_j.\bar{x}_r = \begin{cases} 0 & ; j \neq r \\ 1 & ; j = r \end{cases}$$

The dot product of both sides of the equation for $\bar{A}$ with vector $\bar{x}_r$ gives,

$$\bar{A}.\bar{x}_r = C_1\bar{x}_1.\bar{x}_r + C_2\bar{x}_2.\bar{x}_r + \ldots + C_r\bar{x}_r.\bar{x}_r + \ldots + C_n\bar{x}_n.\bar{x}_r$$

Since $\bar{x}_j.\bar{x}_r = 0$ for $j \neq r$ and equal to 1 for $j = r$ we get all the terms of the form $C_n\bar{x}_n.\bar{x}_r\ (n \neq r)$ are zero, Therefore,

$$\bar{A}.\bar{x}_r = C_r\bar{x}_r.\bar{x}_r = C_r \qquad \ldots (1.3.10)$$

This set of vectors $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ which are mutually perpendicular to each other are called an *orthogonal vector space.* In general, the product $\bar{x}_j.\bar{x}_r$ can be some constant k_r instead of unity. When k_r is unity, the set is called normalized orthogonal set or *orthonormal vector space*. Therefore, in general, for orthogonal vector space we have,

$$\bar{x}_j.\bar{x}_r = \begin{cases} 0 & ; j \neq r \\ k_r & ; j = r \end{cases} \qquad \ldots (1.3.11)$$

Using Eq. (1.3.11) for an orthogonal vector space, the equation for $\bar{A}.\bar{x}_r$ given by Eq. (1.3.10) is modified as,

$$\bar{A}.\bar{x}_r = C_r\bar{x}_r.\bar{x}_r = C_r k_r$$

$$C_r = \frac{\bar{A}.\bar{x}_r}{k_r}$$

If the vector space defined by Eq. (1.3.11) is complete then any vector $\bar{A}$ can be expressed as,

$$\bar{A} = C_1\bar{x}_1 + C_2\bar{x}_2 + \ldots.. + C_n\bar{x}_n = \sum_{r=1}^{n} C_r\bar{x}_r$$

Where,

$$C_r = \frac{\bar{A}.\bar{x}_r}{k_r} = \frac{\bar{A}.\bar{x}_r}{\bar{x}_r.\bar{x}_r} \quad \ldots (1.3.12)$$

1.3.5 Orthogonal Signal Space

The concept of orthogonal vector space can be applied to signal analysis. We have seen that any vector $\bar{A}$ can be expressed as a sum of 'n' mutually orthogonal vectors, provided these vectors form a complete set of co-ordinate axes.

Similarly, in the case of signal any signal x(t) can be expressed as a sum of its components along a set of 'n' mutually orthogonal functions, provided these functions form a complete set. The space bounded by a set of 'n' mutually orthogonal functions forming the co-ordinate system is called orthogonal signal space.

1.3.6 Signal Approximation Using Orthogonal Functions

An arbitrary signal x(t) can be approximated over an interval (t_1, t_2) by a linear combination of 'n' mutually orthogonal signals $(m_1(t), m_2(t), \ldots, m_n(t))$ as,

$$x(t) = C_1m_1(t) + C_2m_2(t) + \ldots.. + C_nm_n(t) = \sum_{r=1}^{n} C_rm_r(t)$$

The set of signals $(m_1(t), m_2(t), \ldots, m_n(t))$ mutually orthogonal to each other are called an orthogonal signal space. Observing the condition of orthogonality for orthogonal vector space $\{\bar{x}_r\}$ given by Eq. (1.3.11), similarly we have for orthogonal signal space, the condition of orthogonality over an interval t_1 to t_2 is defined by,

$$\int_{t_1}^{t_2} m_j(t)m_r(t)\,dt = 0 \quad ; j \neq r$$

And

$$\int_{t_1}^{t_2} m_r^2(t)\,dt = k_r \quad ; j = r$$

For the best approximation we must find the proper values of constants $C_1, C_2, \ldots C_n$ such that ε, the mean square of error $x_e(t)$ is minimized. Error function is defined as,

$$x_e(t) = x(t) - \sum_{r=1}^{n} C_rm_r(t)$$

And
$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[x(t) - \sum_{r=1}^{n} C_r m_r(t) \right]^2 dt$$

This equation shows that ε is a function of $C_1, C_2, \ldots\ldots, C_n$ and to minimize ε, we must have,

$$\frac{\partial \varepsilon}{\partial C_1} = \frac{\partial \varepsilon}{\partial C_2} = \ldots\ldots = \frac{\partial \varepsilon}{\partial C_j} = \ldots\ldots = \frac{\partial \varepsilon}{\partial C_n} = 0$$

Consider the equation, $\frac{\partial \varepsilon}{\partial C_j} = 0$. Since $(t_2 - t_1)$ is a constant, we can write,

$$\frac{\partial}{\partial C_j} \left\{ \int_{t_1}^{t_2} \left[x(t) - \sum_{r=1}^{n} C_r m_r(t) \right]^2 dt \right\} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial C_j} \left\{ \int_{t_1}^{t_2} \left[x^2(t) - 2\sum_{r=1}^{n} x(t) C_r m_r(t) + \sum_{r=1}^{n} C_r^2 m_r^2(t) \right] dt \right\} = 0 \quad \ldots (1.3.13)$$

It can be noticed from The Eq. (1.3.13), the first integration term is independent of C_j. Hence its derivative will be zero. The second and third integration terms will be non-zero only when $r = j$. For $r \neq j$, these terms will be constants (i.e., approximation terms will be $C_1, C_2, \ldots$ excluding C_j) and hence their derivative are zero. Thus the above equation leaves only two non-zero terms as follows,

$$\frac{\partial}{\partial C_j} \int_{t_1}^{t_2} [-2C_j x(t) m_j(t) + C_j^2 m_j^2(t)]\, dt = 0 \qquad \text{(For } r = j\text{)}$$

Changing the order of differentiation and integration, we get,

$$\int_{t_1}^{t_2} \frac{\partial}{\partial C_j} [-2C_j x(t) m_j(t)]\, dt + \int_{t_1}^{t_2} \frac{\partial}{\partial C_j} [C_j^2 m_j^2(t)]\, dt = 0$$

$$\Rightarrow \quad 2\int_{t_1}^{t_2} x(t) m_j(t)\, dt = 2C_j \int_{t_1}^{t_2} m_j^2(t)\, dt$$

$$\therefore \quad C_j = \frac{\int_{t_1}^{t_2} x(t) m_j(t)\, dt}{\int_{t_1}^{t_2} m_j^2(t)\, dt} = \frac{1}{k_j} \int_{t_1}^{t_2} x(t) m_j(t)\, dt \quad \ldots (1.3.14)$$

Thus we can conclude that it is possible to approximate an arbitrary function x(t) over interval (t_1, t_2) by a linear combination of n orthogonal functions, $m_1(t)$, $m_2(t)$,....., $m_n(t)$ as,

$$x(t) = C_1 m_1(t) + C_2 m_2(t) + \ldots + C_n m_n(t) = \sum_{r=1}^{n} C_r m_r(t)$$

Where,

$$C_r = \frac{\int_{t_1}^{t_2} x(t) m_r(t)\, dt}{\int_{t_1}^{t_2} m_r^2(t)\, dt} = \frac{1}{k_r} \int_{t_1}^{t_2} x(t) m_r(t)\, dt \qquad \ldots (1.3.15)$$

Observe the similarity between Eqs. (1.3.12) and (1.3.15) which express C_r for orthogonal vector space and orthogonal signal space respectively.

1.3.7 Mean Square Error

Let us now find the values of ε when optimum values of coefficient C_1, C_2,...., C_n are chosen according to the equation for C_r given by Eq. (1.3.15).

By the definition of mean square error,

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[x(t) - \sum_{r=1}^{n} C_r m_r(t) \right]^2 dt$$

$$= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} x^2(t) dt + \sum_{r=1}^{n} C_r^2 \int_{t_1}^{t_2} m_r^2(t) dt - 2 \sum_{r=1}^{n} C_r \int_{t_1}^{t_2} x(t) m_r(t) dt \right] \qquad \ldots (1.3.16)$$

But from the equation for C_r, we have,

$$\int_{t_1}^{t_2} x(t) m_r(t) dt = C_r \int_{t_1}^{t_2} m_r^2(t) dt = C_r k_r \qquad \ldots (1.3.17)$$

Substituting Eq. (1.3.17) in Eq. (1.3.16), we have,

$$\varepsilon = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} x^2(t) dt + \sum_{r=1}^{n} C_r^2 k_r - 2 \sum_{r=1}^{n} C_r^2 k_r \right] = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} x^2(t) dt - \sum_{r=1}^{n} C_r^2 k_r \right]$$

$$\therefore \quad \varepsilon = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} x^2(t) dt - (C_1^2 k_1 + C_2^2 k_2 + \ldots + C_n^2 k_n) \right] \qquad \ldots (1.3.18)$$

The mean square error can therefore be evaluated using Eq. (1.3.18).

EXAMPLE PROBLEM 1

A rectangular function x(t) is defined by,

$$x(t) = \begin{cases} 1 & ; 0 < t < \pi \\ -1 & ; \pi < t < 2\pi \end{cases}$$

Approximate the above rectangular function by a single sinusoid sint over the interval {0, 2π} such that the mean square error is minimum. Evaluate the mean square error in the approximation. Also show what happens when more number of sinusoidals is used for approximation.

SOLUTION

The given rectangular function x(t) as shown in Fig. 1.3.5(a), can be approximated by a single sinusoid sint as shown in Fig. 1.3.5(b),

$$x(t) \approx C_{12} \sin t$$

The optimum value of C_{12} which will minimize the mean square error is defined as,

$$\therefore \quad C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)x_2(t)dt}{\int_{t_1}^{t_2} x_2^2(t)dt}$$

Let, $$x_1(t) = \begin{cases} 1 & ; \ 0 < t < \pi \\ -1 & ; \ \pi < t < 2\pi \end{cases}$$

And $$x_2(t) = \sin t$$

$$\therefore \quad C_{12} = \frac{\int_0^{2\pi} x(t)\sin t\, dt}{\int_0^{2\pi} \sin^2(t)\, dt} = \frac{\int_0^{\pi}(1)\sin t\, dt + \int_{\pi}^{2\pi}(-1)\sin t\, dt}{\int_0^{2\pi}\left[\frac{1-\cos 2t}{2}\right]dt} = \frac{[-\cos t]_0^{\pi} - [-\cos t]_{\pi}^{2\pi}}{\frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}}$$

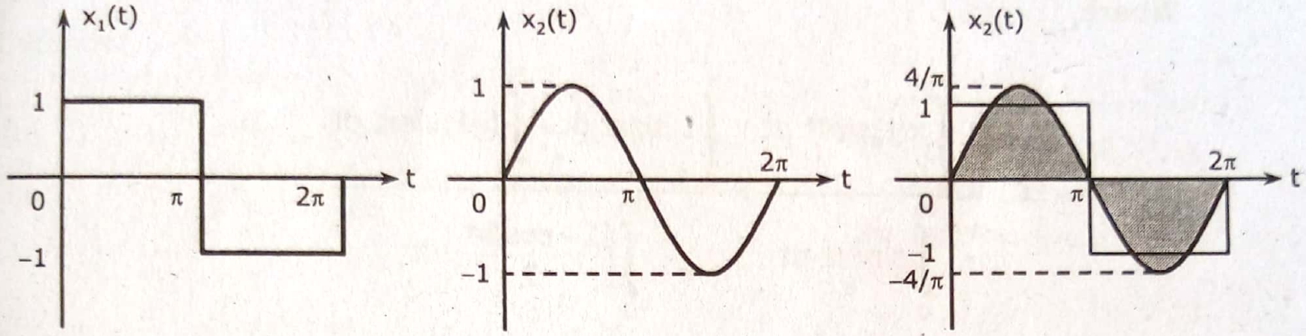
$$= \frac{-[\cos\pi - \cos 0] + [\cos 2\pi - \cos\pi]}{\frac{1}{2}\left[(2\pi - 0 - \frac{\sin 4\pi - \sin 0}{2}\right]} = \frac{-[-1-1]+[1-(-1)]}{\frac{1}{2}[2\pi]} \qquad (\because \sin 4\pi = 0)$$

$$C_{12} = \frac{4}{\pi}$$

$$\therefore \quad x(t) \approx \frac{4}{\pi}\sin t$$

Thus, the best approximation of the rectangular function x(t) by a sinusoidal function sint with minimum mean square error is shown in Fig. 1.3.5(c) and is given as,

$$x(t) \approx \frac{4}{\pi}\sin t$$



(a) Rectangular Function (b) Sine Wave (c) Approximation

Fig. 1.3.5

The mean square error can be evaluated using the formula,

$$\varepsilon = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} x^2(t)\,dt - C_{12}^2 \int_{t_1}^{t_2} \sin^2(t)\,dt\right]$$

Here $t_1 = 0$ and $t_2 = 2\pi$.

$$\therefore \quad \varepsilon = \frac{1}{2\pi - 0}\left[\int_0^{2\pi} 1\,dt - \left(\frac{4}{\pi}\right)^2 \int_0^{2\pi}\left(\frac{1-\cos 2t}{2}\right)dt\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} dt - \frac{16}{\pi^2}\int_0^{2\pi}\left(\frac{1-\cos 2t}{2}\right)dt\right] = \frac{1}{2\pi}\left[[t]_0^{2\pi} - \frac{16}{2\pi^2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left\{(2\pi - 0) - \frac{8}{\pi^2}\left[(2\pi - 0) - \frac{(\sin 4\pi - \sin 0)}{2}\right]\right\}$$

$$= \frac{1}{2\pi}\left[2\pi - \frac{16}{\pi}\right] = 1 - \frac{8}{\pi^2} = 0.189 \qquad [\because \sin 4\pi = 0]$$

Mean square error,

$$\varepsilon = 0.189 = 18.9\%$$

Approximation Using a Finite Series of Sinusoids : When the given rectangular function x(t) is approximated using a finite series of sinusoids, we have,

$$x(t) = \sum_{r=1}^{n} C_r m_r(t) = \sum_{r=1}^{n} C_r \sin rt$$

Where,

$$C_r = \frac{\int_0^{2\pi} x(t)\sin rt\, dt}{\int_0^{2\pi} \sin^2 rt\, dt} = \frac{\int_0^{\pi} 1.\sin rt\, dt + \int_{\pi}^{2\pi} (-1)\sin rt\, dt}{\int_0^{2\pi} \left(\frac{1-\cos 2rt}{2}\right) dt}$$

$$= \frac{\int_0^{\pi} \sin rt\, dt - \int_{\pi}^{2\pi} \sin rt\, dt}{\frac{1}{2}\left[\int_0^{2\pi} (1-\cos 2rt)dt\right]} = \frac{\left[-\frac{\cos rt}{r}\right]_0^{\pi} + \left[\frac{\cos rt}{r}\right]_{\pi}^{2\pi}}{\frac{1}{2}\left[t - \frac{\sin 2rt}{2r}\right]_0^{2\pi}}$$

$$= \frac{\frac{1}{r}[-\cos \pi r + \cos 0 + \cos 2\pi r - \cos \pi r]}{\frac{1}{2}\left[2\pi - \frac{\sin 4\pi r - \sin 0}{2r}\right]}$$

We have,

$$\cos[\pi r] = (-1)^r$$

$$\cos[2\pi r] = 1$$

And $\quad \sin[4\pi r] = 0$

$$\therefore \quad C_r = \frac{\frac{1}{r}[-(-1)^r + 1 + 1 - (-1)^r]}{\frac{1}{2}[2\pi - 0]} = \frac{2}{\pi r}[1 - (-1)^r]$$

$$\therefore \quad C_r = \begin{cases} \frac{4}{\pi r} & ; r = \text{odd} \\ 0 & ; r = \text{even} \end{cases}$$

Thus the approximated function is given by,

$$x(t) = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t + ... = \sum_{r=\text{odd}}^{n} \frac{4}{\pi}\sin rt$$

Now, to evaluate the mean square error, we have,

$$\varepsilon = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} x^2(t)dt - \sum_{r=1}^{n} C_r^2 k_r\right] = \frac{1}{2\pi}\left[\int_0^{2\pi} x^2(t)dt - \sum_{r=1}^{n} C_r^2 k_r\right] \quad \ldots (1.3.18)$$

Consider, $\int_0^{2\pi} x^2(t).dt$

But, $x^2(t) = 1$

$0 < t < 2\pi$

$\therefore$ $$\int_0^{2\pi} x^2(t)\,dt = \int_0^{2\pi} dt = [t]_0^{2\pi} = 2\pi$$

Consider, $$k_r = \int_{t_1}^{t_2} \sin^2 rt\,dt = \int_0^{2\pi} \sin^2 rt\,dt = \int_0^{2\pi}\left(\frac{1-\cos 2rt}{2}\right)dt$$

$$= \frac{1}{2}\left[t - \frac{\sin 2rt}{2r}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[2\pi - 0 - \frac{\sin 4\pi r - \sin 0}{2r}\right]$$

$$= \frac{1}{2}[2\pi - 0] \qquad [\because \sin 4\pi r = 0 \text{ for any value of } r]$$

$\therefore$ $k_r = \pi$

Where, r = 1, 2, 3,........

We have, $$C_r = \begin{cases} \frac{4}{r\pi} & ;r = \text{odd} \\ 0 & ;r = \text{even} \end{cases}$$

Substituting the value of $\int_0^{2\pi} x^2(t)\,dt$, k_r and C_r in Eq. (1.3.18), we get,

$$\varepsilon = \frac{1}{2\pi}\left[2\pi - \sum_{r=\text{odd}}^{n}\left(\frac{4}{\pi r}\right)^2 \pi\right] = 1 - \sum_{r=\text{odd}}^{n} \frac{8}{\pi^2 r^2}$$

CASE I : Approximation of x(t) by a single sinusoid (r = 1),

$$x(t) = \sum_{r=odd}^{n} \frac{4}{\pi} \sin rt$$

$$= \frac{4}{\pi} \sin t$$

And mean square error,

$$\varepsilon = 1 - \sum_{r=odd}^{n} \frac{8}{\pi^2 r^2}$$

$$= 1 - \frac{8}{\pi^2}$$

$$\varepsilon = 0.189$$

CASE II : Approximation of x(t) by three sinusoids (r = 3),

$$x(t) = \sum_{r=odd}^{n} \frac{4}{\pi} \sin rt$$

$$= \frac{4}{\pi} \sin t + \frac{4}{3\pi} \sin 3t$$

And mean square, error,

$$\varepsilon = 1 - \sum_{r=odd}^{n} \frac{8}{\pi^2 r^2} = 1 - \left[\frac{8}{\pi^2} + \frac{8}{(3)^2 \pi^2}\right]$$

$$= 1 - [0.8105 + 0.0900] = 0.0995$$

The given rectangular function x(t) approximated by three sinusoids is shown in Fig. 1.3.6(a).

CASE III : Approximation of x(t) by five sinusoids (r = 5),

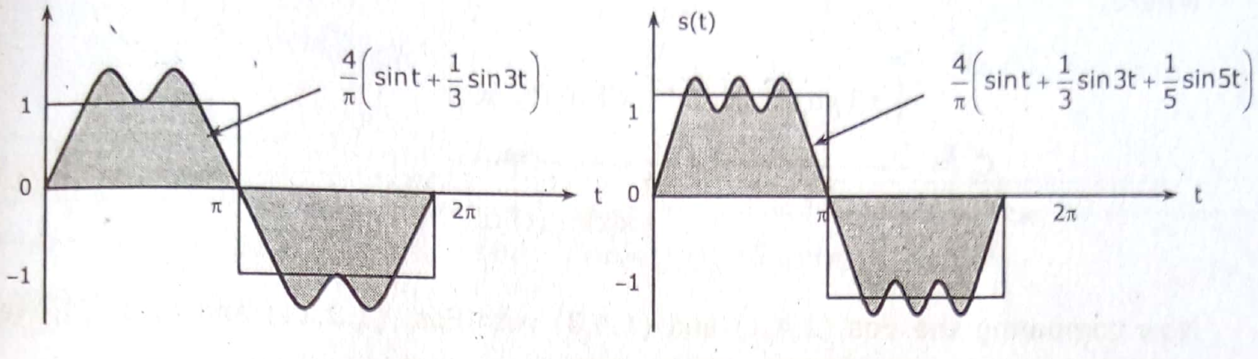
$$x(t) = \sum_{r=odd}^{n} \frac{4}{\pi r} \sin rt = \frac{4}{\pi} \sin t + \frac{4}{3\pi} \sin 3t + \frac{4}{5\pi} \sin 5t$$

And mean square error,

$$\varepsilon = 1 - \sum_{r=odd}^{n} \frac{8}{\pi^2 r^2} = 1 - \left[\frac{8}{\pi^2} + \frac{8}{3^2 \pi^2} + \frac{8}{5^2 \pi}\right]$$

$$= 1 - [0.8105 + 0.0900 + 0.0324] = 0.067$$

The given rectangular function approximated by five sinusoids is shown in Fig. 1.3.6(b).



(a) When Three Sinusoidal Components are Used (b) When Five Sinusoidal Components are Used

Fig. 1.3.6 Approximation of a Rectangular Function by Orthogonal Function

As we increase the number of terms the approximation improves and the mean-square error reduces. For infinite terms the mean-square error is zero.

REVIEW QUESTIONS

(1) *Explain how a function can be approximated by a set of orthogonal functions?*

(2) *Evaluate the mean square error of a signal approximated by a set of orthogonal functions?*

1.4 CLOSED OR COMPLETE SET OF ORTHOGONAL FUNCTIONS

A set of mutually orthogonal functions $m_1(t)$, $m_2(t)$, ... $m_r(t)$ over the interval (t_1, t_2) is said to be a complete or a closed set if there exists no function x(t) for which it is true that,

$$\int_{t_1}^{t_2} x(t)\cdot m_k(t)\,dt = 0 \quad ; \text{ for } k = 1, 2, \ldots.$$

If a function x(t) could be found such that the above integral is zero, then obviously x(t) is orthogonal to each member of the set $[m_r(t)]$ and as a result, it itself a member of the set. So the set cannot be complete without x(t) being its member.

Let for a set $[m_r(t)]$ where (r = 0, 1, 2 . . .) is mutually orthogonal over the interval (t_1, t_2),

$$\int_{t_1}^{t_2} m_j(t) m_r(t)dt = \begin{cases} 0; & \text{if } j \neq r \\ K_j; & \text{if } j \neq r \end{cases} \qquad \text{... (1.4.1)}$$

If this function set is complete, then any function x(t), can be represented as,

$$x(t) = C_1 m_1(t) + C_2 m_2(t) + \ldots + C_r m_r(t) + \ldots \qquad \ldots (1.4.2)$$

Where,

$$C_r = \frac{\int_{t_1}^{t_2} x(t) m_r(t)\, dt}{K_r} = \frac{\int_{t_1}^{t_2} x(t) m_r(t)\, dt}{\int_{t_1}^{t_2} x(t) m_r^2(t)\, dt} \qquad \ldots (1.4.3)$$

Now comparing the Eqs (1.4.1) and (1.4.3) with Eqs. (1.3.11) and (1.3.12), we get the analogy between the vectors and signals.

Therefore we conclude that any vector along mutually orthogonal vectors can be expressed as a sum of its components and forms a complete set. Similarly any function x(t) can be expressed as sum of its components along mutually orthogonal functions and form a closed or complete set.

REVIEW QUESTIONS

(1) *Write complete set of orthogonal functions?*

(2) *Explain when a function is said to be a complete (or) closed set?*

1.5 ORTHOGONALITY IN COMPLEX FUNCTIONS

So far in our discussion we have considered only real functions of real variable. If $x_1(t)$ and $x_2(t)$ are complex functions of real variable t then $x_1(t)$ can be approximated by $C_{12}x_2(t)$ over the interval (t_1, t_2), such that,

$$x_1(t) \approx C_{12} x_2(t)$$

The optimum value of C_{12} to minimize the mean square error magnitude is given by,

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t) x_2^*(t)\, dt}{\int_{t_1}^{t_2} x_2(t) x_2^*(t)\, dt}$$

Where, $x_2^*(t)$ is complex conjugate of $x_2(t)$.

For the complex functions to be orthogonal over the interval (t_1, t_2), $C_{12} = 0$.

$$\int_{t_1}^{t_2} x_1(t) x_2^*(t)\, dt = \int_{t_1}^{t_2} x_1^*(t) x_2(t)\, dt = 0$$

For a set of complex function $\{x_r(t)\}$, (r = 1, 2,.....) mutually orthogonal over the interval (t_1, t_2).

$$\int_{t_1}^{t_2} x_m(t)x_n^*(t)dt = \begin{cases} 0 & ; \quad \text{if } m \neq n \\ k_m & ; \quad \text{if } m = n \end{cases}$$

EXAMPLE PROBLEM 1

Prove that the complex exponential signals are orthogonal functions.

SOLUTION

Consider two complex exponential signals,

$$x_1(t) = e^{jn\omega_0 t} \text{ and } x_2(t) = e^{jm\omega_0 t}$$

Let the interval be t_0 to $t_0 + T$, i.e., from t_0 to $t_0+(2\pi/\omega_0)$. $x_1(t)$ and $x_2(t)$ are orthogonal over the interval t_0 to $t_0+(2\pi/\omega_0)$, if

$$I = \int_{t_0}^{t_0+(2\pi/\omega_0)} x_1(t)x_2^*(t)dt = 0$$

Here $x_1(t) = e^{jn\omega_0 t}$ and $x_2^*(t) = [e^{jm\omega_0 t}]^* = e^{-jm\omega_0 t}$

$$\therefore \quad I = \int_{t_0}^{t_0+(2\pi/\omega_0)} e^{jn\omega_0 t}\, e^{-jm\omega_0 t}\, dt = \int_{t_0}^{t_0+(2\pi/\omega_0)} e^{j(n-m)\omega_0 t}\, dt = \left[\frac{e^{j(n-m)\omega_0 t}}{j(n-m)\omega_0}\right]_{t_0}^{t_0+(2\pi/\omega_0)}$$

$$= \left[\frac{e^{j(n-m)\omega_0[t_0+(2\pi/\omega_0)]} - e^{j(n-m)\omega_0 t_0}}{j(n-m)\omega_0}\right] = \frac{e^{j(n-m)\omega_0 t_0}\, e^{j(n-m)2\pi} - e^{j(n-m)\omega_0 t_0}}{j(n-m)\omega_0}$$

$$= \frac{e^{j(n-m)\omega_0 t_0}(1) - e^{j(n-m)\omega_0 t_0}}{j(n-m)\omega_0}$$

$$[\because e^{j(n-m)2\pi} = \cos(n-m)2\pi + j\sin(n-m)2\pi = 1]$$

$$= 0$$

REVIEW QUESTIONS

(1) *Define orthogonality in a complex functions?*

(2) *Prove that the exponential signals are orthogonal functions.*

1.6 EXPONENTIAL AND SINUSOIDAL SIGNALS

1.6.1 Exponential Signals

1.6.1.1 Real Exponential Signals

The real exponential signals are divided into the following types,

(1) Continuous-Time **Real Exponential Signals** : A continuous-time real exponential signal in its most general form is written as,

$$x(t) = Ke^{at}$$

Where, both K and a are real parameters.

The parameter 'K' is the amplitude of the exponential measured at $t = 0$. The parameter 'a' can be either positive or negative. Depending on the value of 'a' we have three cases.

Case I : If $a = 0$, the signal x(t) is of constant amplitude for all times.

Case II : If a is positive, i.e., $a > 0$, the signal x(t) is a growing exponential signal as t increases.

Case III : If a is negative i.e., $a < 0$, the signal x(t) is a decaying exponential signal as t increases.

These three waveforms are shown in Fig. 1.6.1(a), (b) and (c).

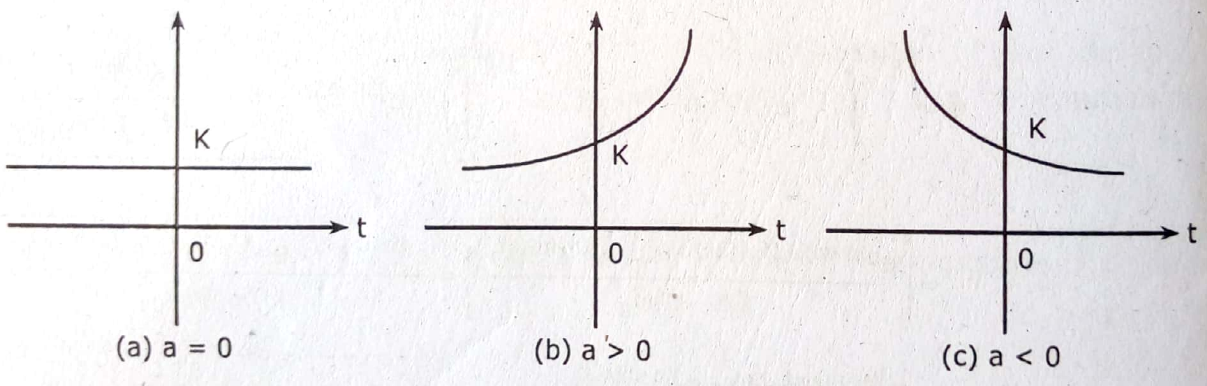


Fig. 1.6.1 Continuous-Time Real Exponential Signals $x(t) = Ke^{at}$

(2) Discrete-Time Real Exponential Signals : A discrete time real exponential signal is expressed as,

$$x[n] = K.a^n \quad \text{... (1.10.9)}$$

Here, a is some real constant. Depending on the values of a, we have four different cases as under,

Case I (a > 1) : Let $a = 3$. Thus, we have, $x[n] = a^n = 3^n$. Graphically, such signal is represented as shown in Fig. 1.6.2(a). Since the signal is exponentially growing, it is called as rising exponential signal.

CASE II ($0 < a < 1$) : Let, a = 1/3. Therefore, we have, $x[n] = a^n = \left(\frac{1}{3}\right)^n$. Graphically, such signal is represented as shown in Fig. 1.6.2(b). Since the signal is exponentially decaying, it is called as decaying exponential signal.

CASE III ($a < -1$) : Let, a = −3 Therefore, $x(n) = (-3)^n$. Graphically, such signal is represented as shown in Fig. 1.6.2(c).

CASE IV ($-1 < a < 0$) : Let $a = -\frac{1}{3}$. Therefore, we have $x[n] = a^n = \left(-\frac{1}{3}\right)^n$. Graphically, such signal is shown in Fig. 1.6.2(d).

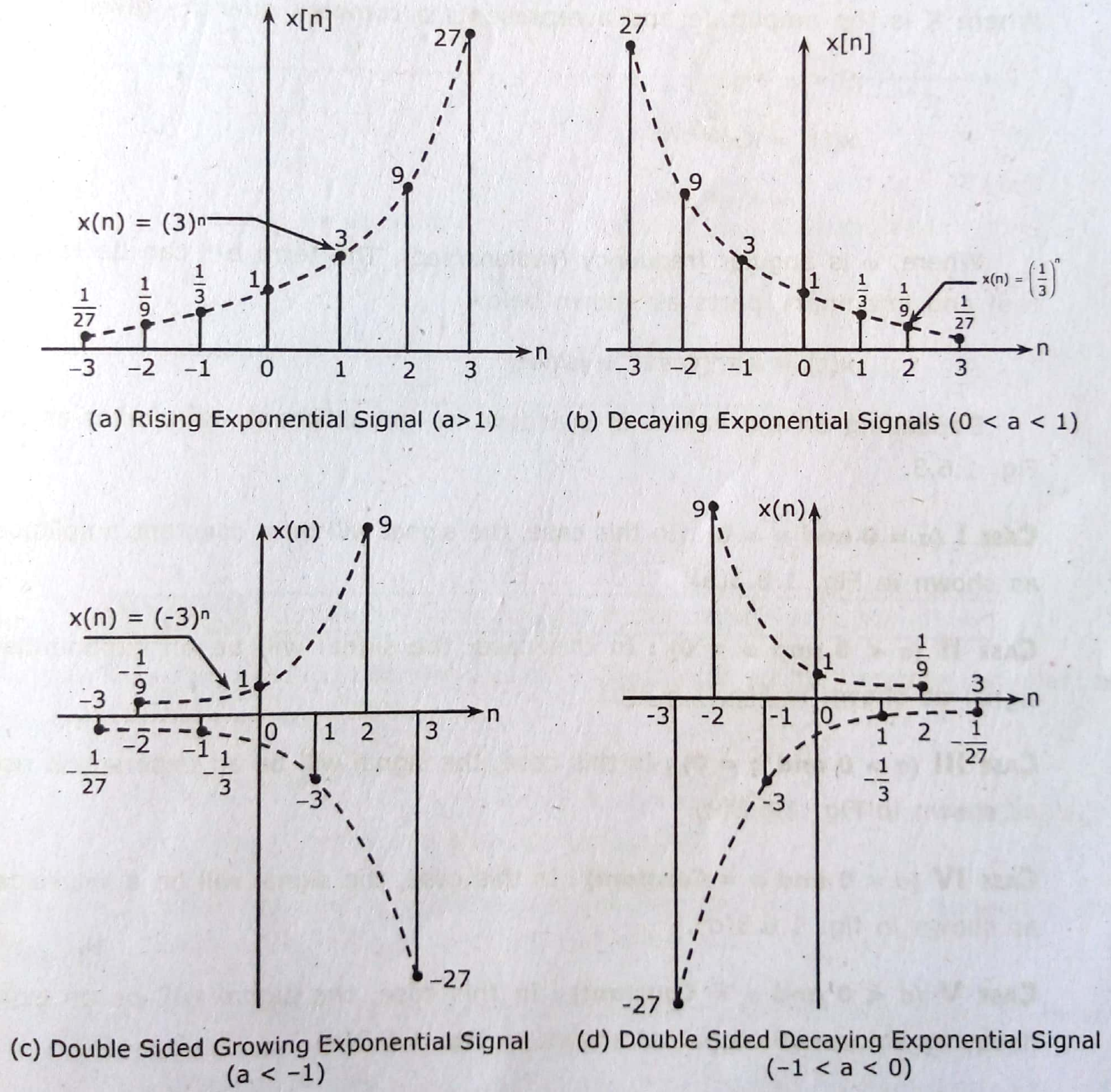


Fig. 1.6.2 Discrete-Time Real Exponential Signals $x[n] = K.a^n$

Complex Exponential Signals

The complex exponential signal is the second most important signal in exponential signals. It is very useful for the representation of signals and processing of signals. It encapsulates both sin and cosine signals by displaying them on real and imaginary components of complex signal. It sub divided into the following types,

(i) Continuous Time Complex Exponential Signals : The complex exponential signal has a general from defined as,

$$x(t) = K.e^{st}$$

Where K is the amplitude and a represents a complex quantity given by,

$$S = \sigma + j\omega$$

$$\therefore \quad x(t) = K.e^{(\sigma+j\omega)t}$$

$$= K.e^{\sigma t}.e^{j\omega t}$$

Where, ω is angular frequency (radians/sec). The term $e^{j\omega t}$ can be resolved into real and imaginary parts as shown below.

$$x(t) = Ke^{\sigma t}(\cos\omega t + j\sin\omega t)$$

Depending on the values of s and w, we get different waveforms as shown in Fig. 1.6.3.

Case I ($\sigma = 0$ and $\omega = 0$) : In this case, the signal will be a constant amplitude signal as shown in Fig. 1.6.3(a).

Case II ($\sigma < 0$ and $\omega = 0$) : In this case, the signal will be an exponential decay signal as shown in Fig. 1.6.3(b).

Case III ($\sigma > 0$ and $\omega = 0$) : In this case, the signal will be an exponential rise signal as shown in Fig. 1.6.3(c).

Case IV ($\sigma = 0$ and ω = Constant) : In this case, the signal will be a sinusoidal signal as shown in fig. 1.6.3(d).

Case V ($\sigma < 0$ and ω = Constant) : In this case, the signal will be an exponential decaying sinusoidal signal as shown in Fig. 1.6.3(e).

Case VI ($\sigma > 0$ and ω = Constant) : In this case, the signal will be an exponential rising sinusoidal signal as shown in Fig. 1.6.3(f).

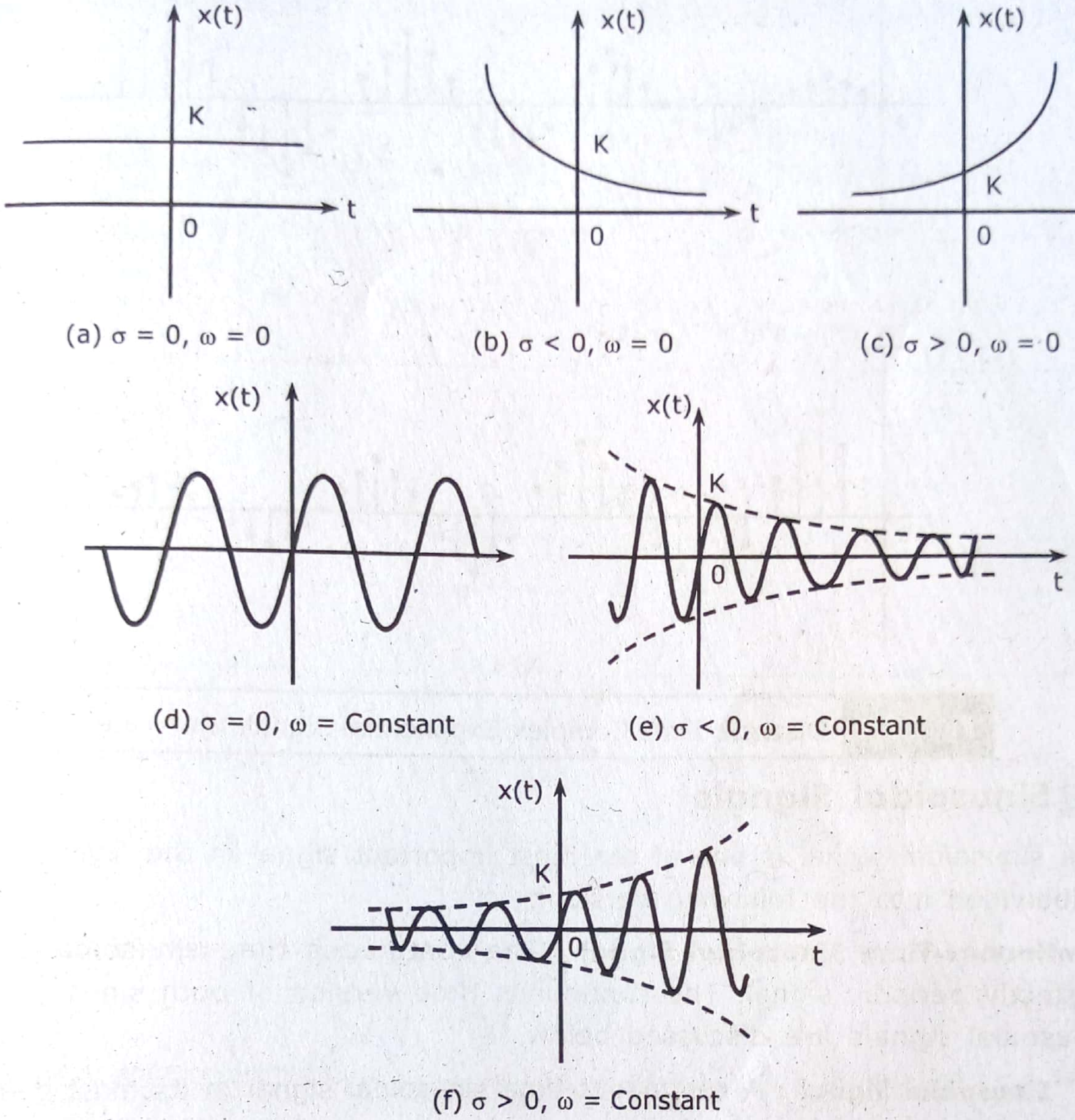


Fig. 1.6.3 Continuous Time Complex Exponential Signals $x(t) = K.e^{st}$

(2) **Discrete-Time Complex Exponential Signals :** The discrete-time complex exponential sequence is defined as,

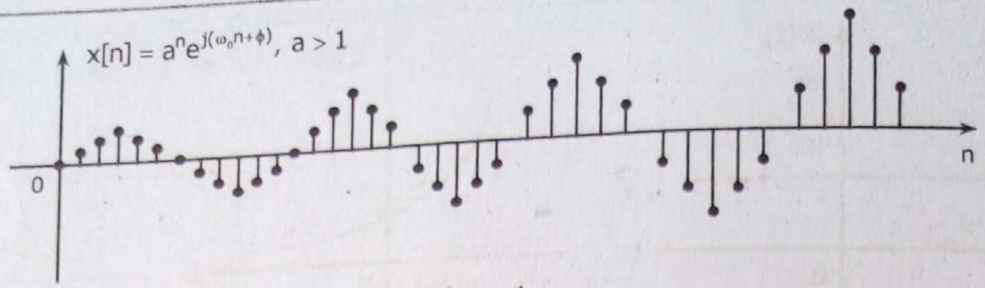
$$x(n) = a^n e^{j(\omega_0 n+\phi)}$$

$$= a^n \cos(\omega_0 n + \phi) + ja^n \sin(\omega_0 n + \phi)$$

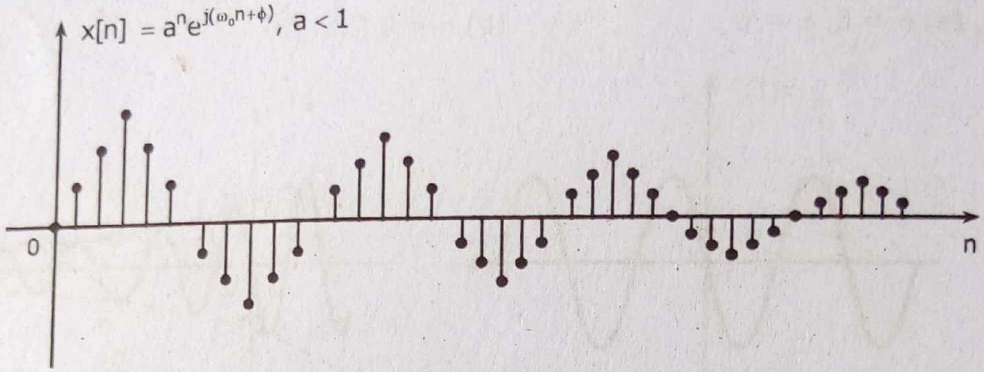
(1) For $|a| = 1$, the real and imaginary parts of complex exponential sequence are sinusoidal.

(2) For $|a| > 1$, the amplitude of the sinusoidal sequence exponentially grows as shown in Fig. 1.6.4(a).

(3) For $|a| < 1$, the amplitude of the sinusoidal sequence exponentially decays as shown in Fig. 1.6.4(b).



(a) a > 1



(b) a < 1

Fig. 1.6.4 Discrete Time Complex Exponential Signals $x[n] = a^n e^{j(\omega_0 n+\phi)}$

1.6.2 Sinusoidal Signals

The sinusoidal signal is one of the most important signal in the signal processing. It is subdivided into the following versions,

(1) Continuous-Time Sinusoidal Signal : The continuous time sinusoidal signal is a perfectly periodic signal. The continuous time version of both sinusoidal and co-sinusoidal signals are discussed below.

(i) Sinusoidal Signal : A continuous-time sinusoidal signal in its most general form is given by,

$$x(t) = A\sin(\omega t + \theta)$$

Where,

A = Amplitude.

ω = Angular frequency (radians/sec) = $2\pi f$.

θ = Phase angle in radians.

When, $\theta = 0$, $x(t) = A\sin\omega t$

θ = positive, $x(t) = A\sin(\omega t + \theta)$

θ = Negative, $x(t) = A\sin(\omega t - \theta)$

Fig. 1.6.5(a) shows the various forms of a sinusoidal signal. The time period of a continuous time sinusoidal signal is given by,

$$T = \frac{2\pi}{\omega}$$

(ii) Co-sinusoidal Signal : A continuous-time co-sinusoidal signal in its most general form is given by,

$$x(t) = A\cos(\omega t + \theta)$$

Fig. 1.6.5(b) shows the various forms of a co-sinusoidal signal.

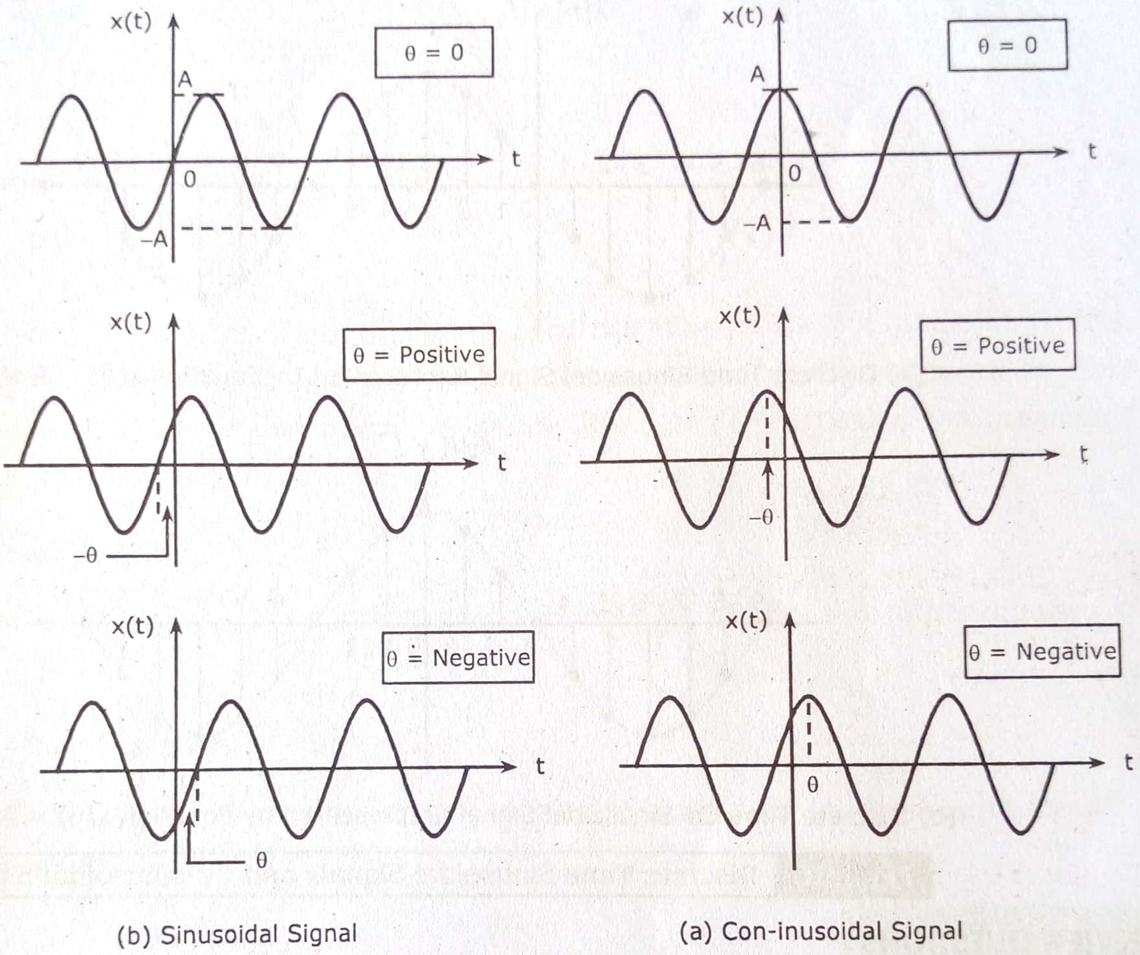


Fig. 1.6.5 Continuous Time Sinusoidal Signals and Co-sinusoidal Signals

(2) Discrete-Time Sinusoidal Signals : The discrete time form of signals of both sinusoidal and co-sinusoidal signals are explained below,

(i) Sinusoidal Signal : The discrete-time sinusoidal signal is given by,

$$x[n] = A\sin(\omega n + \theta)$$

Where A is the amplitude, ω is the angular frequency and θ is phase angle and n is an integer.

The fundamental period of the discrete-time sinusoidal sequence is,

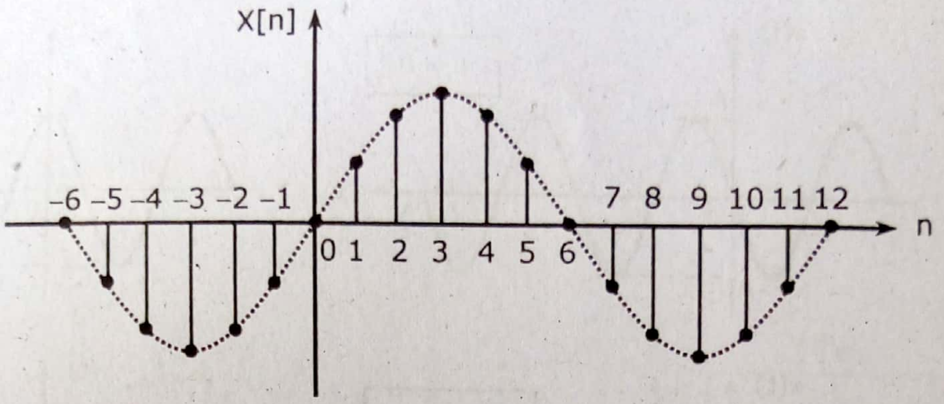
$$N = \frac{2\pi}{\omega}M$$

Where, M and N are integers.

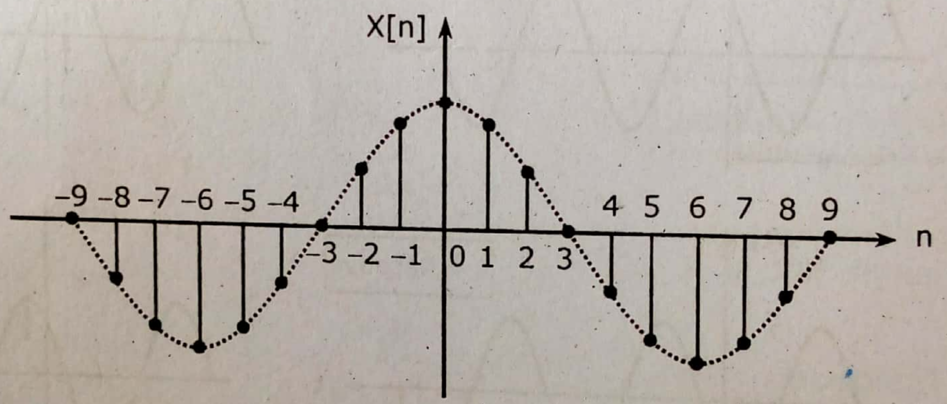
(ii) **Co-Sinusoidal Signal :** The discrete-time co-sinusoidal signal is given by,

$$x[n] = A\cos(\omega n + \theta)$$

Fig. 1.6.6 shows the wave forms of discrete-time sinusoidal and co-sinusoidal signals.



(a) Discrete Time Sinusoidal Signal Represented by Equation $x(n) = A\sin(\omega_0 n)$



(b) Discrete Time Co-Sinusoidal Signal Represented by Equation $x(n) = A\cos(\omega_0 n)$

Fig. 1.6.6 Discrete Time Sinusoidal Signals and Co-sinusoidal Signals

REVIEW QUESTIONS

(1) *Write about real and complex exponential signals?*

(2) *What are sinusoidal signals? Give a brief description with suitable waveform.*

1.7 CONCEPTS OF IMPULSE FUNCTION

The impulse signal is a signal with infinite magnitude and zero duration, but with an area of A. Mathematically, an impulse signal is defined as,

$$\delta(t) = \begin{cases} \infty & ; t = 0 \\ 0 & ; t \neq 0 \end{cases}$$

And $$\int_{-\infty}^{\infty} \delta(t)\, dt = A$$

UNIT IMPULSE SIGNAL

The unit impulse signal is a signal with infinite magnitude and zero duration, but with unit area.

Mathematically, a unit impulse signal is defined as,

$$\delta(t) = \begin{cases} \infty & ; t = 0 \\ 0 & ; t \neq 0 \end{cases}$$

And $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

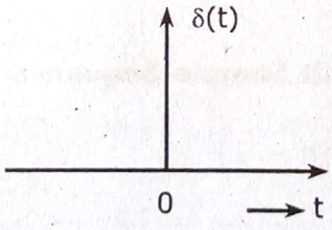
An impulse with infinite magnitude and zero duration does not exist in reality. However a signal with large magnitude and short duration (when compared to time constant of a system) can be considered as an impulse signal. Practically, the magnitude of the impulse is measured by its area.

UNIT SAMPLE SEQUENCE

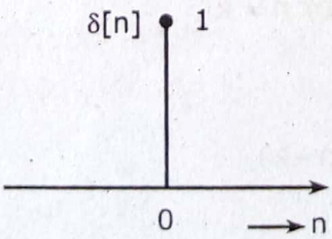
A discrete-time signal is said to be a unit sample sequence or delta sequence if

$$\delta(n) = \begin{cases} 1 & ; \quad \text{for } n = 0 \\ 0 & ; \quad \text{for } n \neq 0 \end{cases}$$

The graphical representation of continuous-time and discrete-time unit impulse function is shown in Fig. 1.7.1.



(a) Continuous Time Unit Impulse Function



(b) Discrete Time Unit Sample Sequence

Fig. 1.7.1 Impulse Function

PROPERTIES OF UNIT IMPULSE FUNCTION

(1) Properties of Continuous-Time Unit Impulse Function

(i) It is an even function of time t,

i.e., $\delta(t) = \delta(-t)$

(ii) $$\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = x(0)$$

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$$

(iii) $$\delta(at) = \frac{1}{|a|}\delta(t)$$

(iv) $$x \; \delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$
$$= x(t_0)$$

(v) $$x(t)\,\delta(t) = x(0)\,\delta(t)$$
$$= x(0)$$

(vi) $$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

(2) Properties of Discrete-Time Unit Sample Sequence

(i) $$\delta[n] = u[n] - u[n-1]$$

(ii) $$\delta[n-k] = \begin{cases} 1 & ;\ \text{for } n = k \\ 0 & ;\ \text{for } n \neq k \end{cases}$$

(iii) $$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

(iv) $$\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-n_0] = x[n_0]$$

EXAMPLE PROBLEM 1

Evaluate the following integrals,

(i) $\int_{-1}^{1} (3t^2 + 1)\, \delta(t)\, dt$

(ii) $\int_{-\infty}^{\infty} (t^2 + \cos \pi t)\, \delta(t - 1)\, dt$

(ii) $\int_{1}^{2} (3t^2 + 1)\, \delta(t)\, dt$

(iv) $\int_{-\infty}^{\infty} e^{-t}\, \delta(2t - 2)\, dt$.

SOLUTION

Shifting property of an impulse function is,

$$\int_{-\infty}^{\infty} x(t)\, \delta(t - t_0)\, dt = x(t_0)$$

Hence,

(i) $\int_{-1}^{+1} (3t^2 + 1)\, \delta(t)\, dt = 3t^2 + 1\Big|_{t=0} = 1$

(ii) $\int_{1}^{2} (3t^2 + 1)\, \delta(t)\, dt = 0$

The integral vanishes due to that the fact the impulse $\delta(t)$ does not appear in the range of integration from 1 to 2.

(iii) $\int_{-\infty}^{\infty} [t^2 + \cos \pi t]\, \delta(t - 1)\, dt = t^2 + \cos \pi t\Big|_{t=1}$

$$= 1 + \cos \pi = 0 \qquad [\because \cos\pi = -1]$$

(iv) $\int_{-\infty}^{\infty} e^{-1}\, \delta(2t - 2)\, dt = \int_{-\infty}^{\infty} e^{-t}\, \delta[2(t - 1)]\, dt$

We get,

$$\int_{-\infty}^{\infty} e^{-t}\, \frac{1}{2}\, \delta(t - 1)\, dt = \frac{1}{2}\, e^{-t}\Big|_{t=1} \qquad \left[\because \delta(at) = \frac{1}{a}\, \delta(t),\ a > 0\right]$$

$$= \frac{1}{2e}$$

1.7.1 Unit Step Function

A continuous-time signal is said to be a unit step signal if,

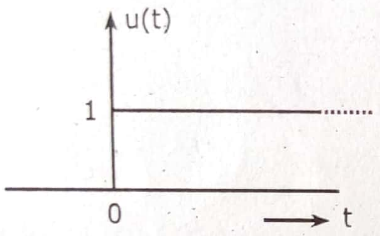
$$u[t] = \begin{cases} 1 & ; \quad \text{for } t > 0 \\ 0 & ; \quad \text{for } t < 0 \end{cases}$$

Conventionally, a continuous-time step signal is represented by u(t). The graphical representation of a continuous-time unit step signal is shown in Fig. 1.7.2(a).

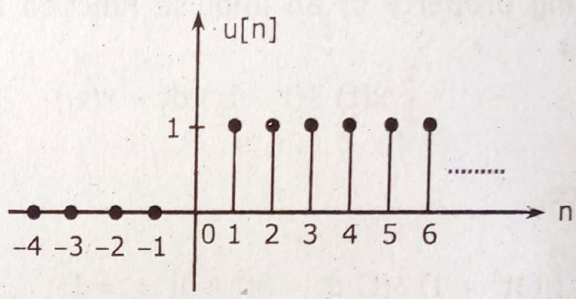
A discrete-time sequence is said to be a unit step signal if,

$$u[n] = \begin{cases} 1 & ; \quad \text{for } n \geq 0 \\ 0 & ; \quad \text{for } n < 0 \end{cases}$$

Conventionally, a discrete-time unit step sequence is represented by u[n]. The graphical representation of a discrete-time unit step sequence is shown in Fig. 1.7.2(b).



(a) Continuous Time Unit Step Signal



(b) Discrete Time Unit Step Sequence

Fig. 1.7.2 Unit Step Function

The integral of unit impulse function is a unit step function and the derivative of unit step function is a unit impulse function.

i.e.,
$$u(t) = \int_{-\infty}^{\infty} \delta(t)\, dt$$

And
$$\delta(t) = \frac{d}{dt} u(t)$$

1.7.2 Ramp Function

A continuous-time ramp function r(t) is mathematically defined by,

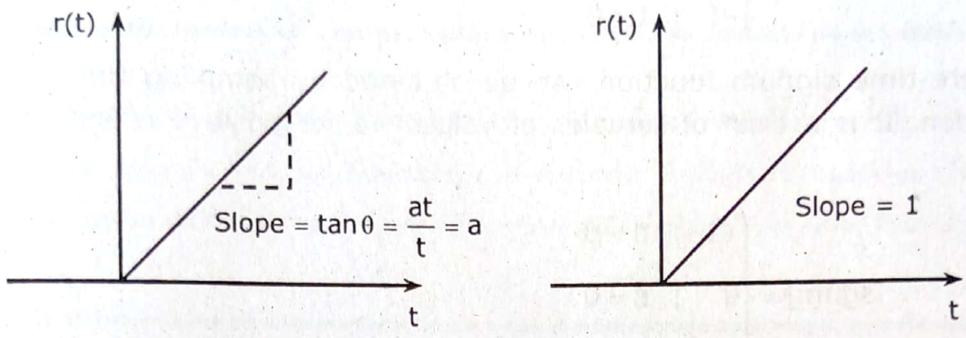
$$r(t) = \begin{cases} at\, u(t) & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \end{cases}$$

Where a represents the slope.

A continuous-time signal r(t) is said to be an unit ramp signal if it has unity slope, i.e., a = 1 mathematically,

$$r(t) = \begin{cases} at.u(t) & ; \; t \geq 0 \\ 0 & ; \; t < 0 \end{cases}$$

The graphical representation of ramp function with slope a and unity slope is shown in Fig. 1.7.3.



(a) Ramp Function with Slope a (b) Unit Ramp Function

Fig. 1.7.3 CT Ramp Function

A discrete-time ramp signal is defined by,

$$r[n] = \begin{cases} an.u[n] & ; \; n \geq 0 \\ 0 & ; \; n < 0 \end{cases}$$

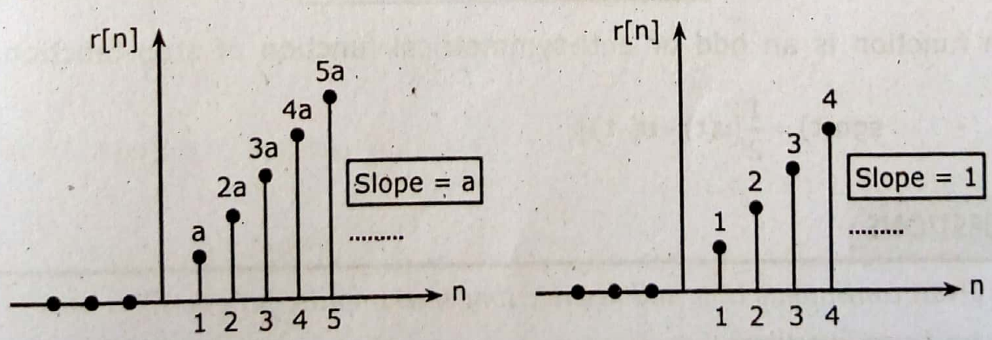
A discrete-time signal, r[n] is said to be an unit ramp sequence, if it has unity slope.

i.e., $a = 1$

Mathematically,

$$r[n] = \begin{cases} n.u[n] & ; \; n \geq 0 \\ 0 & ; \; n < 0 \end{cases}$$

The graphical representation of discrete time ramp signal is shown in Fig. 1.7.4,



(a) Ramp Sequence (b) Unit Ramp Sequence

Fig. 1.7.4 Discrete Time Ramp Sequence

1.7.3 Signum Function

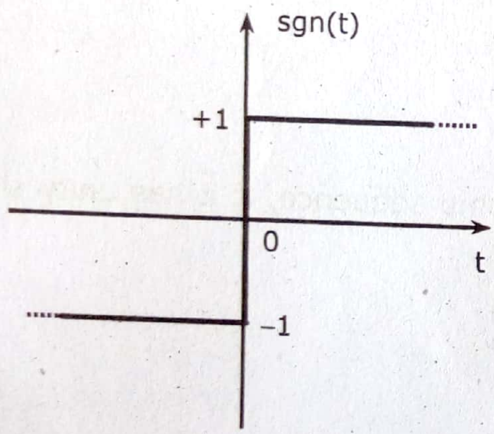
The signum function is defined as the sign of the independent variable t. Therefore, the continuous-time signum signal is defined as,

$$\mathrm{sgn}(t) = \begin{cases} +1 & ; \ t > 0 \\ 0 & ; \ t = 0 \\ -1 & ; \ t < 0 \end{cases}$$

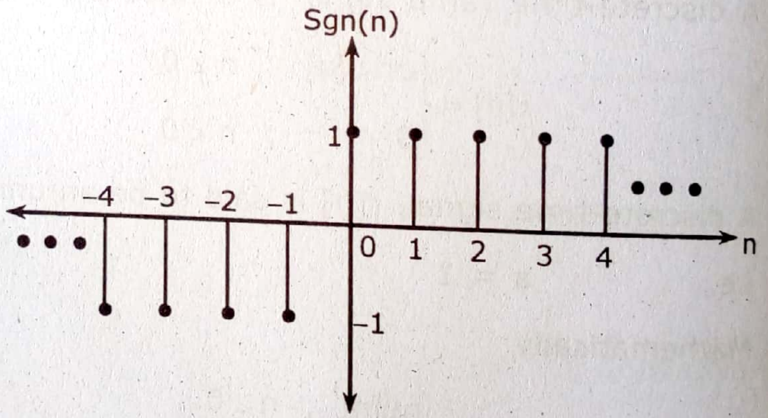
A discrete-time signum function can be obtained by sampling the continuous time signum function. It is a train of samples of values +1 for positive n and −1 for negative n. Therefore,

$$\mathrm{sgn}[n] = \begin{cases} +1 & ; \ n > 0 \\ 0 & ; \ n = 0 \\ -1 & ; \ n < 0 \end{cases}$$

The graphical representation of continuous time and discrete time signum function is shown in Fig. 1.7.5 respectively,



(a) Antinuous Time Signum Function



(b) Discrete Time Signum Function

Fig. 1.7.5 Signum Function

Signum function is an odd or anti-symmetrical function of step-function.

$$\therefore \qquad \mathrm{sgn}(t) = \frac{1}{2}[u(t) - u(-t)]$$

REVIEW QUESTIONS

(1) *Write about continuous time and discrete time unit impulse function?*

(2) *Define unit step function?*

(3) *How can you define the signum function is both continuous time and discrete time sequences. Explain with graphical representation.*