## 1.1 INTRODUCTION

Logic is the study and analysis of the nature of valid argument and the reasoning tool by which valid inferences can be drawn from a set of premises. It is the basis on which all the sciences are built. The axiomatic approach to logic was first introduced by George Boole and it is sometimes called Boolean Logic or Mathematical Logic or Symbolic Logic. Symbolic Logic is now increasingly used in the study of language of a computer.

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments.

In addition to its importance in understanding mathematical reasoning, logic has numerous applications in computer science. These rules are used in the design of computer circuits, the construction of computer programs and the verification of the correctness of programs.

## 1.2 BASIC CONNECTIVES AND TRUTH TABLES

### 1.2.1 Statements and Notations

In symbolic logic we study arguments. The basic building blocks of arguments are declarative sentences called propositions or statements.

The following sentences are some logical statements.

(1) The sun rises in the east.

(2) Smoking is injurious to health.

(3) She is an engineering graduate.

(4) $(a - b)^2 = a^2 - 2ab + b^2$ for all values of a and b.

None of the following is a statement.

(1) How far is Chennai from here ?

(2) What a beautiful day !

(3) Mathematics is fun.

(4) May god bless you.

The sentences (i), (ii), (iii), (iv) are respectively interrogative, exclamatory, subjective and imperative. These sentences are not declarative and it is not possible to say whether any one of them is true or false. Hence, none of these sentences is a statement.

In order to build up the symbolic logic we use some symbols. They are single letters such as A, B, C, ... P, Q, ...... to denote statements.

By truth value of a statement, we mean whether the statement is true or false. If a statement is true, then we say that it has the truth value "True" or "T" and if it is false we say that it has the truth value "False" or "F". Thus truth value is the truth or falsity of a logical statement.

**Examples**

(1) P : 5 is less than 7 and Q : The sum of any two sides of a triangle is equal to third side, then P is true and Q is false.

(2) The sentence $x + 2 = 7$ has no definite truth value, since x is not known. It is called an open sentence, which is not a statement.

(3) Ice melts at $30^0$ C is "F" where as $x^2 \geq 0$ for $x \in R$ is "T".

## 1.2.2 Basic Connectives and their Truth Tables

The statements that we consider initially are simple statements, called atomic or primary statements. Most statements in mathematical logic are combinations of simpler statements, joined by words and phrases like "and", "or", "if . . . . . then", "if and only if" etc. These words and phrases are called logical connectives.

### 1.2.2.1 Negation

The negations of a statement is obtained by adding the word 'not' at appropriate place in the statement.

If P is a statement then the negation of statement is denoted by 7P (or) $\neg P$. If the truth value for the statement P is T then the negation of statement $\neg P$ is F and also if the truth value of the statement $\neg P$ is F then the negation of statement $\neg P$ is T.

**Table 1.2.1 :** Truth Table for Negation

| ***P*** | ***¬P*** |
|---|---|
| T | F |
| F | T |

**Examples**

(1) P : Hyderabad is a city.

The negation of this statement is,

$\neg P$ : Hyderabad is not a city.

(2) P : 11 is a prime number.

The negation of this statement is,

$\neg P$ : 11 is not a prime number.

(3) P : I went to college yesterday.

The negation of this statement is,

$\neg P$ : I did not go to college yesterday.

## EXAMPLE PROBLEM 1

**Find the negation of the proposition.**

**" Today is Friday".**

**And express this in simple English.**

**SOLUTION**

The negation is,

"It is not the case that today is Friday".

This negation can be more simply expressed by,

"Today is not Friday".

### 1.2.2 Disjunction

The disjunction of two statements P and Q is the statement $P \vee Q$ which is read as P or Q. If the truth value of even one out of P and Q is true then that of $P \vee Q$ is true, otherwise then truth value of $P \vee Q$ is false.

**Table 1.2.2 :** Truth Table for Disjunction

| P | Q | $P \vee Q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**Examples**

(1) P : All rectangles are equilateral.

Q : All roses are red.

The solution for this statements is,

$P \vee Q$ : All rectangles are equilateral or All roses are red.

(2) P : Delhi is in India.

Q : Hyderabad is in India.

The solution for this statements is,

$P \vee Q$ : Delhi is in India or Hyderabad is in India.

(3) P : $1 + 1 = 11$ (F).

Q : $2 + 2 = 5$ (F).

Here, the solution for the above statements is,

$P \vee Q$ : $1 + 1 = 11$ (F) or $2 + 2 = 5$ (F).

Here P and Q are having the truth values F and F so, the truth value of $P \vee Q$ is F.

## EXAMPLE PROBLEM 1

**Obtain the truth value of the disjunction of "The earth is flat" and "3 + 5 = 8".**

**SOLUTION**

Let P : The earth is flat.

Q : 3 + 5 = 8.

We know that P is false and Q is true. Therefore, the truth value of $P \vee Q$ is true.

## EXAMPLE PROBLEM 2

**Form the disjunction of P : It is raining today, Q : 2 + 3 = 6.**

**SOLUTION**

It is raining today or 2 + 3 = 6.

### 1.2.2.3 Conjunction

The conjunction of two statements P and Q is the statement $P \wedge Q$ which is read as P and Q. The statement $P \wedge Q$ has the truth value "T" whenever both P and Q have the truth value "T" otherwise it has the truth value "F".

**Table 1.2.3 :** Truth Table for Conjunction

| P | Q | P ? Q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

**Examples**

(1) P : It is raining.

Q : There are 20 chairs in this class room.

The conjunction of the statements P and Q is,

$P \wedge Q$ : It is raining and there are 20 chairs in this class room.

(2) P : Seven is divisible by 3 (F).

Q : 2 + 3 = 5 (T).

Here the solution is,

$P \wedge Q$ : Seven is divisible by 3 and 2 + 3 = 5.

(3) P : He is reading the book.

Q : She is playing with her friends.

The solution for this statements P and Q is,

$P \wedge Q$ : He is reading the book and she is playing with her friends.

### EXAMPLE PROBLEM 1

**Form the conjunction of p : 2 + 3 = 5, Q : Padma is in Bangalore.**

**SOLUTION**

2 + 3 = 5 and Padma is in Bangalore.

### EXAMPLE PROBLEM 2

**Find the conjunction of the propositions p and q where p is the proposition "Today is Friday" and q is the proposition "it is raining today".**

**SOLUTION**

The conjunction of these propositions $p \wedge q$ is the proposition "Today is Friday and it is raining today".

This proposition is true on rainy Fridays and is false on any day that is not a Friday and on Fridays when it does not rain.

## 1.2.2.4 Conditional Connective

Given any two propositions P and Q we denote the statement 'If P then Q' by $P \to Q$. We read this as P implies Q or P is sufficient for Q. Also we call P as the hypothesis and Q as the conclusion. The statement P is called the antecedent and Q is called the consequent of $P \to Q$.

The statement $P \to Q$ has a truth value F when Q has the truth value F and P has the truth value T, otherwise it has truth value T.

**Table 1.2.4 :** Truth Table for Conditional

| P | Q | $P \to Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

**Examples**

(1) P : a = 2.

Q : b is divisible by 3.

The conditional statement is,

$P \to Q$ : If a = 2, then b is divisible by 3.

(2) P : 4 is a prime number (F).

Q : 7 is a prime number (T).

The conditional statement of the above statements is,

$P \to Q$ : If 4 is a prime number, then 7 is a prime number.

## EXAMPLE PROBLEM 1

**Obtain the truth value of the statement "If triangle ABC is equilateral, then it is isosceles".**

**SOLUTION**

Let P : Triangle ABC is equilateral.

Q : Triangle ABC is isosceles.

If Q is true, then $P \rightarrow Q$ is true. If Q is false, then $P \rightarrow Q$ is true only when P is false. So, if triangle ABC is an isosceles triangle, the given statement is always true. Also, if triangle ABC is not isosceles, then it can't be equilateral either. So the given statement is again true.

## EXAMPLE PROBLEM 2

**Write the following statement in symbolic form,**

**If either Jerry takes C++ or Ken takes Java then Larry will take .Net.**

**[Nov./Dec. - 2008]**

**SOLUTION**

Let J(x) : Jerry takes C++.

K(x) : Ken takes Java.

L(x) : Larry takes .Net.

Then the given statement can be written as $J(x) \vee K(x) \rightarrow L(x)$.

### 1.2.2.5 Biconditional Connective

Let P and Q be two propositions. The compound statement $(P \rightarrow Q) \wedge (Q \rightarrow P)$ is the biconditional of P and Q.

We denote it by $P \leftrightarrow Q$ and read it as 'P if and only if Q'. We also say that 'P is necessary and sufficient for Q'. The statement $P \leftrightarrow Q$ has the truth value T whenever both P and Q have identical truth values.

**Table 1.2.5 :** Truth Table for Biondifional

| P | Q | $P \rightarrow Q$ | $Q \rightarrow P$ | $P \leftrightarrow Q$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

**Examples**

(1) P : a = 2.

Q : b is divisible by 3.

Then the biconditional statement of the above statement is,

$P \leftrightarrow Q$ : If a = 2 then b is divisible by 3, and

If b is divisible by 3 then a = 2.

(2) P : 4 is a prime number (F).

Q : 7 is a prime number (T).

Then the biconditional statement is,

$P \leftrightarrow Q$ : If 4 is a prime number then 7 is a prime number and if 7 is a prime number then 4 is a prime number.

Here $P \leftrightarrow Q$ truth value is (F).

EXAMPLE PROBLEM 1

**'a' and 'b' are integers if and only if ab is a rational number. Obtain the truth value of the above statement.**

SOLUTION

P : a is an integer.

Q : b is an integer.

R : ab is a rational number.

The given statement is $(P \wedge Q) \leftrightarrow R$.

$(P \wedge Q) \leftrightarrow R$ will be true if $P \wedge Q$ is true and R is true, or when $P \wedge Q$ is false and R is false.

## 1.2.3 Well Formed Formulas (WFF's)

### 1.2.3.1 Statement Formulas

Statements which do not contain any connectives are called atomic or simple statements. On the other hand, the statements which contain one or more primary statements and atleast one connective are called molecular or composite or compound statements.

**For Example :** Let P and Q be any two simple statements. Some of the compound statements formed by P and Q are,

$\neg P \quad P \vee Q \quad (P \vee Q) \vee (\neg P) \quad P \vee (\neg Q) \quad (P \vee \neg Q) \wedge P$

The above compound statements are called statement formulas derived from the statement variables P and Q. Therefore P and Q are called as components of the statement formulas.

A statement formula alone has no truth value. It has truth value only when the statement variables in the formula are replaced by definite statements and it depends on the truth values of the statement used in replacing the variables.

## 1.2.3.2 Symbols and Properties of Well Formed Formulas

In order to define a formula, we need certain concepts like language, string etc.

Language consists of few symbols given below.

(1) Connectives

(2) Equality denoted by =,

(3) Parenthesis denoted by (,),

(4) Variables denoted by x, y = . . . . . . .

A string is any succession of symbols of the language written one after the other.

A string is called a formula if it has the following properties.

(1) A statement variable standing alone is a well formed formula.

(2) If P is a formula, then $\sim P$ is also a formula.

(3) If P and Q are formulas then $P \wedge Q$, $P \vee Q$, $P \rightarrow Q$ and $P \leftrightarrow Q$ are also formulas.

(4) A string of symbols containing the statement variables, connectives and parenthesis is a well formed formula, iff it can be obtained by finitely many applications of the rules 1, 2, and 3.

A formula is also called well formed formula and is denoted by wff.

**Examples for Wffs**

(1) $\neg(P \wedge Q)$, $(P \rightarrow (P \vee Q))$, $(P \rightarrow (Q \rightarrow R))$.

(2) $(\neg((\neg P) \wedge (\neg Q)))$, $(P \rightarrow (P \vee Q))$, $(((P \vee Q) \vee (P \vee \neg S)) \rightarrow (P \vee R))$, $((P \wedge Q) \rightarrow Q)$.

**Examples for not wffs**

(1) $\neg P \vee Q$, obviously P and Q are wffs. A wff would be either $(\neg P \vee Q)$ or $\neg(P \wedge Q)$

(2) $((P \rightarrow Q) \rightarrow (\wedge Q))$ is not a formula, as $(\wedge Q)$ is not a wff.

(3) $(P \rightarrow Q$ is not a wff as ')' is omitted. Note that $(P \rightarrow Q)$ is a formula.

(4) $(P \wedge Q \rightarrow Q)$. The cause for this not being a wff is that one of the parenthesis in the beginning is missing $((P \wedge Q) \rightarrow Q)$ is wff. While $(P \wedge Q) \rightarrow Q$ is still not a wff.

; possible to introduce some conventions so that the number of parenthesis used can be
.

r the sake of convenience, we shall omit the outer parenthesis. Thus, we write $\vee$ Q for (P $\vee$ Q), (P $\vee$ Q) $\to$ Q in place of ((P $\vee$ Q) $\to$ Q), and (((P $\to$ Q) Q $\to$ R)) $\rightleftarrows$ (P $\to$ R)) instead of ((P $\to$ Q) $\wedge$ (Q $\to$ R)) $\rightleftarrows$ (P $\to$ R).

: shall also assume that $\neg$ connects as little as possible, and that $\wedge$ connects little as possible subject to the condition on $\neg$. For instance, $\neg$(P $\wedge$ Q) $\to$ (($\neg$P) $\wedge$($\neg$R)) n be abbreviated to $\neg$(P $\wedge$ Q) $\to$ ($\neg$P $\wedge$ $\neg$R).

## Truth Tables

ıth tables have already been introduced in the definitions of the connectives. We have :mine the truth value of a statement formula for each possible combination of the truth of the component statements. A table showing all such truth values is called the truth table ormula..

here is only one component, say P, then there are two possible truth values to be considered lse. If the statement formulas have two component statements, namely P and Q. Then e $2^2$ possible combinations of truth values that must be considered. In general, if there istinct components in a statement formula, we need to consider $2^n$ possible combinations values in order to obtain the truth tables.

LE PROBLEM 1

**ınstruct truth table for the following [(P $\vee$ Q) $\wedge$ ($\neg$R)] $\leftrightarrow$ Q.**

ON

t x = (P $\vee$ Q) $\wedge$ ($\neg$R)

**Table 1.2.6 :** Truth Table for [(P $\vee$ Q) $\wedge$ ($\neg$R)] $\leftrightarrow$ Q

| **P** | **Q** | **R** | **¬R** | **P ∨ Q** | **(P ∨ Q) ∨ (¬R) = x** | **x → Q** | **Q → x** | **(x → Q) ∧ (Q → x) = x ↔ Q** |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | T | F | T | F | F |
| T | T | F | T | T | T | T | T | T |
| T | F | T | F | T | F | T | T | T |
| T | F | F | T | T | T | F | T | F |
| F | T | T | F | T | F | T | F | F |
| F | T | F | T | T | T | T | T | T |
| F | F | T | F | F | F | T | T | T |
| F | F | F | T | F | F | T | T | T |

## 1.2.5 Tautology, Contradiction and Contingency

We know that the statement formulas are compound statements derived from some statement variables. Statement formulas are not statements, they become statements when the statement variables used to define them are replaced by definite statements. The truth values of the resulting statement depend on the truth values of the statements substituted for the variables. Thus, the truth table of resulting statement is the summary of all its truth values for all possible assignments of values to the variables appearing in the formula. Therefore, sometimes the truth values of the resulting statement may be 'T' and sometimes 'F'. There are some formulas whose truth values are always 'T' or always 'F' irrespective of the truth values assigned to the variables.

A statement that is true for all possible values of its propositional variables is called a tautology.

A statement that is always false is called a contradiction or an absurdity and a statement that can be either true or false, depending on the truth values of its propositional variables is called a contingency.

**Examples**

(1) The statement $P \vee \neg P$ is a tautology.

(2) The statement $P \wedge \neg P$ is a contradiction.

(3) The statement $(P \to Q) \wedge (P \vee Q)$ is a contingency.

### EXAMPLE PROBLEM 1

**Show that each of the following implications is a Tautology by using truth tables.**

**(i)** $\neg P \to (P \to Q)$

**(ii)** $[(P \vee Q) \wedge (P \to R) \wedge (Q \to R)] \to R$

**SOLUTION**

**(i)** $\neg P \to (P \to Q)$

**Table 1.2.14 :** Truth Table for $\neg P \to (P \to Q)$

| $P$ | $Q$ | $\neg P$ | $P \to Q$ | $\neg P \to (P \to Q)$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | T |
| F | T | T | T | T |
| F | F | T | T | T |

From the last column of the truth table, it follows that $\neg P \to (P \to Q)$ is a tautology.

**EXAMPLE PROBLEM 5**

**Show that $(P \wedge Q) \rightarrow (P \vee Q)$ is a Tautology.**

**SOLUTION**

Given,

$(P \wedge Q) \rightarrow (P \vee Q)$

$\equiv \neg(P \wedge Q) \vee (P \vee Q)$ $[\because P \rightarrow Q \leftrightarrow \neg P \vee Q]$

$\equiv (\neg P \vee \neg Q) \vee (P \vee Q)$ [by DeMorgan's Law]

$\equiv (\neg P \vee P) \vee (\neg Q \vee Q)$ [by Associative and Commutative Laws]

$\equiv T \vee T \equiv T$

## 1.3 LOGICAL EQUIVALENCE

### 1.3.1 The Laws of Logic

Two statement formulae A and B in variables $P_1, P_2 \ldots P_n$ $(n \geq 1)$ are said to be equivalent if they acquire the same truth values for all interpretation i.e., they have identical truth values. Therefore, the statement formulas A and B are equivalent provided $A \leftrightarrow B$ is a Tautology and conversly if $A \leftrightarrow B$ is a tautology, then A and B are equivalent. We shall represent the equivalence of two formulas, say A and B, by writing "$A \Leftrightarrow B$" which is read as "A is equivalent to B".

We state the following list of laws for the algebra of propositions.

For any primitive statements p, q, r any tautology $T_0$, and any contradiction $F_0$,

(1) $\neg\neg P \Leftrightarrow P$ — Law of Double Negation

(2) $\neg(P \vee Q) \Leftrightarrow \neg p \wedge \neg q$ — DeMorgan's Laws

$\neg(P \wedge Q) \Leftrightarrow \neg p \vee \neg q$

(3) $p \vee q \Leftrightarrow q \vee p$ — Commutative Laws

$p \wedge q \Leftrightarrow q \wedge p$

(4) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ — Associative Laws

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

(5) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ — Distributive Laws

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(6) $p \vee p \Leftrightarrow p$ — Idempotent Laws

$p \wedge p \Leftrightarrow p$

(7) $p \vee F_0 \Leftrightarrow p$ Identity Laws

$p \wedge T_0 \Leftrightarrow p$

(8) $p \vee \neg p \Leftrightarrow T_0$ Inverse Laws

$p \wedge \neg p \Leftrightarrow F_0$

(9) $p \vee T_0 \Leftrightarrow T_0$ Domination Laws

$p \wedge F_0 \Leftrightarrow F_0$

(10) $p \vee (p \wedge q) \Leftrightarrow p$ Absorption Laws

$p \wedge (p \vee q) \Leftrightarrow p$

## EXAMPLE PROBLEM 1

**Prove $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$**

**SOLUTION**

We prove this by truth table.

**Table 1.3.1 :** Truth Table for $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

| $P$ | $Q$ | $P \rightarrow Q$ | $\neg P$ | $\neg P \vee Q$ | $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

The truth values for $P \rightarrow Q$ and $\neg P \vee Q$ are same. Therefore, $(P \rightarrow Q) \Leftrightarrow (\sim P \vee Q)$.

## EXAMPLE PROBLEM 2

**Prove that $P \leftrightarrow Q \equiv \neg(P \vee Q) \vee (P \wedge Q)$**

**SOLUTION**

R.H.S.

$\equiv \neg(P \vee Q) \vee (P \wedge Q)$

$\equiv (\neg P \wedge \neg Q) \vee (P \wedge Q)$

$\equiv [(\neg P \wedge \neg Q) \vee P] \wedge [(\neg P \wedge \neg Q) \vee Q]$ [by Distributive Law]

$\equiv [(\neg P \vee P) \wedge (\neg Q \vee P)] \wedge [(\neg P \vee Q) \wedge (\neg Q \vee Q]$

$\equiv (\neg Q \vee P) \wedge (\neg P \vee Q)$

$\equiv (Q \rightarrow P) \wedge (P \rightarrow Q)$

$\equiv P \leftrightarrow Q \equiv$ L.H.S.

## EXAMPLE PROBLEM 11

**Show that $((Q \wedge A) \rightarrow C) \wedge (A \rightarrow (P \vee C)) \Leftrightarrow (A \wedge (P \rightarrow Q)) \rightarrow C$.**

**SOLUTION**

Consider $((Q \wedge A) \rightarrow C) \wedge (A \rightarrow (P \vee C))$

$$\Leftrightarrow (\neg(Q \wedge A) \vee C) \wedge (\neg A \vee (P \vee C))$$

$$\Leftrightarrow ((\neg Q \vee \neg A) \vee C) \wedge (\neg A \vee (P \vee C))$$

$$\Leftrightarrow (\neg A \vee (\neg Q \vee C)) \wedge (\neg A \vee (P \vee C)) \qquad \text{[by Distributive Law]}$$

$$\Leftrightarrow A \vee ((\neg Q \vee C) \wedge (P \vee C))$$

$$\Leftrightarrow \neg A \vee ((P \wedge \neg Q) \vee C)$$

$$\Leftrightarrow \neg A \vee \neg(\neg(P \wedge \neg Q) \wedge \neg C)$$

$$\Leftrightarrow \neg A \vee \neg((\neg P \vee Q) \wedge \neg C)$$

$$\Leftrightarrow \neg A \vee (\neg(\neg P \vee Q)) \vee C$$

$$\Leftrightarrow \neg A \wedge (\neg(P \rightarrow Q)) \vee C$$

$$\Leftrightarrow \neg(A \vee (P \rightarrow Q)) \vee C$$

$$\Leftrightarrow (A \wedge (P \rightarrow Q)) \rightarrow C$$

### 1.3.2 The Principle of Duality

Two formulas A and A* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ and $\vee$ are also called duals of each other. If the formula A contains the special variables T or F, the A*, its dual is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

If the formula A is given by, $A : \neg(P \vee Q) \wedge (P \vee \neg(Q \wedge \neg R))$

Its dual A* is given by, $A^* : \neg(P \wedge Q) \vee (P \wedge \neg(Q \vee \neg R))$

**Result :** Let A and A* be dual formulas and let $P_1, P_2 \ldots . P_n$ be all atomic variables that occur in A and A*. i.e., $A : A(P_1, P_2, \ldots . P_n)$ and $A^* : A^*(P_1, P_2, \ldots P_n)$.

Then using the DeMorgan's Laws,

$$P \wedge Q \Leftrightarrow \neg(\neg P \vee \neg Q)$$

$$(P \vee Q) \Leftrightarrow \neg(\neg P \wedge \neg Q)$$

We can prove, $\neg A(P_1, P_2, \ldots P_n) \Leftrightarrow A^* (\neg P_1, \neg P_2, \ldots \neg P_n)$.

Thus, the negation of a formula is equivalent to its dual in which every variable is replaced by its negation.

Similarly, $A(\neg P_1, \neg P_2, \ldots \neg P_n) \Leftrightarrow \neg A^* (P_1, P_2, \ldots P_n)$.

**Examples**

(1) The dual of $(P \vee Q) \wedge R$ is $(P \wedge Q) \vee R$.

(2) The dual of $\neg(P \vee Q) \wedge (P \vee \neg(Q \wedge \neg S))$ is $\neg(P \wedge Q) \vee (P \wedge \neg(Q \vee \neg S)$

## EXAMPLE PROBLEM 1

**Show that $\neg(\neg P \wedge \neg(Q \vee R)) \Leftrightarrow P \vee (Q \vee R)$.**

**SOLUTION**

Let $A(P, Q, R)$ is $\neg P \wedge \neg(Q \vee R)$.

Then $A^*(P, Q, R)$ is $(\neg P \vee \neg(Q \wedge R))$ and

$A^*(\neg P, \neg Q, \neg R) : \neg\neg P \vee \neg(\neg Q \wedge \neg R) \Leftrightarrow P \vee (Q \vee R)$

On the other hand $\neg A(P, Q, R)$ is $\neg(\neg P \wedge \neg(Q \vee R) \Leftrightarrow P \vee (Q \vee R)$

**Result :** If any two formulas are equivalent, then their duals are also equivalent to each other. i.e., If $A \Leftrightarrow B$ then $A^* \Leftrightarrow B^*$.

(or)

Let $P_1, P_2, \ldots P_n$ be the atomic variables appearing in the formulas A and B. Given that $A \Leftrightarrow B$ means "$A \rightleftarrows B$ is a tautology", then, the following are also tautologies.

$$A(P_1, P_2 \ldots P_n) \leftrightarrows B(P_1, P_2, \ldots P_n) \quad \ldots (1)$$

$$A(\neg P_1, \neg P_2 \ldots \neg P_n) \leftrightarrows B(\neg P_1, \neg P_2, \ldots \neg P_n) \quad \ldots (2)$$

Using (2) we get $\neg A^*(P_1, P_2 \ldots P_n) \leftrightarrows \neg B^*(P_1, P_2, \ldots P_n)$

Hence $A^* \Leftrightarrow B^*$.

## EXAMPLE PROBLEM 4

**Prove the following logical equivalence without using truth table**

**$\neg[\neg((p \vee q) \wedge r) \vee \neg q] \Leftrightarrow q \wedge r$**

**SOLUTION**

$\neg(\neg((p \vee q) \wedge r) \vee \neg q)$

$\Leftrightarrow \neg\neg((p \vee q) \wedge r) \wedge \neg\neg q$ [by DeMorgan's Law]

$\Leftrightarrow (p \vee q) \wedge r() \wedge q$ [by Double Negation]

$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$ [by Associative Law of $\wedge$]

$\Leftrightarrow (p \vee q) \wedge (q \wedge r)$ [by Commutative Law of $\wedge$]

$\Leftrightarrow ((p \vee q) \wedge q) \wedge r$ [by Associative Law of $\wedge$]

$\Leftrightarrow q \wedge r$ [by Absorption Law and Commutative Law for $\wedge$ and $\vee$].

## 1.3.3 Tautological Implications

A statement A is said to be tautologically imply a statement B if and only if A → B is a tautology. In this case, we write A ⇒ B, read as "A implies B".

⇒ is not a connective, A ⇒ B is not a statement formula.

(1) A ⇒B states that A → B is a tautology or A tautologically implies B.

(2) Clearly A⇒B guarantees that B has the truth value T whenever A has the truth value T.

(3) By constructing the truth table of A and B, we can determine whether A ⇒B.

## EXAMPLE PROBLEM 1

**Prove that (P → Q) Þ (¬Q → ¬P)**

**SOLUTION**

We prove this by using the truth table for (P → Q) → (¬ Q → ¬ P).

**Table 1.3.3 :** Truth Table for (P → Q) → (¬Q → ¬P)

| ***P*** | ***Q*** | ***¬P*** | ***¬Q*** | ***P→Q*** | ***¬Q→ ¬P*** | ***(P→ Q) → (¬Q→ ¬P)*** |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

Since all the entries in the last column are true, (P → Q) → (¬ Q → ¬ P) is a tautology.

Hence, (P → Q) ⇒ (¬ Q → ¬ P).

(ii) $P \wedge (Q \rightleftarrows R)$

$$\Leftrightarrow P \wedge ((Q \rightarrow R) \wedge (R \rightarrow Q))$$

$$\Leftrightarrow P \wedge ((\neg Q \vee R) \wedge (\neg R \vee Q))$$

$$\Leftrightarrow P \wedge (\neg(\neg Q \downarrow R) \wedge \neg(\neg R \downarrow Q))$$

$$\Leftrightarrow P \wedge (\neg((\neg Q \downarrow R) \vee (\neg R \downarrow Q)))$$

$$\Leftrightarrow P \wedge ((\neg Q \downarrow R) \downarrow (\neg R \downarrow Q))$$

$$\Leftrightarrow \neg[\neg P \vee \neg(\neg Q \downarrow R) \downarrow R] \downarrow (\neg R \downarrow Q)$$

$$\Leftrightarrow \neg((P \downarrow P) \vee \neg((\neg Q \downarrow R)) \downarrow (\neg R \downarrow Q))$$

$$\Leftrightarrow (P \downarrow P) \downarrow \neg((\neg Q \downarrow R) \downarrow (\neg R \downarrow Q))$$

SOLVED PROBLEM 7

**For any statements p, q prove that,**

**(i)** $\neg(p \downarrow q) \Leftrightarrow (\neg p \uparrow \neg q)$

**(ii)** $\neg(p \uparrow q) \Leftrightarrow (\neg p \downarrow \neg q)$

**SOLUTION**

Using the definition of connective NAND and NOR,

(i) $\neg(p \downarrow q) \Leftrightarrow \neg\{\neg(p \vee q)\}$

$$\Leftrightarrow \neg(\neg p \wedge \neg q) \Leftrightarrow (\neg p) \uparrow (\neg q)$$

(ii) $\neg(p \uparrow q) \Leftrightarrow \neg\{\neg(p \wedge q)\}$

$$\Leftrightarrow \neg(\neg p \vee \neg q)$$

$$\Leftrightarrow (\neg p) \downarrow (\neg q)$$

## 1.3.6 Normal Forms

We shall use the words "product" and "sum" in place of the logical connectives "conjunction" and "disjunction" respectively. In a formula, a product of variables and their negations is called an elementary product. Similarly, a sum of the variables and their negations is called an elementary sum. Let P and Q be any two variables. Then $P$, $\neg P \wedge Q$, $\neg Q \wedge P \wedge \neg P$, $P \wedge \neg P$ and $Q \wedge \neg P$ are some examples of elementary products. $P$, $\neg P \vee Q$, $\neg Q \vee P \vee \neg P$, $P \vee \neg P$ and $Q \vee \neg P$ are some examples of elementary sums of two variables.

A part of the elementary sum or product which is itself an elementary sum or product is called a factor of the original sum or product. The elementary sums or products satisfy the following properties.

(1) An elementary product is identically false iff it contains at least one pair of factors in which one is a negation of the other.

(2) An elementary sum is identically true iff it contains at least one pair of factors in which one is the negation of the other.

**For Example :** We know that for any variable P, $P \wedge \neg P$ is identically false. Hence, if $P \wedge \neg P$ appears in the elementary product, then, the product is identically false.

### 1.3.6.1 Disjunctive Normal Forms (DNF) SOP

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula.

**EXAMPLE PROBLEM 1**

**Obtain the disjunctive normal form of $\neg(P \vee Q) \leftrightarrow P \wedge Q$.**

**SOLUTION**

We have, S.O.P

$\neg(P \vee Q) \leftrightarrow P \wedge Q$

$\equiv (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q))$ $\quad [\because R \leftrightarrow S \equiv (R \wedge S) \vee (\neg R \wedge \neg S)]$

$\equiv (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$

$\equiv (\neg P \wedge \neg Q \wedge P \wedge Q) \vee [((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q)]$

$\equiv (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$

**EXAMPLE PROBLEM 2**

**Obtain the disjunctive normal form for $\neg[\neg(P \leftrightarrow Q) \wedge R]$.**

**SOLUTION**

$\neg[\neg(p \leftrightarrow Q) \wedge R]$

$\equiv \neg[\neg\{(P \to Q) \wedge (Q \to P)\} \wedge R]$

$\equiv \{(P \to Q) \wedge (Q \to P)\} \vee \neg R$

$\equiv [(\neg P \vee Q) \wedge (\neg Q \vee P)] \vee \neg R$

$\equiv (\neg P \vee Q \vee \neg R) \wedge (\neg Q \vee P \vee \neg R)$

$\equiv \{(\neg P \vee Q \vee \neg R) \wedge P\} \vee \{(\neg P \vee Q \vee \neg R) \wedge \neg Q\} \vee \{(\neg P \vee Q \vee \neg R) \wedge \neg R\}$

$\equiv (Q \wedge P) \vee (\neg R \wedge P) \vee (\neg P \wedge \neg Q) \vee (\neg R \wedge \neg Q) \vee (\neg P \wedge \neg R) \vee (Q \wedge \neg R) \vee \neg R$

$\equiv (P \wedge Q) \vee (P \wedge \neg R) \vee (\neg P \wedge \neg Q) \vee (\neg Q \wedge \neg R) \vee (\neg P \wedge \neg R) \vee (Q \wedge \neg R) \vee \neg R$

## 1.3.6.2 Conjunctive Normal Forms (CNF)

A formula which consists of a product of elementary sums and is equivalent to a given formula is called a conjunctive normal form of the given formula.

### EXAMPLE PROBLEM 1

**Obtain a conjunctive normal form of $\neg(P \vee Q) \leftrightarrow P \wedge Q$.** — POS.

**SOLUTION**

$\neg(P \vee Q) \leftrightarrow P \wedge Q$

$\equiv (\neg(P \vee Q) \rightarrow (P \wedge Q)) \wedge ((P \wedge Q) \rightarrow \neg(P \vee Q))$

Using $R \leftrightarrow S$

$\equiv (R \rightarrow S) \wedge (S \rightarrow R)$, $\boxed{P \rightarrow Q \equiv \neg P \vee Q}$ (∨) ∩ (∨)

$\equiv ((P \vee Q) \vee (P \wedge Q)) \wedge ((\neg(P \wedge Q)) \vee (\neg P \wedge \neg Q))$

$\equiv (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge ((\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q))$

$\equiv (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q)$

## 1.3.6.3 Principal Disjunctive Normal Forms (PDNF)

For two statement variables P and Q, construct all possible formulas which consists of conjunctions of P or its negation and conjunctions of Q or its negation such that none of the formulas contain both a variable and its negation. Note that any formula which is obtained by computing the formulas in the conjunction should not be included in this list because any such formula will be equivalent to one included in the list. For example, any one of $P \wedge Q$ or $Q \wedge P$ is included in the list, but not the both. For two variables P and Q, there are 4 formulas included in the list, these are given by,

$P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, $\neg P \wedge \neg Q$.

These formulas are called minterms. Also, no two minterms are equivalent.

Given one formula, an equivalent formula consisting of disjunctions of minterms only is known as its principal disjunctive normal form or sum of products canonical form.

Although our discussion of the principal disjunctive normal form was restricted to formulas containing only two variables, it is possible to define the minterms for three or more variables. Minterms for three variables, P, Q and R are,

$P \wedge Q \wedge R$, $P \wedge Q \wedge \neg R$, $P \wedge \neg Q \wedge R$, $P \wedge \neg Q \wedge \neg R$

$\neg P \wedge Q \wedge R$, $\neg P \wedge Q \wedge \neg R$, $\neg P \wedge \neg Q \wedge R$, $\neg P \wedge \neg Q \wedge \neg R$

In order to obtain the PDNF form, one may first replace the conditionals and biconditionals by their equivalent formulas containing only $\wedge$, $\vee$ and $\neg$. Next, the negations are applied to the variables by using DeMorgan's laws followed by the applications of distributive laws. Any elementary product which is a contradiction is dropped. Minterms are obtained in the disjunctions by introducing missing factors. Identical minterms appearing in the disjunctions are written as one term (since $a \vee a = a \wedge a = a$).

### EXAMPLE PROBLEM 1

**Obtain the principal disjunctive normal form of,**

**(i) $P \to Q$** **(ii) $P \vee Q$**

**SOLUTION**

(i) $P \to Q \equiv (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

(ii) $P \vee Q \equiv (P \vee (Q \vee \neg Q) \vee (P \vee \neg P) \wedge P) \wedge Q$

$\equiv (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$

### EXAMPLE PROBLEM 2

**Obtain the principal disjunctive normal form of,**

**$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$**

**SOLUTION**

To obtain the principal disjunctive normal form, we supply the missing variables. Consider,

$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$\equiv (P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge (Q \vee \neg Q) \wedge R) \vee (Q \wedge R \wedge (P \vee \neg P))$

$\equiv (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

## 1.3.6.4 Principal Conjunctive Normal Forms (PCNF)

Given a number of variables, maxterm of these variables is a formula which consists of disjunctions in which each variable or its negation, but not both, appear only once. Also the maxterms are duals of minterms. For a given formula, an equivalent formula consisting of conjunctions of the maxterms only is known as its principal conjunctive normal form or the product of sums canonical form.

### EXAMPLE PROBLEM 1

**Obtain the principal conjunctive normal form for $(P \to Q) \wedge (Q \leftrightarrow R)$.**

**SOLUTION**

Let $A \equiv (P \to Q) \wedge (Q \leftrightarrow R)$

$\equiv (\neg P \vee Q) \wedge (\neg Q \vee R) \wedge (\neg R \vee Q)$

**ivalent principal disjunctive and principal conjunctive normal forms of Q ∨ R).**

quired forms using truth table.

**Table 1.3.8 :** Truth Table for $(P \wedge Q) \vee (\neg P \vee Q \vee R)$

| $P$ | $\neg P$ | $Q$ | $R$ | $P \wedge Q$ | $\neg P \vee Q$ | $\neg P \vee Q \vee R$ | $(P \wedge Q) \vee (\neg P \vee Q \vee R)$ |
|---|---|---|---|---|---|---|---|
| T | F | T | T | T | T | T | T |
| T | F | T | F | T | T | T | T |
| T | F | F | T | F | F | T | T |
| T | F | F | F | F | F | F | F |
| F | T | T | T | F | T | T | T |
| F | T | T | F | F | T | T | T |
| F | T | F | T | F | T | T | T |
| F | T | F | F | F | T | T | T |

**PDNF**

$(P \to (Q \wedge R)) \wedge (\neg P \to (\neg Q \wedge \neg R))$

$\Leftrightarrow (\neg P \vee (Q \wedge R)) \wedge (P \vee (\neg Q \wedge \neg R)$

$\Leftrightarrow [(\neg P \vee (Q \wedge R)) \wedge P] \vee [(\neg P \vee (Q \wedge R)) \wedge (\neg Q \wedge \neg R)]$

$\Leftrightarrow (\neg P \wedge P) \vee (P \wedge Q \wedge R) \vee [(\neg P \wedge (\neg Q \wedge \neg R)) \vee [(Q \wedge R) \wedge (\neg Q \wedge \neg R)]$

[by Distributive Law]

$\Leftrightarrow (P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$ $[a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)]$

which is the required principal disjunctive normal form.

## 1.4 LOGICAL IMPLICATION

### 1.4.1 Rules of Inferences

The word 'inference' will be used to designate a set of premises accompanied by a suggested conclusion regardless of whether or not the conclusion is a logical consequence of premises. There are faulty inferences and valid inference. Each inference can be written as an implication as follows,

(conjunction of premises) $\to$ conclusion.

Let us take a set of statements $P_1, P_2, P_3 \ldots\ldots P_n$ and a statement q. Then the compound statement is,

$(P_1 \wedge P_2 \wedge P_3 \wedge P_4 \ldots P_n) \to q$ is general form of an argument.

Here n is any positive integer.

The statement $P_1, P_2, P_3 \ldots\ldots P_n$ are called the premises of the argument and q is called a conclusion of the argument.

The above form of argument can be written in a tabular form,

$$\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ \vdots \\ \vdots \\ P_n \\ \hline q \end{array}$$

$\therefore$ Here q means conclusion.

The argument is valid if, the premises are $P_1, P_2, P_3 . . .P_n$ and conclusion is q, if whenever each of premises $P_1, P_2 ..... P_n$ is true then conclusion q is true i.e., the argument $(P_1 \wedge P_2 \wedge P_3 \wedge P_4 ....P_n) \rightarrow q$ is valid when $(P_1 \wedge P_2 \wedge P_3 \wedge P_4 ....P_n) \Rightarrow q$.

In the valid argument the truth value of conclusion is true. If the conclusion value is false then it is invalid argument.

## 1.4.2 Consistency Check for Rules of Inferences

Let the premises $P_1, P_2, P_3 ..... P_n$ of an argument if the conjunction $(P_1 \wedge P_2 \wedge P_3 ..... P_n)$ is a false in every possible situation. Then the arugment is called inconsistent of premises. If the conjunction $P_1 \wedge P_2 \wedge P_3 ..... P_n$ is true in atleast one possible condition then the premises of the argument is called consistent of premises.

**(1) Modus Ponens or Law of Detachment :** If the statement in P is assumed as true, and $P \rightarrow Q$ is accepted as true, then Q is true. Symbolically, we have the following pattern,

$$\begin{array}{l} P \\ P \rightarrow Q \\ \hline \therefore Q \end{array}$$

This rule is called Modus Ponens or Law of Detachment. The rule of detachment is a valid inference, because $[P \wedge (P \rightarrow Q)] \rightarrow Q$ is a tautology.

**Example :** If today is Thursday, ten days from now will be Sunday.

Today is Thursday. Hence ten days from now will be Sunday.

**(2) Rule of Syllogism (or) Transitive Rule :** Whenever the two implications $P \rightarrow Q$ and $Q \rightarrow R$ are accepted as true, we must accept the implication $P \rightarrow R$ as true. In pattern form we write,

$$\begin{array}{l} P \rightarrow Q \\ Q \rightarrow R \\ \hline \therefore P \rightarrow R \end{array}$$

This rule is a valid rule of inference, because the implication $(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is a tautology, this rule is known as Law of Hypothetical Syllogism or Transitive Rule.

The transitive rule can be extended to a large number of implications as follows.

$$\begin{array}{l} P \rightarrow Q \\ Q \rightarrow R \\ R \rightarrow S \\ \hline \therefore P \rightarrow S \end{array}$$

**Example :** If 1960 was a Leap Year, then 1964 was Leap Year. If 1964 was a Leap Year, then 1968 was a Leap Year.

Hence if 1960 was a Leap Year, then 1968 was a Leap Year.

(3) **Modus Tollens (or) Contrapositive Rule :** Contrapositive i.e., $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$.

This is also known as Modus Tollens. We write this in the following pattern,

$$\begin{array}{r} P \rightarrow Q \\ \neg Q \\ \hline \therefore \neg P \end{array}$$

(4) **Rule of Disjunctive Syllogism :** We write it as,

$$\begin{array}{r} P \vee Q \\ \neg P \\ \hline \therefore\ Q \end{array}$$

(5) **Rule of Dilemma :** Dilemma can be written as,

$$\begin{array}{r} P \vee Q \\ P \rightarrow R \\ Q \rightarrow R \\ \hline \therefore\ \rightarrow R \end{array}$$

(6) **DeMorgan's Laws**

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

EXAMPLE PROBLEM 1

## EXAMPLE PROBLEM 2

**Test whether the following is a valid argument,**

**If Sachin hits a Century, then he gets a free Car.**

**Sachin does not get a free car**

**∴ Sachin has not hit a century**

**SOLUTION**

Let p : Sachin hits a century.

q : Sachin gets a free car.

Then the given argument reads,

$$\begin{array}{l} p \to q \\ \underline{\quad \neg q} \\ \therefore \ \neg p \end{array}$$

In view of the Modus Tollens Rule, this is a valid argument.

## EXAMPLE PROBLEM 3

**Test whether the following is a valid argument.**

**If Sachin hits a Century, then he gets a free Car.**

**Sachin get a free car**

**∴ Sachin has hit a century**

**SOLUTION**

Let p : Sachin hits a century.

q : Sachin gets a free car.

Then the given argument reads,

$$\begin{array}{l} p \to q \\ \underline{\quad\ \ q} \\ \therefore \ \ p \end{array}$$

We note that if $p \to q$ and q are true, there is no rule which asserts that p must be true. Indeed, p can be false when $p \to q$ and q are true.

Thus, $[(p \to q) \wedge q] \neq> p$. Therefore, the given argument is not a valid one.

## 1.4.3 Tabulation of Some other Rules along with above Ru

**Table 1.4.1 :** Rules of Inferences

| S.No. | Name of Rule | Rule of Inference | Related Logical Implication |
|---|---|---|---|
| 1) | Rule of Detachment (Modus Ponens) | $\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$ | $[p' \wedge (p \rightarrow q)] \rightarrow q$ |
| 2) | Law of the Syllogism | $\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$ | $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ |
| 3) | Modus Tollens | $\begin{array}{l} p \rightarrow q \\ \underline{\neg q} \\ \therefore \neg p \end{array}$ | $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$ |
| 4) | Rule of Conjunctive Simplification | $\begin{array}{l} p \\ \underline{q} \\ \therefore p \end{array}$ (or) $\begin{array}{l} p \\ \underline{q} \\ \therefore q \end{array}$ | $p \wedge q \rightarrow p$ (or) $p \wedge q \rightarrow q$ |
| 5) | Rule of Disjunctive Simplification | $\dfrac{p}{\therefore p \vee q}$ (or) $\dfrac{q}{\therefore p \vee q}$ | $p \rightarrow p \vee q$ (or) $q \rightarrow p \vee q$ |
| 6) | Rule of Disjunctive Syllogism | $\begin{array}{l} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$ | $[(p \vee q) \wedge \neg p] \rightarrow q$ |

## 1.4.4 Proof by Contradiction (Consistency of Premises and Indirect Method Proof)

If R is any formula, then $R \wedge \neg R$ is a contradiction. A set of formulas $A_1, A_2, \ldots A_m$ is said to be consistent if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in the formulas $A_1, A_2, \ldots A_m$ otherwise, formulas are inconsistent. In other words, $A_1, A_2, \ldots A_m$ are said to be inconsistent if for every assignment of the truth values to the variables, at least one of the formulas $A_1, A_2, \ldots A_m$ is false, i.e., their conjunction is identically false. i.e., $A_1 \wedge A_2 \wedge \ldots \wedge A_m \rightarrow R \wedge \neg R$

The notion of inconsistency is used in a procedure called Proof by Contradiction or Indirect Method of Proof.

In order to prove that a conclusion C follows from a set of premises $A_1, A_2, \ldots A_m$ by using the method proof by contradiction, the procedure is as follows.

Assume that C is false and consider $\neg C$ as an additional premises. If the new set of premises $\neg C, A_1, A_2, \ldots A_m$ is inconsistent, they imply a contradiction, then the assumption that $\neg C$ is true does not hold simultaneously with $A_1, A_2, \ldots A_m$ being true. Therefore, C is true whenever $A_1 \wedge A_2 \wedge \ldots \wedge A_m$ is true. Thus, the conclusion C follows from the premises $A_1, A_2, \ldots A_m$.

Proof by contradiction is sometimes very much convenient. But, if required it can be replaced by another method called a conditional proof which can be given as follows.

$$P \rightarrow (Q \wedge \neg Q) \rightarrow \neg P \qquad \ldots (1)$$

In the proof by contradiction, we show,

$$A_1, A_2, \ldots A_m \rightarrow C$$

By showing $A_1, A_2, \ldots A_m \neg C \Rightarrow R \wedge \neg R$

Now we convert this into,

$$A_1, A_2, \ldots A_m \Rightarrow (R \wedge \neg R) \qquad \ldots (2)$$

From (1) and (2), by using the rule of double negation, one can prove that $A_1, A_2, \ldots A_m \rightarrow C$ and this is the required derivation.

EXAMPLE PROBLEM 1

**Provide a proof by contradiction for the following.**

**For every integer n, if $n^2$ is odd, then n is odd.**

SOLUTION

Here the condition to be proved is $p \to q$, where

$p$ : $n^2$ is odd and

$q$ : n is odd.

We first prove that the contrapositive $\neg q \to \neg p$ is true. Assume that $\neg q$ is true, that is, assume that n is not an odd integer. Then $n = 2k$, where k is an integer. Consequently, $n^r = (2k)^r = 2(2k^2)$ so that $n^2$ is not odd. i.e., p is flase, or equivalently $\neg p$ is true. This proves the contrapositive statement $\neg q \to \neg p$. This proof of the contrapositive $\neg q \to \neg p$ serves as an indirect proof of the given statement $p \to q$.

## 1.4.5 Solved Problems

SOLVED PROBLEM 1

**Test the validity of the following argument.**

**If I study,I will not fail in the examination.**

**If I do not watch TV in the evenings, I will study.**

**I failed in the examination.**

**Therefore, I must have watched TV in the evenings.**

SOLUTION

Let $p$ : I study.

$q$ : I fail in the examination.

$r$ : I watch TV in the evenings.

Then, the given argument reads,

$$\begin{array}{r} p \to \neg q \\ \neg r \to p \\ q \\ \hline \therefore\ r \end{array}$$

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**SOLUTION**

(i) Let $B(x)$ : x is a bird.

$F(x)$ : x can fly.

Then the given statement can be written as $\forall x\, [B(x) \rightarrow F(x)]$ where universe of discourse is the set of all birds.

(ii) Let $B(x)$ : x is a baby.

$I(x)$ : x is illogical.

Then the given statement can be written as $\exists x\, [B(x) \wedge I(x)]$ where the universe of discourse is the set of all babies.

(iii) Let $M(x)$ : x is a man.

$G(x)$ : x is a giant.

Then the given statement can be written as $\exists x\, [M(x) \wedge G(x)]$ where universe of discourse is the set of men in the world.

(iv) Let $S(x)$ : x is student.

$M(x)$ : x likes mathematics.

$H(x)$ : x likes history.

Then the given statement can be written as $\exists x\, [S(x) \wedge M(x) \wedge \neg H(x)]$ where universe of discourse is the set of all students at your college.

## 1.5.1 Free and Bound Variables

When a quantifier is used as a variable x or when we assign a value to this variable, we say that this occurrence of the variable is bound. An occurrence of a variable that is not bound by a quantifier is said to be free. All the variables that occur in a propositional function must be bound to turn it into a proposition. This can be done using a combination of universal quantifiers, existential quantifiers and value assignments.

A statement which includes free variables is not a proposition but only a predicate and a predicate in which all the variables have been bound is in fact a proposition.

The part of the logical expression or predicate formula to which a quantifier is applied is called the scope of the quantifier.

## EXAMPLE PROBLEM 1

**Let P(x, y) : x is elder than y.**

**Q(x, y) : x is stronger than y.**

**R(x) : x is brave.**

**Consider the following symbolic statements and translate them in the english.**

**(i)** $\forall x \exists y [p(x, y) \to Q(x, y) \wedge R(y)]$

**(ii)** $\forall x [\forall y \neg P(x, y) \to R(x)]$

**SOLUTION**

(i) "For all people x, there is a y such that, if x is elder than y, then x is stronger than y and y is brave".

(ii) "For all people x, if the statement x is elder than y is false for all y, then x is brave".

## EXAMPLE PROBLEM 2

**Given R(x, y) : x + y is even and the variables x and y represent integers. Write an English sentence corresponding to each of the following.**

**(i)** $-\forall x \exists y R(x, y)$

**(ii)** $-\exists x \forall y R(x, y)$

**SOLUTION**

(i) for all x their exists y such that x + y is even.

(ii) Their exists x such that for all y, x + y is even.

## 1.5.2 Negations of Quantified Predicates

(1) A universally quantified predicate is false, then there is some value of its variable which makes the predicate false.

(2) An existentially quantified statement is false, then the only v. :y is, the predicate is never true.

(i) $\neg \forall x P(x) \equiv \exists x \neg P(x)$

(ii) $\neg \exists x P(x) \equiv \forall x \neg P(x)$

**Example :** If P(x) is the predicate "x is a good fighter." with UOD of x being a boxer.

The universally quantified statement is,

$\forall x P(x)$ : "All boxers are good fighters".

**(iii)** **There is somebody whom no one loves.**

**(iv)** **There is exactly one person whom everybody loves.**

**(v)** **Everyone loves himself or herself.**

**SOLUTION**

(i) $\forall x \exists y P(x,y)$

(ii) $\forall x \exists y \neg P(x,y)$

(iii) $\exists x \forall y \neg P(y,x)$

(iv) $\exists x(\forall y P(y,x) \wedge \forall z((\forall w P(w,z)) \rightarrow z = x)$

(v) $\forall x\, P(x,x)$

## 1.5.3 Well Formed Formulas for Quantifiers

A string containing a part of the form $\forall$, $\exists$ besides the other connectives is called a formula iff it satisfies all the following properties.

(1) A statement variable standing alone is a well formed formula.

(2) If P is a formula, then $\neg P$ is also a formula.

(3) If P and Q are formulas, then $P \wedge Q, P \vee Q, P \rightarrow Q$ and $P \leftrightarrow Q$ are also formulas.

(4) A string of symbols containing the statement variables, connectives and parenthesis is a well formed formula, iff it can be obtained by finitely many applications of the rules 1, 2 and 3.

(5) If P is a formula and x is a letter free in P then $\forall$ x, P is a formula.

(6) If P is a formula and x is some letter free in P, then $\exists$ x, P is a formula.

As examples, consider the following formulas,

$\forall x P(x,y)$ . . . (1)

$\forall x P(x) \rightarrow Q(x)$ . . . (2)

$\forall x (P(x) \rightarrow \exists y R(x,y))$ . . . (3)

$\forall x (P(x) \rightarrow R(x)) \vee \forall x (P(x) \rightarrow Q(x))$ . . . (4)

$\exists x (P(x) \wedge Q(x))$ . . . (5)

$(\exists\, x) P(x) \wedge Q(x)$ . . . (6)

In (1), both the occurrences of x are bound, while the variable y is free.

In (2), all occurrences of x are bound.

In (3), all the occurrences of both x and y are bound.

In (4), (5) all the occurrences of x are bound.

In (6), the last occurrence of x in Q(x) is free.

Sticktly speaking, generally, when we are considering the content of sentences, we should pay attention to atleast five elements.

(i) The subject

(ii) The predicate

(iii) The quantifier

(iv) The quality

(v) The universe of discourse using the modifier $\neg$ and the quantifiers $\forall$ and $\exists$, we can form eight different expressions involving the open proposition F(x).

**For Example :** $\forall$ x, [$\neg$F(x)] means "for each x in the universe F(x) is false" or "all false". Let us list these eight quantified statements.

**Table 1.5.1 :** Quantified Statements.

| ***Sentence*** | ***Abbreviated Meaning*** |
|---|---|
| $\forall x\, F(x)$ | All true |
| $\exists x\, F(x)$ | Atleast one true |
| $\neg[\exists x\, F(x)]$ | None true |
| $\forall x[\neg F(x)]$ | All false |
| $\exists x\,[\neg F(x)]$ | Atleast one false |
| $\neg\{\exists x\,[\neg F(x)]\}$ | None false |
| $\neg\{\forall x\,[F(x)]\}$ | Not all true |
| $\neg\{\forall x\,[\neg F(x)]\}$ | Not all false |

Now, the eight expressions can be grouped into four groups of two each. Where the two have the same meaning. We list these four types as equivalences.

(1) "all true" $\{\forall x\, F(x)\} \equiv \{\neg[\exists x, \neg F(x)]\}$ none false.

(2) "all false" $\{\forall (x)[\neg F(x)]\} \equiv \{\neg[\exists x\ F(x)]\}$ none true.

(3) "not all true" $\{\neg[\forall x\, F(x)]\} \equiv \{\exists\, x[\neg F(x)]\}$ atleast one false.

(4) "not all false" $\{\neg[\forall x\, (\neg F(x))]\} \equiv \{\exists\, x\, F(x)\}$ atleast one true.

The equivalences also provide information about the negation of this type of quantified statement. In the first statement, we have $\forall x, F(x)$, its negation $\neg[\forall x,\ F(x)]$ occurs in the third statement. Thus, the negation of "all true" is atleast "one false". We list these facts as follows.

**Table 1.5.2 :** Grouping the Quantified Statements

| ***Statement*** | | ***Negation*** | |
|---|---|---|---|
| $\forall x\, F(x)$ | all true | $\exists\ x\ \neg F(x)$ | atleast one false |
| $\exists\, x\, [\neg F(x)]$ | atleast one false | $\forall\ x\ F(x)$ | all true |
| $\forall x\ \neg F(x)$ | all false | $\exists\ x\ F(x)$ | atleast one true |
| $\exists\, x\, F(x)$ | atleast one true | $\forall\ x\ [\neg F(x)]$ | all false |

i.e., to form the negation of a statement involving one quantifier, we need to change only the quantifier from universal to existential or from existential to universal and negate the statement which it quantifies.

## 1.5.4 Solved Problems

SOLVED PROBLEM 1

(iii) $\forall x(P(x) \wedge (q(x) \wedge r(x))) \Leftrightarrow \forall x(P(x) \wedge q(x) \wedge r(x))$

To show that the above statement is logically equivalent, we have to follow as, For each a in the universe then the statement is,

$(P(a) \wedge (q(a) \wedge r(a)))$ and $((P(a) \wedge q(a) \wedge r(a)))$

By the associative law, we have $[P(a) \wedge (q(a) \wedge r(a))] \Leftrightarrow [(P(a) \wedge q(a)) \wedge r(a)]$

Similarly, for the statement functions,

$(P(x) \wedge q(x) \wedge r(x))$ and $(P(x) \wedge q(x) \wedge r(x))$ is

$\therefore \quad \forall x[P(x) \wedge (q(x) \wedge r(x))] \Leftrightarrow [(P(x) \wedge q(x)) \wedge r(x)]$.

## 1.6 QUANTIFIERS, DEFINITIONS AND THE PROOF OF THEOREMS

**[Nov./Dec. - 2008]**

Additional rules of inference are necessary to prove assertions involving open propositions and quantifiers. The following four rules describe when the universal and existential quantifiers can be added to or deleted from an assertion.

**Rule 1**

**Universal Specifications (US) :** If a statement of the form $\forall x \; P(x)$ is assumed to be true, then the universal quantifier can be dropped to obtain P(c) is true for an arbitrary object c in the universe. This rule may be represented as,

$$\frac{\forall x \; P(x)}{\therefore p(c) \text{ for all } c}$$

**Rule 2**

**Universal Generalisation (UG) :** If a statement P(c) is true for each element c of the universe, then the universal quantifier may be prefixed to obtain $\forall x \; P(x)$. In symbols this rule is,

$$\frac{P(c) \text{ for all } c}{\therefore \forall x \; P(x)}$$

**Rule 3**

**Existential Specification (ES) :** If $\exists x \; P(x)$ is assumed to be true, then there is an element c in the universe such that P(c) is true. Symbolically,

$$\frac{\exists x \; P(x)}{\therefore P(c) \text{ for some } c}$$

**Rule 4**

**Existential Generalisation (EG) :** If P(c) is true for some element c in the universe, then $\exists x\, P(x)$ is true. In symbols, we have

$$\frac{P(c) \text{ for some } c}{\therefore \exists x\, P(x)}$$

We shall frequently use some of the equivalences and implications given in Table 1.6.1 for the derivation of a conclusion from a set of premises.

**Table 1.6.1 :** Equivalence and Implications

| | |
|---|---|
| $\exists x(A(x) \vee B(x)) \Leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$ | $E_1$ |
| $(x)(A(x) \wedge B(x)) \Leftrightarrow (x)A(x) \wedge (x)B(x)$ | $E_2$ |
| $\neg(\exists x A(x)) \Leftrightarrow (x)\neg A(x)$ | $E_3$ |
| $\neg(x)A(x) \Leftrightarrow \exists x, \neg A(x)$ | $E_4$ |
| $(x)A(x) \vee (x)B(x) \Leftrightarrow (x)(A(x) \vee B(x))$ | $E_5$ |
| $\exists x(A(x) \wedge B(x)) \Leftrightarrow (\exists x)A(x) \wedge (\exists x)B(x)$ | $E_6$ |
| $(x)(A \vee B(x)) \Leftrightarrow A \vee (x)B(x)$ | $E_7$ |
| $\exists x(A \wedge B(x)) \Leftrightarrow A \wedge \exists x\, B(x)$ | $E_8$ |
| $(x)A(x) \rightarrow B \Leftrightarrow (x)(A(x) \rightarrow B)$ | $E_9$ |
| $\exists x A(x) \rightarrow B \Leftrightarrow \exists(x)(A(x) \rightarrow B)$ | $E_{10}$ |
| $A \rightarrow (x)B(x) \Leftrightarrow (x)(A \rightarrow B(x))$ | $E_{11}$ |
| $A \rightarrow \exists x\, B(x) \Leftrightarrow (\exists x)(A \rightarrow B(x))$ | $E_{12}$ |

EXAMPLE PROBLEM 1

## 1.7 SETS AND SUBSETS

A set is a collection of well defined and well distinguished objects. The objects that make up a set are called the members or elements of the set. It is almost a convention to indicate sets by capital letters like A, B C, or P, Q, R while the elements in the set by smaller letters. Now, to indicate that a particular element or object "belongs to a set" or "a member of the set", we use the greek symbol epsilon "$\in$". For example, if x is the member of a set A, we write it as $x \in A$, if x is not an element of A, then we write $x \notin A$.

**Methods of Describing a Set :** There are two methods to describe a set.

(1) To list the elements called the extension method, or

(2) To indicate the nature or characteristics and the limits with in which the elements lie.

These two approaches have been named variously as Tabular, roaster or enumeration method and Selector, property builder or rule method.

**(1) Tabular Method :** Under this method, we list all the elements of the set.

**Examples**

(i) A set of vowels A = {a, e, i, o, u}.

(ii) A set of odd natural numbers N = {1, 3, 5, 7, . . .}.

**(2) Selector Method :** Under this method, the elements are not listed, but are indicated by description of their characteristics.

**Examples**

(i) A = {x : x is vowel in english alphabet}

(ii) B = {x : x is a natural number and $2 < x < 11$}

### 1.7.1 Types of Sets

**(1) Null Set or Empty Set :** A set having no element is called as an empty set or void set. It is denoted by $\phi$ or { }.

**Examples**

(i) {x : x is an even number not divisible by 2}.

(ii) $\{x : x \in R \text{ and } x^2 < 0\}$.

(2) **Finite Set :** A set having a finite number of elements is called a finite set.

**Examples :**

(i) $A = \{x : x \text{ is an even positive integer} \leq 100\}$.

(ii) A null set $\phi$, is also a finite set, because it has zero number of elements.

(3) **Infinite Set :** A set having infinite number of elements is called an infinite set.

**Examples**

(i) $A = \{x : x \text{ is a natural number}\}$.

(ii) $B = \{x : x \text{ is a positive integer divisible by } 5\}$.

(4) **Subset :** Set A is said to be a subset of B, if each element of set A is also an element of set B. If A is a subset of B, then we write $A \subseteq B$.

So, if $A \subseteq B \Leftrightarrow \{x \in A \Leftrightarrow x \in B\}$. Set A is said to be a proper subset of a set B if

(i) Every element of set A is an element of set B.

(ii) Set B has atleast one element which is not an element of set A. This is expressed by writing $A \subset B$. or $B \supset A$ and read as A is a proper subset of B.

**Examples :** Let $A = \{4, 5, 6\}$

$B = \{4, 5, 7, 8, 6\}$

So, $A \subset B$

(5) **Comparable Sets :** Two sets A and B are said to be comparable, if either of these happens.

(i) $A \subset B$

(ii) $A = B$

(iii) $B \subset A$

**Example :** $A = \{1, 2, 3\}$ and $B = \{1, 2\}$ are comparable, but $A = \{1, 2, 3\}$ and $B = \{2, 3, 6, 7\}$ are incomparable.

(6) **Universal Sets :** Any set which is super set of all the sets under consideration is known as the universal set and is denoted by S or X or U.

**Example**

Let $A = \{1, 2, 3\}$,

$B = \{3, 4, 6, 9\}$, and

$C = \{0, 1\}$

We can take $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as universal set.

The universal set can be chosen arbitrarily for discussion, but once chosen, it is fixed for the discussion.

(7) **Power Set :** The family of all the subsets of a given set A is said to be the power set of A and is expressed by P(A).

So, $P(A) = \{x : x \subseteq A\}$.

**Examples**

If $A = \{1\}$, then $P(A) = \{\phi, \{1\}\}$.

If $A = \{1, 2\}$, then $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$.

If $A = \{1, 2, 3\}$, then $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$.

(i) $\phi \in P(A)$ and $A \in P(A)$ for all sets A.

(ii) The elements of P(A) are subsets of A.

(iii) If A has n elements, then P(A) has $2^n$ elements.

## 1.8 VENN DIAGRAMS

The Venn diagrams are named after English logician John Venn to present pictorial representation of sets. The universal set say, U or X is denoted by a region enclosed by a rectangle and one or more sets say A, B, C are shown through circles or closed curves with in these rectangles. These circles or closed curves intersect eachother if there are any common elements amongst them, if there are no common elements, then, they are shown separately as disjoint sets. Several set relations can be easily shown by these diagrams. These are useful to illustrate the set relations, such as the subset, and the set operations such as intersection, union, complementation.

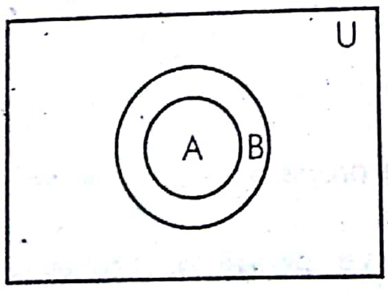



**Fig. 1.8.1 :** $A \subset B$ (Subset)

# 1.9 SET OPERATIONS AND THE LAWS OF SET THEORY

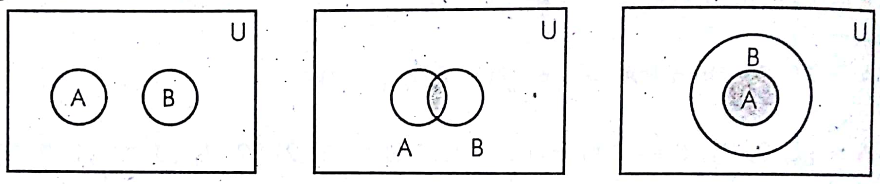
## 1.9.1 Intersection of Two Sets

The intersection of two sets A and B is the set consisting of all the elements which belong to both A and B and is denoted by $A \cap B$.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Two sets are said to be disjoint, if they have no element in common. i.e., $A \cap B = \phi$

The Fig. 1.9.1 shows how the intersection of sets can be expressed by Venn diagrams.




(a) $A \cap B = \phi$ (b) $A \cap B = \boxed{\phantom{x}}$ (b) $A \cap B = \boxed{\phantom{x}} = A$

**Fig. 1.9.1 :** Intersection of Two Sets A and B

**Examples**

(1) Let $A = \{a, b, c, d\}$ and $B = \{b, d, e, f\}$

then $A \cap B = \{b, d\}$

(2) If $A = \{x : x \text{ is an integer}, 1 \le x \le 40\}$ and $B = \{x : x \text{ is an integer}, 21 \le x \le 100\}$

then $A \cap B = \{x : x \text{ is an integer}, 21 \le x \le 40\}$

**Some Important Properties of Intersection**

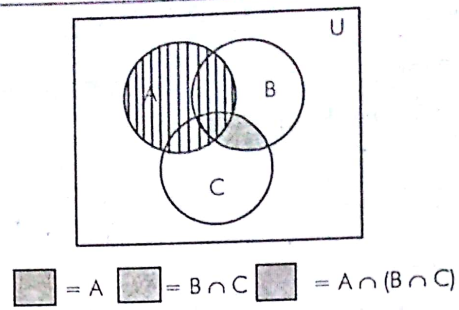
(i) $A \cap B$ is the subset of both the sets A and B i.e., $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

(ii) $A \cap \phi = \phi$ for every set A.

(iii) $A \cap A = A$ for every set A.

(iv) Intersection has commutative property i.e., $A \cap B = B \cap A$.

(v) Intersection has associative property. For any three sets A, B, and C i.e., $A \cap (B \cap C) = (A \cap B) \cap C$.

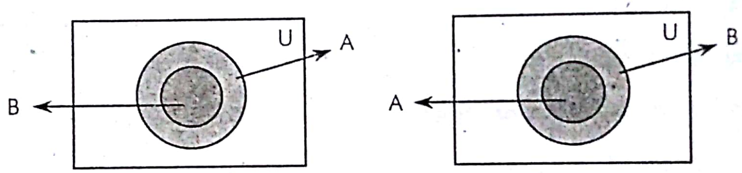



(b)

**Fig. 1.9.2**

(vi) If $B \subseteq A$ then, $A \cap B = B$ and if $A \subseteq B$, then $A \cap B = A$.

This, we will illustrate with the help of a Venn diagram.




(a) $B \subseteq A$ So $A \cap B = B$

(b) $A \subseteq B$ So $A \cap B = A$

**Fig. 1.9.3**

(vii) If $A \subseteq B$ and $B \subseteq C$ then, $A \subseteq (B \cap C)$.

A collection of sets is called a disjoint collection if, for every pair of sets in the collection, the two sets are disjoint. The elements of a disjoint collection are said to be mutually disjoint. Let A be an indexed set, $A = [A_i]_{i \in I}$. The set A is a disjoint collection iff $A_i \cap A_j = \phi$ for all i, j and $i \neq j$.

## 1.9.2 Union of Sets

The union of two sets A and B is the set consisting of all the elements which belong to either A or B or both and is denoted by $A \cup B$.

$A \cup B = \{x : x \in A \text{ or } x \in B, \text{ or } x \in \text{ both } A, B\}$.

The union of two sets A and B is also called the logical sum of A and B.

**Example**

(1) Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 5, 6\}$ and $C = \{3, 4, 6, 8\}$.

Then, $A \cup B = \{1, 2, 3, 4, 5, 6\}$ and

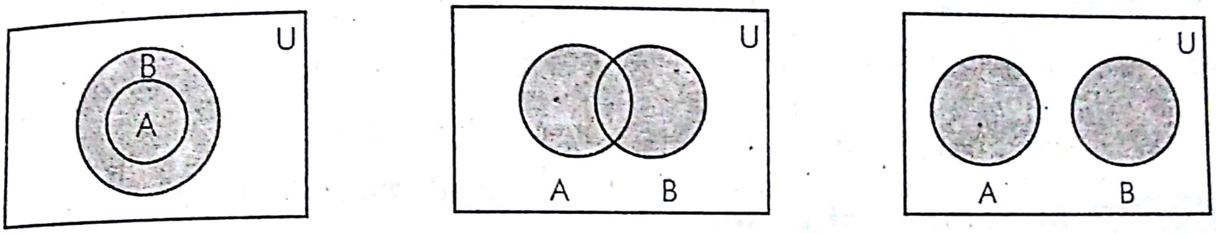
$(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 8\}$,

$B \cup C = \{2, 3, 4, 5, 6, 8\}$, and

$A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6, 8\}$

$A \cup (B \cup C) = (A \cup B) \cup C$

Union of two sets can be illustrated by Venn diagram as shown in Fig. 1.9.4.




(a) $A \cup B = B$ when $A \subset B$ (b) $A \cup B$ when neither $A \subset B$ or $B \subset A$ (c) $A \cup B$ when A and B are disjoint

**Fig. 1.9.4**

**Some Important Properties of Union**

(i) $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

(ii) $A \cup \phi = A$ for every set A.

(iii) $A \cup A = A$ for every set A.

(iv) Union satisfies commutative property $A \cup B = B \cup A$ for any two sets A and B.

(v) It has an associative property i.e., for any three sets A, B and C

$(A \cup B) \cup C = A \cup (B \cup C)$

**Proof :** Let $x \in (A \cup B) \cup C$

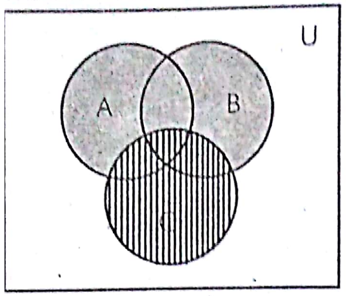
$\Leftrightarrow x \in A \cup B$ or $x \in C$

$\Leftrightarrow x \in A$ or $x \in B$ or $x \in C$

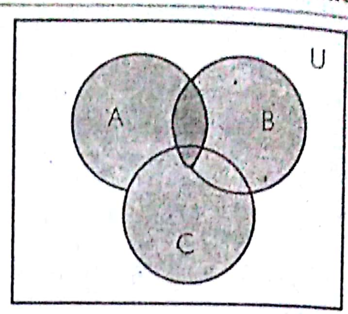
$\Leftrightarrow x \in A$ or $x \in (B \cup C)$

$\Leftrightarrow x \in A \cup (B \cup C)$

$\therefore (A \cup B) \cup C = A \cup (B \cup C)$




ı) Shaded Region : $(A \cup B) \cup C$




(b) Shaded Region : $A \cup (B \cup C)$

**Fig. 1.9.5**

$B \subseteq A$, then, $A \cup B = A$ and if $A \subseteq B$, then $A \cup B = B$.

$\cup B = \phi \Rightarrow A = \phi$ and $B = \phi$. i.e., both A and B are null sets.

$\cap B \subset A \subset A \cup B$.

## Complement of a Set

ıe complement of a set is the set of all those elements which do not belong to the set. rwords, if 'U'be the universal set and A be any set, then the complement of set A is U d is denoted as. $A'$, $A^c$ or $\overline{A}$.

$= U - A = \{x : x \in U \text{ and } x \notin A\}$

**les**

$U = \{1, 3, 5, 9, 10, 18\}$ and $A = \{3, 5, 10\}$

en $\overline{A} = \{1, 9, 18\}$.

ɜnn diagram showing the complement of the set A in set 'U' is given in Fig. 1.9.6.

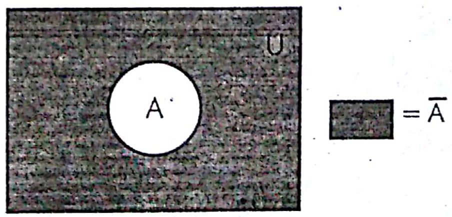



**Fig. 1.9.6 :** Venn Diagram for $\overline{A}$

**Some Important Properties of Complementation**

(1) $A \cap \bar{A} = \phi$

(2) $A \cup \bar{A} = U$

(3) $\bar{U} = \phi$ and $\bar{\phi} = U$

(4) $\overline{(\bar{A})} = A$

(5) If $A \subseteq B$, then $\bar{B} \subseteq \bar{A}$

**Ex. 2)** Complement of B with respect to A : (A – B).

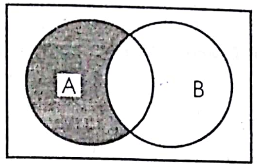



**Fig. 1.9.7 :** Venn Diagram for A – B

## 1.9.4 Difference of Two Sets

The difference of two sets A and B is the set of all those elements which belong to A and not to B and is denoted by A – B.

$A - B = \{x : x \in A \text{ and } x \notin B)$

$B - A = \{x : x \in B \text{ and } x \notin A)$

Difference of two sets can be shown by Venn diagram as shown in Fig. 1.9.8.

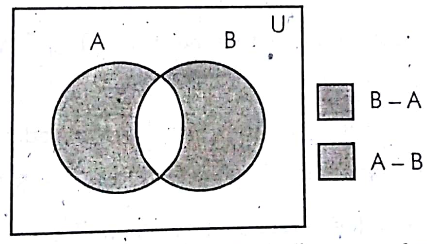



**Fig. 1.9.8 :** Venn Diagram for Differences of Two Sets

**Symmetric Difference :** Let A and B be any two sets. The symmetric difference or Boolean sum of A and B is the set A + B is defined by $A + B = (A - B) \cup (B - A)$.

**Examples**

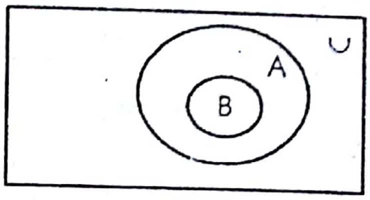
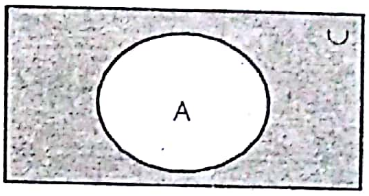
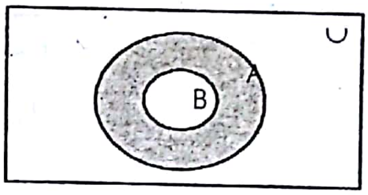
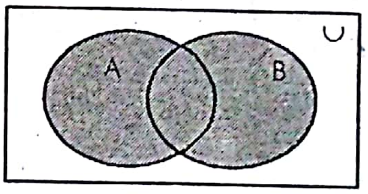
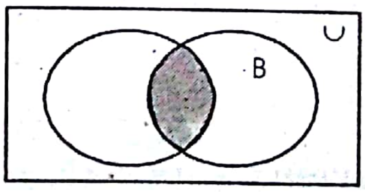
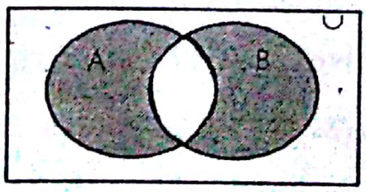
**Ex. 1)** If $A = \{a, b, c, d, e\}$ and $B = \{c, d, e, f, g\}$

then $A - B = \{a, b\}$ and $B - A = \{f, g\}$

$A \Delta B = (A - B) \cup (B - A) = \{a, b\} \cup \{f, g\} = \{a, b, f, g\}$

## 1.9.5 Venn Diagrams of Set Operations

**Table 1.9.1 :** Representation of Venn Diagrams for Set Operations

| Set Operation | Symbol | Venn Diagram |
|---|---|---|
| Set B is Contained in Set A | $B \subset A$ |  |
| Complement of A (Shaded region) | $A^C$ ($A'$ or $\bar{A}$) |  |
| The relative complement of set B w.r.t. A (Shaded region) | $A - B$ |  |
| The Union of two sets A and B (Shaded region) | $A \cup B$ |  |
| The Intersection of sets A and B (Shaded region) | $A \cap B$ |  |
| The Symmetrical difference of sets A and B (Shaded region) | $A \Delta B$ |  |

| | |
|---|---|
| Idempotent Laws | $A \cup A = A$<br>$A \cap A = A$ |
| Associative Laws | $(A \cup B) \cup C = A \cup (B \cup C)$<br>$(A \cap B) \cap C = A \cap (B \cap C)$ |
| Commutative Laws | $A \cup B = B \cup A$<br>$A \cap B = B \cap A$ |
| Distributive Laws | $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$<br>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ |
| Absorption Laws | $A \cup (A \cap B) = A$<br>$A \cap (A \cup B) = A$ |
| Complement and DeMorgans' Laws | $(\overline{A \cup B}) = \bar{A} \cap \bar{B}$<br>$(\overline{A \cap B}) = \bar{A} \cup \bar{B}$<br>$\bar{\phi} = U$<br>$\bar{U} = \phi$<br>$(\bar{\bar{A}}) = A$<br>$A \cup \phi = A$<br>$A \cap U = A$<br>$A \cup U = A$<br>$A \cap \phi = \phi$<br>$A \cup \bar{A} = U$<br>$A \cap \bar{A} = \phi$ |

**ıs and Solved Problems**

(ii)

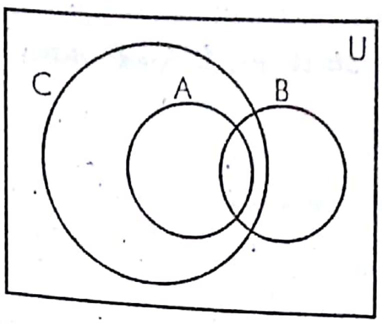



**Fig. 1.9.13 :** $A \cap B \subset A \cap C$ but $B \not\subseteq C$

(iii)

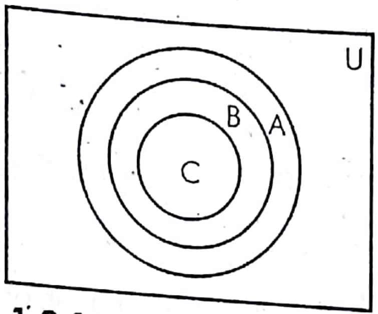



**Fig. 1.9.14 :** $A \cup B = A \cup C$ but $B \neq C$

(iv)

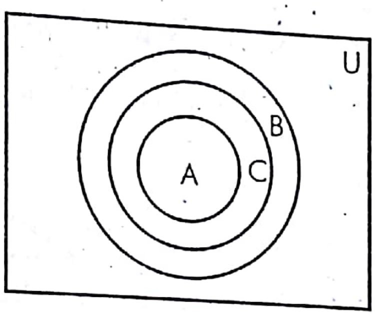



**Fig. 1.9.15 :** $A \cap B = A \cap C$ but $B \neq C$

Here $B \subseteq A \Rightarrow A \cup B = A$

$C \subseteq A \Rightarrow A \cup C = A$

Here $A \subseteq B \Rightarrow A \cap B = A$

$A \subseteq C \Rightarrow A \cap C = A$

## 1.10 COUNTING

There are two fundamental principles of counting called the addition principle or the principle of disjunctive counting, and the multiplication principle called the principle of sequential counting.

### 1.10.1 Sum Rule (or) the Principle of Disjunctive Counting

If a set S is the union of disjoint non empty subsets $A_1, A_2 \ldots A_n$ then $|S| = |A_1| + |A_2| + \ldots + |A_n|$ which also be stated as if $E_1, E_2 \ldots E_n$ are mutually exclusive events and $E_1$ can happen in $e_1$ ways, $E_2$ can happen in $e_2$ ways and so on $E_n$ can happen in $e_n$ ways then either $E_1$ or $E_2$ or . . . or $E_n$ can happen in $e_1 + e_2 + \ldots e_n$ ways.

PROBLEM 1

the number of ways of selecting a card which is either a spade or a diamond?

N

$E_1$ be the event of selecting a spade.

, $E_1$ can be performed in 13 ways.

t $E_2$ be the event of selecting a diamond.

o, $E_2$ can be performed in 13 ways.

lso, $E_1$ and $E_2$ are clearly mutually exclusive.

So, by sum rule $E_1$ or $E_2$ can be performed in $13 + 13 = 26$ ways.

## 0.2 Product Rule (or) the Principle of Sequential (or) Conjunctive Counting

If $S_1, S_2 \ldots S_n$ are non empty sets, then the number of elements in the cartesian product $\times S_2 \times \ldots \times S_n$ is the product $\prod_{i=1}^{n} |S_i|$ i.e., for any two sets A and B if A has m elements and has n elements then the Cartesian product A x B will have mn elements.

Let A = {1, 2, 3, 4}, and B = {a, b, c} then A x B is given by the tree of ig. 1.10.1.

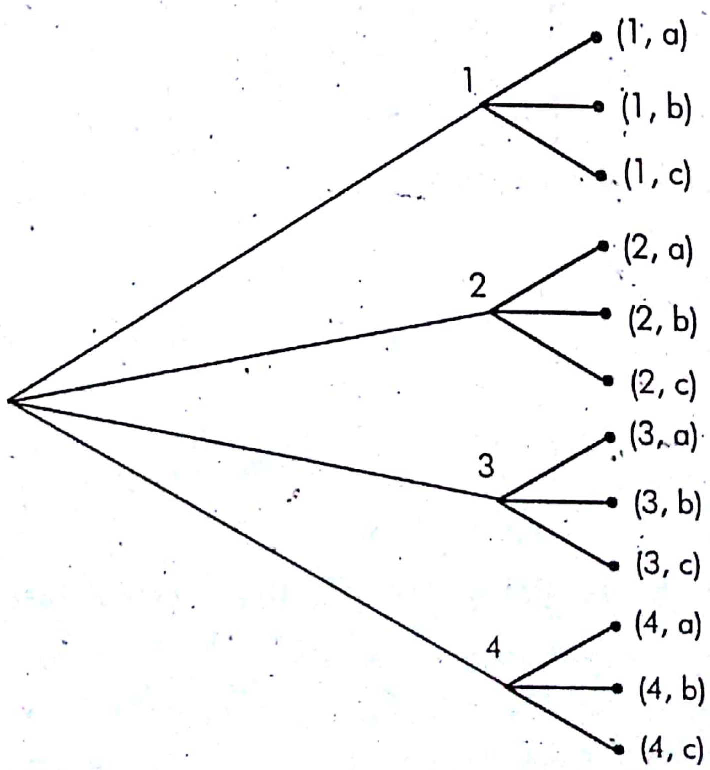



**Fig. 1.10.1 :** Representing the Elements in Tree Form using Product Rule (A × B)

The product can also be stated as "If events $E_1, E_2 \ldots E_n$ can happen $e_1, e_2 \ldots e_n$ ways respectively then the sequence of events $E_1$ first followed by $E_2 \ldots$ followed by $E_n$ can happen in $e_1\ e_2 \ldots e_n$ ways.

EXAMPLE PROBLEM 1

**Suppose a person has 4 shirts and 5 pants. Then find the number of ways that he can wear his dress.**

SOLUTION

Let $E_1$ be the event of wearing a shirt and it can be performed in 4 ways.

Let $E_2$ be the event of wearing a pant and it can be performed in 5 ways. The number of ways the person can wear his dress is the number of ways $E_1$ and $E_2$ can be performed in succession.

i.e., $4 \times 5 = 20$ ways. [by Product Rule]

## 1.10.3 Applications of Counting on Set Theory

The number of distinct elements of finite set is denoted by n(A) or |A| and is also called cardinality of set A.

(1) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(2) $n(A) = n(A - B) + n(A \cap B)$

(3) $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$

(4) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(5) In general, $A_1, A_2, \ldots A_m$ are any sets.

$$n(A_1 \cup A_2 \cup \ldots A_m) = \sum_{i=1}^{m} n(A) - \sum_{i,j}^{n} |A_i \cap A_j| + \sum_{i,j,k}^{n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{m-1} |A_1 \cap A_2 \cap \ldots \cap A_m|$$

$|A_2 \cap A_3 \cap A_4| = 9$

$|A_1 \cap A_2 \cap A_3 \cap A_4| = 1$

$$\therefore \quad |A_1 \cup A_2 \cup A_3 \cup A_4| = (1000 + 900 + 900 + 900) - (100 + 100 + 100 + 90 + 90 + 90) + (10 + 10 + 10 + 9) - 1$$

$$= 3700 - 570 + 39 - 1$$

$$= 3168.$$

SOLVED PROBLEM 17

**Suppose $A_1, A_2 \ldots A_{30}$ are thirty sets each having 5 elements and $B_1, B_2 \ldots B_n$ are n sets each having 3 elements.**

**Let $\bigcup_{i=1}^{30} A_i = \bigcup_{i=1}^{n} B_i = S$ and each element of S belongs to exactly 10 of $A_i$'s and exactly 9 of $B_i$'s. Then, find the value of n.**

SOLUTION

Since element in the union S belongs to 10 of $A_i$'s, we have,

$$S = \bigcup_{i=1}^{30} A_i = \frac{1}{10}(5 \times 30) = 15$$

$$\text{Also } S = \bigcup_{i=1}^{n} B_i = \frac{3n}{9} = \frac{n}{3}$$

$$\therefore \quad \frac{n}{3} = 15 \quad \Rightarrow n = 45$$

## 1.11 THE WELL-ORDERING PRINCIPLE

When $x, y \in Z$, we know that $x + y, xy, x - y \in Z$. Thus we say that the set Z is closed under (the binary operations of) addition, multiplication, and subtraction. Turning to division, however, we find, for example, that $2, 3, \in Z$, but that the rational number 2/3 is not a member of z. So that the set z of all integers is not closed under the binary operation of nonzero division. To cope with this situation, we shall introduce a somewhat restricted form of division for Z and shall concentrate on special elements of $Z^+$ called primes. These primes turn out to be the "building blocks" of the integers.

Given any two distinct integers x, y, we know that we must have either $x < y$ or $y < x$. However, this is also true if, instead of being integers, x and y are rational numbers or real numbers.

Suppose we try to express the subset $Z^+$ of Z, using the inequality symbols $\leq$ and $\geq$.

We find that we can define the set of positive elements of Z as,

$Z^+ = \{x \in Z | x>0\} = \{x \in Z | x \geq 1\}$.

When we try to do likewise for the rational and real numbers, however, we find that

$Q^+ = \{x \in Q | x > 0\}$ and $R^+ = \{x \in R | x > 0\}$,

but we cannot represent $Q^+$ or $R^+$ using $\geq$ as we did for $Z^+$.

The set $Z^+$ is different from the sets $Q^+$ and $R^+$ in that every nonempty subset X of $Z^+$ contains an integer a such that $a \leq x$, for all $x \in X$ – that is, X contains a least (or smallest) element. This is not so for either $Q^+$ or $R^+$. The sets themselves do not contain least elements. There is no smallest positive rational number or smallest positive real number. If q is a positive rational number, then since $0 < q/2 < q$, we would have the smaller positive rational number q/2.

The following property of the set $Z^+ \subset Z$ is,

Every nonempty subset of $Z^+$ contains a smallest element. (We often express this by saying that $Z^+$ is well ordered).

This principle serves to distinguish $Z^+$ from $Q^+$ and $R^+$. It lead to somewhere that is mathematically interesting or useful i.e., it is the basis of a proof technique known as mathematical induction. This technique will often help us to prove a general mathematical statement involving positive integers when certain instances of that statement suggest a general pattern.

## 1.11.1 Mathematical Induction

One of the easiest methods (algorithms) for sorting a list of numbers into increasing order is called selection sort. This algorithm first finds the smallest element in the list and then interchanges it with the first element. After removing the smallest element from further consideration, the algorithm finds and removes from consideration the smallest element remaining (those elements other than the element now first in the list). This process is repeated until the list has just one element remaining. Since finding a smallest element in a set with n elements requires n – 1 comparisons, a selection sort, operating on n + 1 numbers, always makes $n + (n - 1) + (n - 2) + \dots + 1$ comparisons.

Use the following chain of equalities to complete the proof,

$(1 - x)(1 + x + x^2 + x^3 + \ldots + x^n + x^{n+1})$

$= (1 - x)(1 + x + x^2 + x^3 + \ldots + x^n) + (1 - x)x^{n+1}$

[making the formula for n clear]

$= 1 - x^{n+1} + (1 - x)x^{n+1}$ [using the Inductive Hypothesis]

$= 1 - x^{n+1} + x^{n+1} - x^{n+2}$ [simplifying the expression]

$= 1 - x^{n+2}$

Therefore, $n + 1 \in T$.

By the principle of mathematical induction, $T = N$.

## 1.12 RECURSIVE DEFINITIONS (OR) RECURRENCE RELATIONS

A recurrence relation or recurrences, is an equation that expresses a given problem as n - objects interms of the same problem posed for less than n objectives. In this sectionwe discuss in detail several problems that lead to recurrence relations, and give you an indication of how they can be solved.

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses $a_n$ interms of one or more of the previous terms of the sequence namely $a_0, a_1 \ldots a_{n-1}$ for all integers with $n \geq n_0$ where $n_0$ is a non negative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

**Examples**

(1) $S_n = n + S_n - 1$ is a recurrence relation where $S_n$ is the sum of the first n positive integers.

(2) $P_n = rP_{n-1}$ is a recurrence relation. Where $P_n$ is the $n^{th}$ term of a geometric progression with common ratio r.

**Recursive Definition for Conjunction**

Given any statements $p_1, p_2, \ldots p_n, p_{n+1}$,

we define,

(i) The conjunction of $p_1, p_2$ by $p_1 \wedge p_2$ and

(ii) The conjunction of $p_1, p_2, \ldots p_n, p_{n+1}$,

for $n \geq 2$, by $p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge p_{n+1}$

$\Leftrightarrow (p_1 \wedge p_2 \wedge \ldots \wedge p_n) \wedge p_{n+1}$.

[The result in(i) establishes the base for the recursion, while the logical equivalence in (ii) is used to provide the recursive process.

Note that the statement on the right-hand side of the logical equivalence in (ii) is the conjunction of two statements, $p_{n+1}$ and the previoulsy determined statement, $(p_1 \wedge p_2 \wedge \ldots \wedge p_n)$.]

Therefore, we define the conjunction of $p_1, p_2, p_3, p_4$ by

$p_1 \wedge p_2 \wedge p_3 \wedge p_4 \Leftrightarrow (p_1 \wedge p_2 \wedge p_3) \wedge p_4$.

Then by the associative law of $\wedge$, we find that,

$$
\begin{aligned}
&(p_1 \wedge p_2 \wedge p_3) \wedge p_4 \\
&\Leftrightarrow [(p_1 \wedge p_2) \wedge p_3] \wedge p_4 \\
&\Leftrightarrow (p_1 \wedge p_2) \wedge (p_3 \wedge p_4) \\
&\Leftrightarrow p_1 \wedge [p_2 \wedge (p_3 \wedge p_4)] \\
&\Leftrightarrow p_1 \wedge [(p_2 \wedge p_3) \wedge p_4] \\
&\Leftrightarrow p_1 \wedge (p_2 \wedge p_3 \wedge p_4).
\end{aligned}
$$

These logical equivalences show that the truth value for the conjunction of four statements is also independent of the way parentheses might be introduced to indicate how to associate the given statements.

Using the above definition, we now extend our results to the following "Generalized Associative Law for $\wedge$."

Let $n \in Z^+$ where $n \geq 3$, and let $r \in Z^+$ with $1 \leq r < n$. Then

$S(n)$ : For any statements $p_1, p_2, \ldots p_r, p_{r+1}, \ldots p_n$,

$(p_1 \wedge p_2 \wedge \ldots \wedge p_r) \wedge (p_{r+1} \wedge \ldots \wedge p_n) \Leftrightarrow p_1 \wedge p_2 \wedge \ldots \wedge p_r \wedge p_{r+1} \wedge \ldots \wedge p_n$.

**Proof :** The truth of the statement $S(3)$ follows from the associative law for $\wedge$ and this establishes the basis step for our inductive proof.

For the inductive step we assume that $S(k)$ is true for some $k \geq 3$ and all $1 \leq r < k$. That is, we assume the truth of

$S(k) : (p_1 \wedge p_2 \wedge \ldots \wedge p_r) \wedge (p_{r+1} \wedge \ldots \wedge p_k) \Leftrightarrow p_1 \wedge p_2 \wedge \ldots \wedge p_r \wedge p_{r+1} \wedge \ldots \wedge p_k$.

Then we show that $S(k) \Rightarrow S(k+1)$. When we consider $k + 1$ statements, then we must account for all $1 \leq r \leq k + 1$.

(1) If $r = k$, then

$(p_1 \wedge p_2 \wedge \ldots \wedge p_k) \wedge p_{k+1}$

$\Leftrightarrow p_1 \wedge p_2 \wedge \ldots \wedge p_k \wedge p_{k+1}$ (from our recursive definition)

(2) For $1 \leq r < k$,

we have,

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_r) \wedge (p_{r+1} \wedge \ldots \wedge p_k \wedge p_{k+1})$$

$$\Leftrightarrow (p_1 \wedge p_2 \wedge \ldots \wedge p_r) \wedge [(p_{r+1} \wedge \ldots \wedge p_k) \wedge p_{k+1}]$$

$$\Leftrightarrow [(p_1 \wedge p_2 \wedge \ldots \wedge p_r) \wedge (p_{r+1} \wedge \ldots \wedge p_k) \wedge p_{k+1}]$$

$$\Leftrightarrow (p_1 \wedge p_2 \wedge \ldots \wedge p_r \wedge p_{r+1} \wedge \ldots \wedge p_k) \wedge p_{k+1}]$$

$$\Leftrightarrow p_1 \wedge p_2 \wedge \ldots \wedge p_r \wedge p_{r+1} \wedge \ldots \wedge p_k \wedge p_{k+1}.$$

So it follows by the principle of mathematical induction that the open statement S(n) is true for all $n \in Z^+$ where $n \geq 3$.

**Recursive Definition for Disjunction** **[Nov./Dec - 2008]**

We consider the sets $p_1, p_2, \ldots, p_n, p_{n+1}$, where $p_i \subseteq U$ for all $1 \leq i \leq n + 1$, and we define their union recursively as follows,

(1) The union of $p_1$, $p_2$ is $p_1 \vee p_2$. (This is the base for our recursive definition.)

(2) The union of $p_1, p_2, \ldots, p_n, p_{n+1}$,

for $n \geq 2$, is given by $p_1 \vee p_2 \vee \ldots \vee p_n \vee p_{n+1} = (p_1 \vee p_2 \vee \ldots \vee p_n) \vee p_{n+1}$,

where the set on the right-hand side of the set equality is the union of two sets, namely, $p_1 \vee p_2 \vee \ldots \vee p_n$ and $p_{n+1}$. (Here we have the recursive process needed to complete our recursive definition.)

From this definition we obtain the following "Generalized Associative Law for ∨." If $n, r \in Z^+$ where $n \geq 3$, and $1 \leq r < n$, then

$$S(n) : (p_1 \vee p_2 \vee \ldots \vee p_r) \vee (p_{r+1} \vee \ldots \vee p_n) = p_1 \vee p_2 \vee \ldots \vee p_r \vee p_{r+1} \vee \ldots \vee p_n.$$

where $p_i \subseteq U$ for all $1 \leq i \leq n$.

**Proof :** The truth of S(n) for $n = 3$ follows from the associative law of V, thereby providing the basis step needed for this inductive proof.

Assuming the truth of S(k) for some $k \in Z^+$, where $k \geq 3$ and $1 \leq r < k$, we shall now establish our inductive step by showing that $S(k) \Rightarrow S(k+1)$.

When dealing with $k + 1 (\geq 4)$ sets we need to consider all $1 \leq r < k + 1$. We find that,

(1) For $r = k$ we have,

$$(p_1 \vee p_2 \vee \ldots \vee p_k) \vee p_{k+1} = p_1 \vee p_2 \vee \ldots \vee p_k \vee p_{k+1}.$$

This follows from the given recursive definition.

(2) If $1 \leq r < k$, then,

$(p_1 \vee p_2 \vee . . . \vee p_r) \vee (p_{r+1} \vee . . . \vee p_k \vee p_{k+1})$

$\Leftrightarrow (p_1 \vee p_2 \vee . . . \vee p_r) \vee (p_{r+1} \vee . . . \vee p_k \vee p_{k+1})$

$\Leftrightarrow [(p_1 \vee p_2 \vee . . . \vee p_r) \vee (p_{r+1} \vee . . . \vee p_k)] \vee p_{k+1}$

$\Leftrightarrow (p_1 \vee p_2 \vee . . . \vee p_r \vee p_{r+1} \vee . . . \vee p_k) \vee p_{k+1}$

$\Leftrightarrow p_1 \vee p_2 \vee . . . \vee p_r \vee p_{r+1} \vee . . . \vee p_k \vee p_{k+1}$.

So it follows by the principle of mathematical induction that S(n) is true for all integers n $\geq 3$.

## SOLVED PROBLEM 1

**Find the first 5 terms of the sequence defined by $a_n = 6\, a_{n-1}$, $a_0 = 2$.**

**SOLUTION**

$a_1 = 6\, a_0$

$a_1 = 6 \times 2 = 12$

$a_2 = 6\, a_{2-1} = 6\, a_1$

$a_2 = 6. (12) = 72$

$a_3 = 6a_2 = 6 \times 72 = 432$

$a_4 = 6a_3 = 6\,(432) = 2592$

$a_5 = 6a_4 = 6\,(2592) = 15,552$

## SOLVED PROBLEM 2

**Consider the set of all subsets of any non-empty set S, called its power set, and is denoted by P (S). Let us determine a recurrence relation satisfied by $S_n = |P(S)|$ where $|S| = n$.**

**SOLUTION**

Let S = {1, 2, 3 . . . n}.

Now, any subset, A of S either contains the number n or does not. Let us consider these two mutually exclusive cases separately and count the number of such subsets.

If $n \in A$, then $A = \bar{A} \cup \{n\}$. where $\bar{A}$ is a subset of {1, 2, . . . n – 1}.