4.1 ALGEBRAIC SYSTEM - GENERAL EXAMPLES

Let 'S' be a non-empty set. If S is associated with one or more binary operations (f_1, f_2, f_3 . . . f_n binary operations). Then it is called "Algebraic Structure" (or) 'Algebraic System' (or) simply 'Algebra'.

It is denoted by (S, f_1, f_2, f_3 . . . f_n).

(1) If 'o' (circle) is a binary operation in 'S' then the algebraic system is denoted by (S, o).

(2) If '*' is a binary operation in 'S' then the algebraic system is denoted by (S, *).

Example : Let N be the set of all natural numbers and +, • are binary operations. Then (N, +), (N, •) are algebraic systems.

Binary Operation : Let A be a non empty set then the function $f : A \times A \rightarrow A$ is called a binary operation on A. In other words a binary operation on a non empty set A is a rule that assigns to every ordered pair of elements of A unique element of A.

A set A is said to be closed with respect to an operation if this operation on members of A always produces another member of A. Thus, by definition a binary operation has the closure property.

4.1.1 General Properties

Consider the algebraic system (S, +, *) where, S is the set of integers. * and + are the binary operations of multiplication and addition respectively.

Properties Associated with Addition

(1) Associativity : For any x, y, z ∈ S.

$(x + y) + z = x + (y + z)$

(2) Commutativity : For any $x, y \in S$.

$x + y = y + x$

(3) Identity : For any distinguished element $e_1 \in S$ and for any $x \in S$.

$x + e_1 = e_1 + x = x$

(4) Inverse : For all x ∈ S, there is an element in S called negative of x and denoted by –x.

$x + (-x) = e_1$

Properties Associated with Multiplication

(i) **Associativity :** For any $x, y, z \in S$

$(x * y)z = x * (y * z)$

(ii) **Commutativity :** For any $x, y \in S$

$x * y = y * x$

(iii) **Identity :** For any distinguished element $e_2 \in S$ and for any $x \in S$.

$x * e_2 = e_2 * x = x$

(iv) **Idempotent :** For any $x \in S$.

$x * x = x$

(v) **Distributivity :** For any $x, y, z \in S$

$x * (y + z) = (x * y) + (x * z)$

$x + (y * z) = (x + y) * (x + z)$

(vi) **Cancellation Property :** For $x, y, z \in S$ and $x \neq 0$.

$x * y = x * z \Rightarrow y = z$

The algebraic system $(S, +, *)$ should be expressed in the form $(S, +, *, e_1, e_2)$ to emphasize that e_1 and e_2 are distinguished elements of S.

4.1.2 Some other Definitions and it's Properties

(1) Let $(S, *)$ and (H, o) be two algebraic systems then a mapping $f : S \rightarrow H$ from $(S, *)$ to (H, o) satisfying the property that $f(a * b) = f(a) \text{ o } f(b)$ for any $a, b \in S$ is called homomorphism of algebraic system (or) simply morphism.

(2) Let f be a homomorphism from $(S, *)$ to (H, o). If mapping $f : S \rightarrow H$ is onto, then f is called an epimorphism. If $f : S \rightarrow H$ is one-to-one then f is called monomorphism. If $f : S \rightarrow H$ is both one-to-one and onto then f is called an isomorphism.

If $f : S \rightarrow H$ is an isomorphic mapping then $(S, *)$ and (H, o) are called as isomorphic.

(3) Let $(S, *)$ and (H, o) be two algebraic systems such that $H \leq S$. Then a homomorphism f from $(S, *)$ and (H, o) is called an endomorphism.

An isomorphism from $(S, *)$ to (H, o) is called an automorphism if $H = S$.

(4) Let (A, o) be an algebraic system and E be an equivalence relation. The relation E is said to be a congruence relation on (A, o) if it satisfies the substitution property given below.

for all $a_1, a_2 \in A$.

$$(a_1 \in a_1') \wedge (a_2 \in a_2') \Rightarrow (a_1 \circ a_2) \in (a_1' \circ a_2') \text{ where } a_1', a_2' \in A.$$

(5) Let (S, *) be an algebraic system and $A \subseteq S$, if A is closed under the operation * then (A, *) is called sub algebra of (S, *).

(6) Let (S, *) and (H, o) be two algebraic systems. The algebraic system $(S * H, \oplus)$ is called the direct product of the algebraic (S, *) and (H, o) provided by $\oplus$ operation.

For any $S_1, S_2 \in S$ and $h_1, h_2 \in H$.

$$(S_1, h_1) \oplus (S_2, h_2) = (S_1 * S_2, h_1 \circ h_2)$$

In algebraic system $(S * H, \oplus)$.

(S, *) and (H, o) are called the factor algebras of $(S * H, \oplus)$.

4.1.3 Theorems and Solved Problems

THEOREM 1

Let * be a binary relation on X and i_l and i_r be left and right identities with respect to *. Then $i_l = i_r$.

PROOF

Given * be a binary relation on X. i_l and i_r be left and right identities with respect to *.

Since i_l is a left identity, $x * i_r = x$ for all $x \in X$.

For $x = i_l$ this becomes

$$i_l * i_r = i_l \quad \ldots (1)$$

Since i_r is a right identity,

$i_l * x$

For all $x \in X$. For $x = i_r$. This becomes

$$i_l * i_r = i_r \quad \ldots (2)$$

From equation (1) and (2),

$$i_l = i_r$$

4.2 SEMIGROUPS AND MONOIDS

4.2.1 Semigroup

Let (S, *) be any set of algebraic system where S is non empty set and * be a binary relation on S. If the * is associative in 'S' then (S, *) is called semigroup. If for any, a, b, c $\in$ S

$$(a*b)*c = a*(b*c)$$

Example : (N, +) is a semigroup when N is a set of natural numbers and (Q, +) is a semigroup when Q is a set of rational numbers. Because (N, +), (Q, +) both satisfy the associative property.

4.2.2 Monoid

Let (S, *) is a semigroup and it also satisfy the identity property with respect to *, then (S, *) is called a monoid. If for any $a, b, c \in S$ then,

$$a * (b * c) = (a * b) * c$$

and also $e * a = a * e = a$ where e is an identity element.

Example : Consider a semigroup (Q, *) for e $\in$Q, then $e * a = a * e = a \, \forall \, a \in Q$.

So, e is an identity in Q w.r.t. '*'.

$\therefore$ (Q, *) is a monoid.

4.2.3 Semigroup Homomorphism

Let (S, *) and (T, Δ) be two semigroups. A mapping $f : S \to T$ satisfying the property that $f(S_1 * S_2) = f(S_1) \, \Delta \, f(S_2)$ is called a semi group homomorphism, where $S_1, S_2 \in S$.

(1) If f is one-to-one then it is called monomorphism.

(2) If f is onto then it is called epimorphism.

(3) If f is both one-to-one and onto then it is called Isomorphism.

(4) An Isomorphism defined from a semigroup to itself is called automorphism.

4.2.4 Monoid Homomorphism

Let (S, *) and (T, Δ) be two monoids with identity elements e_s and e_T respectively. A mapping $f : S \to T$ is called monoid homomorphism if it satisfy the following properties.

(1) $f(S_1 * S_2) = f(S_1) \, \Delta \, f(S_2)$ and

(2) $f(e_S) = e_T$.

4.2.5 Subsemigroups

Let (S, *) be a semigroup and A be a subset of S(A ⊆ S). Then (A, *) is said to be subsemigroup if (A, *) itself is a semigroup.

Example : (N, +) is a semigroup where N is set of natural numbers and (Q, +) is a semigroup where Q is set of rational numbers. Then,

(N, +) is a subsemigroup of (Q, +).

4.2.6 Submonoid

Let (S, *) be a monoid and A be a subset of S. Then (A, *) is said to be submonoid of (S, *) if (A, *) itself is a monoid.

Example : (Q, *) is a submonoid of (R, *).

Where, R is a set of real numbers.

Q is a set of rational numbers.

4.2.7 Theorems and Solved Problems

THEOREM 1

Let (P, *), (Q, Δ) and (R, ⊕) be any three semigroups and f : P→Q and g : Q→R be semigroup homomorphism. Then gof : P → R is a semigroup homomorphism from (P, *) to (R, ⊕).

PROOF

Let (P, *), (Q, Δ) and (R, ⊕) be any three semigroups and f : P → Q and g : Q→ R be semigroup homomorphism.

Let x, y ∈ P. Then,

$$\begin{aligned} gof(x * y) &= g(f(x * y)) \\ &= g(f(x)\,\Delta\, f(y)) \\ &= g(f(x) \oplus g(f(y))) \\ &= gof(x) \oplus gof(y). \end{aligned}$$

It is proved that gof : P → R is also a semigroup homomorphism from (P, *) to (R, ⊕).

So, we have,

$a = f(x)$

$= f(x *_1 e_1)$

$= f(x) *_2 f(e_1)$

$= a *_2 f(e_1)$ and

$a = f(x)$

$= f(e_1 *_1 x)$

$= f(e_1) *_2 f(x)$

$= f(e_1) *_2 a.$

Thus, for any $a \in S_2$, we have

$a *_2 f(e_1) = f(e_1) *_2 a$

$= a$

So, it means that $f(e_1)$ is the identity element in S_2.

i.e., $f(e_1) = e_2$

$\therefore$ Hence it is proved.

4.3 GROUPS

4.3.1 Definition and Examples

(1) Group : Let (S, *) be an algebraic system, (S, *) is called group if it satisfies the following properties.

(i) Closure Property : i.e., for every $a, b \in S, a * b \in S$.

(ii) Associativity : i.e., for every $a, b, c \in S$ then, $a * (b * c) = (a * b) * c$

(iii) Identity : i.e., for every $a \in S$ there exists an element $a \in S$ then,

$a * e = e * a = a.$

(iv) Inverse : i.e., for every $a \in S$ there exists an element $a^{-1} \in S$ then,

$a * a^{-1} = e.$

The order of a group (S, *) is denoted by $|S|$, is the number of elements of S when S is finite.

(2) Abelian Group : A group (S, *) is said to be abelian if * is commutative in S. i.e., $a * b = b * a \ \forall a, b \in S$.

(3) Permutation Group : A set S is said to be permutation if any one-to-one mapping of a set S onto S.

Consider a set of all permutation of the finite set elements and define a binary operation on them. Then the set of permutations form a group under this defined binary operation. Those groups are called permutation groups.

Example : Let S = {1, 2, 3} be a set and let p denote a permutation of the elements of S. This means $P : S \rightarrow S$ is a bijective mapping. There are two methods of describing the permutation P. The first method is,

Let P(1) = 2, P(2) = 1 and P(3) = 2.

Then P is defined as,

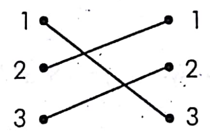
$$P = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Where in the below elements, image of any element of S is entered.

By this notation

$$P = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} \text{and so on.}$$

Another method of describing P is in diagrammatic form.



By permuting the elements of S by an application of the permutation P_i, P_j then we can get the $P_i \lozenge P_j$ permutation where $\lozenge$ is a binary operation on S representing right composition of functions and let * denote left composition of function then,

$P_i \lozenge P_j = P_i * P_j$

for i, j = 1, 2

This is called composition of permutations.

(4) **Symmetric Group :** A permutation group $(S_n, \Diamond)$ is called symmetric group if the set S_n is having all the permutations of n elements. Here $\Diamond$ is a binary operation. The group $(S_n, \Diamond)$ is of order n! and degree n.

Degree of permutation group is the cardinality of the set on which permutations are defined.

Example : $S = \{P_1, P_2\}$

Where, $P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ are two permutations.

Here the degree of the group $(S, \Diamond)$ is 2.

(5) **Dihedral Group :** $(D_n, \Diamond)$ is a permutation group of the degree n and is called a Dihedral group. By considering the set of all regular polygon of n sides under the composition $\Diamond$ where D_n is order of 2n.

Where n = 1, 2, . . .

(6) **Cyclic Group :** Let (G, *) be a group. If there exists $a \in G$, the elements of G can be expressed as some power of 'a' then the group (G, *) is called a cyclic group. i.e., any element is expressed in the form of a^n where n is a positive integer and 'a' is known as the generator of G.

A cyclic group can be defined as,

$$G = \{a^n \mid n \in z\}$$

Then 'a' is called generator of 'G'.

(G, *) is called a cyclic group.

A cyclic group is an abelian group.

Example : Let $G = \{1, -1, i, -i\}$ be a multiplicative group under '*' for any $i \in G$.

So, $i^1 \in G$

So, $i^1 = 1$, $i^2 = -1$, $i^3 = i$, $i^4 = -i$

So, $G = \{i^n \mid n \in z\}$

Then, i. is called generator of 'G', (G, *) is a cyclic group.

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(7) Subgroup

Let (S, *) be a group and H be a subset of S $(H \subseteq S)$. Then (H, *) is called a subgroup of (S, *) if (H, *) itself forms a group i.e., (H, *) also satisfy the closure, associativity, identity and inverse properties to form a group along with H as H is a subset of S. Then, (H, *) is a subgroup of (S, *).

Let (S, *) be a group with identity 'e' then the set of ({e}, *) forms a subgroup of (S, *) and is also a subgroup to itself. Then these two subgroups (S, *) and ({{e}, *}, *) of S are called trivial or improper subgroups of S and all other subgroups are called non-trivial or proper subgroups.

(8) Group Homomorphism : Let (G, *) and (H, Δ) be two groups. A mapping $f : G \to H$ satisfying the property

$f(a * b) = f(a) \Delta f(b) \quad \forall a, b \in G$ is called a homomorphism of a group.

- A homomorphism mapping f is one-to-one then it is called an monomorphism.
- A homomorphism mapping f is onto then it is called an epimorphism.
- A homomorphism mapping f is both one-to-one and onto then it is called an isomorphism.
- A homomorphism mapping f is homomorphism from a group to itself is called an endomorphism.
- A homomorphism mapping f is isomorphism from a group to itself is called an automorphism.

(9) Kernel of a Homomorphism : Let (G, *) and (H, Δ) be two groups and $f : G \to H$ is to be a homomorphism. Let e_G and e_H be identity elements of G, H respectively. Then kernel of homomorphism f is defined as the set of all those elements of G which are mapped to e_H.

$\therefore$ Kernel of $f = \{x \mid x \in G, f(x) = e_H\}$

Kernel f is a subset of group G.

(10) **Order of an Element :** Let (G, *) be a group and $a \in G$. The least positive integer 'n' such that $a^n = e$ is called order of 'a' and it is denoted by O(a) where e is an identity.

(i) The order of identity element in a group is equal to 1.

(ii) If the order of an element is not defined then we say that order of an element a is infinite.

Therefore 'o' is associative in V^*.

$\therefore$ Thus (V^*, o) is a semigroup.

Let us consider that $\wedge \in V^*$,

For any $\alpha \in V^*$

$\alpha \circ \wedge = \wedge \circ \alpha = \alpha$

'$\wedge$' is an identity in V^*.

$\therefore$ (V^*, o) is a monoid, but

$\therefore$ (V^+, o) is only semigroup but not monoid.

4.3.1.1 Theorems and Solved Problems

SOLVED PROBLEM 1

Prove that every subgroup of a cyclic group is cyclic.

SOLUTION

Let G be a cyclic group with generator 'x'. Then $G = \{x^n / n \in Z\}$. Let H be a subgroup of G. Then $H \subseteq G$ and every element of H is integral power of x.

To prove H is cyclic group.

Let m be the smallest positive integer such that $x^m \in H$ and $a \in H$ be any element. Since $H \subseteq G$, $a = x^n$ for some $n \in Z$.

Now, by division algorithm, $\exists$ integer q and r with $0 \le r < m$ such that $n = qm + r$.

Hence, $a = x^n = x^{mq+r} = x^{qm} \cdot x^r = (x^m)^q \cdot x^r$.

Since, $x^m \in H$, and H is subgroup of G, therefore $(x^m)^q \in H$ (because of closure property).

Also, $a \in H$ and hence $x^r \in H$.

Since m is the smallest positive integer such that $x^m \in H$ and since $0 \le r < m$.

Therefore $x^r \in H$ is possible only if $r = 0$, that is $n - qm = 0$, we get $a = x^{qm} = (x^m)^q$.

Thus every element a of H is an integral power of x^m. Hence H is a cyclic group with generator x^m.

4.3.2 Normal Subgroups

A sub-group (H, *) of (G, *) is called a normal sub-group of G if for all $h \in H$ and $g \in G$ $ghg^{-1} \in H$. If H is normal in G then we write $H \triangleleft G$.

Examples

(1) Let (G, *) be a Group, then ({e}, *) is a normal sub-group in G. It is called the improper normal sub-group.

(2) (G, *) is normal (G, *) it is called the improper normal sub-group.

4.3.2.1 Simple Groups

A group (G, *) is called simple group if its only normal sub-groups are G and {e}.

A sub-group (H, *) of a group (G, *) is said to be normal sub-group of (G, *) is for every $g \in G$, $ghg^{-1} \triangleleft H$.

Examples : A group of prime order has no proper sub-groups.

Therefore, every group of prime order is simple. We now give an equivalent definition for the sub-group to be normal.

THEOREM 1

Every sub-group of an abelian group is normal sub-group.

PROOF

Let (G, *) be an abelian group and (H, *) be a sub-group of G.

Now, $g \in G, h \in H$

$\Rightarrow g * h * g^{-1} = h * g * g^{-1}$ [$\because$ G is Abelian and $H \subseteq G$]

$= h * e$

$= h$

$\therefore g * h * g^{-1} = h \in H \ \forall g \in G, h \in H$

(H, *) is normal in G.

SOLUTION

We can verify that $(M, \oplus, \odot)$ and $(Z, +, \times)$ are rings.

Let $M_1 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $M_2 = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$

Now, $f(M_1 \oplus M_2) = f\left(\begin{bmatrix} a+c & b+d \\ b+d & a+c \end{bmatrix}\right)$

$= (a + c) - (b + d)$, by definition.

$= (a - b) + (c - d)$

$= f(M_1) + f(M_2)$

$f(M_1 \oplus M_2) = f\left(\begin{bmatrix} ac+bd & ad+bc \\ ad+bc & ac+bd \end{bmatrix}\right)$

$= (ac + bd) - (ad + bc)$

$= (a - b) \times (c - d)$

$= f(M_1) \times f(M_2)$

Hence f is a ring homomorphism.

4.4 HOMOMORPHISM

4.4.1 Homomorphisms and Isomorphisms

Let G_1 and G_2 be two groups and f be a function from G_1 to G_2. Then, f is called a homomorphism from G_1 to G_2 if $f(ab) = f(a)f(b)$, for all $a, b \in G_1$.

The function $f : G_1 \to G_2$ is called an isomorphism from G_1 into G_2 if,

(1) f is homomorphism from G_1 to G_2, and

(2) f is one-to-one and onto.

The groups G_1 and G_2 are said to be isomorphic if there is an isomorphism from G_1 onto G_2.

Examples

(1) Consider the groups $<R, +>$ and $<R^+, \times>$. Define the function $f : R \to R^+$ by $f(x) = e^x$ for all $x \in R$. Then, for all $a, b \in R$, we have

$f(a + b) = e^{a+b} = e^a\, e^b = f(a)f(b)$

Hence f is a homomorphism.

Next, take any $c \in R^+$. Then, $\log c \in R$ and $f(\log c) = e^{\log c} = f$. Therefore, every element in R^+ has a preimage in R under f. As such, f is onto.

Further, for any $a, b \in R$

$f(a) = f(b) \Rightarrow e^a = e^b \Rightarrow a = b$.

Therefore, f is one-to-one as well. Accordingly, it is an isomorphism.

(2) Consider the groups $<Z, +>$ and $<Z_4, +>$. Define the function $f : Z \to Z_4$ by $f(x) = [x]$ for all $x \in Z$. Then, for all $a, b \in Z$, we have

$f(ab) = f(a + b) = [a + b] = [a] + [b] = f(a) + f(b)$

Therefore, f is a homomorphism.

Since $[x] = \{x + 4k \mid k \in Z\}$ in Z_4, $[x] = f(x + 4k)$ for all $k \in Z$. Thus $[x]$ does not have a single preimage under f. Hence it is not one-to-one.

Consequently, f is not an isomorphism.

(3) Consider the groups $<W_4, .>$ and $<Z_4, +>$. Define the function $f : W_4 \to Z_4$ by $f(1) = [0]$, $f(-1) = [2]$, $f(i) = [1]$, $f(-i) = [3]$

We check that f is one -to-one, onto and a homomorphism. Therefore, f is an isomorphism.

(4) Let G be a group with e as the identity, and $G \neq \{e\}$. Consider the function $f : G \to G$ defined by $f(x) = e$ for all $x \in G$. Then for any $a, b, \in G$, we have $f(a) = e$, $f(b) = e$ and $f(ab) = e$, so that

$f(ab) = e = e \cdot e_1$ $f(a)f(b)$.

Therefore, f is a homomorphism. This is not an isomorphism (because, it cannot be one-to-one or onto).

4.4.1.1 Theorems and Solved Problems

THEOREM 1

Let f be a homomorphism from a group G_1 to a group G_2.

Prove that,

(i) If e_1 is the identity in G_1 and e_2 is the identity in G_2, we have $f(e_1) = e_2$

(ii) $f(a^{-1}) = (f(a))^{-1}$ for all $a \in G_1$.

(iii) If H_1 is a subgroup of G_1 and $H_2 = f(H_1)$, then H_2 is a subgroup of G_2.

(iv) If f is an isomorphism from G_1 onto G_2, then f^{-1} is an isomorphism from G_2 onto G_1.

PROOF

(i) We have, in G_2,

$$e_2\, f(e_1) = f(e_1), \text{ because } e_2 \text{ is the identity in } G_2$$
$$= f(e_1\, e_1), \text{ because } e_1 e_1 = e_1$$
$$= e_1 \text{ in } G_1$$
$$= f(e_1)\, f(e_1), \text{ because } f \text{ is a homomorphism.}$$

Therefore, $e_2 = f(e_1)$ by the cancellation law.

(ii) For any $a \in G_1$, we have,

$$f(a)\, f(a^{-1}) = f(aa^{-1}) = f(e_1) = e_2$$

and $f(a^{-1})\, f(a) = f(a^{-1}\, a) = f(e_1) = e_2$

These show that $f(a^{-1})$ is the inverse of $f(a)$ in G_2. That is $f(a^{-1}) = [f(a)]^{-1}$.

(iii) $H_2 = f(H_1)$ is the image of H_1 under f; this is a subset of G_2. Take any $x, y \in H_2$. Then, $x = f(a)$, $y = f(b)$ for some $a, b \in H_1$. Since H_1 is a subgroup of G_1, we have $ab^{-1} \in H_1$. Consequently,

$$xy^{-1} = f(a)\, [f(b)]^{-1} = f(a)\, f(b^{-1}) = f\,(ab^{-1}) \in f(H_1) = H_2.$$

Accordingly, H_2 is a subgroup of G_2.

Since, H_1 is a subgroup of G_1, it follows that $f(H_1) = H_2$ is a subgroup of G_2.

(iv) Since $f : G_1 \to G_2$ is an isomorphism, f is one-to-one and onto. Therefore, the function $f^{-1} : G_2 \to G_1$ exists and is one-to-one and onto.

Take any $x, y \in G_2$. Then $xy \in G_2$ and there exist $a, b \in G_1$ such that $x = f(a)$, $y = f(b)$. Therefore,

$$f^{-1}(xy) = f^{-1}(f(a)\, f(b))$$
$$= f^{-1}(f(ab)), \text{ because f is a homomorphism.}$$
$$= ab, \text{ because } f^{-1} f \text{ is the identity function.}$$
$$= f^{-1}(x)\, f^{-1}(y).$$

This shows that $f^{-1} : G_2 \to G_1$ is a homomorphism as well. Thus, f^{-1} is an isomorphism.

4.5 RESIDUE ARITHMETIC

Number systems may be classed into non-positional and positional types. The Roman numeral are a classical example of a nonpositional system. In these systems, the position of a digit in a number implies a certain weight by which this digit is multiplied. Thus any nonnegative integer x in a general positional number system can be represented as,

$$x = a_n w_n + a_{n-1} w_{n-1} + \ldots a_1 w_1 + a_0 w_0 = \sum_{i=0}^{n} a_i w_i$$

Where the a_i's are a set of permissible digits and the w_i are a set of weights.

If we let the values of w_i be successive powers of some fixed number, then the number system is said to have a fixed-based or fixed-radix.

We obtain familiar decimal or binary number system respectively. The binary number system is used in computers much more frequently than its decimal. A number system which is not fixed-base is said to be mixed-base. A mixed-base number system is called the residue number system.

[Nov./Dec. - 2008]

A number system has the uniqueness property if each number in this system can be represented in only one way. The decimal and binary number systems for non-negative integers are unique. If we extend the set of a numbers to be represented to all integers, then the sign-magnitude representation of zero can be +00 . . 0 or –00 . . .0.

Similarly, using 1s complement representation in the binary system yields two possible representations for zero, namely 11. . . 1 and 000 . . .0.

A number system is said to be redundant if there exist fewer numbers than the number of possible combinations of the digits.

An example of a weighted mixed-based number system is given in Table 4.5.1 with $w_0 = 1$, $w_1 = 2$, $w_2 = 6$, $0 \le a_0 < 2$, $0 \le a_1 < 3$ and $0 < a_2 < 5$.

Table 4.5.1 : Weighted Mixed-based Number System

No.	a_2	a_1	a_0
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	0	2	0
5	0	2	1
6	1	0	0
7	1	0	1
8	1	1	0
9	1	1	1
10	1	2	0
29	4	2	1

Certain weighted number systems are important because the magnitude comparison of two numbers can be performed by simply comparing their corresponding digits according to position. Consider, the number system defined by,

$x = 5a_4 + 4a_3 + 3a_2 + 2a_1 + a_0$

Where $0 \le a_i < 10$. Let x and y be the following numbers.

$x = 2 * 5 + 1 * 4 + 0 * 3 + 0 * 2 + 0 * 1$

$y = 0 * 5 + 2 * 4 + 2 * 3 + 0 * 2 + 0 * 1$

It is not possible to compare these numbers algebraically by a digit-by-digit comparison.

The decimal and binary number systems have been used to perform arithmetic on digital computers because of the following.

(1) Algebraic comparison of two numbers can be easily mechanized.

(2) The range of these number systems can be easily extended by adding more digit positions.

(3) Multiplication and division by the fixed radix can be achieved by shifting digit positions in memory.

(4) The logic required for performing a particular arithmetic operation (such as addition) is more or less the same for all digit positions.

(5) Overflow detection is easy.

Definition : The residue number system as an alternative to the binary and decimal number systems, fixed-based number systems in digital computer have many advantages, these systems have disadvantages that restrict the speed of performing arithmetic operations.

The residue number system is not a weighted system and consequently, it does not have many of the advantages of fixed-base systems.

Disadvantages of the residue number system when compared to the fixed-base number system are as follows.

(1) Comparison of numbers is difficult.

(2) It is difficult to determine whether an overflow occurred.

(3) Division is complex.

(4) The residue number system is not convenient for the representation of fractions.

(5) Residue arithmetic can be justified if efficient means of conversion into and out of the residue system are available.

Let m be a positive integer. By the unique factorization theorem, it is possible to factorize m as,

$m = p_1^{n_1}, p_2^{n_2}, \ldots, p_r^{n_r} = m_1 m_2 \ldots m_r$

Where $p_1, p_2, \ldots p_r$ are prime numbers, $n_1, n_2, \ldots, n_r$ are positive integers and $m_1 = p_1^{n_1}$, $m_2 = p_2^{n_2}, \ldots,$ and $m_r = p_r^{n_r}$ are pairwise relatively prime, that is $GCD(m_i, m_j) = 1$ for $i \ne j$.

Example : Let m = 30, so that $m_1 = 2$, $m_2 = 3$ and $m_3 = 5$ with $Z^{*}_{30} = Z_2 \times Z_3 \times Z_3$.

The residue representations of the numbers in Z_{30} are given in the Table 4.5.2.

We now define operations of addition, subtraction and multiplication on Z^{*}_{m} interms of the corresponding operations in Z_m, for i = 1, 2, . . .r and denote these operations by $\oplus_m$, Θ_m and $\otimes_m$ respectively.

$(Z_m, +_m)$ and $<Z_{mi}, +_{mi}>$ are cyclic groups. Let us now define an operation $\oplus_m$ on Z^{*}_{m} such that for any two numbers $< x_1, x_2, . . . x_r >$ and $< y_1, y_2, . . . y_r >$ in Z^{*}_{m} which are residue representations of $x, y \in Z_m$.

$$<x_1, . . ., x_r) \oplus_m <y_1, . . ., y_r) > = g(x) \oplus_m g(y)$$

$$= <x_1 + m_1 y_1, . . . , x_r + m_r y_r>$$

Table 4.5.2 : Residue Representation of the Numbers in Z_{30}

Residue Digits Moduli				Residue Digits Moduli			
×	2	3	5	×	2	3	5
0	0	0	0	15	1	0	0
1	1	1	1	16	0	1	1
2	0	2	2	17	1	2	2
3	1	0	3	18	0	0	3
4	0	1	4	19	1	1	4
5	1	2	0	20	0	2	0
6	0	0	1	21	1	0	1
7	1	1	2	22	0	1	2
8	0	2	3	23	1	2	3
9	1	0	4	24	0	0	4
10	0	1	0	25	1	1	0
11	1	2	1	26	0	2	1
12	0	0	2	27	1	0	2
13	1	1	3	28	0	1	3
14	0	2	4	29	1	2	4

Clearly $(Z^{*}_{m}, \oplus_m)$ is a cyclic group.

Similar equations can be given for subtraction and multiplication, as follows.

$$\langle x_1, \ldots x_r \rangle \ominus_m \langle y_1, \ldots, y_r \rangle = g(x) \ominus_m g(y)$$

$$\langle x_1, \ldots, x_r) \otimes_m \langle y_1, \ldots, y_r \rangle = g(x) \otimes_m g(y)$$

$$= \langle x_1 \times_m y_1, \ldots, x_r \times_m y_r)$$

A model for addition, subtraction and multiplication can be represented by Fig. 4.5.1.

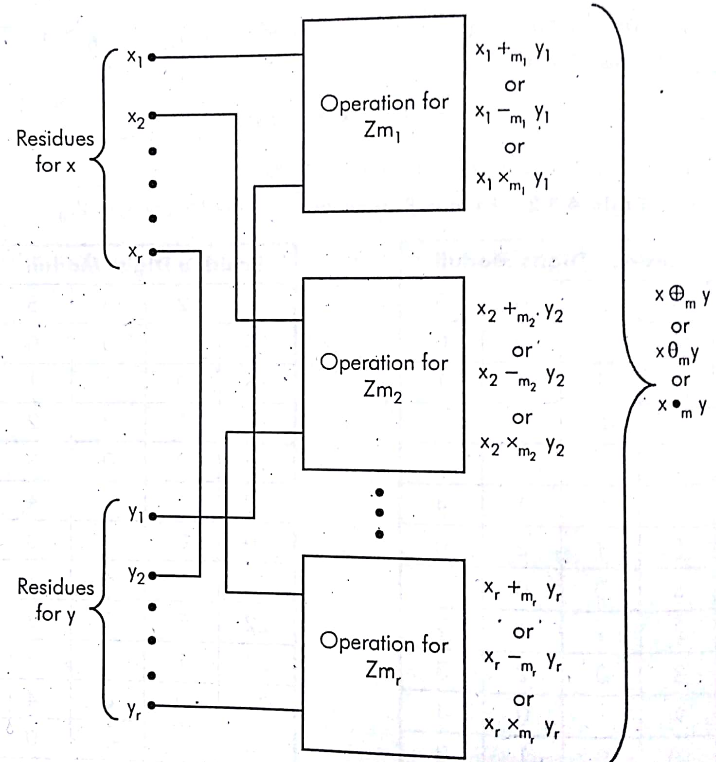


Fig. 4.5.1 : A Residue System for Performing Addition, Subtraction or Multiplication

An important property of modular arithmetic is the cancellation law of multiplication. If the generated common divisor of an element $c \in Z_m$ and m is 1, that is GCD(c, m) = 1 then for any two elements $a, b \in Z_m$.

(ca) mod m = (cb) mod m

$\Rightarrow$ a mod m = b mod m,

By definition of congruence, $ca = pm + r_1$ and $cb = qm + r_2$

Since (ca) mod m = (cb) mod m, then $r_1 = r_2$, and it therefore follows that

$ca - pm - cb = -qm$ (or)

$c(a - b) = (p - q)m$

Consequently, $c(a - b)$ must be divisible by m, and because GCD (c, m) = 1, it then follows that a – b is a multiple of m, that is,

a mod m = b mod m

As a consequence of this fact, we can say that if GCD (a, m) = 1, then an equation of the form,

(ax) mod m = b mod m

Has a unique solution for x mod m.

THEOREM 1

The quality a' exists and is unique if and only if GCD (a, m) = 1 and $a \neq 0$.

PROOF

Consider the set of numbers $\{(a^n \bmod m \mid n > 0\}$ which consists of atmost m distinct elements. There must exist two positive integers n_1 and n_2 with $n_1 > n_2$ such that

$a^{n}{}_1 \bmod m = a^{n}{}_2 \bmod m$

However, this equation can be rewritten as,

$(a^{n}{}_1{}^{-n}{}_2 . a^{n}{}_2) \bmod m = a^{n2} \bmod m$

Since GCD (a, m) = 1, it directly follows that GCD $(a^{n2}, n) = 1$. Furthermore, by the cancellation law of multiplication, we obtain

$a^{n1 - n2} \bmod m = 1$ and $(a^{n}{}_1{}^{-n}{}_2{}^{-1})$ a mod m = 1

Which implies that $a^{n1 - n2 - 1}$ mod m is the inverse of a. This inverse is unique by the cancellation law of multiplication.

Assume to the contrary that GCD(a, m) = q where q > 1. Then a and m can be written as,

$a = c_1q$ and $m = c_2q$, where c_1 and c_2 are integers.

It therefore follows that,

$(a'\ a) \bmod m = (a'\ c_1q) \bmod m = (a'\ c_1\ q) \bmod c_2q = q(a'c_1) \bmod c_2$.

Since $(c_1\ q) \bmod c_2\ q = q(c_1 \bmod c_2)$. Since q > 1 by assumption, it is impossible for (a' a) mod m to be 1, and consequently a has no inverse.

5.1 DEFINITIONS AND EXAMPLES OF GRAPH THEORY

5.1.1 Graph

A graph 'G' is a pair of sets (V, E).

Where, V is a set of vertices and E is a set of edges.

The most common representation of a graph is a diagram with vertices and edges.

(1) Vertices (or) nodes are represented as points (or) small circles.

(2) Edges are represented as line segments (or) curve joinings of its end vertices.

Example

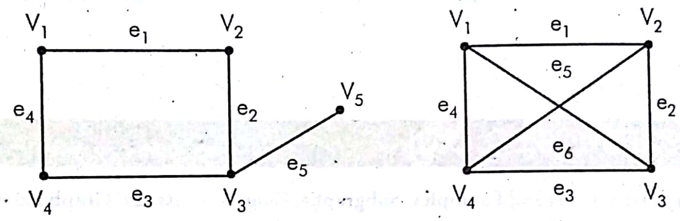


Fig. 5.1.1 : Representation of a Graph with Vertices and Edges

In the Fig. 5.1.1, the first graph consists of five vertices and five edges. So, $V = \{V_1, V_2, V_3, V_4, V_5\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$. In the second graph, there are four vertices and six edges.

So, $V = \{V_1, V_2, V_3, V_4\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.

5.1.2 Null Graph

A graph in which number of edges is zero is called as null graph.

Example

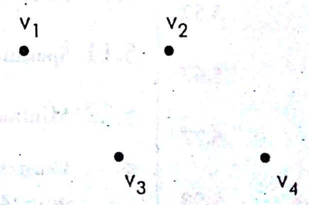


Fig. 5.1.2 : Null Graph

The null graph with four vertices and zero edges.

5.1.3 Self Loop

An edge joining a vertex to itself is called as self loop.

Example

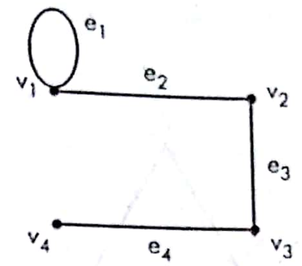


Fig. 5.1.3 : A Graph with a Self Loop

In this graph, edge e_1 is a self loop.

5.1.4 Parallel (or) Multiple Edges

In a graph it may be possible to have more than one edge with a single pair of vertices, such edges are called parallel edges.

Example

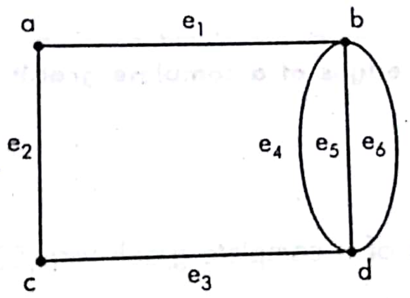


Fig. 5.1.4 : Representation of Graph with Parallel Edges

In this example e_4, e_5, e_6 are parallel edges.

5.1.5 Simple Graph

A graph which contains neither self loop nor parallel edges is called a simple graph.

Example

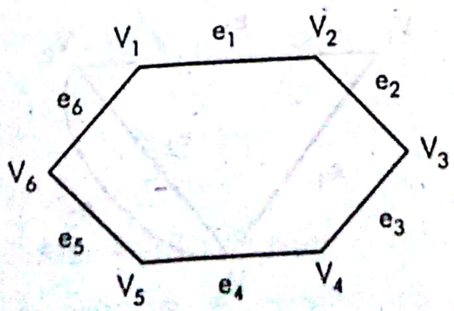


Fig. 5.1.5 : Simple Graph

5.1.6 Complete Graph

A simple graph in which there is exactly one edge between each pair of distinct vertices, is called a complete graph.

Example

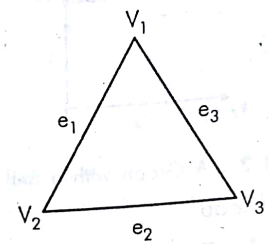


Fig. 5.1.6 : Complete Graph

The number of edges in complete graph with n vertices $= \frac{n(n-1)}{2}$.

EXAMPLE PROBLEM 1

Find the total number of edges of a complete graph with 50 vertices.

[Nov./Dec. - 2008]

SOLUTION

The total number of edges of a complete graph with 50 vertices

$$= {}^nC_2 = {}^{50}C_2 = \frac{50(50-1)}{2} = 1,225.$$

5.1.7 Multigraph

A graph which contain parallel edges is called multigraph.

Example

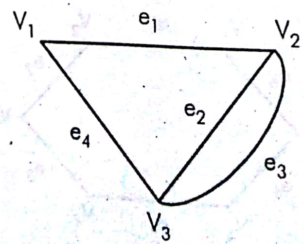


Fig. 5.1.7 : Multigraph

5.1.8 Order and Size of a Graph

The number of vertices in a graph 'G' is called order of the graph and is denoted by |V(G)|.

The number of edges in a graph 'G' is called size of the graph and is denoted by |E(G)|.

Example

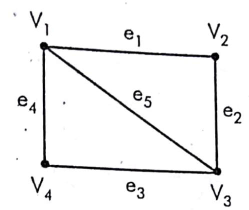


Fig. 5.1.8 : Order and Size of a Graph

In the Fig. 5.1.8, order of the graph is |V(G)| = 4 and size of the graph is |E(G)| = 5.

5.1.9 Directed Graph

The graph in which the elements of the edge set are ordered pairs of vertices is called directed graph or diagraph.

Here order pair (v_i, v_j) denotes an edge from vertex v_i to vertex v_j.

(v_j, v_i) denotes an edge from v_j to v_i.

Example

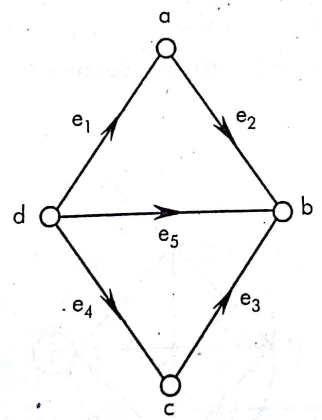


Fig. 5.1.9 : Directed Graph

Here in this graph, elements edge set are ordered pair of vertices is,

e_1 = (d, a), e_2 = (a, b), e_3 = (c, b), e_4 = (d, c), e_5 = (d, b)

5.1.10 Non-Directed Graph

A graph in which the elements of the edge set are unordered pair of vertices is called a non-directed graph.

Here (v_i, v_j) denotes an edge between v_i, v_j.

Example

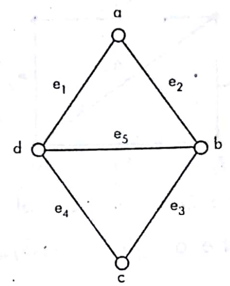


Fig. 5.1.10 : Non-directed Graph

Then $e_1 = \{a, d\}$, $e_2 = \{a, b\}$, $e_3 = \{c, b\}$, $e_4 = \{c, d\}$, $e_5 = \{b, d\}$.

5.1.11 Degree of Vertex in a Non-Directed Graph and Degree Sequence

The degree of a vertex V of a graph G is the number of edges of G, which are incident with v. The degree of a vertex v in a graph G is denoted by $deg_G(v)$. A vertex of degree zero is called an isolated vertex. A vertex with degree one is a pendant vertex. A vertex of odd degree is an odd vertex and a vertex of even degree is an even vertex.

If $v_1, v_2 \ldots v_n$ are the vertices of G, then the sequence $\{d_1, d_2 \ldots d_n\}$ where $d_i = deg_G(v_i)$ is the degree sequence of G.

Example

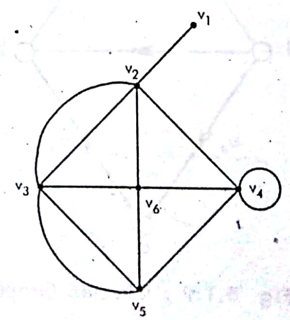


Fig. 5.1.11 : Degree of Vertex in a Non-directed Graph

Degree sequence is given by {1, 4, 5, 5, 4, 4}.

5.1.12 Degree of Vertex in a Directed Graph

The number of edges incident to a vertex is called the in-degree of the vertex and the number of edges incident from it is called its out-degree for a digraph. The in-degree of a vertex v in a graph G is denoted by $deg_{G+}(v)$ and the out-degree by $deg_{G-}(v)$. The degree of a vertex is determined by counting each loop incident on it twice and each other edge once.

The minimum of all the degrees of the vertices of a graph G is denoted by $\delta(G)$ and the maximum of all the degrees of the vertices of G is denoted by $\Delta(G)$.

Example

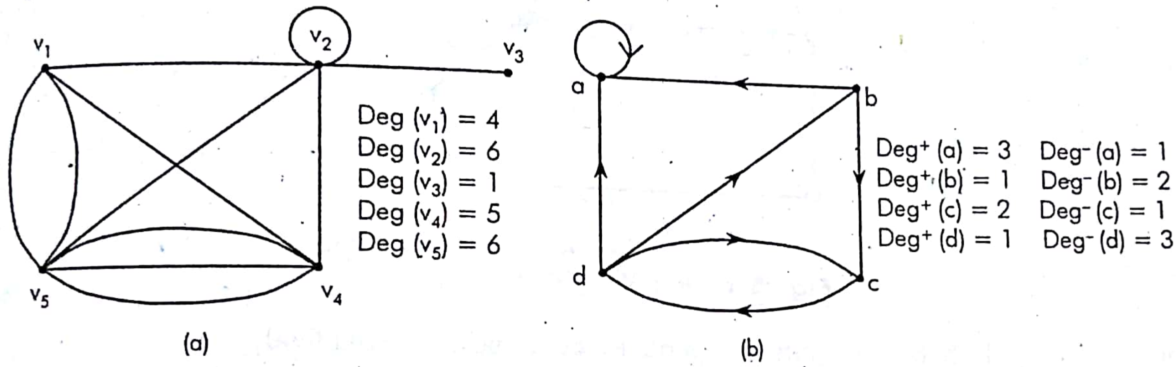
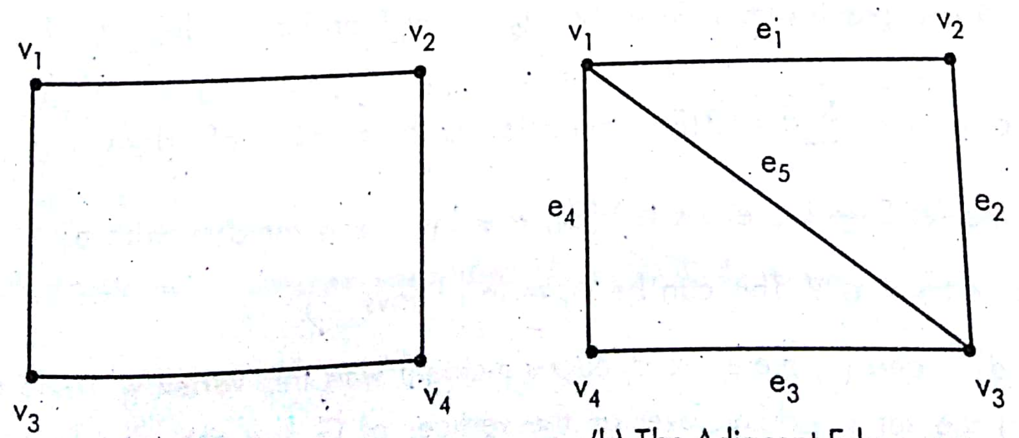


Fig. 5.1.12 : Degree of Vertex in a Directed Graph

5.1.13 Adjacent Vertices and Adjacent Edges

Two vertices u and v are said to be adjacent if there exists an edge between them. For a graph G = (V, E) the relation 'adjacency' is non reflexive and symmetric relation on V. Similarly, if distinct edges e_1 and e_2 of G have a vertex in common then e_1 and e_2 are called adjacent edges.

Example



(a) The Adjacent Vertices are v_1, v_2 and v_2, v_4 etc.

(b) The Adjacent Edges are e_1, e_2 and e_3, v_4 etc.

Fig. 5.1.13 : Representation of Graph with Adjacent Vertices and Adjacent Edges

5.1.14 Finite and Infinite Graph

A graph is finite if both its vertex set and the edge set are finite. Otherwise it is an infinite graph.

Hereafter by a graph G means only a finite graph (for which V(G) is a nonempty finite set).

5.1.15 Weighted Graph

A graph in which weights are assigned to every edge is called a weighted graph.

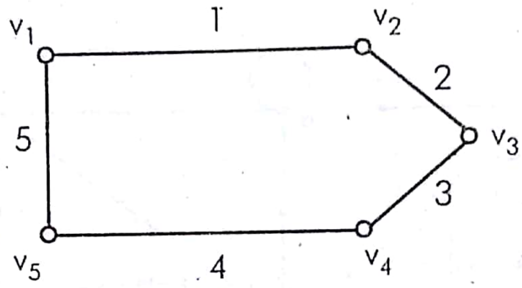


Fig. 5.1.14 : Weighted Graph

Here 1, 2, 3, 4, 5 are weights assigned to each edge respectively.

5.1.16 The Fundamental Theorem of Graph Theory

THEOREM 1

Now, choose an edge e in E(G). This can be done in |E| ways. This edge has precisely two end vertices and they give two elements of S. Summing over every edge $e \in E(G)$, we get,

$$|S| = 2|E| \quad \ldots (2)$$

From (1) and (2) we get,

$$\sum_{i=1}^{n} d_i = 2|E|$$

i.e., the sum of the degrees of all the vertices of any graph is even. This is also known as "Hand Shaking Lemma".

Result 1 : Any graph has even number of odd degree vertices.

Let W be the set of vertices of odd degree and U be the set of vertices of even degree.

Then,

$$\sum_{v \in V(G)} \deg v = \sum_{v \in W} \deg v + \sum_{v \in U} \deg v = 2|E| \text{ but,}$$

$\sum_{v \in U} \deg v$ is even.

Hence $\sum_{v \in W} \deg v$ is even.

Result 2 : If $k = \delta(G)$ is the minimum degree of all the vertices of a graph G then

$$k|V| \leq \sum_{v \in V} \deg v = 2|E|.$$

In particular, if G is a k regular graph then $k|V| = \sum_{v \in V(G)} \deg v = 2|E|$.

EXAMPLE PROBLEM 1

Is there a graph with degree sequence (1, 3, 3, 3, 5, 6, 6).

SOLUTION

It is not possible to draw a graph with a given degree sequence because the number of vertices of odd degree must be even.

EXAMPLE PROBLEM 2

Is there a simple graph with a degree sequence (1, 1, 3, 3, 3, 4, 6, 7).

SOLUTION

It is not possible to draw a simple graph with a given degree sequence.

Suppose that there is a simple graph with a given sequence then the vertex of degree 7 must be adjacent to all the remaining vertices, so in particular it is adjacent to both the vertices of degree one. So the vertex of degree 6 cannot be adjacent to either of these two vertices of degree 1 and also the graph is simple and the vertex of degree 6 cannot be adjacent to itself. So, there are only five vertices left which can be adjacent to vertex of degree 6 and this is a contradiction.

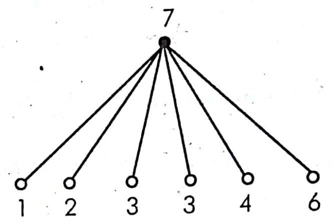


Fig. 5.1.15 : Degree Sequence of Seventh Vertex

EXAMPLE PROBLEM 3

Is there a simple non-directed graph or determine the degree sequence is a graph or not.

(i) (2, 3, 3, 4, 4, 5)

(ii) (2, 3, 4, 4, 5)

(iii) (1, 3, 3, 3)

(iv) (2, 3, 3, 4, 5, 6, 7)

SOLUTION

(i) It is not possible to draw a simple graph with a given degree sequence because the number of odd degree vertices must be even.

(ii) It is not possible to draw a simple graph with a given degree sequence.

Because in order to locate a vertex of degree 5 it should be adjacent to 5 vertices but as it is a simple graph self loops are not allowed. So, only 4 vertices are left that can be adjacent to vertex of degree 5 which is a contradiction.

Similarly,

(iii) (1, 3, 3, 3), (iv) (2, 3, 3, 4, 5, 6, 7) both (iii), (iv) are also not possible to draw a graph.

5.1.17 Path

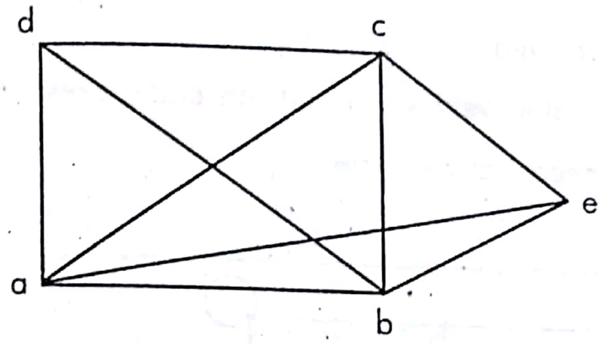
In a non-directed graph G, a sequence 'P' of zero or more edges of the form $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}$ or $v_0 v_1 \ldots v_n$ is called a path from v_0 to v_n. Where v_0 is the initial vertex and v_n is the terminal vertex of the path P.

(1) In a path, vertices and edges may be repeated any number of times.

(2) The number of edges in a path is called length of the path.

(3) A path of length zero is called trivial path.

Example : Conisder the following graph and it's path and length.



Path	Length
a-b-c-d	3
a-b-a	2
a-b-c-d-a	4
a-b-c-d-b-e-a	6
a	0
a-b	1

Fig. 5.1.16 : Representation of Path and Length using a given Graph

(i) Open Path : A path in which initial and terminal vertices are distinct is called open path. In above example a – b – c – d is open path.

(ii) Closed Path : A path in which initial and terminal vertices are same is called closed path. In the above example,

a, a – b – a, a – b – c – d – a, a – b – c – d – b – e – a, are closed paths.

A trivial path is taken as a closed path.

5.1.18 Simple Path

A path is said to be simple if all the edges and vertices on path are different except possibly at the end points.

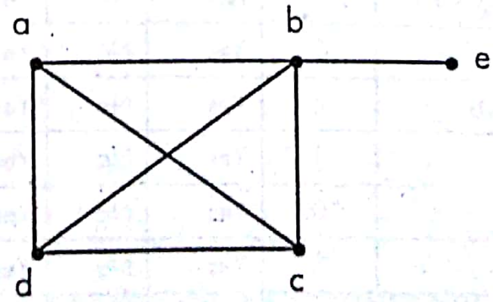


Fig. 5.1.17 : Simple Path Representation

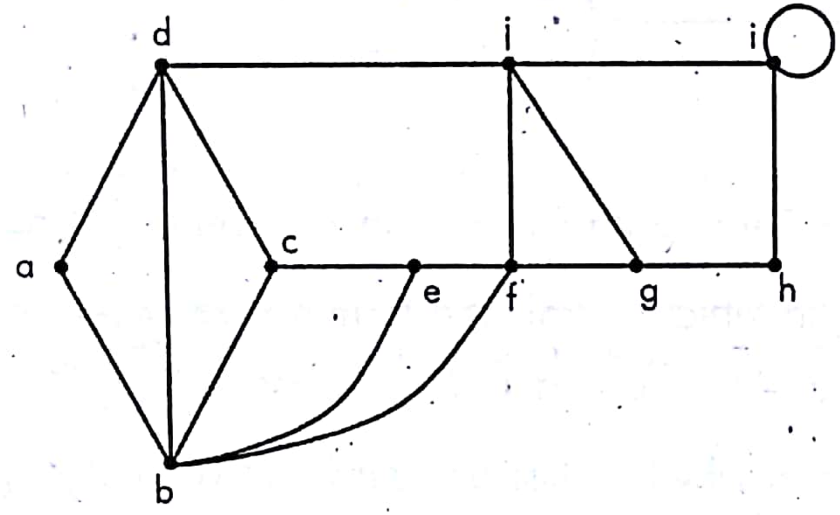


Fig. 5.1.18 : Graph(G)

1.18 we can identify different paths. They are given in the Table 5.1.1.

Table 5.1.1 : Representation of different Paths using given Graph

S.No.	Path	Length	Simple	Open	Closed	Circuit	Cycle
1)	a–b–c–e–f–j–d–a	7	Yes	No	Yes	Yes	Yes
2)	b–c–e–f–g–i–f–b	7	No	No	Yes	Yes	No
3)	a–b–a	2	Yes	No	Yes	No	No
4)	a–b–c–b–a	4	Yes	No	Yes	No	No
5)	i – i	1	Yes	No	Yes	Yes	Yes
6)	a	0	Yes	No	Yes	No	No
7)	d–b–c–d	3	Yes	No	Yes	Yes	Yes
8)	e–f–g–j–f–b	5	No	Yes	No	No	No

EXAMPLE PROBLEM 2

In the undirected graph,

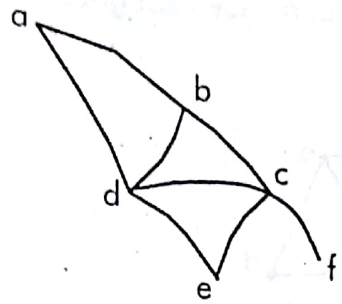


Fig. 5.1.19 : Undirected Graph

Find,

(i) An a-a circuit of length 6.

(ii) An a-a cycle of maximum length.

SOLUTION

(i) From graph G given in Fig. 5.1.19 an a-a circuit of length 6 is {a, b, d, c, e, d, a}.

(ii) An a-a cycle of maximum length is,

{a, b, c, e, d, a}

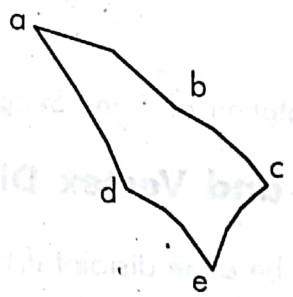


Fig. 5.1.20 : An a-a Cycle of Maximum Length

EXAMPLE PROBLEM 3

Let G be the following graph shown in Fig. 5.1.21 then,

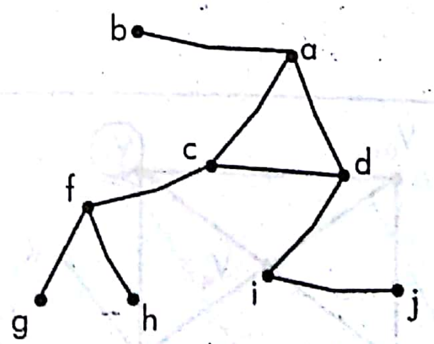


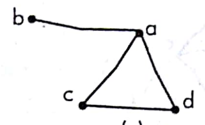
Fig. 5.1.21 : Graph(G)

How many connected subgraphs of G have four vertices and include a cycle? Also write these subgraphs.

SOLUTION

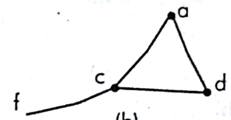
From Fig. 5.1.21 graph G has 3 connected subgraphs and four vertices and include a cycle. They are as follows.

{b, a, c, d}



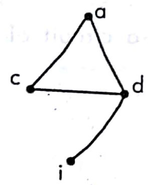
(a)

{f, c, d, a}



(b)

{i, d, a, c}



(c)

Fig. 5.1.22 : Representation of Three Subgraphs using Graph(G)

5.1.20 Edge Disjoint Paths and Vertex Disjoint Paths

Two paths in a graph are said to be edge disjoint if they have no common edges, but they have a common vertices.

Two paths in a graph are said to be vertex disjoint if they have no common vertices (once they have no common vertices, they cannot have edges also).

Example

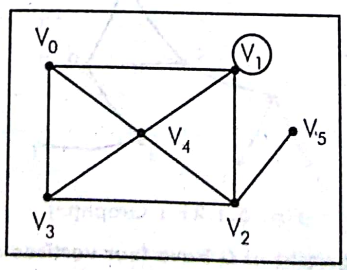


Fig. 5.1.23 : Representation of Edge Disjoint and Vertex Disjoint Path in a Graph

(1) $\{(V_0, V_4), (V_4, V_3), (V_1, V_4), (V_4, V_2)\}$ are edge-disjoint, since they have no common edges.

(2) $\{(V_0, V_3), (V_3, V_2), (V_4, V_2)\}$ are edge disjoint.

(3) $\{(V_0, V_1)\}$ is edge disjoint.

(4) $\{(V_0, V_3)\}, \{(V_1, V_2) (V_2, V_5)\}$ are vertex disjoint.

(5) $\{(V_3, V_2) (V_2, V_5)\}$ and $\{(V_0, V_4) (V_4, V_1)\}$ are the vertex disjoint.

5.1.21 Connected Graph

An undirected graph is connected if there is a path between every pair of distinct vertices of the graph.

A graph that is not connected is the union of two or more connected subgraphs, each pair of which has no vertex in common. These disjoint connected subgraphs are called the connected components of the graph.

Examples

(1) A graph with 7 vertices.

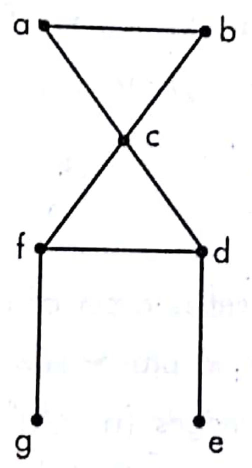


Fig. 5.1.24 : Graph(G_1)

(2) A graph with 11 vertices and 3 components.

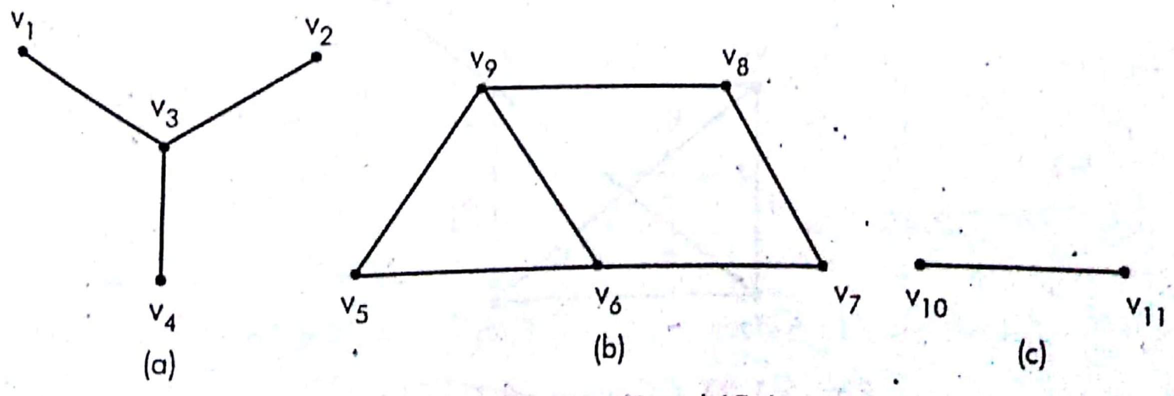


Fig. 5.1.25 : Graph(G_2)

5.1.22 Cut Vertex

If a graph G is connected and e is an edge such that G – e is not connected, then e is said to be a bridge or cut edge. i.e., the removal of cut edge produces a graph with more connected components than in the original graph. If v is a vertex of G such that G – v is not connected, then, v is a cut vertex.

Example : Consider the graph,

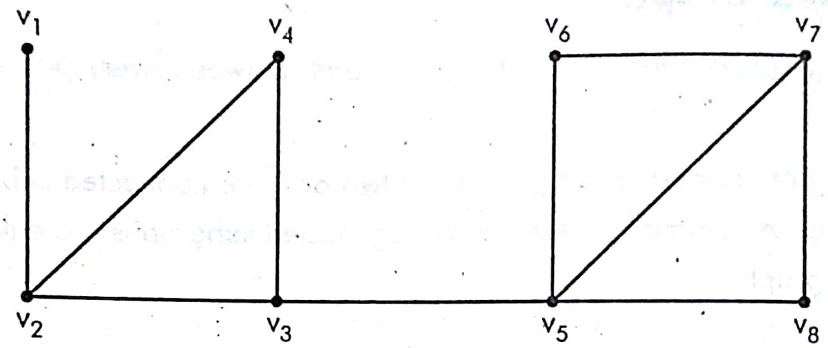


Fig. 5.1.26 : Graph(G)

In the Fig. 5.1.26, cut vertices are v_2, v_3, v_5. The removal of one of these vertices disconnects the graph, cut edges are $\{v_1, v_2\}$ and $\{v_3, v_5\}$ removing either one of these edges disconnects G.

5.1.23 Cut Set

In a connected graph G, a cut set is a set of edges whose removal from G leaves G disconnected, provided removal of no proper subset of these edges disconnects G. For example, in the Fig. 5.1.27, the set of edges {a, c, d, f} is a cut set. The other cut sets are {a, b, g}, {a, b, e, f}. Where as the set {a, c, h, d} is not a cut set because one of its proper subsets {a, c, h} is a cut set.

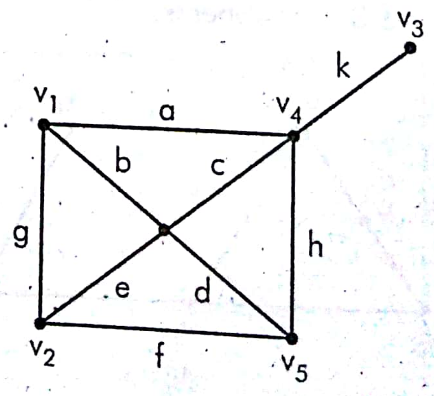


Fig. 5.1.27 : Connected Graph(G)

5.1.24 Edge Connectivity

Each cut set of a connected graph G consists of a certain number of edges. The number of edges in the smallest cut-set is defined as the edge connectivity of G or the edge connectivity $\lambda(G)$ of a connected graph G is the smallest number of edges whose removal disconnects G.

We know that, in G if the removal of e disconnects G and an edge is a bridge iff e does not belong to any cycle of G. The edge connectivity of any graph with a bridge is 1.

EXAMPLE PROBLEM 1

Find the edge connectivity of the graph G given in the Fig. 5.1.28.

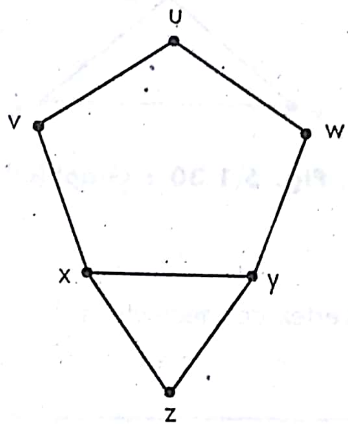


Fig. 5.1.28 : Graph(G)

SOLUTION

Clearly, this graph does not have any bridges. Therefore, its edge connectivity is more than 1. Now, if we remove the edges xz, zy, then, the graph gets disconnected. Similarly, there are other sets of two edges, namely {x v, v u} and {u w, w y} whose removal disconnects G. Therefore, we get that edge connectivity as 2.

Bridge : Bridge is an edge which connects the two vertices and removal of such edge disconnects the graph into two disjoint graphs.

Example

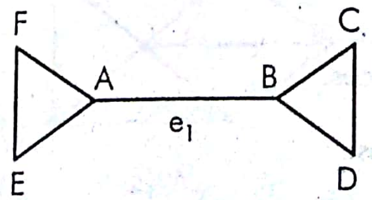


Fig. 5.1.29 : Edge Connectivity of Two Disjoint Graphs.

Here edge e_1 is the bridge for the graph, removal of which makes the graph disconnected.

Scanned by CamScanner

5.1.25 Vertex Connectivity

The vertex connectivity of a connected graph G is the smallest number of vertices whose removal disconnects G and is denoted by k(G).

EXAMPLE PROBLEM 1

Find the vertex connectivity of the graph in Fig. 5.1.30.

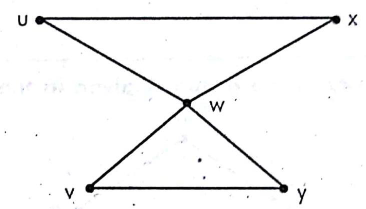


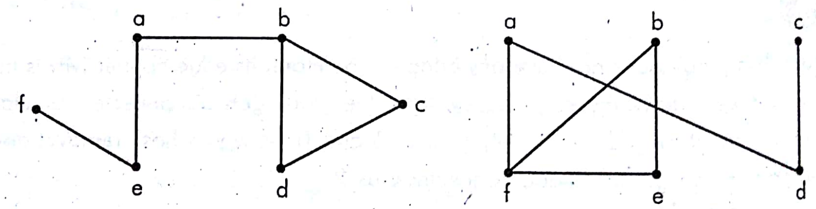
Fig. 5.1.30 : Graph(G)

SOLUTION

Here the cut vertex is w, so vertex connectivity is 1.

EXAMPLE PROBLEM 2

Identify the cut vertices (if any) in the graphs shown in Fig. 5.1.31.



(a) Graph(G_1) (b) Graph(G_2)

Fig. 5.1.31

SOLUTION

(a) b and a are cut vertices.

(b) a, d, f are cut vertices.

For a connected graph G, k(G) ≥ 1 and a connected graph G is said to be k-connected if k(G) = k.

5.1.26 Cycle Graph

A cycle graph of order n is a connected graph whose edges forms a cycle of length 'n' and is denoted by C_n.

(1) In a cycle graph of order 'n' vertices will have n vertices and n edges.

(2) In C_n, $\deg(v_i) = n - 1 \;\forall i$ and every C_n is a regular graph.

Example

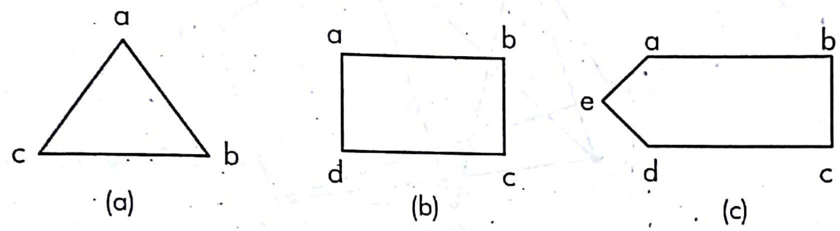


Fig. 5.1.32 : Examples of Cycle Graph

5.1.27 Wheel Graph

A wheel graph of order 'n' is a graph obtained by joining a single new vertex to each vertex of cycle graph (C_{n-1}) of order $(n - 1)$. It is denoted by W_n.

In W_n, $|V| = n$,

$|E| = 2(2-1)$ $\quad\quad [\because (n-1) + (n-1) = 2(n-1)]$

Example : Consider the following graph,

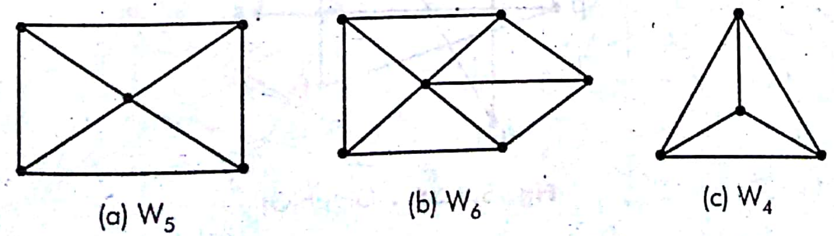


Fig. 5.1.33

Scanned by CamScanner

5.1.28 Bipartite Graph

A graph G is said to be bipartite if its vertex set V can be partitioned into two non empty sets X and Y such that every edge in the graph connects a vertex in X and a vertex in Y so that no edge in G connects either two vertices in X or two vertices in Y.

EXAMPLE PROBLEM 1

Consider the graph.

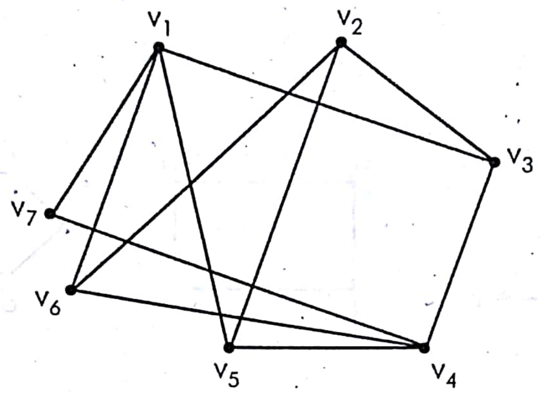


Fig. 5.1.34 : Graph(G)

SOLUTION

The graph G is bipartite since the vertex is the union of two disjoint sets $\{v_3, v_5, v_6, v_7\}$.

EXAMPLE PROBLEM 2

Consider the graph.

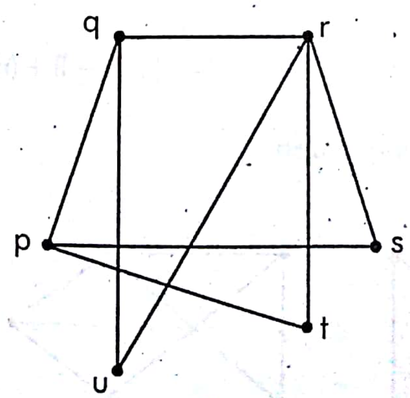


Fig. 5.1.35 : Graph(G)

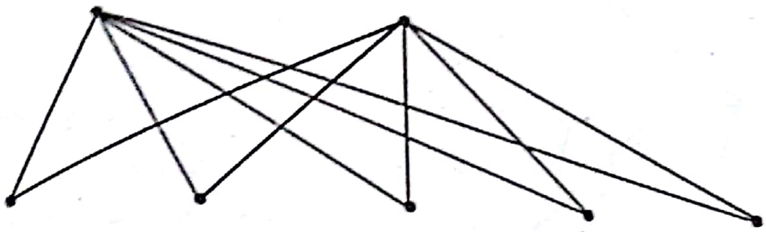
SOLUTION

This graph is not bipartite, since we cannot partition the vertex set into disjoint sets.

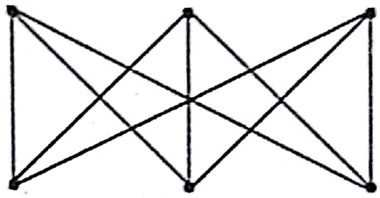
5.1.29 Complete Bipartite Graph

A bipartite graph is said to be a complete bipartite graph, if every vertex in X is connected to every vertex in Y. A complete bipartite graph with bipartition $|X| = m$, $|Y| = n$, $X \cup Y = V$ contains mn edges and m + n vertices.

Example



(a) $K_{2,5}$



(b) $K_{3,3}$

Fig. 5.1.36

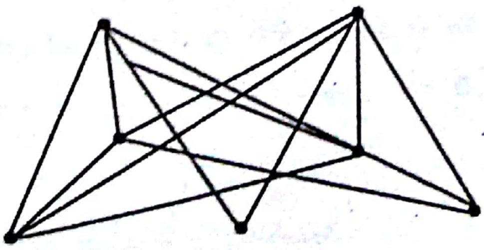
5.1.30 K-Partite Graph

A tri-partite graph is a graph where the set of vertices may be partitioned into three subsets. So, that no edge has incident vertices in the same subset.

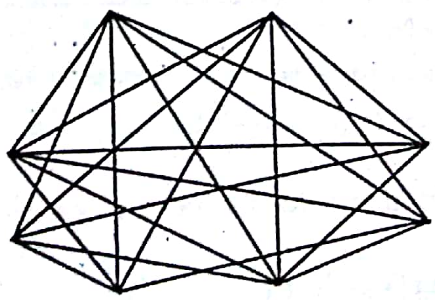
In general a graph G is K-partite if it is possible to partition the vertex set of G into k disjoint subsets $V_1, V_2 \ldots\ldots V_k$ such that each edge has one end point in one subset V_i and other end point in from V_j $(i \neq j)$.

A complete K-partite graph is a simple K-partite graph with the additional property that each vertex in V_i is adjacent to each vertex in V_j where $i \neq j \; \forall i$ and j if $n_i = |V_i|$ then the complete K-partite graph is denoted by $K_{n_1, n_2, n_3 \ldots\ldots n_k}$.

Example



(a) $K_{2,2,3}$



(b) $K_{2,2,2,2}$

Fig. 5.1.37

THEOREM 6

The vertex connectivity of a graph can never exceed its edge connectivity.

PROOF

Let $\lambda = \lambda(G)$ be the edge connectivity of a graph G.

Then, there exists a cut set S in G with λ edges. Let v_1 and v_2 be the parts into which G is broken by S. Then every edge in S has one end vertex in v_1 and the other end vertex in v_2. Therefore the graph G becomes disconnected if the end vertices of the edges of S that belong to v_1 (or v_2) are removed. The number of such vertices is λ and by the definitions of vertex connectivity, this number cannot be less than k(G) i.e., $\lambda(G) \geq k(G)$.

From the above two theorems we can conclude that for any connected graph $k(G) \leq \lambda(G) \leq \delta(G)$. For example consider the connected graph shown in Fig. 5.1.50.

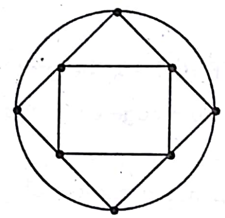


Fig. 5.1.50 : Connected Graph(G)

For this graph, $k(G) = \lambda(G) = \delta(G) = 4$.

5.2 SUBGRAPHS, COMPLEMENTS AND FUSION OF GRAPHS

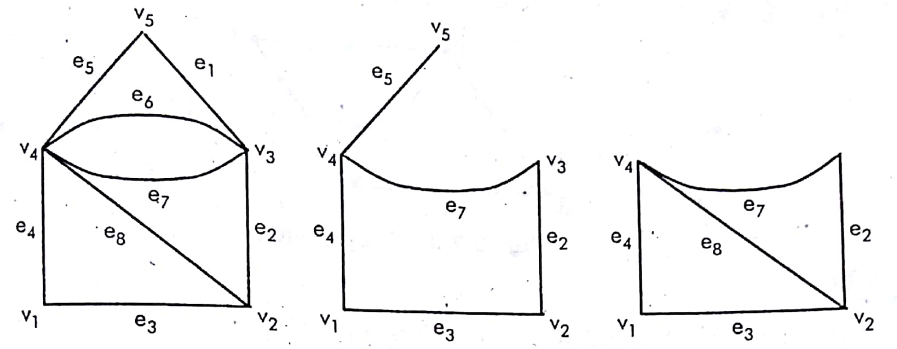
5.2.1 Subgraphs

If G and H are two graphs with vertex sets V(H) and V(G) and edge sets E(H) and E(G) respectively such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we call H as a subgraph of G.

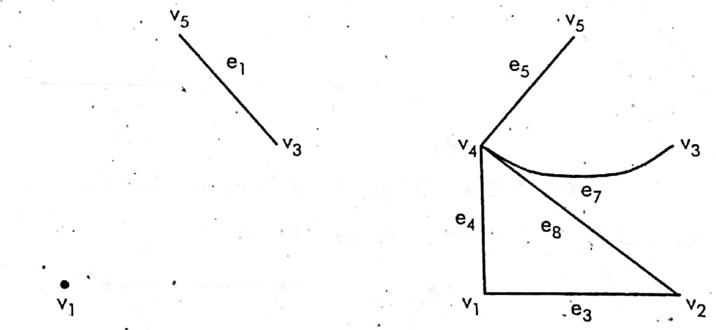
If $V(H) \subset V(G)$ and $E(H) \subset E(G)$ then H is a proper subgraph of G and if $V(H) = V(G)$ then we say that H is spanning subgraph of G.

Suppose that V_1 is a non-empty subset of V. The subgraph of G whose vertex set is V_1 and whose edge set is the set of those edges of G that have both ends in V_1 is called the subgraph of G induced by V_1 and is denoted by $G - V_1$ i.e., $G - V_1$ is the subgraph obtained from G by deleting the vertices in V_1 together with their incident edges. Similarly $G - E_1$ is the subgraph obtained from G deleting the edges in E_1 where E_1 is any one empty subset of E. $G - \{E_1\}$ is an edge induced subgraph of G.

By deleting from a graph G all loops and in each collection of parallel edges all edges but the one we obtain is a simple spanning subgraph of G, called the underlying simple graph of G.



(a) Graph G (b) A Spanning Subgraph of G (c) Proper Subgraph of G



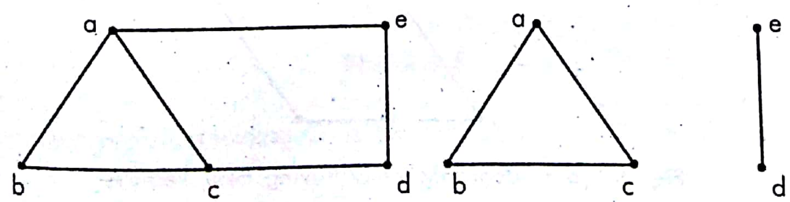
(d) The Vertex Induced Subgraph G.{v_2 v_4} (e) The Edge Induced Subgraph of G

Fig. 5.2.1

If G and H are graphs then H is a subgraph of G iff;

(i) $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

(ii) For every $e' \in E'$, if e' is incident on V' and W' then, $V', W' \in E'$.



(a) Graph(G_1) (b) Graph(G_2) (c) Graph(G_3)

Fig. 5.2.2

EXAMPLE PROBLEM 1

The graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ shown in Fig. 5.2.3 since $V' \subseteq V$ and $E' \subseteq E$.

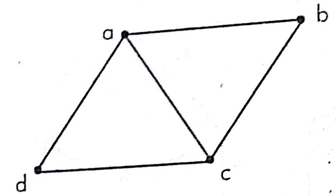


Fig. 5.2.3 : Subgraph(G)

SOLUTION

The subgraphs of Fig. 5.2.3 with two vertices are,

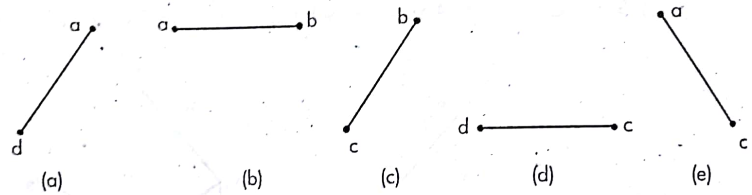


Fig. 5.2.4 : Subgraphs of G having Two Vertices

The subgraphs of Fig. 5.2.3 with three vertices are,

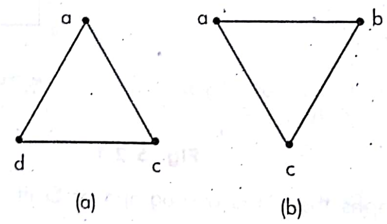


Fig. 5.2.5 : Subgraphs of G having Three Vertices

The subgraphs of Fig. 5.2.3 with four vertices are,

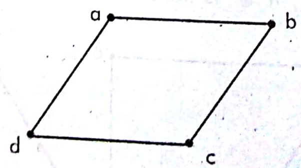
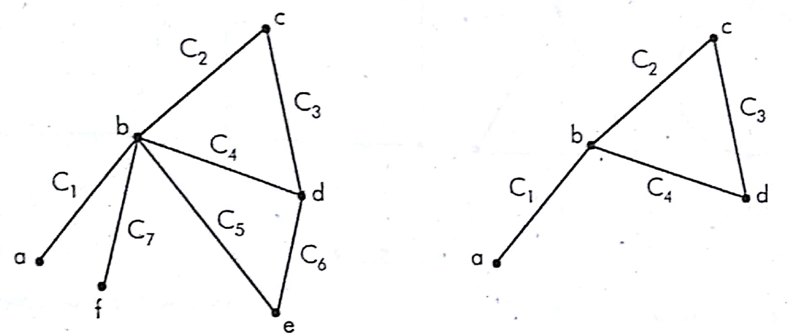


Fig. 5.2.6 : Subgraphs of G having Four Vertices

All the above subgraphs are having atleast two vertices. There are totally 8 subgraphs possible.

5.2.2 Component

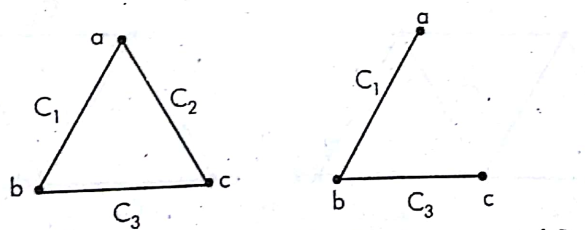
Let G be a graph and let v be a vertex in G. The subgraph G′ of G consisting of all edges and vertices in G that are contained in some path beginning at v is called component of G containing V.



(a) Graph(G) (b) G′ is a Component of G

Fig. 5.2.7

Example : Let G be the graph given in Fig. 5.2.8.



(a) Graph(G) (b) G′ is a Component of G

Fig. 5.2.8

The component of G containing G′ is the subgraph,

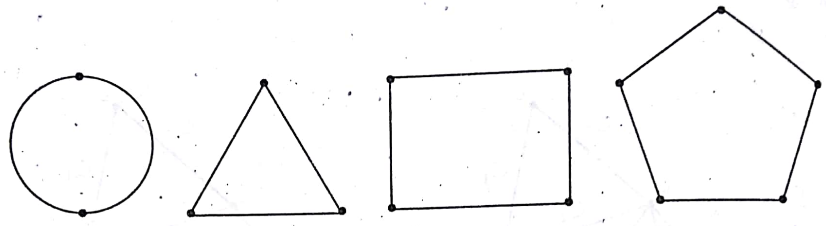
G′ = (V′, E′) where,

V′ = (a, b, c), E′ = {C_1, C_3}.

5.2.3 Euler Cycle

A cycle graph of order n is a connected graph whose edges form a cycle of length 'n' and is denoted by C_n.

Example



(a) C_2 (b) C_3 (c) C_4 (d) C_5

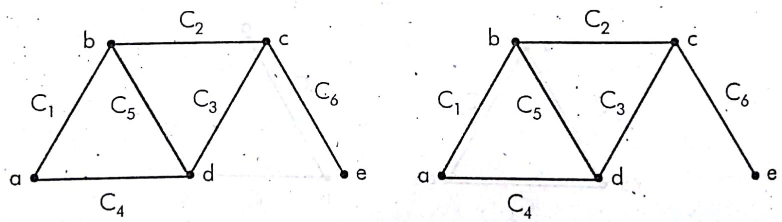
Fig. 5.2.9

A cycle (or circuit) is a path of non-zero length from v to v with no repeated edges.

A simple cycle is a cycle from v in which, except for the begining and ending vertices that are both equal to v. There are no repeated vertices.

Simple path is the path with no repeated vertices.

Example : For the following graph with the path specified state whether there is simple path, cycle or simple cycle ?



(a) Graph(G_1) (b) Graph(G_2)

Fig. 5.2.10

Table 5.2.1

Path	Simple Path	Cycle	Simple Cycle
a, b, c, d	Yes	No	No
a, b, c, d, a	No	Yes	Yes
a, b, c, d, c, e	Yes	No	No

Euler Cycle : An Euler cycle in a graph G is a cycle that includes all of the edges and all of the vertices of G.

Necessary and Sufficient Condition for an Euler Cycle : A graph G has an Euler cycle if and only if G is connected and the degree of every vertex is even.

THEOREM 1

If a graph G has an Euler cycle, then G is connected and every vertex has even degree.

PROOF

Suppose that G has an Euler cycle.

Every vertex in G has even degree.

If v and w are vertices in G, the portion of the Euler cycle that takes use from v to w serves as a path from v to w.

∴ G is connected.

Note

(1) If G is connected graph and every vertex has even degree, then G has an Euler cycle.

(2) If G consists of one vertex v and no edges. We call the path (v) an Euler cycles for G.

SOLVED PROBLEM 1

Let G be the graph as shown in Fig. 5.2.11.

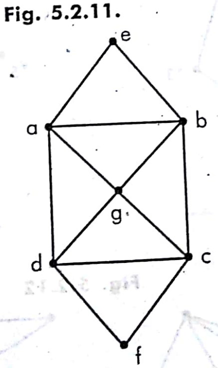


Fig. 5.2.11 : Graph(G)

SOLUTION

Observe that G is connected and that the degree of the vertices a, b, c, d and G are given below.

$\delta(a) = \delta(b) = \delta(c) = \delta(d) = 4 = \delta(g)$

Degree of e and f are, $\delta(e) = \delta(f) = 2$

Since the degree of every vertex is even by the theorem (If G is a connected graph and every vertex has even degree then G has an Euler cycle).

G has an Euler cycle.

By observation, we find the Euler cycle (d, a, e, b, c, d, g, b, a, g, c, f, d).

5.2.4 Complement of a Graph

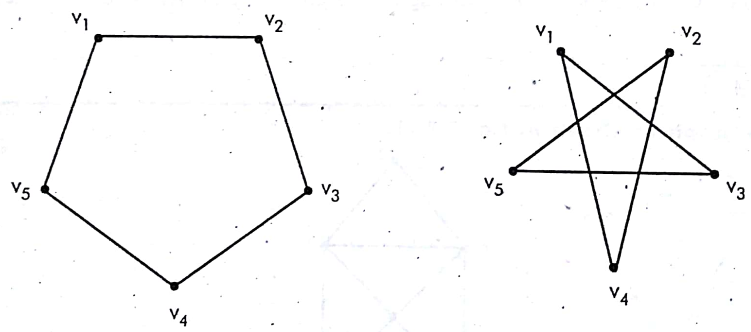
The complement $\bar{G}$, of a graph G = (V, E) is the graph with the vertex set V such that two vertices are adjacent in $\bar{G}$ if and only if these vertices are non adjacent in G. i.e., G is a graph with $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{(x, y) : (x, y) \notin E(G)\}$.

Suppose G is a graph with n-vertices and m-edges.

Then the number of edges in $\bar{G} = \frac{n(n-1)}{2} - m$. Since in a set V with n elements there can be ${}^nC_2 = \frac{n(n-1)}{2}$ edges.

Examples : A graph G and its complement.

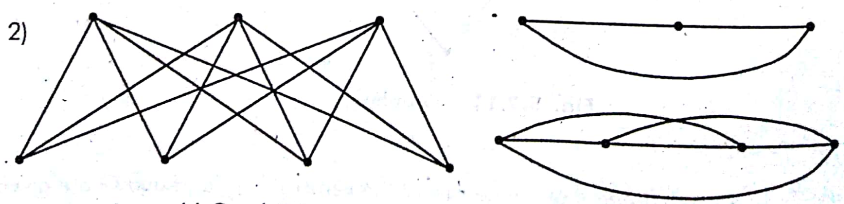
1)



(a) Graph(G) (b) Graph($\bar{G}$)

Fig. 5.2.12

2)



(c) Graph(G) (d) Graph($\bar{G}$)

Fig. 5.2.13

SOLUTION

Clearly graph G as shown in Fig. 5.2.15 has 8 vertices and 12 edges. So,

$\bar{G} = {}^{8}C_2 - 12 = 28 - 12 = 28 - 12 = 16$ edges.

The complement of the graph is,

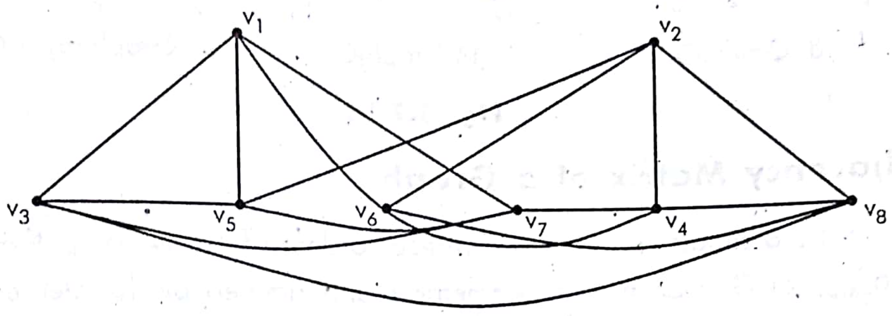


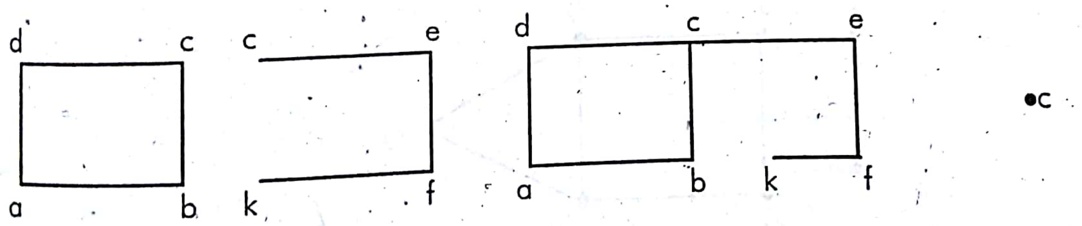
Fig. 5.2.16 : Complement of G(i.e., $\bar{G}$)

5.2.5 Union and Intersection of Graphs

If G and G' are two graphs then the union of these graphs are obtained by taking the union of the vertex sets and the union of the edges. This is denoted by $G \cup G'$.

Let G and G' are two graphs then the intersection of these graphs is denoted by $G \cap G'$ and obtained by taking the intersection of vertex set and intersection of edge.

Example : Consider the G and G′ given in Fig. 5.2.17.



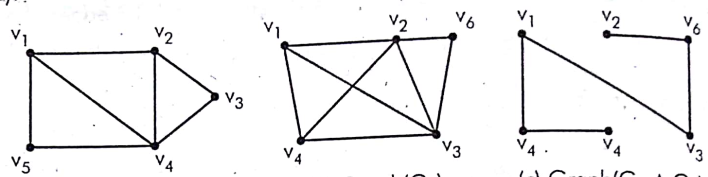
(a) Graph(G) (b) Graph(G′) (c) Graph($G \cup G'$) (d) Graph($G \cap G'$)

Fig. 5.2.17

Ring Sum of Graphs : Suppose we consider the graph whose vertex set is $V_1 \cap V_2$ and the edge set is $E_1 \Delta E_2$, where $E_1 \Delta E_2$ is the symmetric difference of E_1 and E_2. This graph is called the ring sum of G_1 and G_2, it is denoted by $G_1 \Delta G_2$. Thus,

$$G_1 \Delta G_2 = (V_1 \cap V_2, E_1 \Delta E_2).$$

For the two graphs G_1 and G_2 shown in Fig. 5.2.18(a) and (b), the ring sum is shown in Fig. 5.2.18(c).



(a) Graph(G_1) (b) Graph(G_2) (c) Graph($G_1 \Delta G_2$)

Fig. 5.2.18

5.2.6 Adjacency Matrix of a Graph

Let G (V, E) be a simple graph with n vertices ordered from v_1 to v_n, then, the adjacency matrix $A = [a_{ij}]_{n \times n}$ of G is an $n \times n$ symmetric matrix defined by the elements.

$$m_{ij} = \begin{cases} 1 \text{ when } v_i \text{ is adjacent to } v_j \\ 0 \text{ otherwise} \end{cases}$$

It is denoted by A(G) or A_G.

(1) An adjacency matrix of a graph is based on the ordering chosen for the vertices. Since there are n! different orderings of n-vertices, we have n! different adjacency matrices for a graph with n vertices.

(2) The sum of entries in a column of the adjacency matrix for an undirected graph is deg v.

Example : Consider the graph G of Fig. 5.2.19 and its adjacency matrix.

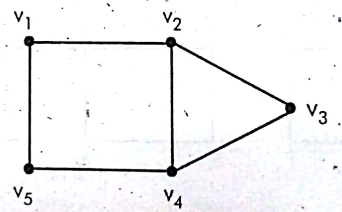


Fig. 5.2.19 : Graph(G)

$$A_G = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

5.2.7 Fusion of Graphs

Let u and v be distinct vertices of a graph G. We can construct a new graph G_1 by fusing the two vertices by replacing them with a single new vertex x such that every edge that was incident with either u or v in G is now incident with x i.e., the end u and the end v become end x. Thus the new graph has one less vertex than G but the same number of edges as G and the degree of the vertex x is the sum of degree of u and v.

Examples

(1)

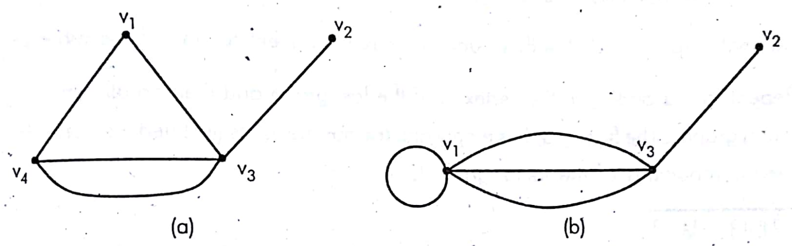


Fig. 5.2.20 : v_1 is Fused with v_4

(2)

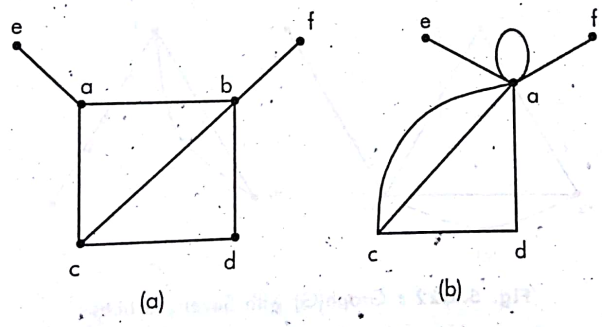


Fig. 5.2.21 : a is Fused with b

Algorithm to Find New Adjacency Matrix after Fusion

Step 1 : Change u's row to the sum of u's row with v's row and change u's column to the sum of u's column and v's column.

Step 2 : Delete the row and column corresponding to v. The resulting matrix is the adjacency matrix of the new graph G.

Algorithm for Connectedness

Step 1 : Replace G by its underlying simple graph. To get the adjacency matrix of new graph just replace all non zero entries off the diagonal by 1 and make all entries on the diagonal by 0. Denote the underlying simple graph also as G.

Step 2 : Fuse vertex v_1 to the first of the vertices $v_2, v_3, \ldots, v_n$ with which it is adjacent to give a new graph, also denoted by G, in which the new vertex is also denoted by v_1.

Step 3 : The above two step process gives the adjacency matrix A(G).

Step 4 : Repeat steps 1 and 2 with v_1 until v_1 is not adjacent to any of the other vertices.

Step 5 : Repeat step 2 and 4 on the vertex v_2 of the last graph and then on all remaining vertices of the resulting graphs. The final graph is empty and the number of its (isolated) vertices is the number of connected components of the initial graph G.

SOLVED PROBLEM 1

Given below is the adjacency matrix of graph G with seven vertices listed $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ use fusion algorithm to check the connectedness.

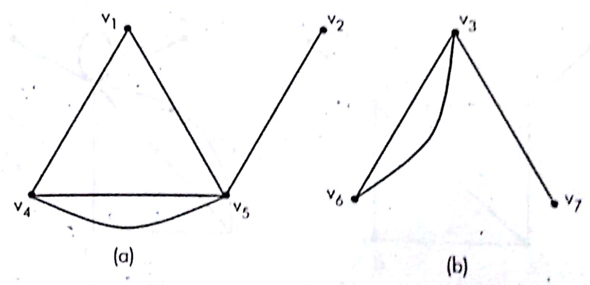


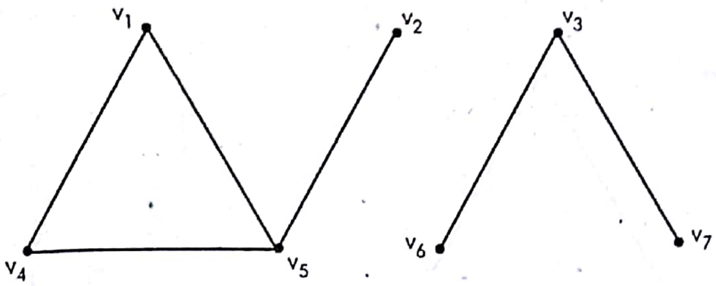
Fig. 5.2.22 : Graph(G) with Seven Vertices

Adjacency matrix of graph

$$G = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

SOLUTION

Step 1 : Replacing graph G shown in Fig. 5.2.22 by its underlying simple graph.



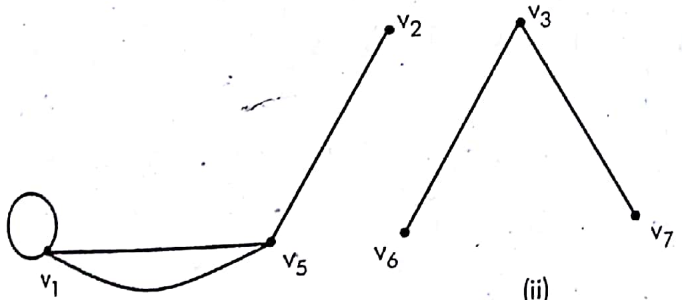
(a) (b)

Fig. 5.2.23 : Simple Graph

Adjacency matrix is given by,

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2 : Fusing v_1 with v_4.

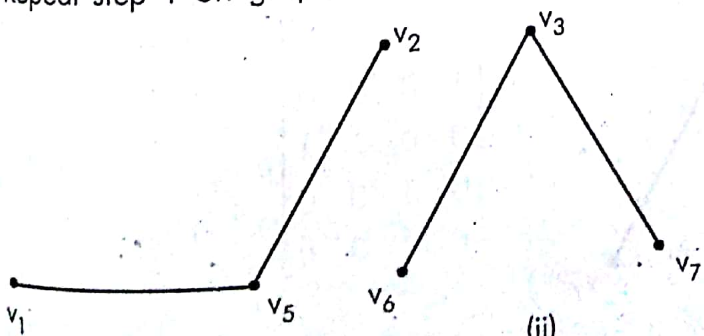


(i) (ii)

(a) Fusing v_1 with v_4

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Repeat step 1 on graph obtained in step 2.

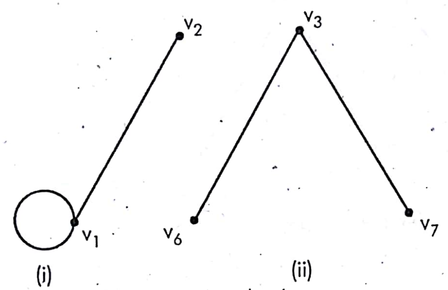


(i) (ii)

(b)

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

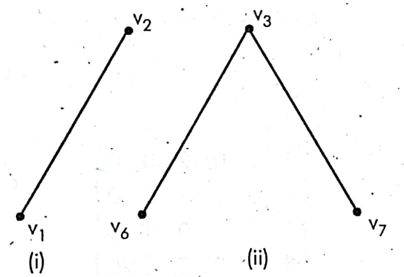
Fusing v_1 with v_5.



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(c) Fusing v_1 with v_5

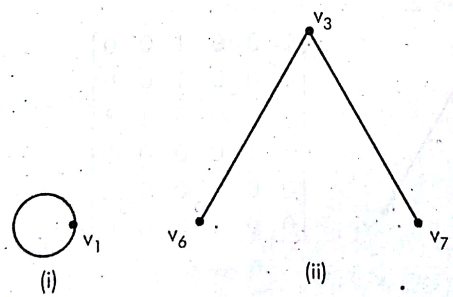
Repeat step 1.



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(d)

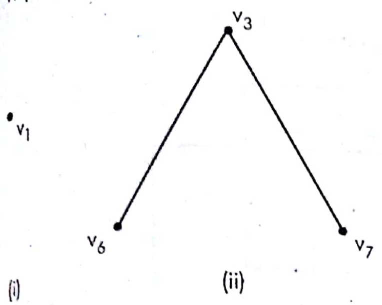
Fusing v_1 with v_2.



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(e) Fusing v_1 with v_2

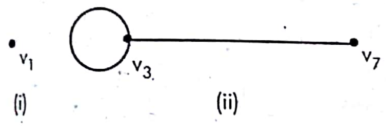
Repeating step 1.



$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(f)

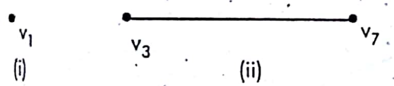
Fusing v_3 with v_6.



$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(g) Fusing v_3 with v_6

Repeating step 1.



$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(h)

Fusing v_3 with v_7.

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(i) Fusing v_3 with v_7

Repeating step 1

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(j)

Fig. 5.2.24

Since the final adjacency matrix is a 2×2 null matrix, we conclude that original graph G has 2 connected components.

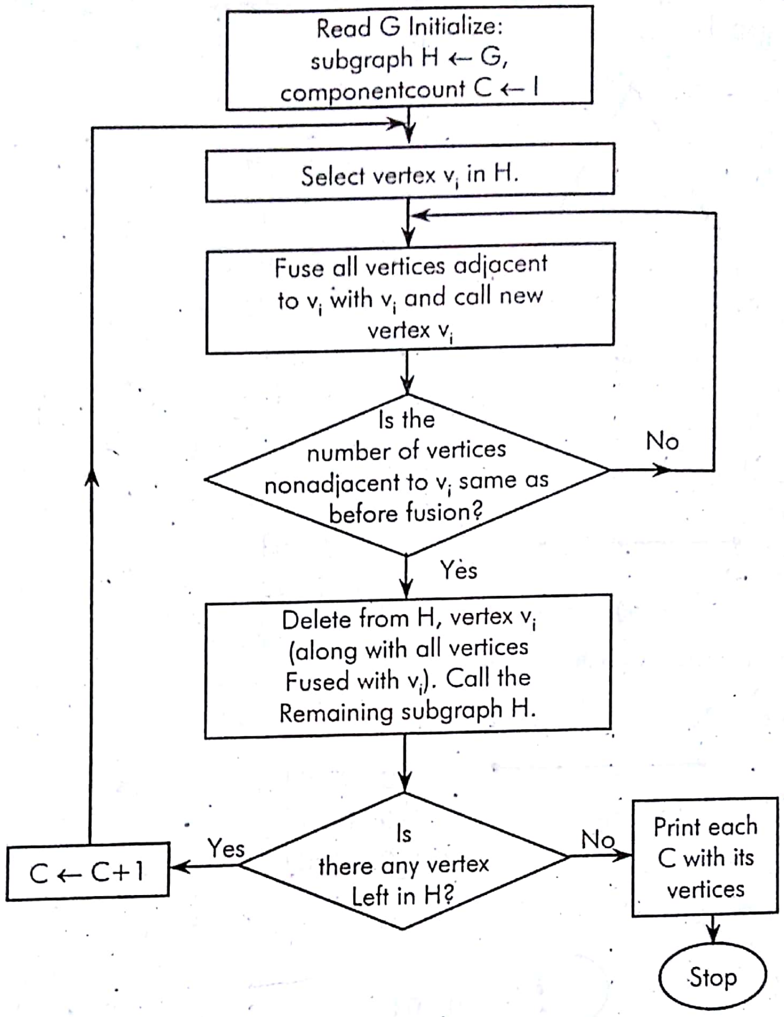


Fig. 5.2.25 : Flowchart

5.3 ISOMORPHISM

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. A function $f : G_1 \to G_2$ is called an isomorphism if,

(1) f is one to one.

(2) f is onto.

(3) $(x, y) \in E(G_1)$ if and only if $f(x), f(y) \in E(G_2)$ i.e., two vertices x and y are adjacent in G_1 iff f(x) and f(y) are adjacent in G_2. If the graph G_1 is isomorphic to G_2 then we write $G_1 \cong G_2$.

5.3.1 Properties of Isomorphism

(1) If two graphs G_1 and G_2 are isomorphic then,

(i) $|V(G_1)| = |V(G_2)|$

(ii) $|E(G_1)| = |E(G_2)|$

(iii) If $v \in V(G_1)$ then $\deg G_1(v) = \deg G_2(V)$

i.e., the degree sequences of G_1 and G_2 are the same.

(iv) If (u, u) is a loop in G_1, then $(f(u), f(u))$ is a loop in G_2.

(v) If $v_0 - v_1 - v_2 \ldots v_{n-1} - v_n - v_0$ is cycle of length n in G_1 then

$f(v_0) - f(v_4) \ldots f(v_{n-1}) - f(v_n) - f(v_0)$ is a cycle of length n in G_2.

(vi) If two graphs are isomorphic then their adjacency matrices are same.

(2) The isomorphism of simple graphs is an equivalence relation.

A graph G is isomorphic to itself by the identity function.

So, isomorphism is reflexive.

Suppose that the graph G_1 is isomorphic to G_2. Then there exists a function f which is one to one and onto and that preserves adjacency. We know that whenever a function $f : G_1 \rightarrow G_2$ is a bijection then $f^{-1} : G_2 \rightarrow G_1$ is also a bijection. So there exists $f^{-1} : G_2 \rightarrow G_1$ which is one to one and onto, and preserves adjacency and non-adjacency.

So, G_2 is isomorphic to G_1.

Hence isomorphism is symmetric.

Suppose $G_1 \cong G_2$ and $G_2 \cong G_3$. Then there are one to one correspondences f and g from G_1 to G_2 and G_2 to G_3.

Then there are one to one correspondences f and g from G_1 to G_2 and G_2 to G_3 that preserves adjacency and non adjacency.

If follows that gof is a one to one mapping from G_1 to G_3 that preserves adjacency and non adjacency.

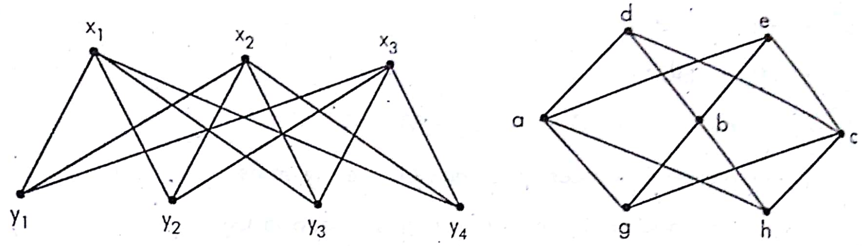
Therefore G_1 is isomorphic to G_3.

Hence isomorphism is transitive.

Therefore isomorphism is an equivalence relation.

EXAMPLE PROBLEM 1

Consider the following two graphs.



(a) Graph(G) (b) Graph(H)

Fig. 5.3.1

SOLUTION

Define a map $f : V(G) \rightarrow V(H)$ as follows.

$f(x_1) = a$, $f(x_2) = b$, $f(x_3) = c$, $f(y_1) = d$

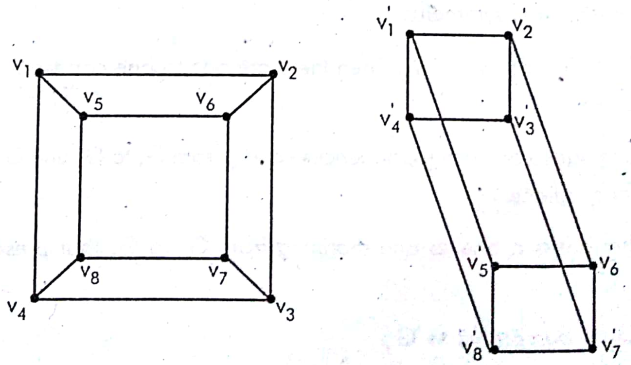
$f(y_2) = e$, $f(y_3) = g$, $f(y_4) = h$ then clearly f is one one and onto.

Also if $(x, y) \in E(G)$ then $(f(x), f(y)) \in E(H)$.

Therefore the two graphs given in Fig. 5.3.1 are isomorphic.

EXAMPLE PROBLEM 2

Consider the following two graphs.



(a) Graph(G) (b) Graph (G′)

Fig. 5.3.2

SOLUTION

The 2 graphs given in Fig. 5.3.2 are Isomorphic.

Conditions for isomorphism,

(i) Both graphs should have same number of vertices.

(ii) Both graphs should have same number of edges.

(iii) Both graphs should have equal number of vertices with same degree.

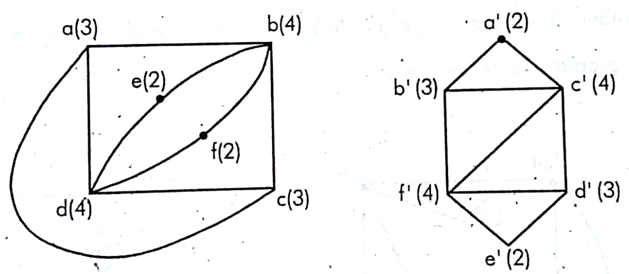
(iv) Both graphs should have same cycle vector (C_1, C_2, C_3 . . . . . . .C_n).

Where, C_i = number of cycles of length i.

5.3.2 Theorem and Solved Problems

SOLVED PROBLEM 1

(a) Determine whether the following graphs are isomorphic.



(a) Graph(G) (b) Graph(G')

Fig. 5.3.3

SOLUTION

(i) (V(G)) = (V(G'))

= 6

(ii) (E(G)) = (E(G'))

= 9

(iii) Degree of sequence of (G) = Degree of sequence of (G') (i.e., (2, 2, 3, 3, 4, 4) in G, (2, 2, 3, 3, 4, 4) in G').

SOLVED PROBLEM 11

Prove that C_5 is the only cycle graph isomorphic to its complement.

SOLUTION

[Nov./Dec. - 2008]

Let C_n, $\bar{C}_n$ contains same number of edges and we know that, the number of edges in C_n is n, also the number of edges in C_n + the number of edges in $\bar{C}_n = \frac{n(n-1)}{2}$.

$$n + n = \frac{n(n-1)}{2}$$

$$\Rightarrow \quad 4n = n^2 - n$$

$$\Rightarrow \quad n^2 - 5n = 0$$

$$\Rightarrow \quad n = 5$$

$\therefore$ C_5 is the only cycle graph isomorphic to its complement.

5.4 VERTEX DEGREE

5.4.1 Multigraph

Multigraphs consists of vertices and undirected edges between these vertices with multiple edges between pairs of vertices allowed.

Every simple graph is also a multigraph. However, not all multigraphs are simple graphs, since in a multigraph two or more edges may connnect the same pair of vertices.

A multigraph G = (V, E) consists of a set V of vertices, a set E of edges, and a function f from E to {u, v} u, v $\in$ V, u $\neq$ v.

The edges e_1 and e_2 are called multiple or parallel edges if $f(e_1) = f(e_2)$.

Multigraph is a triple G = (V, E, f) consisting of a set of vertices V, a set of edges E and on incident function f that maps each edge in E to pair of vertices in 'V'. Here,

V = {a, b, c, d}

E = {1, 2, 3, 4, 5}

f = {1(a, b), 2(b, c), 3(c, d), 4(d, a), 5(a, c)}

(1) Edges are uniquely identified and can determine the multiplicity of an edge between any two vertices.

(2) Vertices and edges can be labeled and edges can be assigned capacities and/or weights.

(3) Maximum flow and shortest paths algorithms may be applied to a multigraph.

(4) It is possible to determined if a multigraph is a planar.

Example

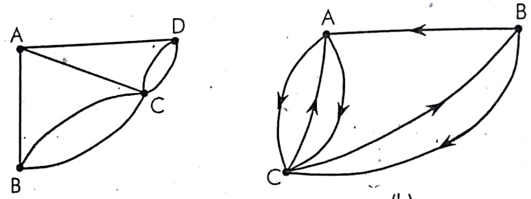


Fig. 5.4.1 : Multigraphs

5.4.2 Euler's Path

An Euler's path in a multigraph is a path that travels all edges exactly once and vertices can be travelled atleast once.

In any multigraph, we can get an Euler's path only if there are two vertices of odd degree and remaining vertices degree is even.

Further, the Euler's path will always start at odd degree, vertex and end at other odd degree vertex.

Example : The Euler's path in Fig. 5.4.2 is,

{(a, b), (b, c), (c, d), (d, e), (e, a), (a, d), (d, b), (b, e)}

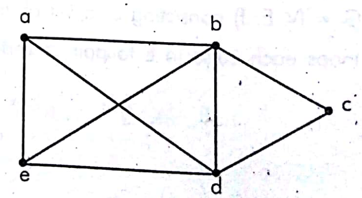


Fig. 5.4.2 : Euler's Path in a Multigraph

5.4.3 Euler's Circuit

An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

If the starting and ending vertices of Euler's path are same, then such a path is called Euler's circuit. There cannot be odd degree vertices in the graph in which case the circuit can start from any vertex and end on the same vertex.

EXAMPLE PROBLEM 1

Consider the graph in the Fig. 5.4.3.

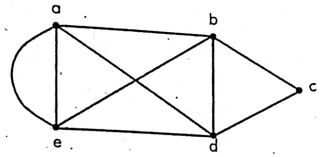


Fig. 5.4.3 : Graph(G)

SOLUTION

The Euler's circuit in Fig. 5.4.3 is,

{(a, b), (b, c), (c, d), (d, e), (e, a), (a, d), (d, b), (b, e), (e, a)}

EXAMPLE PROBLEM 2

Consider the graph shown in the Fig. 5.4.4.

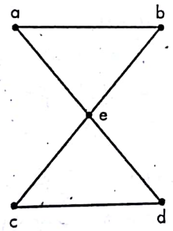


Fig. 5.4.4 : Graph(G)

SOLUTION

The graph has Euler circuit given by, a, e, c, d, e, b, a.

EXAMPLE PROBLEM 3

Write a note on "Konigsberg-bridge" problem.

SOLUTION

This problem is stated as, two islands C and D formed by pregetriven in Konigsberg were connected each other and two banks A and B with seven bridges as shown in the Fig. 5.4.5.

SOLVED PROBLEM 3

Consider the directed graph given in the Fig. 5.4.11. Construct an Euler circuit if it exists.

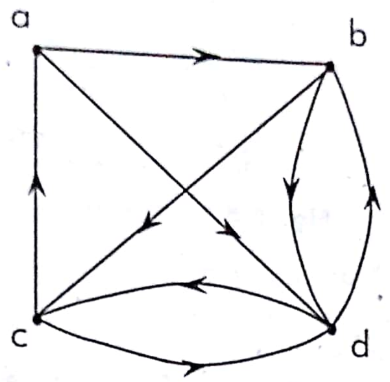


Fig. 5.4.11 : Directed Graph

SOLUTION

In-degree (a) = 1

Out-degree (a) = 2

In-degree (b) = 2

Out-degree (b) = 2

In-degree (c) = 2

Out-degree (c) = 2

In-degree (d) = 3

Out-degree (d) = 2

There is no Euler circuit because in-degree (a) ≠ out-degree (a) and the vertices a and d have odd-degree. However there exists Euler path a, b, d, b, c, d, c, a, d.

5.5 PLANAR GRAPHS

A graph G is called planar, if it can be drawn on the plane in such a way that no two edges cross eachother at any point, except possibly at the common end vertex. Such a drawing is called a plane drawing. A graph may be planar, even if it is usually drawn with crossings, since it may be possible to draw it in a different way without crossings.

Examples

(1) Consider the graph shown in the Fig. 5.5.1.

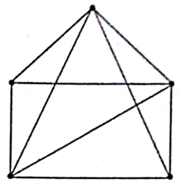
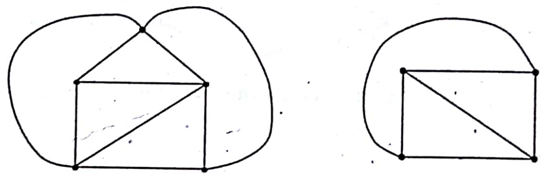


Fig. 5.5.1 : Graph(G)

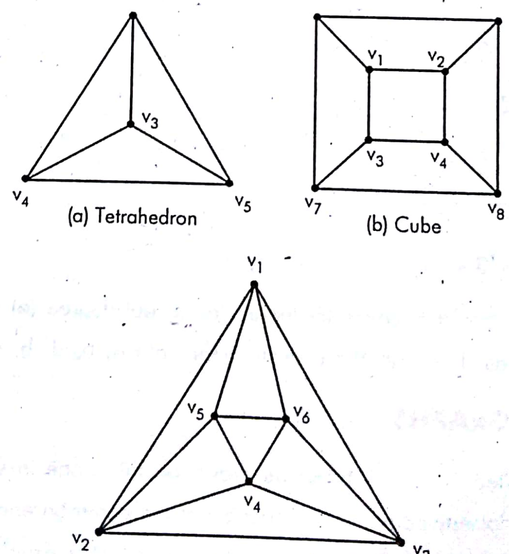
Planar graphs of Fig. 5.5.1 are G_1 and G'.



(a) Planar Graph(G_1) (b) Planar Graph(G')

Fig. 5.5.2

(2) Graphs of regular solids.



(a) Tetrahedron (b) Cube

(c) Octahedron

Fig. 5.5.3 : Regular Solids

5.6 HAMILTONIAN PATHS AND CYCLES

[Nov./Dec. - 2008]

A path $v_0, v_1, v_2 \ldots v_{n-1}, v_n$ in the graph G = (V, E) is called a Hamiltonian path if $V = \{v_0, v_1, \ldots v_{n-1}, v_n\}$ and $x_i \neq x_j$ for $0 \leq i \leq j \leq n$. i.e., a Hamiltonian path is defined as a simple path that contains all vertices of G but the end points may be different.

A graph G is said to be Hamiltonian if there exists a cycle containing every vertex of G. Such a cycle is referred to as a Hamiltonian cycle.

EXAMPLE PROBLEM 1

The Hamiltonian cycle is p, q, r, s, t, p

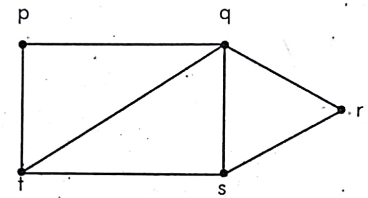


Fig. 5.6.1 : Hamiltonian Graph(G)

SOLUTION

A Hamiltonian graph contains a cycle containing all the vertices. So Hamiltonian graph cannot have cut vertices or pendant vertices.

5.6.1 Some Basic Rules for Constructing Hamiltonian Paths and Cycles

(1) If G has n vertices, then a Hamiltonian path must contain exactly n – 1 edges, and a Hamiltonian cycle must contain exactly n edges.

(2) There cannot be three or more edges incident with one vertex in a Hamiltonian cycle.

(3) When building a Hamiltonian path or cycle we should not leave any vertex.

(4) One of the Hamiltonian cycle we are building passed through a vertex V, then all other unused edges incident on V can be deleted because only two edges incident on V can be included in a Hamiltonian cycle.

EXAMPLE PROBLEM 1

Construct a non-Hamiltonian graph on 5 vertices.

SOLUTION

Since e is a cut vertex, there is no Hamiltonian path.

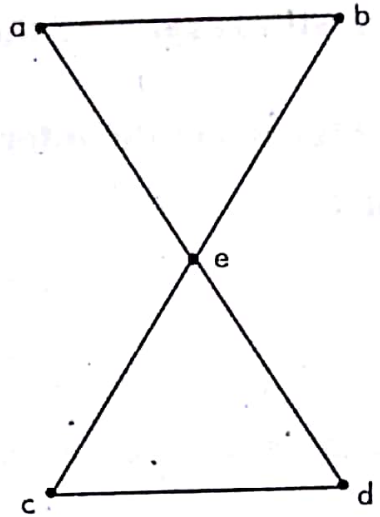


Fig. 5.6.3 : Non-Hamiltonian Graph on Vertices

EXAMPLE PROBLEM 2

Show that $K_{m,n}$ is not Hamilton when m + n is odd.

SOLUTION

Since $K_{m,n}$ is bipartite, it does not have cycle of odd length. Since it has an odd number of vertices, a Hamiltonian cycle in this graph if exists, must be of odd length. So $K_{m,n}$ is not Hamiltonian when m + n is odd.

EXAMPLE PROBLEM 3

Number of regions of degree 6 = 6 (5 + total hexagon).

$r_4 + r_4' = 3 - (1)$ ($\because$ r_4 means region with degree 4).

$r_6 + r_6' = 6 - (2)$ ($\because$ r_6 means region with degree 6).

By Grinberg's theorem if a Hamiltonian cycle existed, then we would have

$$\sum_{i=4}^{6} (i-2)(r_i - r_i') = 0$$

$$\Rightarrow (4-2)(r_4 - r_4') + (6-2)(r_6 - r_6') = 0$$

$$\Rightarrow 2(r_4 - r_4') + 4(r_6 - r_6') = 0$$

$$\Rightarrow (r_4 - r_4') = -2(r_6 - r_6')$$

But then $r_4 - r_4'$ must be an even integer.

However since $r_4 + r_4' = 3$, the only possibility for r_4 and r_4' are 0 and 3 and 1 and 2.

Neither of these possibilities is such that their difference is even.

$\therefore$ Assumption that there was a Hamiltonian cycle led to a contradiction.

$\therefore$ G_6 has no Hamiltonian cycle.

5.7 GRAPH COLORING

Given a planar or non-planar graph G, if we assign colors (colours) to its vertices in such a way that no two adjacent vertices have (receive) the same color, then we say that the graph G is properly colored. In other words, proper coloring of a graph means assigning colors to its vertices such that adjacent vertices have different colors.

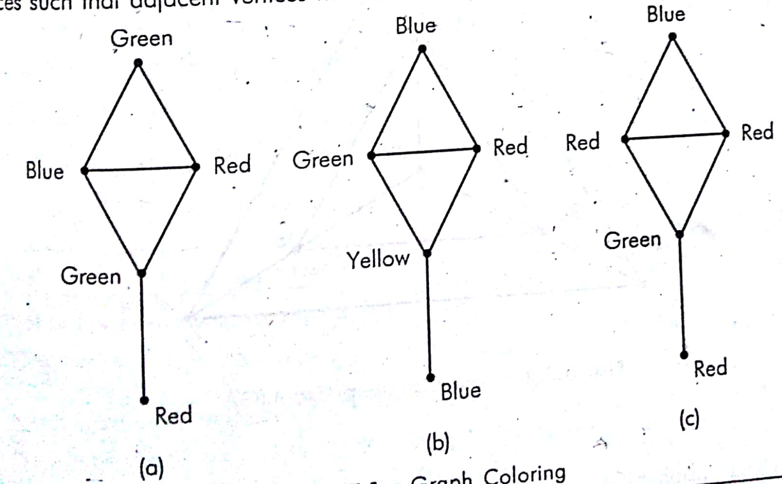


Fig. 5.7.1 : Graph Coloring

The Fig. 5.7.1 shows the first two graphs are properly colored where as the third graph is not properly colored.

By examining the first two graphs in Fig. 5.7.1 which are properly colored, we note the following.

(1) A graph can have more than one proper coloring.

(2) Two non-adjacent vertices in a properly colored graph can have the same color.

5.7.1 Chromatic Numbers

A k-vertex coloring of a graph G is an assignment of k colours to the vertices of G in such a way that no two adjacent vertices have the same color.

A graph G is k – colorable if there is a k - vertex coloring. The minimum number of colors required to color a graph G is called the vertex chromatic number of G and is denoted by χ (G).

EXAMPLE PROBLEM 1

Consider the graph,

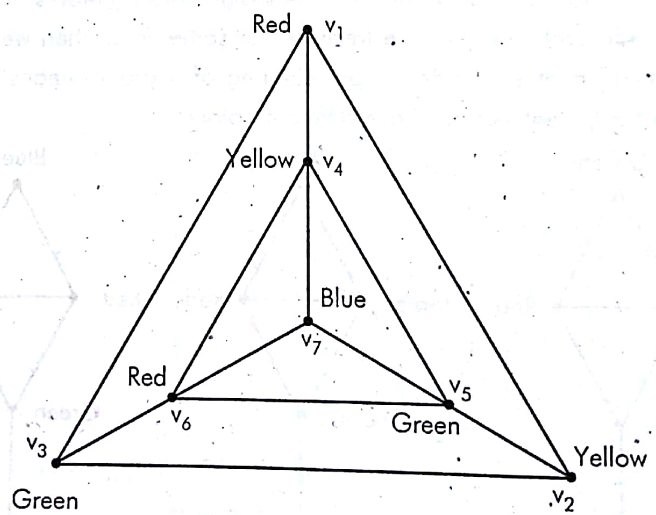


Fig. 5.7.2 : Four Colouring Graph(G)

SOLUTION

4 coloring of graph $\chi(G) = 4$.

5.7.1.1 Rules to Find Chromatic Numbers of a Graph G

Rule 1 : $\chi(G) \leq n$ where n is the number of vertices of G.

Rule 2 : A complete graph k_n, on n vertices require n colors.

Rule 3 : If H is a subgraph of G then $\chi(G) \geq \chi(H)$.

Rule 4 : If deg (v) = d, then atmost d colors are required to color the vertices adjacent to v.

Rule 5 : $\chi(G) = \max\{\chi(C) : C$ is a connected component of $G\}$.

Rule 6 : Every k-chromatic graph has atleast k colors for vertices v such that degree (v) $\geq k - 1$.

Rule 7 : For any graph G, $\chi(G) \leq 1 + \Delta(G)$ where $\Delta(G)$ is the largest degree of any vertex of G.

Rule 8 : If $\delta(G)$ the minimum degree of any vertex of (G) then,

$$\chi(G) \geq \frac{|V|}{|V| - \delta(G)} \quad \text{i.e.,} \quad \chi(G) = \frac{n}{n - \delta(G)}$$

Where n is the number of vertices in G.

Note

(1) A graph G is called a k-critical graph if $\chi(G) = k$ and $\chi(G - v) < \chi(G)$ for each vertex v of G.

(2) Since bipartite graph cannot contain circuits of odd length, it is 2-chromatic. The following three for a graph G are equivalent.

(i) G is chromatic.

(ii) G is bipartite.

(iii) G has no circuits of odd length.

EXAMPLE PROBLEM 1

What is the chromatic number of a tree with atleast two vertices?

SOLUTION

We know that bipartite graphs are graphs without odd cycles. Trees are a cyclic graphs i.e., they do not contain cycles as subgraphs. So trees are bipartite. Since trees are connected and we have assumed that it has atleast two vertices, hence it has the chromatic number 2.

Let G_1 be the subgraph induced by the vertices w, x, y, z. Let G_2 be the complete graph K_3 with vertices v, w and x. Then,

$G_1 \cap G_2$ is the edge {w, x}, so $G_1 \cap G_2 = K_2$.

Therefore,

$$P(G, \lambda) = \frac{P(G_1,\lambda)\cdot P(G_2,\lambda)}{\lambda^{(2)}}$$

$$= \frac{\lambda^{(4)}\cdot\lambda^{(3)}}{\lambda^{(2)}}$$

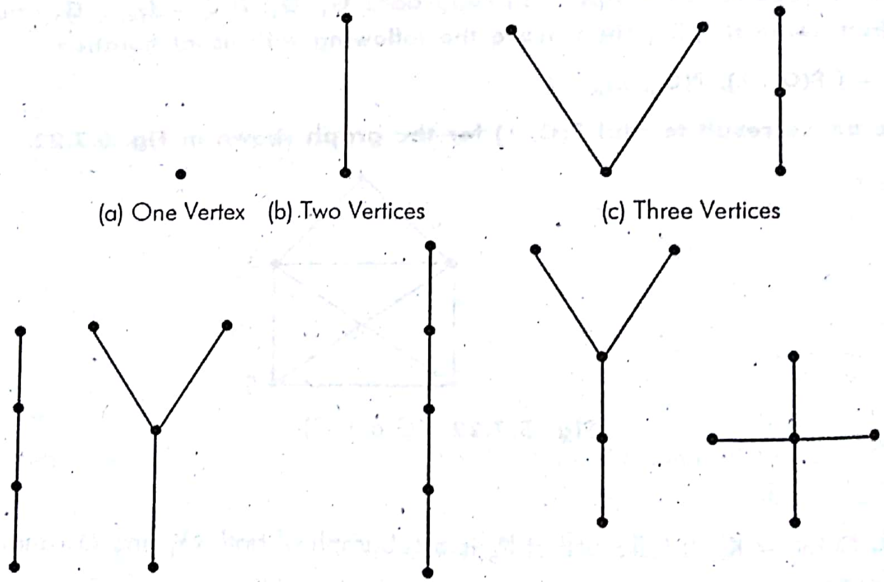
$$= \frac{\lambda^2(\lambda-1)^2(\lambda-2)^2(\lambda-3)}{\lambda(\lambda-1)}$$

$$= \lambda(\lambda-1)(\lambda-2)^2(\lambda-3).$$

5.8 DEFINITIONS OF TREE

A tree is a simple graph G such that there is a unique simple non-directed path between each pair of vertices of G or a tree is connected graph without any circuits.

Example



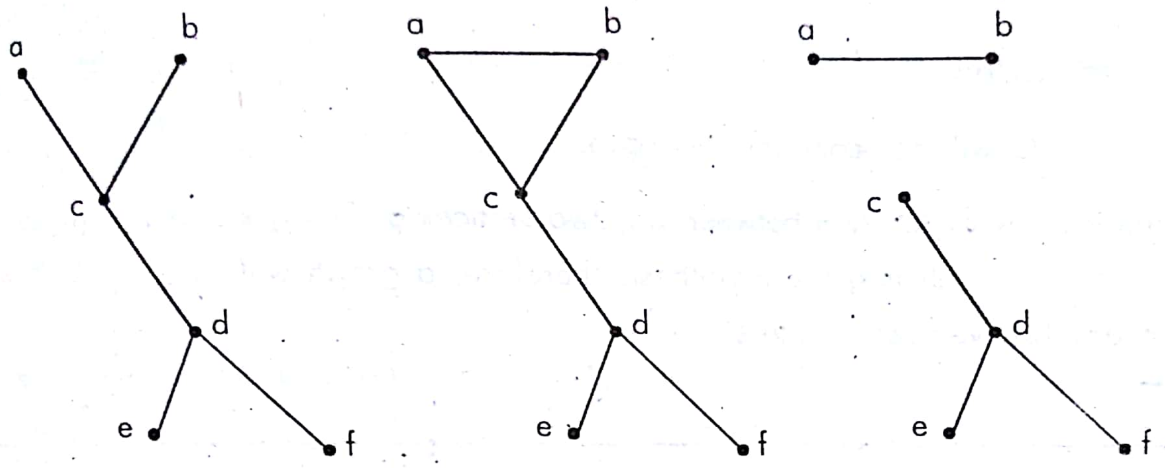
(a) One Vertex (b) Two Vertices (c) Three Vertices

(d) Four Vertices (e) Five Vertices

Fig. 5.8.1 : Example of Trees with different Number of Vertices

Let G = (V, E) be a loop-free undirected graph. The graph G is called a tree if G is connected and contains no cycles.

In Fig. 5.8.2 the graph G_1 is a tree, but the graph G_2 is not a tree because it contains the cycle {a, b}, {b, c}, {c, a}. The graph G_3 is not connected, so it cannot be a tree. However, each component of G_3 is a tree and in this case we call G_3 a forest.



(G_1)
a) Spanning Tree of G_2

(G_2)
b) Graph

(G_3)
c) Spanning Forest for G_2

Fig. 5.8.2

In Fig. 5.8.2 we see that G_1 is a subgraph of G_2 where G_1 contains all the vertices of G_2 and G_1 is a tree. In this situation G_1 is a spanning tree for G_2. Hence, a spanning tree for a connected graph is a spanning subgraph that is also a tree. We may think of a spanning tree as providing minimal connectivity for the graph and as a minimal skeletal framework holding the vertices together. The graph G_3 provides a spanning forest for the graph G_2.

5.9 PROPERTIES AND EXAMPLES OF TREES

THEOREM 1