UNIT 2

FOURIER TRANSFORM

PROFESSIONAL'S
The Destination Towards Knowledge & Success

SYLLABUS

Deriving Fourier Transform from Fourier Series, Fourier Transform of Arbitrary Signal, Fourier Transform of Standard Signals, Fourier Transform of Periodic Signals, Properties of Fourier Transforms, Fourier Transforms Involving Impulse Function and Signum Function, Introduction to Hilbert Transform.

PART - A
SHORT QUESTIONS WITH ANSWERS

Q1) How the Fourier transform can be represented in terms of magnitude and phase responses?

Ans. : In general, $X(\omega)$ is a complex quantity, Thus it can be written as,

$$X(\omega) = X_R(\omega) + j\, X_I(\omega)$$

Where,

$X_R(\omega)$ = Real part of $X(\omega)$.

$X_I(\omega)$ = Imaginary part of $X(\omega)$.

The magnitude of $X(\omega)$ is given by,

$$|X(\omega)| = \sqrt{X_R(\omega)^2 + X_I(\omega)^2}$$

The phase of $X(\omega)$ is given by,

$$\angle X(\omega) = \tan^{-1}\left(\frac{X_I(\omega)}{X_R(\omega)}\right)$$

The plot of $|X(\omega)|$ versus ω is known as amplitude (magnitude) spectrum and the plot of $\angle X(\omega)$ versus ω is known as phase spectrum. The amplitude spectrum and phase spectrum together is called frequency spectrum.

Q2) Define Fourier transform pair.

(or)

Define Fourier transform and its inverse transform.

Ans. : Fourier transform is a transformation technique which transforms signals from the continuous-time domain to the corresponding frequency-domain and vice-versa and which is applicable for both periodic as well as aperiodic signals. In other words, Fourier transform, overcomes the limitation of Fourier series. *Fourier transform of a non-periodic function can be developed by finding the Fourier series of a periodic function and then tending time period to infinity.*

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$X(\omega)$ or $X(j\omega)$ is called the Fourier transform or Fourier integral of $x(t)$.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Hence, above equation is called the inverse Fourier transform of $x(\omega)$. It is also called as synthesis equation.

These equations are known as Fourier transform pair and can be denoted as,

$$X(\omega) = F[x(t)]$$

And $$x(t) = F^{-1}[X(\omega)]$$

Where, 'F' denotes fourier transform operator.

Q3) Write down the conditions for the existence of Fourier transform.

Ans. : A condition which the Fourier transform has to satisfy for its transformability is called as Dirichlet's condition. Dirichlet's condition for the existence of Fourier transform or convergence of Fourier transform is as follows,

CONDITION I : A signal $x(t)$ should be absolutely integrable.

i.e., $$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

CONDITION II : A signal $x(t)$ should have a finite number of maxima and minima within any finite interval.

CONDITION III : A signal $x(t)$ should have a finite number of discontinuities within any finite interval and each of these discontinuities is finite.

Q4) Write the Fourier transform for unit impulse function.

Ans. : The impulse function shown in Figure (a) is mathematically defined as,

$$x(t) = \delta(t) = \infty \quad ; t = 0$$
$$= 0 \quad ; t \neq 0$$

The Fourier transform of x(t) is given by,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t}\Big|_{t=0} \qquad [\because \delta(t) \text{ exists only at } t = 0]$$

i.e.,
$$\int_{-\infty}^{\infty} \delta(t)dt = 1 = e^{-0} = 1$$

$$\therefore \quad \delta(t) \xleftrightarrow{F.T} 1$$

Above equation implies that the unit impulse has a Fourier transform consisting of equal distributions at all frequencies. The unit impulse function and its Fourier transform is shown in Figures (a) and (b) respectively.

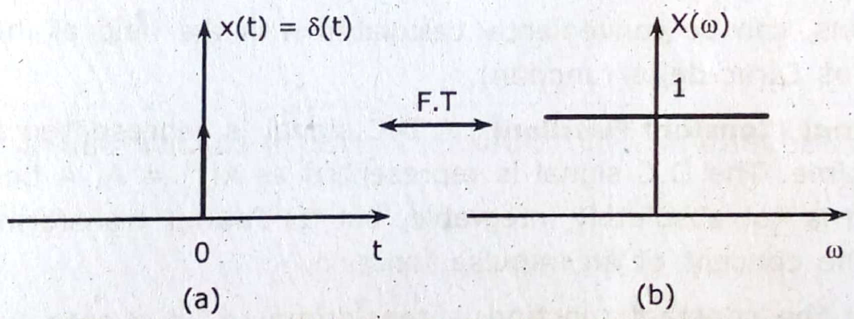

Figure Impulse Function and Its Fourier Transform

Q5) Define Hilbert transform. What are the Hilbert transform pair?

Ans. : When the phase angle of all frequency components of a given signal x(t) are shifted by ±90°, the resulting function, in time-domain is known as Hilbert Transform of the signal.

Hilbert transform is denoted by $\hat{x}(t)$ and can be written in terms of x(t) as,

$$\hat{x}(t) = x(t) * \frac{1}{\pi t} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} \cdot d\tau$$

[Here sign * represents convolution]

The inverse Hilbert Transform is given by,

$$x(t) = \frac{-1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{x}(\tau)}{t-\tau} \cdot d\tau$$

The functions $\hat{x}(t)$ and x(t) are said to constitute Hilbert transform pair.

Q6) Find the Fourier transform of,

$$\mathbf{x(t) = \delta(t + t_0) + \delta(t - t_0)}$$

Ans. : Given Data, $x(t) = \delta(t + t_0) + \delta(t - t_0)$

By the definition of Fourier transform,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} [\delta(t + t_0) + \delta(t - t_0)] e^{-j\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t + t_0) e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt$$

By shifting property we have,

$$\int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = x(t_0)$$

Then, $X(\omega) = e^{-j\omega(-t_0)} + e^{-j\omega(+t_0)} = e^{j\omega t_0} + e^{-j\omega t_0}$

Q7) How can you find the Fourier transform of a function x(t), which does not satisfy the Dirichlet's conditions?

Ans. : The Fourier transforms of functions which do not satisfy directly the Dirichlet's conditions, can be conveniently calculated with the help of impulse function (also known as Dirac-delta function).

D.C Signal (Constant Function) : A D.C signal is represented by a constant value for all time. The D.C signal is represented as $x(t) = A$, A being a constant. This function is not absolutely integrable, but its Fourier transform can be determined using the concept of an impulse function.

Let the constant function is considered to be a gate function of amplitude A having width 2T in the limit $T \to \infty$.

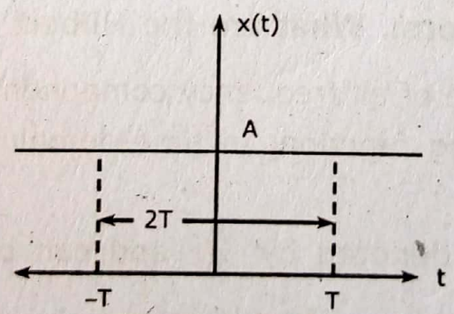

Figure D.C Signal Represented by a Constant Function

$$A\,\mathrm{Rect}\left(\frac{t}{2T}\right) \xleftrightarrow{F.T} 2AT\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right)$$

$$X(\omega) = F[A] = \lim_{T\to\infty} F\left[A\,\mathrm{Rect}\left(\frac{t}{2T}\right)\right] = \lim_{T\to\infty} 2AT\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right) = 2\pi A\left[\lim_{T\to\infty} \frac{T}{2\pi}\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right)\right]$$

As the limit $T \to \infty$, the sampling function approaches a delta function $\delta(\omega)$

$\therefore$ $F[A] = 2pa\delta(\omega)$

Q6) Find the Fourier transform of,

$$x(t) = \delta(t + t_0) + \delta(t - t_0)$$

Ans. : Given Data, $x(t) = \delta(t + t_0) + \delta(t - t_0)$

By the definition of Fourier transform,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} [\delta(t + t_0) + \delta(t - t_0)] e^{-j\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t + t_0) e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt$$

By shifting property we have,

$$\int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = x(t_0)$$

Then, $X(\omega) = e^{-j\omega(-t_0)} + e^{-j\omega(+t_0)} = e^{j\omega t_0} + e^{-j\omega t_0}$

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Let the constant function is considered to be a gate function of amplitude A having width 2T in the limit $T \to \infty$.

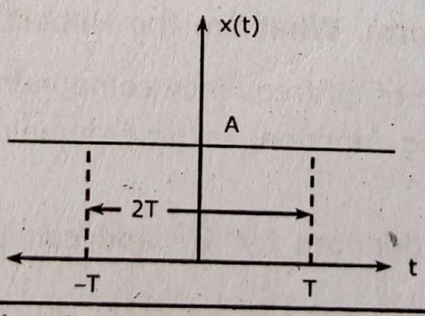

Figure D.C Signal Represented by a Constant Function

$$A\,\text{Rect}\left(\frac{t}{2T}\right) \xleftrightarrow{F.T} 2AT\,\text{sinc}\left(\frac{\omega T}{\pi}\right)$$

$$X(\omega) = F[A] = \lim_{T\to\infty} F\left[A\,\text{Rect}\left(\frac{t}{2T}\right)\right] = \lim_{T\to\infty} 2AT\,\text{sinc}\left(\frac{\omega T}{\pi}\right) = 2\pi A\left[\lim_{T\to\infty} \frac{T}{2\pi}\,\text{sinc}\left(\frac{\omega T}{\pi}\right)\right]$$

As the limit $T \to \infty$, the sampling function approaches a delta function $\delta(\omega)$

$\therefore$ $F[A] = 2pa\delta(\omega)$

Q8) State and prove frequency shifting property of Fourier transform.

Ans. : Frequency shifting property states that the multiplication in time domain by a complex sinusoid corresponds to frequency shift in $X(\omega)$ i.e.,

If $\quad x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $\quad x(t)\, e^{j\omega_0 t} \xleftrightarrow{F.T} X(\omega - \omega_0)$

PROOF

The Fourier transform of $x(t)\, e^{j\omega_0 t}$ is given by,

$$F[x(t)\, e^{j\omega_0 t}] = \int_{-\infty}^{\infty} [x(t)\, e^{j\omega_0 t}]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega - \omega_0)t}\, dt = X(\omega - \omega_0)$$

$\therefore \quad x(t) e^{j\omega_0 t} \xleftrightarrow{F.T} X(\omega - \omega_0)$

Similarly, $\quad x(t) e^{-j\omega_0 t} \xleftrightarrow{F.T} X(\omega + \omega_0)$

Q9) Prove that the signal x(t) and its Hilbert transform $\hat{x}(t)$ are having the same magnitude.

(or)

Prove that $|\hat{X}(\omega)| = |X(\omega)|$ by using Hilbert transform property.

Ans. : A signal x(t) and its Hilbert transform $\hat{x}(t)$ have the same magnitude spectrum.

$$|\hat{X}(\omega)| = |X(\omega)|$$

PROOF

Given that, $\quad \hat{x}(t) = HT[x(t)]$

As we have,

$$\hat{X}(f) = -j\,\text{sgn}(f) X(f)$$

$\Rightarrow \quad |\hat{X}(f)| = |-j\,\text{sgn}(f) X(f)|$

$\Rightarrow \quad |\hat{X}(f)| = |-j\,\text{sgn}(f)|\,|X(f)|$

Since $|-j\,\text{sgn}(f)| = 1$ for all f, we have,

$$|\hat{X}(f)| = |X(f)|$$

Hence, $\quad |\hat{X}(\omega)| = |X(\omega)|$

Q10) State and prove the Parseval's relation to Fourier transform.

Ans. : Let x(t) be an energy signal and

If $x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Parseval's theorem states that the, signal energies of an energy signal and its Fourier transform are equal.

PROOF

Consider the L.H.S of equation, we have,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t)x^*(t) \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} \, d\omega \right)^* dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left(\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} \, dt \right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, X(\omega) \, d\omega$$

$$\therefore \quad E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

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PART - B

ESSAY QUESTIONS WITH REFERENCES

Q1) Write aperiodic signal representation by Fourier integral. **[Refer Section No. 2.2]**

Q2) List the Dirichlets condition? **[Refer Section No. 2.2.1]**

Q3) Write down the existence of Fourier transform? And write the magnitude and phase representation of Fourier transform. **[Refer Section Nos. 2.2.1 and 2.2.2]**

Q4) What is the Fourier transform of an arbitrary signal? **[Refer Section No. 2.3]**

Q5) Find Fourier transform of a unit step function and draw its spectral density function. **[Refer Section No. 2.4.8]**

Q6) What is the Fourier transform of singularity functions (unit impulse and unit step function). **[Refer Section Nos. 2.4.1 and 2.4.8]**

Q7) State and prove the time differentiation theorem of Fourier transform? **[Refer Section No. 2.6.6]**

Q8) State and explain the properties of Fourier transform. **[Refer Section No. 2.6]**

Q9) How can you derive the Fourier transform by using impulse function and signum function? **[Refer Section No. 2.7]**

Q10) Explain Hillbert transform? And what are its properties? **[Refer Section Nos. 2.8 and 2.8.1]**

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2.1 INTRODUCTION

Fourier transform is a transformation technique which transforms signals from the continuous-time domain to the corresponding frequency-domain and vice-versa and which is applicable for both periodic as well as aperiodic signals. In other words, Fourier transform, overcomes the limitation of Fourier series. *Fourier transform of a non-periodic function can be developed by finding the Fourier series of a periodic function and then tending time period to infinity.*

The Fourier transform is an extremely useful mathematical tool and is extensively used in the analysis of linear time-invariant (LTI) systems, cryptography, signal processing, astronomy, etc. Several applications ranging from RADAR to spread spectrum communication also uses Fourier transform.

2.2 DERIVING FOURIER TRANSFORM FROM FOURIER SERIES (OR) APERIODIC SIGNAL REPRESENTATION BY FOURIER INTEGRAL

We may derive the Fourier transform of a non-periodic signal from the Fourier series of a periodic signal by describing a non-periodic signal as the limiting case of a periodic signal where period of the signal T, approaches infinity.

Let x(t) be a non-periodic signal and $x_p(t)$ be periodic with period T and their relation is given by,

$$x(t) = \underset{T \to \infty}{Lt} [x_p(t)] \quad \dots (2.2.1)$$

The exponential Fourier series representing a periodic signal $x_p(t)$ over an interval $\left[\frac{-T}{2}, \frac{T}{2}\right]$ is given by,

$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \quad \dots (2.2.2)$$

Where, $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x_p(t)\, e^{-jn\omega_0 t}\, dt$$

$$\Rightarrow \quad TC_n = \int_{-T/2}^{T/2} x_p(t)\, e^{-jn\omega_0 t}\, dt$$

Defining $TC_n = X(n\omega_0)$ and allowing the fundamental period T, tends to infinity, i.e., $T \to \infty$, we get,

$$TC_n = X(n\omega_0) = \underset{T \to \infty}{Lt} \int_{-T/2}^{T/2} x_p(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\underset{T \to \infty}{Lt}\, x_p(t)\right] e^{-jn\omega_0 t}\, dt \quad \dots (2.2.3)$$

As $T \to \infty$, we have $\omega = \left(\frac{2\pi}{T}\right) \to 0$, thus the angular frequency moves from being a discrete variable to becoming a continuous variable, i.e., $n\omega_0 \to \omega$ as $T \to \infty$

Therefore, Eq. (2.2.3) becomes,

$$X(\omega) = \int_{-\infty}^{\infty} \left[\underset{T \to \infty}{\text{Lt}} x_p(t) \right] e^{-j\omega t} dt \quad \text{... (2.2.4)}$$

Substituting Eq. (2.2.1) in Eq. (2.2.4), we get,

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt} \quad \text{... (2.2.5)}$$

$X(\omega)$ or $X(j\omega)$ is called the Fourier transform or Fourier integral of $x(t)$. Eq. (2.2.5) is also called as analysis equation. $X(\omega)$ represents the frequency spectrum of $x(t)$.

Consider Eq. (2.2.2),

$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{T}\, e^{jn\omega_0 t} \qquad \left(\because C_n = \frac{TC_n}{T} = \frac{X(\omega)}{T} \text{ as } T \to \infty\right)$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(\omega)\, e^{jn\omega_0 t}\, \omega_0 \qquad \left(\because T = \frac{2\pi}{\omega_0}\right) \quad \text{... (2.2.6)}$$

Substituting Eq. (2.2.6) in Eq. (2.2.1), we get,

$$x(t) = \underset{T \to \infty}{\text{Lt}} \left[x_p(t)\right]$$

$$= \underset{T \to \infty}{\text{Lt}} \left[\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(\omega)\, e^{jn\omega_0 t}\, \omega_0 \right]$$

As $T \to \infty$, $\omega_0 = \left(\frac{2\pi}{T}\right)$ becomes infinitesimally small and may be represented by $d\omega$. Also the summation becomes integration. Also $n\omega_0 \to \omega$ as $T \to \infty$.

$$\therefore \quad \boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega} \quad \text{... (2.2.7)}$$

Hence Eq. (2.2.7) is called the inverse Fourier transform of $x(\omega)$. It is also called as synthesis equation. The integral on the right side is called as Fourier integral.

The equations,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

These equations are known as Fourier transform pair and can be denoted as,

$$X(\omega) = F[x(t)]$$

And $$x(t) = F^{-1}[X(\omega)]$$

Where, 'F' denotes fourier transform operator.

The other notation that can be used to represent the Fourier transform pair is,

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

Sometimes the Fourier transform is expressed as a function of cycle frequency (f), rather than angular frequency (ω), then we have the Fourier transform pair defined as,

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df$$

2.2.1 Dirichlets Condition (OR) Existence of Fourier Transform

A condition which the Fourier transform has to satisfy for its transformability is called as Dirichlet's condition. Dirichlet's condition for the existence of Fourier transform or convergence of Fourier transform is as follows,

Condition I : A signal x(t) should be absolutely integrable.

i.e., $$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

Condition II : A signal x(t) should have a finite number of maxima and minima within any finite interval.

Condition III : A signal x(t) should have a finite number of discontinuities within any finite interval and each of these discontinuities is finite.

Dirichlet's condition is a sufficient condition but not a necessary condition. There are several functions that do not satisfy the above conditions, yet have Fourier transforms in the limit. For example, a step function or sinusoidal function violates the Dirichlet's conditions yet have Fourier transform in the limit. Such transforms are however, obtained in the limit with the help of the .dealized function known as impulse function.

2.2.2 Magnitude and Phase Representation of Fourier Transform

In general, $X(\omega)$ is a complex quantity, Thus it can be written as,

$$X(\omega) = X_R(\omega) + j\, X_I(\omega)$$

Where,

$X_R(\omega)$ = Real part of $X(\omega)$.

$X_I(\omega)$ = Imaginary part of $X(\omega)$.

The magnitude of $X(\omega)$ is given by,

$$|X(\omega)| = \sqrt{X_R(\omega)^2 + X_I(\omega)^2}$$

The phase of $X(\omega)$ is given by,

$$\angle X(\omega) = \tan^{-1}\left(\frac{X_I(\omega)}{X_R(\omega)}\right)$$

The plot of $|X(\omega)|$ versus ω is known as amplitude (magnitude) spectrum and the plot of $\angle X(\omega)$ versus ω is known as phase spectrum. The amplitude spectrum and phase spectrum together is called frequency spectrum.

REVIEW QUESTIONS

(1) *Derive the Fourier transform from Fourier series?*

(2) *Explain about Dirichlets condition?*

(3) *Write down the magnitude and phase representation of Fourier transform?*

2.3 FOURIER TRANSFORM OF ARBITRARY SIGNAL

We can find the Fourier Transform (FT) of an arbitrary signal in two ways.

In the first method, a function x(t) can be expressed over a finite interval of time $\left(\frac{-T}{2} < t < \frac{T}{2}\right)$ and then let T go infinite, while in second method we chose only cycle of interval of time i.e., $(-\infty < t < \infty)$. This second method is more convenient.

Let us consider a function x(t) over a time interval, $(-\infty < t < \infty)$ as shown in Fig. 2.3.1.

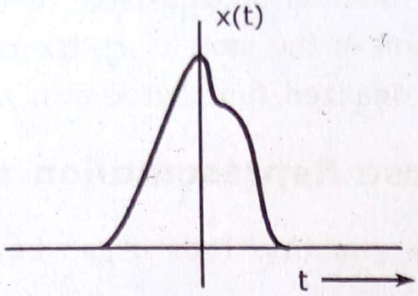


Fig. 2.3.1 Function x(t)

For this we need to construct a new periodic function $x_T(t)$ with period 'T', where function x(t) repeats itself every T seconds as shown in Fig. 2.3.2.

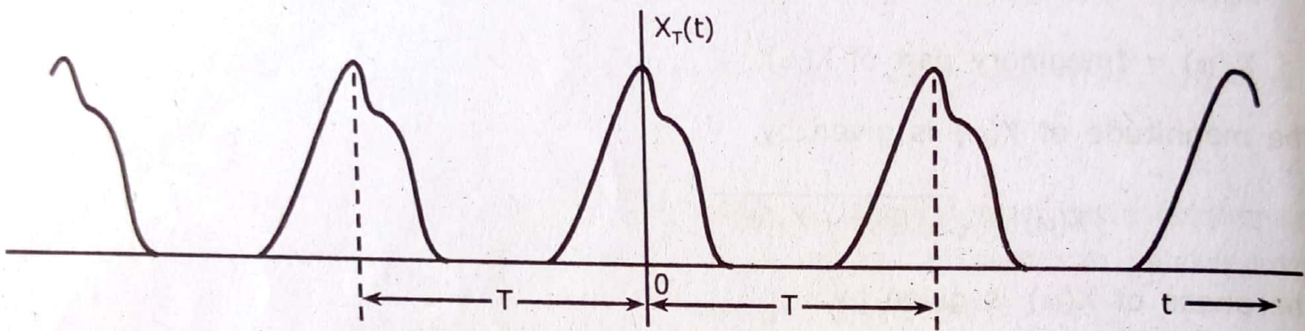


Fig. 2.3.2 Function $X_T(t)$ with Period T

In the limit if T becomes infinite, then pulses in the function repeat after an infinite interval. Hence, $T \to \infty$, $x_T(t)$ and x(t) are identical.

i..e, $\lim_{T \to \infty} x_T(t) = x(t)$

The exponential Fourier series for $x_T(t)$ can be represented as,

$$x_T(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$X_n = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) e^{-jn\omega_0 t} dt \quad \ldots (2.3.1)$$

Where,

X_n = Amplitude of component of frequency $n\omega_0$.

ω_0 = Fundamental frequency.

If T increases, then ω_0 will decrease and amplitude will also decreases.

When $T = \infty$, the magnitude becomes infinitesimally small, but now the spectrum exists for every value of ω. To illustrate this point,

Let, $n\omega_0 = \omega_n$ ... (2.3.2)

Then, X_n is a function of ω_n,

$$X_n = X_n(\omega_n)$$

Further let,

$$TX_n(\omega_n) = X(\omega_n) \quad \text{... (2.3.3)}$$

Then, $$x_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega_n)e^{j\omega_n t} \quad \text{... (2.3.4)}$$

From Eqs. (2.3.1) and (2.3.3), we have,

$$X(\omega_n) = TX_n$$

$$= \int_{-T/2}^{T/2} x_T(t)e^{-j\omega_n t}dt \quad \text{... (2.3.5)}$$

Substituting the value $T = \frac{2\pi}{\omega_0}$ in Eq. (2.3.4),

$$x_T(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(\omega_n)e^{j\omega_n t}\omega_0 \quad \text{... (2.3.6)}$$

The Eq. (2.3.6) represents that $x_T(t)$ can be expressed as sum of exponential signal of frequencies $\omega_1, \omega_2, \omega_3 \dots \omega_n$. The graphical representation of Eq. (2.3.6) is shown in Fig. 2.3.3,

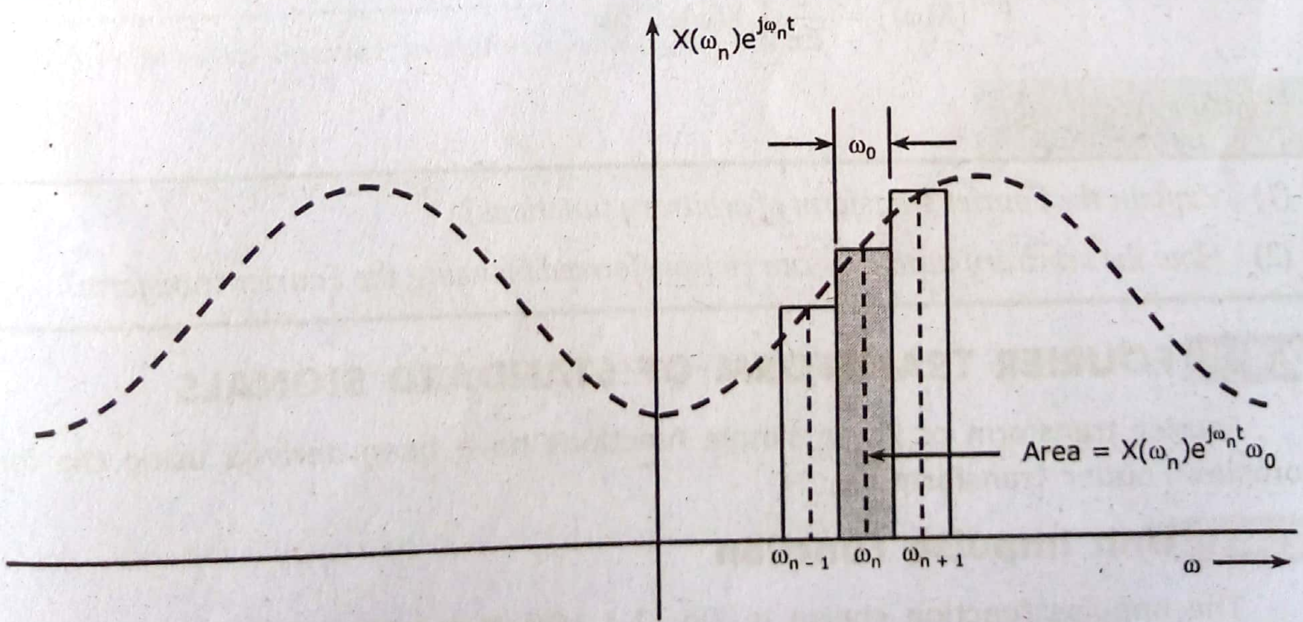


Fig. 2.3.3 Graphical Representation of $X_T(t)$ as a Function of ω

In Fig. 2.3.3, each frequency component is separated by a distance ω_0.

Therefore the area of shaded rectangle is $X(\omega_n)e^{j\omega_n t}\omega_0$.

The dotted curve represents the sum of rectangular areas. The curve is a continuous function of ω and is given by $F(\omega)e^{j\omega t}$.

As $T \rightarrow \infty$, the function $x_T(t) \rightarrow x(t)$ and Eqs. (2.3.6) and (2.3.5) becomes,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \qquad \text{... (2.3.7)}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{... (2.3.8)}$$

Hence, we have successfully represented a non-periodic function x(t) in terms of exponential functions over a time interval $(-\infty < t < \infty)$. Eq. (2.3.7) represents the sum of exponential functions with frequencies lying in the interval $(-\infty < \omega < \infty)$. The amplitude of any frequency ω is proportional to $X(\omega)$. Therefore $X(\omega)$ is called as spectral density function. The spectral density function can be evaluated by using Eq. (2.3.8).

$$X(\omega) = F[x(t)]$$

And $$x(t) = F^{-1}[X(\omega)] \qquad \text{... (2.3.9)}$$

Thus, $X(\omega)$ is and the direct Fourier transform of x(t) and x(t) is the inverse Fourier transform of $X(\omega)$,

$$F[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{... (2.3.10)}$$

$$F^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \qquad \text{... (2.3.11)}$$

REVIEW QUESTIONS

(1) *Explain the Fourier transform of arbitrary functions?*

(2) *How the arbitrary functions can be transformed by using the Fourier transform?*

2.4 FOURIER TRANSFORM OF STANDARD SIGNALS

Fourier transform of some simple functions have been derived using the formula of complex Fourier transform.

2.4.1 Unit Impulse Function

The impulse function shown in Fig. 2.4.1(a) is mathematically defined as,

$$x(t) = \delta(t) = \infty \quad ; t = 0$$
$$= 0 \quad ; t \neq 0$$

The Fourier transform of x(t) is given by,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} \delta(t)\,e^{-j\omega t}\,dt$$

$$= e^{-j\omega t}\Big|_{t=0} \qquad [\because \delta(t) \text{ exists only at } t = 0]$$

i.e.,
$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$= e^{-0} = 1$$

$$\therefore \quad \delta(t) \xleftrightarrow{F.T} 1 \qquad \ldots (2.4.1)$$

Eq. (2.4.1) implies that the unit impulse has a Fourier transform consisting of equal distributions at all frequencies. The unit impulse function and its Fourier transform is shown in Fig. 2.4.1(a) and Fig. 2.4.1(b) respectively.

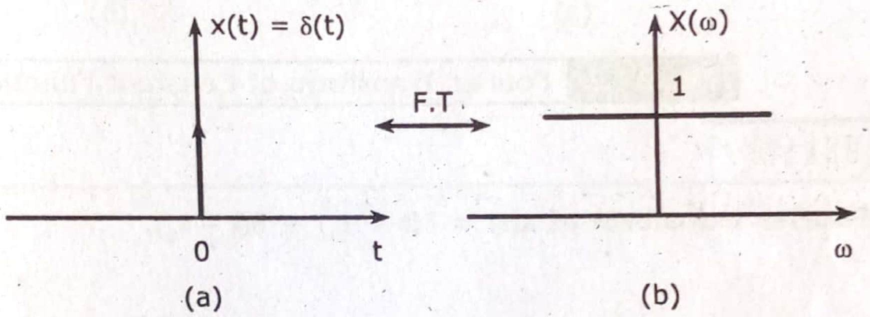


Fig. 2.4.1 Impulse Function and Its Fourier Transform

EXAMPLE PROBLEM 1

Find the inverse Fourier transform of $X(\omega) = \delta(\omega)$.

SOLUTION

The inverse Fourier transform is defined as,

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,d\omega$$

$$\Rightarrow \quad F^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\,e^{j\omega t}\,d\omega$$

From the sampling property of the impulse function i.e.,

$$\int_{-\infty}^{\infty} x(t).\delta(t) = x(t)\Big|_{t=0}$$

We have,

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi}\, e^{j\omega t}\Big|_{\omega=0} = \frac{1}{2\pi}$$

(or) $$\delta(\omega) = F\left[\frac{1}{2\pi}\right]$$

$$\therefore \quad \frac{1}{2\pi} \xleftrightarrow{F.T} \delta(\omega) \qquad \ldots (2.4.2)$$

(or) $$1 \xleftrightarrow{F.T} 2\pi\delta(\omega) \qquad \ldots (2.4.3)$$

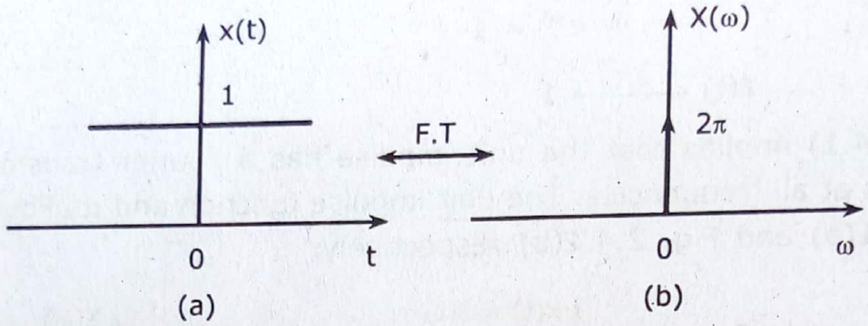


Fig. 2.4.2 Fourier Transform of Constant Function

EXAMPLE PROBLEM 1

Find the Fourier transform of $x(t) = \delta(t + t_0) + \delta(t - t_0)$.

SOLUTION

Given Data,

$$x(t) = \delta(t + t_0) + \delta(t - t_0)$$

By the definition of Fourier transform,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} [\delta(t + t_0) + \delta(t - t_0)]e^{-j\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t + t_0)e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t} dt$$

By shifting property we have,

$$\int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t} dt = x(t_0)$$

Then, $$X(\omega) = e^{-j\omega(-t_0)} + e^{-j\omega(+t_0)}$$

$$X(\omega) = e^{j\omega t_0} + e^{-j\omega t_0}$$

2.4.2 Single-Sided Exponential Signal

A single sided decaying exponential signal as shown in Fig. 2.4.3(a) is mathematically expressed as,

$$x(t) = {}^{-at} \quad ; t \geq 0$$

$$= 0 \quad ; t < 0$$

The Fourier transform of the signal x(t) is given by,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow \quad X(\omega) = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a + j\omega)t}\, dt$$

$$= \left[\frac{e^{-(a + j\omega)t}}{-(a + j\omega)}\right]_0^{\infty} = \left[\frac{e^{-(a + j\omega)\infty} - e^{0}}{-(a + j\omega)}\right]$$

$$= \frac{0 - 1}{-(a + j\omega)} = \frac{1}{a + j\omega}$$

$$\therefore \quad e^{-at}u(t) \xleftrightarrow{\text{F.T}} \frac{1}{a + j\omega} \qquad \dots (2.4.4)$$

Consider, $$X(\omega) = \frac{1}{a + j\omega} = \frac{a - j\omega}{(a + j\omega)(a - j\omega)} = \frac{a - j\omega}{a^2 + \omega^2}$$

$$= \left(\frac{a}{a^2 + \omega^2}\right) - j\left(\frac{\omega}{a^2 + \omega^2}\right)$$

Therefore X(ω) is a complex quantity, with magnitude and phase given by,

Magnitude, $$|X(\omega)| = \sqrt{X_R(\omega)^2 + X_I(\omega)^2}$$

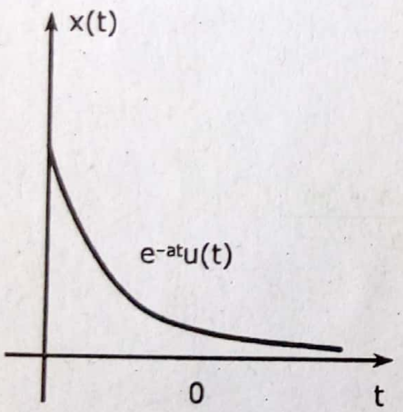
$$= \sqrt{\left(\frac{a}{a^2 + \omega^2}\right)^2 + \left(\frac{\omega}{a^2 + \omega^2}\right)^2} = \sqrt{\frac{a^2 + \omega^2}{(a^2 + \omega^2)^2}} = \frac{1}{\sqrt{a^2 + \omega^2}}$$

Phase, $$\angle X(\omega) = \tan^{-1}\left(\frac{X_I(\omega)}{X_R(\omega)}\right) = \tan^{-1}\left(\frac{\frac{-\omega}{a^2 + \omega^2}}{\frac{a}{a^2 + \omega^2}}\right) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

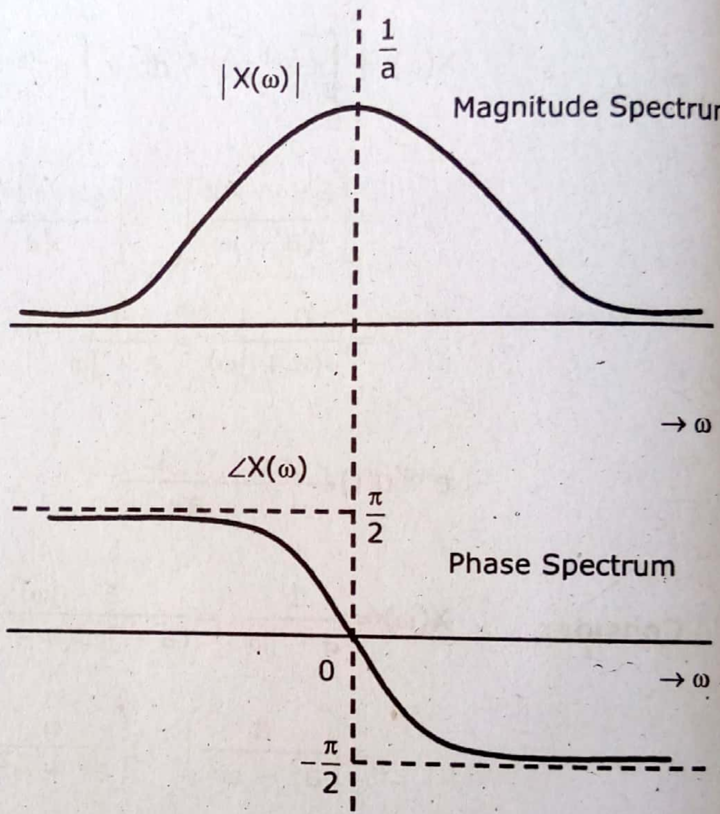
The magnitude and phase values at various frequencies are plotted in Table 2.4.1 and their graph is shown in Fig. 2.4.3(b).

Table 2.4.1 Magnitude and Phase Spectrum Calculation of Fig. 2.4.3(b)

ω	$-\infty$	$-10a$	$-5a$	$-a$	$-\frac{a}{2}$	0	$\frac{a}{2}$	a	$5a$	$10a$	∞
$\lvert X(\omega)\rvert$	0	$\frac{0.1}{a}$	$\frac{0.2}{a}$	$\frac{0.7}{a}$	$\frac{0.9}{a}$	$\frac{1}{a}$	$\frac{0.9}{a}$	$\frac{0.7}{a}$	$\frac{0.2}{a}$	$\frac{0.1}{a}$	0
$\angle X(\omega)$	90°	84.3°	78.7°	45°	26.6°	0	−26.6°	−45°	−78.7°	−84.3°	−90°



(a) Single Sided Exponential Function



(b) Frequency Spectrum

Fig. 2.4.3

2.4.3 Double-Sided Exponential Signal

A double sided exponential signal as shown in Fig. 2.4.4(a) is mathematically expressed as,

$$x(t) = e^{-a|t|} = \begin{cases} e^{-at} & ; t \geq 0 \\ e^{at} & ; t \leq 0 \end{cases}$$

The Fourier transform of the signal x(t) is given by,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \left[\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{0}^{\infty}$$

$$= \frac{e^{0} - e^{-\infty}}{(a-j\omega)} + \frac{e^{-\infty} - e^{0}}{-(a+j\omega)}$$

$$= \frac{1-0}{(a-j\omega)} + \frac{0-1}{-(a+j\omega)}$$

$$\therefore \quad e^{-a|t|} \xleftrightarrow{\text{F.T}} \frac{2a}{a^2+\omega^2} \qquad \text{... (2.4.5)}$$

Magnitude, $|X(\omega)| = \left|\dfrac{2a}{a^2+\omega^2}\right|$

Phase, $\angle X(\omega) = \tan^{-1}(0) = 0^{o}$

Figs. 2.4.4(b) and (c) shows the magnitude and phase spectrum of x(t).

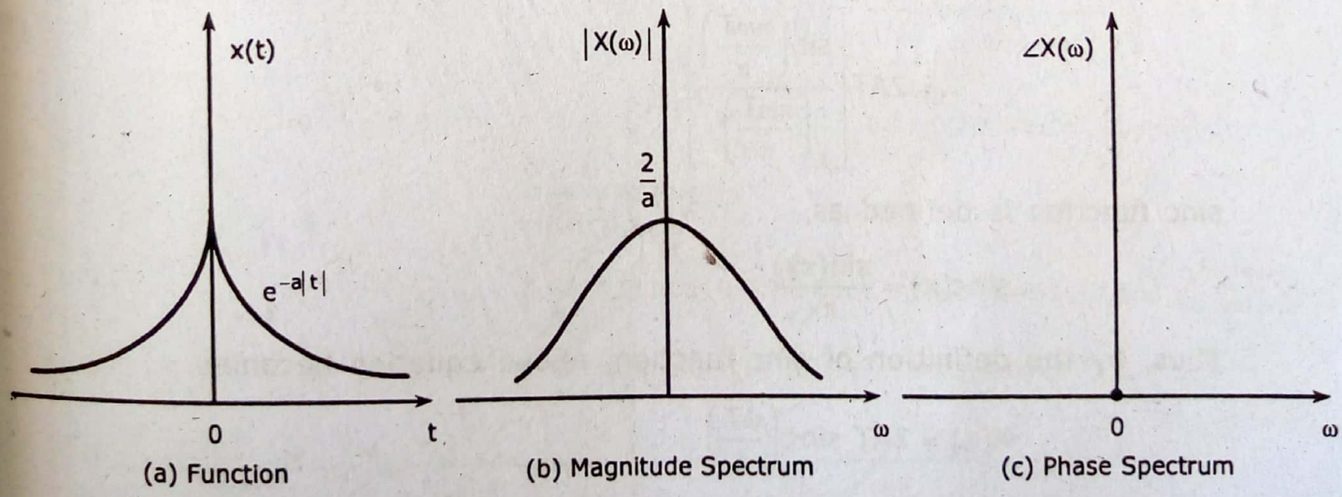


Fig. 2.4.4 Double-sided Exponential Function and its Frequency Spectra

2.4.4 Gate Function

A gate function shown in Fig. 2.4.5 is mathematically defined as,

$$x(t) = A\,\text{Rect}\left(\frac{t}{2T}\right) = AT\left(\frac{t}{2T}\right) = \begin{cases} A & ; |t| < T \\ 0 & ; |t| > T \end{cases}$$

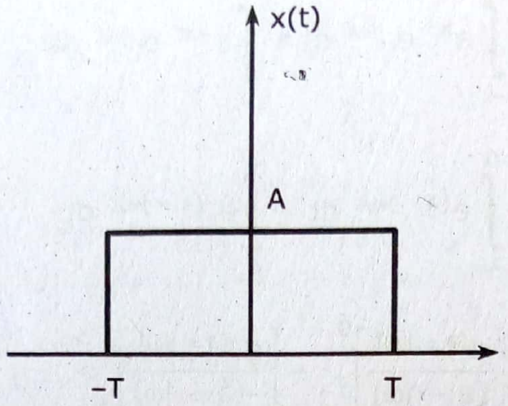


Fig. 2.4.5 Rectangular Pulse or Gate Function

Fourier Transform of function x(t) is defined as,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-T}^{T} Ae^{-j\omega t}\,dt$$

$$= \left[\frac{A}{-j\omega}e^{-j\omega t}\right]_{-T}^{T} = \frac{A}{-j\omega}\left(e^{-j\omega T} - e^{j\omega T}\right)$$

$$= \frac{A}{j\omega}\left(e^{j\omega T} - e^{-j\omega T}\right) = \frac{2A}{\omega}\left(\frac{e^{j\omega T} - e^{-j\omega T}}{2j}\right)$$

$$= \frac{2A}{\omega}\sin(\omega T) \qquad \left(\because \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta\right)$$

$$= 2AT\left[\frac{\sin\left(\frac{\pi\omega T}{\pi}\right)}{\left(\frac{\pi\omega T}{\pi}\right)}\right]$$

sinc function is defined as,

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Thus, by the definition of sinc function, above equation becomes,

$$X(\omega) = 2AT\,\text{sinc}\left(\frac{\omega T}{\pi}\right)$$

$$\therefore \quad A\,\text{Rect}\left(\frac{t}{2T}\right) \xleftrightarrow{F.T} 2AT\,\text{sinc}\left(\frac{\omega T}{\pi}\right) \qquad \text{... (2.4.6)}$$

In terms of frequency,

$$A\,\text{Rect}\left(\frac{t}{2T}\right) \xleftrightarrow{F.T} 2AT\,\text{sinc}(2fT) \qquad \ldots (2.4.7)$$

Sinc(x) function is zero at $x = \pm n$, where n = 1, 2, 3, 4. So $\text{sinc}\left(\frac{\omega T}{\pi}\right)$ becomes zero at $\frac{\omega T}{\pi} = \pm n$, i.e., $\omega = \pm\frac{n\pi}{T}$. To find the value of $X(\omega)$ at $\omega=0$, put $\omega=0$, i.e.,

$$X(0) = \int_{-T}^{T} A e^{j\omega t}\,dt = A[t]_{-T}^{T}$$

$$= 2AT$$

$$\therefore \quad X(\omega) = \begin{cases} 2AT & ; \omega = 0 \\ 2AT\,\text{sinc}\left(\frac{\omega T}{\pi}\right) & ; \omega \neq 0 \end{cases}$$

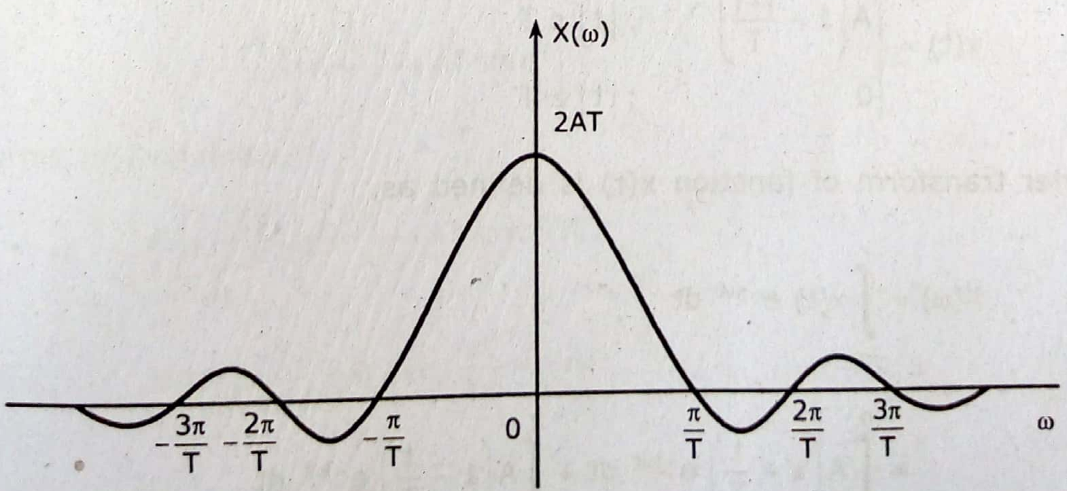
Magnitude spectrum,

$$|X(\omega)| = \begin{cases} |2AT| & ; \omega = 0 \\ \left|2AT\,\text{sinc}\left(\frac{\omega T}{\pi}\right)\right| & ; \omega \neq 0 \end{cases}$$

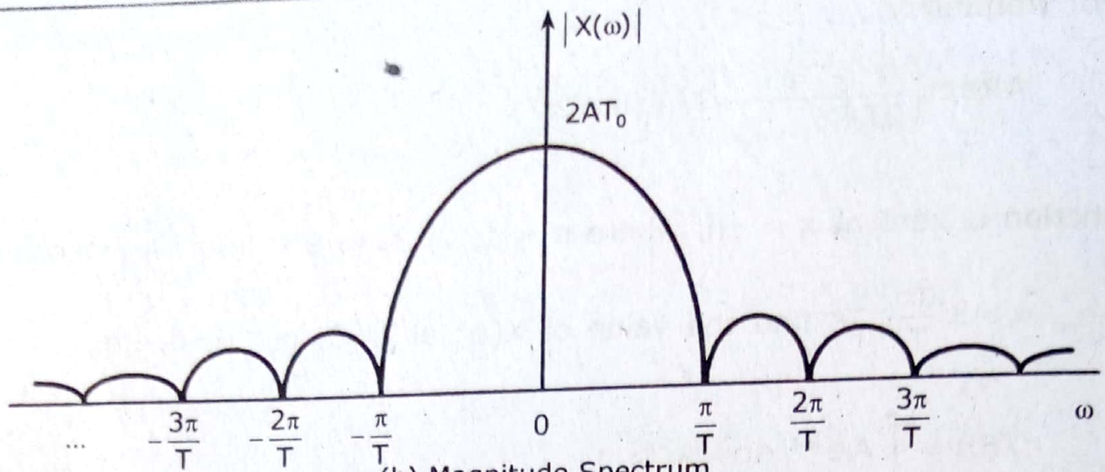
Phase spectrum,

$$\angle X(\omega) = \begin{cases} 0 & ; \omega = 0 \\ -\pi & ; \text{sinc}(\omega) < 0 \text{ and } \omega > 0 \\ +\pi & ; \text{sinc}(\omega) < 0 \text{ and } \omega < 0 \end{cases}$$

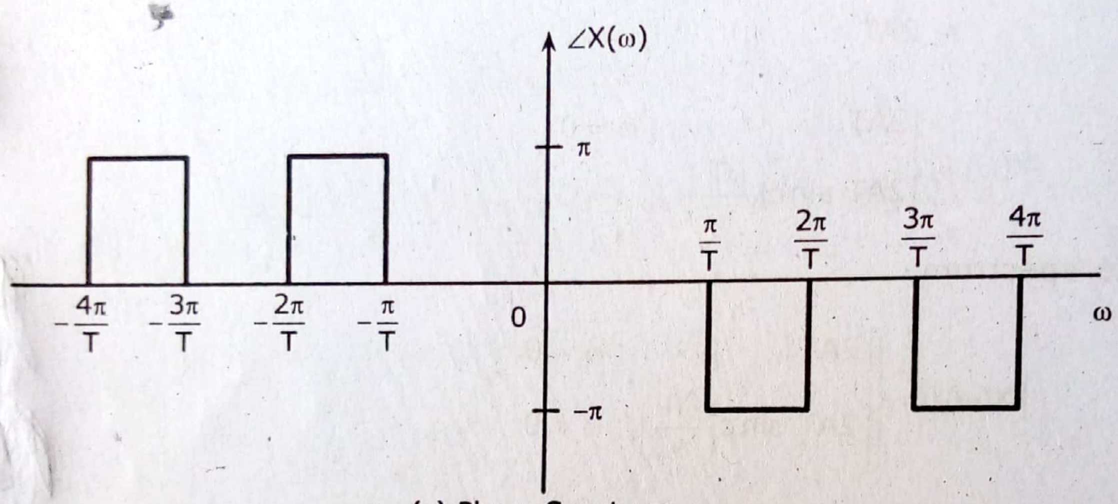
Fig. 2.4.6 shows the Fourier transform of gate function and its magnitude, phase spectrum.



(a) Fourier Transform a Gate Pulse



(b) Magnitude Spectrum



(c) Phase Spectrum

Fig. 2.4.6

2.4.5 Triangular Pulse

A triangular pulse shown in Fig. 2.4.7(a) is mathematically defined as,

$$x(t) = \begin{cases} A\left(1 - \frac{|t|}{T}\right) & ; |t| \leq T \\ 0 & ; |t| \geq T \end{cases}$$

The Fourier transform of function x(t) is defined as,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T}^{0} A\left(1 + \frac{t}{T}\right) e^{-j\omega t}\, dt + \int_{0}^{T} A\left(1 - \frac{t}{T}\right) e^{-j\omega t}\, dt$$

Changing the order of integration, we get,

$$X(\omega) = \int_0^T A\left(1-\frac{t}{T}\right) e^{j\omega t}\,dt + \int_0^T A\left(1-\frac{t}{T}\right) e^{-j\omega t}\,dt$$

$$= 2A\int_0^T \left(1-\frac{t}{T}\right) \frac{e^{j\omega t}+e^{-j\omega t}}{2}\,dt$$

$$= 2A\int_0^T \left(1-\frac{t}{T}\right) \cos(\omega t)\,dt \qquad \left(\because \frac{e^{j\theta}+e^{-j\theta}}{2} = \cos\theta\right)$$

$$= \left\{\left[2A\left(1-\frac{t}{T}\right)\frac{\sin(\omega t)}{\omega}\right]_0^T - \frac{2A}{\omega}\int_0^T \left(\frac{-1}{T}\right)\sin(\omega t)\,dt\right\}$$

$$= \left\{\left[2A\left(1-\frac{T}{T}\right)\frac{\sin\omega T}{\omega}\right] - \left[2A\frac{\sin 0}{\omega}\right]\right\} + \frac{2A}{\omega T}\left[\frac{-\cos(\omega t)}{\omega}\right]_0^T$$

$$= \frac{2A}{\omega^2 T}[-\cos\omega T + 1] = \frac{2A}{\omega^2 T}(1-\cos\omega T)$$

$$= \frac{4A}{\omega^2 T}\sin^2\left(\frac{\omega T}{2}\right) = \frac{AT}{\left(\frac{\omega^2 T^2}{4}\right)}\sin^2\left(\frac{\omega T}{2}\right) \qquad \left(\because 1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)\right)$$

$$= AT\left(\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}}\right)^2 = AT\left(\frac{\sin\left(\frac{\pi\omega T}{2\pi}\right)}{\left(\frac{\pi\omega T}{2\pi}\right)}\right)^2$$

$$= AT\,\mathrm{sinc}^2\left(\frac{\omega T}{2\pi}\right)$$

$$\therefore \quad A\left[1-\frac{|t|}{T}\right] \xleftrightarrow{F.T} AT\,\mathrm{sinc}^2\left(\frac{\omega T}{2\pi}\right) \qquad \ldots (2.4.8)$$

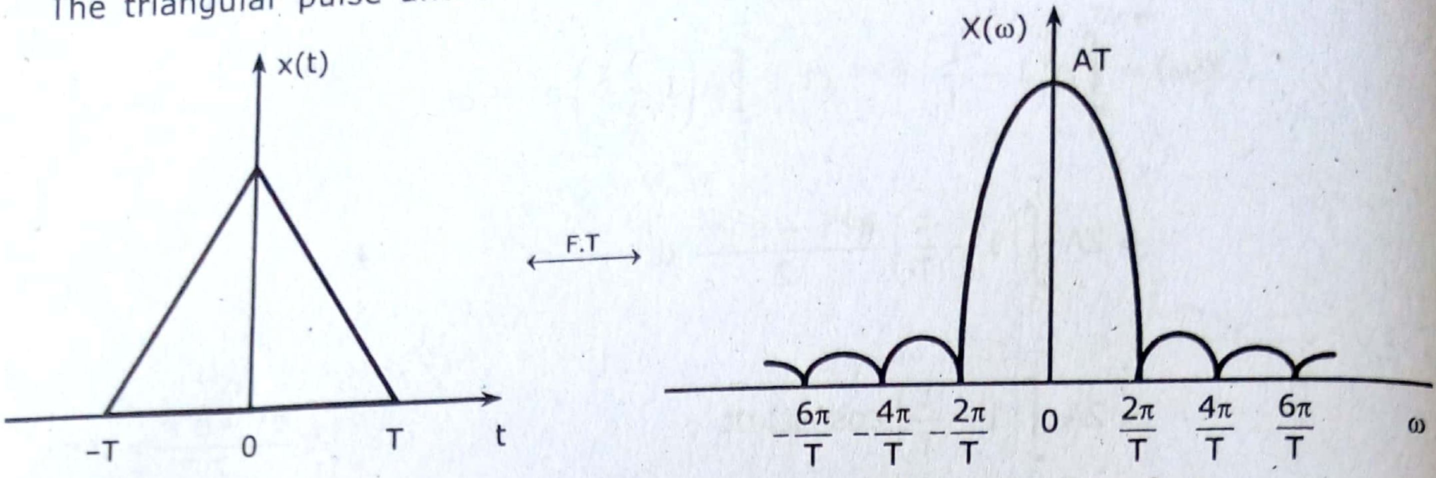
In terms of frequency, f,

$$A\left[1-\frac{|t|}{T}\right] \xleftrightarrow{F.T} AT\,\mathrm{sinc}^2(fT) \qquad \ldots (2.4.9)$$

$$\mathrm{sinc}^2\left(\frac{\omega T}{2\pi}\right) = 0 \text{ for } \frac{\omega T}{2\pi} = \pm n,$$

i.e., $\omega = \frac{\pm 2n\pi}{T}$, Where n = 1, 2, 3, ...

The triangular pulse and its Fourier transform $X(\omega)$ is shown in Fig. 2.4.7.



(a) Triangular Pulse (b) Fourier Transform

Fig. 2.4.7

2.4.6 Gaussian Pulse

Gaussian pulse shown in Fig. 2.4.8(a) is mathematically defined as,

$$x(t) = e^{-\pi t^2}$$

The Fourier transform of signal x(t) is given by,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-\pi t^2}\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-(\pi t^2 + j\omega t)}\, dt$$

Substituting,

$$\pi t^2 + j\omega t = \left(\sqrt{\pi}\, t + \frac{j\omega}{2\sqrt{\pi}}\right)^2 + \frac{\omega^2}{4\pi} \text{ gives}$$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-\left(\sqrt{\pi}\, t + \frac{j\omega}{2\sqrt{\pi}}\right)^2} e^{\frac{-\omega^2}{4\pi}}\, dt$$

$$= e^{\frac{-\omega^2}{4\pi}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\pi}\, t + \frac{j\omega}{2\sqrt{\pi}}\right)^2} dt$$

Let $u = \sqrt{\pi}\, t + \frac{j\omega}{2\sqrt{\pi}}$ which gives $dt = \frac{1}{\sqrt{\pi}} du$, $u \to -\infty$ as $t \to -\infty$ and $u \to \infty$ as $t \to \infty$.

$$\therefore \quad X(\omega) = e^{\frac{-\omega^2}{4\pi}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{\sqrt{\pi}}$$

$$= e^{\frac{-\omega^2}{4\pi}} \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u^2}\, du \qquad [\because \text{ guassian function is even function}]$$

Since we know that,

$$\int_0^\infty e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$$

$$X(\omega) = e^{\frac{-\omega^2}{4\pi}} \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2}$$

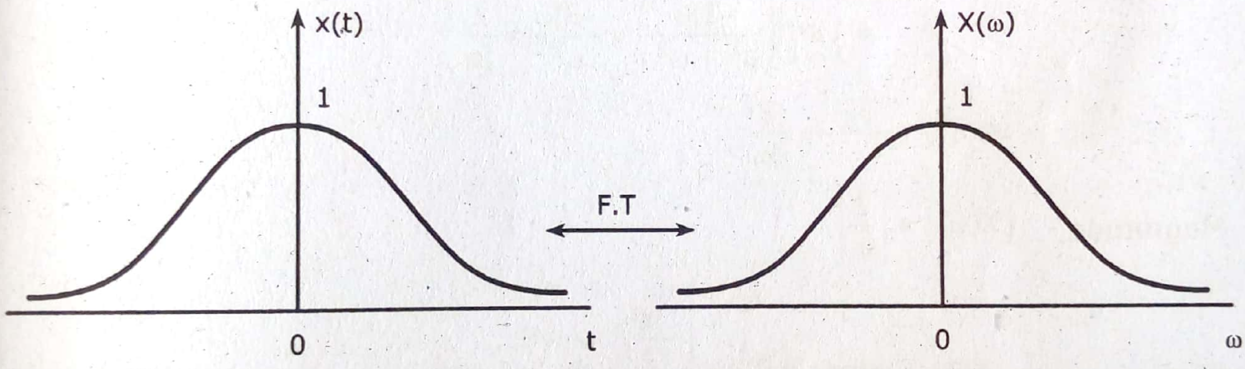
$$X(\omega) = e^{\frac{-\omega^2}{4\pi}}$$

$$\therefore \quad e^{-\pi t^2} \xleftrightarrow{F.T} e^{\frac{-\omega^2}{4\pi}} \qquad \text{... (2.4.10)}$$

In terms of frequency,

$$e^{-\pi t^2} \xleftrightarrow{F.T} e^{-\pi f^2} \qquad \text{... (2.4.11)}$$

The Fourier transform $X(\omega)$ of a Gaussian pulse and its frequency response is shown in Fig. 2.4.8.



(a) Gaussian Pulse (b) Fourier Transform

Fig. 2.4.8

2.4.7 Signum Function

Signum function shown in Fig. 2.4.9, is mathematically defined as,

$$\text{sgn}(t) = \begin{cases} 1 & ; t > 0 \\ 0 & ; t = 0 \\ -1 & ; t < 0 \end{cases}$$

This signal is not absolutely integrable. To make it absolutely integrable, this function is expressed as a sum of two one-sided exponentials, $e^{-at}\, u(t) - e^{+at}\, u(-t)$ in the limit $a \to 0$. Thus,

$$\text{sgn}(t) = \lim_{a \to 0} \left[e^{-at}\, u(t) - e^{at}\, u(-t)\right]$$

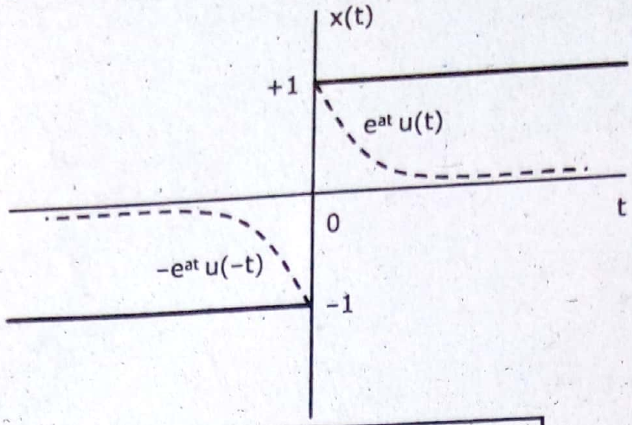


Fig. 2.4.9 Signum Function

The Fourier transform of sgn(t) is given by,

$$F[\text{sgn}(t)] = \lim_{a \to 0} \left[F\left[e^{-at}\,u(t) - e^{at}\,u(-t)\right]\right]$$

$$= \lim_{a \to 0} \left[F\left[e^{-at}\,u(t)\right] - F\left[e^{at}\,u(-t)\right]\right]$$

$$= \lim_{a \to 0} \left(\frac{1}{a + j\omega} - \frac{1}{a - j\omega}\right)$$

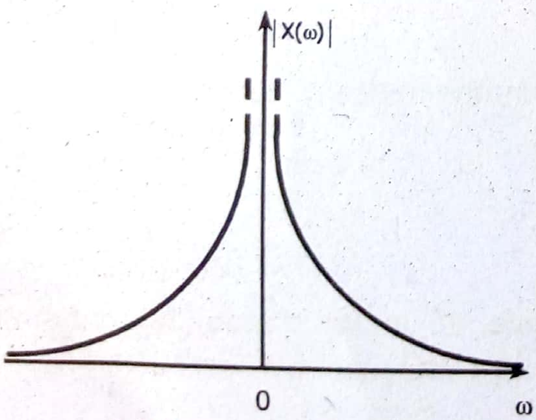
$$= \lim_{a \to 0} \left(\frac{-2j\omega}{a^2 + \omega^2}\right) = \frac{-2j\omega}{\omega^2} = \frac{2}{j\omega}$$

$$\therefore \quad \text{sgn}(t) \xleftrightarrow{\text{F.T}} \frac{2}{j\omega} \quad \text{... (2.4.1}$$

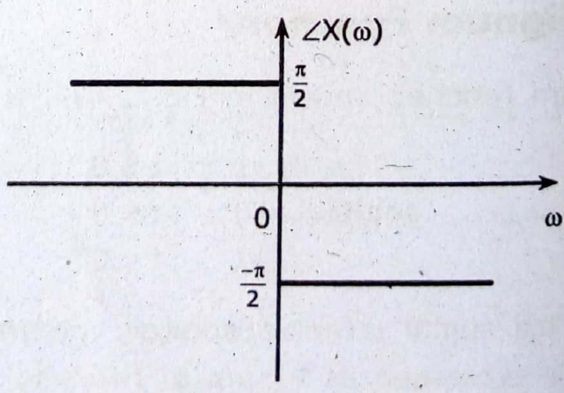
Magnitude, $|X(\omega)| = \left|\frac{2}{\omega}\right|$

Phase, $\angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{0}\right) = \begin{cases} -\frac{\pi}{2}, & \omega > 0 \\ \frac{\pi}{2}, & \omega < 0 \end{cases}$

Fig. 2.4.10 shows the magnitude and phase spectrum of signum function.



(a) Magnitude Spectrum

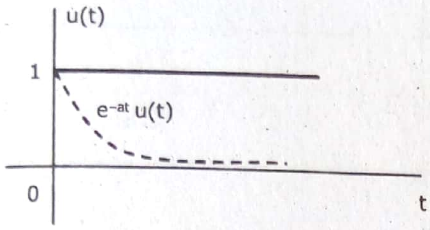


(b) Phase Spectrum of the Signum Function

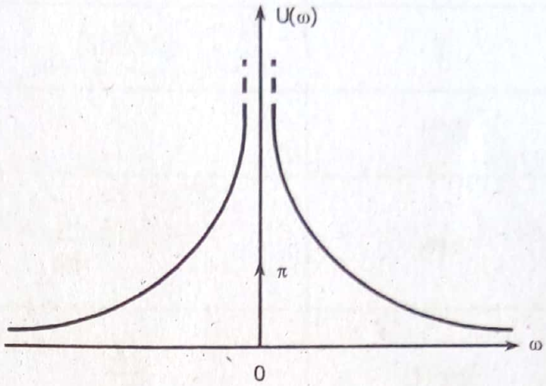
Fig. 2.4.10

2.4.8 Unit Step Function

The unit step signal shown in Fig. 2.4.11(a) is not absolutely integrable. So we approach this problem by considering u(t) to be a decaying exponential e^{-at} u(t) in the limit as $a \to 0$. Therefore,



(a) Unit Step Function



(b) Frequency Spectrum

Fig. 2.4.11

$$u(t) = \lim_{a \to 0} [e^{-at} u(t)]$$

Fourier transform of u(t) is given by,

$$F[u(t)] = U(\omega) = F\left[\lim_{a \to 0} e^{-at} u(t)\right] = \lim_{a \to 0} F[e^{-at} u(t)] = \lim_{a \to 0} \left[\frac{1}{a + j\omega}\right]$$

Expressing the RHS in terms of its real and imaginary parts gives,

$$U(\omega) = \lim_{a \to 0} \left[\frac{a}{a^2 + \omega^2} - j\frac{\omega}{a^2 + \omega^2}\right] = \lim_{a \to 0} \left[\frac{a}{a^2 + \omega^2}\right] - \frac{j}{\omega} = \lim_{a \to 0} \left[\frac{a}{a^2 + \omega^2}\right] + \frac{1}{j\omega}$$

The function $\frac{a}{a^2 + \omega^2}$ has interesting properties. First, the area under this function is π regardless of the value of a,

i.e.,
$$\int_{-\infty}^{\infty} \frac{a}{a^2 + \omega^2} d\omega = \left[\tan^{-1} \frac{\omega}{a}\right]_{-\infty}^{\infty} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Second, when $a \to 0$, this function approaches zero for all $\omega \neq 0$ and all its area (π) is concentrated at a single point $\omega = 0$. Clearly, this function approaches as impulse of strength π, i.e.,

$$\lim_{a \to 0} \left[\frac{a}{a^2 + \omega^2}\right] = \pi\delta(\omega)$$

$$U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\therefore \quad u(t) \xleftrightarrow{F.T} \pi\delta(\omega) + \frac{1}{j\omega} \qquad \text{... (2.4.13)}$$

Table 2.4.2 lists the Fourier transform of some standard signals.

Table 2.4.2 Fourier Transform Pairs

x(t)	X(ω)	X(f)
1	$2\pi\delta(\omega)$	$\delta(f)$
$\delta(t)$	1	1
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\mathrm{Sgn}(t)$	$\frac{2}{j\omega}$	$\frac{1}{j\pi f}$
$\mathrm{Rect}(t)$	$\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$	$\mathrm{sinc}(f)$
$\Delta(t)$	$\mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right)$	$\mathrm{sinc}^2(f)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	$\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)$; $f_0=\frac{\omega_0}{2\pi}$
$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	$\frac{2}{j}[\delta(f-f_0)-\delta(f+f_0)]$
$e^{-at}u(t)$	$\frac{1}{a+j\omega}$	$\frac{1}{a+j2\pi f}$
$e^{at}u(-t)$	$\frac{1}{a-j\omega}$	$\frac{1}{a-j2\pi f}$
$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$	$\frac{2a}{a^2+(2\pi f)^2}$

REVIEW QUESTIONS

(1) *Find the Fourier transform of a unit step function and draw its spectral density function?*

(2) *Find Fourier transform of singularity functions (unit impulse and unit step function).*

2.5 FOURIER TRANSFORM OF PERIODIC SIGNALS

The Fourier transform is developed as a limiting case of Fourier series by extending its period to infinite. And also the Fourier series is just a limiting case of Fourier transform. Firstly as we know that the Fourier transform of periodic function does not exist, since it fails to satisfy the condition of absolute integrability. For any periodic function x(t),

$$\int_{-\infty}^{\infty} |x(t)|\, dt = \infty \quad \text{... (2.5.1)}$$

As shown in the Eq. (2.5.1), Fourier transform of periodic function does not exist, but it exists in limits i.e., only in the finite time interval $\left(\frac{-\tau}{2}, \frac{\tau}{2}\right)$, where τ becomes infinite.

The Fourier transform of periodic function is the sum of Fourier transform of its individual components.

Here expressing a periodic function x(t) with period T as,

$$X(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \quad \text{... (2.5.2)}$$

Where, $\omega_0 = \frac{2\pi}{T}$

Taking Fourier transform on both sides of Eq. (2.5.2), we have,

$$F[x(t)] = F \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} X_n F(e^{jn\omega_0 t}) \quad \text{... (2.5.3)}$$

Here, we know that the transform of $e^{jn\omega_0 t}$ is given as,

$$F\left[e^{jn\omega_0 t}\right] = 2\pi\delta(\omega - n\omega_0) \quad \text{... (2.5.4)}$$

Now, substituting Eq. (2.5.4) in Eq. (2.5.3) we get,

$$F[x(t)] = 2\pi \sum_{n=-\infty}^{\infty} X_n \delta(\omega - n\omega_0) \quad \text{... (2.5.5)}$$

Eq. (2.5.5) states that the Fourier transform of periodic signal consist of impulses located at harmonic frequencies, where each impulse is equal to 2π time the value of corresponding coefficient in exponential Fourier series.

Hence, the result is the periodic function which contains components only of discrete harmonic frequencies.

EXAMPLE PROBLEM 1

Find Fourier transform of a sequence of equidistant impulses of unit strength and separated by T seconds as shown in Fig. 2.5.1.

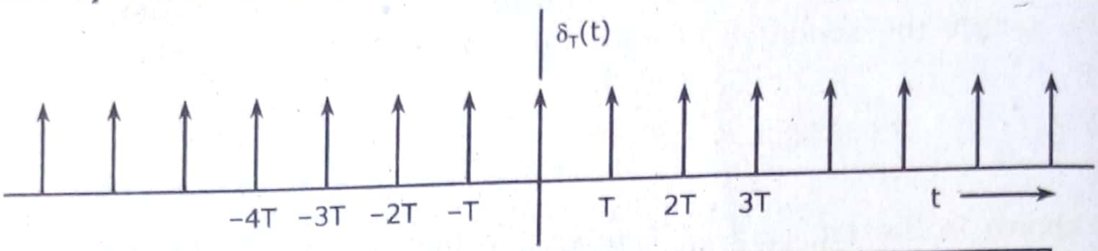


Fig. 2.5.1 The Sequence of a Uniform Equidistant Impulse Function

SOLUTION

This function is very important in sampling theory and hence it is convenient to denote this function by a special symbol $\delta_T(t)$.

Thus,

$$\delta_T(t) = \delta(t) + \delta(t - T) + \delta(t - 2T) + \ldots + \delta(t - nT) + \ldots + \delta(t + T) + \delta(t + 2T) + \ldots + \delta(t - nT) + \ldots$$

$$= \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \ldots (2.5.6)$$

The Eq. (2.5.6) is a periodic function with period T. The Fourier series for this function would be,

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

Where,
$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t) e^{-jn\omega_0 t} dt$$

Function $\delta_T(t)$ in interval $\left(\frac{-T}{2}, \frac{T}{2}\right)$ is simply $\delta(t)$. Hence,

$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t} dt \quad \ldots (2.5.7)$$

We know that sampling property of impulse function can be expressed as,

$$\int_{0^-}^{0^+} x(t)\delta(t)dt = x(0) \quad \ldots (2.5.8)$$

Hence, according to Eq. (2.5.8) the Eq. (2.5.7) will be reduced to,

$$X_n = \frac{1}{T}$$

X_n is a constant $\left(\frac{1}{T}\right)$. And which contains the frequencies as,

$$\omega = 0, \pm\omega_0, \pm 2\omega_0 \ldots \pm n\omega_0 \ldots \text{etc.}$$

Where, $\omega_0 = \frac{2\pi}{T}$

Hence, $$\delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

Hence, to find Fourier transform of $\delta_T(t)$, use Eq. (2.5.4).

$$\therefore \quad X_n = \frac{1}{T}$$

$$F[\delta_T(t)] = 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T}\delta(\omega - n\omega_0)$$

$$= \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

$$= \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

$$= \omega_0 \delta_{\omega_0}(\omega) \qquad \ldots (2.5.9)$$

Thus Eq. (2.5.9) states that the impulse train function is its own transform.

This sequence of impulses with periods $T = \frac{1}{2}$ and T = 1 seconds and their respective transform is shown in Fig. 2.5.2,

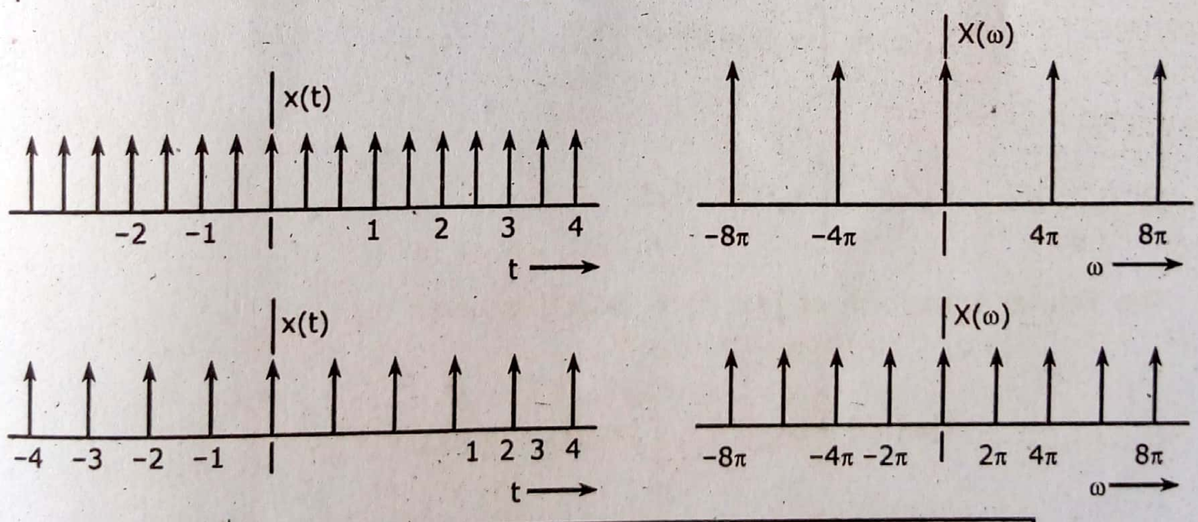


Fig. 2.5.2 Periodic Impulse Functions and their Transforms

REVIEW QUESTIONS

(1) *Explain the Fourier transform of periodic signals?*

(2) *How would you find the Fourier transform of a periodic signal?*

2.6 PROPERTIES OF FOURIER TRANSFORM

Fourier transform possess a number of properties which are very useful for determining in the transforms of certain functions which cannot be evaluated easily. Moreover these properties are useful in developing conceptual insights into the relationship between the time-domain and frequency-domain descriptions of a signal. We will use a shorthand notation,

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

to indicate the relationship between a time-domain signal x(t) and its Fourier transform $X(\omega)$.

2.6.1 Linearity

Linearity property states that, the Fourier transform of a weighted sum of two signals is equal to the weighted sum of individual Fourier transform, i.e.,

If $\quad x_1(t) \xleftrightarrow{F.T} X_1(\omega)$

And $\quad x_2(t) \xleftrightarrow{F.T} X_2(\omega)$

Then, $\quad ax_1(t) + bx_2(t) \xleftrightarrow{F.T} aX_1(\omega) + bX_2(\omega) \quad$ ... (2.6.1)

PROOF

By definition of Fourier transform,

$$X_1(\omega) = \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt$$

And

$$X_2(\omega) = \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

The Fourier transform of $[ax_1(t) + bx_2(t)]$ is given by,

$$F[ax_1(t) + bx_2(t)] = \int_{-\infty}^{\infty} [ax_1(t) + bx_2(t)]\, e^{-j\omega t}\, dt$$

$$F[ax_1(t) + bx_2(t)] = a \underbrace{\int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t}\, dt}_{x_1(\omega)} + b \underbrace{\int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt}_{x_2(\omega)}$$

$$F[ax_1(t) + bx_2(t)] = aX_1(\omega) + bX_2(\omega)$$

EXAMPLE PROBLEM 1

Find the Fourier transform of the double-sided exponential by using the Fourier transforms of the decaying and rising exponentials and linearity property of Fourier transforms.

SOLUTION

The double-sided exponential is given by,

$$x(t) = e^{-a|t|} = \begin{cases} e^{at} & ; \quad t < 0 \\ e^{-at} & ; \quad t > 0 \end{cases}$$

This function can be written as

$$x(t) = e^{-at}\, u(t) + e^{at}\, u(-t)$$

Using the linearity property we can write

$$X(\omega) = F[e^{-at}\, u(t)] + F[e^{at}\, u(-t)]$$

$$\therefore \quad X(\omega) = \frac{1}{a + j\omega} + \frac{1}{a - j\omega}$$

$$= \frac{2a}{a^2 + (\omega)^2}$$

$$\therefore \quad e^{-a|t|} \xleftrightarrow{\text{F.T}} \frac{2a}{a^2 + \omega^2}$$

This result same as that obtained earlier Eq. (2.4.5).

2.6.2 Time Shifting

If $\qquad x(t) \xleftrightarrow{\text{F.T}} X(\omega)$

Then, $\qquad x(t - t_0) \xleftrightarrow{\text{F.T}} X(\omega)\, e^{-j\omega t_0} \qquad \text{... (2.6.2)}$

When a signal is shifted in time-domain, the magnitudes of its Fourier transform remain unchanged, but introduces into its Fourier transform a phase shift, ωt_0, which is a linear function of ω.

PROOF

The Fourier transform of $x(t - t_0)$ is given by,

$$F[x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0)\, e^{-j\omega t}\, dt$$

Let $\tau = (t - t_0)$, which gives $d\tau = dt$, $\tau \to -\infty$, as $t \to -\infty$ and $\tau \to \infty$ as $t \to \infty$.

$$F[x(t - t_0)] = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega(\tau + t_0)}\, d\tau$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau = e^{-j\omega t_0} X(\omega)$$

$\therefore \quad x(t - t_0) \xleftrightarrow{F.T} X(\omega) e^{-j\omega t_0}$

Similarly, $\quad x(t + t_0) \xleftrightarrow{F.T} X(\omega) e^{+j\omega t_0}$

EXAMPLE PROBLEM 1

Obtain the Fourier transform of, a trapezoidal function f(t) shown in Fig. 2.6.1.

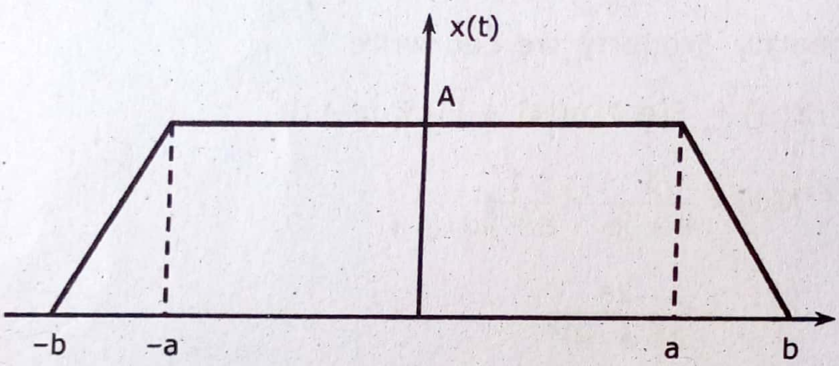
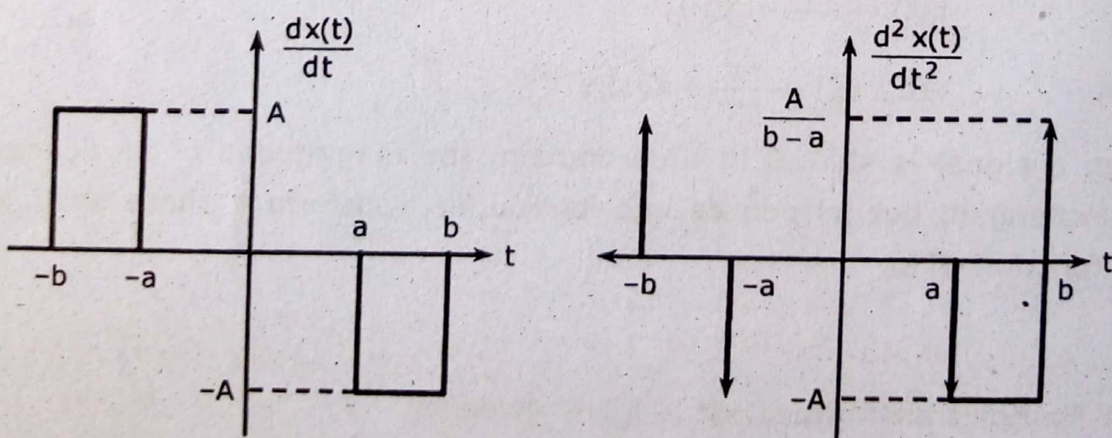


Fig. 2.6.1 Trapezoidal Function

SOLUTION

By the differentiation of the trapezoidal function, we get the waveform as shown in Fig. 2.6.2(a). By the differentiation of the waveform obtained in Fig. 2.6.2(a), we get the impulse function as shown in Fig. 2.6.2(b).



(a) Plot of $\frac{dx(t)}{dt}$ (b) Plot of $\frac{d^2x(t)}{dt^2}$

Fig. 2.6.2

From Fig. 2.6.2(b),

$$\frac{d^2 x(t)}{dt^2} = [A\delta(t+b) + A\delta(t-b) - A\delta(t+a) - A\delta(t-a)]\left(\frac{1}{b-a}\right)$$

$$= \frac{A}{b-a}[\delta(t+b) + \delta(t-b) - \delta(t+a) - \delta(t-a)] \quad \text{... (2.6.3)}$$

Using time-shifting theorem, i.e.,

$$\delta(t-t_0) \xleftrightarrow{F.T} e^{j\omega t_0}$$

And property of differentiation, i.e.,

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{F.T} (j\omega)^n X(\omega)$$

Eq. (2.6.3) becomes,

$$(j\omega)^2 X(\omega) = \frac{A}{b-a}[e^{-j\omega b} + e^{j\omega b} - e^{-j\omega a} - e^{j\omega a}]$$

$$-\omega^2 X(\omega) = \frac{A}{b-a}[(e^{j\omega b} + e^{-j\omega b}) - (e^{j\omega a} + e^{-j\omega a})]$$

$$= \frac{2A}{b-a}\left[\left(\frac{e^{j\omega b} + e^{-j\omega b}}{2}\right) - \left(\frac{e^{j\omega a} + e^{-j\omega a}}{2}\right)\right]$$

$$\therefore \quad X(\omega) = \frac{-2A}{(b-a)\omega^2}[\cos\omega a - \cos\omega b]$$

2.6.3 Frequency Shifting

If $\quad x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $\quad x(t)\, e^{j\omega_0 t} \xleftrightarrow{F.T} X(\omega - \omega_0) \quad \text{... (2.6.4)}$

Frequency shifting property states that the multiplication in time domain by a complex sinusoid corresponds to frequency shift in $X(\omega)$.

PROOF

The Fourier transform of $x(t)\, e^{j\omega_0 t}$ is given by,

$$F[x(t)\, e^{j\omega_0 t}] = \int_{-\infty}^{\infty} [x(t)\, e^{j\omega_0 t}]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega - \omega_0)t}\, dt = X(\omega - \omega_0)$$

$$\therefore \quad x(t)e^{j\omega_0 t} \xleftrightarrow{F.T} X(\omega - \omega_0)$$

Similarly $\quad x(t)e^{-j\omega_0 t} \xleftrightarrow{F.T} X(\omega + \omega_0)$

EXAMPLE PROBLEM 1

Find the Fourier transform of,

(i) $x(t) = \cos(\omega_0 t)$

(ii) $x(t) = \sin(\omega_0 t)$ using Frequency shifting property.

SOLUTION

(i) From Euler's formula, we have,

$$x(t) = \cos(\omega_0 t)$$

$$= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

Taking Fourier transform, we get,

$$F[x(t)] = F[\cos(\omega_0 t)]$$

$$= F\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

$$= \frac{1}{2}\left[F(e^{j\omega_0 t}) + F(e^{-j\omega_0 t})\right] \qquad \text{... (2.6.5)}$$

Using Fourier transform of constant unit amplitude function defined by Eq. (2.4.3), we have,

$$1 \xleftrightarrow{F.T} 2\pi\delta(\omega)$$

From frequency shifting property,

$$\left.\begin{aligned} 1e^{j\omega_0 t} &\xleftrightarrow{F.T} 2\pi\delta(\omega - \omega_0) \\ 1e^{-j\omega_0 t} &\xleftrightarrow{F.T} 2\pi\delta(\omega + \omega_0) \end{aligned}\right\} \qquad \text{... (2.6.6)}$$

Using Eq. (2.6.6) in Eq. (2.6.5), we get,

$$F[x(t)] = X(\omega)$$

$$= \frac{1}{2}[2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)]$$

$$\therefore \quad \cos(\omega_0 t) \xleftrightarrow{F.T} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \qquad \text{... (2.6.7)}$$

The frequency spectrum of $\cos(\omega_0 t)$ consists of two impulses at ω_0 and $-\omega_0$, as shown in Fig. 2.6.3(a).

(ii) From Euler's formula, we have,

$$x(t) = \sin\omega_0 t$$

$$= \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

By taking Fourier transform, we have,

$$F[x(t)] = X(\omega)$$

$$= F\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right]$$

$$= \frac{1}{2j}\left[F(e^{j\omega_0 t}) - F(e^{-j\omega_0 t})\right] \quad \text{... (2.6.8)}$$

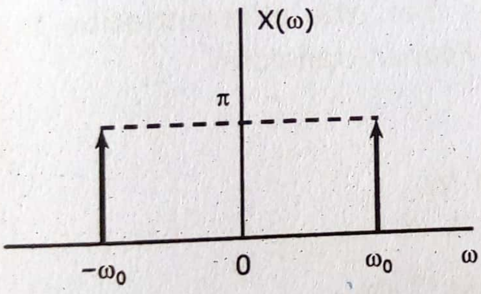
Using Eq. (2.6.6) in Eq. (2.6.8), we get,

$$X(\omega) = \frac{1}{2j}[2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)]$$

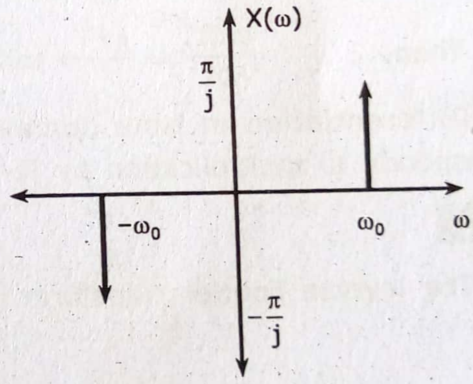
$$= \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

$$\therefore \quad \sin(\omega_0 t) \xleftrightarrow{F.T} \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] \quad \text{... (2.6.9)}$$

The Fourier transform of $\sin(\omega_0 t)$ is as shown in Fig. 2.6.3(b)



(a) Frequency Spectrum of $\cos(\omega_0 t)$



(b) Frequency Spectrum of $\sin(\omega_0 t)$

Fig. 2.6.3

2.6.4 Time Scaling

If $\quad x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $\quad x(at) \xleftrightarrow{F.T} \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \quad \text{... (2.6.10)}$

The time scaling property states that time compression of a signal results in its spectral expansion and time expansion of the signal results in its spectral compression.

2.6.5 Time Reversal

If $x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $x(-t) \xleftrightarrow{F.T} X(-\omega)$ ... (2.6.11)

The time-reversal property states that reversing a signal in time also reverses its Fourier transform.

PROOF

Substituting a = −1 in Eq. (2.6.10), we have,

$$F[x(-t)] = \frac{1}{|-1|} X\left(\frac{\omega}{-1}\right)$$

$$F[x(-t)] = X(-\omega)$$

An interesting consequence of the time-reversal property is that if x(t) is even, then its Fourier transform is also even, i.e.,

$$\text{if } x(-t) = x(t), \text{ then } X(-\omega) = X(\omega) \quad ...\ (2.6.12)$$

Similarly, if x(t) is odd, then its Fourier transform is also odd i.e.,

$$\text{if } x(-t) = -x(t), \text{ then } X(-\omega) = -X(\omega) \quad ...\ (2.6.13)$$

2.6.6 Differentiation in Time Domain

If $x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $\frac{dx(t)}{dt} \xleftrightarrow{F.T} j\omega X(\omega)$

Differentiation in time domain states that, the differentiation in time domain, corresponds to multiplication by $j\omega$ in the Fourier transform.

PROOF

The inverse Fourier transform is given by,

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$$

Differentiating above equation with respect to 't' gives,

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega X(\omega)]\, e^{j\omega t}\, d\omega \quad ...\ (2.6.14)$$

$$\Rightarrow \quad \frac{dx(t)}{dt} = F^{-1}[j\omega X(\omega)]$$

$$\therefore \quad \frac{dx(t)}{dt} \xleftrightarrow{F.T} j\omega X(\omega) \quad ...\ (2.6.15)$$

PROOF

The Fourier transform of x(at) is given by,

$$F[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

CASE I : For a positive real constant a,

$$F[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Let $\tau = at$, which gives $d\tau = a\,dt$ and $\tau \to -\infty$ as $t \to -\infty$ and $\tau \to \infty$ as $t \to \infty$. Therefore,

$$F[x(at)] = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau$$

$$= \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

CASE II : For a negative real constant −a,

$$F[x(-at)] = \int_{-\infty}^{\infty} x(-at)\, e^{-j\omega t}\, dt$$

Let $\tau = -at$, which gives $d\tau = -a\,dt$ and $\tau \to \infty$ as $t \to -\infty$ and $\tau \to -\infty$ as $t \to \infty$. Therefore,

$$F[x(-at)] = -\frac{1}{a}\int_{\infty}^{-\infty} x(\tau)\, e^{j\left(\frac{\omega}{a}\right)\tau}\, d\tau$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(-\frac{\omega}{a}\right)\tau}\, d\tau$$

$$= \frac{1}{a} X\left(-\frac{\omega}{a}\right)$$

Combining the two cases, we have,

$$F[x(at)] = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Similarly, $x\left(\frac{t}{a}\right) \xleftrightarrow{F.T} |a|\, X(a\omega)$

Differentiating Eq. (2.6.14) w.r.t 't', we get,

$$\frac{d^2x(t)}{dt^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}[j\omega X(\omega)]\,[j\omega e^{j\omega t}]\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}[(j\omega)^2 X(\omega)]e^{j\omega t}\,d\omega$$

$$= F^{-1}[(j\omega)^2X(\omega)]$$

Similarly, the differentiation property can be extended, to give,

$$\frac{d^nx(t)}{dt^n} \xleftrightarrow{F.T} (j\omega)^nX(\omega) \qquad \text{... (2.6.16)}$$

2.6.7 Differentiation in Frequency Domain

If $\qquad x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $\qquad -jtx(t) \xleftrightarrow{F.T} \frac{dX(\omega)}{d\omega} \qquad \text{... (2.6.17)}$

(or) $\qquad tx(t) \xleftrightarrow{F.T} j\frac{dX(\omega)}{d\omega}$

Differentiation in frequency domain property states that, the differentiation in frequency domain is equivalent to multiplying the time domain by t.

PROOF

The Fourier transform of x(t) is defined as,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt$$

Differentiating both sides w.r.t. ω gives,

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty}[-jtx(t)]\,e^{-j\omega t}\,dt$$

$$= F[-jtx(t)]$$

$$\therefore \qquad -jtx(t) \xleftrightarrow{F.T} \frac{dX(\omega)}{d\omega}$$

(or) $\qquad tx(t) \xleftrightarrow{F.T} j\frac{dX(\omega)}{d\omega}$

EXAMPLE PROBLEM 1

Find the Fourier transform of the signal $x(t) = t\,e^{-at}\,u(t)$ using differentiation in frequency-domain property.

SOLUTION

Given signal is,

$$x(t) = te^{-at}\,u(t)$$

Let $\quad f(t) = e^{-at}\,u(t)$

So, $\quad x(t) = tf(t)$

Using Frequency domain differentiation property defined by Eq. (2.6.17), we have,

$$F[x(t)] = F[t\,f(t)] = j\frac{dF(\omega)}{d\omega}$$

Here, $\quad f(t) = e^{-at}\,u(t)$

And its Fourier transform,

$$F(\omega) = \frac{1}{(a + j\omega)}$$

$$X(\omega) = F[t\,e^{-at}\,u(t)] = j\frac{d}{d\omega}\left(\frac{1}{a + j\omega}\right)$$

$$= j\left[\frac{0 - j}{(a + j\omega)^2}\right] = \frac{1}{(a + j\omega)^2}$$

$$\therefore \quad t\,e^{-at}\,u(t) \xleftrightarrow{F.T} \frac{1}{(a + j\omega)^2}$$

2.6.8 Convolution in Time-domain

If $\quad x_1(t) \xleftrightarrow{F.T} X_1(\omega)$

And $\quad x_2(t) \xleftrightarrow{F.T} X_2(\omega)$

Then, $\quad x_1(t) * x_2(t) \xleftrightarrow{F.T} X_1(\omega)\,X_2(\omega)$

The convolution property states that, convolution of two signals in the time domain corresponds to product of their respective Fourier transforms in the frequency domain.

PROOF

The Fourier transform of $[x_1(t) * x_2(t)]$ is given by,

z(t)

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\,e^{-j\omega t}\,dt$$

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Using the definition of convolution, that is,

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\, d\tau$$

We get,

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)]\, d\tau\right) e^{-j\omega t}\, dt$$

Interchanging the order of integration and noting that $x_1(\tau)$ does not depend on t gives,

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\left(\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t}\, dt\right) d\tau$$

Using the time-shifting property of Fourier transform, defined by Eq. (2.6.2), the above equation becomes,

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\left(X_2(\omega)\, e^{-j\omega\tau}\right) d\tau$$

$$= X_2(\omega)\int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau = X_1(\omega).X_2(\omega)$$

$$\therefore \quad x_1(t) * x_2(t) \xleftrightarrow{F.T} X_1(\omega)\, X_2(\omega) \qquad \text{... (2.6.18)}$$

2.6.9 Multiplication in Time-Domain

If $\quad x_1(t) \xleftrightarrow{F.T} X_1(\omega)$

And $\quad x_2(t) \xleftrightarrow{F.T} X_2(\omega)$

Then, $\quad x_1(t)\, x_2(t) \xleftrightarrow{F.T} \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$

$$x_1(t)\, x_2(t) \xleftrightarrow{F.T} \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)X_2(\omega-\lambda)\, d\lambda$$

Multiplication in time-domain property states that the multiplication of two signals in time-domain is equivalent to the convolution of their Fourier transforms in the frequency domain.

PROOF

The Fourier transform of $x_1(t)\, x_2(t)$ is given by,

$$F[x_1(t)x_2(t)] = \int_{-\infty}^{\infty} [x_1(t)x_2(t)]\, e^{-j\omega t}\, dt$$

Expressing $x_2(t)$ as the inverse Fourier transform of $X_2(\omega')$, we get,

$$F[x_1(t)x_2(t)] = \int_{-\infty}^{\infty} x_1(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\omega') e^{+j\omega' t}\, d\omega' \right) e^{-j\omega t}\, dt$$

Interchanging the order of integration,

$$F[x_1(t)x_2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\omega')\, d\omega' \left[\int_{-\infty}^{\infty} x_1(t)\, e^{-j(\omega - \omega')}\, dt \right]$$

Using the frequency-shifting property, the bracketed term is $X_1(\omega - \omega')$. Substituting this into the above equation gives,

$$F[x_1(t)x_2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\omega')\, X_1(\omega - \omega')\, d\omega'$$

Putting $\omega - \omega' = \lambda$, in above equation, we get,

$$F[x_1(t)x_2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\omega - \lambda) X_1(\lambda)\, d\lambda$$

$$= \frac{1}{2\pi} [X_1(\omega) * X_2(\omega)]$$

$$\therefore \quad x_1(t)x_2(t) \xleftrightarrow{F.T} \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)] \qquad \dots (2.6.19)$$

2.6.10 Integration in Time-Domain

If $\quad x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $\quad \int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{F.T} \frac{1}{j\omega} X(\omega)$

Integration in time-domain property states that, the integration in time domain is equivalent to the multiplication of frequency domain by $1/j\omega$.

PROOF

Let, $$r(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Differentiating above equation w.r.t 't', we get,

$$\frac{dr(t)}{dt} = x(t)$$

$$\Rightarrow \quad FT[x(t)] = FT\left[\frac{d}{dt} r(t)\right]$$

From differentiation in time domain we have,

$$X(\omega) = jwR(\omega)$$

$$\Rightarrow \quad R(\omega) = \frac{1}{j\omega} X(\omega)$$

$$\therefore \quad F.T\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j\omega} X(\omega) \qquad \text{... (2.6.20)}$$

2.6.11 Duality

If $x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $X(t) \xleftrightarrow{F.T} 2\pi x(-\omega)$

Duality property tells us that if x(t) has a Fourier transform $X(\omega)$, then if we form a new function of time that has the functional form of the transform, x(t), it will have a Fourier transform $X(\omega)$ that has the functional form of the original time function, but is a function of frequency.

PROOF

The inverse Fourier transform is given by,

$$F^{-1}[X(\omega)] = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow \quad 2\pi x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Replacing t with −t in the above equation gives,

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

Now interchanging the variables t and ω yields,

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

$$2\pi x(-\omega) = F[X(t)]$$

$$\therefore \quad X(t) \xleftrightarrow{F.T} 2\pi x(-\omega) \qquad \text{... (2.6.21)}$$

EXAMPLE PROBLEM 1

Applying the duality property, calculate the inverse Fourier transform of one-sided exponential function given by,

$$X(\omega) = \begin{cases} e^{-\alpha\omega} & ; \omega > 0 \\ 0 & ; \omega < 0 \end{cases}$$

SOLUTION

The Fourier transform of single-sided exponential function, $e^{+at}u(-t)$ is given by,

$$e^{+at}u(-t) \xleftrightarrow{F.T} \frac{1}{a - j\omega}$$

Using the duality property, we can write,

$$\frac{1}{a - jt} \xleftrightarrow{F.T} 2\pi e^{-a\omega}u(\omega)$$

Therefore, the inverse Fourier transform of the given function $X(\omega)$ is,

$$x(t) = \left(\frac{1}{a - jt}\right)\frac{1}{2\pi}$$

EXAMPLE PROBLEM 2

Find the Fourier transform x(w) of the signal shown in Fig. 2.6.4(a).

$$\mathbf{x(t) = \frac{1}{\pi t}}$$

SOLUTION

Using the result of Fourier transform of signum function defined by Eq. (2.4.12), we have,

$$\text{Sgn}(t) \xleftrightarrow{F.T} \frac{2}{j\omega}$$

Using linearity property, we have,

$$\frac{j}{2\pi}\text{Sgn}(t) \xleftrightarrow{F.T} \frac{1}{\pi\omega}$$

Applying the duality property, we get,

$$\frac{1}{\pi t} \xleftrightarrow{F.T} \left(\frac{j}{2\pi}\,\mathrm{sgn}(-\omega)\right) 2\pi$$

$$\therefore \quad F\left[\frac{1}{\pi t}\right] = -j\,\mathrm{sgn}(\omega) \qquad [\because \mathrm{sgn}(-\omega) = \mathrm{sgn}(\omega)]$$

The frequency spectrum $X(\omega)$ is shown in Fig. 2.6.4(b).

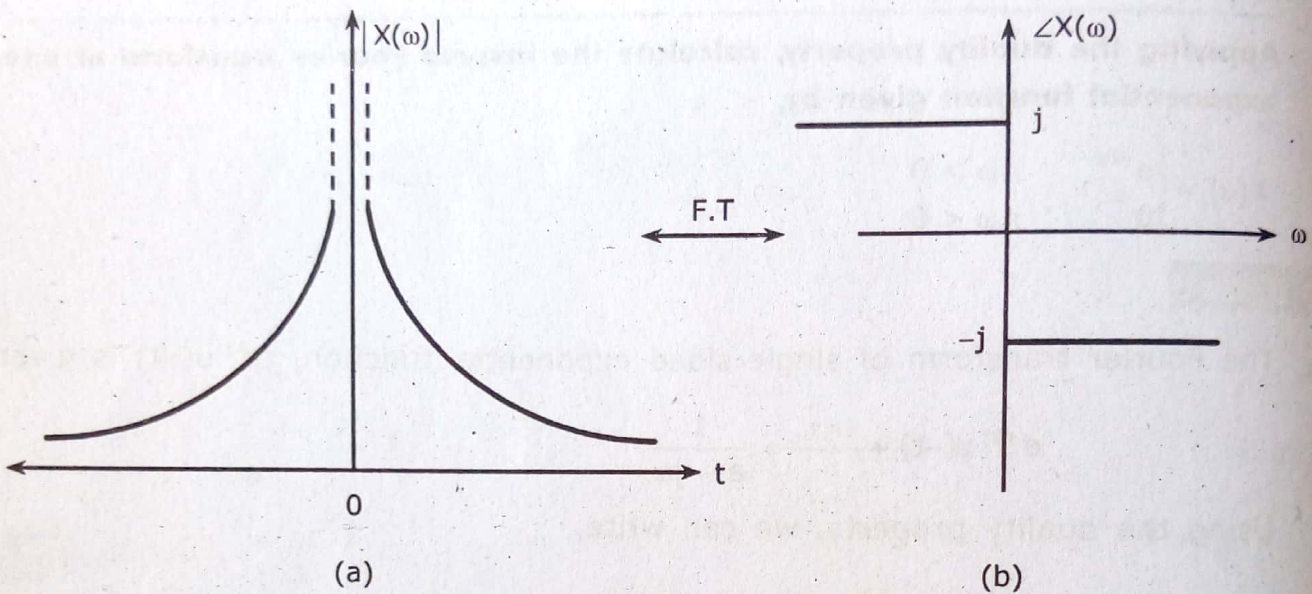


Fig. 2.6.4 Plots of given function x(t) and its Fourier Transform $X(\omega)$

2.6.12 Conjugation and Conjugate Symmetry

If $\quad x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $\quad x^*(t) \xleftrightarrow{F.T} X^*(-\omega)$

It states that, the Fourier transform of a conjugation of x(t) is equivalent to frequency reversal of conjugation of $X(\omega)$.

PROOF

The Fourier transform of $x^*(t)$ is given by,

$$F[x^*(t)] = \int_{-\infty}^{\infty} x^*(t)\, e^{-j\omega t}\, dt = \left[\int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt\right]^*$$

$$= \left[\int_{-\infty}^{\infty} x(t)\, e^{-j(-\omega)t}\, dt\right]^* = [X(-\omega)]^*$$

$$\therefore \quad x^*(t) \xleftrightarrow{F.T} x^*(-\omega) \qquad \text{... (2.6.22)}$$

2.6.13 Area Under x(t)

If $\quad x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $$\int_{-\infty}^{\infty} x(t)\, dt = X(0) \qquad \dots (2.6.23)$$

That is, the area of a signal x(t) is equal to the value of its Fourier transform $X(\omega)$ at $\omega = 0$.

PROOF

Fourier transform of a signal x(t) is,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Then, area under x(t) is defined as $X(\omega)$ at $\omega = 0$ is,

$$X(\omega)\big|_{\omega=0} = X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

2.6.14 Area Under X(ω)

If $\quad x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega \qquad \dots (2.6.24)$$

That is, the value of the function x(t) at t = 0 is equal to the area under its Fourier transform $\frac{X(\omega)}{2\pi}$.

PROOF

The inverse Fourier transform of x(t) is given as,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Then, area under signal $X(\omega)$ as x(t) at t = 0, we have,

$$x(t)\big|_{t=0} = x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega$$

2.6.15 Parseval's Relation

Let x(t) be an energy signal and

If $x(t) \xleftrightarrow{F.T} X(\omega)$

Then, $$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega \qquad \text{... (2.6.25)}$$

Parseval's theorem states that the, signal energies of an energy signal and its Fourier transform are equal.

PROOF

Consider the L.H.S of Eq. (2.6.25), we have,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$= \int_{-\infty}^{\infty} x(t)x^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega\right)^* dt$$

$$= \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t}\, d\omega\right) dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\left(\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, X(\omega)\, d\omega$$

$$\therefore \quad E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

2.6.16 Summary of Properties of Fourier Transform

Let, $F\{x(t)\} = X(\omega)$; $F\{x_1(t)\} = X_1(\omega)$; $F\{x_2(t)\} = X_2(\omega)$

Table 2.6.1 Summary of Properties of Fourier Transform

S.No.	Property	x(t)	X(ω)	X(f)
(1)	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$	$aX_1(f) + bX_2(f)$
(2)	Time Shift	$x(t - t_0)$	$e^{-j\omega t_0}X(\omega)$	$e^{-j2\pi f t_0}X(f)$
(3)	Frequency Shift	$e^{j\omega_0 t}x(t)$	$X(\omega - \omega_0)$	$X\left(f - \frac{\omega_0}{2\pi}\right)$
(4)	Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$
(5)	Time Reversal	$x(-t)$	$X(-\omega)$	$X(-f)$
(6)	Time Differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$	$j2\pi f X(f)$
(7)	Frequency Differentiation	$-jtx(t)$	$\frac{dX(\omega)}{d\omega}$	$\frac{1}{2\pi}\frac{dX(f)}{df}$
(8)	Convolution in Time Domain	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$	$X_1(f)X_2(f)$
(9)	Multiplication in Time-Domain	$x_1(t) . x_2(t)$	$\frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$	$X_1(f) * X_2(f)$
(10)	Integration	$\int_{-\infty}^{t} x(t)\, dt$	$\frac{1}{j\omega}X(\omega)$	$\frac{1}{j2\pi f}X(f)$
(11)	Duality	$X(t)$	$2\pi x(-\omega)$	$x(-f)$
(12)	Conjugation	$x^*(\pm t)$	$X^*(\mp\omega)$	$X^*(\mp f)$

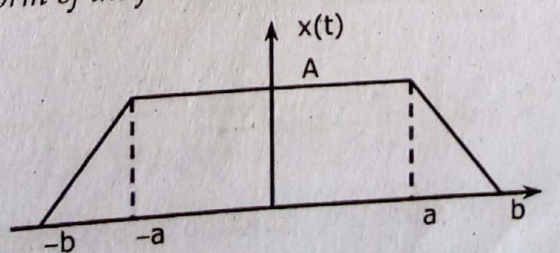
REVIEW QUESTIONS

(1) *Write the different properties of Fourier transform?*

(2) *State and prove the time differentiation theorem of Fourier transform.*

(3) *State and prove the time frequency shifting property of the Fourier transform.*

(4) *(a) Find the Fourier transform of the function x(t) shown in figure.*



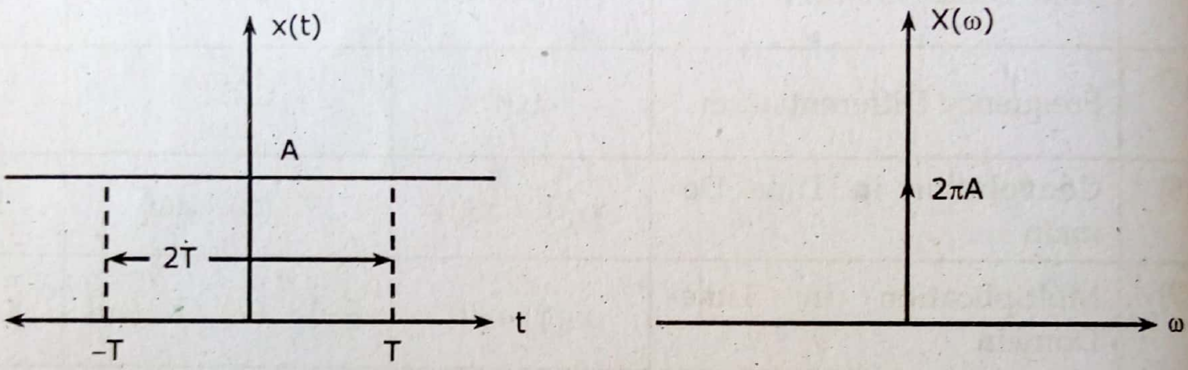
(b) State and prove the property used.

2.7 FOURIER TRANSFORMS INVOLVING IMPULSE FUNCTION AND SIGNUM FUNCTION

The Fourier transforms of functions which do not satisfy directly the Dirichlet's conditions, can be conveniently calculated with the help of impulse function (also known as Dirac-delta function).

(1) **D.C Signal (Constant Function) :** A D.C signal is represented by a constant value for all time, t as shown in Fig. 2.7.1(a). The D.C signal is represented as $x(t) = A$, A being a constant. This function is not absolutely integrable, but its Fourier transform can be determined using the concept of an impulse function.

Let the constant function is considered to be a gate function of amplitude A, having width 2T in the limit $T \to \infty$.



(a) D.C Signal Represented by a Constant Function (b) Fourier Spectrum of that D.C Signal

Fig. 2.7.1

Using the Fourier transform of gate function defined by Eq. (2.4.6), i.e.,

$$A\,\mathrm{Rect}\left(\frac{t}{2T}\right) \xleftrightarrow{F.T} 2AT\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right)$$

$$X(\omega) = F[A]$$

$$= \lim_{T\to\infty} F\left[A\,\mathrm{Rect}\left(\frac{t}{2T}\right)\right]$$

$$= \lim_{T\to\infty} 2AT\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right)$$

$$= 2\pi A\left[\lim_{T\to\infty}\frac{T}{2\pi}\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right)\right]$$

As the limit $T \to \infty$, the sampling function approaches a delta function $\delta(\omega)$,

$\therefore$ $$F[A] = 2pa\delta(\omega)$$

(2) **Unit Step Function :** The unit step function shown in Fig. 3.7.2(a) is defined as,

$$u(t) = \begin{cases} 1 & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \end{cases}$$

The Fourier transform of u(t) can be derived using the Fourier transform of sgn(t). It can be easily seen that,

$$\text{sgn}(t) = 2u(t) - 1$$

$$u(t) = \frac{1}{2}(\text{sgn}(t) + 1)$$

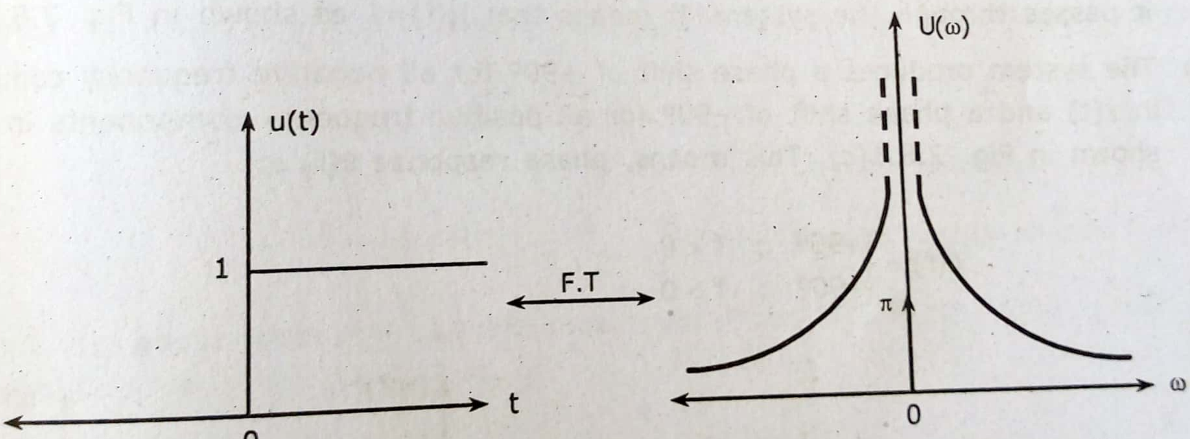
Applying Fourier transform on both sides of above equation, we get,

$$U(\omega) = \frac{1}{2}F[1] + \frac{1}{2}F[\text{sgn}(t)]$$

$$= \frac{1}{2}(2\pi\delta(\omega)) + \frac{1}{2}\left(\frac{2}{j\omega}\right) = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\left(\because \text{sgn}(t) \xleftrightarrow{F.T} \frac{2}{j\omega} \text{ Using Eq. (3.3.12)}\right)$$

The spectrum of unit step function is as shown in Fig. 2.7.2(b). It can be seen that the spectrum of the unit-step function contains a delta function weighted by π occurring at zero angular frequency.



(a) Unit Step Function

(b) Spectrum of Unit Step Function

Fig. 2.7.2

REVIEW QUESTIONS

(1) *What is a Fourier transform of an impulse function?*
(2) *Write down the Fourier transform of a D.C signal (constant function).*
(3) *Explain the Fourier transform of unit step function along with appropriate diagram?*

2.8 INTRODUCTION TO HILBERT TRANSFORM

When the phase angle of all frequency components of a given signal x(t) are shifted by ±90°, the resulting function, in time-domain is known as Hilbert Transform of the signal.

Hilbert transform is denoted by $\hat{x}(t)$ and can be written in terms of x(t) as,

$$\hat{x}(t) = x(t) * \frac{1}{\pi t} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} \cdot d\tau$$

[Here sign * represents convolution]

The inverse Hilbert Transform is given by,

$$x(t) = \frac{-1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{x}(\tau)}{t-\tau} \cdot d\tau$$

The functions $\hat{x}(t)$ and x(t) are said to constitute Hilbert transform pair.

PROOF

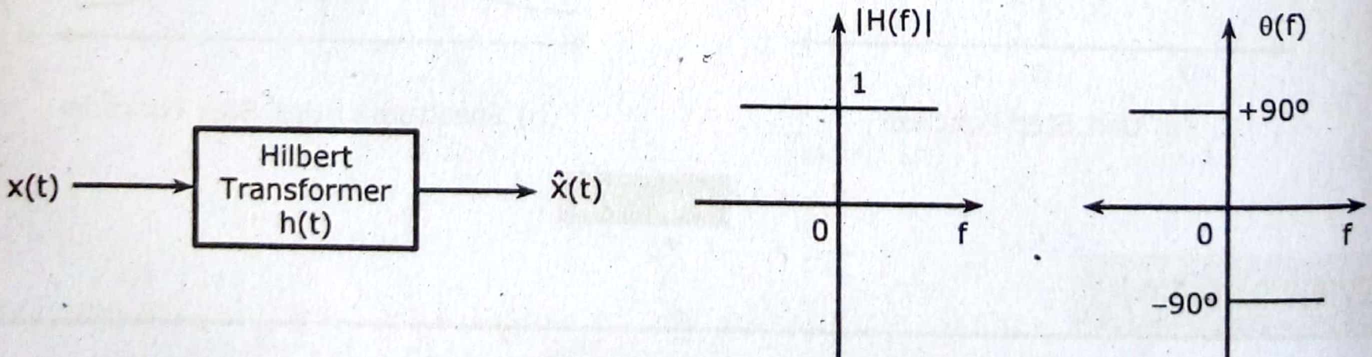
Consider a linear time invariant system as shown in Fig. 2.8.1(a), having x(t) as an input and the output of the system is the Hilbert transform of the signal and has been denoted by $\hat{x}(t)$.

The characteristics of this system may be specified as follows,

(1) The magnitude of frequency components present in x(t) remains unchanged when it passes through the system. It means that H(f)=1 as shown in Fig. 2.8.1(b).

(2) The system produces a phase shift of +90° for all negative frequency components in x(t) and a phase shift of −90° for all positive frequency components in x(t) as shown in Fig. 2.8.1(c). This means, phase response θ(f) is,

$$\theta(f) = \begin{cases} +90^\circ & ; \quad f < 0 \\ -90^\circ & ; \quad f > 0 \end{cases}$$



(a) Hilbert Transformer (b) Magnitude Response (c) Phase Response

Fig. 2.8.1 Hilbert Transform

The transfer function of LTI system shown in Fig. 2.8.1(a) is thus given by,

$$H(f) = |H(f)|\, e^{j\theta(f)} = 1 \cdot e^{j\theta(f)}$$

$$H(f) = \begin{cases} 1.e^{j\frac{\pi}{2}} & ; \quad f < 0 \\ 1.e^{-j\frac{\pi}{2}} & ; \quad f > 0 \end{cases}$$

$$e^{j\frac{\pi}{2}} = j$$

And $$e^{-j\frac{\pi}{2}} = -j$$

Hence, $$\frac{H(f)}{j} = \begin{cases} 1 & ; \quad f < 0 \\ -1 & ; \quad f > 0 \end{cases} = -\text{sgn}(f)$$

$\therefore$ $$H(f) = -j\text{sgn}(f) \qquad \text{... (2.8.1)}$$

Where sgn(f) is the signum function defined by,

$$\text{sgn}(f) = \begin{cases} 1 & ; \quad f < 0 \\ -1 & ; \quad f > 0 \end{cases}$$

Using the Fourier transform pair of signum function defined by,

$$\text{sgn}(t) \xleftrightarrow{\text{F.T}} \frac{1}{j\pi f} \qquad \text{... (2.8.2)}$$

Using the duality property of Fourier transform to the above transform pair, we have,

$$\frac{1}{j\pi t} \xleftrightarrow{\text{F.T}} \text{sgn}(-f) \qquad \text{... (2.8.3)}$$

$\Rightarrow$ $$\frac{1}{j\pi t} \xleftrightarrow{\text{F.T}} -\text{sgn}(f)$$ ($\because$ signum function is an odd function)

Using the above transform pair the impulse response, h(t) of the Hilbert transformer can be obtained from Eq. (2.8.1) as,

$$h(t) = F^{-1}\{H(f)\}$$

$$= F^{-1}[-j\text{sgn}(f)]$$

$$h(t) = \frac{1}{\pi t} \qquad \text{... (2.8.4)}$$

The output of the Hilbert transformer can be obtained by convolving the input signal x(t) with the impulse response of the system defined by Eq. (2.8.4). Thus the Hilbert transform $\hat{x}(t)$ can be written in terms of x(t) as,

$$\hat{x}(t) = x(t) * \frac{1}{\pi t} \qquad \text{... (2.8.5)}$$

Using the definition of convolution, Eq. (2.8.5), can also be written as,

$$\therefore \quad \hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}\,d\tau \qquad \text{... (2.8.6)}$$

It is clearly seen that Hilbert transform x(t) is a linear operation. The inverse Hilbert transform is a mean for recovering x(t) from the transformed signal $\hat{x}(t)$. The inverse Hilbert transform is given by,

$$\therefore \quad x(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\hat{x}(\tau)}{t-\tau}\,d\tau \qquad \text{... (2.8.7)}$$

The functions x(t) and $\hat{x}(t)$ are said to constitute ***Hilbert transform pair.***

Now we proceed to calculate the Fourier transform of the Hilbert transform of the signal. Denoting the Fourier transforms of x(t) and $\hat{x}(t)$ by X(f) and $\hat{X}(f)$ respectively and taking Fourier transform on both sides of Eq. (2.8.5) we write.

$$\hat{X}(f) = F\left[x(t) * \frac{1}{\pi t}\right] \qquad \text{... (2.8.8)}$$

Using the convolution property of Fourier transforms and the transform pair of Eq. (2.8.3) we get,

$$\hat{X}(f) = -j\,\mathrm{sgn}(f)\,X(f) \qquad \text{... (2.8.9)}$$

2.8.1 Properties of Hilbert Transform

PROPERTY I : A signal x(t) and its Hilbert transform $\hat{x}(t)$ have the same magnitude spectrum.

$$|\hat{X}(\omega)| = |X(\omega)|$$

PROOF

Given that,

$$\hat{x}(t) = HT[x(t)]$$

From Eq. (2.8.9), we have,

$$\hat{X}(f) = -j\mathrm{sgn}(f)X(f)$$

$$\Rightarrow \quad |\hat{X}(f)| = |-j\mathrm{sgn}(f)X(f)|$$

$$\Rightarrow \quad |\hat{X}(f)| = |-j\mathrm{sgn}(f)|\,|X(f)|$$

Since $|-j\text{sgn}(f)| = 1$ for all f, we have,

$$|\hat{X}(f)| = |X(f)| \quad \ldots (2.8.10)$$

Hence, $$|\hat{X}(\omega)| = |X(\omega)|$$

PROPERTY II : If $\hat{x}(t)$ is the Hilbert transform of x(t), then the Hilbert transform of $\hat{x}(t)$ is $-x(t)$, i.e.,

If $$\hat{x}(t) = HT[x(t)] \quad \ldots (2.8.11)$$

Then $$-x(t) = HT[\hat{x}(t)]$$

PROOF

From Fig. 2.8.1(a), we have,

$$H(f) = \frac{\hat{X}(f)}{X(f)}$$

$$\Rightarrow \quad \hat{X}(f) = X(f)\, H(f)$$

$$X(f) = \frac{1}{H(f)}\hat{X}(f)$$

But we have,

$$H(f) = -j\text{sgn}(f) = \begin{cases} -j & ; \quad 0 < f \leq \infty \\ j & ; \quad -\infty \leq f < 0 \end{cases}$$

$$\therefore \quad \frac{1}{H(f)} = -H(f) = j\text{sgn}(f) = \begin{cases} j & ; \quad 0 \leq f \leq \infty \\ j & ; \quad -\infty \leq f < 0 \end{cases}$$

$$\therefore \quad X(f) = -H(f)\, \hat{X}(f)$$

$$-X(f) = H(f)\, \hat{X}(f)$$

Taking the inverse Fourier transform, we obtain,

$$-x(t) = \hat{x}(t) * h(t)$$

$$= \hat{x}(t) * \frac{1}{\pi t} \quad \ldots (2.8.12)$$

$$\therefore \quad -x(t) = HT[\hat{x}(t)]$$

PROPERTY III : A signal x(t) and its Hilbert transform $\hat{x}(t)$ are orthogonal, i.e., if x(t) is an energy signal, then,

$$\int_{-\infty}^{\infty} x(t)\, \hat{x}(t)\, dt = 0$$

PROOF

For a real signal x(t) multiplied by its Hilberts transform $\hat{x}(t)$ we may write,

$$\int_{-\infty}^{\infty} x(t)\, \hat{x}(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, \hat{X}^*(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, \hat{X}(-\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} x(t)\, \hat{x}(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)[-j\,sgn(-\omega)\, X(-\omega)]\, d\omega$$

Since, $sgn(-\omega) = -sgn(\omega)$

And $X(-\omega) = X^*(\omega)$

We have,

$$\int_{-\infty}^{\infty} x(t)\, \hat{x}(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)[j\,sgn(\omega)\, X^*(\omega)]\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} sgn(\omega)\, X(\omega)\, X^*(\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} x(t)\, \hat{x}(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} sgn(\omega)\, |X(\omega)|^2\, d\omega$$

The integrand in the RHS of the above equation is an odd function of ω, being the product of the odd function $sgn(\omega)$ and the even function $|X(\omega)|^2$.

Hence, the integral is zero, yielding the final result,

$$\int_{-\infty}^{\infty} x(t)\, \hat{x}(t)\, dt = 0$$

This shows that an energy signal x(t) and its Hilbert transform $\hat{x}(t)$ are orthogonal over the entire interval $(-\infty, \infty)$. Similarly, a power signal x(t) and its Hilbert transform $\hat{x}(t)$ are orthogonal over one period, as shown by,

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, \hat{x}(t)\, dt = 0$$

PROPERTY IV : An energy signal x(t) and its Hilbert transform $\hat{x}(t)$ have the same energy.

$$E_X = \int_{-\infty}^{\infty} |X(t)|^2\, dt$$

$$= \int_{-\infty}^{\infty} |\hat{x}(t)|^2\, dt$$

PROOF

Using the Parseval's theorem of energy signals, we have,

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)^2|\, d\omega \qquad \text{... (2.8.13)}$$

Similarly, we have,

$$\int_{-\infty}^{\infty} |\hat{x}(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{X}(\omega)^2|\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |-j\text{sgn}(\omega)\, X(\omega)|^2\, d\omega$$

$$\int_{-\infty}^{\infty} |\hat{x}(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |-j\text{sgn}(\omega)|^2 |X(\omega)|^2\, d\omega$$

Since $|-j\text{sgn}(\omega)|^2 = 1$ for all ω, we obtain,

$$\int_{-\infty}^{\infty} |\hat{x}(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

By using Eq. (2.8.14), we get,

$$\int_{-\infty}^{\infty} |\hat{x}(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Similarly, a power signal x(t) and its Hilbert transform $\hat{x}(t)$ have the same power.

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |\hat{x}(t)|^2 dt$$

REVIEW QUESTIONS

(1) *Define Hilbert transform and proof its magnitude and phase responses?*

(2) *Write down the various properties of Hilbert transform?*

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