3.1 GENERATING FUNCTION, IT'S PROPERTIES AND INTRODUCTORY EXAMPLES

Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series.

Generating functions can be used to solve many types of counting problems, such as the number of ways to select or distribute objects of different kinds, subject to a variety of constraints, and the number of ways to make change for a dollar using coins of different dominations.

Generating functions can be used to solve recurrence relations by translating a recurrence relation for the terms of sequence into an equation involving a generating function. This equation then can be solved to find a closed form for the generating function.

From this closed form forms the coefficients of the power series for the generating function, solving the original recurrence relation.

Generating functions can also be used to prove combinatorial identities by taking advantage of relatively simple relationships between functions that can be translated into identities involving the terms of sequences.

Generating functions are a helpful tool for studying many properties of sequences beside those described in this section, such as their use for establishing asymptotic formulae for the terms of a sequence.

To the sequence $A = \{a_r\}_{r=0}^{\infty}$ assign the symbol $A(x) = a_0 + a_1 x + ... + a_n x^n$

$$A(x) = \sum_{r=0}^{\infty} a_r x^r$$

The expression A(x) is called a formal power series, a_i is the coefficient of x^i, the term $a_i x^i$ is the term of degree i, and the term a_0 is called constant term. The symbol x^i is simply a device for locating the coefficient a_i and for this reason, the formal power series.

$f(x) = \sum_{r=0}^{\infty} a_r x^r$ is called an (ordinary) generating function for the sequence $A = \{a_r\}_{r=0}^{\infty}$.

Examples

(1) Let us take the sequences (1, 3, 9, 27 .... 3^r) and is denoted by $A = [3^r]_{r=0}^{\infty}$. Then the generating function of this sequence is $A(x) = \sum_{r=0}^{\infty} 3^r x^r$.

(2) Let us take a sequence $B = \{b_r\}_{r=0}^{\infty}$ where,

$$b_r = \begin{cases} 0 & \text{if } 0 \le r \le 3 \\ 1 & \text{if } 4 \le r \le 6 \\ 2 & \text{if } r = 7 \\ 3 & \text{if } 8 \le r \end{cases}$$

∴ is the sequence where,

$b_0 = b_1 = b_2 = b_3 = 0,$

$b_4 = b_5 = b_6 = 1,$

$b_7 = 2, b_8 = 3,$

$b_r = 0, 0, 0, 0, 1, 1, 1, 2, 3.......$

Then the generating function for this sequence,

$B = \{b_r\}_{r=0}^{\infty}$ is

$B(x) = x^4 + x^5 + x^6 + 2x^7 + 3x^8 + ...$

(3) Let us take the sequence $P = \{P_r\}_{r=0}^{\infty}$ where $P_r = r + 1$ for each value of r then the sequence is 1, 2, 3, 4, 5 .... then for this sequence the function is defined as,

$$P(x) = \sum_{r=0}^{\infty} (r+1)\, x^r.$$

(4) Let us take a sequence $Q = \{Q_r\}_{r=0}^{\infty}$ where for $Q_r = r^3$. The sequence is 0, 1, 8, 27, 64 .... then for this sequence the function is defined as,

$$Q(x) = \sum_{r=0}^{\infty} (r^3)\, x^r.$$

3.1.1 Generating Function Properties

Let $A(x) = \sum_{i=0}^{\infty} a_i x^i$, $B(x) = \sum_{j=0}^{\infty} b_j x^j$ and $C(x) = \sum_{k=0}^{\infty} C_k x^k$ be 3 formal power series then we define the following concepts.

(1) **Equality :** $A(x) = B(x)$ iff $a_n = b_n$ for each $n \ge 0$.

(2) **Multiplication by a Constant Number (C) :** $CA(x) = \sum_{r=0}^{\infty} (Ca_r)\, x^r.$

(3) Sum : $A(x) + B(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$.

(4) Product : $A(x) . B(x) . C(x) = \sum_{n=0}^{\infty} P_n x^n$ where $P_n = \sum_{i+j+k=n} a_i . b_j . c_k . x^n$.

Examples

Ex. 1) Let in general

$$A = \{2^r\}_{r=0}^{\infty}$$

$$B = \{b_r\}_{r=0}^{\infty} \text{ where,}$$

$$b_r = \begin{cases} 0 & \text{if } 0 \le r \le 4 \\ 2 & \text{if } 5 \le r \le 9 \\ 3 & \text{if } r = 10 \\ 4 & \text{if } 11 \le r \end{cases}$$

$$C = \{C_r\}_{r=0}^{\infty} \text{ where, } C_r = r + 1$$

$$D = \{d_r\}_{r=0}^{\infty} \text{ where } d_r = r^2$$

The generating functions of A, B, C, D are as follows.

$$A(x) = \sum_{r=0}^{\infty} 2^r x^r$$

$$B(x) = 2x^5 + 2x^6 + 2x^7 + 2x^8 + 2x^9 + 3x^{10} + 4x^{11} + ...$$

$$C(x) = \sum_{r=0}^{\infty} (r+1)x^r$$

$$D(x) = \sum_{r=0}^{\infty} (r^2)x^r$$

Generates the sequences A, B, C and D.

Ex. 2) Let $A(x) = \sum_{r=0}^{\infty} a x^r$

$B(x) = \sum_{S=0}^{\infty} b_S x^S$ be two formal power series, then the following holds good.

(1) **Equality :** $A(x) = B(x)$ iff $a_n = b_n$ for each $n \geq 0$.

(2) **Multiplication by a Scalar Number C :** $CA(x) = \sum_{r=0}^{\infty}(Ca_r)x^r$.

(3) **Sum :** $A(x) + B(x) = \sum_{n=0}^{\infty}(a_n + b_n)x^n$.

(4) **Product :** $A(x)\ B(x) = \sum_{n=0}^{\infty} P_n x^n$ where, $P_n = \sum_{j+k=n} a_j b_k x^n$.

Which means that the constant term a_0 A(x) is multiplied with the coefficient of b_n of x^n in B(x). Then proceed to the coefficient a_1 of x in A(x) and coefficient b_{n-1} of x^{n-1} in B(x), the coefficient of increasing powers of x in A(x) and decreasing powers of x in B(x) are multiplied as below to get P_n.

$$P_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

$$= \sum_{i=0}^{n} a_i b_{n-i}$$

$$\therefore \quad A(x)\ B(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)\, x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \ldots + (a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n b_0)x^n + ..$$

Ex. 3) $A(x) = a_0 + a_3 x^3 + a_4 x^4 + a_8 x^8$ and

$B(x) = b_0 + b_4 x^4 + b_5 x^5 + b_8 x^8$.

The coefficient of x^r in A(x) B(x) is found by considering the powers $\{x^0, x^3, x^4, x^8\}$ from A(x) and power $\{x^0, x^4, x^5, x^8\}$ from B(x).

Since the highest power is 8. Thus, the coefficient of x^8 in the product A(x) B(x) is such that,

$P_8 = a_0 b_8 + a_3 b_5 + a_4 b_4 + a_8 b_0$

$\because$ (0, 8), (3, 5), (4, 4) and (8, 0) are the only pairs of exponents of A(x) and B(x) whose sum is 8.

3.1.2 Generation of Function Modes

In generation of function modes if a problem is given to count the number of non-negative internal solution to an equation.

$e_1 + e_2 + \ldots + e_n = r$ with constraints on each e_i then we can build a generating function.

$A_1(x)\ A_2(x) \ldots A_n(x)$ whose coefficient is x^r.

We have shifted the summation in the next–to–last equality by setting $t = n+r$ so that $t = n$ when $r = 0$ and $n + r - 1 = t - 1$, and then we replaced t by r as the index of summation in the last equality to return to our original notation.

Hence there are $C(r-1, r-n)$ ways to select r objects of n different kinds if we must select atleast one object of each kind.

3.2 DEFINITIONS AND EXAMPLES – CALCULATIONAL TECHNIQUES

3.2.1 Multiplicative Inverse

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is a formal power series, then A(x) is said to have a multiplicative inverse if there is a formal power series $B(x) = \sum_{k=0}^{\infty} b_k x^k$ such that $A(x)\, B(x) = 1$.

If A(x) has a multiplicative inverse, then $a_0 b_0 = 1$ so that a_0 must be non zero. The converse is also true.

If $a_0 \neq 0$, then we can determine the coefficients of B(x) by writing down the coefficients of successive powers of X in A(x) B(x) from the definition of product of 2 power series and then equating these to power series 1.

We have, $a_0 b_0 = 1$ . . . (1)

$a_0 b_1 + a_1 b_0 = 0$ . . . (2)

$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0$ . . . (3)

$a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = 0$ . . . (4)

$a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0 = 0$ and so on.

from (1) we have $b_0 = \dfrac{1}{a_0}$

from (2) we have $b_1 = \dfrac{-a_1 b_0}{a_0} = \dfrac{-a_1}{a_0^2}$ $\quad \left(\because b_0 = \dfrac{1}{a_0}\right)$

from (3) we have $b_2 = \dfrac{-a_1 b_1 - a_2 b_0}{a_0} = \dfrac{a_1^2 - a_2 a_0}{a_0^3}$

from (4) we have $b_3 = \dfrac{-a_1 b_2 - a_2 b_1 - a_3 b_0}{a_0}$

Substituting lower expressions to get higher expressions. Thus, a formal power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ has a multiplicative inverse iff the constant term a_0 is different from zero.

3.2.2 Division of Formal Power Series

In $A(x)$ and $C(x)$ are power series we say that $A(x)$ divides $C(x)$ if there is a formal power series $D(x)$ such that $C(x) = A(x)\ D(x)$.

Therefore we can write $D(x) = \frac{C(x)}{A(x)}$.

If $A(x) = \sum_{n=0}^{r} a_n x^n$ is such that $a_0 \neq 0$, then $A(x)$ has a multiplicative inverse $B(x) = \frac{1}{A(x)}$ and then $A(x)$ divides any $C(x)$–just let $D(X) = C(x)$, $B(x) = C(x)\ \frac{1}{A(x)}$.

3.2.3 Geometric Series

Let us use what we have learned to determine the multiplicative inverse for $A(x) = 1 - x$.

Let $B(x) = \frac{1}{A(x)} = \sum_{n=0}^{r} b_n x^n$. Solving successively for $b_0, b_1, \ldots,$ as above we see that ($a_0, a_1, a_2, \ldots, a_n$ values are the coefficients of powers of x in $A(x)$).

$$b_0 = \frac{1}{a_0} = 1$$

$$b_1 = \frac{-a_1 b_0}{a_0} = \frac{-(-1)(1)}{(1)} = 1$$

$$b_2 = \frac{-a_1 b_1 - a_2 b_0}{a_0} = \frac{-(-1)(1)-(0)(1)}{(1)} = 1$$

$$b_3 = \frac{-a_1 b_2 - a_2 b_1 - a_3 b_0}{a_0} = 1$$

and so on. We see that each $b_i = 1$ so that we have an expression for the geometric series

$$\frac{1}{1-x} = \sum_{r=0}^{\infty} x^r \quad \ldots(1)$$

If we replace in the above expression x by ax where a is a real number, then we see that

$$\frac{1}{1-ax} = \sum_{r=0}^{\infty} a^r x^r$$

It is called as geometric series.

In particular, let $a = -1$, then we get

$$\frac{1}{1+x} = \sum_{r=0}^{\infty}(-1)^r x^r = 1 - x + x^2 - x^3$$

This is called an alternating generating series. Likewise,

$$\frac{1}{1+ax} = \sum_{r=0}^{\infty}(-1)^r a^r x^r$$

For n positive integer.

$$\frac{1}{(1-x)^n} = \left(\sum_{r=0}^{\infty} x^k\right)^n = \sum_{r=0}^{\infty} C(n-1+r, r)x^r \quad \ldots (2)$$

The first equality follows from the above comments and the fact that $\sum_{k=0}^{\infty} x^k$ is multiplicative inverse of $1 - x$. The equality $\frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} C(n-1+r, r)\, x^r$ could also be proved by mathematical induction $C(n + r - 1, r) = C(n + r, r)$.

By replacing x by $-x$ in the above, we get the following identity for a positive integer.

$$\frac{1}{(1+x)^n} = \sum_{r=0}^{\infty} C(n-1+r, r)(-1)^r x^r \quad \ldots (3)$$

Following this pattern, replace x by ax in (3) and (2) to obtain,

$$\frac{1}{(1-ax)^n} = \sum_{r=0}^{\infty} C(n-1+r, r)\, a^r x^r$$

$$\frac{1}{(1+ax)^n} = \sum_{r=0}^{\infty} C(n-1+r, r)(-a)^r x^r$$

Likewise, replace x by x^k in (1) to get for 'k' a positive integer.

$$\frac{1}{1-x^k} = \sum_{r=0}^{\infty} x^{kr} = 1 + x^k + x^{2k} + \ldots$$

and $\frac{1}{1+x^k} = \sum_{r=0}^{\infty}(-1)^r x^{kr}$

If a is a non zero real number,

$$\frac{1}{a-x} = \frac{1}{a}\left(\frac{1}{1-\frac{x}{a}}\right) = \frac{1}{a}\sum_{r=0}^{\infty}\frac{x^r}{a^r}$$

and $$\frac{1}{x-a} = \frac{-1}{a-x} = -\frac{1}{a}\sum_{r=0}^{\infty}\frac{x^r}{a^r}$$

Other identities that are frequently used are, if n is a positive integer,

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \dots + x^n$$

$$(1 + x)^n = 1 + ({}^nC_1)x + ({}^nC_2)x^2 + \dots + ({}^nC_n)x^n$$

$$(1 + x^k)^n = 1 + ({}^nC_1)x^k + ({}^nC_2)x^{2k} + \dots + ({}^nC_n)x^{nk}$$

$$(1 - x)^n = 1 - ({}^nC_1)x + ({}^nC_2)x^2 + \dots + (-1)^n ({}^nC_n)x^n$$

$$(1 - x^k)^n = 1 - ({}^nC_1)x^k + ({}^nC_2)x^{2k} + \dots + (-1)^n ({}^nC_n)x^{nk}$$

$$(1 + ax)^n = ({}^nC_0) + ({}^nC_1)ax + \dots\dots ({}^nC_n)a^nx^n$$

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \dots\dots \sum_{i=0}^{\infty}x^i$$

$$\frac{1}{1-ax} = 1 + (ax) + (ax)^2 + (ax)^3 + \dots\dots$$

$$= \sum_{i=0}^{\infty}(ax)^i = \sum_{i=0}^{\infty}a^ix^i$$

$$= 1 + ax + a^2x^2 + a^3x^3 + \dots$$

$$\frac{1}{(1+x)^n} = ({}^{-n}C_0) + ({}^{-n}C_1)x + ({}^{-n}C_2)x^2 + \dots$$

$$= \sum_{i=0}^{\infty}({}^{-n}C_i)x^i$$

$$= 1 + (-1)(n + 1 - {}^1C_1)x + (-1)^2(n + 2 - {}^1C_2)x^2 + \dots\dots$$

$$= \sum_{i=0}^{\infty}(-1)^i (n+i-{}^1C_i)x^i$$

$$\frac{1}{(1-x)^n} = (1-x)^{-n}$$
$$= ({}^{-n}C_0) + ({}^{-n}C_1)(-x) + ({}^{-n}C_2)(-x)^2 + \ldots\ldots$$
$$= \sum_{i=0}^{\infty} ({}^{-n}C_i)(-x)^i$$
$$= 1 + (-1)(n + 1 - {}^1C_1)(-x) + (-1)^2 (n + 2 - {}^1C_2)(-x)^2 + \ldots\ldots$$
$$= \sum_{i=0}^{\infty} ({}^{n+i-1}C_i)x^i$$

3.2.4 Use of Partial Function Decomposition

If A(x) and C(x) are polynomials, we show how to compute $\frac{C(x)}{A(x)}$ by using the above identities and partial fractions. We know that from algebra if A(x) is a product of linear factors.

$A(x) = a_n(x-\alpha_1)^{r_1}, (x-\alpha_2)^{r_2} \ldots (x-\alpha_k)^{r_k}$, and if C(x) is any polynomials of degree less than the degree of A(x), then $\frac{C(x)}{A(x)}$ can be written as the sum of elementary fractions as follows.

$$\frac{C(x)}{A(x)} = \frac{A_{11}}{(x-\alpha_1)^{r_1}} + \frac{A_{12}}{(x-\alpha_1)^{r_1-1}} + \ldots\ldots + \frac{A_{1r_1}}{(x-\alpha_1)^{r_1}} + \frac{A_{21}}{(x-\alpha_2)^{r_2}} + \frac{A_{22}}{(x-\alpha_2)^{r_2-1}}$$
$$+ \ldots\ldots + \frac{A_{2r_2}}{(x-\alpha_2)^{r_2}} + \frac{A_{k1}}{(x-\alpha_k)^{r_k}} + \frac{A_{k2}}{(x-\alpha_k)^{r_k-1}} + \ldots\ldots + \frac{A_{kr_k}}{(x-\alpha_k)^{r_k}}$$

To find the numbers A_{11}, ... A_{kr_k} we multiply both sides of the last equation by $(x-\alpha_1)^{r_1}(x-\alpha_2)^{r_2} \ldots\ldots (x-\alpha_k)^{r_k}$ to clear off denominators and then we equate coefficients of the same power of x.

3.2.5 Linearity

Ordinary generating functions posses an important transformation property, namely linearity. Linearity implies that, if any two sequences of numbers $\{a\}_{r=0}^{\infty}$ and $\{b\}_{r=0}^{\infty}$ and scalars D and E are given, then sequence $\{c\}_{r=0}^{\infty}$ formed by the linear combination $C_r = D\,a_r + E\,b_r$, has the generating function,

$$C(x) = \sum_{r=0}^{\infty} (Da_r + Eb_r)x^r = DA(x) + EB(x)$$

Where $A(x) = \sum_{r=0}^{\infty} a_r x^r$ and $B(x) = \sum_{r=0}^{\infty} b_r x^r$ are the generating functions for $\{a\}_{r=0}^{\infty}$ and $\{b\}_{r=0}^{\infty}$ respectively.

Then we substitute the values in the expression,

$$\frac{1}{(x-3)(x-2)^2}$$

$$= \frac{1}{x-3} - \frac{1}{x-2} - \frac{1}{(x-2)^2}$$

$$= \left(\frac{-1}{3}\right)\left(\frac{1}{1-(x/3)}\right) + \frac{1}{2}\left(\frac{1}{1-(x/2)}\right) + \left(\frac{-1}{4}\right)\left(\frac{1}{(1-(x/2))^2}\right)$$

$$= \left(\frac{-1}{3}\right)\sum_{i=0}^{\infty}(x/3)^i + \frac{1}{2}\sum_{i=0}^{\infty}(x/2)^i + \left(\frac{-1}{4}\right)\left[{}^{-2}C_0 + {}^{-2}C_1(-x/2) + \ldots\right]$$

Then the coefficient of x^8 is,

$$\left(\frac{-1}{3}\right)\left(\frac{1}{3}\right)^8 + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^8 + \left(\frac{-1}{4}\right)({}^{-2}C_8)\left(\frac{-1}{2}\right)^8 = -\left[\left(\frac{1}{3}\right)^9 + 7\left(\frac{1}{2}\right)^{10}\right].$$

$$\left[\because {}^{-n}C_r = (-1)^r\,{}^{(n+r-1)}C_r\right]$$

3.3 PARTITIONS OF INTEGERS

3.3.1 Characteristic Property of Numeric Function

(1) Suppose a be a numeric function. By the asymptotic behaviour of a, we mean how the volume of the function a varies for large r.

For example, $a_r = 3r^2, r \geq 0$

The value of the numeric function increases for increasing r and for, $b_r = \frac{2}{r}, r > 0$

The value of the function decreases for increasing r. Finally for, $c_r = 7r^3, r \geq 0$

The value of the numeric function remains constant for increasing r.

(2) Suppose a and b be two numeric functions, and if there exists two positive constants r and k such that $|b_r| \geq a_i\, a_r$ for $r \geq k$, then we say that the numeric function a asymptotically dominates b, or the numeric function b is dominated by a.

Example : Let a and b be two numeric functions such that,

$$\begin{Bmatrix} a_r = 3r^2 + r + 1, r \geq 0 \\ b_r = 3r^2 - r^{1/2} - 7, r \geq 0 \end{Bmatrix}$$

then "a dominates b".

If |a| asymptotically dominates a if b is asymptotically dominated by a, then,

λ, λb is also dominated by a (λ-constant).

Note : If a is given numeric function, then the set of all numeric function that are dominated by a is called order of a or "big-oh a" and is denoted by O(a).

3.3.2 Generating Function by Partitions of Integers

By partition of a positive integer n into positive summands is meant, the number of sets in each of which the sum of its elements is equal to n. It is denoted by P(n). Alternatively, we can say that a partition of a positive integer n is an unordered collection of positive integers (or parts) whose sum is n. Generating functions play a great role in the theory of partition.

Notations

P(n) = Number of distinct partitions of n.

$p_N(n)$ = Number of paritItions of n into parts atmost equal to N.

= Number of solution is non-negative integers of $1u_1+2u_2+3u_3+...+Nu_N = n$.

$q_N(n)$ = Number of partitions of n into atmost N parts.

= Number of distributions of n identical objects (1s) among N identical places, empty places being permitted.

EXAMPLE PROBLEM 1

Determine the generating function for the sequence a_0, a_1 ...... where a_n is the number of partitions of the non-negative integer n into,

***(i)* Even summands.**

***(ii)* Distinct even summands.**

***(iii)* Distinct odd summands.**

SOLUTION

(i) $f(x) = \left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^6}\right)$

$= \pi_{i=1}^{\infty} \frac{1}{1-x^{2i}}$

(ii) $g(x) = (1 + x^2)\ (1 + x^4) + (1 + x^6)$

$= \pi_{i=1}^{\infty}(1+x^{2i})$

(iii) $h(x) = (1 + x)\ (1 + x^3)\ (1 + x^5)$

$= \pi_{i=1}^{\infty}(1+x^{(2i-1)})$

3.4 EXPONENTIAL GENERATING FUNCTION (EGF)

Let us consider the generating function given by,

$$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n$$

This will generate the binomial coefficients ${}^nC_0, {}^nC_1, \ldots {}^nC_n$.

For any $0 \le r \le n$

$$\begin{aligned} {}^nC_r &= \frac{(n)!}{(r)!(n-r)!} \\ &= \frac{1}{(r)!}\frac{(n)!}{(n-r)!} \\ &= \frac{1}{(r)!}{}^nP_r \end{aligned}$$

$$(1+x)^n = {}^nP_0 + {}^nP_1x + {}^nP_2\frac{x^2}{(2)!} + \ldots + {}^nP_r\frac{x^r}{(r)!} + \ldots + {}^nP_n\frac{x^n}{(n)!}$$

So if we take the coefficient of $\frac{x^r}{(r)!}$ in above, it gives the sequence ${}^nP_r$.

Definition (Exponential Generating Function) : For the sequence $a_0, a_1, a_2\ldots$ of real numbers,

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2\frac{x^2}{(2)!} + a_3\frac{x^3}{(3)!} + \ldots \\ &= \sum_{k=0}^{\infty} a_k x^k \end{aligned}$$

is called the exponential generating function for the given sequence.

Example

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{(2)!} + \frac{x^3}{(3)!} + \ldots \\ &= \sum_{k=0}^{\infty}\frac{x^k}{(k)!} \end{aligned}$$

So e^x is the exponential generating function of 1, 1, 1, . . .

Whereas e^x is the generating function of the sequence,

$$1, 1, \frac{1}{(2)!}, \frac{1}{(3)!}, \frac{1}{(4)!}, \ldots$$

The exponential generating function helps in counting certain types of arrangements.

Expansion Series

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

$$e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\ldots$$

$$\frac{e^x+e^{-x}}{2} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$$

$$\frac{e^x-e^{-x}}{2} = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$$

3.4.1 Solved Problems

SOLVED PROBLEM 1

Determine the sequences generated by each of the following exponential generating functions.

(i) $f(x) = 5e^{5x}$

(ii) $f(x) = 7e^{8x} - 4e^{3x}$

(iii) $f(x) = 2e^x + 3x^2$

(iv) $f(x) = e^{3x} - 28x^3 - 6x^2 + 9x$

(v) $3x^3 + e^{2x}$

SOLUTION

(i) $f(x) = 5e^{5x} = 5\sum_{n=0}^{\infty}\frac{(5x)^n}{n!}$, so f(x) is the exponential generating function for the sequence $5, 5^2, 5^3, 5^4, \ldots$

(ii) $f(x) = 7e^{8x} - 4e^{3x} = 7\sum_{n=0}^{\infty}\frac{(8x)^n}{n!}-4\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}$. The sequence is $7(8)^n - 4(3)^n$ with $n = 0, 1, 2, 3, \ldots$ i.e., 3, 44, 412, 3476, . . .

(iii) $f(x) = 2e^x + 3x^2 = 2\sum_{n=0}^{\infty}\frac{x^n}{n!}+3x^2$. So the sequence is 2, 2, (2 + 3), 2, 2, 2, . . .

(iv) $f(x) = e^{3x} - 28x^3 + 6x^2 + 9x = \sum_{n=0}^{\infty}\frac{3^n x^n}{n!}-28x^3-6x^2+9x$. So the sequence is 3^0, $3^1 + 9, 3^2 - 6, 3^3 - 28, 3^4, 3^5, 3^6, \ldots$ i.e., $1, 12, 3, -1, 3^4, 3^5, 3^6, \ldots$

(b) $x\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right)$. Now we see that in order to get $x^{12}/12!$ we need to consider the term $x[(3x)^{11}/11!] = 3^{11}\ (x^{12}/11!) = (12)\ (3^{11})\ (x^{12}/12!)$ and here the coefficient of $x^{12}/12!$ is $(12)\ (3^{11})$ and

(c) $(x^2/2)\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right)$. For this last summand we observe that $(x^2/2)\ [(3x)^{10}/10!] = (1/2)\ (3^{10})\ (x^{12}/10!) = (1/2)\ (12)\ (11)\ (3^{10})\ (x^{12}/12!)$ where this time the coefficient of $x^{12}/12!$ is $(1/2)\ (12)\ (11)\ (3^{10})$.

Consequently, the number of 12 flag signals with atleast three white flags, or none at all, is

$4^{12} - 12(3^{11}) - (1/2)\ (12)\ (11)\ (3^{10}) = 10{,}754{,}218.$

3.5 SUMMATION OPERATOR

Let f(x) and g(x) be two generating functions and let $g'(x) = f(x)\ g(x)$ . . . (1)

If we choose $g(x) = \frac{1}{(1-x)}$

then $g'(x) = \frac{f(x)}{(1-x)}$ [using (1)]

$$= \sum_{k=0}^{\infty} p_k x^k$$

where $p_k = \sum_{k=0}^{\infty} a_k$

$p_0 = a_0,$

$p_1 = a_o + a_1,$

$p_2 = a_0 + a_1 + a_2,$

So the coefficient p_n of x_n will be the sum of the first n terms of the sequence given by the generating function f(x) i.e., $f(x) = \sum_{k=0}^{\infty} p_k x^k$.

Example : $\frac{x(1+x)}{(1-x)^3}$ generates a sequence.

Let $\frac{x(1+x)}{(1-x)^3}\ \frac{1}{(1-x)} = \frac{x(1+x)}{(1-x)^4}$ is generating function for the sequence $0^2,\ 0^2 + 1^2,\ 0^2 + 1^2 + 2^2,\ 0^2 + 1^2 + 2^2 + 3^2,$ ......

So the coefficient of x^n is,

$$({}^{-4}C_{n-1})\,(-1)^{n-1} + ({}^{-4}C_{n-2})(-1)^{n-2}$$

$$= (-1)^{n-1}\,({}^{4+(n-1)-1}C_{n-1})\,(-1)^{n-1} + (-1)^{n-2}\,({}^{4+(n-2)-1}C_{n-2})\,(-1)^{n-2}$$

$$= ({}^{n+2}C_{n-1}) + ({}^{n+1}C_{n-2})$$

$$= \frac{(n+2)!}{3!(n-1)!} + \frac{(n+1)!}{3!(n-2)!}$$

$$= \frac{1}{6}[(n+2)(n+1)(n)+(n+1)(n)(n-1)]$$

$$= \frac{1}{6}(n)(n+1)[(n+2)+(n-1)]$$

$$= \frac{n(n+1)(2n+1)}{6}$$

3.6 FIRST-ORDER LINEAR RECURRENCE RELATION

[NOV./DEC. - 2008]

The general form of first order linear recurrence relation can be written as $a_{n+1} = da_n,\ n \geq 0$ where d is a constant.

The general first-order linear recurrence relation with constant coefficients has the form $a_{n+1} + ca_n = f(n),\ n \geq 0$, where c is a constant and f(n) is a function on the set N of nonnegative integers.

When $f(n) = 0$ for all $n \in N$, the relation is called homogeneous, otherwise it is called nonhomogeneous.

Let us consider the equation $a_{n+1} = 3a_n,\ n \geq 0$ is a recurrence relation because the value of a_{n+1} (the present consideration) is dependent on a_n (a prior consideration). Since a_{n+1} depends only on its immediate predecessor, the relation is said to be of first order.

Values such as a_0 or a_1, given in addition to the recurrence relations, are called boundary conditions. The expression $a_0 = A$, where A is a constant, is also referred to as an initial condition. Our examples show the importance of the boundary condition in determining the unique solution.

Let us return now to the recurrence relation,

$$a_{n+1} = 3a_n,\ n \geq 0,\ a_0 = 5.$$

The first four terms of this sequence are,

$a_0 = 5,$

$a_1 = 3a_0 = 3(5),$

$a_2 = 3a_1 = 3(3a_0) = 3^2(5),$ and

$a_3 = 3a_2 = 3(3^2(5)) = 3^3(5).$

These results suggest that for each $n \geq 0$, $a_n = 5(3^n)$. This is the unique solution of the given recurrence relation. In this solution, the value of a_n is a function of n and there is no longer any dependence on prior terms of the sequence, once we define a_0. To compute a_{10}, for example, we simply calculate $5(3^{10}) = 295,245$, there is no need to start at a_0 and build upto a_9 inorder to obtain a_{10}.

The unique solution of the recurrence relation,

$a_{n+1} = da_n$, where $n \geq 0$, d is a constant, and $a_0 = A$, is given by

$a_n = Ad^n, \quad n \geq 0.$

EXAMPLE PROBLEM 1

Find the first 5 terms of the sequence defined by $a_n = 6\,a_{n-1}$, $a_0 = 2$.

SOLUTION

$a_1 = 6\,a_0$

$a_1 = 6 \times 2 = 12$

$a_2 = 6\,a_{2-1} = 6\,a_1$

$a_2 = 6.\,(12) = 72$

$a_3 = 6a_2 = 6 \times 72 = 432$

$a_4 = 6a_3 = 6\,(432) = 2592$

$a_5 = 6a_4 = 6\,(2592) = 15,552$

EXAMPLE PROBLEM 2

Consider the set of all subsets of any non-empty set S, called its power set, and is denoted by P (S). Let us determine a recurrence relation satisfied by $S_n = |P(S)|$ where $|S| = n$.

SOLUTION

Let $S = \{1, 2, 3 \ldots n\}$.

Now, any subset, A of S either contains the number n or does not. Let us consider these two mutually exclusive cases separately and count the number of such subsets.

EXAMPLE PROBLEM 6

Find a recurrence relation with initial condition that uniquely determines each of the following sequences that begin with the given terms

(i) **3, 7, 11, 15, 19, . . .** **(ii)** $8, \frac{24}{7}, \frac{72}{49}, \frac{216}{343},$

SOLUTION

(i) Here first term is $a_0 = 3$ and increases by 4. So the recurrence relation is $a_n = a_{n-1} + 4$ for $n > 1$.

(ii) Here $a_0 = 8$ and each term gets multiplied by $\frac{3}{7}$. So the recurrence relation is $a_n = \frac{3}{7}a_{n-1}$ for $n > 1$.

3.7 THE NTH AND SECOND ORDER LINEAR HOMOGENEOUS RECURRENCE RELATION WITH CONSTANT COEFFICIENTS

3.7.1 nth Order Linear Relation

Let S be a sequence of numbers with domain $k \geq 0$. An nth order linear recurrence relation on S with constant coefficients is a recurrence relation that can be written in the form,

$$S(k) + c_1S(K-1) + c_2S(k-2) + \ldots + c_nS(k-n) = f(k) \quad (k \geq n)$$

Where $c_1, c_2 \ldots c_n$ are numbers and f is a numeric function that is defined for $k \geq n$.

3.7.2 nth and Second Order Homogeneous Recurrence Relation

An nth order linear relation is a homogeneous recurrence relation if $f(k) = 0$ for all k. For each recurrence relation, $S(k) + c_1S(k-2) + \ldots + c_nS(k-n) = f(k)$, associated homogeneous relation is $S(k) + c_1S(k-1) + \ldots + c_nS(k-n) = 0$. If $n = 2$, then it is second order, $S(k) + c_1S(k-1) + c_2S(k-2) = 0$.

Consider the second order homogeneous relation $S(k) - 7\ S(k-1) + 12\ S(k-2) = 0$ together with the initial conditions $S(0) = 4$ and $S(1) = 4$. We can predict that the solution to this relation involves terms of the form ba_k where b and a are non zero constants that must be determined. If the solutions were to equal to this quantity exactly, then $S(k) = ba^k$, $S(k-1) = ba^{k-1}$, $S(k-2) = ba^{k-2}$. If we substitute the exponential expressions into the original equation, we get $ba^k - 7ba^{k-1} + 12\ ba^{k-2} = 0$.

Each term on the left-hand side has a factor of ba^{k-2}, which is non zero. Dividing through by this common factor we get,

$$a^2 - 7a + 12 = (a-3)(a-4) = 0$$

This equation is called the characteristic equation of the recurrence relation. The only possible values of a are 3 and 4. So general solution of the given equation will be of the form,

$S(k) = b_1 3^k + b_2 4^k$, where b_1 and b_2 are real numbers.

This set of sequence is called the general solution of the recurrence relation. If we didn't have initial conditions for S, we would stop here. The initial conditions make it possible for us to obtain definite values for b_1 and b_2.

$$\left.\begin{array}{l} S(0) = 4 \\ S(1) = 4 \end{array}\right\} \Rightarrow \begin{array}{l} b_1 3^0 + b_2 4^0 = 4 \\ b_1 3^1 + b_2 4^1 = 4 \end{array}$$

$$b_1 + b_2 = 4$$

$$\Rightarrow 3b_1 + 4b_2 = 4$$

Solving we get $b_1 = 12$ and $b_2 = -8$ the solution is $S(k) = 12\ (3^k) - 8\ (4^k)$.

3.7.3 Characteristic Equation

The characteristic equation of the homogeneous n^{th} order linear relation $S(k) + c_1\ S(k-1) + \ldots c_n\ S(k-n) = 0$ is the n^{th} degree polynomial equation.

$a^n + c_1\ a^{n-1} + \ldots + c_{n-1}\ a + c_n = 0.$

The left hand side of this equation is called the characteristic polynomial.

Examples

(1) The characteristic equation of $F(k) - F(k-1) - F(k-2) = 0$ is $a^2 - a - 1 = 0$.

(2) The characteristic equation of

$Q(k) + 2Q(k-1) - 3Q(k-2) - 6Q(k-4) = 0$ is $a^4 + 2a^3 - 3a^2 - 6 = 0$.

3.7.4 Working Rule for Solving Homogeneous Relation

The process of determining a closed form expression for the terms of a sequence from its recurrence relation is called solving the relation.

(1) Write the characteristic equation of the relation $S(k) + c_1\ S(k-1) + \ldots + c_n\ S(k-n) = 0$ which is $a^n + c_1\ a^{n-1} + \ldots + c_{n-1}\ a + c_n = 0$.

(2) Find all roots of the characteristic equation called characteristic roots.

(3) If there are n distinct characteristic roots, $a_1, a_2 \ldots a_n$, then the general solution of the recurrence relation is $S(k) = b_1 a_1^k + b_2 a_2^k + \ldots + b_n a_n^k$.

If there are fewer than n characteristic roots, then atleast one root is a multiple root.

If a_j is a double root, then $b_j a_j^k$ is replaced with $(b_{j0} + b_{j1} k)\ a_j^k$.

In general, if a_j is a root of multiplicity p then $b_j a_j^k$ is replaced with $(b_{j0} + b_{j1}k + \ldots + b_{j(p-1)} k^{p-1}) a_j^k$.

(4) If n initial conditions are given. Obtain n linear equations in n unknowns by substitution. Solve these equations.

Note : For solving recursive relation by substitution, the relation $P(n) = f(n) P(n-1)$, $n \geq 0$ with $P(0) = f(0)$ where f is a function that is defined for all $n \geq 0$ has the solution, $P(n) = f(0).(f(1).(f(2)\ldots f(n) = \pi_{k=0}^{n} f(n), n \geq 0$.

EXAMPLE PROBLEM 1

Solve S (k) – 7S (k – 2) + 6S (k – 3) = 0 where S (0) = 8, S (1) = 6 and S (2) = 22.

SOLUTION

Let the solution will be of the form $S(k) = ba^k$

$S(k-1) = ba^{k-1}$, $S(k-2) = ba^{k-2}$

$S(k-3) = ba^{k-3}$ substituting these values in

$S(k) - 7.S(k-2) + 6S(k-3) = 0$ we get,

$ba^k - 7.\ ba^{k-2} + 6.\ ba^{k-3} = 0$

$\Rightarrow ba^{k-3}(a^3 - 7a + 6) = 0$

$\Rightarrow a^3 - 7a + 6 = 0$

Therefore the characteristic equation is,

$a^3 - 7a + 6 = 0$

$\Rightarrow (a-1)(a-2)(a+3) = 0$

$\Rightarrow a = 1, 2$ and -3

The general solution is,

$S(k) = b_1 (1)^k + b_2 (2)^k + b_3 (-3)^k$

$S(0) = 8 \Rightarrow b_1 + b_2 + b_3 = 8$

$S(1) = 6 \Rightarrow b_1 + 2b_2 - 3b_3 = 6$

$S(2) = 22 \quad \Rightarrow b_1 + 4b_2 + 9b_3 = 22$

Solving these three equations we get,

$b_1 = 5$, $b_2 = 2$ and $b_3 = 1$.

Therefore $S(k) = 5 + 2(2^k) + 1(-3)^k = 5 + 2^{k+1} + (-3)^k$.

The roots of this equation are $k_1 = 1/2$, $k_2 = 1$, $k_3 = -1$, which are real and distinct. Consequently, we take the general solution for a_n as,

$$a_n = A \times (1/2)^n + B \times 1^n + C \times (-1)^n \quad \ldots (1)$$

where A, B, C are arbitrary constants.

To determine A, B, C we use the given values (initial conditions) $a_0 = 0$, $a_1 = 1$, $a_2 = 2$. Putting these values into the equation(1), we get

$$0 = a_0 = A \times \left(\frac{1}{2}\right)^0 + B \times 1^0 + C \times (-1)^0,$$

$$1 = a_1 = A \times \left(\frac{1}{2}\right)^1 + B \times 1^1 + C \times (-1)^1,$$

$$2 = a_2 = A \times \left(\frac{1}{2}\right)^2 + B \times 1^2 + C \times (-1)^2.$$

These can be rewritten as,

$A + B + C = 0$, $A + 2B - 2C = 2$, $A + 4B + 4C = 8$.

Solving these, we get $A = -8/3$, $B = 5/2$ and $C = 1/6$. Putting these into (1), we get

$$a_n = -\frac{8}{3} \times \left(\frac{1}{2}\right)^n + \frac{1}{6} \times (-1)^n + \frac{5}{2}$$

This is the required solution.

3.8 NON-HOMOGENEOUS RECURRENCE RELATION

3.8.1 Working Rule for Solving Non-Homogeneous Finite Order Relations

To solve the recurrence relation $S(k) + c_1 S(k-1) + \ldots + C_n S(k-n) = f(k)$.

(1) Write the associated homogeneous relation and find its general solution. Call this the homogeneous solution, denote this by $S^h(k)$.

(2) Find particular solution with the help of the Table 3.8.1.

Table 3.8.1 : Particular Solutions for Given Right-hand Sides

S.No	***Right Hand Side***	***Form of Particular Solution***
1)	A constant, q	A constant d
2)	A linear function $q_0 + q_1 k$	A linear function, $d_0 + d_1 k$
3)	An m^{th} degree polynomial $q_0 + q_1 k + \ldots + q_m k^m$	An m^{th} degree polynomial $d_0 + d_1 k + \ldots + d_m k^m$
4)	An exponential function $q \cdot a^k$	An exponential function $d \cdot a^k$

(3) Substitute the required form of particular solution in the recurrence relation and find the unknown coefficients.

(4) The general solution of the recurrence relation is the sum of the homogeneous and particular solutions. If no initial conditions are given, then $S(k) = S^h(k) + S^p(k)$. If n initial conditions are given, obtain 'n' linear equations in n unknowns and solve the system to get a complete solution.

SOLVED PROBLEM 1

Solve the recurrence $S(k) + 5\,S(k-1) = 9$, $S(0) = 6$.

SOLUTION

The characteristic equation is $a + 5 = 0$

Therefore $a = -5$

The homogeneous solution is $S^h(k) = b_1(-5)^k$

Since the right hand side is constant, the particular solution will be constant, say d.

If we substitute $S^p(k)$ in the recurrence relation we get,

$d + 5d = 9$

$\Rightarrow\ 6d = 9$

$\Rightarrow\ d = 9/6 = 1.5$

The general solution is,

$S(k) = S^h(k) + S^p(k)$

$\quad = b_1\ (-5)^k + 1.5$

$S(0) = 6$

$\Rightarrow\ b_1\ (-5)^0 + 1.5 = 6$

$\Rightarrow\ b_1 + 1.5 = 6$

$\Rightarrow\ b_1 = 4.5$

and $S(k) = 4.5\ (-5)^k + 1.5$

3.8.2 Linear Non-Homogeneous Recurrence Relations

A recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$, where $c_1, c_2 \ldots c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n.

If $\{a_n^{(p)}\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$. Then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$ where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}.$$

EXAMPLE PROBLEM 1

Find the complete solution of the recurrence $a_n = 3a_{n-1} - 4n,\ n \geq 1$.

SOLUTION

The associated linear homogeneous recurrence relation is $a_n = 3a_{n-1}$.

Characteristic equation is $t = 3$.

Solution is $a_n = \alpha_1.3^n$ where α_1 is a constant i.e., $\{a_n^{(h)}\} = \alpha_1.3^n$.

Now consider $a_n = 3a_{n-1} - 4^n$.

Let us see if a_n can be of the form $An + B$ where A, B are constants.

$$An+B = 3[A(n-1)+B] - 4n$$

$$= n(3A-4) - 3A + 3B$$

Comparing the coefficients of n we have $A = 3A - 4$ and $B = 3B - 3A$.

i.e., $A = 2$ and $B = 3$ i.e., $An + B$ is a solution if and only if $A = 2$ and $B = 3$.

So, $a_n = 2n + 3$ is a particular solution of the given recurrence.

Therefore, total solution of the recurrence will be $a_n = \alpha_1.3^n + 2n + 3$.

3.9 DIVIDE AND CONQUER ALGORITHM

[Nov./Dec. - 2008]

This is the decomposition algorithm that solves a problem of size $n \in Z^+$ by,

(1) Breaking it up into a number of instances of the same kind of problem with smaller input parameter.

(2) Solving these sub problems.

(3) Use their solutions to construct a solution for the original problem of size n.

SOLVED PROBLEM 1

In a tennis tournament, each entrant plays a match in the first round. Next all winners from the first round play a second round match. Winners continue to move on to the next round, until finally only one player is left as the tournament winner. Assuming that tournament always involve $n=2^k$ players, for some k, find the recurrence relation for the number of rounds in a tournament of n players.

SOLUTION

The recurrence relation for a_n, the number of rounds is $a_n = a_{n/2} + 1$. Since after $a_{n/2}$ rounds there remains only two players, the winners of the sub tournament of the first n/2 players and the sub tournament of the second n/2 players. One more round picks the tournament winner from the two remaining players. Here $a_1 = 0$ since with one player, no tournament.

SOLVED PROBLEM 2

To multiply two n-digit numbers, one must do normally n^2 digit - times - digit multiplications. Use a divide and conquer algorithm to do better when n is a power of 2.

SOLUTION

Let n be a power of 2. Let the two n- digit numbers be A and B. We split each of these numbers into two $\frac{n}{2}$ - digit parts,

$A = A_1 10^{n/2} + A_2$ and

$B = B_1 10^{n/2} + B_2$ (like $1235 = 12 \times 100 + 35$)

$A \cdot B = A_1 B_1 \; 10^n + A_1 B_2 \; 10^{n/2} + A_2 B_1 \; 10^{n/2} + A_2 B_2$

We need only to make three $\frac{n}{2}$ - digit multiplications A_1B_1, A_2B_2 and $(A_1 + A_2)$. $(B_1 + B_2)$ to determine A . B

Since $A_1B_2 + A_2B_1 = (A_1 + A_2).(B_1 + B_2) - A_1B_1 - A_2B_2$

If a_n denotes the number of digit-times-digit multiplications needed to multiply two n-digit numbers by the above procedure, this gives the recurrence relation.

$a_n = 3a_{n/2}$.

SOLVED PROBLEM 3

There are algorithms that multiply two n × n matrices, when n is even, using seven multiplications each of two $\left(\frac{n}{2}\right)\times\left(\frac{n}{2}\right)$ matrices and 15 additions of $\left(\frac{n}{2}\right)\times\left(\frac{n}{2}\right)$ matrices. Hence if f(n) is the number of operations used, it follows that

$$f(n) = 7f\left(\frac{n}{2}\right) + \frac{15.n^2}{4}$$

SOLUTION

(1) Let f be an increasing function that satisfies the recurrence relation $f(n) = af\left(\frac{n}{b}\right) + C$ whenever n is divisible by b where $a \geq 1$, b is an integer >1 and c is a positive real number, then

$$f(n) = 0\,(n^{\log_b^a}) \text{ if } a > 1$$

$$= 0\,(\log n) \text{ if } a = 1$$

(2) Let f be an increasing function that satisfies the recurrence relation $f(n) = af\left(\frac{n}{b}\right) + c.n^d$ whenever $n = b^k$ where $k \in z^+$, a, b ≥ 1 and c, d $\in R^+$. Then,

$$f(n) = 0\,(n^d) \quad \text{if } a < b^d$$

$$= 0\,(n^d \log n) \quad \text{if } a = b^d$$

$$= (n^{\log_b^a}) \quad \text{if } a > b^d$$

SOLVED PROBLEM 4

Solve the divide and conquer relation $a_n - 7a_{n-1} = 2n$ where $n = 3^k$ for $k \geq 1$ and $a_1 = 5/2$

SOLUTION

We employ the change of variable $b_k = a_n = a_{3^k}$. Then the transformed relation is,

$b_k - 7b_{k-1} = 2.3^k$ and . . . (1)

$a_1 = b_0 = 5/2.$

The corresponding linear recurrence relation is $b_k - 7\, b_{k-1} = 0$, characteristic equation is $t - 7 = 0$ its root is 7.

Therefore $b_k^{(h)} = B.7^k$ for some constant B.

Moreover, a particular solution of the non homogenous relation takes of the form

$b_k^{(p)} = A.3^k$

By substituting $b_k^{(p)}$ in (1) we get the value of $A = -\frac{3}{2}$ so that

$b_k = -\frac{3}{2}(3^k) + B.7^k$

$b_0 = 5/2 \Rightarrow B = 4$

Thus, $b_k = -\frac{3}{2}(3^k) + 4\,(7^k)$

Now, $n = 3^k$

$\Rightarrow k = \log_3^n$ so

$b_k = a_n = \left(-\frac{3}{2}\right) n + 4.\ 7^{\log_3^n}$

Since $7^{\log_3^n} = n^{\log_3^7}$

Therefore, $a_n = \left(-\frac{3}{2}\right) n + (4)\ n^{\log_3^7}.$

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