UNIT 3

SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

PROFESSIONAL'S
The Destination Towards Knowledge & Success

SYLLABUS

Linear System, Impulse Response, Response of a Linear System, Linear Time Invariant (LTI) System, Linear Time Variant (LTV) System, Transfer Function of a LTI System, Filter Characteristics of Linear Systems, Distortion Less Transmission through a System, Signal Bandwidth, System Bandwidth, Ideal LPF, HPF and BPF Characteristics, Causality and Paly-Wiener Criterion for Physical Realization, Relationship between Bandwidth and Rise Time.

PART - A
SHORT QUESTIONS WITH ANSWERS

Q1) What is a system? How the systems are classified?

Ans. : A system may be defined as "a set of elements or functional blocks which are connected together to produce an output in response to an input signal". Thus, a system can be thought of as a process that involves in transforming input signals into output signals.

$$y(n) = T[x(n)]$$

The systems are classified as follows,

(1) Continuous-time and discrete-time systems.

(2) Linear and non-linear systems.

(3) Time invariant and time variant systems.

(4) Static (memoryless) and dynamic (memory) systems.

(5) Causal and non-causal systems.

(6) Stable and unstable systems.

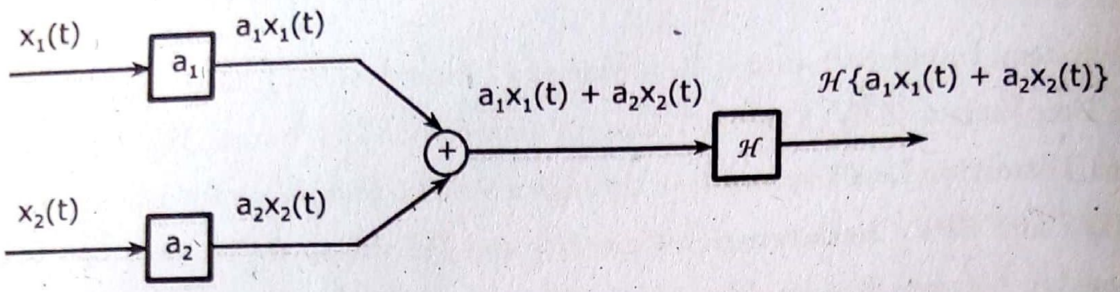
(7) Lumped-parameter and distributed-parameter systems.

(8) Invertible and non-invertible systems.

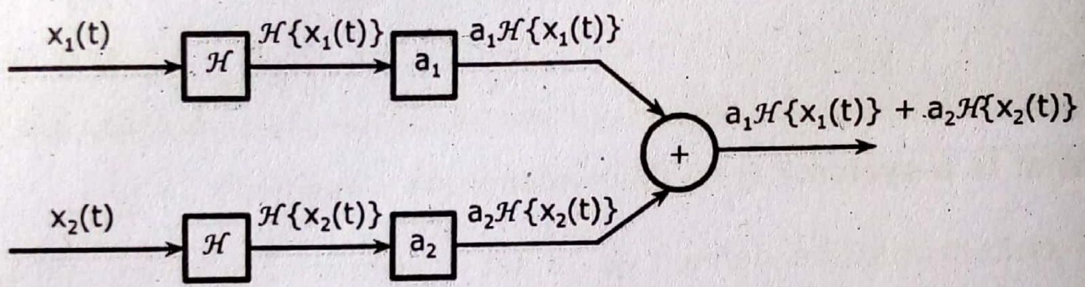
Q2) Define linear system.

Ans. : A system is said to be linear system if it satisfies the principle of super position. If a system does not satisfy the principle of superposition, then it is said to be a non-linear system.

Superposition rule states that response due to sum of weighted inputs is same as the sum of weighted responses.



(a) Weighted Sum of Inputs



(b) Weighted Sum of Responses

Figure Diagrammatic Explanation of Linearity

Q3) How can you find that the system is linear or not?

Ans. : Following steps illustrate the procedure to test for linearity,

(1) Let $x_1(t)$ and $x_2(t)$ be two inputs to the system $\mathcal{H}$ and $y_1(t)$ and $y_2(t)$ be the corresponding responses.

(2) Consider a signal, $x_3(t) = a_1x_1(t) + a_2x_2(t)$ which is a weighted sum of $x_1(t)$ and $x_2(t)$.

(3) Let $y_3(t)$ be the response to the $x_3(t)$.

(4) Check whether $y_3(t) = a_1y_1(t) + a_2y_2(t)$. If equal then the system is linear, otherwise it is nonlinear.

Similar procedure is applicable for discrete-time systems.

Q4) Check whether the function $y(t) = e^{x}(t)$ is linear or not?

Ans. : Let,

$$y_1(t) = e^{x_1(t)}$$

And $$y_2(t) = e^{x_2(t)}$$

Output due to weighted sum of inputs,

$$y_3(t) = \mathcal{H}[x_3(t)]$$

$$= \mathcal{H}[a_1x_1(t) + a_2x_2(t)]$$

$$= e^{a_1x_1(t) + a_2x_2(t)}$$

$$y_3(t) = e^{a_1x_1(t)}e^{a_2x_2(t)}$$

Output due to individual weighted sum of outputs,

$$a_1y_1(t) + a_2y_2(t) = e^{a_1x_1(t)} + e^{a_2x_2(t)}$$

Since output due to weighted sum of inputs not equal to individual weighted sum of outputs, hence non-linear system.

$y_1(t) = e^{x_1(t)}$
$y_2(t) = e^{x_2(t)}$
$y' = e^{x_1(t)} + e^{x_2(t)}$
$y'' = e^{x_1(t)+x_2(t)}$
$y' \neq y''$
non-linear

Q5) Define impulse response of an LTI system.

Ans. : Impulse response is defined as "the response of a system to the impulse signal as a input or excitation of a system".

Consider a LTI continuous-time system with input x(t) as the impulse signal at t = 0, y(t) be the response (or output) of the system and h(t) be the impulse response (i.e., response of system for impulse input).

The response of a system is given by convolution of input and impulse response, that is,

$$y(t) = x(t) * h(t) = \delta(t) * h(t)$$

Where, * denotes the convolution operator.

impulse response is output of the system when input is impulse signal applied at time t = 0.

Q6) What is the Linear Time Variant (LTV) system?

Ans. : A system which is linear but time variant is called a linear time variant system. That is, a system for which the principle of superposition and the principle of homogeneity are valid but the input/output characteristics change with time is called a LTV system.

For a linear time variant system at least one of the coefficients of the differential equation describing the system vary with time.

Q7) How can you obtain the response of a linear system?

Ans. : Output (or response) of a linear system can be obtained by convolving impulse response of the system with input.

In case of a continuous-time system, the response y(t) of the system for an arbitrary input x(t) is given by convolution of input x(t) with impulse response h(t) of the system is expressed as,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

In case of a discrete-time system,

$$y[n] = x[n] * h[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$$

Q8) What is an LTI system? Write its properties.

Ans. : A system which is linear as well as time invariant is called a linear time invariant system. That is, a system for which the principle of superposition and the principle of homogeneity are valid and in addition, the input/output characteristics do not change with time is called a LTI system.

Some of the most basic and important properties of continuous-time LTI systems are,

(1) The commutative property.

(2) The distributive property.

(3) The associative property.

(4) Systems with and without memory.

(5) Causality.

(6) Stability.

(7) Invertibility.

Q9) Define signal bandwidth of a system.

Ans. : The frequency components of a signal extend from $-\infty$ to ∞. Any practical signal has finite energy. As a result, the frequency components approach zero as ω tends to ∞. Therefore, we neglect the frequency components which have negligible energy and select only a band of frequency components which have most of the signal energy. This band or range of frequencies that contain most of the signal energy is known as the signal bandwidth. Normally, the frequency band is selected such that it contains around 95% of total energy depending on the precision.

Q10) What is an ideal LPF?

Ans. : An ideal low-pass filter allows transmission of the signals of frequencies below a certain frequency ω_c radians per second without any distrotion. The signals of frequencies above ω_c radians/second are completely attenuated. Here ω_c is called the cutoff frequency. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal LPF has magnitude response given by,

$$|H(\omega)| = \begin{cases} 1 & ; \ |\omega| < \omega_c \\ 0 & ; \ |\omega| > \omega_c \end{cases}$$

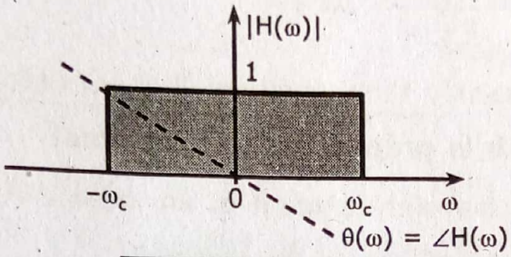


Figure Ideal LPF

Q11) What is an ideal BRF?

Ans. : An ideal band rejection filter rejects totally all of the signals of frequencies within a certain frequency band $(\omega_2 - \omega_1)$ radians/second and transmits all signals of frequencies outside this band without any distortion. Here $(\omega_2 - \omega_1)$ is the rejection band. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal BRF is specified by,

$$|H(\omega)| = \begin{cases} 0 & ; \ |\omega_1| < \omega < |\omega_2| \\ 1 & ; \ \omega < |\omega_1| \text{ and } \omega > |\omega_2| \end{cases}$$

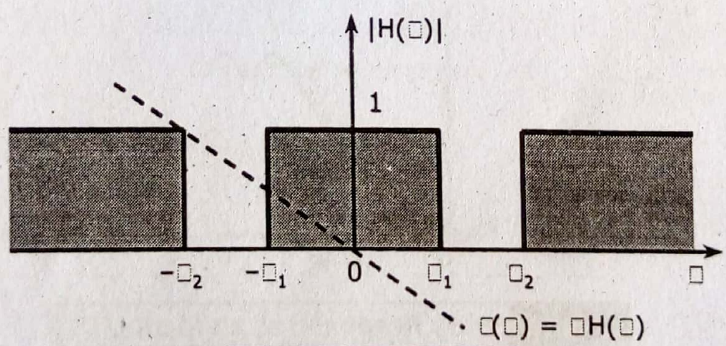


Figure Ideal BRF

Q12) What do you mean by a causal system?

Ans. : A system is said to be causal if it does not produce an output before the input is applied. For an LTI system to be causal, the condition to be satisfied is its impulse response must be zero for t less than zero, i.e.,

$$h(t) = 0 \text{ for } t < 0$$

Q13) What is the time domain criterian of physical realizability?

Ans. : Physical realizability implies that it is physically possible to construct that system in real time. For a physically realizable system the unit impulse response h(t) must be causal i.e., $h(t) = 0$ for $t < 0$. This is the time domain criterion of physical realizability.

Q14) What is a Paley-wiener criterian?

Ans. : The Paley-wiener criteria States that the necessary and sufficient condition for the magnitude response $|H(\omega)|$ to be realizable is,

$$\int_{-\infty}^{\infty} \frac{|\ln||H(\omega)|}{1+\omega^2}\, d\omega < \infty$$

If $H(\omega)$ does not satisfy this condition, it is unrealizable.

Q15) How the bandwidth is proportional to rise time?

Ans. : We know that the transfer function of an ideal LPF is given by,

$$H(\omega) = |H(\omega)|\, e^{-j\omega t_o}$$

Here,

$$|H(\omega)| = \begin{cases} 1 & ; \ |\omega| < \omega_c \\ 0 & ; \ |\omega| > \omega_c \end{cases}$$

Where, ω_c is called the cutoff frequency.

$$H(\omega) = e^{-j\omega t_o} \quad ; \ -\omega_c \le \omega \le \omega_c, \text{ i.e., } |\omega| \le \omega_c$$

$$= 0 \quad ; \ |\omega| > \omega_c$$

The rise time t_r is defined as the time required for the response to reach from 0% to 100% of the final value. To find it, draw a tangent at $t = t_0$ with the line $y(t) = 0$ and $y(t) = 1$. From figure, we have,

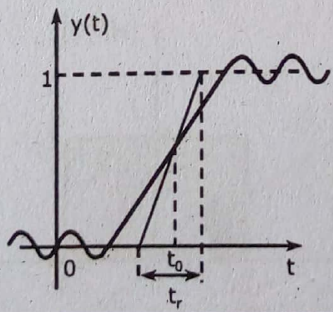


Figure Step Response of an Ideal LPF

$$t_r = \frac{\pi}{\omega_c}$$

For a low-pass filter,

Cut off frequency = Bandwidth of a system.

So the rise time is inversely proportional to the bandwidth.

Bandwidth × Rise time = Constant

PART - B
ESSAY QUESTIONS WITH REFERENCES

Q1) Write the concepts of linear and non-linear systems? **[Refer Section No. 3.2.2]**

Q2) Explain impulse response of an LTI system what is the response of a linear system? **[Refer Section Nos. 3.3 and 3.4]**

Q3) What are the various properties of LTI systems? **[Refer Section No. 3.5]**

Q4) Explain the concepts of linear time invariant system and linear time variant system? **[Refer Section No. 3.5]**

Q5) What is the transfer function of a linear time invariant system? **[Refer Section No. 3.6]**

Q6) Write the filter characteristics of a linear system with suitable example? **[Refer Section No. 3.7]**

Q7) What are the characteristics of a distortionless transmission system? **[Refer Section No. 3.8]**

Q8) Explain signal bandwidth and system bandwidth? **[Refer Section Nos. 3.9 and 3.10]**

Q9) Is it possible to construct a system with infinite bandwidth? Why? Give a reason to your answer? **[Refer Section No. 3.10]**

Q10) What are the characteristics of LPF, HPF and BPF? **[Refer Section No. 3.11]**

Q11) What is poly-wiener criterion? Explain its role in physical realization of a system? **[Refer Section No. 3.12]**

Q12) How the system bandwidth is proportional to rise time? **[Refer Section No. 3.13]**

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3.1 INTRODUCTION

A system may be defined as "a set of elements or functional blocks which are connected together to produce an output in response to an input signal". Thus, a system can be thought of as a process that involves in transforming input signals into output signals.

The operation performed by a system on input sequence x[n] to produce an output sequence y[n] is mathematically expressed as,

$$y(n) = T[x(n)] \quad \text{... (3.1.1)}$$

Such a system is as shown in Fig. 3.1.1.

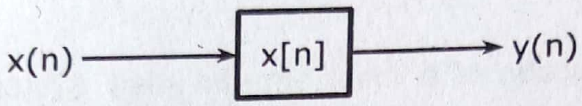


Fig. 3.1.1 A System that Transforms an Input Sequence x(n) into an Output Sequence y(n)

3.2 CLASSIFICATION OF SYSTEMS

The systems are classified as follows,

(1) Continuous-time and discrete-time systems.

(2) Linear and non-linear systems.

(3) Time invariant and time variant systems.

(4) Static (memoryless) and dynamic (memory) systems.

(5) Causal and non-causal systems.

(6) Stable and unstable systems.

(7) Lumped-parameter and distributed-parameter systems.

(8) Invertible and non-invertible systems.

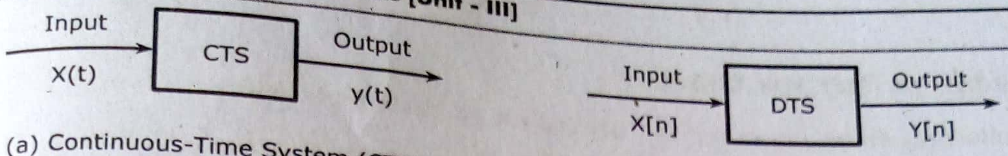
3.2.1 Continuous-Time and Discrete-Time Systems

CONTINUOUS-TIME SYSTEMS (CTS)

A continuous-time system is one in which continuous-time input signals are transformed into continuous-time output signals. Such system is represented pictorially as shown in Fig. 3.2.1(a), where x(t) is the input and y(t) is the output.

DISCRETE-TIME SYSTEMS (DTS)

A discrete-time system is one in which discrete-time input signals are transformed into discrete-time output signals. Such a system is shown in Fig. 3.2.1(b), where x[n] is the input and y[n] is the output.



(a) Continuous-Time System (CTS)

(b) Discrete-Time System (DTS)

Fig. 3.2.1 Systems

3.2.2 Linear and Non-Linear Systems

A system is said to be linear system if it satisfies the principle of super position. If a system does not satisfy the principle of superposition, then it is said to be a non-linear system.

Superposition rule states that response due to sum of weighted inputs is same as the sum of weighted responses. In symbolic terms linearity property for both continuous-time and discrete-time systems is defined as,

For continuous-time system,

$$\mathcal{H}\{a_1x_1(t) + a_2x_2(t)\} = a_1y_1(t) + a_2y_2(t)$$

For discrete-time system,

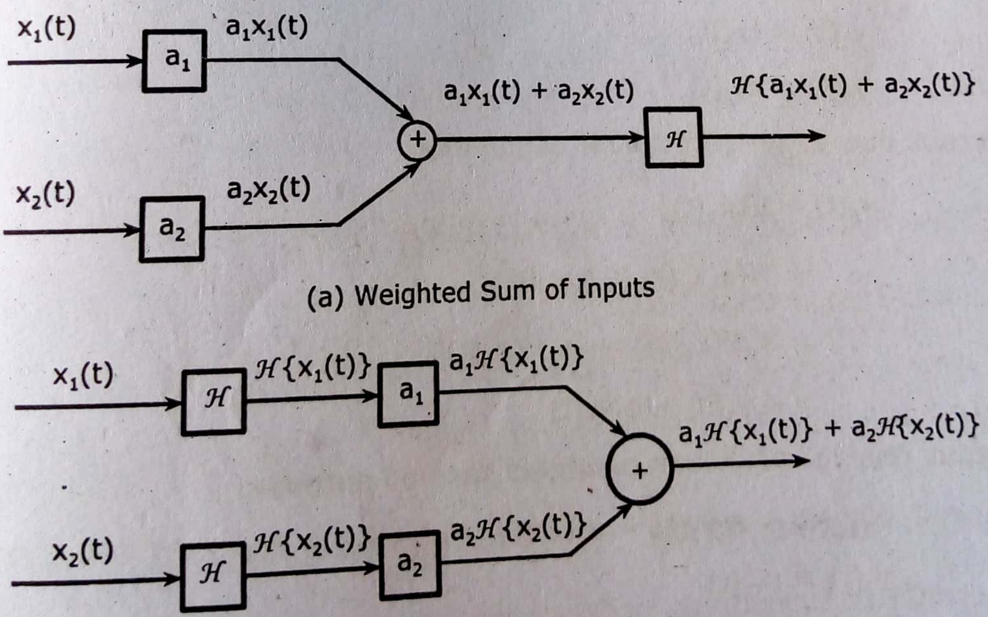
$$\mathcal{H}\{a_1x_1[n] + a_2x_2[n]\} = a_1y_1[n] + a_2y_2[n]$$

Where,

a_1, a_2 = Constants.

$y_1(t)$ $(y_1[n])$, $y_2(t)$ $(y_2[n])$ = Outputs of the system when $x_1(t)$ $(x_1[n])$ and $x_2(t)$ $(x_2[n])$ are respective inputs.

The diagrammatic explanation of linearity property is shown in Fig. 3.2.2.



(a) Weighted Sum of Inputs

(b) Weighted Sum of Responses

Fig. 3.2.2 Diagrammatic Explanation of Linearity

Procedure to Test for Linearity

Following steps illustrate the procedure to test for linearity,

(1) Let $x_1(t)$ and $x_2(t)$ be two inputs to the system $\mathcal{H}$ and $y_1(t)$ and $y_2(t)$ be the corresponding responses.

(2) Consider a signal, $x_3(t) = a_1x_1(t) + a_2x_2(t)$ which is a weighted sum of $x_1(t)$ and $x_2(t)$.

(3) Let $y_3(t)$ be the response to the $x_3(t)$.

(4) Check whether $y_3(t) = a_1y_1(t) + a_2y_2(t)$. If equal then the system is linear, otherwise it is nonlinear.

Similar procedure is applicable for discrete - time systems.

EXAMPLE PROBLEM 1

Determine whether the following systems are linear or non-linear,

(i) $y = t.x(t)$

(ii) $y(t) = e^{x(t)}$

(iii) $\frac{dy(t)}{dt} + 10y(t) = 2x(t)$

(iv) $y[n] = 2x^2[n]$

SOLUTION

(i) $y = t.x(t)$,

Consider,

$$y_1(t) = tx_1(t)$$

And $y_2(t) = tx_2(t)$

Output due to weighted sum of inputs,

$$\begin{aligned} y_3(t) &= \mathcal{H}\{x_3(t)\} \\ &= \mathcal{H}\{a_1x_1(t) + a_2x_2(t)\} \\ &= a_1\mathcal{H}\{x_1(t)\} + a_2\mathcal{H}\{x_2(t)\} \\ &= a_1y_1(t) + a_2y_2(t) \end{aligned}$$

Output due to individual weighted sum of outputs,

$$a_1y_1(t) + a_2y_2(t) = a_1tx_1(t) + a_2tx_2(t) = t\{a_1x_1(t) + a_2x_2(t)\}$$

$$\therefore \quad y_3(t) = tx_3(t)$$

Since output due to weighted sum of inputs is equal to individual weighted sum of outputs, hence linear system.

(ii) $y(t) = e^{x(t)}$,

Let,

$$y_1(t) = e^{x_1(t)}$$

And $$y_2(t) = e^{x_2(t)}$$

Output due to weighted sum of inputs,

$$y_3(t) = \mathcal{H}[x_3(t)]$$

$$= \mathcal{H}[a_1x_1(t) + a_2x_2(t)]$$

$$= e^{a_1x_1(t) + a_2x_2(t)}$$

$$y_3(t) = e^{a_1x_1(t)}e^{a_2x_2(t)}$$

Output due to individual weighted sum of outputs,

$$a_1y_1(t) + a_2y_2(t) = e^{a_1x_1(t)} + e^{a_2x_2(t)}$$

Since output due to weighted sum of inputs is not equal to individual weighted sum of outputs, hence non-linear system.

(iii) $\dfrac{dy(t)}{dt} + 10y(t) = 2x(t)$,

The output due to weighted sum of inputs is given as,

$$a_1\frac{dy_1(t)}{dt} + a_2\frac{dy_2(t)}{dt} + 10[a_1y_1(t) + a_2y_2(t)] = 2[a_1x_1(t) + a_2x_2(t)]$$

$$\frac{d}{dt}[a_1y_1(t) + a_2y_2(t)] + 10[a_1y_1(t) + a_2y_2(t)] = 2[a_1x_1(t) + a_2x_2(t)]$$

Output due to individual weighted sum of outputs,

$$\frac{d}{dt}[a_1y_1(t)] + 10a_1y_1(t) = 2a_1x_1(t)$$

$$\frac{d}{dt}[a_2y_2(t)] + 10a_2y_2(t) = 2a_2x_2(t)$$

Adding the above two equations we get,

$$\frac{d}{dt}[a_1y_1(t) + a_2y_2(t)] + 10[a_1y_1(t) + a_2y_2(t)] = 2[a_1x_1(t) + a_2x_2(t)]$$

Since output due to weighted sum of inputs is equal to individual weighted sum of outputs, hence linear system.

(iv) $y[n] = 2x^2[n]$,

Consider that,

$$y_1[n] = 2x_1^2[n]$$

And $$y_2[n] = 2x_2^2[n]$$

Output due to weighted sum of inputs,

$$y_3[n] = \mathcal{H}[x_3[n]] = \mathcal{H}[a_1x_1[n] + a_2x_2[n]]$$

$$= 2[a_1x_1[n] + a_2x_2[n]]^2$$

$$= 2(a_1^2x_1^2[n] + a_2^2x_2^2[n] + 2a_1a_2x_1[n]x_2[n])$$

$$y_3[n] = 2a_1^2x_1^2[n] + 2a_2^2x_2^2[n] + 4a_1a_2x_1[n]x_2[n]$$

Output due to individual weighted sum of outputs,

$$a_1y_1[n] + a_2y_2[n] = 2a_1x_1^2[n] + 2a_1x_2^2[n]$$

Since output due to weighted sum of inputs is not equal to individual weighted sum of outputs, hence the given system is non-linear.

3.2.3 Time Invariant and Time Variant Systems

A system is said to be Time-Invariant if its input-output characteristics do not change with respect to time i.e., if input x(t) is delayed by time τ and output is also delayed by time τ, then the system is called time-invariant system, otherwise it is said to be time-variant system.

Let x(t) (x[n]) be the input of the system and y(t) (y[n]) be the output of the system, then a system (either a continuous-time or discrete time system), is said to be time-invariant if it satisfies the condition.

For continuous-time system,

If, $$x(t) \xleftrightarrow{\mathcal{H}} y(t)$$

Then, If $$x(t-\tau) \xleftrightarrow{\mathcal{H}} y(t-\tau)$$

For discrete-time system,

If, If $$x[n] \xleftrightarrow{\mathcal{H}} y[n]$$

Then, If $$x[n-n_0] \xleftrightarrow{\mathcal{H}} y[n-n_0]$$

A system is said to be time-variant system if it is does not satisfy the above condition.

PROCEDURE TO TEST FOR TIME INVARIANCE

Following steps illustrates the procedure to test for time invariance,

STEP I : Delay the input signal by τ units of time and determine the response of the system for this delayed input signal. Let this response be $y_1(t)$.

STEP II : Delay the response of the system without shifting input by τ units of time. Let this delayed response be $y_2(t)$.

STEP III : Check whether $y_1(t) = y_2(t)$. If they are equal then the system is time invariant. Otherwise the system is time variant.

Similar procedure is applicable for discrete-time systems.

EXAMPLE PROBLEM 1

Determine whether the following systems are time-variant or time-invariant ?

(i) $y(t) = tx(t)$

(ii) $y(t) = x(t^2)$

(iii) $y(t) = \sin x(t)$

(iv) $y[n] = x^2(n)$

SOLUTION

(i) $y(t) = tx(t)$,

Step I : Delay input signal $x(t)$ by τ units of time and response $y_1(t)$ for this delayed input will be,

$$y_1(t) = x(t - \tau)$$

Step II : Delay the response of the system without shifting input by τ units of time, Let the response be $y_2(t - \tau)$.

i.e., $$y_2(t - \tau) = (t - \tau)x(t - \tau)$$

Since, $$y_1(t) \neq y_2(t - \tau)$$

Hence system is time-variant.

(ii) $y(t) = x(t^2)$,

Step I : Delay input signal $x(t)$ by τ units of time and response be $y_1(t)$

$$\therefore \quad y_1(t) = x(t^2 - \tau)$$

Step II : Delay the response of the system without shifting input by τ units of time, Let the response be $y_2(t - \tau)$.

i.e., $y_2(t - \tau) = x[(t - \tau)^2]$

Since, $y_1(t) \neq y_2(t - \tau)$

Hence system is time variant.

(iii) $y(t) = \sin x(t)$

Step I : Delay input signal $x(t)$ by τ units of time and response be $y_1(t)$.

$$y_2(t) = \sin x(t - \tau)$$

Step II : Delay the response of the system without shifting input by τ units of time, Let the response be $y_2(t - \tau)$.

$$y_2(t - \tau) = \sin x(t - \tau)$$

Since $y_1(t) = y_2(t - \tau)$

Hence, the system is said to be time-invariant.

(iv) $y[n] = x^2(n)$,

Step I : Delay input signal $x[n]$ by N units of intervals and response be $y_1[n]$.

$$y_1[n] = x^2[n - N]$$

Step II : Delay the response of the system without shifting input by N units of intervals, let the response be $y_2[n - N]$.

$$y_2[n] = x^2[n - N]$$

Since $y_1[n] = y_2[n - N]$

Hence time invariant system.

3.2.4 Static (Memory Less) and Dynamic (Memory) Systems

Static System

A system is said to be static or memoryless system if the output at any instant t or n depends on the input at that instant t or n but not on the past or future values of the input.

Examples

(1) $y(t) = x^2(t)$

(2) $y[n] = 2x[n]$

DYNAMIC SYSTEM

A system is said to be dynamic or memory system if the output at any instant depends not only on the input at that instant but also on past and future values.

Examples

(1) $y(t) = x(t - 3) + 2x(t + 2)$

(2) $y[n] = x[n - 2] + x[n - 1]$

Note : *A purely resistive electrical circuit is a static system, whereas an electric circuit having inductors and/or capacitors is a dynamic system.*

EXAMPLE PROBLEM 1

Find whether the following systems are static or dynamic,

(i) $y(t) = x(-t)$ (ii) $y(t) = x(2t)$

(iii) $\frac{d^2x(t)}{dt^2} + 3x(t)$ (iv) $y[n] = x2[n]$

SOLUTION

(i) $y(t) = x(-t)$,

When, $t = 1$, $y(1) = x(-1) \Rightarrow$ The output at $t = 1$ depends on the past input $x(-1)$.

When, $t = 0$, $y(0) = x(0) \Rightarrow$ The output at $t = 0$ depends on the present input $x(0)$.

When, $t = -1$, $y(-1) = x(1) \Rightarrow$ The output at $t = -1$ depends on the present input $x(1)$.

From the above analysis we can say that the output for any value of t (except $t = 0$) depends on past, present and future values of input. Hence the system is dynamic.

(ii) $y(t) = x(2t)$,

When, $t = -1$, $y(-1) = x(-2) \Rightarrow$ Output at $t = -1$ depends on the past input $x(-2)$.

When, $t = 0$, $y(0) = x(0) \Rightarrow$ Output at $t = 0$ depends on the present input $x(0)$.

When, $t = 1$, $y(1) = x(2) \Rightarrow$ Output at $t = 1$ depends on the future input $x(2)$.

From the above analysis we can say that the output for any value of t(except $t = 0$) depends on past, present and future values of input. Hence the system is dynamic.

(iii) $\frac{d^2x(t)}{dt^2} + 3x(t)$,

Given that,

$$y(t) = \frac{d^2x(t)}{dt^2} + 3x(t)$$

Note : *Any continuous-time system described by a differential equation or any discrete-time system described by a difference equation is always a dynamic system.*

Since the given system is described by a differential equation. Therefore, the system is dynamic.

(iv) $y[n] = x^2[n]$

When, $n = 1$, $y[-1] = x^2[-1]$ $\Rightarrow$ The output at $n = -1$ depends on input $x^2[-1]$

When, $n = 0$, $y[0] = x^2[0]$ $\Rightarrow$ The output at $n = 0$ depends on input $x^2[0]$

When, $n = 1$, $y[1] = x^2[1]$ $\Rightarrow$ The output at $n = 1$ depends on input $x^2[1]$

From the above analysis, we can say that output at any instant depends only on the input at that instant. Hence the system is static.

3.2.5 Causal and Non-Causal Systems

Causal System

A system is said to be causal or non-anticipative if the output of the system at any time t depends only on the present and past values of the input but not on future inputs. Causal systems are real time systems. They are physically realizable.

The impulse response of a causal system is zero for t or n < 0, since $\delta(t)$ or $\delta[n]$ exists only at t or n = 0.

i.e., $h(t) = 0$ for $t < 0$ and $h[n] = 0$ for $n < 0$

Examples $y(t) = x(t - 2) + 2x(t)$

$y(t) = tx(t)$

$y[n] = nx[n]$

$y[n] = x[n - 2] + x[n - 1] + x[n]$

NON-CAUSAL SYSTEM

A system is said to be non-causal (anticipative) if the output of the system at any time t depends on future inputs. They do not exist in real time. They are not physically realizable.

Examples

$$y(t) = x(t + 2) + 2x(t)$$

$$y(t) = x^2(t) + tx(t + 1)$$

$$y[n] = x[n] + x[2n]$$

$$y[n] = x^2[n] + 2x[n + 2]$$

EXAMPLE PROBLEM 1

Find whether the following systems are causal or non-causal?

(i) $y(t) = 2x(t) + 3x(2 - t)$

(ii) $y(t) = 3x(t) + \frac{2\, dx(t)}{dt}$

(iii) $y[n] = x[n^2]$

SOLUTION

(i) $y(t) = 2x(t) + 3x(2 - t)$,

When $t = -1$, $y(-1) = 2x(-1) + 3x(3)$ ⇒ The output at $t = -1$, i.e., $y(-1)$ depends on the present input $x(-1)$ and future input $x(3)$.

When $t = 0$, $y(0) = 2x(0) + 3x(2)$ ⇒ The output at $t = 0$, i.e., $y(-1)$ depends on the present input $x(0)$ and future input $x(2)$.

When $t = 1$, $y(1) = 2x(1) + 3x(1)$ ⇒ The output at $t = 1$, i.e., $y(1)$ depends on the present input $x(1)$.

When $t = 2$, $y(2) = 2x(2) + 3x(-1)$ ⇒ The output at $t = 2$, i.e., $y(2)$ depends on the present input $x(2)$ and past input $x(-1)$.

From the above analysis we can say that for $t < 1$, the system output depends on present and future inputs. Hence the system is non-causal.

(ii) $3x(t) + \frac{2dx(t)}{dt}$

$$y(t) = 3x(t) + 2\frac{dx(t)}{dt} = 3x(t) + 2\left[\underset{\Delta t \to 0}{Lt} \frac{x(t) - x(t - \Delta t)}{\Delta t}\right]$$

In the above equation, for any value of t, the x(t) is present input and x(t − Δt) is the past input.

Therefore we can say that the response for any value of t depends on present and past input. Hence the system is causal.

(iii) $y[n] = x[n^2]$

When n = −1, y[−1] = x[−1] ⇒ The response at n=−1, i.e., y[−1] depends on the future input x[−1].

When n = 0, y[0] = x[0] ⇒ The response at n = 0, i.e., y[0] depends on the present input x[0].

When n = 1, y[1] = x[1] ⇒ The response at n = 1, i.e., y[1] depends on the present input x[1].

When n = 2, y[2] = x[4] ⇒ The response at n = 2, i.e., y[2] depends on the future input x[4].

From the above analysis we can say that the response for any value of n(except n = 0 and n = 1) depends on future inputs. Hence the system is non-causal.

3.2.6 Stable and Unstable Systems

A system is said to be stable Bounded Input-Bounded Output (BIBO) system if and only if every bounded input produces a bounded output.

Let x(t) be the input of continuous time system and y(t) be the response or output for x(t). The term bounded input refers that input x(t) is bounded in magnitude such that,

$$|x(t)| \leq M_x \text{ and } M_x < \infty, \text{ for all } t,$$

Where,

M_x is the constant.

Also the term bounded output refers that output g(t) is bounded in magnitude such that,

$$|y(t)| \leq M_y \text{ and } M_y < \infty, \text{ for all } t,$$

Where, M_y is the constant.

CONDITION FOR STABLE OF AN LTI SYSTEM

For a Linear Time Invariant (LTI) system, the condition for BIBO stability can be transformed to a condition on impulse response, h(t) or h[n],

(1) For BIBO stability of an LTI continuous-time system, the integral of impulse response should be finite.

i.e.,
$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

(2) For BIBO stability of an LTI discrete-time system, the summation of impulse response should be finite.

i.e.,
$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

(3) A system is unstable if its impulse response h(t) (h[n]) grows without bound, i.e., approaches infinite after a sufficiently long time.

EXAMPLE PROBLEM 1

State whether the following systems are stable or unstable?

(i) $h(t) = e^{-at}u(t)$

(ii) $h(t) = e^{at}\, u(t)$

(iii) $h[n] = b^n \; ; n < 0 = a^n \; ; n \geq 0.$

SOLUTION

(i) $h(t) = e^{-at}\, u(t)$

For stability, $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

$$\therefore \int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} |e^{-at}u(t)|\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at}u(t)\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, dt$$

$$= \left[\frac{e^{-at}}{-a}\right]_0^{\infty}$$

$$= \frac{e^{-\infty}}{-a} - \frac{e^0}{-a}$$

$$= 0 - \frac{1}{a}$$

$$= -\frac{1}{a}$$

Here, $\int_{-\infty}^{\infty} |h(t)|\, dt = -\frac{1}{a}$ constant, less than infinity.

Hence the system is stable.

(ii) $h(t) = e^{at}u(t)$,

For stability, $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

$$\therefore \int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} |e^{at}u(t)|\, dt$$

$$= \int_{-\infty}^{\infty} e^{at}u(t)dt$$

$$= \int_{0}^{\infty} e^{at}dt$$

$$= \left[\frac{e^{at}}{a}\right]_0^{\infty}$$

$$= \frac{e^{\infty}}{a} - \frac{e^0}{a}$$

$$= \infty - \frac{1}{a}$$

$$= \infty$$

Here, $\int_{-\infty}^{\infty} |h(t)|\, dt = \infty$.

Hence the system is unstable.

(iii) h[n] = b^n ; n < 0 = a^n ; n ³ 0

The condition to be satisfied for the stability of the system is,

$$\sum_{n=0}^{\infty} |h[n]| < \infty$$

$$\therefore \sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{-1} |b|^n + \sum_{n=0}^{\infty} |a|^n$$

$$= \sum_{n=1}^{\infty} |b|^{-n} + \sum_{n=0}^{\infty} |a|^n$$

$$= \sum_{n=0}^{\infty} |b|^{-n} - 1 + \sum_{n=0}^{\infty} |a|^n \qquad [\because |b| = 1]$$

$$= \sum_{n=0}^{\infty} (|b|^{-1})^n - 1 + \sum_{n=0}^{\infty} |a|^n$$

The summation of infinite terms in the above equation converges if, $0 < |a| < 1$ and $0 < |b| < 1$. Hence by using infinite geometric series formula,

$$\sum |h(n)| = \frac{1}{1 - |b|^{-1}} - 1 + \frac{1}{1 - |a|}$$

$$\left[\because \text{Infinite geometric series sum of formula} \quad \sum_{n=0}^{\infty} c^n = \frac{1}{1-c} \text{ if } 0 < |c| < 1\right]$$

= Constant

Therefore, the system is stable if $|a|<1$ and $|b|>1$

3.2.7 Lumped Parameter and Distributed Parameter Systems

In lumped-parameter systems, each component is lumped at one-point in space. These systems are described by ordinary differential equations. In distributed-parameter systems, the signals are functions of space as well as time. These systems are described by partial differential equations.

3.2.8 Invertible and Non-Invertible Systems

A system is said to be invertible if there is unique output for every unique input. Fig. 3.2.3 shows this concept.

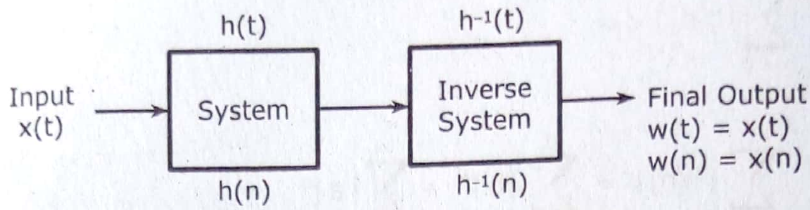


Fig. 3.2.3 Invertible System

If the system is invertible, there exists an inverse system. If these two systems are cascaded as shown in Fig. 3.2.4, then final output is same as input. If the system is denoted by H, then its inverse is denoted by H^{-1}. The cascading of the two systems gives,

$$\mathcal{H}\mathcal{H}^{-1} = 1$$

EXAMPLE PROBLEM 1

Determine whether the following systems are invertible or non-invertible,

(i) y(t) = 2x(t)

(ii) y[t] = x^n(t)

SOLUTION

(i) y(t) = 2x(t),

For this system, the inverse system will be,

$$w(t) = \frac{1}{2} y(t)$$

Hence, if the two systems are cascaded, input is first multiplied by 2. Then the output of the first system is divided by 2. Therefore, original input is obtained back.

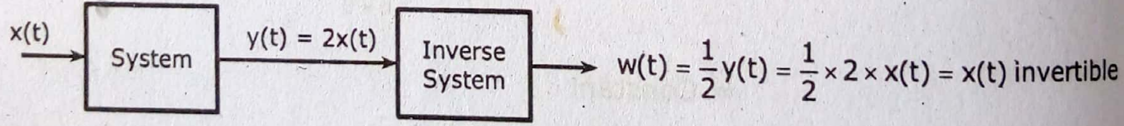
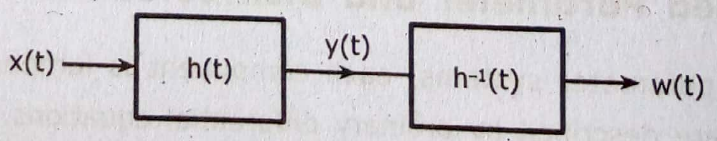


Fig. 3.2.4 Invertible System

(ii) y[t] = x^n(t)



h(t) is a Squarer

Fig. 3.2.5 Non-Invertible System

This system square the input. Hence, inverse system take square root,

$$w(t) = \sqrt{y(t)}$$

$$= \sqrt{x^2(t)}$$

$$w(t) = \pm\, x(t)$$

Thus two outputs are possible, x(t) and −x(t). This means there is no unique output for unique input. Hence, the system is non-invertible.

REVIEW QUESTIONS

(1) *Explain about linear systems? How do you find that the system is linear or not?*
(2) *What are the concepts of linear and non-linear systems?*
(3) *Define stable and invertible systems?*

3.3 IMPULSE RESPONSE

Impulse response is defined as ***"the response of a system to the impulse signal as a input or excitation of a system".***

Consider a LTI continuous-time system with input x(t) as the impulse signal at t = 0, y(t) be the response (or output) of the system and h(t) be the impulse response (i.e., response of system for impulse input) as shown in Fig. 3.3.1.

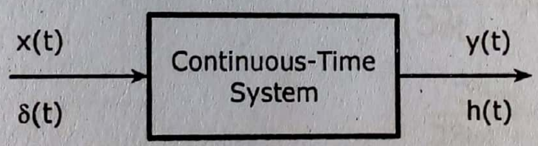


Fig. 3.3.1 Continuous-time System

The response of a system is given by convolution of input and impulse response, that is,

$$y(t) = x(t) * h(t) = \delta(t) * h(t) \qquad \text{... (3.3.1)}$$

Where, * denotes the convolution operator. Applying Fourier transform on both sides of Eq. (3.3.1) and using the property, $x_1(t) * x_2(t) \leftrightarrow X_1(\omega).X_2(\omega)$ we have,

$$F[y(t)] = Y(\omega) = 1.H(\omega) \qquad \left[\because \delta(t) \xleftrightarrow{F.T} 1\right]$$

Applying inverse Fourier transform, we get,

$$F^{-1}[Y(\omega)] = y(t) = F^{-1}[H(\omega)] = h(t)$$

From above analysis, we can say that, *"impulse response is output of the system when input is impulse signal applied at time t = 0".*

EXAMPLE PROBLEM 1

Find the impulse response of the system shown in Fig. 3.3.2.

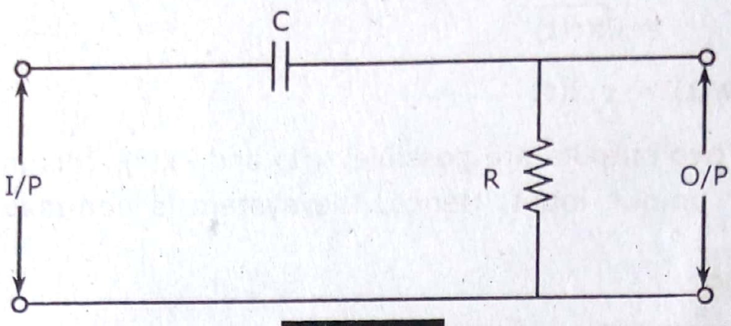


Fig. 3.3.2

Find the transfer function. What would be its frequency response? Sketch the response.

SOLUTION

Let x(t) be the input, y(t) be the output, h(t) be the impulse response,

The transfer function,

$$H(\omega) = \frac{\text{Output}}{\text{Input}}$$

$$= \frac{Y(\omega)}{X(\omega)}$$

$$= \frac{R}{R + \left(\frac{1}{j\omega C}\right)}$$

$$\therefore \quad H(\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

Impulse response,

$$h(t) = F^{-1}[H(\omega)]$$

$$= F^{-1}\left[\frac{j\omega RC}{1 + j\omega RC}\right]$$

$$= F^{-1}\left[\frac{j\omega}{\left(\frac{1}{RC}\right) + j\omega}\right]$$

$$= F^{-1}\left[1 - \frac{\frac{1}{RC}}{j\omega + \left(\frac{1}{RC}\right)}\right]$$

$$\therefore \quad h(t) = \delta(t) - \frac{1}{RC}e^{-t/RC}$$

To sketch the response, we have,

$$h(t) = F^{-1}[H(\omega)]$$

$$H(\omega) = \frac{j\omega RC}{1 + j\omega RC} = \left(\frac{j\omega RC}{1 + j\omega RC}\right)\left(\frac{1 - j\omega RC}{1 - j\omega RC}\right)$$

$$= \frac{j\omega RC + (\omega RC)^2}{1 + (\omega RC)^2}$$

Magniutde, $|H(\omega)| = \dfrac{\omega RC}{\sqrt{1 + (\omega RC)^2}}$

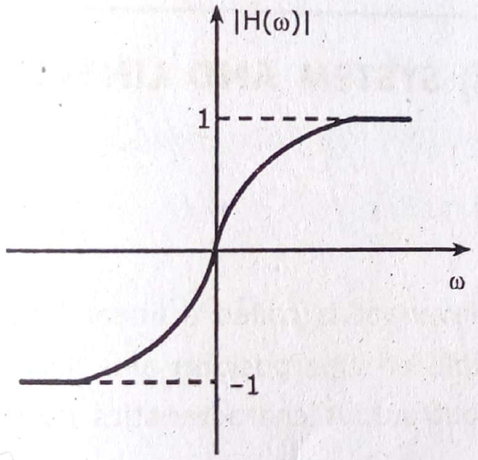
And, phase, $\theta(\omega) = \dfrac{\pi}{2} - \tan^{-1}(\omega RC)$

Let, $RC = 1$

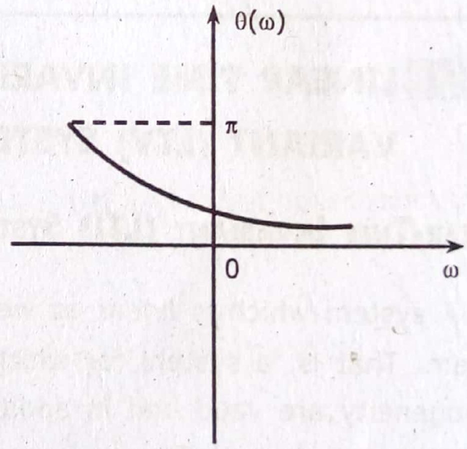
$\therefore$ $|H(\omega)| = \dfrac{\omega}{\sqrt{1 + \omega^2}}$

And $\theta(\omega) = \dfrac{\pi}{2} - \tan^{-1}(\omega)$

The magnitude and phase response plots are given in Fig. 3.3.3(a) and Fig. 3.3.3(b) respectively.



(a) Magnitude Response Plot



(b) Phase Response Plot

Fig. 3.3.3

REVIEW QUESTIONS

(1) *Describe impulse response of a signal?*
(2) *State and explain the impulse response of a continuous time system?*

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3.4 RESPONSE OF A LINEAR SYSTEM

Output (or response) of a linear system can be obtained by convolving impulse response of the system with input.

In case of a continuous-time system, the response y(t) of the system for an arbitrary input x(t) is given by convolution of input x(t) with impulse response h(t) of the system is expressed as,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau$$

In case of a discrete-time system, the output (or response) is given by convolution sum of input x[n] with impulse h[n] of the system is given by,

$$y[n] = x[n] * h[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n - K]$$

REVIEW QUESTIONS

(1) *What are the responses of system in both continuous time and discrete time systems? And give an expression for each?*

(2) *Explain the response of a system which is linear?*

3.5 LINEAR TIME INVARIANT (LTI) SYSTEM AND LINEAR TIME VARIANT (LTV) SYSTEM

Linear-Time Invariant (LTI) System

A system which is linear as well as time invariant is called a linear time invariant system. That is, a system for which the principle of superposition and the principle of homogeneity are valid and in addition, the input/output characteristics do not change with time is called a LTI system.

For an LTI system, the output due to delayed input [x(t − τ)], i.e., y(t, τ) must be equal to the delayed output y(t − τ).

i.e., $$y(t, \tau) = y(t - \tau)$$

That is if the input is delayed by τ units the corresponding output will also be delayed by t units. For a linear time invariant system, all the coefficients of the differential equation describing the system are constants.

Linear Time Variant (LTV) System

A system which is linear but time variant is called a linear time variant system. That is, a system for which the principle of superposition and the principle of homogeneity are valid but the input/output characteristics change with time is called a LTV system.

For an LTV system, the output due to delayed input $[x(t - \tau)]$, i.e., $y(t, \tau)$ is not equal to the delayed output $y(t - \tau)$.

i.e., $$y(t, \tau) \neq y(t - \tau)$$

That is for an LTV system, if the input is delayed by τ units then the corresponding output will not be delayed by exactly τ units.

For a linear time variant system at least one of the coefficients of the differential equation describing the system vary with time.

Some of the most basic and important properties of continuous-time LTI systems are,

(1) The commutative property.

(2) The distributive property.

(3) The associative property.

(4) Systems with and without memory.

(5) Causality.

(6) Stability.

(7) Invertibility.

3.5.1 The Commutative Property

A basic property of convolution in continuous-time is that it is a commutative operation.

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t - \tau)\, d\tau$$

i.e., the output of an LTI system with input $x(t)$ and unit impulse response $h(t)$ is identical to the output of an LTI system with input $h(t)$ and impulse response $x(t)$.

3.5.2 The Distributive Property

Distributive property states that,

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

i.e., two systems with impulse responses $h_1(t)$ and $h_2(t)$ connected in parallel can be replaced by a single system with impulse response $h_1(t) + h_2(t)$.

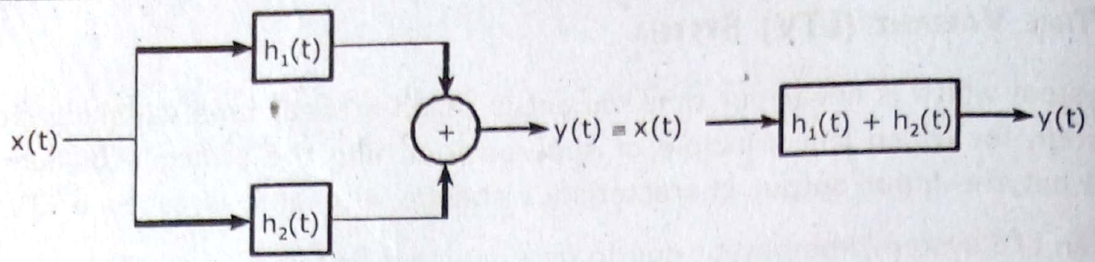


Fig. 3.5.1 The Distributive Property

3.5.3 The Associative Property

Associative property states that,

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

i.e., it is immaterial in which order the signals are convolved.

According to the associative property, the series interconnection of two systems is equivalent to a single system. Fig. 3.5.2 illustrates the associative property.

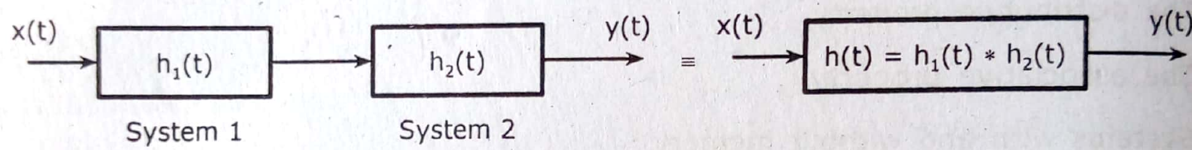


Fig. 3.5.2 The Associative Property

The impulse response of the cascade of two LTI systems is the convolution of their individual impulse responses.

3.5.4 Systems with and without Memory

A system is said to be static or memoryless if its output at any time depends only on the value of the input at that time. A continuous-time LTI system is memoryless if $h(t) = 0$ for $t \neq 0$ and such a memoryless LTI system has the form,

$$y(t) = kx(t)$$

For some constant k and has the impulse response.

$$h(t) = k\delta(t)$$

Therefore, if $h(t) \neq 0$ for $t \neq 0$, The continuous-time has memory. A memory system is also called a dynamic system.

If $k = 1$

$$h(t) = \delta(t)$$

Then the system becomes an identity system.

3.5.5 Causality

A causal system is non anticipatory and does not produce an output before an input is applied. Its output depends only on the present and past values of input but not on future inputs.

Therefore for a causal LTI system we have,

$$h(t) = 0 \text{ for } t < 0$$

The output of a causal LTI system for a non-causal input signal is,

$$y(t) = \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

The output of a causal LTI system for a causal input signal is,

$$y(t) = \int_{0}^{t} h(\tau)\, x(t-\tau)\, d\tau = \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

A non-causal system is anticipatory and $h(t) \neq 0$ for $t < 0$

3.5.6 Stability

A system is stable if every bounded input produces a bounded output. The BIBO stability of an LTI system can be easily determined from its impulse response. For a continuous-time LTI system to be BIBO stable, its impulse response h(t) must be absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

3.5.7 Invertibility

A continuous-time LTI system with impulse response h(t) is said to be invertible, if an inverse system with impulse response $h_1(t)$ which when connected in series with the original system produces an output equal to the input of the first system.

i.e., $$h(t) * h_1(t) = \delta(t)$$

Fig. 3.5.3 illustrates invertibility of system.

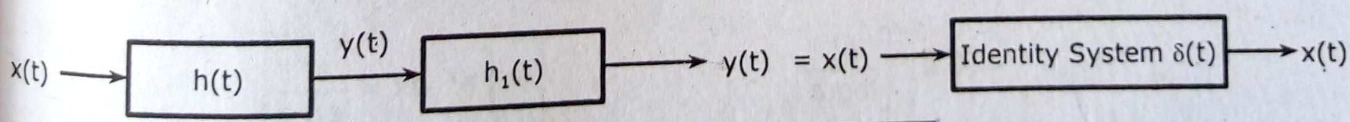


Fig. 3.5.3 Invertibility of System

Note : Similar properties of convolution are applicable for discrete-time LTI systems also.

EXAMPLE PROBLEM 1

Let the system function of a LTI system be 1/jω + 2. What is the output of the system for an input (0.8)ᵗ.

SOLUTION

Given data : LTI system function,

$$H(\omega) = \frac{1}{j\omega + 2}$$

Input, $x(t) = (0.8)^t\, u(t)$

The impulse response,

$$h(t) = F^{-1}[H(\omega)] = F^{-1}\left(\frac{1}{j\omega + 2}\right) = e^{-2t}\, u(t) \qquad \left(\because e^{-at}\, u(t) \xleftrightarrow{F.T} \frac{1}{a + j\omega}\right)$$

We know that the output y(t) is the convolution of input x(t) and impulse response h(t).

$$\therefore \quad y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\tau}\, u(\tau)\, (0.8)^{t-\tau}\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\tau}\, (0.8)^t\, (0.8)^{-\tau}\, d\tau$$

$$= (0.8)^t \int_0^{\infty} (0.8e^2)^{-\tau}\, d\tau$$

Let, $(0.8e^2)^{-1} = a$

$$\therefore \quad y(t) = (0.8)^t \int_0^{\infty} a^{\tau}\, d\tau$$

$$= (0.8)^t \left[\frac{a^{\tau}}{\log a}\right]_0^{\infty} \qquad \left[\because \int a^x\, dx = \frac{a^x}{\log a}\right]$$

$$= (0.8)^t \left[\frac{(0.8e^2)^{-\tau}}{\log (0.8e^2)^{-1}}\right]_0^{\infty} = \frac{(0.8)^t}{\log [(0.8e^2)^{-1}]}[(0.8e^2)^{-\infty} - 1]$$

$$= \frac{(0.8)^t}{-\log(0.8e^2)}[0-1]$$

$$= \frac{(0.8)^t}{\log(0.8e^2)}$$

$$= \frac{(0.8)^t}{(\log 0.8 + 2)} \qquad [\because \log e^2 = 2]$$

Output of the system,

$$y(t) = \frac{(0.8)^t}{(\log 0.8 + 2)}$$

REVIEW QUESTIONS

(1) *What are the properties of an LTI system?*

(2) *Explain LTI systems? What are the basic properties of continuous time LTI systems?*

(3) *Define the term "LTV systems".*

3.6 TRANSFER FUNCTION OF LTI SYSTEM

The transfer function of a LTI system shown in Fig. 3.6.1 is defined as the ratio of the output to the input in frequency domain.

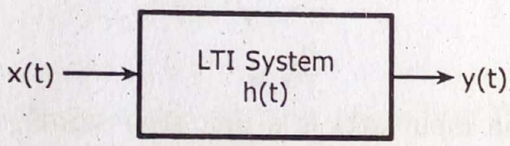


Fig. 3.6.1 LTI System

The transfer function of a continuous-time LTI system may be defined using Fourier transform or Laplace transform. The transfer function is defined only under zero initial conditions.

- The transfer function of a LTI system $H(\omega)$ is defined as the ratio of the Fourier transform of the output signal to the Fourier transform of the input signal when the initial conditions are zero. Or simply we can say that the transfer function $H(\omega)$ of a LTI system is the Fourier transform of its impulse response.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Where $Y(\omega)$ represents frequency spectrum of $y(t)$ and $X(\omega)$ represents the frequency spectrum of input $x(t)$.

$H(\omega)$ is a complex quantity having magnitude and phase.

i.e., $$H(\omega) = |H(\omega)|e^{j\theta(\omega)}$$

Where,

$|H(\omega)|$ = Amplitude (Magnitude) response of a LTI system.

$\theta(\omega)$ = Phase response of a LTI system.

The impulse response h(t) of a system is the inverse Fourier transform of its transfer function (frequency response) $H(\omega)$.

$$H(\omega) = F|h(t)|$$

(or) $$h(t) = F^{-1}|H(\omega)|$$

The transfer function of a system in s-domain (Laplace domain) is defined as the ratio of the Laplace transform of the output of the system to the Laplace transform of the input of the system when the initial conditions are neglected. The transfer function of a system can also be defined as the Laplace transform of the impulse response of the system.

$$H(s) = \frac{Y(s)}{X(s)}$$

The impulse response is nothing but the inverse Laplace transform of the transfer function H(s).

i.e., $$h(t) = L^{-1}[H(s)]$$

Unit Step Response

In an LTI system, if the input x(t) is a unit step signal, then the response is called a unit step response (s(t)). The step response s(t) of a continuous-time LTI system (represented by $\mathcal{H}$) is defined as the response of the system when the input is u(t), that is,

$$s(t) = \mathcal{H}\{u(t)\} \quad \text{... (3.6.1)}$$

In many applications, the step response s(t) is also a useful characterization of the system. The step response s(t) can be easily determined by using convolution integral, that is,

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\tau} h(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\tau} h(\tau)\, d\tau \qquad [\because u(t-\tau) \text{ is defined for } \tau < t < \infty] \quad \text{... (3.6.2)}$$

Thus, the step response s(t) can be obtained by integrating the impulse response h(t). Differentiating Eq. (3.6.2) with respect to t, we get,

$$h(t) = s'(t) = \frac{ds(t)}{dt} \quad \text{... (3.6.3)}$$

Thus, the impulse response h(t) can be determined by differentiating the step response s(t).

REVIEW QUESTIONS

(1) *What is the transfer function of a linear time invariant system?*

(2) *Explain about the transfer function of an LTI system? What is the unit step response of an LTI system?*

3.7 FILTER CHARACTERISTICS OF LINEAR SYSTEMS

Consider a system with an applied input signal x(t). Let this system processes the signal x(t) and give rise to a response signal y(t) in a way that is characteristic of the system given by,

$$Y(\omega) = H(\omega).X(\omega) \quad \text{... (3.7.1)}$$

Where,

$Y(\omega)$ = Spectral density function of the output signal y(t).

$X(\omega)$ = Spectral density function of the input signal x(t).

$H(\omega)$ = Transfer function of the system.

The system, therefore modifies the spectral density function of the input signal. It is clear that the system acts as a kind of filter to various frequency components. Some frequency components are boosted in strength, some are attenuated and some may remain unaffected. Similarly, each frequency component undergoes a different amount of phase shift in the process of transmission.

Thus a system modifies the spectral density function of the input according to its filter characteristics. The modification is carried out according to the transfer function $H(\omega)$, acts as a weighting function or spectral shaping function to the different frequency components in the input signal. An LTI system, therefore, acts as a filter.

The band of frequency that is allowed by the filter is called pass-band and the band of frequency that is severely attenuated and not allowed to pass through the filter is called stop-band or rejection-band.

A filter is basically a frequency selective network. The filter characteristics of an LTI system are given.

LTI system which allow the transmission of only low frequency components and rejects (stops) all high frequency components are called low-pass filters (LPFs).

LTI systems which allow the transmission of only high frequency components and rejects (stops) all low frequency components are called high-pass filters (HPFs).

LTI system which allow transmission of only a particular band of frequencies and rejects (stops) all other frequency components are called band-pass filters (BPFs).

LTI system which reject only a particular band of frequencies and allow all other frequency components are called band-rejection filters (BRFs).

Let us consider a simple R-C network shown in Fig. 3.7.1,

(1) In Fig. 3.7.1(a), a square-pulse is applied at input aa' and the output $V_0(t)$ is taken across the output terminals Fig. 3.7.1(b).

(2) A spectral density i.e., the rectangular pulse is shown in Fig. 3.7.1(c).

(3) The transfer function $H(\omega)$ of the network relating to output voltage to the input voltage would be $\frac{1}{\omega + 1}$.

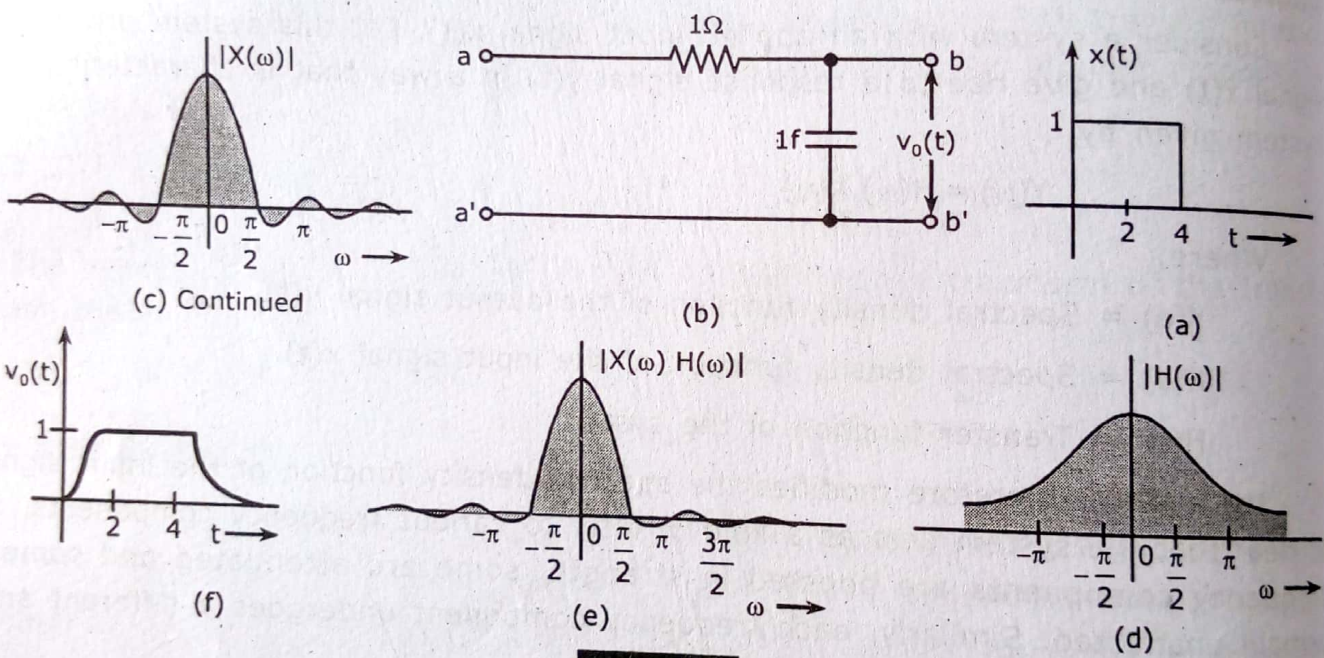


Fig. 3.7.1

Hence,

$$H(\omega) = \frac{1}{\omega + 1}$$

(4) The filter characteristics of this magnitude plot $|H(\omega)|$ is shown in Fig. 3.7.1(d).

(5) This network is known as the simplest form of low pass filter.

(6) The Fig. 3.7.1(e) shows the magnitude $|X(\omega)H(\omega)|$ of the response spectral density function.

REVIEW QUESTIONS

(1) *What are the various filter characteristics of a linear system?*

(2) *Write the characteristics of filter of a linear system with suitable example.*

3.8 DISTORTION LESS TRANSMISSION THROUGH A SYSTEM

The change in the shape of the signal when it is transmitted through a system is called distortion. A signal transmitted through a system is said to be distortionless, if the response (or output) is an exact replica of the input signal. This replica, may of course, have a different magnitude and may also have different time delay. Mathematically, a signal x(t) when transmitted through a system is said to be distortionless if the output is,

$$y(t) = Kx(t - t_0) \quad \text{... (3.8.1)}$$

Where,

K = Change in amplitude.

t_0 = Delay time (t_d).

Typical input and output waveforms of a distortionless system are as shown in Fig. 3.8.1.

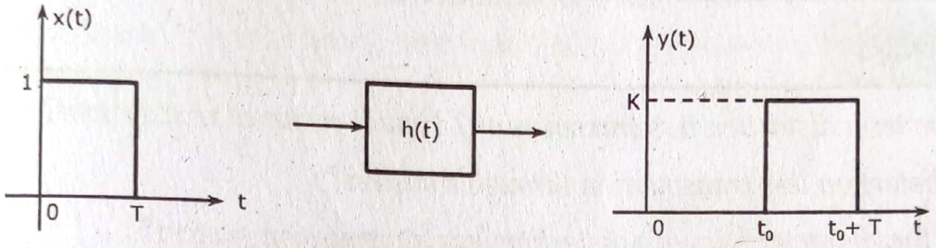


Fig. 3.8.1 Distortionless LTI System

Applying Fourier transform on both sides of the Eq. (3.8.1) and using the shifting property, i.e., $\delta(t - t_0) \xleftrightarrow{F.T} e^{-j\omega t_0}$ we have,

$$Y(\omega) = ke^{-j\omega t_d}\, X(\omega) \qquad (\because t_0 = t_d)$$

Therefore, for distortionless transmission, the transfer function of the system must be of the form.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = ke^{-j\omega t_d}$$

Taking inverse Fourier transform, the corresponding impulse response must be,

$$h(t) = k\delta(t - t_0)$$

Magnitude, $|H(\omega)| = k$

Phase shift,

$$\theta(\omega) = \angle H(\omega) = -\omega t_0 = -\omega t_d \qquad [\because t_0 = t_d]$$

So for distortionless transmission of a signal through a system.

The magnitude $|H(\omega)|$ should be a constant, i.e., all the frequency components of the input signal must undergo the same amount of amplification or attenuation, i.e., the system bandwidth is infinite the phase spectrum should be proportional to frequency.

But, in practice, no system can have infinite bandwidth and hence distortionless conditions can never met exactly.

The magnitude and phase characteristics of a distortionless system is shown in Fig. 3.8.2.

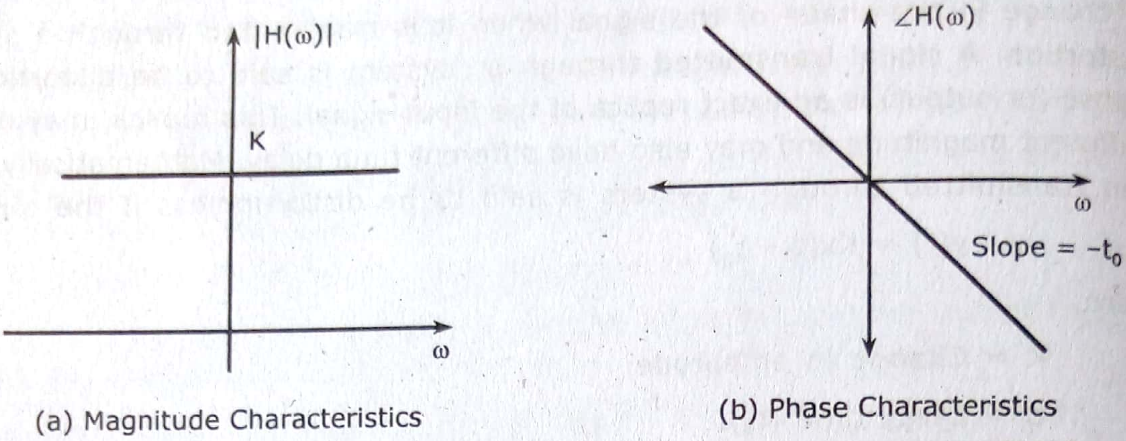


Fig. 3.8.2 Characteristics of a Distortionless Transmission System

REVIEW QUESTIONS

(1) *Define the term distortion less transmission? How it occurred in a system?*

(2) *Explain distortion less transmission through a system?*

(3) *What are the characteristics of a distortion less transmission system?*

3.9 SIGNAL BANDWIDTH

The frequency components of a signal extend from $-\infty$ to ∞. Any practical signal has finite energy. As a result, the frequency components approach zero as ω tends to ∞. Therefore, we neglect the frequency components which have negligible energy and select only a band of frequency components which have most of the signal energy. This band or range of frequencies that contain most of the signal energy is known as the signal bandwidth. Normally, the frequency band is selected such that it contains around 95% of total energy depending on the precision.

EXAMPLE PROBLEM 1

Determine the maximum bandwidth of signals that can be transmitted through the low-pass RC filter shown in fig. 3.9.1, if over this bandwidth, the gain variation is to be within 10% and the phase variation is to be within 7% of the ideal characteristics.

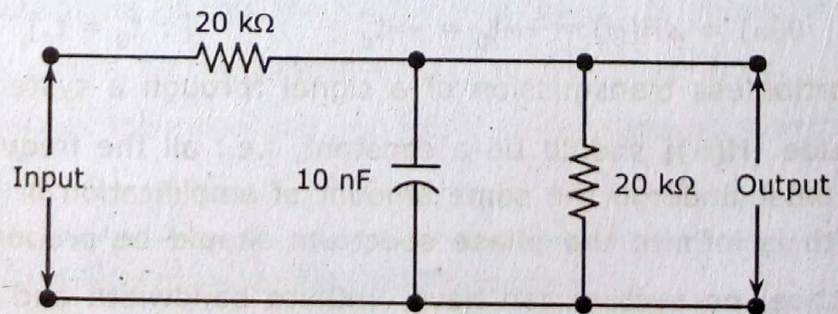


Fig. 3.9.1 Circuit

SOLUTION

Let R = 20 kΩ and C = 10 nF the given RC network transformed into frequency-domain can be represented as shown in Fig. 3.9.2(a). Combining the parallel R and C at the output side, the equivalent circuit is shown in fig. 3.9.2(b)

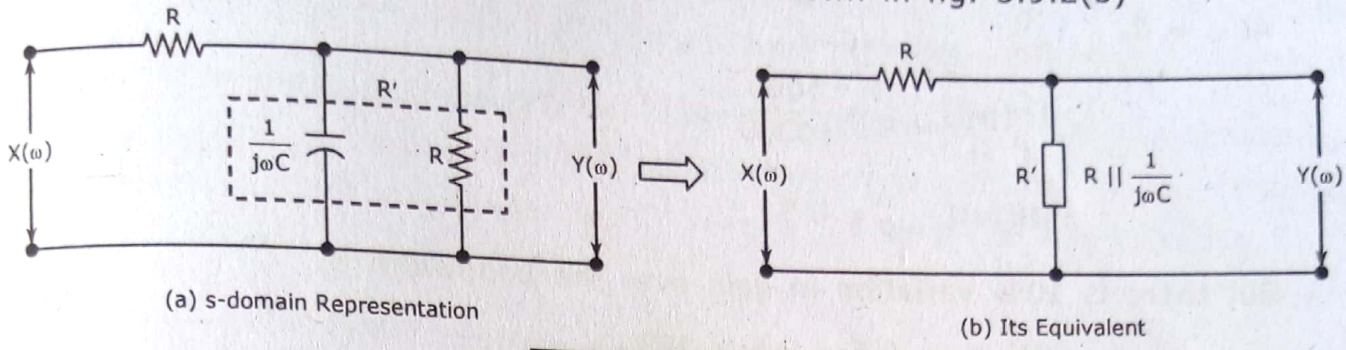


Fig. 3.9.2 Circuit

From the basic circuit theory, the input/output relationship (transfer function) of the given circuit is,

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{\left[R \,||\, \frac{1}{j\omega C}\right]}{R + \left[R \,||\, \frac{1}{j\omega C}\right]} = \frac{\left[\dfrac{R\,\square\,1/j\omega C}{R + \dfrac{1}{j\omega C}}\right]}{R + \left[\dfrac{R\,\square\,1/j\omega C}{R + \dfrac{1}{j\omega C}}\right]}$$

$$= \frac{\dfrac{R}{1 + j\omega RC}}{R + \left[\dfrac{R}{1 + j\omega RC}\right]} = \frac{R}{2R + j\omega R^2 C} = \frac{1}{2 + j\omega RC}$$

But, given R = 20 kΩ, C = 10nF

Now,

$$H(j\omega) = \frac{1}{2 + j\omega(10 \times 10^{-9} \times 20 \times 10^3)} = \frac{1}{2 + j\omega(2 \times 10^{-4})}$$

$$= \frac{1}{2 + (2j\omega / 10^4)} = \frac{10^4}{2 \times 10^4 + 2j\omega}$$

$$\therefore \quad H(j\omega) = \frac{5000}{j\omega + 10000}$$

Magnitude, $|H(j\omega)| = \dfrac{5000}{\sqrt{\omega^2 + 10000}}$

Phase angle,

$$\angle H(jw) = \theta(\omega)$$

$$= -\tan^{-1}\left(\frac{\omega}{10000}\right)$$

At $\omega = 0$,

$$|H(j\omega)|_{\omega=0} = \frac{5000}{10000}$$

$$|H(j\omega)|_{\omega=0} = 0.5$$

But there is 10% variation in gain over the bandwidth B.

$$\therefore \quad |H(\omega)| = 0.5 - 0.5 \times 10\%$$

$$= 0.45$$

But, $$|H(j\omega)| = \frac{5000}{\sqrt{B^2 + 10^8}}$$ [$\because$ Bandwidth B = ω]

$$\Rightarrow \quad B^2 + 10^8 = \left(\frac{5000}{0.45}\right)^2$$

$$\Rightarrow \quad B^2 = 23.46 \times 10^6$$

$$\therefore \quad B = 4.84 \text{ kHz}$$

But, $$B = 2\pi f$$

$$f = \frac{B}{2\pi}$$

$$= \frac{4.84 \times 10^3}{2\pi}$$

$$f = 770.8 \text{ Hz}$$

$$f = \frac{B}{2\pi} = \frac{4.84 \times 10^3}{2\pi} = 770.8 \text{ Hz}$$

Phase at frequency,

$$f = 770.8 \text{ Hz is,}$$

$$\theta(\omega) = -\tan^{-1}\left(\frac{4.84}{10}\right) = -25.83\%$$ (7% of ideal value)

REVIEW QUESTIONS

(1) Explain about signal bandwidth?

(2) How a frequency band is selected in a linear system?

3.10 SYSTEM BANDWIDTH

To transmit a signal without any distortion, we need a system with infinite bandwidth. A system with infinite bandwidth is impossible to construct because of its physical limitations. Actually, a satisfactory distortionless transmission can be achieved by a system with finite, but fairly large bandwidths, if the magnitude $|H(\omega)|$ is constant over this band.

The band of frequencies, for which the magnitude $|H(\omega)|$ in a system is constant is usually specified by its bandwidth. System bandwidth is defined as the range of frequencies over which the magnitude $|H(\omega)|$ remains within $1/\sqrt{2}$ times (within 3 dB) of its value at midband.

The bandwidth of a system whose $|H(\omega)|$ plot is shown in Fig. 3.10.1 is $(\omega_2 - \omega_1)$ where ω_2 is called the upper cutoff frequency or upper 3 dB frequency or upper half power frequency and ω_1 is called the lower cutoff frequency or lower 3 dB frequency or lower half power frequency.

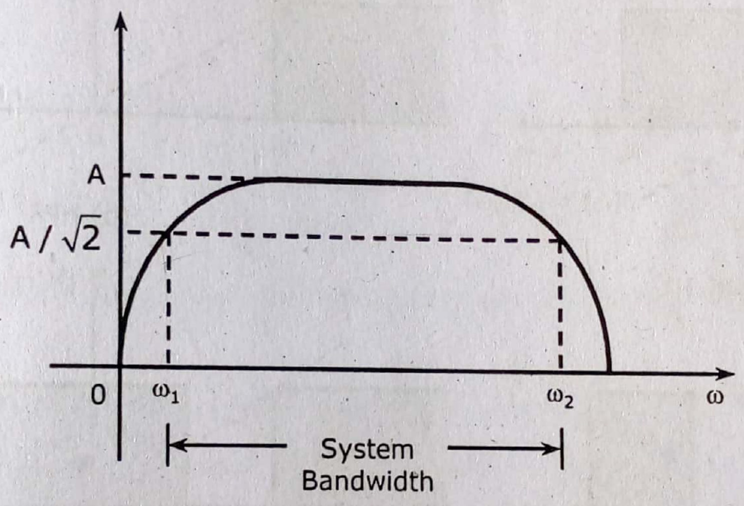


Fig. 3.10.1 System Bandwidth

The band limited signals can be transmitted without distortion, if the system bandwidth is atleast equal to the signal bandwidth.

REVIEW QUESTIONS

(1) *Define system bandwidth. How limited signals can be distorted without transmission?*

(2) *Is it possible to construct a system with infinite bandwidth? Why? Give a reason to your answer?*

(3) *What happens if the system bandwidth is equal to the signal bandwidth?*

3.11 IDEAL FILTER CHARACTERISTICS (LPF, HPF, BPF)

A filter is a frequency selective network which allows transmission of signals of certain frequencies without attenuation or with very little attenuation and it rejects or heavily attenuates signals of all other frequencies.

An ideal filter has very sharp cutoff characteristics and it passes signals of certain specified band of frequencies exactly and totally rejects signals of frequencies outside this band. Its phase spectrum is linear.

Based on the frequency response characteristics, filters are categorized as low-pass filter (LPF), high-pass filter (HPF), band-pass filter (BPF), band-elimination or band-stop or band-reject filter (BEF, BSF, BRF) and All Pass Filter (APF). Ideal versions of these filters are explained below and their magnitude responses are shown in Fig. 3.11.1.

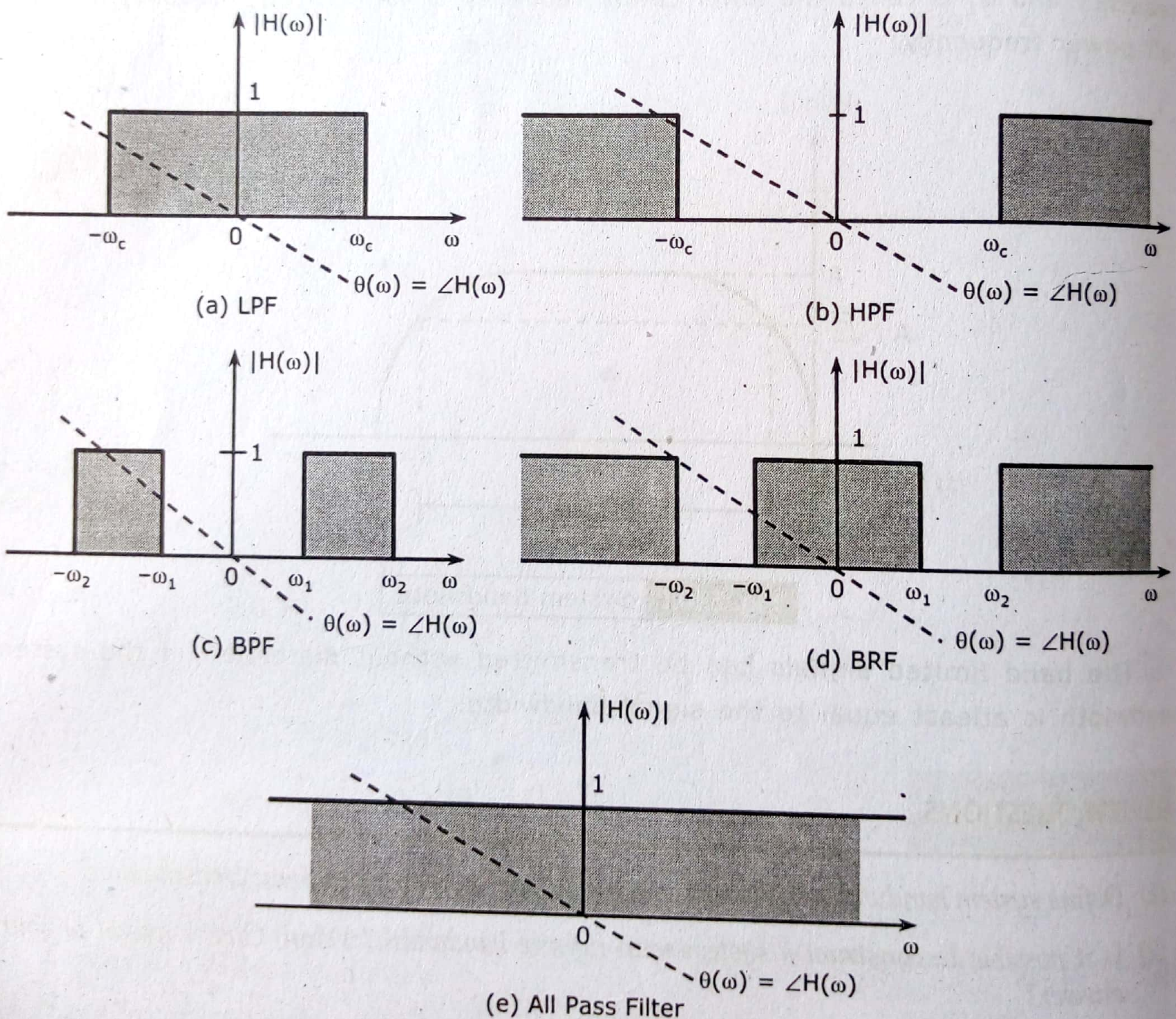


Fig. 3.11.1 Ideal Filters

(1) **Ideal LPF :** An ideal low-pass filter allows transmission of the signals of frequencies below a certain frequency ω_c radians per second without any distrotion. The signals of frequencies above ω_c radians/second are completely attenuated. Here ω_c is called the cutoff frequency. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal LPF has magnitude response given by,

$$|H(\omega)| = \begin{cases} 1 & ; \quad |\omega| < \omega_c \\ 0 & ; \quad |\omega| > \omega_c \end{cases}$$

The frequency response characteristics of an ideal LPF are shown in Fig. 3.11.1(a). It is a gate function.

(2) **Ideal HPF :** An ideal high-pass filter allows transmission of the signals of frequencies above a certain frequency ω_c radians/second without any distrotion and completely attenuates the signals of frequencies below ω_c radians/second. Here ω_c is called the cutoff frequency. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal HPF has magnitude response given by,

$$|H(\omega)| = \begin{cases} 0 & ; \quad |\omega| < \omega_c \\ 1 & ; \quad |\omega| > \omega_c \end{cases}$$

The frequency response characteristics of an ideal HPF are shown in Fig. 3.11.1(b).

(3) **Ideal BPF :** An ideal band-pass filter allows transmission of the signals of frequencies within a certain frequency band $(\omega_2 - \omega_1)$ radians/second without any distrotion and completely attenuates the signals of frequencies outside this band. Here $(\omega_2 - \omega_1)$ is the bandwidth of the band-pass filter. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal BPF has magnitude response by,

$$|H(\omega)| = \begin{cases} 1 & ; \quad |\omega_1| < \omega < |\omega_2| \\ 0 & ; \quad \omega < |\omega_1| \text{ and } \omega > |\omega_2| \end{cases}$$

The frequency response characteristics of an ideal BPF are shown in Fig. 3.11.1(c).

(4) **Ideal BRF :** An ideal band rejection filter rejects totally all of the signals of frequencies within a certain frequency band $(\omega_2 - \omega_1)$ radians/second and transmits all signals of frequencies outside this band without any distortion. Here $(\omega_2 - \omega_1)$ is the rejection band. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal BRF is specified by,

$$|H(\omega)| = \begin{cases} 0 & ; \quad |\omega_1| < \omega < |\omega_2| \\ 1 & ; \quad \omega < |\omega_1| \text{ and } \omega > |\omega_2| \end{cases}$$

The frequency response characteristics of an ideal BRF are shown in Fig. 3.11.1(d).

(5) **All Pass Filter :** An all pass filter transmits signals of all frequencies without any distortion, that is, its bandwidth is ∞ as shown in Fig. 3.11.1(e). The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal all pass filter is specified by,

$$|H(\omega)| = 1 \qquad \text{[For all frequencies]}$$

NOTE : All ideal filters are non-causal systems. Hence none of them is physically realizable.

REVIEW QUESTIONS

(1) What are the ideal characteristics of filters in a system?

(2) Explain ideal LPF and HPF characteristics of a system?

(3) By using a diagram explain all pass filter characteristics of a system?

3.12 CAUSALITY AND PALEY-WIENER CRITERION FOR PHYSICAL REALIZATION

A system is said to be causal if it does not produce an output before the input is applied. For an LTI system to be causal, the condition to be satisfied is its impulse response must be zero for t less than zero, i.e.,

$$h(t) = 0 \text{ for } t < 0$$

Physical realizability implies that it is physically possible to construct that system in real time. For a physically realizable system the unit impulse response h(t) must be causal i.e., $h(t) = 0$ for $t < 0$. This is the time domain criterion of physical realizability.

In the frequency domain, this condition is equivalent to the well-known Paley-Wiener criterion, which states that the necessary and sufficient condition for the magnitude response $|H(\omega)|$ to be realizable is,

$$\int_{-\infty}^{\infty} \frac{|\ln||H(\omega)||}{1+\omega^2} d\omega < \infty \qquad \text{... (3.12.1)}$$

If $H(\omega)$ does not satisfy this condition, it is unrealizable.

The magnitude function $|H(\omega)|$ of a realizable system can be zero at some discrete frequencies, but it cannot be zero over any finite band of frequencies. This means that if $|H(\omega)| = 0$ over any finite band, then $|\ln|H(\omega)|| = \infty$ over that band and consequently $H(\omega)$ is unrealizable.

If $|H(\omega)|$ decays exponentially (or at a higher rate) with ω, the integral in Eq. (3.12.1) goes to infinity and $|H(\omega)|$ cannot be realized. Clearly, $|H(\omega)|$ cannot decay too fast with ω.

REVIEW QUESTIONS

(1) *Explain causality and physical realization in a system?*

(2) *Define poly-wiener criterion? And explain its role in physical realization of a system?*

3.13 RELATIONSHIP BETWEEN BANDWIDTH AND RISE TIME

We know that the transfer function of an ideal LPF is given by,

$$H(\omega) = |H(\omega)|\, e^{-j\omega t_0}$$

Here,
$$|H(\omega)| = \begin{cases} 1 & ; \ |\omega| < \omega_c \\ 0 & ; \ |\omega| > \omega_c \end{cases}$$

Where, ω_c is called the cutoff frequency.

$$H(\omega) = e^{-j\omega t_0} \quad ; \ -\omega_c \le \omega \le \omega_c, \text{ i.e., } |\omega| \le \omega_c$$
$$= 0 \quad ; \ |\omega| > \omega_c$$

The impulse response h(t) of the LPF is obtained by taking the inverse Fourier transform of the transfer function $H(\omega)$.

$$\therefore \quad h(t) = F^{-1}[H(\omega)] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega t_0}\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_0)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(t-t_0)}}{j(t-t_0)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c(t-t_0)} - e^{-j\omega_c(t-t_0)}}{j(t-t_0)}\right]$$

$$= \frac{1}{\pi(t-t_0)}[\sin\omega_c(t-t_0)] \qquad \left[\because \sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}\right]$$

$$= \frac{\omega_c}{\pi}\left[\frac{\sin\omega_c(t-t_0)}{\omega_c(t-t_0)}\right]$$

The impulse response of the ideal LPF is shown in Fig. 3.13.1. The impulse response has a peak value at $t = t_0$. This value ω_c/π is proportional to cutoff frequency ω_c. The width of the main lobe is $2\pi/\omega_c$. As $\omega_c \to \infty$, the LPF becomes an all pass filter. As $t_0 \to 0$, the output response peak $\to \infty$, that is, the output response approaches input.

Further, the impulse response h(t) is non-zero for $t < 0$, even through the input $\delta(t)$ is applied at $t = 0$. That is, the impulse response begins before the input is applied. In real life, no system exhibits such type of characteristics. Hence we can conclude that ideal LPF is physically not realizable.

If the impulse response is known, the step response can be obtained by convolution.

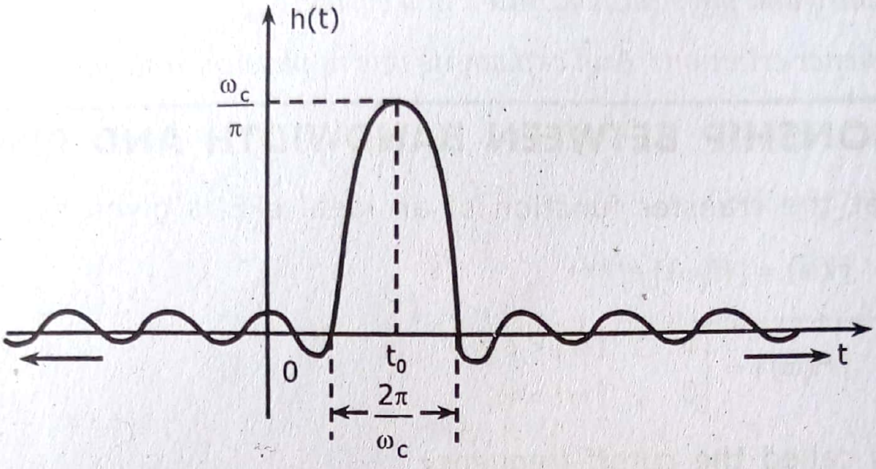


Fig. 3.13.1 Impulse Response of the Ideal LPF

The step response,

$$y(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau)\, d\tau \qquad \left[\because u(t-\tau) = 1 \ ; \quad t \geq \tau \atop \qquad\quad = 0 \ ; \quad t < \tau\right]$$

We have,

$$h(t) = \frac{\omega_c}{\pi}\left[\frac{\sin\omega_c(t - t_0)}{\omega_c(t - t_0)}\right]$$

$$\Rightarrow \qquad h(\tau) = \frac{\omega_c}{\pi}\left[\frac{\sin\omega_c(\tau - t_0)}{\omega_c(\tau - t_0)}\right]$$

$$\therefore \qquad y(t) = \int_{-\infty}^{t} \frac{\omega_c}{\pi}\left[\frac{\sin\omega_c(\tau - t_0)}{\omega_c(\tau - t_0)}\right] d\tau$$

Let, $\qquad x = \omega_c(\tau - t_0)$

$\therefore \qquad dx = \omega_c\, d\tau$

(or) $\qquad d\tau = \frac{dx}{\omega_c}$

$x \to -\infty$ as $\tau \to -\infty$ and $x \to \omega_c(t - t_0)$ as $\tau \to t$.

$$\therefore \quad y(t) = \int_{-\infty}^{\omega_c(t-t_0)} \frac{\omega_c}{\pi}\frac{\sin x}{x}\frac{dx}{\omega_c} = \frac{1}{\pi}\int_{-\infty}^{\omega_c(t-t_0)} \frac{\sin x}{x}dx$$

$$= \frac{1}{\pi}\left[Si(x)\right]_{-\infty}^{\omega_c(t-t_0)}$$

Where Si is the sine integral function.

The properties of sine integral functions are,

(1) $Si(x)$ is an odd function, that is $Si(-x) = -Si(x)$.

(2) $Si(0) = 0$

(3) $Si(\infty) = \pi/2$ and $Si(-\infty) = -(\pi/2)$

A sketch of $Si(x)$ is shown in Fig. 3.13.2(a).

The step response can be expressed as,

$$y(t) = \frac{1}{\pi}[Si[\omega_c(t-t_0)] - Si(-\infty)]$$

$$= \frac{1}{\pi}\left[Si[\omega_c(t-t_0)] + \frac{\pi}{2}\right] \qquad \left[\because Si(-\infty) = -\frac{\pi}{2}\right]$$

$$= \frac{1}{2} + \frac{\pi}{2}Si[\omega_c(t-t_0)]$$

If $\omega_c \to \infty$, then the response is,

$$y(t) = \frac{1}{2} + \frac{\pi}{2}Si(\infty) = \frac{1}{2} + \frac{1}{\pi}\left(\frac{\pi}{2}\right) = 1$$

If $\omega_c \to -\infty$, then the response is,

$$y(t) = \frac{1}{2} + \frac{\pi}{2}Si(-\infty) = \frac{1}{2} + \frac{1}{\pi}\left(-\frac{\pi}{2}\right) = 0$$

The step response of ideal LPF is shown in Fig. 3.13.2(b).

From Fig. 3.13.2(b), we can observe that y(t) approaches a delayed unit step $u(t - t_0)$. But the abrupt rise of input corresponds to more gradual rise of the output.

The rise time t_r is defined as the time required for the response to reach from 0% to 100% of the final value. To find it, draw a tangent at $t = t_0$ with the line $y(t) = 0$ and $y(t) = 1$. From Fig. 3.13.2(b), we have,

$$\left.\frac{dy(t)}{dt}\right|_{t=t_0} = \frac{1}{t_r} = \frac{\omega_c}{\pi}\frac{d}{dt}\left.\left(\frac{\sin\omega_c(t-t_0)}{\omega_c(t-t_0)}\right)\right|_{t=t_0} = \frac{\omega_c}{\pi}$$

$$\therefore \quad t_r = \frac{\pi}{\omega_c}$$

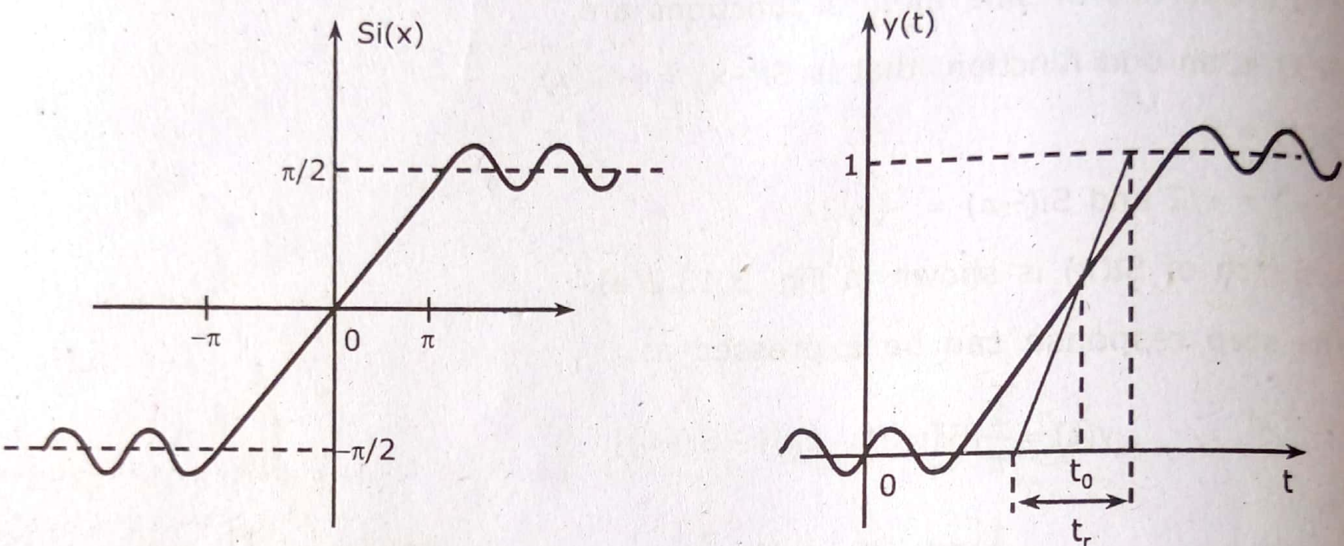
For a low-pass filter,

Cut off frequency = Bandwidth of a system.

So the rise time is inversely proportional to the bandwidth.

Bandwidth × Rise time = Constant

i.e., $\omega_c \times t_r = \pi$ (constant)



(a) Si Function

(b) Step Response of an Ideal LPF

Fig. 3.13.2

REVIEW QUESTIONS

(1) *Derive a relationship between bandwidth and rise time?*

(2) *How the system bandwidth is proportional to rise time?*

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